High order hydrodynamical equations

F. Sala, J. Torra and P. Cortés
Departament de Física de la Terra i del Cosmos, Universitat de Barcelona,
Av. Diagonal, 645, 08028 Barcelona, Spain

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Summary. — Hydrodynamical equations act as a link between the local observed magnitudes of galactic motion and the general ones accounting for the behaviour of the Galaxy as a whole. Constraints are set usually in order to use them even in the lower order hierarchy. We present in this paper the complete expressions up to their fourth order. These equations will be used in the next future in their general form taking into account both the expected increase of kinematic data that the astrometric mission Hipparcos will provide, and some recent results indicating the possibility to obtain estimates for the momenta gradients.

Key words: galactic dynamics — statistical astronomy.

1. Introduction.

The hydrodynamical equations obtained by taking momenta of the collisionless Boltzmann equation establish relations among the stellar density derivatives, the velocity of the local standard of rest, their derivatives, the galactic force field, the momenta of the residual velocity distribution and their derivatives.

To our knowledge only Eelsalu (1979, 1982) has published incomplete fourth order equations for an axi-symmetric system, being the usual procedure to consider suitable hypothesis to reduce the number of variables (Turon, 1971; Vandervoort, 1975; Núñez, 1982). It is also possible to introduce new variables, e.g. King (1967) pseudomomenta, in which the low order equations are separable. Hydrodynamical equations have been also used when considering dynamical problems in external galaxies, e.g. Sargent et al. (1977), and also in the basis of the development of the Lin theory (Rohlfs, 1976).

On the other hand, Erickson (1975) has shown the possibility of determining with reasonable accuracy the central momenta up to the 4th order of the velocity distribution. Other determinations have been done by Núñez and Torra (1982). Results obtained by Torra (1984) using a rough estimation of the magnitudes involved from a compiled catalogue of kinematic data confirm the presence of local irregularities, also determined by other authors (Menge da Freitas, 1980, 1982; Oblak, 1983) and ratify that at least some of the terms involving momenta gradients cannot be considered null. As stated by Crézé (1978), one of the goals of the Hipparcos mission will be the determination of the momenta gradients.

Then, it will be possible to work with improved observational data in the next future using the hydrodynamical equations in their complete form, as well as they can be used in order to build up theoretical models either in their complete form or assuming hypotheses that make null some of the terms involved.

2. Derivation of the hydrodynamical equations.

As it is well known, stellar systems can be described by a distribution function \( f(t, \mathbf{r}, \mathbf{V}) \), or density in the phase space which gives the number of stars with positions in the range \((\mathbf{r}, \mathbf{r} + d\mathbf{r})\), and velocities in the range \((\mathbf{V}, \mathbf{V} + d\mathbf{V})\) at a certain time \(t\). Applying Liouville’s theorem to this stellar system we can write the so called fundamental equation of the stellar dynamics:

\[
\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \mathbf{V} \cdot \nabla f - \mathbf{V} \cdot \nabla u = 0
\]

where \(U(t, \mathbf{r})\) is the potential that governs the system, and where we have neglected the effect of stellar encounters which can be included as a perturbational term in the right hand side of the fundamental equation.

If we use cartesian coordinates:

\[
\mathbf{r} = (x, y, z) \quad \mathbf{V} = (X, Y, Z)
\]

the expression of the fundamental equation is:

\[
\frac{\partial f}{\partial t} + X \frac{\partial f}{\partial x} + Y \frac{\partial f}{\partial y} + Z \frac{\partial f}{\partial z} - \frac{\partial U}{\partial x} \frac{\partial f}{\partial x} - \frac{\partial U}{\partial y} \frac{\partial f}{\partial y} - \frac{\partial U}{\partial z} \frac{\partial f}{\partial z} = 0. \tag{1}
\]

Send offprint requests to: F. Sala.
The hydrodynamical equations of stellar dynamics are obtained when performing the different momenta

$$\int_{v} V^n \frac{Df}{Dt} dV = 0 \quad (2)$$

of the fundamental equation (1), assuming that \( f \) and its products by \( V^n \) vanish in the boundary of the integration domain.

Let us consider the velocity of the local standard of rest \( V_0 = (X_o, Y_o, Z_o) \) defined by

$$V_0(t, r) = N(t, r) \int_{v} V f(t, r, V) dV$$

being

$$N(t, r) = \int_{v} f(t, r, V) dV$$

the stellar density.

We introduce now

$$A_{pqr} = \int_{v} X^p Y^q Z^r f dV$$

and express it and the distribution function in terms of the residual velocity :

$$v = V - V_0.$$ 

After some algebra we can write

$$A_{pqr} = \sum_{a=0}^{p} \sum_{b=0}^{q} \sum_{c=0}^{r} \binom{p}{a} \binom{q}{b} \binom{r}{c} \times X_0^{p-a} Y_0^{q-b} Z_0^{r-c} P_{abc} \quad (3)$$

where

$$P_{abc} = \int_{v} (X - X_o)^a (Y - Y_o)^b (Z - Z_o)^c f dv$$

is the so called pressure, related to the corresponding central momentum \( \mu_{abc} \) of the residual velocities distribution function by :

$$P_{abc} = N(t, r) \mu_{abc}.$$ 

Substituting (1) in (2) and taking into account the expression for \( A_{pqr} \) we obtain for the \( n = p + q + r \) order equation :

$$\frac{\partial A_{pqr}}{\partial t} + \frac{\partial A_{(p+1)qr}}{\partial x} + \frac{\partial A_{pq(r+1)}}{\partial y} + \frac{\partial A_{pq(r+1)}}{\partial z} + p \frac{\partial U}{\partial x} A_{(p-1)qr} + q \frac{\partial U}{\partial y} A_{pq(q-1)r} + r \frac{\partial U}{\partial z} A_{pq(r-1)} = 0 \quad (5)$$

If, instead of cartesian, cylindrical coordinates

$$r = (\omega, \theta, z) \quad V = (\Pi, \Theta, Z)$$

are used the fundamental equation is :

$$\frac{\partial f}{\partial t} + \Pi \frac{\partial f}{\partial \omega} + \Theta \frac{\partial f}{\partial \theta} + Z \frac{\partial f}{\partial z} - \left( \frac{\partial U}{\partial \omega} - \Theta \frac{\partial U}{\partial \theta} \right) \frac{\partial f}{\partial \Pi} - \left( \frac{1}{\omega} \frac{\partial U}{\partial \theta} + \frac{\Pi \Theta}{\omega} \right) \frac{\partial f}{\partial \Theta} - \frac{\partial U}{\partial z} \frac{\partial f}{\partial Z} = 0.$$ 

Then, the general expression for the \( n \)-order equation is :

$$\frac{\partial A_{pqr}}{\partial t} + \frac{\partial A_{(p+1)qr}}{\partial x} + \frac{1}{\omega} \frac{\partial A_{pq(q+1)r}}{\partial \theta} + \frac{\partial A_{pq(r+1)}}{\partial z} + p \frac{\partial U}{\partial \omega} A_{(p-1)qr} + q \frac{\partial U}{\partial \theta} A_{pq(q-1)r} + r \frac{\partial U}{\partial z} A_{pq(r-1)} - \frac{p}{\omega} A_{(p-1)q+2} + \frac{q+1}{\omega} A_{p+1} = 0 \quad (6)$$

being now :

$$A_{pqr} = \sum_{a=0}^{p} \sum_{b=0}^{q} \sum_{c=0}^{r} \binom{p}{a} \binom{q}{b} \binom{r}{c} \Pi_0^{p-a} \Theta_0^{q-b} Z_0^{r-c} P_{abc}$$

where

$$P_{abc} = \int_{V} (\Pi - \Pi_o)^a (\Theta - \Theta_o)^b (Z - Z_o)^c f dv$$

being the central momentum \( \mu_{abc} \) defined also by (4).

General expressions (5) and (6) allow us to extend the hierarchy of hydrodynamical equations to the \( n \)-order. Taking into account (4) we can express these equations in terms of central momenta of the residual velocities distribution. In Appendices A and B we present the extended form of these equations up to the fourth order in both
cartesian and cylindrical coordinates respectively. These extended equations up to the fourth order are also available, on request, in terms of pressures.

3. Concluding remarks.

When deriving from (5) and (6) the equations of the Appendices A and B there is the possibility to express the resulting 3rd and 4th order equations including the partial derivatives of the potential, or substitute them taking into account the 2nd order equations.

The system formed by the hydrodynamical equations up to the \(n\)-th order is constituted by:

\[
\sum_{i=0}^{n} \binom{i + 2}{2} = \binom{n + 3}{3}
\]

equations, being the number of components of the velocity of the local standard of rest and momenta of order higher than 2

\[
\sum_{i=0}^{n} \binom{i + 3}{2} = \binom{n + 4}{3}
\]

These \(\binom{n + 4}{3}\) magnitudes in addition to density and potential appear also derived with respect to time and position.

So, the system is far away to be closed by increasing the number of equations. Difference between magnitudes and equations is

\[
2 + \binom{n + 4}{3} - \binom{n + 3}{3} = 2 + \binom{n + 3}{2}.
\]

Any observational work that determines enough magnitudes can allow us to close the system and determine the rest of them by solving it. For instance, if estimates of high order momenta and their gradients are obtained, and assuming the values for the density gradient, the system becomes linear and can be treated by a least squares method to obtain the components of the galactocentric velocity of the local standard of rest and the components of the galactic force.

Another possibility is, as usual, to consider a set of simplifying hypotheses that enable to solve the differential system and to determine the form of the unknown functions involved. Both possibilities will be used in forthcoming papers.

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References

\[ \frac{\partial \psi_{103}}{\partial t} + \frac{\partial \psi_{103}}{\partial \theta} + \frac{\partial \psi_{103}}{\partial \phi} + \frac{\partial Z_0}{\partial z} - 3 \frac{\psi_{102}}{\partial \theta} \frac{\psi_{101}}{\partial \phi} + \frac{1}{\partial \psi_{101}} + \frac{1}{\partial \psi_{102}} + \frac{1}{\partial \psi_{103}} \]

\[ + \frac{\partial \psi_{101}}{\partial \theta} \frac{\partial \psi_{110}}{\partial \phi} + \frac{1}{\partial \psi_{110}} + \frac{\psi_{101}}{\partial \phi} + \frac{\psi_{100} \psi_{102}}{\partial \phi} + 2 \frac{\partial Z_0}{\partial \phi} + \frac{3}{\partial \psi_{102}} \]

\[ + \frac{3}{\partial \psi_{112}} \frac{\partial \psi_{110}}{\partial \phi} + \frac{\partial \psi_{110}}{\partial \phi} + \frac{\partial Z_0}{\partial \phi} + \frac{\partial \psi_{111}}{\partial \phi} \]

\[ + \frac{\partial \psi_{111}}{\partial \phi} + \frac{\partial \psi_{103}}{\partial \phi} + \frac{2 \partial \psi_0}{\partial \phi} + \frac{\partial \psi_{104}}{\partial \phi} \]

\[ + \frac{\partial \psi_{031}}{\partial \theta} \frac{\partial \psi_{021}}{\partial \phi} + \frac{1}{\partial \psi_{021}} + \frac{1}{\partial \psi_{031}} + \frac{1}{\partial \psi_{041}} + \frac{1}{\partial \psi_{032}} + \frac{4}{\partial \psi_{131}} = 0 \]

\[ \frac{\partial \psi_{031}}{\partial \theta} \frac{\partial \psi_{021}}{\partial \phi} + \frac{1}{\partial \psi_{021}} + \frac{1}{\partial \psi_{031}} + \frac{1}{\partial \psi_{041}} + \frac{1}{\partial \psi_{032}} + \frac{4}{\partial \psi_{131}} = 0 \]