PHYSICAL REVIEW A VOLUME 33, NUMBER 4 APRIL 1986

Spectrum of transmitted light in optical bistability: Effects of phase fluctuations of the driving laser

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(Received 24 June 1985)

Time-dependent correlation functions and the spectrum of the transmitted light are calculated for absorptive optical bistability taking into account phase fluctuations of the driving laser. These fluctuations are modeled by an extended phase-diffusion model which introduces non-Markovian effects. The spectrum is obtained as a superposition of Lorentzians. It shows qualitative differences with respect to the usual calculation in which phase fluctuations of the driving laser are neglected.

I. INTRODUCTION

Optical bistability[1] can be considered a prototype of many nonequilibrium phenomena. Models of optically bistable devices are good candidates to demonstrate general properties and problems associated with the description of systems driven away from equilibrium. One such question is the role played by fluctuations in the behavior of a nonequilibrium system. A stochastic theory of optical bistability taking into account fluctuations associated with the spontaneous-emission process (quantum noise) was developed some years ago.[1,2] However, in practical situations, quantum and thermal noise are not the dominant types of noise. Parametric or external sources of noise, such as fluctuations in the laser driving the optically bistable device, are likely to be the major causes of noise in the system. This point of view is clearly discussed in a recent paper by Lugiato and Horowickz[3] in which the possible different sources of parametric noise are analyzed. In this sense, models of optical bistability are also of great interest for the study of the effect of external noise in nonequilibrium systems.[4] Partial aspects of the parametric noise problem in optical bistability have been addressed by several authors.[5-11]

In this paper we wish to discuss the effect of the phase fluctuations of the driving laser on the spectrum of the transmitted light. Our calculations are carried out in the good-cavity limit for absorptive optical bistability. The spectrum of the transmitted light when only quantum noise is considered was discussed in detail by Bonifacio and Lugiato.[12] In the same way as in that classical analysis, we do not consider here tunneling effects that occur for long times in the bistable domain. We calculate the correlation functions of the transmitted light in the cooperative branch and in the one-atom branch but for times smaller than the escape time from one branch to the other. However, we have to go beyond the linear analysis to find amplitude fluctuations of the transmitted light induced by the phase fluctuations of the driving laser. Our calculations are based on a phenomenological model which neglects all possible sources of noise except those associated with a finite phase linewidth of the driving laser. These phase fluctuations are taken to be the dominant ones.[8,9] They are modeled by an extended phase-diffusion model[13,14] in which a random frequency is modeled by a Gaussian noise with a finite correlation time. As a consequence we address the problem of calculating correlation functions of a non-Markovian process driven by colored noise.[15-19] The stationary distribution in amplitude and phase of our stochastic model is not known. We calculate the correlation functions by a perturbative expansion in a small parameter which measures the strength of fluctuations. The problem of calculating time-dependent correlation functions of the transmitted light in the presence of a fluctuating driving laser has also been studied by Willis,[9,10] who precisely considered a finite phase linewidth of the driving laser. Other studies which take into account phase fluctuations of the driving laser such as those of Refs. 3 and 7 do not address this problem.[21] Our work has two important differences with respect to that of Willis.[9,10] First, our analysis is not a linear theory. Second, we allow for a nonwhite frequency noise. This considerably complicates the dynamical problem.

It follows from our calculations that the spectrum of the transmitted light shows important qualitative differences with respect to the spectrum of an ideal situation in which only quantum or thermal noise is considered. The spectrum is dominated by phase fluctuations and the correlation time of the frequency fluctuations of the driving laser plays an important role in those phase fluctuations. Amplitude fluctuations appear only as a second-order effect, but they also show qualitative differences with respect to an ideal situation.

The stochastic model used in our calculations is presented and discussed in Sec. II. Section III contains the perturbative calculations of the correlation functions and spectrum. A short discussion and summary of results are given in Sec. IV. Some mathematical details are given in the Appendix.

II. A MODEL WITH FINITE PHASE LINEWIDTH IN THE DRIVING LASER

Our starting point is the Maxwell-Bloch equations for optical bistability in the rotating-wave approximation and in the mean-field limit as first proposed by Bonifacio and Lugiato.[22] We also take the good-cavity limit in which the cavity damping constant κ is much smaller than the decay constant for the polarization $\gamma_\perp$ and population difference $\gamma_{\parallel}$, so that these two variables follow instantaneously the evolution of the electric field. In this limit the equation for the normalized transmitted field E becomes[2,22]

$$\dot{E}^{\pm}=(\mp i\delta-1)E^{\pm}-\frac{\Gamma^2(1\pm i\Delta)E^{\pm}}{1+|E|^2}+\overline{E}^{\pm}\ , \qquad (2.1)$$

where $\overline{E}$ is the normalized incident field and $\Gamma^2=-g^2N/\kappa\gamma_\perp(1+\Delta^2)$. g is the coupling constant proportional to the dipole moment; N the number of atoms in the sample; $\Delta=(\omega-\nu)/\gamma_\perp$, where ω is the frequency of the driving laser; ν the frequency of the two-level atoms in the cavity; and $\delta=(\omega_0-\omega)/\kappa$, where ω_0 is the cavity frequency. Equation (2.1) is written in a dimensionless time measured in units of κ.

Introducing the amplitude and phase of the fields $E,\overline{E}$ by

$$E^{\pm}=r\exp(\pm i\phi),\quad \overline{E}^{\pm}=\overline{r}\exp(\pm i\overline{\phi})\ , \qquad (2.2)$$

Eq. (2.1) can be written as a set of coupled equations for r and ϕ:

$$\dot{r}=-r-\frac{\Gamma^2 r}{1+r^2}+\overline{r}\cos(\phi-\overline{\phi})\ , \qquad (2.3)$$

$$\dot{\phi}=-\delta-\frac{\Gamma^2\Delta}{1+r^2}-\frac{\overline{r}}{r}\sin(\phi-\overline{\phi})\ . \qquad (2.4)$$

Equations (2.3) and (2.4) give a deterministic description of the system. The mean-field equation of state is given by the stationary solution of (2.3) and (2.4). In the particular case of absorptive optical bistability in which $\Delta=\delta=0$, the equation of state becomes

$$\phi=\overline{\phi},\quad r=\overline{r}-\frac{\Gamma^2 r}{1+r^2}\ . \qquad (2.5)$$

Bistability occurs for $\Gamma^2>8$. In this domain of parameters and for $\overline{r}_m<\overline{r}<\overline{r}_M$, (2.5) has three solutions for r corresponding to two locally stable states and one unstable state. The spinodal points r_m and r_M are, respectively, the terminal points of the one-atom and cooperative branches.

Equations (2.3) and (2.4) are usually supplemented with random terms modeling different sources of noise. Quantum fluctuations and thermal noise are associated with internal fluctuations of the system. Here, we concentrate on the effect on the transmitted light of the fluctuations present in the incident laser. The incident laser is ordinarily included in the theory as a constant parameter. In this sense our analysis corresponds to a parametric or external-noise problem in which that parameter is replaced by a stochastic process. Amplitude fluctuations of a standard laser are very small except close to the spinodal points, while phase fluctuations are usually the major cause of a finite linewidth of the laser. We consider only such phase fluctuations of the incident field $\overline{E}$.[9] To this end we rewrite (2.3) and (2.4) in terms of a relative phase variable $\Omega=\phi-\overline{\phi}$:

$$\dot{r}=-r-\frac{\Gamma^2 r}{1+r^2}+\overline{r}\cos\Omega\ , \qquad (2.6)$$

$$\dot{\Omega}=-\delta-\frac{\Gamma^2\Delta}{1+r^2}-\frac{\overline{r}}{r}\sin\Omega-\dot{\overline{\phi}}\ . \qquad (2.7)$$

Phase fluctuations of the incident laser are now introduced considering its frequency $\dot{\overline{\phi}}(t)$ as a stochastic quantity. We set

$$\dot{\overline{\phi}}(t)=\epsilon^{1/2}\dot{\theta}(t)\ , \qquad (2.8)$$

where ϵ measures the strength of the fluctuations. In this paper we will only consider the case of absorptive bistability. However, the consideration of frequency fluctuations implies that the parameters δ and Δ in (2.7) are not strictly zero but rather fluctuating quantities around a mean value zero. The fluctuations of δ simply add to those of $\dot{\overline{\phi}}$. We consider them to be included in the value of ϵ. Fluctuations of Δ in (2.7) caused by frequency fluctuations lead, in time measured in units of κ, to terms proportional to $\kappa/\gamma_\perp$. These terms are neglected in the good-cavity limit considered here. In conclusion, our final stochastic model for absorptive bistability is given by (2.6) and

$$\dot{\Omega}=-\frac{\overline{r}}{r}\sin\Omega-\epsilon^{1/2}\dot{\theta}(t)\ , \qquad (2.9)$$

where $\Gamma^2=-g^2N/\kappa\gamma_\perp$. A complete definition of this model requires that one specify the properties of the random term $\dot{\theta}(t)$.

A standard model of phase fluctuations is the phase-diffusion model (PDM) in which the frequency $\dot{\theta}(t)$ is modeled by a Gaussian white noise so that $\theta(t)$ is a diffusion (Wiener) process. Such a model neglects the finite correlation time of the frequency fluctuations. To take into account this effect we use an extended PDM (Refs. 13 and 14) in which $\dot{\theta}(t)$ is a Gaussian process of zero mean and correlation

$$\langle\dot{\theta}(t)\dot{\theta}(t')\rangle=\tau^{-1}\exp\left[-\frac{|t-t'|}{\tau}\right]\ . \qquad (2.10)$$

The correlation of the driving field is then given by

$$\langle\overline{E}^{+}(t+s)\overline{E}^{-}(t)\rangle =\overline{r}^{\,2}\exp(-\epsilon\{\,|s|+[\exp(-s/\tau)-1]\tau\})\ . \qquad (2.11)$$

In the limit in which the correlation time $\tau\to 0$, we recover the ordinary PDM in which the spectrum of $\overline{E}$ is a Lorentzian of half-width ϵ. For $\epsilon\tau\ll 1$ the spectrum associated with (2.11) is still a Lorentzian of half-width ϵ but with a frequency cutoff at τ^{-1}. The PDM has been justified[13] as arising from spontaneous-emission fluctuations. However, the main contribution to the linewidth of a real laser seems to be due to the jitter of the resonator. We take here (2.11) as an appropriate model for the phase fluctuations of the driving laser.[23] The value of the parameter ϵ is the appropriate one for a real laser and it is several orders of magnitude larger than the noise strength associated with spontaneous-emission fluctuations. On the other hand, we do not consider here the phase fluctuations of the field E due to the jitter of the cavity pumped by $\overline{E}$.

The two noise parameters left in our theory are ϵ and τ. We recall that (2.6) and (2.9) are equations for dimensionless variables, so that if D is the linewidth of the driving laser, the parameter ϵ relates a decay rate of the driving laser and the cavity lifetime:

$$\epsilon=\frac{D}{\kappa} . \tag{2.12}$$

A typical lower value of D for a usual laser is $D\sim 10$ kHz and $\kappa\sim 10^6-10^7$ Hz^{-1} so that $\epsilon\sim 10^{-2}-10^{-3}$. The parameter τ is given in terms of the correlation time in ordinary units τ_0 by $\tau=\kappa\tau_0$. An important point should be noted in connection with (2.6), (2.9), and (2.10). Equations (2.6) and (2.9) are based on the adiabatic elimination of the polarization and population difference variables. The consistent consideration of a nonvanishing value of τ in (2.10) requires that the eliminated variables be faster than the phase fluctuations. This condition is met when $\gamma_{\parallel}^{-1},\gamma_{\perp}^{-1}\ll\tau_0$ which is usually satisfied.

A main difficulty of the model defined by (2.6), (2.9), and (2.10) is that the stationary distribution cannot be easily calculated because detailed balance is not satisfied. A desirable possibility is to reduce the problem to a one-variable model. In the domain of $\overline{r}$ for which bistability occurs, the long-time dynamics is dominated by tunneling effects. In this long-time scale a separation of time scales is possibly feasible because the amplitude r undergoes an activated process and therefore has a final slow evolution. However, linearizing the deterministic limit of (2.6)–(2.9) around a solution r_0 of the mean-field equation of state (2.5),

$$\delta\dot{r}=-\lambda_r\delta r,\quad \dot{\phi}=-\lambda_\phi\phi , \tag{2.13}$$

where $\delta r=r-r_0$, and the relaxation rates λ_r and λ_ϕ,

$$\lambda_r=\left.\frac{d\overline{r}}{dr}\right|_{r=r_0}=\left[1+\frac{\Gamma^2(1-r_0^2)}{(1+r_0^2)^2}\right],$$
$$\lambda_\phi=\frac{\overline{r}}{r_0}=1+\frac{\Gamma^2}{1+r_0^2} , \tag{2.14}$$

are of the same order of magnitude. As a consequence, outside the domain of bistability and also inside it but in the time domain in which tunneling processes are still very rare, a separation of fast and slow variables is not possible. We are precisely interested in this time domain in which metastable states can be considered stable. Our aim is to calculate the fluctuations of the transmitted light caused by the driving-laser phase fluctuations. Since no reduction of the problem seems easy to achieve, our strategy is based on the possibility of a perturbative expansion in the small parameter ϵ. A perturbative calculation of the spectrum of the transmitted light is presented in Sec. III.

III. CORRELATION FUNCTIONS AND SPECTRUM OF TRANSMITTED LIGHT

A. Calculational scheme

We wish to calculate the fluctuations of the transmitted light around a stable or metastable state. These are given by the locally stable solutions of (2.5). To this end we have to confront two basic difficulties. The first one is the nonlinearity of the equations and the second one is its non-Markovian character due to the nonvanishing value of the parameter τ. One may think that fluctuations around a locally stable state are well described by a linearized analysis. However, as shown below in a linear analysis of our model, the equations for amplitude and phase decouple and the amplitude becomes a nonfluctuating quantity. Nonlinearities are here taken into account by a perturbative calculation to second order in the small parameter ϵ. The additional difficulty associated with non-Markovian features is treated by following ideas already developed and applied in Refs. 15–18. Essentially, this amounts again to an expansion in ϵ and, when necessary, in the parameter τ.

Our calculation is based on the expansion

$$r(t)=r_0+\epsilon^{1/2}r_1(t)+\epsilon r_2(t)+\epsilon^{3/2}r_3(t)+\epsilon^2 r_4(t)+\cdots , \tag{3.1}$$

$$\Omega(t)=\Omega_0+\epsilon^{1/2}\Omega_1(t)+\epsilon\Omega_2(t)+\epsilon^{3/2}\Omega_3(t)+\epsilon^2\Omega_4(t)+\cdots . \tag{3.2}$$

$r_i(t)$ and $\Omega_i(t)$ satisfy stochastic differential equations obtained by substituting (3.1) and (3.2) in (2.6) and (2.9) and collecting terms of the same order in ϵ. The point r_0,Ω_0 which we expand around is a stationary solution of the deterministic limit of (2.6) and (2.9). The relative phase is $\Omega_0=0$ which corresponds to a phase locking between the transmitted and driving lasers in the limit of vanishing linewidth of the driving laser. The amplitude r_0 is a solution of (2.5) which we take as a locally stable one. We find to order $\epsilon^{1/2}$

$$\dot{r}_1=-\lambda_r r_1 , \tag{3.3}$$

$$\dot{\Omega}_1=-\lambda_\phi\Omega_1-\dot{\theta} ; \tag{3.4}$$

to order ϵ

$$\dot{r}_2=-\lambda_r r_2+R_1 r_1^2-\frac{\overline{r}}{2}\Omega_1^2 , \tag{3.5}$$

$$\dot{\Omega}_2=-\lambda_\phi\Omega_2+\frac{\lambda_\phi^2}{\overline{r}}r_1\Omega_1 ; \tag{3.6}$$

to order $\epsilon^{3/2}$

$$\dot{r}_3=-\lambda_r r_3+2R_1 r_1 r_2+R_2 r_1^3-\overline{r}\Omega_1\Omega_2 , \tag{3.7}$$

$$\dot{\Omega}_3=-\lambda_\phi\Omega_3+\frac{\lambda_\phi^2}{\overline{r}}(r_1\Omega_2+r_2\Omega_1)-\frac{\lambda_\phi^3}{\overline{r}^2}r_1^2\Omega_1+\frac{\lambda_\phi}{6}\Omega_1^3 ; \tag{3.8}$$

and to order ϵ^2

$$\dot{r}_4 = -\lambda_r r_4 + 2R_1 r_1 r_3 + R_1 r_2^2 + 3R_2 r_1^2 r_2 + R_3 r_1^4 - \frac{\bar{r}}{2}(2\Omega_1\Omega_3 + \Omega_2^2 - \tfrac{1}{12}\Omega_1^4) , \tag{3.9}$$

$$\dot{\Omega}_4 = -\lambda_\phi \Omega_4 + \frac{\lambda_\phi}{2}\Omega_1^2\Omega_2 + \frac{\lambda_\phi^2}{\bar{r}}(r_1\Omega_3 + r_2\Omega_2 + r_3\Omega_1 - \tfrac{1}{6}r_1\Omega_1^3) - \frac{\lambda_\phi^3}{\bar{r}^2}(r_1^2\Omega_2 + 2r_1r_2\Omega_1) + \frac{\lambda_\phi^4}{\bar{r}^3}r_1^3\Omega_1 ; \tag{3.10}$$

where λ_r and λ_ϕ are given in (2.14) and the constants R_1, R_2, and R_3 are defined as

$$R_1 = \frac{\Gamma^2 r_0(3 - r_0^2)}{(1 + r_0^2)^3} , \tag{3.11}$$

$$R_2 = \frac{\Gamma^2[1 + r_0^2(r_0^2 - 6)]}{(1 + r_0^2)^4} , \tag{3.12}$$

$$R_3 = \frac{\Gamma^2 r_0[r_0^2(10 - r_0^2) - 5]}{(1 + r_0^2)^5} . \tag{3.13}$$

Introducing a new variable $\Delta r = r - r_0$, the equations for Δr and Ω to order ϵ^2 obtained from (3.3)–(3.10), or directly expanding (2.6)–(2.9), are

$$\Delta\dot{r} = -\lambda_r \Delta r + R_1 \Delta r^2 + R_2 \Delta r^3 + R_3 \Delta r^4 - \frac{\bar{r}}{2}\Omega^2 + \frac{\bar{r}}{24}\Omega^4 , \tag{3.14}$$

$$\dot{\Omega} = -\lambda_\phi \Omega + \frac{\lambda_\phi^2}{\bar{r}}\Delta r\Omega - \frac{\lambda_\phi^3}{\bar{r}^2}\Delta r^2\Omega + \frac{\lambda_\phi}{6}\Omega^3 + \frac{\lambda_\phi^4}{\bar{r}^3}\Delta r^3\Omega - \frac{\lambda_\phi^2}{6\bar{r}}\Delta r\Omega^3 - \epsilon^{1/2}\dot{\theta} . \tag{3.15}$$

Correlations in the transmitted light are given by the steady-state correlation functions $C_{\Delta r\Delta r}(s), C_{\Omega\Omega}(s)$. These are defined by

$$C_{\Delta r\Delta r}(s) = \lim_{t_0 \to -\infty} \langle \Delta r(s)\Delta r(0)\rangle , \tag{3.16}$$

$$C_{\Omega\Omega}(s) = \lim_{t_0 \to -\infty} \langle \Omega(s)\Omega(0)\rangle , \tag{3.17}$$

where t_0 is the preparation time in which initial conditions have been specified. It must be stressed that due to the non-Markovian character of the problem these correlation functions do not coincide with $\langle \Delta r(s)\Delta r(0)\rangle$ and $\langle \Omega(s)\Omega(0)\rangle$ defined with stationary initial conditions specified at $t_0 = 0$.[15–17] In the following we always assume implicitly the limit $t_0 \to -\infty$. These correlation functions satisfy a complicated hierarchy of coupled equations which is obtained from (3.14) and (3.15). The hierarchy can be closed to a given order in ϵ. The resulting set of closed equations is somewhat simpler to solve using the auxiliary equations (3.3)–(3.10). The equations for the correlation functions must be supplemented with stationary moments which enter as initial conditions. The stationary moments are also calculated closing to a given order in ϵ the hierarchy for the moments, which follows from (3.14) and (3.15). Terms of non-Markovian origin appear in the hierarchies for the stationary moments and for the correlation functions, through correlations of $\dot{\theta}(s)$ with the variables Δr and Ω as seen explicitly below. These terms can also be calculated perturbatively in the noise parameters.

B. First-order calculation

To lowest order in ϵ, (3.14) and (3.15) reduce to the linear decoupled equations (3.3) and (3.4) for $\Delta r = \epsilon^{1/2} r_1$ and $\Omega = \epsilon^{1/2}\Omega_1$. Therefore, the phase fluctuations of the driving laser have no effect on the amplitude of the transmitted light in this order of calculation.[24] From Eq. (3.4) for Ω we have[15]

$$\frac{d}{ds}\langle \Omega^2(s)\rangle = -2\lambda_\phi\langle \Omega^2(s)\rangle - 2\epsilon^{1/2}\langle \Omega(s)\dot{\theta}(s)\rangle , \tag{3.18}$$

where we have calculated $\langle \Omega(s)\dot{\theta}(s)\rangle$ through Novikov's formula[25]

$$\langle \Omega(s)\dot{\theta}(s)\rangle = \int_{-\infty}^{s} ds_1 \left\langle \frac{\delta\Omega(s)}{\delta\dot{\theta}(s_1)}\right\rangle \langle \dot{\theta}(s)\dot{\theta}(s_1)\rangle = -\frac{\epsilon^{1/2}}{1 + \tau\lambda_\phi} . \tag{3.19}$$

In the same way

$$\frac{d}{ds}\langle \Omega(s)\Omega(0)\rangle = -\lambda_\phi\langle \Omega(s)\Omega(0)\rangle + \frac{\epsilon}{1 + \tau\lambda_\phi}\exp(-s/\tau) . \tag{3.20}$$

Equations (3.18) and (3.20) are solved as

$$\langle \Omega(s)\Omega(0)\rangle = \langle \Omega^2\rangle \left[\frac{\exp(-\lambda_\phi s)}{1 - \tau\lambda_\phi} - \frac{\tau\lambda_\phi}{1 - \tau\lambda_\phi}\exp(-s/\tau) \right] , \tag{3.21}$$

$$\langle \Omega^2\rangle = \frac{\epsilon}{\lambda_\phi(1 + \tau\lambda_\phi)} , \tag{3.22}$$

where $\langle \Omega^2\rangle$ is the stationary value of $\langle \Omega^2(t)\rangle$. We note that no approximation is made in obtaining the results (3.21) and (3.22) from (3.4), so that within the lowest-order expansion in ϵ the effect of $\tau \neq 0$ is contained in an exact way. The most characteristic non-Markovian dynamical effect comes from the existence of the term proportional to $\exp(-s/\tau)$ in (3.20).[15–17] This gives rise to the second term in (3.21). It implies that even in a linear analysis $\langle \Omega(s)\Omega(0)\rangle$ does not relax through a simple exponential. The second term in (3.21) causes a slow initial decay of the correlation function. In fact, it follows from (3.21) that the slope of $\langle \Omega(s)\Omega(0)\rangle$ as a function of s at $s = 0$ is zero.[27] In the limit of $\tau \to 0$, (3.21) reduces to the result of Willis.[9,28]

We next consider the spectrum of the transmitted light. In order to eliminate the part of the field associated with the driving laser we introduce a slowly varying component

$$E'^{\pm} = E^{\pm}\exp(\mp i\epsilon^{1/2}\theta) . \tag{3.23}$$

The correlation function of $E'^{\pm}$ is given by

$$\langle E'^{+}(s)E'^{-}(0)\rangle = r_0^2\langle \exp\{i[\Omega(s)-\Omega(0)]\}\rangle$$
$$= r_0^2\exp\{-\tfrac{1}{2}\langle[\Omega(s)-\Omega(0)]^2\rangle\}\ , \quad (3.24)$$

where in the last step we have used Kubo's formula[29] and the Gaussian property of $\langle\Omega\rangle$. Substituting (3.21) and (3.22) in (3.24) we find

$$\langle E'^{+}(s)E'^{-}(0)\rangle = r_0^2\exp\left[-\langle\Omega^2\rangle\left[1-\frac{1}{1-\tau\lambda_\phi}[\exp(-\lambda_\phi s)-\tau\lambda_\phi\exp(-s/\tau)]\right]\right]\ . \quad (3.25)$$

Consistent with our calculation to first order in ϵ and given that $\langle\Omega^2\rangle$ is of order ϵ, we can expand the exponential in (3.25) to order ϵ. Recalling, in addition, that due to the Gaussian property of Ω, $\langle E'^{\pm}\rangle = r_0\exp(-\langle\Omega^2\rangle/2)$, and defining $\delta E'^{\pm}=E'^{\pm}-\langle E'^{\pm}\rangle$, we finally obtain

$$\langle \delta E'^{+}(s)\delta E'^{-}(0)\rangle = \frac{r_0^2\langle\Omega^2\rangle}{1-\tau\lambda_\phi}[\exp(-\lambda_\phi s)-\tau\lambda_\phi\exp(-s/\tau)]\ . \quad (3.26)$$

The spectrum associated with (3.26) is

$$S(\omega)=\frac{1}{\pi}\mathrm{Re}\left[\int_0^\infty ds\,\exp(-i\omega s)\langle\delta E'^{+}(s)\delta E'^{-}(0)\rangle\right]$$
$$=\frac{r_0^2\epsilon}{\pi(1-\tau^2\lambda_\phi^2)}\left[\frac{1}{\omega^2+\lambda_\phi^2}-\frac{1}{\omega^2+(1/\tau)^2}\right]\ . \quad (3.27)$$

It consists in the superposition of two Lorentzians of linewidths λ_ϕ and τ^{-1}. The second Lorentzian is a novel feature of the spectrum which appears due to the finite correlation time of the frequency fluctuations. The weight of the two Lorentzians is the same but with opposite sign. The common weight depends on τ through a factor $(1-\tau^2\lambda_\phi^2)^{-1}$. In the transmitting state $\lambda_\phi\sim 1$ and in the cooperative branch $\lambda_\phi\gg 1$. Therefore, for small values of τ the two linewidths can become comparable in the cooperative branch, while in the transmitting state the second Lorentzian has a large linewidth. The opposite happens when $\tau\lesssim 1$ (we recall that there is here no restriction to small τ), namely, the two linewidths become comparable in the transmitting state. When the two linewidths are comparable, the common weight of the two Lorentzians becomes larger. In the limiting case $\lambda_\phi\tau=1$, in which the two relevant time scales coincide, the two Lorentzians collapse to a single one with a weight $r_0^2\langle\Omega^2\rangle$. In situations with $\lambda_\phi\tau\ll 1$ the relative effect of the second Lorentzian will be more important on the tails of $S(\omega)$. This modification of the tails of $S(\omega)$ is associated with the slow initial decay of the correlation function.

In this first-order calculation, amplitude fluctuations do not appear. The spectrum is dominated by phase fluctuations. In our model, amplitude fluctuations are a second-order effect. However, it is necessary to consider such fluctuations to account for the line-narrowing effect absent in (3.27). This is done next, calculating the amplitude-fluctuations spectrum to order ϵ^2.

C. Second-order calculation

A calculation of correlation functions to order ϵ^2 introduces a dynamical coupling of amplitude and phase absent in the linear approximation. To this order in ϵ, nonlinear couplings give rise to amplitude fluctuations induced by the driving-laser frequency fluctuations.

Equations for $\langle\Delta r(s)\Delta r(0)\rangle$ and $\langle\Omega(s)\Omega(0)\rangle$ to order ϵ^2 can be obtained from (3.14) and (3.15):

$$\frac{d}{ds}\langle\Delta r(s)\Delta r(0)\rangle+\lambda_r\langle\Delta r(s)\Delta r(0)\rangle$$
$$=R_1\langle\Delta r^2(s)\Delta r(0)\rangle$$
$$+R_2\langle\Delta r^3(s)\Delta r(0)\rangle-\frac{\bar r}{2}\langle\Omega^2(s)\Delta r(0)\rangle\ , \quad (3.28)$$

$$\frac{d}{ds}\langle\Omega(s)\Omega(0)\rangle+\lambda_\phi\langle\Omega(s)\Omega(0)\rangle$$
$$=\frac{\lambda_\phi^2}{\bar r}\langle\Delta r(s)\Omega(s)\Omega(0)\rangle-\frac{\lambda_\phi^3}{\bar r^2}\langle\Delta r^2(s)\Omega(s)\Omega(0)\rangle$$
$$+\frac{\lambda_\phi}{6}\langle\Omega^3(s)\Omega(0)\rangle+\epsilon\exp(-s/\tau)\ . \quad (3.29)$$

The last term has been obtained using Novikov's formula[25] and an expansion in τ:

$$\langle\dot\theta(s)\Omega(0)\rangle=-\epsilon^{1/2}\exp(-s/\tau)$$
$$+O(\tau\exp(-s/\tau))\ . \quad (3.30)$$

We are now considering τ as an additional small parameter, and we neglect possible terms of order ϵ and ϵ^2 but of order higher than the leading one in τ. For example, terms of order $\tau\exp(-s/\tau)$ in (3.30) are neglected. They would contribute to order τ^2 to the correlation functions. The last term on the right-hand side (rhs) of (3.29) is a characteristic non-Markovian term which vanishes for $\tau=0$ but becomes important at finite τ and early times. Such early-time contribution in (3.29) modifies the long-time behavior of the correlation function through the solution of (3.28) and (3.29) from $s=0$ onwards.

In order to solve Eqs. (3.28) and (3.29) we close the hierarchy of equations obtaining the nonlinear correlation functions on the rhs of (3.28) and (3.29) to order ϵ^2. This is done in the Appendix, where these correlation functions are calculated using the auxiliary variables r_i and Ω_i. Substituting these correlation functions in (3.28) and (3.29), the final results obtained for $\langle\Delta r(s)\Delta r(0)\rangle$ and $\langle\Omega(s)\Omega(0)\rangle$ are

$$\langle \Delta r(s)\Delta r(0)\rangle = (\langle \Delta r^2\rangle - \langle \Delta r\rangle^2)\exp(-\lambda_r s) + \langle \Delta r\rangle^2 + \frac{\bar{r}}{2(\lambda_r - 2\lambda_\phi)}(\langle \Delta r\Omega^2\rangle - \langle \Delta r\rangle\langle \Omega^2\rangle)[\exp(-\lambda_r s) - \exp(-2\lambda_\phi)] , \tag{3.31}$$

$$\langle \Omega(s)\Omega(0)\rangle = \langle \Omega^2\rangle\{\exp(-\lambda_\phi s) + \tau\lambda_\phi[\exp(-\lambda_\phi s) - \exp(-s/\tau)]\} + \frac{\lambda_\phi^2}{\lambda_r}\left[\frac{1}{\bar{r}}(\langle \Omega^2\Delta r\rangle - 3\langle \Omega^2\rangle\langle \Delta r\rangle) + \frac{\tau\epsilon^2\langle \Omega^2\rangle}{\lambda_r}\right]\{\exp(-\lambda_\phi s) - \exp[-(\lambda_r + \lambda_\phi)s]\} + \lambda_\phi\langle \Omega^2\rangle\left[(3\lambda_\phi - \lambda_r)\frac{\langle \Delta r\rangle}{\bar{r}} - \frac{\epsilon\tau}{2\lambda_r}(5\lambda_\phi - \lambda_r)\right] s\exp(-\lambda_\phi s) . \tag{3.32}$$

In these equations, stationary moments enter through the initial conditions at $s=0$ needed to solve (3.28) and (3.29). Stationary moments can also be calculated closing to order ϵ^2 the corresponding hierarchy which follows from (3.14) and (3.15). This is also done in the Appendix again using the auxiliary variables r_i, Ω_i. From that calculation we find

$$\langle \Delta r\rangle^2 = \frac{\epsilon^2\bar{r}^2}{4\lambda_r^2}\left[\frac{1}{\lambda_\phi^2} - \frac{2\tau}{\lambda_\phi}\right] , \tag{3.33}$$

$$\langle \Delta r^2\rangle - \langle \Delta r\rangle^2 = \frac{\epsilon^2\bar{r}^2}{2\lambda_r\lambda_\phi^2(\lambda_r + 2\lambda_\phi)} , \tag{3.34}$$

$$\langle \Omega^2\rangle = \epsilon\left[\frac{1}{\lambda_\phi} - \tau\right] + \frac{\epsilon^2}{2\lambda_\phi\lambda_r}\left[\frac{\lambda_r^2}{\lambda_\phi(\lambda_r + 2\lambda_\phi)} - 1 + \tau(\lambda_\phi - \lambda_r)\right] , \tag{3.35}$$

$$\langle \Omega^2\Delta r\rangle - \langle \Omega^2\rangle\langle \Delta r\rangle = -\frac{\epsilon^2\bar{r}}{\lambda_\phi^2(\lambda_r + 2\lambda_\phi)} . \tag{3.36}$$

The correlation function for the amplitude is a superposition of two exponentials with decays rates λ_r and $2\lambda_\phi$. In our perturbative treatment these decay rates do not depend on the noise parameters. The presence of the second exponential is an important qualitative difference from the case in which only quantum fluctuations around the locally stable states is considered.[12] In this last case only the decay rate λ_r appears in the lowest-order contribution to the amplitude correlation function. In this sense the existence of a second decay rate can be considered as a nonlinear effect due to the coupling of amplitude and phase fluctuations. The phase correlation function contains in addition to the exponentials in (3.21) a new decay rate $\lambda_r + \lambda_\phi$ and a characteristic term proportional to $s\exp(-\lambda_\phi s)$. These new terms are associated with nonlinear effects. The weights of the different terms in (3.31) and (3.32) are proportional to ϵ^2. In this order of approximation the value of τ only affects the amplitude correlation function through the stationary moments $\langle \Delta r^2\rangle$ and $\langle \Delta r\rangle$. The phase correlation function depends on τ also through exponential terms, as already found out in the linear calculation.

We now consider the spectrum of the amplitude fluctuations. Using (3.31), (3.33), (3.34), and (3.36) we find

$$S_r(\omega) = \frac{1}{\pi}\mathrm{Re}\left[\int_0^\infty ds\,\exp(-i\omega s)\langle \delta r(s)\delta r(0)\rangle\right] = \frac{\epsilon^2\bar{r}^2}{\pi\lambda_\phi(4\lambda_\phi^2 - \lambda_r^2)}\left[\frac{1}{\omega^2 + \lambda_r^2} - \frac{1}{\omega^2 + 4\lambda_\phi^2}\right] , \tag{3.37}$$

where $\delta r = r - \langle r\rangle$. To the order of approximation of our calculation, $S_r(\omega)$ in (3.37) turns out to be independent of τ. An explicit representation of (3.37) is shown in Figs. 1 and 2 for the two branches of the hysteresis cycle. These figures parallel the analogous ones in Ref. 12 for the spectrum calculated when considering only quantum noise. We note, however, that (3.37) is the spectrum only of the amplitude fluctuations. The spectrum (3.37) is the superposition of two Lorentzians. As mentioned in connection with the correlation function, the existence of the second Lorentzian for amplitude fluctuations is a novel feature

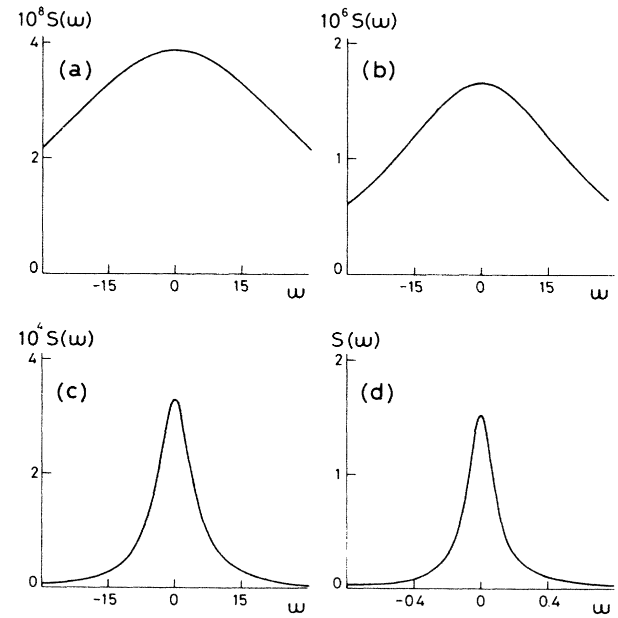

FIG. 1. Spectrum $S(\omega)$ of the transmitted light for $\Gamma^2=40$. $S(\omega)$ is given in units of ϵ^2/π. Points (a)—(d) correspond to the cooperative branch of the hysteresis cycle. Point (d) is the terminal point of the cooperative branch. (a) $\bar{r}=4.06$, (b) $\bar{r}=14.193$, (c) $\bar{r}=20.589$, (d) $\bar{r}=21.026$.

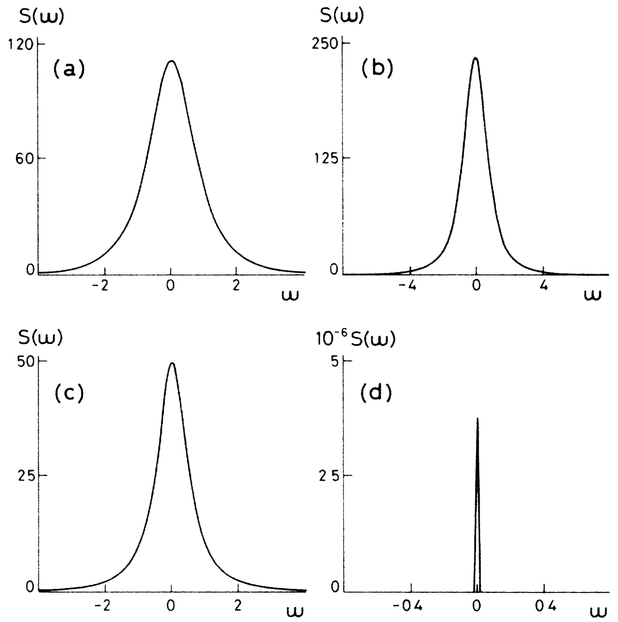

FIG. 2. Spectrum $S(\omega)$ of the transmitted light for $\Gamma^2=40$. $S(\omega)$ is given in units of ϵ^2/π. Points (a)–(d) correspond to the one-atom branch. Point (d) is the terminal point of the one-atom branch. (a) $\bar{r}=21.995$, (b) $\bar{r}=31.33$, (c) $\bar{r}=13.39$, (d) $\bar{r}=12.48$.

that comes out of our calculation. The weights of the two Lorentzians in (3.37) are the same, and therefore $S_r(\omega)$ has a single maximum at $w=0$. In the quantum-noise calculation[12] the complete spectrum also has two Lorentzians associated, respectively, with amplitude and phase. In this case the two Lorentzians have weights of different sign in the cooperative branch, giving rise to a spectrum with the shape of a doublet. Figures 1 and 2 show that $S_r(\omega)$ exhibits line narrowing and growth of the value at $\omega=0$ when $\bar{r}$ comes close to the spinodal or turning points. At these points $\lambda_r \to 0$ and from (3.37) we see that $S(\omega=0)\sim\lambda_r^{-2}$. Also, an effective linewidth $\bar{\omega}$ defined by $S(\bar{\omega})=S(\omega=0)/2$ is proportional to λ_r for $\lambda_r \ll 1$. The characteristic phenomenon of line narrowing discussed in the case of quantum noise[12] is here modified for the amplitude fluctuations by the presence of the second Lorentzian in (3.37).

IV. SUMMARY AND DISCUSSION

In this paper we have calculated the spectrum of the transmitted light in the good-cavity limit of absorptive optical bistability taking into account phase fluctuations of the driving laser modeled by an extended phase-diffusion model. As a main qualitative difference from earlier calculations we find new Lorentzians in the spectrum which do not appear when only quantum noise is considered[12] or when white-noise fluctuations for the incident phase are assumed.[9] Being more precise, our first-order calculation, which coincides with a linear analysis, leads to a phase correlation function which contains two exponentials. The spectrum is a superposition of two Lorentzians, both associated with phase fluctuations. The first Lorentzian has the linewidth of the usual one.[12] This first Lorentzian is the only one which appears with an ordinary phase-diffusion model for the incident laser.[9] The second Lorentzian appears as a consequence of the characteristic non-Markovian effect due to the finite correlation time of the frequency fluctuations of the driving laser. The linewidth of this second Lorentzian can in many cases be comparable to the linewidth of the first one. When the inverse relaxation rate λ_ϕ^{-1} is large in comparison with the correlation time of the incident phase fluctuations, the effect of the second Lorentzian is most noticeable in the tails of the spectrum. In the first-order calculation the amplitude of the transmitted laser does not fluctuate. Amplitude fluctuations due to incident phase noise only appear in the second-order calculation due to nonlinear couplings between amplitude and phase. The spectrum of amplitude fluctuations in this approximation is also a superposition of two Lorentzians. The existence of two Lorentzians (none of them associated now with non-Markovian properties) is an interesting difference from the case in which phase fluctuations of the incident laser are neglected.[12] In the latter case the lowest-order contribution to the spectrum contains a single Lorentzian. The amplitude spectrum exhibits the well-known phenomenon of line narrowing at the extremes of the hysteresis cycle[12] but with modifications due to the superposition of two Lorentzians.

Our calculation is based on a perturbative expansion in the parameter ϵ defined as the ratio of the linewidth of the driving laser to the cavity lifetime. This calculation scheme permits us to include nonlinearities and non-Markovian effects without reducing the problem to an approximate single-variable model. Limitations of the method are the ordinary ones in this sort of perturbative analysis. A more serious limitation of our results for a direct comparison with possible experiments is that we have neglected other noise sources. The question of which are the real sources of noise and their order of magnitude is an important but not well understood problem.[30] Parametric noise sources are discussed in Ref. 3. Along with this reference, our strategy here is to study separately the effect of a given noise source, in this case phase fluctuations of the incident laser, and to look for its signature or characteristic consequences. At present this seems to be the only feasible way of obtaining explicit results. It is generally accepted that phase fluctuations of the incident laser, beside being unavoidable, dominate over amplitude fluctuations and quantum or thermal noise.[7–10,30] In fact, quantum noise is only of practical relevance in miniaturized devices. Probably the more important effect neglected in our calculations is the phase fluctuations of the transmitted field due to the jitter of the cavity. In some circumstances this could be comparable to the fluctuations calculated here. In addition, unforeseen effects due to the combination of noise sources might appear. Finally, we mention that although we have only dealt here with absorptive bistability, there is no problem, in principle, in using the same method to consider dispersive bistability. However, the algebra becomes considerably more complicated due to the additional terms to be kept in (2.9). We hope to report results for dispersive bistability in subsequent work.

ACKNOWLEDGMENT

Financial support from Comisión Asesora de Investigación Científica y Técnica Project No. 361/84 (Spain) is acknowledged.

APPENDIX

1. Stationary moments

The stationary moments which appear in the calculation of the stationary-state correlation functions in Sec. III C can be expressed in terms of the stationary moments of the auxiliary variables r_i and Ω_i. We obtain to order ϵ^2

$$\langle \Delta r\rangle^2=\epsilon^2\langle r_2\rangle^2+2\epsilon^2\langle r_1r_3\rangle \;, \tag{A1}$$

$$\langle \Delta r^2\rangle=2\epsilon^{3/2}\langle r_1r_2\rangle+\epsilon^2\langle r_2^2\rangle+2\epsilon^2\langle r_1r_3\rangle \;, \tag{A2}$$

$$\langle \Delta r\Omega^2\rangle=\epsilon^{3/2}\langle r_1\Omega_1^2\rangle+\epsilon^2\langle r_2\Omega_1^2\rangle+\epsilon^2\langle r_1\Omega_1\Omega_2\rangle \;, \tag{A3}$$

$$\langle \Omega^2\rangle=\epsilon\langle \Omega_1^2\rangle+2\epsilon^{3/2}\langle \Omega_1\Omega_2\rangle +\epsilon^2\langle \Omega_2^2\rangle+2\epsilon^2\langle \Omega_1\Omega_3\rangle \;. \tag{A4}$$

To calculate the moments on the rhs of (A1)—(A4) we consider a generic set of variables q_i, $i=1,2,\ldots,n$, obeying the stochastic differential equations

$$\dot q_i=v_i(\mathbf{q}(s))+g_i\dot\theta(s) \;, \tag{A5}$$

where $v_i(\mathbf{q}(s))$ are general functions of the variables q_i, and g_i are a set of constants independent of q_i. The equation for a moment $\langle f(\mathbf{q}(s))\rangle$, where $f(\mathbf{q}(s))$ is a general function of q_i and $\mathbf{q}=(q_i,q_2,\ldots,q_n)$, is obtained from (A5) as

$$\frac{d}{ds}\langle f(\mathbf{q}(s))\rangle=\sum_j \langle \partial_j f(\mathbf{q}(s))v_j(\mathbf{q}(s))\rangle +\sum_j \langle \partial_j f(\mathbf{q}(s))g_j\dot\theta(s)\rangle \;. \tag{A6}$$

The last term on the rhs of (A6) can be calculated using Novikov's formula:[25]

$$\langle \partial_j f(\mathbf{q}(s))g_j\dot\theta(s)\rangle =\sum_m \int_{-\infty}^{s} ds_1\left\langle \partial_m\partial_j f(\mathbf{q}(s))\frac{\delta q_m(s)}{\delta\dot\theta(s_1)}\bigg|_{q_m(s)=q_m} g_j\right\rangle \times\langle \dot\theta(s)\dot\theta(s_1)\rangle \;. \tag{A7}$$

Substituting (2.10) in (A7) and by successive partial integrations, (A6) becomes to first order in τ

$$\frac{d}{ds}\langle f(\mathbf{q}(s))\rangle=\sum_j \langle \partial_j f(\mathbf{q}(s))v_j(\mathbf{q}(s))\rangle+\sum_{m,j}[\langle \partial_m\partial_j f(\mathbf{q}(s))g_m g_j\rangle-\tau\langle \partial_m\partial_j f(\mathbf{q}(s))M_m(\mathbf{q}(s))g_j\rangle] \;, \tag{A8}$$

where

$$\frac{\delta q_m(s)}{\delta\dot\theta(s_1)}\bigg|_{\substack{q(m)=q_m\\ s_1=s}}=g_m \;,$$

$$\frac{d}{ds_1}\left[\frac{\delta q_m(s)}{\delta\dot\theta(s_1)}\right]\bigg|_{\substack{q(m)=q_m\\ s_1=s}} =M_m(\mathbf{q}(s))=\sum_p -\partial_p v_m(\mathbf{q}(s))g_p \;. \tag{A9}$$

From Eq. (A8) and taking into account that for the stationary moments $(d/ds)\langle f(\mathbf{q})\rangle=0$, we obtain a closed set of algebraic equations. Specializing (A5) to the set of equations (3.3)—(3.10), we obtain algebraic equations for the moments appearing on the rhs of (A1)—(A4). The solution of these equations leads to (3.33)—(3.36).

2. Stationary correlation functions

The stationary correlation functions which appear on the rhs of the equations (3.28) and (3.29) for $\langle \Delta r(s)\Delta r(0)\rangle$ and $\langle \Omega(s)\Omega(0)\rangle$ can also be expressed in terms of the stationary correlation functions of the auxiliary variables r_i and Ω_i. We obtain to order ϵ^2

$$\langle \Delta r^2(s)\Delta r(0)\rangle=\epsilon^{3/2}\langle r_1^2(s)r_1(0)\rangle +\epsilon^2[2\langle r_1(s)r_2(s)r_1(0)\rangle +\langle r_1^2(s)r_1(0)\rangle] \;, \tag{A10}$$

$$\langle \Delta r^3(s)\Delta r(0)\rangle=\epsilon^2\langle r_1^3(s)r_1(0)\rangle \;, \tag{A11}$$

$$\langle \Omega^2(s)\Delta r(0)\rangle=\epsilon^{3/2}\langle \Omega_1^2(s)r_1(0)\rangle +\epsilon^2[2\langle \Omega_1(s)\Omega_2(s)r_1(0)\rangle +\langle \Omega_1^2(s)r_2(0)\rangle] \;, \tag{A12}$$

$$\langle \Omega^3(s)\Omega(0)\rangle=\epsilon^2\langle \Omega_1^3(s)\Omega_1(0)\rangle \;, \tag{A13}$$

$$\langle \Delta r^2(s)\Omega(s)\Omega(0)\rangle=\epsilon^2\langle r_1^2(s)\Omega_1(s)\Omega_1(0)\rangle \;, \tag{A14}$$

$$\langle \Delta r(s)\Omega(s)\Omega(0)\rangle=\epsilon^{3/2}\langle r_1(s)\Omega_1(s)\Omega_1(0)\rangle +\epsilon^2[\langle r_1(s)\Omega_1(s)\Omega_2(0)\rangle +\langle r_1(s)\Omega_2(s)\Omega_1(0)\rangle +\langle r_2(s)\Omega_1(s)\Omega_1(0)\rangle] \;. \tag{A15}$$

To calculate the stationary-state correlation functions on the rhs of (A10)—(A15) we consider again the generic set of stochastic differential equations (A5). We have

$$\frac{d}{ds}\langle f(\mathbf{q}(s))f'(\mathbf{q}(0))\rangle=\sum_i\langle \partial_i f(\mathbf{q}(s))v_i(\mathbf{q}(s))f'(\mathbf{q}(0))\rangle+\sum_i\langle \partial_i f(\mathbf{q}(s))g_i\dot{\theta}(s)f'(\mathbf{q}(0))\rangle\ , \tag{A16}$$

where $f(\mathbf{q})$ and $f'(\mathbf{q})$ are general functions of the variables q_i. The last term in (A16) is again calculated using Novikov's formula:[25]

$$\begin{aligned}\langle \partial_i f(\mathbf{q}(s))g_i\dot{\theta}(s)f'(\mathbf{q}(0))\rangle\\ &=\sum_m\int_{-\infty}^{s}ds_1\left\langle \partial_m\partial_i f(\mathbf{q}(s))\frac{\delta q_m(s)}{\delta\dot{\theta}(s_1)}\bigg|_{q_m(s)=q_m} g_i f'(\mathbf{q}(0))\right\rangle\langle\dot{\theta}(s)\dot{\theta}(s_1)\rangle\\ &\quad+\sum_m\int_{-\infty}^{0}ds_1\left\langle \partial_i f(\mathbf{q}(s))g_i\partial_m f'(\mathbf{q}(0))\frac{\delta q_m(0)}{\delta\dot{\theta}(s_1)}\bigg|_{q_m(0)=q_m'}\right\rangle\langle\dot{\theta}(s)\dot{\theta}(s_1)\rangle\ .\end{aligned} \tag{A17}$$

The last term in (A17) is zero in the Markovian case. This is due to the presence of a δ correlation function for $\langle\dot{\theta}(s)\dot{\theta}(s_1)\rangle$ in this case and to the fact that $s_1<0<s$. This term is the most characteristic of the non-Markovian dynamics. It introduces important differences in the behavior of the correlation functions and the spectrum. Substituting (A17) in (A16) and by successive partial integration, we obtain to first order in τ

$$\begin{aligned}\frac{d}{ds}\langle f(\mathbf{q}(s))f'(\mathbf{q}(0))\rangle=&\sum_i\langle \partial_i f(\mathbf{q}(s))v_i(\mathbf{q}(s))f'(\mathbf{q}(0))\rangle\\ &+\sum_{i,m}[\langle\partial_m\partial_i f(\mathbf{q}(s))g_m g_i f'(\mathbf{q}(0))\rangle-\tau\langle\partial_m\partial_i f(\mathbf{q}(s))M_m(\mathbf{q}(s))g_i f'(\mathbf{q}(0))\rangle\\ &+\exp(-s/\tau)\langle\partial_i f(\mathbf{q}(s))g_i\partial_m f'(\mathbf{q}(0))g_m\rangle]\ ,\end{aligned} \tag{A18}$$

where we have neglected a term proportional to $\tau\exp(-s/\tau)$ because it introduces contributions of order τ^2 for the correlation function.

Then, specializing (A18) to the set of equations (3.3)–(3.10), we obtain a closed set of equations for the stationary correlation functions on the rhs of (A10)–(A15). The final solution is

$$\langle\Omega^2(s)\Delta r(0)\rangle=(\langle\Omega^2\Delta r\rangle-\langle\Omega^2\rangle\langle\Delta r\rangle)\exp(-2\lambda_\phi s)+\langle\Omega^2\rangle\langle\Delta r\rangle\ , \tag{A19}$$

$$\langle\Omega^3(s)\Omega(0)\rangle=(\langle\Omega^4\rangle+3\epsilon\tau\langle\Omega^2\rangle)\exp(-\lambda_\phi s)-3\epsilon\tau\langle\Omega^2\rangle\exp(-s/\tau)\ , \tag{A20}$$

$$\begin{aligned}\langle\Delta r(s)\Omega(s)\Omega(0)\rangle=&\left[(\langle\Omega^2\Delta r\rangle-3\langle\Omega^2\rangle\langle\Delta r\rangle)+\epsilon\tau\left[\frac{5\bar{r}}{2\lambda_r}\langle\Omega^2\rangle+\langle\Delta r\rangle\right]\right]\exp[-(\lambda_r+\lambda_\phi)s]\\ &+\left[3\langle\Omega^2\rangle\langle\Delta r\rangle-\epsilon\tau\frac{5\bar{r}}{2\lambda_r}\langle\Omega^2\rangle\right]\exp(-\lambda_\phi s)-\epsilon\tau\langle\Delta r\rangle\exp(-s/\tau)\ .\end{aligned} \tag{A21}$$

The functions (A10), (A11), and (A14) vanish in order ϵ^2. Substituting (A19)–(A21) in (3.28) and (3.29) we obtain a closed set of equations whose solutions are given by (3.31) and (3.32).

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