First-passage times for non-Markovian processes: Correlated impacts on bound processes

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Our previously developed stochastic trajectory analysis technique has been applied to the calculation of first-passage time statistics of bound processes. Explicit results are obtained for linearly bound processes driven by dichotomous fluctuations having exponential and rectangular temporal distributions.

I. INTRODUCTION

In many applications in the physical sciences and engineering it is necessary to evaluate the *first-passage time*, i.e., the time at which a stochastic process first reaches a "critical value."[1–8] Examples arise in "false alarm" problems in electrical engineering, in mechanical engineering, in chemical physics, and in laser physics.[8]

The numerical analysis, by means of simulations or Monte Carlo methods, of first-passage times and other extreme events is usually very expensive and time consuming since such events are rare and require a large number of long runs to provide reliable statistics.[9,10] The importance of analytic methods for calculating extrema statistics is therefore clear. Nevertheless, these analytic methods are only available in a limited number of cases such as for independent random processes[1,8] and diffusive one-dimensional Markov processes, i.e., processes described by a one-dimensional Fokker-Planck equation.[3–8]

Recently, efforts have been devoted to extending the theory of first-passage times (and other extrema statistics) to non-Markovian processes.[11,12] However, the practical application of these theories depends on the construction of operators for which a general prescription is not available. This construction has only been implemented for simple cases with dichotomous Markovian fluctuations.[12]

In three recent papers[13–15] we have obtained first-passage times for systems driven by dichotomous (not necessarily Markov) fluctuations. Specifically, we have studied simple systems defined by a variable $X(t)$ whose dynamical evolution is given by the stochastic differential equation

$$\dot{X}=F(t)\ . \tag{1.1}$$

Thus, $X(t)$ is an unconstrained "Einstein" process that describes a free process subjected to random impacts. $F(t)$ is a dichotomous random process taking on the values a and $-b$ $(a,b>0)$. The times that $F(t)$ retains the values a and $-b$ are respectively governed by the "switching" distributions $\psi_a(t)$ and $\psi_b(t)$.[16] We have obtained an integral equation that governs the evolution of the first-passage time probability density for arbitrary distributions $\psi_{a,b}(t)$. From this integral equation we have been able to derive closed and exact expressions for the mean first-passage time for several choices of $\psi_{a,b}(t)$.

In this paper our goal is to extend the preceding results to more general one-dimensional bound processes driven by external dichotomous fluctuations

$$\dot{Y}(t)=G(Y)+g(Y)F(t)\ , \tag{1.2}$$

where $G(Y)$ and $g(Y)$ are smooth functions and $F(t)$ is the dichotomous random process defined above. As is well known the change of variables

$$X=\int^{Y}\frac{dy'}{g(y')} \tag{1.3}$$

transforms Eq. (1.2) into an equation with additive fluctuations, i.e., into an equation of the form

$$\dot{X}(t)=f(X)+F(t)\ . \tag{1.4}$$

Therefore we can study the first-passage time for processes whose dynamical evolution is given by equations of the form (1.4) and relate these results directly to the more general processes (1.2) if the relation between X and Y is monotonic.

In Sec. II we detail the dynamics of the system. Section III is devoted to a discussion of the statistical quantities that we need later. In Sec. IV we obtain the equations satisfied by the first-passage time probability density. The formalism is applied to various specific examples in Sec. V, and the conclusions are presented in Sec. VI.

II. DYNAMICS OF THE SYSTEM

We consider a one-dimensional dynamical system driven by external dichotomous fluctuations. The system is specified by the variable $X(t)$ whose dynamical evolution is given by the stochastic differential equation (1.4) where $F(t)$ is a dichotomous (not necessarily Markov) random variables alternately taking on the values a and $-b$, with $a,b>0$, and $\psi_a(t)$ and $\psi_b(t)$ are the probability distributions of the "time of residence" in the states

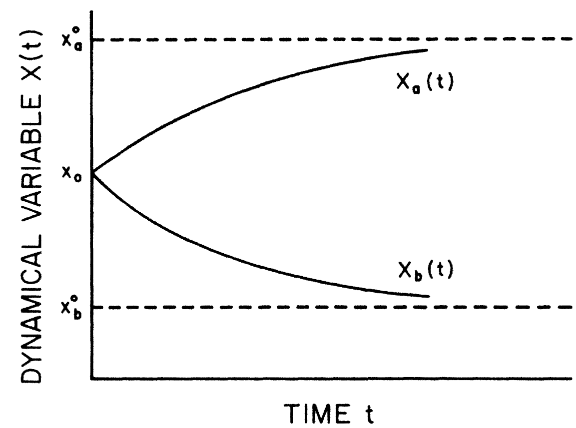

FIG. 1. Dynamical variable as a function of time for two values of $F(t)$. $X_a(t)$ is the trajectory when $F(t)=a$, and $X_b(t)$ is the trajectory when $F(t)=-b$. The trajectories approach their respective asymptotic fixed points x_a^0 and x_b^0.

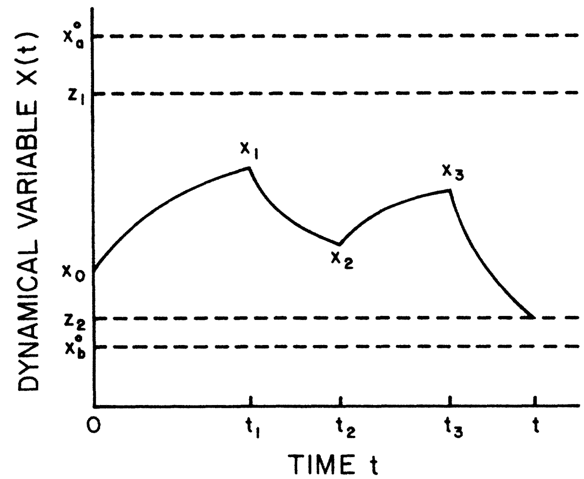

FIG. 2. The trajectory shown is for a random function $F(t)$ that has switched values three times in the interval $(0,t)$. The absorbing boundaries z_1 and z_2 lie within the natural boundaries (asymptotically stable fixed points) x_a^0 and x_b^0.

$F(t)=a$ and $-b$, respectively. We assume that $f(X)$ is smooth and such that *the solution $X(t)$ of Eq. (1.4) does not become infinite in a finite time.*

Let $X_a(t)$ be the solution of Eq. (1.4) when $F(t)=a$ and analogously for $X_b(t)$. Since $f(X)+a \geq f(X)-b$ for all X, we have by the *comparison theorem*[17] that

$$X_a(t) \geq X_b(t) \tag{2.1}$$

for all t. Let x^0 be an asymptotically stable fixed point of Eq. (1.4), e.g., when $x^0=x_a^0$ then

$$f(x_a^0)+a=0 \tag{2.2a}$$

and

$$\lim_{t\to\infty} X_a(t)=x_a^0 , \tag{2.2b}$$

with similar relations for $F(t)=-b$ and $x^0=x_b^0$. Then, from the comparison theorem, Eq. (2.1), we have

$$x_a^0 \geq x_b^0 \tag{2.3}$$

(see Fig. 1). Thus when the process represented by Eq. (1.4) has at least two asymptotically stable fixed points [one for $F(t)=a$ and the other for $F(t)=-b$] there exist two "natural barriers," $X=x_a^0$ and x_b^0, that the system cannot exceed. Therefore if we are interested in finding the mean first-passage time when the process (1.4) reaches certain values, say z_1 or z_2, these values must lie inside the natural barriers (see Fig. 2)

$$x_b^0 < z_2 \leq z_1 < x_a^0 . \tag{2.4}$$

If Eq. (1.4) has only one asymptotically stable fixed point [for example, for $F(t)=a$] we shall assume that $f(X)$ is such that $X_b(t)$ is a decreasing function of time. In this case the restriction (2.4) on the critical values z_1 and z_2 is

$$z_2 \leq z_1 < x_a^0 . \tag{2.5}$$

Finally, when Eq. (1.4) has no asymptotically stable fixed points [as in the case $f(X)=0$] we shall assume that $f(X)$ is such that $X_a(T)$ $[X_b(t)]$ is an increasing (decreasing) function of t. Now no restrictions apply upon the critical values z_1,z_2.

For $F(t)=a$ the solution of the differential equation (1.4) is

$$t=\int_{x_0}^{X} dX \frac{1}{f(X)+a} , \tag{2.6}$$

where

$$x_0=X(t=0) . \tag{2.7}$$

Defining the function

$$\phi_a^{-1}(X) \equiv \int^X dX' \frac{1}{f(X')+a} \tag{2.8}$$

we have from Eq. (2.6) that

$$X_a(t)=\phi_a(t+\phi_a^{-1}(x_0)) . \tag{2.9}$$

The solution of Eq. (1.4) for $F(t)=-b$ is similarly given by

$$X_b(t)=\phi_b(t+\phi_b^{-1}(x_0)) , \tag{2.10}$$

where

$$\phi_b^{-1}(X) \equiv \int^X dX' \frac{1}{f(X')-b} . \tag{2.11}$$

Therefore if we take $F(0)=a$ we have the following trajectory (Fig. 2):

$$X(t)=\phi_a(t+\phi_a^{-1}(x_0)), \quad 0 \leq t \leq t_1 ; \tag{2.12a}$$

$$X(t)=\phi_b(t+\phi_b^{-1}(\phi_a(t_1+\phi_a^{-1}(x_0)))), \quad t_1 \leq t \leq t_1+t_2 ; \tag{2.12b}$$

$$X(t)=\phi_a(t+\phi_a^{-1}(\phi_b(t_2+\phi_b^{-1}(\phi_a(t_1+\phi_a^{-1}(x_0)))))), \quad t_1+t_2 \leq t \leq t_1+t_2+t_3 \tag{2.12c}$$

and so on. For this realization the distribution $\psi_a(t)$ governs the odd time intervals $t_1, t_3, \ldots, t_{2n-1}, \ldots$, and $\psi_b(t)$ governs the even ones. In the following we shall use the notation

$$x_{2n-1} \equiv X(t_{2n-1}) = \phi_a(t_{2n-1} + \phi_a^{-1}(\phi_b(t_{2n-2} + \phi_b^{-1}(\phi_a(t_{2n-3} + \cdots))))) , \tag{2.13a}$$

$$x_{2n} \equiv X(t_{2n}) = \phi_b(t_{2n} + \phi_b^{-1}(\phi_a(t_{2n-1} + \phi_a^{-1}(\phi_b(t_{2n-2} + \cdots))))) \tag{2.13b}$$

for the odd and even intervals of the process. Thus when $F(0)=a$ we have

$$x_{2n-1} = \phi_a(t_{2n-1} + \phi_a^{-1}(x_{2n-2})) , \tag{2.14a}$$

$$x_{2n} = \phi_b(t_{2n} + \phi_b^{-1}(x_{2n-1})) \tag{2.14b}$$

$(n = 0,1,2,\ldots)$, where t_{2n-1} is governed by $\psi_a(t)$ and t_{2n} by $\psi_b(t)$. On the other hand if the process starts with $F(0) = -b$ we have

$$x_{2n-1} = \phi_b(t_{2n-1} + \phi_b^{-1}(x_{2n-2})) , \tag{2.15a}$$

$$x_{2n} = \phi_a(t_{2n} + \phi_a^{-1}(x_{2n-1})) \tag{2.15b}$$

$(n = 0,1,2,\ldots)$, where now t_{2n-1} is governed by $\psi_b(t)$ and t_{2n} by $\psi_a(t)$.

III. FIRST-PASSAGE TIME PROBABILITY DENSITY

Our goal is to calculate the conditional first-passage time probability density $p(t;x_0)$ defined as follows:

$p(t;x_0)dt \equiv$ Probability that the process $X(\tau)$ [given that $X(0)=x_0$]
crosses z_1 or z_2 in the time range $t \le \tau \le t+dt$
without ever having crossed either of these levels
during the time span $0 \le \tau < t$. (3.1)

To calculate $p(t;x_0)$ it is useful to denote each time range t_n between switches as an "interval" and to define the auxiliary probability

$p_n(t;x_0)dt =$ Probability that the first crossing of z_1 or z_2
occurs during the nth interval in the time range $(t,t+dt)$. (3.2)

Clearly, the first-passage time probability density is

$$p(t;x_0) = \sum_{n=1}^{\infty} p_n(t;x_0) . \tag{3.3}$$

The existence of two realizations of the stochastic process $F(t)$ in (2.1) leads one to define the two probability densities

$$p^{(a)}(t;x_0) \equiv p(t;x_0)\,|_{F(0)=a} \tag{3.4}$$

and

$$p^{(-b)}(t;x_0) \equiv p(t;x_0)\,|_{F(0)=-b} \tag{3.5}$$

[the same definitions apply for $p_n^{(a,-b)}(t;x_0)$]. Therefore if $w_0(a\,|\,x_0)$ $[w_0(-b\,|\,x_0)]$ is the probability that $F(0)=a$ $[F(0)=-b]$, given that $X(0)=x_0$, then

$$p(t;x_0) = p^{(a)}(t;x_0)w_0(a\,|\,x_0) + p^{(-b)}(t;x_0)w_0(-b\,|\,x_0) . \tag{3.6}$$

The distributions $p^{(a)}(t;x_0)$ and $p^{(-b)}(t;x_0)$ are not independent of one another since there exist symmetry relations between them. Indeed, from the description of the model in Sec. II, it is easy to see that if in the expressions for $p^{(a)}(t;x_0)$ we make the replacements

$$\begin{gathered} a \to -b , \\ -b \to a , \\ \psi_a \to \psi_b , \\ \psi_b \to \psi_a , \\ z_1 \to z_2 , \\ z_2 \to z_1 , \end{gathered} \tag{3.7}$$

we obtain $p^{(-b)}(t;x_0)$ and similarly, given $p^{(-b)}(t;x_0)$ one can obtain $p^{(a)}(t;x_0)$.

The probability densities $p_n(t;x_0)$ can be constructed explicitly from the trajectories (2.12). Let us consider a realization that begins with $F(0)=a$, as detailed in Eq. (2.12). We wish to insure that no crossing of the levels z_1 and z_2 has occurred in the first $(n-1)$ intervals and that a crossing does occur during the nth interval. During the first interval no crossing occurs if

$$x_1 = \phi_a(t_1 + \phi_a^{-1}(x_0)) < z_1 \tag{3.8a}$$

or, equivalently, if

$$t_1 < \int_{x_0}^{z_1} dX \frac{1}{f(X)+a} \equiv \tau_1 . \tag{3.8b}$$

The probability that the inequality (3.8b) holds is

$$\text{Prob}(t_1<\tau_1)=\int_0^{\tau_1}\Phi_a(t_1)dt_1 , \qquad (3.9)$$

where $\Phi_a(t)$ is the probability density for the first interval. For example, in a "modified renewal process"[16] the usual choice is

$$\Phi_a(t)=\lambda_a\int_{-\infty}^{0}\psi_a(t-\tau)d\tau . \qquad (3.10)$$

In general, $\Phi_a(t)$ depends on the preparation of the system. No crossing during the second interval occurs if

$$x_2=\phi_b(t_2+\phi_b^{-1}(x_1))>z_2 , \qquad (3.11a)$$

i.e., if

$$t_2<\int_{x_1}^{z_2}dX\frac{1}{f(X)-b}\equiv\tau_2 . \qquad (3.11b)$$

The probability that this inequality is satisfied is

$$\text{Prob}(t_2<\tau_2)=\int_0^{\tau_2}\psi_b(t_2)dt_2 . \qquad (3.12)$$

Similar conditions can be written for the probability that each successive interval up to and including the $(n-1)$st does not lead to a crossing. The explicit expressions are

$$\text{Prob}(t_{2i-1}<\tau_{2i-1})=\int_0^{\tau_{2i-1}}\psi_a(t_{2i-1})dt_{2i-1} ,\quad i=2,3,4,\ldots, \qquad (3.13)$$

$$\text{Prob}(t_{2i}<\tau_{2i})=\int_0^{\tau_{2i}}\psi_b(t_{2i})dt_{2i},\quad i=1,2,3,\ldots, \qquad (3.14)$$

where

$$\tau_{2i-1}\equiv\int_{x_{2i-2}}^{z_1}dX\frac{1}{f(X)+a} \qquad (3.15)$$

and

$$\tau_{2i}\equiv\int_{x_{2i-1}}^{z_2}dX\frac{1}{f(X)-b} . \qquad (3.16)$$

As can be seen from Eq. (2.13) the quantities τ_i are functions of the switching times $t_{i-1},t_{i-2},\ldots,t_1$:

$$\tau_i=\tau_i(x_{i-1})\equiv\tau(t_{i-1},t_{i-2},\ldots,t_2,t_1) \qquad (3.17)$$

for $i=2,3,4,\ldots,$ and

$$\tau_1=\tau(x_0) . \qquad (3.18)$$

To proceed with our explicit construction we must choose the parity of n: If it is odd then a crossing during the nth interval can only occur at z_1, while an even n can only lead to a crossing at the level z_2 [and the converse if $F(0)=-b$, see Fig. 2]. We select $n=2m$ and note that the z_2 level *will* be crossed during the $2m$th interval if

$$x_{2m}=\phi_b(t_{2m}+\phi_b^{-1}(x_{2m-1}))<z_2 , \qquad (3.19a)$$

i.e., if

$$t_{2m}>\int_{x_{2m-1}}^{z_2}dX\frac{1}{f(X)-b}\equiv\tau_{2m} . \qquad (3.19b)$$

The probability that this inequality is satisfied is

$$\text{Prob}(t_{2m}>\tau_{2m})=\int_{\tau_{2m}}^{\infty}\psi_b(t)dt . \qquad (3.20)$$

Finally, we must specify *when* during the $2m$th interval the crossing actually occurs. For the crossing to occur at time t it is necessary that

$$X(t)=\phi_b(\Delta(t)+\phi_b^{-1}(x_{2m-1}))=z_2 \qquad (3.21)$$

where

$$\Delta(t)\equiv t-(t_1+t_2+\cdots+t_{2m-1}) . \qquad (3.22)$$

Taking into account Eq. (2.12) we can write Eq. (3.21) as follows:

$$\Delta(t)=\int_{x_{2m-1}}^{z_2}dX\frac{1}{f(X)-b}\equiv\tau_{2m} \qquad (3.23)$$

and therefore Eq. (3.21) is equivalent to

$$t=t_1+t_2+\cdots+t_{2m-1}+\tau_{2m} . \qquad (3.24)$$

The probability density for this crossing event is the delta function

$$\delta(t-(t_1+t_2+\cdots+t_{2m-1}+\tau_{2m})) . \qquad (3.25)$$

Collecting the results (3.9), (3.13), (3.14), (3.20), and (3.25) immediately gives the following integral expression for the probability density $p_{2m}^{(a)}(t;x_0)$:

$$p_{2m}^{(a)}(t;x_0)=\int_0^{\tau_1}dt_1\,\Phi_a(t_1)\times\int_0^{\tau_2}dt_2\,\psi_b(t_2)\cdots\int_0^{\tau_{2m-1}}dt_{2m-1}\,\psi_a(t_{2m-1})\times\int_{\tau_{2m}}^{\infty}dt_{2m}\,\psi_b(t_{2m})\delta(t-(t_1+t_2+\cdots+t_{2m-1}+\tau_{2m})) \qquad (3.26)$$

for $m\geq 1$. For odd n similar reasoning leads to

$$p_{2m-1}^{(a)}(t;x_0)=\int_0^{\tau_1}dt_1\,\Phi_a(t_1)\times\int_0^{\tau_2}dt_2\,\psi_b(t_2)\cdots\int_0^{\tau_{2m-2}}dt_{2m-2}\,\psi_b(t_{2m-2})\times\int_{\tau_{2m-1}}^{\infty}dt_{2m-1}\,\psi_a(t_{2m-1})\delta(t-(t_1+t_2+\cdots+t_{2m-2}+\tau_{2m-1})) \qquad (3.27)$$

for $m \geq 2$, and

$$p_1^{(a)}(t;x_0)=\delta(t-\tau_1)\int_{\tau_1}^{\infty} dt_1\, \psi_a(t_1)\ . \tag{3.28}$$

Similar expressions can be obtained for $p_n^{(-b)}(t;x_0)$ [see Eq. (3.7)].

IV. INTEGRAL EQUATION FOR THE EVOLUTION OF THE FIRST-PASSAGE TIME PROBABILITY DENSITY

Our next step is to Laplace transform Eqs. (3.26)—(3.28) according to the definition

$$\tilde{p}(s;x_0)\equiv\int_0^{\infty} dt\, e^{-st}p(t;x_0)\ , \tag{4.1}$$

and to establish an integral recursion relation to connect the nth and $(n+2)$nd densities. The recursion relation leads to an integral equation for the Laplace transform $\tilde{p}(s;x_0)$ of the first-passage time probability density. From Eqs. (3.26)—(3.28) and (4.1) we have

$$\tilde{p}_1^{(a)}(s;x_0)=e^{-s\tau_1}\int_{\tau_1}^{\infty} dt_1\, \Phi_a(t_1)\ , \tag{4.2}$$

$$\tilde{p}_{2m-1}^{(a)}(s;x_0)=\int_0^{\tau_1} dt_1\, \Phi_a(t_1) \times\int_0^{\tau_2} dt_2\, \psi_b(t_2)\cdots\int_0^{\tau_{2m-2}} dt_{2m-2}\, \psi_b(t_{2m-2}) \times\int_{\tau_{2m-1}}^{\infty} dt_{2m-1}\, \psi_a(t_{2m-1})e^{-s(t_1+t_2+\cdots+t_{2m-2}+\tau_{2m-1})}\ , \tag{4.3}$$

$$\tilde{p}_{2m}^{(a)}(s;x_0)=\int_0^{\tau_1} dt_1\, \Phi_a(t_1)\int_0^{\tau_2} dt_2\, \psi_b(t_2)\cdots\int_0^{\tau_{2m-1}} dt_{2m-1}\, \psi_a(t_{2m-1}) \times\int_{\tau_{2m}}^{\infty} dt_{2m}\, \psi_b(t_{2m})e^{-s(t_1+t_2+\cdots+t_{2m-1}+\tau_{2m})}\ . \tag{4.4}$$

If we define the auxiliary functions

$$\tilde{I}_{2m-2}^{(a)}(s;x_1)=\int_0^{\tau_2} dt_2\, \psi_b(t_2)\int_0^{\tau_3} dt_3\, \psi_a(t_3)\cdots\int_0^{\tau_{2m-2}} dt_{2m-2}\, \psi_b(t_{2m-2}) \times\int_{\tau_{2m-1}}^{\infty} dt_{2m-1}\, \psi_a(t_{2m-1}) \times e^{-s(t_2+t_3+\cdots+t_{2m-2}+\tau_{2m-1})} \tag{4.5a}$$

and

$$\tilde{I}_{2m-1}^{(a)}(s;x_1)=\int_0^{\tau_2} dt_2\, \psi_b(t_2)\int_0^{\tau_3} dt_3\, \psi_a(t_3)\cdots\int_0^{\tau_{2m-1}} dt_{2m-1}\, \psi_a(t_{2m-1}) \times\int_{\tau_{2m}}^{\infty} dt_{2m}\, \psi_b(t_{2m})e^{-s(t_2+t_3+\cdots+t_{2m-1}+\tau_{2m})} \tag{4.5b}$$

in terms of which

$$\tilde{p}_n^{(a)}(s;x_0)=\int_0^{\tau_1} dt_1\, \Phi_a(t_1)e^{-st_1}\tilde{I}_{n-1}^{(a)}(s;x_1(t_1)),\quad n\geq 2\ , \tag{4.6}$$

we obtain the *recursion relation*

$$\tilde{I}_{n+2}^{(a)}(s;x_0)=\int_0^{\tau_2} dt_2\, e^{-st_2}\psi_b(t_2)\int_0^{\tau_3} dt_3\, e^{-st_3}\psi_a(t_3)\tilde{I}_n^{(a)}(s;x_3(t_3,t_2)),\quad n\geq 1\ . \tag{4.7}$$

Summing this recursion relation from $n=1$ to $n=\infty$ leads to the integral equation

$$\tilde{I}^{(a)}(s;x_1)=\tilde{I}_1^{(a)}(s;x_1)+\tilde{I}_2^{(a)}(s;x_1)+\int_0^{\tau_2} dt_2\, e^{-st_2}\psi_b(t_2)\int_0^{\tau_3} dt_3\, e^{-st_3}\psi_a(t_3)\tilde{I}^{(a)}(s;x_3(t_3;t_2)) \tag{4.8}$$

where

$$\tilde{I}^{(a)}(s;x)\equiv\sum_{n=1}^{\infty}\tilde{I}_n^{(a)}(s;x)\ . \tag{4.9}$$

In terms of this function the Laplace transform of the first-passage time probability density [for $F(0)=a$] is

$$\tilde{p}^{(a)}(s;x_0)=\tilde{p}_1^{(a)}(s;x_0)+\int_0^{\tau_1} dt_1\Phi_a(t_1)e^{-st_1}\tilde{I}^{(a)}(s;x_1(t_1))\ . \tag{4.10}$$

By means of the equivalence given in Eq. (3.7), it is straightforward to find the analogous expressions for $\tilde{I}^{(-b)}(s;x)$ and $\tilde{p}^{(-b)}(s;x_0)$:

$$\tilde{I}^{(-b)}(s;x_1)=\tilde{I}_1^{(-b)}(s;x_1)+\tilde{I}_2^{(-b)}(s;x_1)+\int_0^{\bar{\tau}_2} dt_2\, e^{-st_2}\psi_a(t_2)\int_0^{\bar{\tau}_3} dt_3\, e^{-st_3}\psi_b(t_3)\tilde{I}^{(-b)}(s;x_3(t_3,t_2)) \tag{4.11}$$

and

$$\tilde{p}^{(-b)}(s;x_0)=\tilde{p}_1^{(-b)}(s;x_0)+\int_0^{\bar{\tau}_1} dt_1\,\Phi_b(t_1)e^{-st_1}\tilde{I}^{(-b)}(s;x_1(t_1)) \tag{4.12}$$

where

$$\bar{\tau}_{2m-1}\equiv\int_{x_{2m-2}}^{z_2} dX\frac{1}{f(X)-b}\ , \tag{4.13}$$

$$\bar{\tau}_{2m}\equiv\int_{x_{2m-1}}^{z_1} dX\frac{1}{f(X)+a}\ . \tag{4.14}$$

[Note that now the trajectories x_m are given by Eq. (2.15).] Finally, the Laplace transform of the first-passage time probability density [Eq. (3.6)] *for arbitrary* $F(0)$ is

$$\tilde{p}(s;x_0)=\tilde{p}^{(a)}(s;x_0)w_0(a\mid x_0)+\tilde{p}^{(-b)}(s;x_0)w_0(-b\mid x_0)\ . \tag{4.15}$$

The entire problem has been reduced to solving the integral equation (4.8). However, as has been pointed out previously,[13-15] exact solutions of such integral equations cannot be found for arbitrary forms of the $\psi_{a,b}(t)$, even in the simplest case $f(X)=0$, although one can construct approximation schemes for specific forms of these functions. Nevertheless there *are* situations when the integral equations can be solved *exactly* for special, but relevant, forms of $\psi_{a,b}(t)$. In the next section we give two such examples.

We should note a further simplification in these results when the system, as often happens, is prepared in such a way that $\Phi_{a,b}(t)=\psi_{a,b}(t)$ (this is the case of an "ordinary renewal process"[16]). In this case the auxiliary functions are $\tilde{I}(s;x)=\tilde{p}(s;x)$ and the probability densities $\tilde{p}^{(a)}(s;x_0)$ and $\tilde{p}^{(-b)}(s;x_0)$ themselves satisfy the integral equations

$$\tilde{p}^{(a)}(s;x_0)=\tilde{p}_1^{(a)}(s;x_0)+\tilde{p}_2^{(a)}(s;x_0)+\int_0^{\tau_1} dt_1\,e^{-st_1}\psi_a(t_1)\int_0^{\tau_2} dt_2\,e^{-st_2}\psi_b(t_2)\tilde{p}^{(a)}(s;x_2(t_2,t_1)) \tag{4.16}$$

and

$$\tilde{p}^{(-b)}(s;x_0)=\tilde{p}_1^{(-b)}(s;x_0)+\tilde{p}_2^{(-b)}(s;x_0)+\int_0^{\bar{\tau}_1} dt_1\,e^{-st_1}\psi_b(t_1)\int_0^{\bar{\tau}_2} dt_2\,e^{-st_2}\psi_a(t_2)\tilde{p}^{(-b)}(s;x_2(t_2,t_1))\ . \tag{4.17}$$

V. APPLICATIONS

In this section we evaluate the first-passage time of the bound process (1.4) for various forms of $\psi_{a,b}(t)$ for which the integral equation (4.8) can be converted to an equivalent differential equation that can be solved analytically with the appropriate boundary conditions. For simplicity we take $\Phi(t)=\psi(t)$, although other forms of $\Phi(t)$ can be easily incorporated.

A. Dichotomous Markov process $F(t)$

If $F(t)$ is a dichotomous Markov process, then the distributions $\psi_{a,b}(t)$ are exponential,

$$\psi_\rho(t)=\lambda_\rho e^{-\lambda_\rho t},\quad \rho=a,b\ , \tag{5.1}$$

where λ_a^{-1} and λ_b^{-1} are the average residence times in the states $F(t)=a$ and $-b$, i.e., λ_a^{-1} and λ_b^{-1} are average times between switches. In order to insure that the fluctuations are zero centered we impose the condition $a\lambda_b=b\lambda_a$. For these fluctuations we show in the Appendix that the integral equation (4.16) is equivalent to the second-order differential equation

$$\frac{d^2\tilde{p}^{(a)}(s;x_0)}{dx_0^2}+\left[\frac{f'(x_0)}{f(x_0)+a}-\frac{s+\lambda_b}{f(x_0)-b}-\frac{s+\lambda_a}{f(x_0)+a}\right]\frac{d\tilde{p}^{(a)}(s;x_0)}{dx_0}+\frac{s(s+\lambda_a+\lambda_b)}{[f(x_0)+a][f(x_0)-b]}\tilde{p}^{(a)}(s;x_0)=0 \tag{5.2}$$

together with the boundary conditions

$$\text{(i)}\quad \tilde{p}^{(a)}(s;z_1)=1\ , \tag{5.3}$$

$$\text{(ii)}\quad \left.\frac{d\tilde{p}^{(a)}(s;x_0)}{dx_0}\right|_{x_0=z_2}=\frac{1}{f(z_2)+a}[-\lambda_a+(s+\lambda_a)\tilde{p}^{(a)}(s;z_2)]\ . \tag{5.4}$$

The first boundary condition insures that a process initiated at the upper boundary with positive slope is immediately trapped with certainty. The second boundary condition is not of the usual form for a Fokker-Planck process in which trapping at the lower boundary is also guaranteed if the process starts there [i.e., $\tilde{p}^{(a)}(s;z_2)=1$]. The physical interpretation of condition (5.4) is not straightforward.

A differential equation for the mean first-passage time $T_1^{(a)}(x_0)$ can be obtained using the defining relation

$$T_1^{(a)}(x_0) \equiv \int_0^\infty dt\, t p^{(a)}(t;x_0) = -\frac{\partial}{\partial s}\tilde{p}^{(a)}(s;x_0)\bigg|_{s=0} . \tag{5.5}$$

The derivative of Eq. (5.2) with respect to s evaluated at $s=0$ yields the differential equation

$$\frac{d^2T_1^{(a)}(x_0)}{dx_0^2} + \left[\frac{f'(x_0)}{f(x_0)+a} - \frac{\lambda_b}{f(x_0)-b} - \frac{\lambda_a}{f(x_0)+a}\right]\frac{dT_1^{(a)}(x_0)}{dx_0} = \frac{\lambda_a+\lambda_b}{[f(x_0)+a][f(x_0)-b]} . \tag{5.6}$$

The boundary conditions for (5.6) are similarly obtained from (5.3) and (5.4):

$$\text{(i)}\quad T_1^{(a)}(z_1)=0 \tag{5.7}$$

$$\text{(ii)}\quad \frac{dT_1^{(a)}(x_0)}{dx_0}\bigg|_{x_0=z_2} = \frac{1}{f(z_2)+a}[\lambda_a T_1^{(a)}(z_2)-1] . \tag{5.8}$$

The solution of the problem (5.6)–(5.8) is straightforward and is given by

$$T_1^{(a)}(x_0) = \int_{z_1}^{x_0} V^{(a)}(x)e^{-M^{(a)}(x)}dx + C^{(a)}\int_{z_1}^{x_0} e^{-M^{(a)}(x)}dx \tag{5.9}$$

where

$$M^{(a)}(x) \equiv \int^x \left[\frac{f'(x)}{f(y)+a} - \frac{\lambda_b}{f(y)-b} - \frac{\lambda_a}{f(y)+a}\right]dy , \tag{5.10}$$

$$V^{(a)}(x) \equiv \int^x \frac{\lambda_a+\lambda_b}{[f(y)+a][f(y)-b]}e^{M^{(a)}(y)}dy , \tag{5.11}$$

and

$$C^{(a)} \equiv \frac{-V^{(a)}(z_2)e^{-M^{(a)}(z_2)} + \dfrac{1}{f(z_2)+a}\left[\lambda_a\displaystyle\int_{z_1}^{z_2} V^{(a)}(x)e^{-M^{(a)}(x)}dx - 1\right]}{e^{-M^{(a)}(z_2)} - \dfrac{\lambda_a}{f(z_2)+a}\displaystyle\int_{z_1}^{z_2} e^{-M^{(a)}(x)}dx} . \tag{5.12}$$

Following a similar method we can easily obtain closed analytic expressions for all the first-passage time moments, $T_n(x_0)$. *We can therefore assert that the problem of extrema statistics for unidimensional stochastic processes driven by dichotomous Markov noise has been completely solved.*

Before closing this section we evaluate the mean first-passage time for two particular forms of the drift $f(X)$. One is the previously treated case of no drift, i.e., $f(X)=0$. (This is the only case which has been treated by a variety of methods in the literature.[12–15]) The second example is that of a linear drift $f(X)=-\mu X$.

1. No drift (Refs. 13–15)

In this case

$$f(X)=0 . \tag{5.13}$$

For simplicity we choose: $a=b$, $\lambda_a=\lambda_b\equiv\lambda$, $z_1=z$, $z_2=-z$; $w_0(a\,|\,x_0)=\frac{1}{2}$, and $w_0(-a\,|\,x_0)=\frac{1}{2}$. Then from Eqs. (5.9)–(5.12) we obtain

$$T_1(x_0) \equiv \tfrac{1}{2}[T_1^{(a)}(x_0)+T_1^{(-a)}(x_0)] = \frac{\lambda}{a^2}(z^2-x_0^2)+\frac{z}{a} \tag{5.14}$$

which agrees with previous results.[12–15] An average of $T_1(x_0)$ over a uniform distribution of initial conditions yields the averaged mean first-passage time[14,15]

$$T_1 \equiv \frac{1}{2z}\int_{-z}^{z} T_1(x_0)dx_0 = \frac{2}{3}\frac{\lambda}{a^2}z^2+\frac{z}{a} . \tag{5.15}$$

For comparison, we note that the corresponding results for a diffusive process with diffusion coefficient D are[14,15]

$$T_1(x_0)=(z^2-x_0^2)/2D \tag{5.16}$$

and

$$T_1=\frac{z^2}{3D} . \tag{5.17}$$

These results are shown in Figs. 3 and 4.

2. Linear drift

Here we take

$$f(X)=-\mu X \quad (\mu>0) . \tag{5.18}$$

In this case the solutions of the differential equation (1.4) for $F(t)=a,-b$ are

$$X_a(t)=\frac{a}{\mu}-(a-\mu x_0)^{-\mu t} \tag{5.19}$$

and

$$X_b(t)=-\frac{b}{\mu}+(b+\mu x_0)^{-\mu t} . \tag{5.20}$$

We see from these equations that (a/μ) $[-(b/\mu)]$ is the asymptotically stable fixed point of $X_a(t)$ $[X_b(t)]$. There-

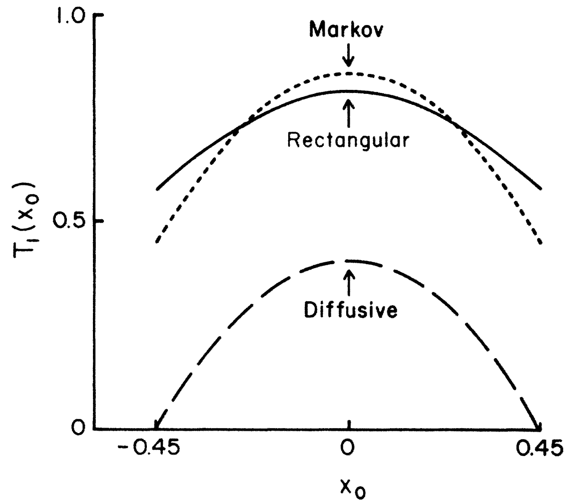

FIG. 3. Mean first-passage time $T_1(x_0)$ as a function of the initial location x_0 with parameter values $\lambda=2$, $a=1$, and $z=0.45$. The driftless processes depicted are diffusive (— — —), dichotomous Markov (- - -), and dichotomous rectangular (——).

fore the process $X(t)$ can only reach the critical values z_1 and z_2 if they lie within the interval $(-b/\mu, a/\mu)$, i.e., if

$$-\frac{b}{\mu} < z_2 \le z_1 < \frac{a}{\mu} \ . \tag{5.21}$$

Introducing Eq. (5.18) into Eqs. (5.9)–(5.12) yields an expression for the mean first-passage time $T_1(x_0)$ in terms of hypergeometric functions. The resulting expression is not particularly instructive in its most general form. In order to obtain a more transparent form we note that the hypergeometric functions reduce to elementary functions for certain choices of the parameter values. In particular, with the choices $a=b=1$, $\lambda_a=\lambda_b=2$, $\mu=1$, $z_1=z$, $z_2=-z$ $(|z|<1)$, $w_0(a\,|\,x_0)=\frac{1}{2}$, and $w_0(-a\,|\,x_0)=\frac{1}{2}$, Eq. (5.9) gives the simple form

$$T_1(x_0)=\tfrac{1}{2}\ln\left[\frac{1-x_0^2}{1-z^2}\right]+\frac{2}{3}\frac{(x_0^2-2)}{(1-x_0^2)^2}-\frac{(z^2+z-4)}{3(1+x)(1-z)^2} \ . \tag{5.22}$$

The averaged mean first-passage time for a uniform initial distribution is

$$T_1=\frac{-3z^3+2z^2+3z}{3(1+z)(1-z)^2} \ . \tag{5.23}$$

The corresponding results for a diffusive process are[18]

$$T_1(x_0)=8\int_{x_0}^{z} du\, e^{2u^2}\int_0^u e^{-2y^2}dy \tag{5.24}$$

and

$$T_1=2\left[\frac{e^{2z^2}}{z}\int_0^z dy\, e^{-2y^2}-1\right] \ . \tag{5.25}$$

These results are compared in Figs. 5 and 6.

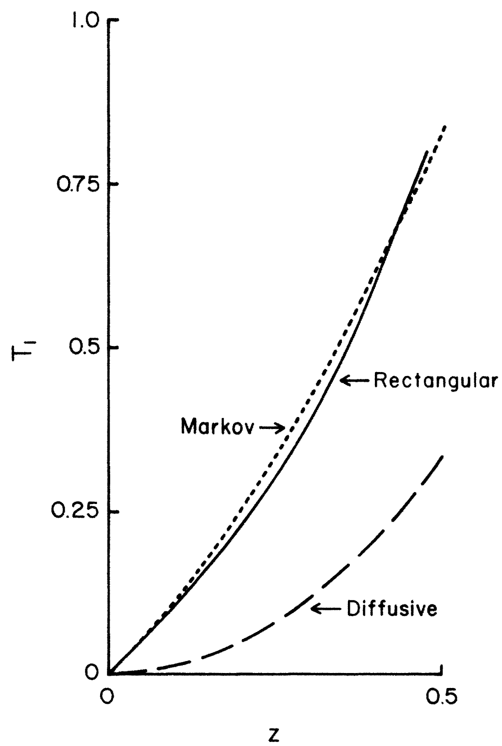

FIG. 4. Mean first-passage time T_1 as a function of separation between absorbing boundaries ($2z$) for various driftless processes. The processes shown are diffusive (— — —), dichotomous Markov (- - -), and dichotomous rectangular (——).

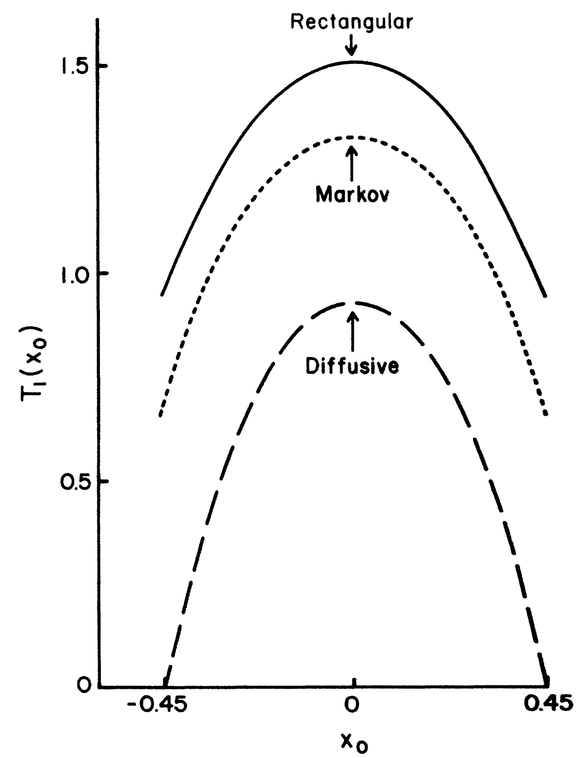

FIG. 5. Mean first-passage time $T_1(x_0)$ as a function of the initial location x_0 with parameter values $\lambda=2$, $a=1$, and $z=0.45$, The linearly bound processes shown are diffusive (— — —), dichotomous Markov (- - -), and dichotomous rectangular (——).

B. Non-Markovian fluctuations: The rectangular process $F(t)$

One example of a simple non-Markovian dichotomous process is a rectangular process in which the distributions $\psi_p(t)$, $p=a,b$, are given by

$$\psi_p(t)=\begin{cases}\dfrac{\lambda_p}{2}, & 0\le t\le \dfrac{2}{\lambda_p}\\ 0, & \text{otherwise}\,.\end{cases} \tag{5.26}$$

Here once again λ_a^{-1} and λ_b^{-1} are the mean times between switches. In order to allow the process a finite probability of reaching the boundary z_1 or z_2 even in the first time interval we restrict our analysis to the parameter domains

$$\frac{1}{\lambda_b}>\tfrac{1}{2}\int_{z_1}^{z_2}dX\frac{1}{f(X)-b}\ . \tag{5.27a}$$

and

$$\frac{1}{\lambda_b}>\tfrac{1}{2}\int_{z_1}^{z_2}dX\frac{1}{f(X)-b}\ . \tag{5.27b}$$

In the Appendix we show that in this case the transform $\tilde{p}^{(a)}(s;x_0)$ satisfies the differential equation

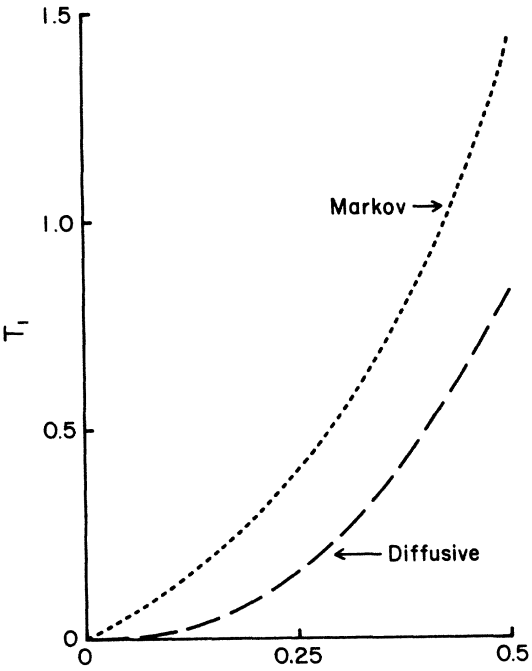

FIG. 6. Mean first-passage time T_1 as a function of separation between absorbing boundaries $(2z)$ for linearly bound processes: diffusive (— — —) and dichotomous Markov (- - -).

$$\frac{d^2\tilde{p}^{(a)}(s;x_0)}{dx_0^2}+\left[\frac{f'(x_0)}{f(x_0)+a}-\frac{s}{f(x_0)-b}-\frac{s}{f(x_0)+a}\right]\frac{d\tilde{p}^{(a)}(s;x_0)}{dx_0}+\frac{s^2-\lambda_a\lambda_b/4}{[f(x_0)+a][f(x_0)-b]}\tilde{p}^{(a)}(s;x_0)=G^{(a)}(s;x_0) \tag{5.28}$$

where

$$G^{(a)}(s;x_0)=\frac{\lambda_a/2}{f(x_0)+a}\left[s\left[\frac{1}{f(x_0)+a}-\frac{1}{f(x_0)-b}\right]e^{-st_a(z_1,x_0)}-\frac{\lambda_b/2}{f(x_0)-b}e^{-st_b(z_2,x_0)}\right] \tag{5.29}$$

and

$$t_a(u,v)\equiv\int_v^u dX\frac{1}{f(X)+a}\ , \tag{5.30a}$$

$$t_b(u,v)\equiv\int_v^u dX\frac{1}{f(X)-b}\ . \tag{5.30b}$$

The boundary conditions for Eq. (5.28) are

$$\text{(i)}\ \ \tilde{p}^{(a)}(s;z_1)=1\ , \tag{5.31}$$

$$\text{(ii)}\ \ \left.\frac{d\tilde{p}^{(a)}(s;x_0)}{dx_0}\right|_{x_0=z_2}=\frac{1}{f(z_2)+a}\left[\frac{\lambda_a}{2}(e^{-st_a(z_1,z_2)}-1)+s\tilde{p}^{(a)}(s;z_2)\right]\ . \tag{5.32}$$

The equation for the mean first-passage time obtained from (5.28) is

$$\frac{d^2T_1^{(a)}(x_0)}{dx_0^2}+\frac{f'(x_0)}{f(x_0)+a}\frac{dT_1^{(a)}(x_0)}{dx_0}-\frac{\lambda_a\lambda_b/4}{[f(x_0)+a][f(x_0)-b]}T_1^{(a)}(x_0)$$
$$=\frac{-\lambda_a/2}{[f(x_0)+a]^2}+\frac{\lambda_a/2}{[f(x_0)+a][f(x_0)-b]}\left[1-\frac{\lambda_b}{2}\int_{x_0}^{z_2}\frac{dX}{f(X)-b}\right]\ , \tag{5.33}$$

with boundary conditions

(i) $T_1^{(a)}(z_1)=0$, (5.34)

$$\text{(ii)}\quad \left.\frac{dT_1^{(a)}(x_0)}{dx_0}\right|_{x_0=z_2}=\frac{1}{f(z_2)+a}\left[\frac{\lambda_a}{2}\int_{z_2}^{z_1}\frac{dx}{f(x)+a}-1\right] . \tag{5.35}$$

1. No drift

With $f(X)=0$, $z_1=z$, and $z_2=-z$ we get the differential equation[13–15]

$$\frac{d^2T_1^{(a)}(x_0)}{dx_0^2}+\frac{\lambda_a\lambda_b}{4ab}T_1^{(a)}(x_0)=\frac{-\lambda_a}{2a}\left[\frac{a+b}{ab}\right]+\frac{\lambda_a\lambda_b}{4ab^2}(z+x_0) \tag{5.36}$$

with boundary conditions

(i) $T_1^{(a)}(z)=0$, (5.37)

$$\text{(ii)}\quad \left.\frac{dT_1^{(a)}(x_0)}{dx_0}\right|_{x_0=-z}=\frac{\lambda_a z}{a^2}-\frac{1}{a} . \tag{5.38}$$

The solution of (5.36)–(5.38) with $a=b=1$, $\lambda_a=\lambda_b=2$, and $|z|<1$ gives

$$T_1(x_0)=\frac{2(1-z)\cos x_0}{\cos z-\sin z}-2+z . \tag{5.39}$$

Eq. (5.39) coincides with the expression of $T_1(x_0)$ that we have given elsewhere.[13,14] An average of (5.39) over a uniform initial distribution gives

$$T_1=\frac{2(1-z)\sin z}{z(\cos z-\sin z)}-2+z . \tag{5.40}$$

These results are shown in Figs. 3 and 4.

2. Linear drift

For the case

$$f(X)=-\mu X , \tag{5.41}$$

with $a=b=1$, $\lambda_a=\lambda_b=2$, $\mu=1$, $z_2=-z$, and $z_1=z$ $[\,|z|<(e-1)/(e+1)]$, we obtain the equation

$$(1-x_0^2)\frac{d^2T_1^{(a)}(x_0)}{dx_0^2}-(1+x_0)\frac{dT_1^{(a)}(x_0)}{dx_0}+T_1^{(a)}(x_0)=\frac{-2}{1-x_0}+\ln\left[\frac{1+x_0}{1-z}\right] , \tag{5.42}$$

with

(i) $T_1^{(a)}(z)=0$, (5.43)

$$\text{(ii)}\quad \left.\frac{dT_1^{(a)}(x_0)}{dx_0}\right|_{x_0=-z}=\frac{1}{1+z}\left[\ln\left[\frac{1+z}{1-z}\right]-1\right] . \tag{5.44}$$

In this case the mean first-passage time is given by

$$T_1(x_0)=\tfrac{1}{2}[(1+x_0)\alpha(x_0)+(1-x_0)\alpha(-x_0)]+\tfrac{1}{2}[\beta(x_0)+\beta(-x_0)]+\tfrac{1}{4}[\beta(x_0)(1+x_0)-\beta(-x_0)(1-x_0)+2c_2x_0]\ln\left[\frac{1-x_0}{1+x_0}\right]+c_1+c_2 , \tag{5.45}$$

where

$$\alpha(x_0)\equiv-\int_0^{x_0}\left[1-\tfrac{1}{2}(1-x)\ln\left[\frac{1+x}{1-z}\right]\right]\left[1+\tfrac{1}{2}(1+x)\ln\left[\frac{1-x}{1+x}\right]\right]dx , \tag{5.46}$$

$$\beta(x_0)\equiv\tfrac{1}{6}(1+x_0)^2\left[(x_0-2)\ln\left[\frac{1+x_0}{1-z}\right]-\tfrac{1}{3}(x_0-\tfrac{25}{2})\right] , \tag{5.47}$$

and

$$c_1=-\frac{1}{2}\left\{[\alpha(z)+\alpha(-z)]+[\beta(z)-\beta(-z)]\left[\frac{1}{1+z}+\tfrac{1}{2}\ln\left[\frac{1-z}{1+z}\right]\right]+\frac{1}{1+z}\left[1+\ln\left[\frac{1-z}{1+z}\right]\right]\right\} , \tag{5.48}$$

$$c_2=-\frac{1}{2}\left\{[\beta(z)+\beta(-z)]+\frac{1}{\frac{1}{1+z}+\frac{1}{2}\ln\left[\frac{1-z}{1+z}\right]}\left[\alpha(z)-\alpha(-z)-\frac{1}{1+z}-\frac{1}{1+z}\ln\left[\frac{1-z}{1+z}\right]\right]\right\} . \tag{5.49}$$

VI. CONCLUSIONS

Herein we have considered the problem of the extrema statistics of one-dimensional non-Markovian processes stimulated by dichotomous fluctuations. The first-passage time problem has been reduced to solving an integral equation for the Laplace transform of the first-passage time probability density function. This integral equation is valid for dichotomous fluctuations possessing arbitrary correlated properties. The method we have constructed for the treatment of such problems[13,15] relies on the explicit construction of trajectories. We are able to select those trajectories that reach the critical values at a given time and to properly weigh their contribution to the first-passage time distribution.

In some cases it is possible to convert the above integral equation for the first-passage time probability density to a differential equation. From this differential equation one can directly construct a differential equation for the mean first-passage time. This result, which generalizes a similar one obtained for unbounded processes,[13–15] bears no resemblance to those generated by Fokker-Planck[5,8,19] or master equation processes.[8,12] In this paper we have considered two particular examples where the integral representation of the first-passage time density can be converted to a differential one.

Let us now summarize the principal results that can be deduced from the examples considered in Sec. V. For an unbound process the first-passage time from an initial location x_0 to z or $-z$ is shorter if the process is diffusive than it is for a process driven by correlated dichotomous fluctuations (cf. Fig. 3).[13–15] The trapping efficiency of the dichotomously driven process depends on the correlation properties of the fluctuations. Thus, for the value of z used in Fig. 3 ($z=0.45$), the relative trapping efficiency of the Markov and rectangular processes depends on the value of x_0. For smaller (larger) values of z the rectangular result lies below (above) the Markov one for all values of x_0. This behavior is also reflected in the mean first-passage time averaged over a uniform initial distribution (cf. Fig. 4).

When the process is linearly bound to the origin, we see from Fig. 5 that the mean first-passage time is larger than that of the unbound process depicted in Fig. 3. It should be emphasized that the results in Fig. 5 are the first *exact* mean first-passage times reported for bound non-Markovian processes. We note that the diffusive result still lies below those with memory. The relative ordering of bound processes is again seen to be dependent on the type of memory. The crossover between the rectangular Markov fluctuations occurs for smaller z ($z<0.45$) in the bound process than in the free process. In Fig. 6 we compare the mean first-passage times averaged over a uniform initial distribution for the diffusive and dichotomous Markov bound processes. Note the difference in scales in Figs. 4 and 6, and that T_1 for the bound processes is therefore about twice as long as for the free ones.

The explicit trajectory calculations constitute a formally tractable method for determining the extrema statistics of non-Markovian processes. We are presently extending this formalism to systems of more than one variable and to fluctuations with more general statistics.

ACKNOWLEDGMENTS

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APPENDIX: DERIVATION OF DIFFERENTIAL EVOLUTION EQUATIONS

It is convenient to introduce new variables of integration into Eqs (4.16) and (4.17). We make a change of variables suggested by the dynamics of the system:

$$x_1=\Phi_a(t_1+\Phi_a^{-1}(x_0))\,, \tag{A1a}$$

$$x_2=\Phi_b(t_2+\Phi_b^{-1}(x_1))\,, \tag{A1b}$$

or, equivalently, by

$$t_1=\int_{x_0}^{x_1} dX\frac{1}{f(X)+a}\equiv t_a(x_1;x_0)\,, \tag{A2a}$$

$$t_2=\int_{x_1}^{x_2} dX\frac{1}{f(X)-b}\equiv t_b(x_2;x_1)\,. \tag{A2b}$$

In terms of these new variables we can write Eq. (4.16) as

$$\begin{aligned}\tilde{p}^{(a)}(s;x_0)&=\tilde{p}_1^{(a)}(s;x_0)+\tilde{p}_2^{(a)}(s;x_0)\\&\quad+\int_{x_0}^{z_1}dx_1\frac{\psi_a(t_a(x_1;x_0))}{f(x_1)+a}e^{-st_a(x_1;x_0)}\int_{x_1}^{z_2}dx_2\frac{\psi_b(t_b(x_2;x_1))}{f(x_2)-b}e^{-st_b(x_2;x_1)}\tilde{p}^{(a)}(s;x_2)\,.\end{aligned} \tag{A3}$$

A similar replacement can be made for (4.17).

1. Derivation of Eqs. (5.2)–(5.4) for Markov $F(t)$

When ψ_a,ψ_b are the exponential forms (5.1), Eq. (A3) becomes

$$\tilde{p}^{(a)}(s;x_0)=h^{(a)}(s;x_0)+\lambda_a\lambda_b\int_{x_0}^{z_1}dx_1\frac{e^{-(s+\lambda_a)t_a(x_1,x_0)}}{f(x_1)+a}\int_{x_1}^{z_2}dx_2\frac{e^{-(s+\lambda_b)t_b(x_2,x_1)}}{f(x_2)-b}\tilde{p}^{(a)}(s;x_2) \tag{A4}$$

where

$$h^{(a)}(s;x_0)\equiv p_1^{(a)}(s;x_0)+p_2^{(a)}(s;x_0)\ . \tag{A5}$$

The x_0 derivative of (A4) is

$$\frac{d}{dx_0}\tilde{p}^{(a)}(s;x_0)=\frac{d}{dx_0}h^{(a)}(s;x_0)+\frac{(\lambda_a+s)}{f(x_0)+a}[\tilde{p}^{(a)}(s;x_0)-h^{(a)}(s;x_0)]-\frac{\lambda_a\lambda_b}{f(x_0)+a}\int_{x_0}^{z_2}dx_2\frac{e^{-(s+\lambda_b)t_b(x_2,x_0)}}{f(x_2)-b}\tilde{p}^{(a)}(s;x_2)\ . \tag{A6}$$

Another x_0 derivative and reorganization of terms yields

$$\left[\frac{d^2}{dx_0^2}+\left[\frac{f'(x_0)}{f(x_0)+a}-\frac{\lambda_b+s}{f(x_0)-b}-\frac{\lambda_a+s}{f(x_0)+a}\right]\frac{d}{dx_0}+\frac{s(s+\lambda_a+\lambda_b)}{[f(x_0)+a][f(x_0)-b]}\right]\tilde{p}^{(a)}(s;x_0)$$
$$=\left[\frac{d^2}{dx_0^2}+\left[\frac{f'(x_0)}{f(x_0)+a}-\frac{\lambda_b+s}{f(x_0)-b}-\frac{\lambda_a+s}{f(x_0)+a}\right]\frac{d}{dx_0}+\frac{(\lambda_a+s)(\lambda_b+s)}{[f(x_0)+a][f(x_0)-b]}\right]h^{(a)}(s;x_0)\ . \tag{A7}$$

Using the explicit forms

$$\tilde{p}_1^{(a)}(s;x_0)=e^{-s\tau_1}\int_{\tau_1}^{\infty}dt_1\,\psi_a(t_1) \tag{A8}$$

and

$$\tilde{p}_2^{(a)}(s;x_0)=\int_0^{\tau_1}dt_1\,e^{-s(t_1+\tau_2)}\psi_a(t_1)\int_{\tau_2}^{\infty}dt_2\,\psi_b(t_2) \tag{A9}$$

we get

$$h^{(a)}(s;x_0)=e^{-(\lambda_a+s)t_a(z_1,x_0)}+\lambda_a\int_{x_0}^{z_1}dx_1\frac{1}{f(x_1)+a}\exp\{-[(\lambda_a+s)t_a(x_1;x_0)+(\lambda_b+s)t_b(z_2,x_1)]\}\ . \tag{A10}$$

One easily shows that the right-hand side of Eq. (A7) vanishes identically, thus yielding Eq. (5.2).

One boundary condition is obtained by setting $x_0=z_1$ in Eq. (A4). The integrated term vanishes and $h^{(a)}(s;z_1)=1$, so that

$$\tilde{p}^{(a)}(s;z_1)=1\ . \tag{A11}$$

The second boundary condition is obtained by setting $x_0=z_2$ in Eq. (A6):

$$\left.\frac{d\tilde{p}^{(a)}(s;x_0)}{dx_0}\right|_{x_0=z_2}=\frac{1}{f(z_2)+a}[-\lambda_a+(\lambda_a+s)\tilde{p}^{(a)}(s;z_2)]\ . \tag{A12}$$

2. Derivation of Eqs. (5.27)–(5.32) for rectangular $F(t)$

If ψ_a and ψ_b are of the rectangular forms (5.26) then in order to allow the process a finite probability of reaching the boundary z_1 or z_2 even in the first time interval we must have, for the quantities $t_a(x_1,x_0)$ and $t_b(x_2,x_1)$ that appear in Eq. (A3), the following bounds:

$$t_a(x_1;x_0)<\frac{2}{\lambda_a}$$

and

$$t_b(x_2,x_1)<\frac{2}{\lambda_b}\ . \tag{A13}$$

On the other hand, observing that $X_a(t)$ $[X_b(t)]$ is an increasing [decreasing] function of time we also have

$$t_a(x_1;x_0)>\int_{z_2}^{z_1}dX\frac{1}{f(X)+a}$$

and

$$t_b(x_2,x_1)>\int_{z_1}^{z_2}dX\frac{1}{f(X)-b}\ . \tag{A14}$$

Therefore, if

$$\frac{1}{\lambda_a}>\tfrac{1}{2}\int_{z_2}^{z_1}dX\frac{1}{f(X)+a}$$

and

$$\frac{1}{\lambda_b}>\tfrac{1}{2}\int_{z_1}^{z_2}dX\frac{1}{f(X)-b}\ , \tag{A15}$$

the integral equation (A3) becomes

$$\tilde{p}^{(a)}(s;x_0)=h^{(a)}(s;x_0)+\frac{\lambda_a\lambda_b}{4}\int_{x_0}^{z_1}dx_1\frac{e^{-st_a(x_1,x_0)}}{f(x_1)+a}\int_{x_1}^{z_2}dx_2\frac{e^{-st_b(x_2,x_1)}}{f(x_2)-b}\tilde{p}^{(a)}(s;x_2)\ . \tag{A16}$$

The x_0 derivative of (A16) is

$$\frac{d}{dx_0}\tilde{p}^{(a)}(s;x_0)=\frac{d}{dx_0}h^{(a)}(s;x_0)+\frac{s}{f(x_0)+a}[\tilde{p}^{(a)}(s;x_0)-h^{(a)}(s;x_0)]-\frac{\lambda_a\lambda_b/4}{f(x_0)+a}\int_{x_0}^{z_2}dx_2\frac{e^{-st_b(x_2,x_0)}}{f(x_2)-b}\tilde{p}^{(a)}(s;x_2)\ . \tag{A17}$$

Another x_0 derivative and reorganization of terms yields

$$\left[\frac{d^2}{dx_0^2}+\left[\frac{f'(x_0)}{f(x_0)+a}-\frac{s}{f(x_0)-b}-\frac{s}{f(x_0)+a}\right]\frac{d}{dx_0}+\frac{s^2-\lambda_a\lambda_b/4}{[f(x_0)+a][f(x_0)-b]}\right]\tilde{p}^{(a)}(s;x_0)$$
$$=\left[\frac{d^2}{dx_0^2}+\left[\frac{f'(x_0)}{f(x_0)+a}-\frac{s}{f(x_0)-b}-\frac{s}{f(x_0)+a}\right]\frac{d}{dx_0}+\frac{s^2}{[f(x_0)+a][f(x_0)-b]}\right]h^{(a)}(s;x_0)\ . \tag{A18}$$

The inhomogeneous term $h^{(a)}(s;x_0)$ has the explicit form

$$h^{(a)}(s;x_0)=\left[1-\frac{\lambda_a}{2}t_a(z_1,x_0)\right]e^{-st_a(z_1,x_0)}+\frac{\lambda_a\lambda_b}{4}\int_{x_0}^{z_1}dx_1\frac{\left[\frac{2}{\lambda_b}-t_b(z_2,x_1)\right]}{f(x_1)+a}e^{-s[t_a(x_1;x_0)+t_b(z_2;x_1)]}\ . \tag{A19}$$

Using (A19), the right-hand side of Eq. (A18) becomes

$$\frac{\lambda_a/2}{f(x_0)+a}\left[s\left[\frac{1}{f(x_0)+a}-\frac{1}{f(x_0)-b}\right]e^{-st_a(z_1;x_0)}-\frac{\lambda_b}{2}\frac{e^{-st_b(z_2,x_0)}}{f(x_0)-b}\right]\equiv G^{(a)}(s;x_0)\ . \tag{A20}$$

As before, one boundary condition is obtained by setting $x_0=z_1$ in Eq. (A16):

$$\tilde{p}^{(a)}(s;z_1)=1\ . \tag{A21}$$

The second boundary condition follows from (A17) if we set $x_0=z_2$:

$$\left.\frac{d\tilde{p}^{(a)}(s;x_0)}{dx_0}\right|_{x_0=z_2}=\frac{1}{f(z_2)+a}\left[\frac{\lambda_a}{2}(e^{-st_a(z_1,z_2)}-1)+s\tilde{p}^{(a)}(s;z_2)\right]\ . \tag{A22}$$

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