Fréedericksz transition in a periodic magnetic field

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We study the Fréedericksz transition in a twist geometry under the effect of a periodic modulation of the magnitude of the applied magnetic field. We find a shift of the effective instability point and a time-periodic state with anomalously large orientational fluctuations. This time-periodic state occurs below threshold and it is accompanied by a dynamically stabilized spatial pattern. Beyond the instability the emergence of a transient pattern can be significantly delayed by a fast modulation, allowing the observation of pattern selection by slowing down the reorientational dynamics.

I. INTRODUCTION

The magnetic Fréedericksz transition in the nematic phase of liquid crystals is the result of a competition between elastic and magnetic torques. For a magnetic field larger than a critical one the molecules align following the applied magnetic field. Recent, renewed interest in this phenomenon is partly due to the appearance of a transient spatial pattern which appears in some circumstances during the reorientation.[1] The study of the formation of this transient pattern belongs to the class of problems dealing with pattern-selection mechanisms. The dynamics of pattern formation in the Fréedericksz transition has been described[2] using an analogy with the problem of spinodal decomposition. A natural question to address in this context is the possibility of stabilizing those patterns by some mechanism. Motivated by this idea, we study in this paper the behavior of the system when the magnitude of the magnetic field is periodically changed by using values that are alternately larger and smaller than the critical value.

Studies of instabilities under a periodic modulation of the control parameter exist for several physical systems. In particular, thermal convection under external modulation[3] and periodic spinodal decomposition caused by periodic temperature quenches[4] have been considered in some detail. Comparisons of the effects of periodic and stochastic modulations are also available.[5] These studies reveal some interesting features of modulated instabilities which are also worth studying in the case of the Fréedericksz transition. Three of these features are, respectively, the effective shifts in the instability point, the existence of anomalously large fluctuations, and a delay in the observation of the instability or postponement in the onset time.

The shift of the instability point can be found in a deterministic description of systems with more than one relevant variable.[3] It can also occur in the slow modulation limit as a combined effect of fluctuations, nonlinearities, and modulation for a single-variable system.[6] In our case we find for the Fréedericksz transition a shift in the instability point in a deterministic description, which also involves a single variable. This is a consequence of the nonlinearity of the model in the control parameter. Anomalously large fluctuations are commonly associated with critical points or transient states. Under a periodic modulation it is possible to predict time-periodic stable states with large fluctuations caused by a periodic enhancement. Such a state, described in this paper, is found below the shifted instability point. It exhibits large spatially distributed fluctuations in the orientation of the molecules for slow modulations. These fluctuations are described by a related structure factor. Finally, we also find that above the shifted instability, and for fast modulations, the onset time for the observation of the instability can be significantly increased. This results in a delay in the emergence of the transient spatial pattern. Consequently, such a modulation procedure provides us with a powerful method to increase the reorientational dynamics time interval, thus allowing the observation of pattern-selection mechanisms and subsequent pattern annihilations during considerably larger intervals of time. These results clearly suggest the convenience of performing experiments carried out under these circumstances. Concerning the possibility of pattern stabilization, we predict the existence of time-periodic solutions with an associated spatial pattern. These solutions occur below the shifted instability point.

The paper is organized as follows. Section II contains the basic equations and parameters used in the analysis of the Fréedericksz transition in a periodic field. In Sec. III a mean-field analysis of the instability is given. Section IV considers the problem of the spatially distributed orientational fluctuations and addresses the question concerning pattern formation.

II. EQUATIONS FOR THE FRÉEDERICKSZ TRANSITION IN A PERIODIC FIELD

We consider a nematic sample in the twist geometry as is shown in Fig. 1. The sample is contained between two plates perpendicular to the z axis. Boundary conditions corresponding to strong anchoring of the director are assumed at $z=\pm d/2$. The director is initially aligned along the x axis $[\mathbf{n}^0=(1,0,0)]$ and the magnetic field $\mathbf{H}$ is aligned the y axis. Its direction is constant but its magnitude will be a periodic function of time. We assume homogeneity in the y direction and only the macroscopic flow v_y along that direction is retained. We also assume that the director reorientation takes place essentially in the x-y plane,

$$n_x(x,z)=\cos\phi(x,z) , \tag{2.1}$$

$$n_y(x,z)=\sin\phi(x,z) . \tag{2.2}$$

Under these hypotheses the stochastic coupled equations for $\phi(x,z)$ and $v_y(x,z)$ can be written in a minimal coupling approximation[2] as

$$d_t\phi(x,z)=-\frac{1}{\gamma_1}\frac{\delta F}{\delta\phi}+\tfrac{1}{2}(1+\lambda)\partial_x v_y+\xi(x,z,t) , \tag{2.3}$$

$$d_t v_y(x,z)=\frac{1}{2\rho}(1+\lambda)\partial_x\frac{\delta F}{\delta\phi}+\frac{1}{\rho}(\nu_2\partial_z^2 v_y+\nu_3\partial_x^2 v_y) +\partial_x\Omega_{yx}+\partial_z\Omega_{yz} , \tag{2.4}$$

where F is the Oseen-Frank distortion-free energy supplemented with the magnetic contribution,

$$\frac{\delta F}{\delta\phi}=-\left[k_{22}\partial_z^2\phi+k_{33}\partial_x^2\phi+\chi_a H^2\left[\phi-\tfrac{2}{3}\phi^3\right]\right] , \tag{2.5}$$

k_{22} and k_{33} are the elastic constants associated, respectively, with twist and bend deformations, χ_a is the anisotropic part of the magnetic susceptibility, ρ is the mass density, γ_1, ν_2, and ν_3 are viscosity coefficients, λ is a dimensionless number depending also on the viscosities of the material, and $\xi(x,z,t)$ and $\Omega_{y\alpha}(x,z,t)$ are Gaussian random forces satisfying fluctuation-dissipation relations,

$$\langle\xi(x,z,t)\xi(x',z',t')\rangle =2\frac{k_BT}{\gamma_1 L_y}\delta(x-x')\delta(z-z')\delta(t-t') , \tag{2.6}$$

$$\langle\Omega_{y\alpha}(x,z,t)\Omega_{y\beta}(x',z',t')\rangle =2\frac{k_BT}{\rho^2 L_y}\nu_\alpha\delta_{\alpha\beta}\delta(x-x')\delta(z-z')\delta(t-t') , \quad \alpha,\beta=\{x,z\} \tag{2.7}$$

where $\nu_x\equiv\nu_3$, $\nu_z\equiv\nu_2$, L_y is the y-linear dimension of the sample, and T the temperature.

We consider a periodic modulation of the magnetic field H in (2.5) around a mean value denoted by H_0 and with period T. For simplicity we assume instantaneous changes of $H(t)$ at time t_j,

$$H(t)=H_0+\bar{H}(t) , \tag{2.8}$$

$$\bar{H}(t)=\begin{cases} H_1, & t_{2j-1}<t<t_{2j}=t_{2j-1}+T/2 \\ -H_1, & t_{2j}<t<t_{2j+1} . \end{cases} \tag{2.9}$$

Equations (2.3) and (2.4) are better analyzed by using a Fourier mode analysis,

$$\phi(x,z,t)=\sum_m \theta_m(x,t)\cos\left[(2m+1)\frac{\pi z}{d}\right] =\sum_m\sum_{q_x}\theta_{m,q_x}(t)\cos\left[(2m+1)\frac{\pi z}{d}\right]e^{iq_x x} . \tag{2.10}$$

The linear-stability analysis of the modes $\theta_{m,q_x}(t)$ indicates that m modes become unstable for

$$H>H_c=(k_{22}\pi^2/\chi_a d^2)^{1/2} .$$

Only the $m=0$ mode is unstable for $H_c<3H_c$. In addition, linear instability of q_x modes is predicted for[2]

$$\frac{\chi_a H_c^2}{k_{22}}\left[\frac{H^2}{H_c^2}-(2m+1)^2\right]>\frac{k_{33}}{k_{22}}q_x^2 . \tag{2.11}$$

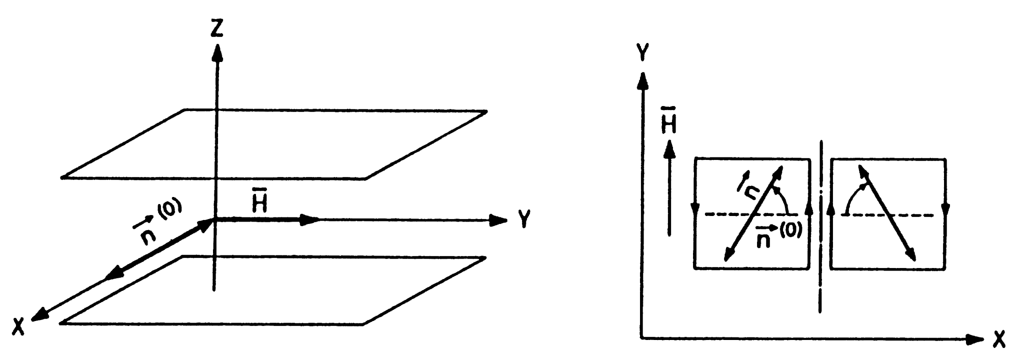

FIG. 1. Schematic representation of the geometry of the nematic sample. Flows generated by oppositely rotating zones which explain the appearance of transient structures are also schematically displayed.

These linear stability results are independent of the dynamical coupling between director and velocity fields.

As can be noticed from (2.3)–(2.5), the control parameter governing the instability is the square of the magnetic field. The periodic modulation of $H^2(t)$ is shown in Fig. 2 in terms of the squared reduced field $h^2(t)=H^2(t)/H_c^2$ ($h_0=H_0/H_c$, $h_1=H_1/H_c$). The effects of this modulation can be completely characterized in terms of three dimensionless parameters,

$$r_0=1-(h_0^2+h_1^2)\gtrless 0 , \quad (2.12)$$

$$r_1=2h_0h_1>0 , \quad (2.13)$$

$$\mu=r_1T>0 . \quad (2.14)$$

r_0 measures the difference between the mean value of $h^2(t)$ and the instability point ($h^2=1$) in the absence of modulation. The parameter r_1 measures the amplitude of the modulation of $h^2(t)$ and μ measures the strength of the modulation. When $|r_0|/r_1<1$ the system is periodically driven through the instability point.

A first study of the stability properties of the system under a periodic modulation can be done by considering the dynamical evolution of the mode $\theta_{m=0,\, q_x\to 0}(t)\equiv\theta(t)$. This represents a strict mean-field analysis for the most unstable m mode corresponding to the spatially averaged quantity

$$\theta_m(t)=(1/L_x)\int dx\, \theta_m(x,t) .$$

In this approximation hydrodynamic effects do not enter into the description. Neglecting the coupling of the $m=0$ mode with other m modes, the equation for $\theta(t)$ reads

$$\partial_s\theta(s)=-\theta(s)+h^2(s)\theta(s)[1-\tfrac{1}{2}\theta^2(s)]+\xi(s) , \quad (2.15)$$

where we have introduced a dimensionless time scale $s=(\chi_a H_c^2/\gamma_1)t$ and $\xi(s)$ is a Gaussian white noise of zero mean value and correlations

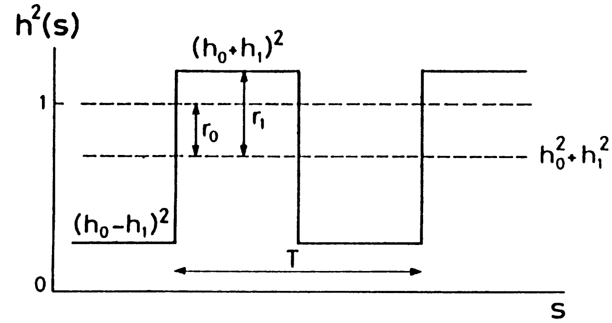

FIG. 2. Modulation of $h^2(s)$ taking alternative values $(h_0+h_1)^2$ and $(h_0-h_1)^2$ with mean value $h_0^2+h_1^2$.

$$\langle\xi(s)\xi(s')\rangle=2\varepsilon\delta(s-s'), \quad \varepsilon=\frac{k_BT/V}{\chi_a H_c^2} \quad (2.16)$$

with V being the volume of the sample. The period T is from now on also measured in units of s. The analysis of the solutions of (2.15) is postponed to Sec. III.

Equation (2.15) cannot describe the possible emergence of the experimentally observed striped like spatial pattern perpendicular to the initial orientation of the director in the sample. To address this question we have to retain spatial inhomogeneities along the x direction and introduce the effect of the hydrodynamic coupling.[1,2] The formation of transient patterns when the field H is switched from $H_i<H_c$ to a final fixed value $H_f>H_c$ has been studied[2] by analyzing the dynamical evolution of the structure factor associated with the spatially distributed orientational fluctuations associated with the most unstable m mode, i.e., the $m=0$ mode. This structure factor is defined as

$$C(Q,t)=\langle\theta_{m=0,\,q_x}(t)\theta_{m=0,\,-q_x}(t)\rangle .$$

In the approximation of negligible inertia ($d_tv_y=0$), one obtains from (2.3) and (2.4)

$$d_sC(Q,s)=\frac{2}{f(Q)}\left[\left[h^2(s)-1-\frac{k_{33}}{k_{22}}Q^2\right]C(Q,s)-\frac{h^2(s)}{2}\sum_{q_{x_1},q_{x_2}}\langle\theta_{q_{x_1}}(s)\theta_{q_{x_2}}(s)\theta_{q_x-(q_{x_1}+q_{x_2})}(s)\theta_{-q_x}(s)\rangle\right]+\frac{2}{f(Q)}\varepsilon , \quad (2.17)$$

where

$$Q^2=(q_xd/\pi)^2 ,$$

$$f(Q)=1-\frac{\bar{\alpha}}{1+\bar{\eta}Q^{-2}}, \quad \bar{\alpha}=\frac{+\frac{1}{4}\gamma_1(1+\lambda)^2}{\nu_3+\frac{1}{4}\gamma_1(1+\lambda)^2} , \quad (2.18)$$

$$\bar{\eta}=\frac{\nu_2}{\nu_3+\frac{1}{4}\gamma_1(1+\lambda)^2} .$$

At this point we recall that a viscosity γ_1 has been already used in the time adimensionalization proposed after (2.15), so that in view of (2.17) we conclude that the effective viscosity coefficient appropriate to the dynamics of $C(Q,t)$ is given by $\bar{\gamma}_1=\gamma_1 f(Q)$. The replacement of γ_1 by $\bar{\gamma}_1$ contains the whole effect of the hydrodynamic coupling. The wave-number dependence of $\bar{\gamma}_1$ implies that $Q=0$ is not the most unstable q_x mode. This is the origin of the occurrence of the spatial pattern. The evolution of the structure factor under a periodic modulation of the magnetic field and its implications on pattern formation are discussed in Sec. IV.

III. MEAN-FIELD SOLUTIONS: SHIFTED INSTABILITY AND PERIODIC SOLUTION

Neglecting fluctuations, (2.15) can be solved by a change of variables. Introducing $x(s)\equiv\theta^{-2}(s)$, it is immediately found that

$$\theta^{-2}(s)=\left[\theta^{-2}(0)+\int_0^s ds' h^2(s') \times\exp\left[2\int_0^{s'} ds''[h^2(s'')-1]\right]\right] \times\exp\left[-2\int_0^s ds'[h^2(s')-1]\right] . \tag{3.1}$$

The long-time behavior of this solution is determined by the sign of

$$\int_0^s ds'[h^2(s')-1] .$$

We have

$$h^2(s)=(1-r_0)+f(s) , \tag{3.2}$$

$$f(s)=\begin{cases} r_1, & s_{2j-1}<s<s_{2j}=s_{2j-1}+T/2 \\ -r_1, & s_{2j}<s<s_{2j+1} . \end{cases} \tag{3.3}$$

Due to the secular factor $e^{2r_0 s}$, $\theta^2(s)\to 0$ as $s\to\infty$ for $r_0>0$, while a nontrivial periodic solution appears for $r_0<0$ and $s\to\infty$. This indicates an instability at $r_0=0$. For $r_0>0$ no homogeneous reorientation occurs for long times. For $r_0<0$ an x homogeneous periodic reorientation of the director occurs after some transient. The instability point $r_0=0$ corresponds to $H_0^2=H_c^2-H_1^2$. Therefore, for a fixed value of the amplitude of modulation H_1, the instability occurs for an averaged value of $H(t),H_0$, which is smaller than the critical value H_c in absence of modulation. This shift in the instability point is an interesting effect of the modulation and it is a consequence of the fact that the control parameter H appears nonlinearly (H^2) in the dynamical equation (2.15). Within a deterministic analysis other types of shifts of instability points can be found in systems described by more than one relevant variable.[3]

The periodic solution $\theta_\infty(s)$ for $r_0<0$ can be written as

$$\theta_\infty^{-2}(s)=\frac{1}{1-e^{2\mu\alpha}}\int_0^T ds' h^2(s-s') \times\exp\left[2\mu\alpha\frac{s'}{T}-2\int_{s-s'}^{s} ds'' f(s'')\right] , \tag{3.4}$$

where we have introduced a convenient parameter $\alpha=r_0/r_1$. Equation (3.4) is manifestly periodic with period T. The behavior of (3.4) is better understood considering the values of $\theta_\infty(s)$ at the end of semiperiods in some limiting cases. Taking $s_{2j}=nT$ and $s_{2j+1}=nT+T/2$, we find in the limit of small modulation, $\mu\ll 1$,

$$\theta_\infty^2(nT)=2\left[1-\frac{1}{h_0^2+h_1^2}\right]+2\frac{1}{(r_0-1)^2T}\frac{(1-e^{r_0T})}{(1+e^{r_0T})}\mu +O(\mu^2) , \tag{3.5}$$

$$\theta_\infty^2(nT+T/2)=2\left[1-\frac{1}{h_0^2+h_1^2}\right] -2\frac{1}{(r_0-1)^2T}\frac{(1-e^{r_0T})}{(1+e^{r_0T})}\mu+O(\mu^2) . \tag{3.6}$$

The first term, $2[1-1/(h_0^2+h_1^2)]$, is the equilibrium value for a field $h^2=h_0^2+h_1^2$. Equations (3.5) and (3.6) indicate that $\theta_\infty(s)$ oscillates with a small amplitude around the equilibrium value associated with the mean value of $h^2(s)$. In the opposite limit of large modulation, $\mu\gg 1$, we find the following. For $|\alpha|>1$,

$$\theta_\infty^2(nT)=2\left[1-\frac{1}{(h_0+h_1)^2}\right] -\frac{4r_1}{\left[r_1+\frac{1}{1-\sigma}\right]^2(\sigma^2-1)}e^{\mu(\sigma-1)} , \tag{3.7}$$

$$\theta_\infty^2(nT+T/2)=2\left[1-\frac{1}{(h_0-h_1)^2}\right] +\frac{4r_1}{\left[r_1-\frac{1}{1+\sigma}\right]^2(\sigma^2-1)}e^{\mu(\sigma+1)} , \tag{3.8}$$

and for $|\sigma|<1$,

$$\theta_\infty^2(nT)=2\left[1-\frac{1}{(h_0+h_1)^2}\right]+\frac{2r_1\left[r_1-\frac{1}{1+\sigma}\right]}{\left[r_1+\frac{1}{1-\sigma}\right]^2}e^{2\mu\sigma} , \tag{3.9}$$

$$\theta_\infty^2(nT+T/2)=r_1(1-\sigma^2)e^{-\mu(\sigma+1)} . \tag{3.10}$$

When $\sigma<0$, $|\sigma|>1$, $H(s)$ always remains larger than H_c. The two first terms in (3.7) and (3.8) are the equilibrium values associated with the values of $H(s)$ in each semiperiod. The second terms are small corrections which decrease exponentially with μ. When $|\sigma|<1$, we find a similar result for $s=nT$, but for $s=nT+T/2$, $H(s)$ has adopted a value smaller than H_c during the last semiperiod and θ_∞ is thus exponentially close to zero. In summary, for $\mu\gg 1$, $\theta(s)$ follows the modulation nearly reaching equilibrium in each semiperiod.

When including fluctuations in the description of the system an additional shift of the instability point can be found. This is an effective stabilizing shift which acts in an opposite sense of the one discussed previously. It occurs as a result of the interplay of modulation, nonlinearities, and fluctuations and it is due to the mixing, in a short time scale, of the two symmetric solutions for θ. The phenomenon was analyzed for a general model in Ref. 6 and it only occurs for slow modulation. The essential idea is that for $r_0<0$, $\mu \gg 1$, and $|\sigma|<1$, a small fluctuation acting at times close to $nT+T/2$ takes a positive θ_∞ to a negative one and vice versa. The long-time average (for long times) then gives a zero value for θ_∞. The value of the parameters beyond which this mechanism becomes efficient determines the shifted instability. Such a value can be estimated making (3.10) equal to the noise intensity measured by ε/r_1. We will no longer discuss this phenomenon in this paper. For $r_0<0$ we will consider here effects which appear for fast modulation ($\mu \ll 1$). In this limit only the deterministic destabilizing shift takes place with the instability point located at $r_0=0$.

IV. STRUCTURE FACTOR AND PATTERN FORMATION

In this section we consider the spatial dependence of the orientational fluctuations as described by the structure factor. Equation (2.17) for $C(Q,s)$ is solved in a linear approximation,

$$\partial_s C(Q,s)=\frac{2}{f(Q)}\left[h^2(s)-1-\frac{k_{33}}{k_{22}}Q^2\right]C(Q,s)+\frac{2\varepsilon}{f(Q)} . \tag{4.1}$$

The stability of the different Q modes is determined by the sign of the redefined parameter σ,

$$\sigma \equiv \left[r_0+\frac{k_{33}}{k_{22}}Q^2\right]\Big/ r_1 . \tag{4.2}$$

This is an obvious generalization of the parameter σ introduced after (3.4), which becomes now Q dependent. When $\sigma>0$, $h_0^2+h_1^2-1-(k_{33}/k_{22})Q^2<0$ and $C(Q,s)$ tends, for long times, to a periodic solution. But if $\sigma<0$, $C(Q,s)$ diverges for the modes for which $h_0^2+h_1^2-1-(k_{33}/k_{22})Q^2>0$. These conclusions follow from the general solution of (4.1). Such a solution can be written in terms of a parameter τ ($0\le\tau\le T/2$), which measures time from the end of each semiperiod. We take $s=mT+\tau$ for odd semiperiods and $s=mT+T/2+\tau$ for even semiperiods. The index m indicates the number of periods elapsed ($m=0,1,2,\dots$). We find

$$C_1(m,\tau)=\frac{\varepsilon/r_1}{1+\sigma}\exp[-2f^{-1}(Q)\sigma m\mu] \\ -P_1(\tau)\{\exp[-2f^{-1}(Q)\sigma m\mu]-1\} , \tag{4.3}$$

$$C_2(m,\tau)=\frac{\varepsilon/r_1}{\sigma-1}\left[1-\frac{2}{\sigma+1}\exp[2f^{-1}(Q)(1-\sigma)r_1\tau]\right] \\ \times\exp[-2f^{-1}(Q)\sigma m\mu] \\ -P_2(\tau)\{\exp[-2f^{-1}(Q)\sigma m\mu]-1\} , \tag{4.4}$$

where

$$P_1(\tau)=\frac{\varepsilon/r_1}{1+\sigma}\left[1-\frac{2}{1-\sigma}\frac{1-\exp[f^{-1}(Q)(1-\sigma)\mu]}{1-\exp[-2f^{-1}(Q)\sigma\mu]} \right. \\ \left. \times\exp[-2f^{-1}(Q)(1+\sigma)r_1\tau]\right] , \tag{4.5}$$

$$P_2(\tau)=\frac{\varepsilon/r_1}{\sigma-1}\left[1-\frac{2}{1+\sigma}\frac{1-\exp[-f^{-1}(Q)(1+\sigma)\mu]}{1-\exp[-2f^{-1}(Q)\sigma\mu]} \right. \\ \left. \times\exp[2f^{-1}(Q)(1-\sigma)r_1\tau]\right] . \tag{4.6}$$

$C_1(m,\tau)$ and $C_2(m,\tau)$ are the solutions for odd and even semiperiods, respectively. They satisfy the obvious continuity conditions $C_1(m,\tau=T/2)=C_2(m,\tau=0)$ and $C_2(m,\tau=T/2)=C_1(m+1,\tau=0)$. In Eqs. (4.3) and (4.4) one clearly identifies a systematic evolution given by

$$\exp[-2f^{-1}(Q)\sigma m\mu]$$

which is modulated by other functions defined in each semiperiod. As we anticipated, the sign of σ determines the existence of a periodic solution for $m\to\infty$ or a divergence of the structure factor.

The two situations leading to a periodic solution or divergence can be understood in terms of our previous nonlinear analysis of $\theta(s)$. For $r_0>0$, $\sigma>0$ and a global instability (in the sense of an homogeneous reorientation for long times) does not occur. The periodic solution for $C(Q,s)$ describes the long-time evolution in which, however, large although finite periodic orientational fluctuations exist. In this time periodic state nonlinearities play no essential role. In this regime we find that a spatial pattern develops under some circumstances to be discussed below. This might be understood as a stabilization of the transient pattern which occurs when changing H from an initial value $H_i<H_c$ to a final fixed value $H_f>H_c$.[2] Here the pattern appears for a mean value of $H^2(s)$ smaller than H_c^2 ($r_0>0$). In the other situation in which $r_0<0$ the divergent linear solution of (4.1) cannot describe the final approach to the state characterized by a homogeneous periodic reorientation of $\theta(s)$. However, it does describe the early stages of the transient evolution towards that state. The time domain of validity of the linear approximation becomes larger with faster modulation. During this transient evolution we will see the occurrence of a spatial pattern whose characteristic wavelength oscillates in time. The emergence of the pattern and subsequent evolution can be significantly delayed by decreasing the period of modulation.

A. $r_0>0$: Periodic solution

When $r_0>0$, $\sigma>0$ and we find from (4.3) and (4.4) stationary periodic solutions in the long-time-limit regime,

$$C_1^\infty(\tau)\equiv C_1(m=\infty,\tau)=P_1(\tau) , \tag{4.7}$$

$$C_2^\infty(\tau)\equiv C_2(m=\infty,\tau)=P_2(\tau) . \tag{4.8}$$

A typical example of the behavior of this solution is shown in Fig. 3. This figure makes explicit the important difference between the cases $\sigma>1$ and $\sigma<1$. For $\sigma(Q)>1$ the mode Q is always stable. For $\sigma(Q)<1$ the system is still globally stable, but the mode Q is unstable during the semiperiods in which $(h_0+h_1)^2>1$. In these semiperiods it has an exponential growth. As a consequence we find a periodic amplification of fluctuations to anomalously large values for a noncritical stationary state. These anomalous fluctuations can be described in terms of the function which modulates the equilibrium fluctuations associated with the mean value of h^2, $\langle h^2\rangle=h_0^2+h_1^2$,

$$C_i^\infty(\tau)=C_{\text{eq}}(h^2=h_0^2+h_1^2)F_i(\tau),\quad i=1,2 \tag{4.9}$$

since

$$C_{\text{eq}}(h^2=h_0^2+h_1^2)=\varepsilon/(\sigma r_1)\ ,$$

$$F_i(\tau)=(\sigma r_1/\varepsilon)P_i(\tau)\ .$$

In the limit of no modulation ($\mu=0$) $F_i=1$. The effect of modulation in the fluctuations is more clearly displayed in the opposite limit $\mu\to\infty$. In this limit and at the end of the semiperiods we obtain

$$F_1(\tau=T/2)=\frac{\sigma}{\sigma+1}\ , \tag{4.10}$$

$$F_2(\tau=T/2)=\begin{cases}\dfrac{\sigma}{\sigma-1}, & \sigma>1\\[2mm] \dfrac{2\sigma}{1-\sigma^2}\exp[f^{-1}(Q)(1-\sigma)\mu], & \sigma<1\ .\end{cases} \tag{4.11}$$

These results just indicate that for $\sigma>1$ equilibrium fluctuations are modulated by a finite periodic quantity. However, for $\sigma<1$ fluctuations are greatly enhanced, the modulating function grows exponentially with μ, and its maximum value is reached at the end of the semiperiod in which Q is unstable.

The preceding discussion on the amplification of fluctuations is qualitatively independent of hydrodynamic effects. Indeed, neglecting the coupling of the director and velocity fields amounts to formally setting $f(Q)=1$ in the preceding formulas. In particular, (4.10) and (4.11) are not modified when there is no instability ($\sigma>1$), and for $\sigma<1$ we just find a smaller amplification since $f^{-1}(Q)\geq 1$. However, the inclusion of hydrodynamic effects permits us now to address the question of the formation of spatial structures. For a time-independent magnetic field these structures appear as a transient phenomenon caused by hydrodynamic effects. They are associated with a most unstable mode $Q\neq 0$. The question now consists in the possible appearance and stability of these structures for the modulating conditions we are dealing with here. Of course, such structures could only appear when there is some kind of instability in the system. Therefore we restrict our discussion to $\sigma<1$. The spatial structure may be expected to appear during the semiperiod in which $h^2>1$. A necessary condition for the formation of such a structure is then that $C_2^\infty(\tau)$ has a maximum at a value $Q\neq 0$ for some time $\tau<T/2$. The emergence of this maximum at $Q\neq 0$ identifies the development of a periodic structure. The requirements for the fulfillment of this condition are more easily determined in the limit $\mu\to\infty$, in which the structure could most likely occur. In this limit $C_2^\infty(\tau)$ can be rewritten as

$$\frac{C_2^\infty(\tau)}{\varepsilon}=\frac{C_{\text{eq}}(h^2=(h_0-h_1)^2)}{\varepsilon}e^{\omega(Q)\tau}+\frac{1}{(h_0+h_1)^2-1-(k_{33}/k_{22})Q^2}(e^{\omega(Q)\tau}-1)\ , \tag{4.12}$$

where

$$\omega(Q)=2f^{-1}(Q)[(h_0+h_1)^2-1-(k_{33}/k_{22})Q^2]\ . \tag{4.13}$$

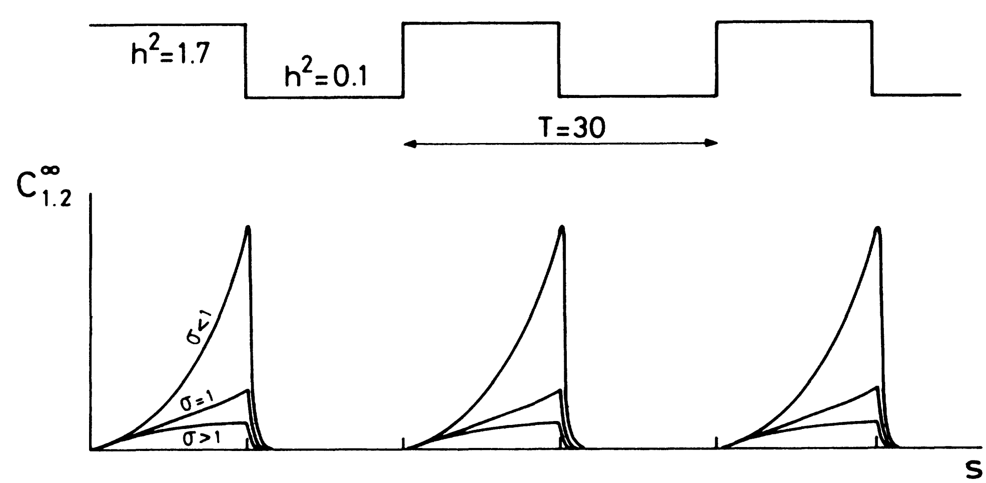

FIG. 3. Long-time-limit results for the periodically modulated structure factor in the regime $r_0>0$. The three well-differentiated behaviors corresponding to the value of the instability parameter σ are depicted.

Equation (4.12) corresponds to the evolution of the structure factor under a constant magnetic field $h^2=(h_0+h_1)^2$ starting at $\tau=0$ with the equilibrium initial condition associated with a field $h^2=(h_0-h_1)^2$. In other words, in this limit the system reaches equilibrium at the end of each semiperiod in which $h=h_0-h_1$. A necessary condition for the occurrence of a periodic pattern can now be taken from the results of Ref. 2,

$$(h_0+h_1)^2>1+\frac{k_{33}}{k_{22}}\frac{\bar{\eta}}{\bar{\alpha}} . \qquad (4.14)$$

In terms of the parameters r_1 and r_0 used here (4.14) becomes

$$r_1-r_0>\frac{k_{33}}{k_{22}}\frac{\bar{\eta}}{\bar{\alpha}} . \qquad (4.15)$$

Properly speaking, (4.14) is a condition to have a most unstable mode specified by a nonhomogeneous mode $Q\neq 0$. This instability would manifest itself as a maximum of the structure factor only if the semiperiod of instability lasts long enough. This is always the case in the limit under consideration, $\mu=r_1T\to\infty$, since the requirements $\sigma(Q=0)<1$ and $h^2(s)>0$ imply that r_1 is bounded; $1-r_0>r_1>r_0$.

In the case of a constant magnetic field the condition to obtain a spatial pattern can be always satisfied by taking a large enough magnetic field. However, in the case of a modulated magnetic field, (4.15) can be only fulfilled if a certain unequality among the material parameters is satisfied. Indeed, the previously mentioned requirements on r_1 and (4.15) imply that

$$1>\frac{k_{33}}{k_{22}}\frac{\bar{\eta}}{\bar{\alpha}} . \qquad (4.16)$$

In the limit $\mu\gg 1$, Eqs. (4.15) and (4.16) give conditions for the occurrence of the spatial pattern. A different question is the persistence of the pattern through the whole period of modulation. Such persistence would imply that the maximum of the structure factor remains at a value of Q different from zero during the whole period. It can be checked from (4.12) that this is not the case, indicating that the pattern is not persistent in the limit $\mu\to\infty$. Outside this limit numerical results for $C_1^\infty(\tau)$ show that the same conclusion applies for finite μ. A typical form of the evolution of the maximum of the structure factor obtained from (4.7)–(4.8) for finite μ is shown in Fig. 4.[7] Larger values of μ give a more intense pattern structure in the sense of earlier formation, longer persistence, and larger values of $Q^2_{\max}$. Decreasing the value of the period the pattern may not even be formed in the periodic solution (4.7), (4.8). Physically it corresponds to situations in which T becomes comparable with the time required for the instability to become apparent. This of the order of the lifetime of the unstable state measured by a mean first-passage time (MFPT).[2,8].

B. $r_0<0$: Transient evolution

We now consider the general solution (4.3)–(4.6) for $r_0<0$. We have already mentioned that the solution only makes sense during a transient regime. This transient regime is at least of the order of the MFPT. We can get an idea of this MFPT by using the appropriate formulas in the absence of modulation

$$T_{\mathrm{MFPT}}\simeq\frac{1}{\dfrac{2}{f(Q)}\left[h^2-1-\dfrac{k_{33}}{k_{22}}Q^2\right]}\ln\frac{1}{2\varepsilon} . \qquad (4.17)$$

In general, this MFPT sets an upper bound of validity of a linear theory. In our situation here this restricts our analysis to periods $T\leq T_{\mathrm{MFPT}}$, although it may be reasonably expected that for fast modulations the range of validity of a linear approach actually extends beyond that MFPT. We are interested in the situation in which the magnetic field $h^2(s)$ periodically crosses the instability

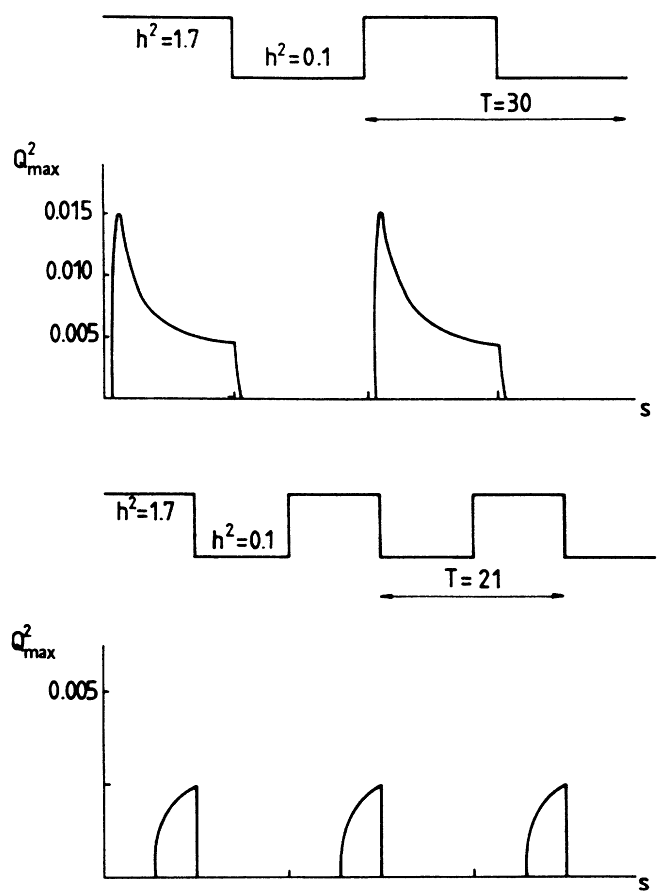

FIG. 4. Wave number corresponding to the maximum of the structure factor vs time for two different values of the modulation period, $T=30$ and $T=21$. Times are measured in units of $\gamma_1/(\chi_a H_c^2)\simeq 10$ sec, corresponding to typical samples ($S=1$ cm^2, $d=10^{-2}$ cm). In the two cases $(h_0+h_1)^2=1.7$, $(h_0-h_1)^2=0.1$. The values for the material parameters are $\bar{\alpha}=0.74$, $\bar{\eta}=0.20$, and $k_{33}/k_{22}=2.5$. No pattern appears for $T\leq 18.9$.

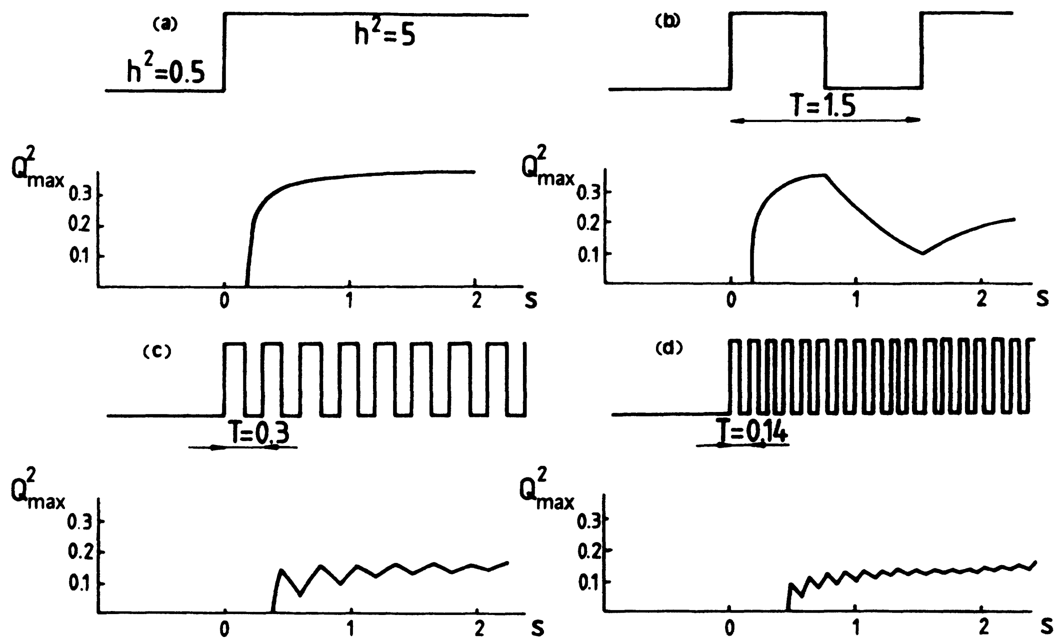

FIG. 5. Evolution of the maximum of the structure factor vs time for different values of the modulation period. For all cases $(h_0+h_1)^2=5$, $(h_0-h_1)^2=0.5$, and material parameters adopt the values presented in Fig. 4. For these parameters the MFPT, Eq. (4.17), is $T_{\rm MFPT}\simeq 1.7$ and the periods of modulation are 1.5, 0.3, and 0.14. The asymptotic value of $Q^2_{\rm max}$ is 0.13.

$h^2(s)=1$. This corresponds to $1+|r_0|\geq r_1>|r_0|$. In particular, we focus on the behavior of the Q modes which are periodically unstable. These modes are those for which $0>\sigma>-1$. The systematic unstable evolution of these modes is given by the exponential terms

$$\exp[-2f^{-1}(Q)\sigma m\mu]$$

in (4.3) and (4.4). Such systematic evolution is the dominant one for fast modulation ($\mu\ll 1$). The emergence of a spatial pattern is then associated with the existence of a mode $Q\neq 0$ of maximum systematic growth. It can be shown that in our conditions such a mode exists for

$$|r_0|>\frac{k_{33}}{k_{22}}\frac{\bar{\eta}}{\bar{\alpha}} \tag{4.18}$$

or

$$h_0^2+h_1^2>1+\frac{k_{33}}{k_{22}}\frac{\bar{\eta}}{\bar{\alpha}} . \tag{4.19}$$

This is the same condition as in the absence of modulation but now for a field h^2, which is the average of $h^2(s)$. Since (4.18) is independent of r_1, there is no restriction on material parameters as we found in (4.16).

Figure 5 shows the evolution of the maximum of the structure factor for different values of the period of modulation as computed from (4.3)–(4.6).[9] It describes how the transient evolution proceeds via the formation of a spatial pattern. The characteristic wavelength of the pattern is determined by $Q_{\rm max}$. As a consequence, it has an oscillatory behavior. $Q^2_{\rm max}$ tends asymptotically to a periodic function of time oscillating around a mean value which coincides with the asymptotic $Q^2_{\rm max}$ predicted for a fixed field, whose magnitude was the average value $h_0^2+h_1^2$ in the actual modulation situation. The most noticeable feature of the results in Fig. 5 is that the emergence of the pattern can be significantly delayed by reducing the period of modulation. The emergence of the pattern occurs at an onset time which is identified with the time at which $Q^2_{\rm max}$ becomes different from zero. This happens to be a rather well-defined time. Here we are dealing with a pattern selection mechanism which is supposedly based on the dominance of the mode of fastest growth. The fast modulation of the control parameter slows down significantly the process of pattern formation. Therefore it gives an efficient way of studying this phenomenon during larger time domains and to describe it within linear theory. The delay in the onset time for pattern formation is seen to occur when the semiperiod of modulation becomes comparable with this onset time in the absence of modulation. This is 0.17 in the units of Fig. 5 and for the parameter values corresponding to this figure. In these circumstances the system has no time to decay from its unstable state during the semiperiod of instability and the time interval of the reorientational dynamics, here manifested in the pattern occurrence, increases.

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[7]The vertical line for Q^2_{max} up to $Q^2_{max}=0.0105$ in Fig. 4 when $T=30$ indicates that Q^2_{max} changes discontinuously from zero to a nonzero value. This happens for $22<T<38$. For these values of T and during a very short time interval the structure factor has a relative maximum at $Q^2\neq 0$ that rapidly dominates the one at $Q=0$. For other values of T the more typical situations of a structure factor with a single maximum occurs for all times. A double peaked structure factor may also appear, for some parameter values, during the transient evolution following an instantaneous change of h to a fixed final value.

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[9]Note that in this figure, contrary to the rest of our discussion, we take the field $h(s)>1$ during the first semiperiod of modulation. This is done for a better comparison with the situation in Fig. 5(a).