First-passage times for non-Markovian processes: Shot noise

Jaume Masoliver
Department of Chemistry (B-014) and Institute for Nonlinear Science (R-002), University of California, San Diego, La Jolla, California 92093
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The stochastic-trajectory-analysis technique is applied to the calculation of the mean—first-passage-time statistics for processes driven by external shot noise. Explicit analytical expressions are obtained for free and bound processes.

I. INTRODUCTION

In several recent papers exact expressions of mean first-passage times have been obtained for systems driven by dichotomous (not necessarily Markov) fluctuations. There have been two different approaches to the problem. In one approach,[1] based on the explicit construction of trajectories, one obtains exact analytic expressions for the mean first-passage time (MFPT) statistics even when the fluctuations are not Markovian. The other approach,[2,3] based on the construction of evolution operators, only deals with Markovian fluctuations. The aim of the present paper is to apply the trajectory techniques to processes driven by external shot noise, i.e., the noise that arises in vacuum tubes and crystals because of the random emission and motion of electrons in such devices.[4]

We shall study the extreme events for general one-dimensional process of the form

$$\dot{Y}(t)=G(Y)+g(Y)F(t)\ , \tag{1.1}$$

where $G(Y)$ and $g(Y)$ are smooth functions and $F(t)$ is the random driving process

$$F(t)=\sum_i \gamma_i \delta(t-\tau_i)\ , \tag{1.2}$$

where $\{(\gamma_i,\tau_i),\ i=1,2,3,\dots\}$ is a sequence of random points in the plane (γ,t) with $\{\gamma_i\}$ independent of $\{\tau_i\}$. We take the quantities γ_i to be positive and uncorrelated. Therefore, the random process $F(t)$ may be viewed as a sequence of pulses at random times τ_i, each pulse having an independent random weight γ_i. The effect of this input noise on the output of the system (1.1) is a series of independent random jumps in the trajectory of the system—that is, a discontinuous trajectory $Y(t)$, with the discontinuities of independent random heights occurring at random times.

We assume that the random times $\{\tau_i;\ i=1,2,3,\dots\}$ are such that the time intervals $t_i\equiv\tau_i-\tau_{i-1}$ constitute an ordinary renewal process[5] with a given "switch" distribution $\psi(t)$. When $\psi(t)=\lambda e^{-\lambda t}$, the random times $\{\tau_i\}$ form a Poisson sequence of random points and the process (1.2) is delta correlated.[4] When $\psi(t)$ is an arbitrary distribution, $F(t)$ is in general not delta correlated[6] and the random process $Y(t)$ given by Eq. (1.1) is non-Markovian. Here we should note that the term "shot noise" has traditionally been applied only when $\{\tau_i\}$ is a Poisson sequence of random points. However, in this paper we use the term shot noise in a broader sense, applying it to the process (1.2) even if the random times $\{\tau_i\}$ are not a Poisson sequence, i.e., even if $\psi(t)$ is not exponential.

As is well known, the change of variables

$$X=\int^{Y}\frac{dy'}{g(y')} \tag{1.3}$$

turns Eq. (1.1) into an equation of the form

$$\dot{X}(t)=f(X)+F(t)\ . \tag{1.4}$$

Therefore, we can study the first-passage time for processes whose dynamical evolution is given by an equation of the form (1.4) and relate these results to the more general process (1.1) when the relation (1.3) between X and Y is monotonic.

This paper is organized as follows. In Sec. II we detail the dynamics of the system. In Sec. III we obtain the integral equations for the mean first-passage time and its probability distribution. In Secs. IV and V we obtain closed exact expressions for the MFPT for free processes and bound processes, respectively. The conclusions are drawn in Sec. VI.

II. DYNAMICS OF THE SYSTEM

Let $X(t)$ be the process described by Eq. (1.4), where $F(t)$ is the shot noise given by Eq. (1.2). $\{\gamma_i\}$ ($\gamma_i\geq 0$, $i=01,2,\dots$) is a set of independent random variables with probability distributions $\chi(\gamma_i)$. The set of random times $\{\tau_i\}$ is such that the time intervals $t_k\equiv\tau_k-\tau_{k-1}$ form a renewal process with a given distribution $\psi(t_k)$. We assume that $f(X)$ in Eq. (1.4) is smooth and such that the solution $X(t)$ of Eq. (1.4) never becomes infinite in a finite time.

We assume that $\gamma_0=0$. The solution of the differential equation (1.4) during the time interval $0\leq t<t_1$ is then given by

$$t=\phi^{-1}(X)-\phi^{-1}(x_0)\ , \tag{2.1}$$

where

$$x_0\equiv X(t=0) \tag{2.2}$$

and

$$\phi^{-1}(X) \equiv \int^{X} \frac{dX'}{f(X')} \;. \tag{2.3}$$

Equation (2.1) can be written in the form

$$X(t) = \phi(t+\phi^{-1}(x_0)), \quad 0 \le t < t_1 \;. \tag{2.4}$$

During the time interval $t_1 \le t < t_1 + t_2$, Eq. (1.4) is equivalent to the integral equation

$$X(t) = x_1 + \gamma_1 + \int_{t_1}^{t} f(X(t))dt \;, \tag{2.5}$$

where

$$x_1 \equiv \phi(t_1 + \phi^{-1}(x_0)) \;. \tag{2.6}$$

Using Eq. (2.3) we can write the solution of Eq. (2.5) in the form

$$X(t) = \phi(t+\phi^{-1}(\gamma_1 + \phi(t_1 + \phi^{-1}(x_0)))) \;, \quad t_1 \le t < t_1 + t_2 \;. \tag{2.7}$$

Proceeding by induction we construct the following *discontinuous trajectory:*

$$X(t) = \phi(t+\phi^{-1}(x_0)), \quad 0 \le t < t_1 \;, \tag{2.8a}$$

$$X(t) = \phi(t+\phi^{-1}(\gamma_1 + \phi(t_1+\phi^{-1}(x_0)))), \quad t_1 \le t < t_1 + t_2 \;, \tag{2.8b}$$

$$X(t) = \phi(t+\phi^{-1}(\gamma_2 + \phi(t_2+t_1+\phi^{-1}(\gamma_1+\phi(t_1+\phi^{-1}(x_0)))))), \quad t_1+t_2 \le t < t_1+t_2+t_3 \;, \tag{2.8c}$$

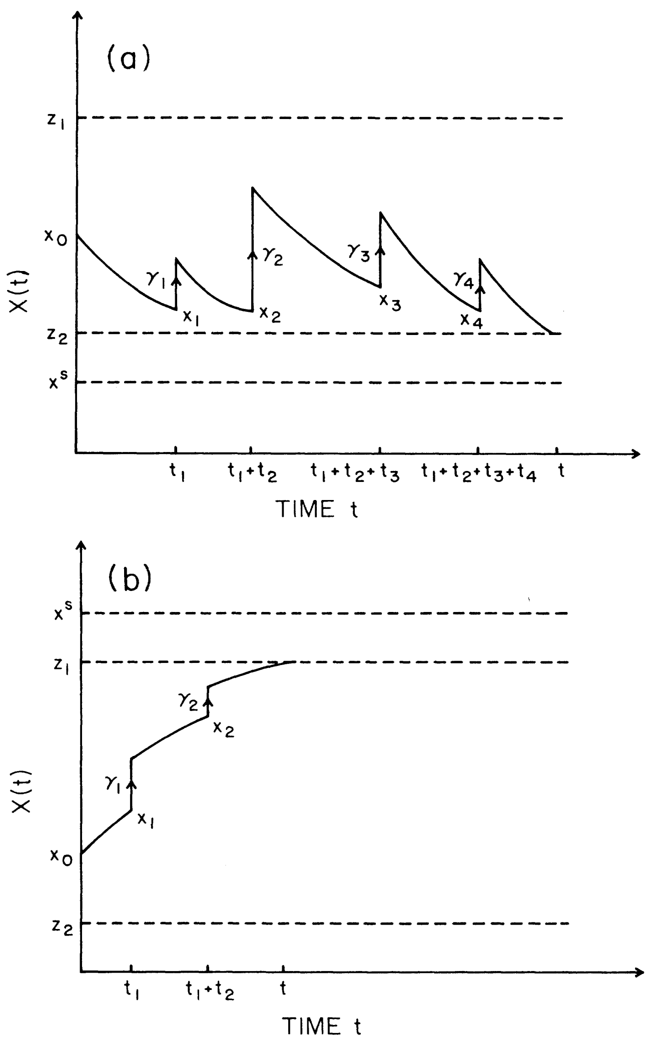

FIG. 1. (a) trajectory when the critical levels z_1 and z_2 are both greater than the fixed point x^s, (b) trajectory when the critical levels z_1 and z_2 are lower than the fixed point x^s. Observe that in this case the lower boundary z_2 cannot be reached.

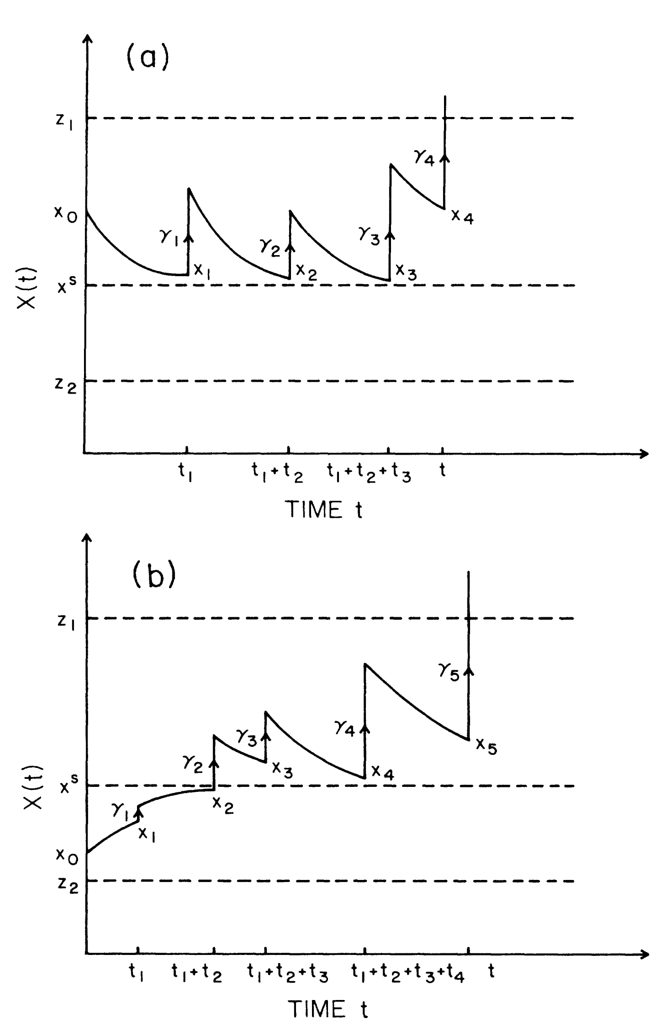

FIG. 2. (a) Trajectory when the critical levels are at distinct sides of the fixed point x^s. The initial location x_0 is greater than x^s. The lower boundary z_2 cannot be reached. (b) Trajectory when the critical levels are at distinct sides of the fixed point x^s. The initial location x_0 is lower than x^s. The lower boundary z_2 cannot be reached.

and so on. The use of the notation

$$x_k \equiv \phi(t_k + \phi^{-1}(\gamma_{k-1} + x_{k-1})) \tag{2.9}$$

allows us to write the expression

$$X(t) = \phi(t + \phi^{-1}(\gamma_{n-1} + x_{n-1})) ,\quad t_1 + \cdots + t_{n-1} \le t < t_1 + \cdots + t_{n-1} + t_n \tag{2.10}$$

for the nth term of the trajectory (2.8).

Let x^s be an asymptotically stable fixed point of the deterministic equation $\dot{X}(t) = f(X)$, i.e., x^s is such that

$$f(x^s) = 0 \tag{2.11}$$

and

$$\lim_{t\to\infty} X(t) = x^s . \tag{2.12}$$

If we are interested in finding the MFPT when the process (1.4) first reaches certain "critical" values z_1 and z_2 (with $z_2 \le z_1$), we must know these values relative to the fixed point x^s. Two different situations arise: (a) z_1 and z_2 are either both greater or both less than x^s,

$$z_1 \ge z_2 > x^s \quad\text{or}\quad x^s > z_1 \ge z_2 \tag{2.13}$$

[Figs. 1(a) and 1(b)]. (b) z_1 and z_2 surround the fixed point x^s,

$$z_1 > x^s > z_2 \tag{2.14}$$

(Fig. 2). In this case the system can only reach the upper level z_1 which turns the level z_2 into a mere lower bound for x_0. This case is therefore equivalent to the problem of MFPT to one critical level.

Finally, when the deterministic equation has no fixed points, $f(X)$ always has a definite sign and the deterministic solution is monotonic. From a dynamical point of view this case is completely equivalent to case (a) above.

III. INTEGRAL EQUATIONS FOR THE MEAN FIRST-PASSAGE TIME AND ITS PROBABILITY DISTRIBUTION

We define the first-passage-time probability density $p(t;x_0)$ as follows:

$P(t;x_0)dt \equiv$ Probability that the process $X(\tau)$ [given that $X(0) = x_0$] crosses z_1 or z_2 in the time range $t \le \tau \le t + dt$ without ever having crossed either of these levels during the time span $0 \le \tau < t$. (3.1)

In terms of this probability density the MFPT is given by

$$T(x_0) = \int_0^\infty dt\, t\, p(t;x_0) . \tag{3.2}$$

Following Ref. 1 it is useful to denote the time between two jumps as an "interval" and to define the auxiliary

$p_n(t;x_0)dt \equiv$ Probability that the first crossing of z_1 or z_2 occurs during the nth interval in the time range $(t, t+dt)$. (3.3)

The first-passage-time probability density is

$$p(t;x_0) = \sum_{n=1}^{\infty} p_n(t;x_0) . \tag{3.4}$$

Now in order to evaluate $p(t;x_0)$ we must proceed separately according to the location of the levels z_1 and z_2 relative to the fixed point x^s.

A. Critical levels at one side of the fixed point

(i) Let us assume first that both levels are greater than the fixed point [Fig. 1(a)]

$$z_1 \ge z_2 > x^s . \tag{3.5}$$

To construct the probability densities $p_n(t;x_0)$ from the trajectory (2.7), we have to insure that no crossing of the levels z_1 or z_2 has occurred in the first $(n-1)$ intervals and that a crossing *does* occur during the nth interval.

If no crossing has occurred in the first $(k-1)$ intervals, then during the kth interval no crossing occurs if *simultaneously*

$$x_k = \phi(t_k + \phi^{-1}(\gamma_{k-1} + x_{k-1})) > z_2 \tag{3.6}$$

and

$$x_k + \gamma_k < z_1 . \tag{3.7}$$

Condition (3.6) is equivalent to

$$t_k < \int_{x_{k-1}+\gamma_{k-1}}^{z_2} dx \frac{1}{f(x)} \equiv \bar{t}_k . \tag{3.8}$$

The probability that the inequalities (3.7) and (3.8) hold is

$$\text{Prob}\{x_k > z_2; x_k + \gamma_k < z_1 \mid x_i > z_2; x_i + \gamma_i < z_1 \ (i<k)\} = \int_0^{\bar{t}_k} dt_k\, \psi(t_k) \int_0^{z_1 - x_k} d\gamma_k\, \chi(\gamma_k) . \tag{3.9}$$

A crossing occurs during the nth interval if $x_n < z_2$ (i.e., $t_n > \bar{t}_n$) or $x_n + \gamma_n > z_1$ (with $x_n > z_2$). The probability of this crossing event is given by

$$\text{Prob}\{x_n < z_2 \text{ or } (x_n + \gamma_n > z_1; x_n > z_2)\} = \int_{\bar{t}_n}^{\infty} dt_n\, \psi(t_n) + \int_0^{\bar{t}_n} dt_n\, \psi(t_n) \int_{z_1 - x_n}^{\infty} d\gamma_n\, \chi(\gamma_n) . \tag{3.10}$$

The probability density for the "continuous" crossing event $(x_n < z_2)$ is

$$\delta(t - (t_1 + t_2 + \cdots + t_{n-1} + \bar{t}_n)) \tag{3.11}$$

and the probability density for the "jump" crossing event $(x_n + \gamma_n > z_1)$ is

$$\delta(t - (t_1 + t_2 + \cdots + t_n)) . \tag{3.12}$$

Collecting the results (3.9), (3.10), (3.11), and (3.12) gives the expression for the probability density $p_n(t;x_0)$:

$$p_n(t;x_0)=\int_0^{\bar{t}_1} dt_1\,\psi(t_1)\int_0^{z_1-x_1} d\gamma_1\,\chi(\gamma_1)\cdots\int_0^{\bar{t}_{n-1}} dt_{n-1}\,\psi(t_{n-1})\int_0^{z_1-x_{n-1}} d\gamma_{n-1}\,\chi(\gamma_{n-1})$$
$$\times\left[\int_{\bar{t}_n}^{\infty} dt_n\,\psi(t_n)\delta(t-(t_1+\cdots+t_{n-1}+\bar{t}_n))+\int_0^{\bar{t}_n} dt_n\,\psi(t_n)\int_{z_1-x_n}^{\infty} d\gamma_n\,\chi(\gamma_n)\delta(t-(t_1+\cdots+t_{n-1}+t_n))\right]. \tag{3.13}$$

(In writing Eq. (3.13) [as we did in writing Eq. (3.9)] we have assumed that the system is initially prepared in such a way that we deal with an ordinary renewal process[5] for the time intervals, i.e., the distribution for the first interval $\psi(t)(0\le t<t_1)$ is the same as for subsequent ones.)

From Eq. (3.13) we see that the Laplace transform of $p_n(t;x_0)$

$$\hat{p}_n(s;x_0)\equiv\int_0^{\infty} dt\,e^{-st}p_n(t;x_0) \tag{3.14}$$

satisfies the recurrence relation

$$\hat{p}_{n+1}(s;x_0)=\int_0^{\bar{t}_1} dt_1\,e^{-st_1}\psi(t_1)\times\int_0^{z_1-x_1} d\gamma_1\,\chi(\gamma_1)\hat{p}_n(s;x_1+\gamma_1) \tag{3.15}$$

$(n=1,2,3,\ldots)$ which immediately leads to the following integral equation:

$$\hat{p}(s;x_0)=\hat{p}_1(s;x_0)+\int_0^{\bar{t}_1} dt_1\,e^{-st_1}\psi(t_1)\times\int_0^{z_1-x_1} d\gamma_1\,\chi(\gamma_1)\hat{p}(s;x_1+\gamma_1)\,. \tag{3.16}$$

Here

$$\hat{p}(s;x_0)=\sum_{n=1}^{\infty}\hat{p}_n(s;x_0) \tag{3.17}$$

is the Laplace transform of the first-passage-time probability density defined in (3.1),

$$\bar{t}_1=\int_{x_0}^{z_2} dx\frac{1}{f(x)}\ , \tag{3.18}$$

and

$$\hat{p}_1(s;x_0)=e^{-st_1}\int_{t_1}^{\infty} dt_1\,\psi(t_1)+\int_0^{\bar{t}_1} dt_1\,e^{-st_1}\psi(t_1)\int_{z_1-x_1}^{\infty} d\gamma_1\,\chi(\gamma_1)\,. \tag{3.19}$$

In terms of $\hat{p}(s;x_0)$ the mean first-passage time $T(x_0)$ defined in Eq. (3.2) is given by

$$T(x_0)=-\left.\frac{\partial\hat{p}(s;x_0)}{\partial s}\right|_{s=0}. \tag{3.20}$$

Applying Eq. (3.20) to Eq. (3.16) and taking into account the normalization of $p(t;x_0)$, i.e.,

$$\hat{p}(0;x_0)=\int_0^{\infty} dt\,p(t;x_0)=1\ ,$$

we arrive at the integral equation for the MFPT:

$$T(x_0)=\bar{t}_1\int_{\bar{t}_1}^{\infty} dt_1\,\psi(t_1)+\int_0^{\bar{t}_1} dt_1\,t_1\psi(t_1)+\int_0^{\bar{t}_1} dt_1\,\psi(t_1)\int_0^{z_1-x_1} d\gamma_1\,\chi(\gamma_1)T(x_1+\gamma_1)\,. \tag{3.21}$$

(ii) Let us assume now that the critical levels are both lower than the fixed point

$$z_2\le z_1<x^s\,. \tag{3.22}$$

In this case the trajectory (2.7) is an increasing function of t [see Fig. 1(b)] and the system is only able to reach the upper level z_1. No crossing occurs during the kth interval if [see Eqs. (3.7) and (3.8)]

$$t_k<\int_{x_{k-1}+\gamma_{k-1}}^{z_1} dx\frac{1}{f(x)}\equiv\bar{t}\,'_k \tag{3.23}$$

and

$$x_k+\gamma_k<z_1\,. \tag{3.24}$$

Therefore, the Laplace transform $\hat{p}(s;x_0)$ of the first-passage-time probability distribution and the MFPT satisfy in this case the integral equations (3.10) and (3.21), respectively, but with $\bar{t}\,'_1$ instead of $\bar{t}_1$.

Finally, we can write the integral equation for the MFPT in the form

$$T(x_0)=h(x_0)+\int_0^{\bar{\tau}} dt_1\,\psi(t_1)\int_0^{z_1-x_1} d\gamma_1\,\chi(\gamma_1)T(x_1+\gamma_1)\ , \tag{3.25}$$

where

$$h(x_0)\equiv\bar{\tau}\int_{\bar{\tau}}^{\infty} dt_1\,\psi(t_1)+\int_0^{\bar{\tau}} dt_1\,t_1\psi(t_1) \tag{3.26}$$

and $\bar{\tau}$ stands for $\bar{t}_1$ ($\bar{t}\,'_1$) if the critical values are both greater (lower) than the fixed point x^s.

B. Critical levels at distinct sides of the fixed point

In this case the system cannot reach the lower level z_2 and the upper level z_1 can only be crossed by a suitable jump and not by dynamical evaluation (see Fig. 2). Therefore, instead of Eq. (3.13) for the auxiliary probabilities $p_n(t;x_0)$, we now have the expression

$$p_n(t;x_0)=\int_0^\infty dt_1\,\psi(t_1)\int_0^{z_1-x_1}d\gamma_1 X(\gamma_1)\cdots\int_0^\infty dt_{n-1}\,\psi(t_{n-1})\int_0^{z_1-x_{n-1}}d\gamma_{n-1}X(\gamma_{n-1})\int_0^\infty dt_n\,\psi(t_n)\times\int_{z_1-x_n}^\infty d\gamma_n X(\gamma_n)\delta(t-(t_1+\cdots+t_n)) \quad (3.27)$$

and the Laplace transform $\hat{p}(s;x_0)$ given by Eq. (3.17) satisfies the integral equation

$$\hat{p}(s;x_0)=\hat{p}_1(s;x_0)+\int_0^\infty dt_1\,e^{-st_1}\psi(t_1)\int_0^{z_1-x_1}d\gamma_1 X(\gamma_1)\hat{p}(s;x_1+\gamma_1)\ , \quad (3.28)$$

where

$$\hat{p}_1(s;x_0)=\int_0^\infty dt_1\,e^{-st_1}\psi(t_1)\int_{z_1-x_1}^\infty d\gamma_1 X(\gamma_1)\ . \quad (3.29)$$

The mean first-passage time $T(x_0)$ now satisfies the integral equation

$$T(x_0)=\tau_m+\int_0^\infty dt_1\,\psi(t_1)\int_0^{z_1-x_1}d\gamma_1 X(\gamma_1)T(x_1+\gamma_1)\ , \quad (3.30)$$

where

$$\tau_m\equiv\int_0^\infty dt\,t\psi(t) \quad (3.31)$$

is the mean time between jumps.

IV. MEAN FIRST-PASSAGE TIMES FOR FREE PROCESSES

We will apply the results of Sec. III to a "free process," i.e., a process $X(t)$ given by the equation

$$\dot{X}(t)=F(t)\ , \quad (4.1)$$

where $F(t)$ is given by Eq. (1.2). In this case the trajectory $X(t)$ is given by the sequence of step functions

$$X(t)=\gamma_{n-1}+x_{n-1}\ ,\quad t_1+\cdots+t_{n-1}\le t<t_1+\cdots+t_{n-1}+t_n\ , \quad (4.2)$$

where

$$x_k=x_0+\gamma_1+\gamma_2+\cdots+\gamma_{k-1} \quad (4.3)$$

$(k=1,2,3,\ldots)$. The time intervals $\bar{t}_k$ defined in Eq. (3.8) are now

$$\bar{t}_k=\infty\ . \quad (4.4)$$

Therefore, the integral equation (3.16) for the MFPT probability density reads

$$\hat{p}(s;x_0)=\hat{p}_1(s;x_0)+\hat{\psi}(s)\int_0^{z_1-x_0}d\gamma_1 X(\gamma_1)\hat{p}(s;x_0+\gamma_1)\ , \quad (4.5)$$

where

$$\hat{\psi}(s)\equiv\int_0^\infty dt_1\,e^{-st_1}\psi(t_1) \quad (4.6)$$

and

$$\hat{p}_1(s;x_0)=\hat{\psi}(s)\int_{z_1-x_0}^\infty d\gamma_1 X(\gamma_1)\ . \quad (4.7)$$

The integral equation for the mean first-passage time itself is

$$T(x_0)=\tau_m+\int_0^{z_1-x_0}d\gamma_1 X(\gamma_1)T(x_0+\gamma_1)\ . \quad (4.8)$$

The definition of the non-negative variable

$$u\equiv z_1-x_0\quad (x_0\ge 0) \quad (4.9)$$

allows us to write Eq. (4.5) in the following convolution form:

$$\hat{p}(s;z_1-u)=\hat{p}_1(s;z_1-u)+\hat{\psi}(s)\int_0^u dv\,X(u-v)\hat{p}(s;z_1-v)\ . \quad (4.10)$$

The general solution of this integral equation is

$$\hat{p}(s;z_1-u)=\hat{\psi}(s)L^{-1}\left[\frac{1-L\{X(u)\}}{p[1-\hat{\psi}(s)L\{X(u)\}]}\right]\ , \quad (4.11)$$

where

$$L\{X(u)\}\equiv\int_0^\infty du\,e^{-pu}X(u) \quad (4.12)$$

is the Laplace transform of $X(u)$ and L^{-1} stands for the inverse transform. Equation (4.11) represents the most general solution to the problem since no specific forms of $\psi(t)$ and $X(\gamma)$ have been used and all the first-passage-time moments can easily be derived from it. In particular the mean first-passage time is

$$T(x_0)=\tau_m L^{-1}\left[\frac{1}{p[1-L\{X(z_1-x_0)\}]}\right]\ , \quad (4.13)$$

where τ_m is the mean time between jumps.

Before closing this section we will give the explicit expression of $T(x_0)$ for two important special forms of the jump distribution $X(u)$: the exponential distribution and the δ distribution.

A. Jumps sizes exponentially distributed

For this case we have

$$X(u)=\frac{1}{\gamma}e^{-u/\gamma}\quad (u\ge 0)\ , \quad (4.14)$$

where γ is the mean size of the jumps. The introduction of Eq. (4.14) into Eq. (4.13) yields

$$T(x_0)=\tau_m\left[1+\frac{z_1-x_0}{\gamma}\right]\ . \quad (4.15)$$

B. Jumps of the same size

In this case

$$X(u)=\delta(u-\gamma) \quad (4.16)$$

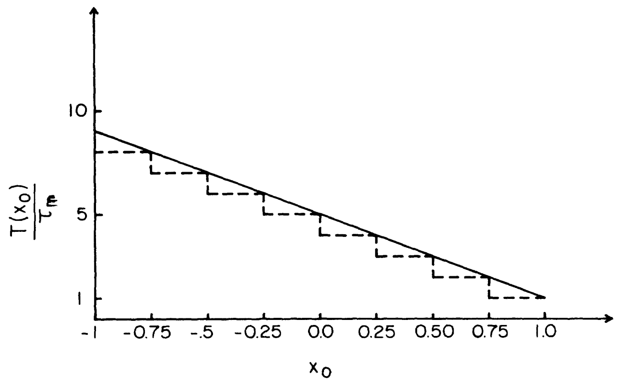

FIG. 3. Mean first-passage time $T(x_0)$ as a function of the initial location x_0 for a free process. τ_m is the mean time between jumps. Mean jump size: $\gamma=0.25$. Solid line, exponentially distributed jumps; dashed line, jumps of the same size.

and from Eq. (4.13) we get the obvious solution

$$T(x_0)=\tau_m\left[1+\sum_{n=1}^{\infty}\Theta(z_1-x_0-n\gamma)\right], \tag{4.17}$$

where $\Theta(x)$ is the Heaviside unit step function. These results are shown in Fig. 3.

V. MEAN FIRST-PASSAGE TIME FOR BOUND PROCESSES

In this section we evaluate the mean first-passage time $T(x_0)$ for the bound process (1.4). Explicit expressions of $T(x_0)$ are given when the time-interval distribution $\psi(t)$ is exponential and the jump distribution is either exponential or a δ function.

As we mentioned in Sec. I, if the set of random times $\{\tau_i\}$ is a Poisson sequence of random points, then the time intervals $t_i=\tau_i-\tau_{i-1}$ are exponentially distributed, that is,

$$\psi(t_i)=\lambda e^{-\lambda t_i}, \tag{5.1}$$

where $\lambda^{-1}=\tau_m$ is the average time between two consecutive jumps. Now we will write the integral equations founded in Sec. III for this particular time distribution.

A. Critical levels at one side of the fixed point

In this case the integral equation for the MFPT is given by Eq. (3.23). When $\psi(t)$ is the exponential form (5.1), Eq. (3.25) becomes

$$T(x_0)=\frac{1}{\lambda}(1-e^{\lambda\bar{\tau}})+\lambda\int_0^{\bar{\tau}}dt_1\,e^{-\lambda t_1}\int_0^{z_1-x_1}d\gamma_1\,\mathcal{X}(\gamma_1)T(x_1+\gamma_1), \tag{5.2}$$

where $\bar{\tau}\equiv\bar{t}_1(\bar{t}'_1)$ is given by Eq. (3.18) [Eq. (3.23)] if the critical levels are greater (lower) than the fixed point.

1. Exponentially distributed jumps sizes

In this case

$$\mathcal{X}(\gamma_1)=\frac{1}{\gamma}e^{-\gamma_1/\gamma} \tag{5.3}$$

and Eq. (5.2) reads

$$T(x_0)=\frac{1}{\lambda}(1-e^{-\lambda\bar{\tau}})+\frac{\lambda}{\gamma}\int_0^{\bar{\tau}}dt_1\,e^{-\lambda t_1}\int_0^{z_1-x_1}d\gamma_1\,e^{-\gamma_1/\gamma}T(x_1+\gamma_1). \tag{5.4}$$

We show in Appendix A that Eq. (5.4) is equivalent to the following differential equation:

$$\frac{d^2T(x_0)}{dx_0^2}+\left[\frac{f'(x_0)}{f(x_0)}-\frac{\lambda}{f(x_0)}-\frac{1}{\gamma}\right]\frac{dT(x_0)}{dx_0}=\frac{1/\gamma}{(f(x_0)} \tag{5.5}$$

with boundary conditions

$$\text{(i) } T(y)=0, \tag{5.6}$$

$$\text{(ii) } \left.\frac{dT(x_0)}{dx_0}\right|_{x_0=z_1}=\frac{1}{f(z_1)}[\lambda T(z_1)-1], \tag{5.7}$$

where $y=z_2(z_1)$ if the critical levels are both greater (lower) than x^s. The solution of the problem (5.5)–(5.7) is straightforward and is given by

$$T(x_0)=\int_y^{x_0}dx\frac{e^{M(x)}}{f(x)}\left[\frac{1}{\gamma}\int^x dx'\,e^{-M(x')}+C\right], \tag{5.8}$$

where

$$M(x)\equiv\frac{x}{\gamma}+\lambda\rho(x), \tag{5.9}$$

$$\rho(x)\equiv\int^x\frac{dx'}{f(x')}, \tag{5.10}$$

and

$$C=\frac{-1+\frac{\lambda}{\gamma}\int_y^{z_1}dx\frac{e^{M(x)}}{f(x)}\int^x dx\,e^{-M(x)}-\frac{1}{\gamma}e^{M(z_1)}\int^{z_1}dx\,e^{-M(x)}}{e^{M(z_1)}-\lambda\int_y^{z_1}dx\frac{e^{M(x)}}{f(x)}}. \tag{5.11}$$

As an example we will evaluate the MFPT for a linear drift of the form

$$f(x)=-x \quad (x^s=0) . \tag{5.12}$$

For simplicity we take $\lambda=1$. In this case Eqs. (5.8) and (5.11) give

$$T(x_0)=\gamma\left[\frac{1}{y}-\frac{1}{x_0}\right]+\ln\left|\frac{x_0}{y}\right|-C\Phi(x_0,y) , \tag{5.13}$$

where

$$C=\left[\frac{\gamma}{y}+\ln\left|\frac{z_1}{y}\right|\right]\left[\frac{e^{z_1/\gamma}}{z_1}+\Phi(z_1,y)\right]^{-1} \tag{5.14}$$

and

$$\Phi(u,v)\equiv\frac{e^{v/\gamma}}{v}-\frac{e^{u/\gamma}}{u}+\frac{1}{\gamma}\left[\mathrm{Ei}\left[\frac{u}{\gamma}\right]-\mathrm{Ei}\left[\frac{v}{\gamma}\right]\right] , \tag{5.15}$$

where $\mathrm{Ei}(u)$ is the exponential-integral function.[7] These results are shown in Fig. 4.

2. Jumps of the same size

Now $\mathcal{X}(\gamma_1)$ is the δ distribution

$$\mathcal{X}(\gamma_1)=\delta(\gamma-\gamma_1) . \tag{5.16}$$

We show in Appendix B that the integral equation (5.2) with $\mathcal{X}(\gamma_1)$ given by Eq. (5.16) is equivalent to the following "initial"-boundary value problem for a differential equation with deviating arguments

$$\frac{dT(x_0)}{dx_0}+\frac{\lambda}{f(x_0)}[T(x_0+\gamma)-T(x_0)]=\frac{-1}{f(x_0)} , \quad x_0<z_1-\gamma \tag{5.17}$$

$$T(x_0)=\frac{1}{\lambda}(1+C^{(0)}e^{\lambda\rho(x_0)}), \quad x_0>z_1-\gamma \tag{5.18}$$

$$T(y)=0 , \tag{5.19}$$

where $\rho(x_0)$ is defined in Eq. (5.10), $C^{(0)}$ is constant to be determined, and $y=z_2$ (z_1) if $z_1>x^s$ $(z_1<x^s)$.

The most general method to exactly solve the problem (5.17)–(5.19) is the *method of steps*.[8] Following this method the *exact solution* of (5.17)–(5.19) can be written in the form

$$T^{(0)}(x_0)=\frac{1}{\lambda}+C^{(0)}e^{\lambda\rho(x_0)} , \tag{5.20a}$$

$$T^{(k)}(x_0)=\frac{1}{\lambda}+e^{\lambda\rho(x_0)}\left[C^{(k)}-\lambda\int^{x_0}\frac{dx}{f(x)}e^{-\lambda\rho(x)}T^{(k-1)}(x+\gamma)\right], \quad k=1,2,\ldots,N \tag{5.20b}$$

where

$$N\equiv\left[\frac{z_1-z_2}{\gamma}\right] \tag{5.21}$$

is the next lower integer portion of $(z_1-z_2)/\gamma$, and

$$T^{(k)}(x_0)\equiv T(x_0) \text{ when } z_1-(k+1)\le x_0\le z_1-k\gamma , \quad k=0,1,\ldots,N-1 , \tag{5.22}$$

$$T^{(N)}(x_0)\equiv T(x_0) \text{ when } z_1\le x_0\le z_1-N\gamma . \tag{5.23}$$

The solution (5.20) is continuous[8] at the points

$$x_0=z_1-N\gamma, \quad k=1,2,\ldots,N .$$

Therefore, the $(N+1)$ constants $C^{(0)},C^{(1)},\ldots,C^{(N)}$ can be determined by the N relations of continuity

$$T^{(k)}(z_1-k\gamma)=T^{(k-1)}(z_1-k\gamma), \quad k=1,2,\ldots,N , \tag{5.24}$$

and the boundary condition (5.19).

As an example we consider again the case of the linear drift (5.12), with $\lambda=1$ and $z_1=z_1-2\gamma$, i.e., $N=2$. In this case Eq. (5.20) yields

$$T(x_0)=\begin{cases}1+\dfrac{C^{(0)}}{|x_0|}, & z_1-\gamma\le x_0\le z_1\\ 2+\dfrac{1}{|x_0|}(C^{(1)}+C^{(0)}\ln|x_0+\gamma|) , & z_1\le x_0<z_1-\gamma ,\end{cases} \tag{5.25}$$

where

$$C^{(0)}=\begin{cases}(z_1-\gamma-2z_2)\left[1+\ln\left[\dfrac{z_2+\gamma}{z_1}\right]\right]^{-1}, & \text{if } z_2\ge z_2>0\\ z_1, \quad \text{if } z_2\le z_1<0\end{cases} \tag{5.26}$$

and

$$C^{(1)}=\begin{cases}[2z_2(\ln z_1-1)-(z_1-\gamma)\ln(z_2+\gamma)]\left[1+\ln\left[\dfrac{z_2+\gamma}{z_1}\right]\right]^{-1}, & \text{if } z_1\ge z_1>0\\ z_1(2-\ln|z_1|)-\gamma, \quad \text{if } z_2\le z_1<0 .\end{cases} \tag{5.27}$$

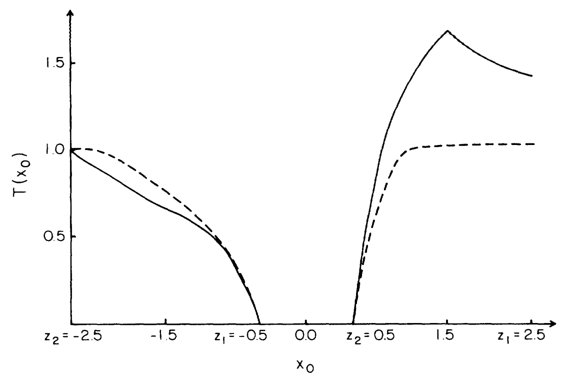

FIG. 4. Mean first-passage time $T(x_0)$ as a function of the initial location x_0 with parameter values $\lambda=1$, $\gamma=1$, $z_1=2.5$ (-0.5), and $z_2=0.5$ (-2.5). The linearly bound processes shown have exponentially distributed jumps (— — —) and δ-distributed jumps (——).

These results are shown in Fig. 4.

B. Critical levels at distinct sides of the fixed point

In this case we have

$$z_2 < x^s < z_1 \tag{5.28}$$

and the integral equation for the MFPT is given by Eq. (3.30). When the time-interval distribution $\psi(t)$ is exponential [Eq. (5.1)], Eq. (3.30) takes the form

$$T(x_0)=\frac{1}{\lambda}+\lambda\int_0^\infty dt_1\, e^{-\lambda t_1}\int_0^{z_1-x_1} d\gamma_1 \mathcal{X}(\gamma_1)T(x_1+\gamma_1)\ . \tag{5.29}$$

The comparison of Eq. (5.29) with Eq. (5.2) shows, as we already mentioned, that the replacement of $\overline{\tau}$ by $+\infty$ in Eq. (5.29) will only produce changes in the boundary conditions. In fact, when $x_0=x^s$, then $x_1=x^s$ and Eq. (5.29) becomes

$$T(x^s)=\frac{1}{\lambda}+\int_0^{z_1-x^s} d\gamma_1 \mathcal{X}(\gamma_1)T(x^s+\gamma_1) \tag{5.30}$$

which confirms the intuition about the behavior of $T(x_0)$ at the fixed point [compare Eq. (5.30) with Eq. (4.8)]. Equation (5.30) is the general boundary condition for this case.

1. Exponentially distributed jumps sizes

When $\mathcal{X}(\gamma)$ is given by Eq. (5.3), the integral equation (5.29) is equivalent to the differential equation (Appendix A)

$$\frac{d^2T(x_0)}{dx_0^2}+\left[\frac{f'(x_0)}{f(x_0)}-\frac{\lambda}{f(x_0)}-\frac{1}{\gamma}\right]\frac{dT(x_0)}{dx_0}=\frac{1/\gamma}{f(x_0)}\ , \tag{5.31}$$

which is the same as Eq. (5.50) but with different boundary conditions. In this case the boundary conditions are

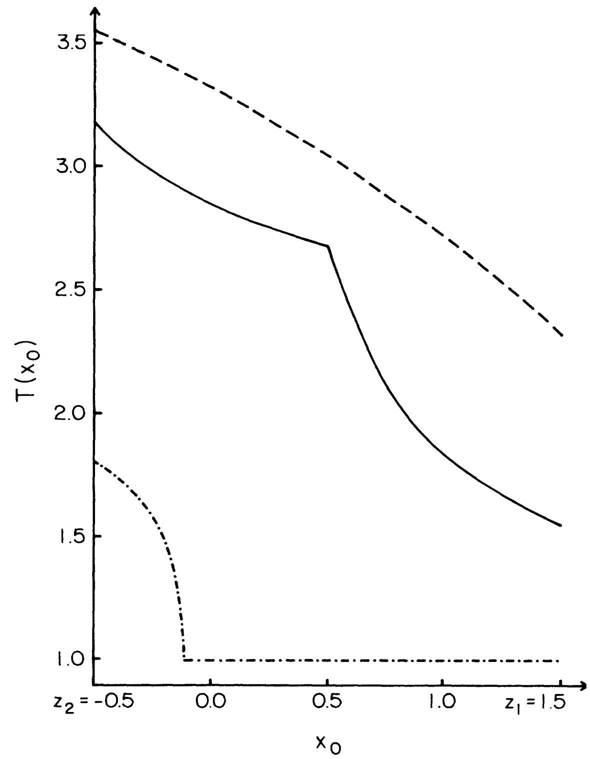

FIG. 5. Mean first-passage time $T(x_0)$ as a function of the initial location x_0 with parameter values $\lambda=1$, $z_1=1.5$, and $z_2=-0.5$. The linearly bound process shown has exponentially distributed jumps with $\gamma=1$ (— — —), δ-distributed jumps with $\gamma=1$ (——), and δ-distributed jumps with $\gamma=1.6$ (—·—·—·).

$$\text{(i)}\ T(x^s)=\frac{1}{\lambda}+\frac{1}{\gamma}\int_0^{z_1-x^s} d\gamma_1\, e^{-\gamma_1/\gamma}T(x^s+\gamma_1)\ , \tag{5.32}$$

$$\text{(ii)}\ \left.\frac{dT(x_0)}{dx_0}\right|_{x_0=z_1}=\frac{\lambda}{f(z_1)}\left[T(z_1)-\frac{1}{\lambda}\right]. \tag{5.33}$$

The general solution of Eq. (5.31) is

$$T(x_0)=\frac{1}{\gamma}\int^{x_0} dx\frac{1}{f(x)}e^{M(x)}\int^x dx'\, e^{-M(x')} +C_1\int^x dx\frac{1}{|f(x)|}e^{M(x)}+C_2\ , \tag{5.34}$$

where $M(x)$ is defined in Eq. (5.9). The constants C_1 and C_2 are determined by introducing Eq. (5.34) into Eqs. (5.32) and (5.33).

For the case of the linear drift (5.12), with $\lambda=1$ and $z_1>0>z_2$, Eqs. (5.32)–(5.34) give

$$T(x_0)=C+\ln|x_0|+\frac{\gamma}{x_0}(e^{x_0/\gamma}-1)-\mathrm{Ei}\left[\frac{x_0}{\gamma}\right], \tag{5.35}$$

where Ei(x) is the exponential-integral function and

$$C \equiv -\ln z_1 + \mathrm{Ei}(z_1/\gamma) . \quad (5.36)$$

This result is shown in Fig. 5.

2. Jumps of the same size

Let $\chi(\gamma_1)=\delta(\gamma-\gamma_1)$; then according to the size of γ of the jump we have to consider the cases $\gamma > z_1 - x^s$ and $\gamma < z_1 - x^s$.

(a) Let us assume first that $\gamma > z_1 - x^s$. In this case it is easy to convince oneself that when $x_0 > z_1 - \gamma$ the system necessarily crosses the level z_1 in the first jump (see Fig. 2); therefore, $T(x_0)$ is the mean time between jumps

$$T(x_0)=\frac{1}{\lambda} .$$

When $x_0 < z_1 - \gamma$, the integral equation (5.2) leads to a differential equation with deviating arguments (Appendix B). We thus have the "inital"-value problem

$$\frac{dT(x_0)}{dx_0}+\frac{\lambda}{f(x_0)}[T(x_0+\gamma)-T(x_0)]=\frac{-1}{f(x_0)} , \quad x_0 < z_1-\gamma \quad (5.37)$$

$$T(x_0)=\frac{1}{\lambda}, \quad x_0 > z_1-\gamma . \quad (5.38)$$

The exact solution of this problem given by the method of steps reads

$$T^{(0)}(x_0)=\frac{1}{\lambda} , \quad (5.39)$$

$$T^{(k)}(x_0)=\frac{1}{\lambda}+e^{\lambda\rho(x_a)}\left[C^{(k)}-\lambda\int^{x_0}dx\frac{1}{f(x)}e^{-\lambda\rho(x)}T^{(k-1)}(x+\gamma)\right] , \quad (5.40)$$

$(k=1,2,\ldots,N)$ where $\rho(x)$, N, and $T^{(k)}(x)$ are defined by Eqs. (5.10), (5.21), and (5.22), respectively. The constants $C^{(1)},\ldots,C^{(N)}$ are determined by continuity [Eq. (5.24)].

For the case $f(x)=-x(x^s=0)$ with $\lambda=1$, $\gamma > z_1$, and $z_2=z_1-2\gamma$ $(n=2)$, we have the exact solution

$$T(x_0)=\begin{cases} 1, & z_1-\gamma \le x_0 \le z_1 \\ 2-\left|\dfrac{z_1-\gamma}{x_0}\right|, & z_1-2\gamma \le x_0 \le z_1-\gamma \end{cases} \quad (5.41)$$

(Fig. 5).

(b) Finally, we study the case $\gamma < z_1 - x^s$. The MFPT is now given by the solution of the "initial"-value problem

$$\frac{dT(x_0)}{dx_0}+\frac{\lambda}{f(x_0)}[T(x_0+\gamma)-T(x_0)]=\frac{-1}{f(x_0)} , \quad x_0 \le z_1-\gamma \quad (5.42)$$

$$T(x_0)=\frac{1}{\lambda}\left[1+C^{(0)}e^{\lambda t(x_0,z_1-\gamma)}\right], \quad x_0 \ge z_1-\gamma , \quad (5.43)$$

where

$$t(u,v)\equiv\int_v^u dx\frac{1}{f(x)}=\rho(u)-\rho(v) . \quad (5.44)$$

The exact solution of (5.42) and (5.43) reads

$$T^{(k)}(x_0)=\frac{1}{\lambda}+e^{\lambda\rho(x_0)}\left[C^{(k)}-\lambda\int^{x_0}dx\frac{1}{f(x)}e^{-\lambda\rho(x)}T^{(k-1)}(x+\gamma)\right] \quad (k=1,2,\ldots,N) . \quad (5.45)$$

The $(N+1)$ constants $C^{(0)},C^{(1)},\ldots,C^{(N)}$ are determined by continuity [Eq. (5.24)] and by the boundary condition

$$T^{(m)}(x^s)=\frac{1}{\lambda}+T^{(m-1)}(x^s+\gamma) , \quad (5.46)$$

where m is such that

$$z_1-(m+1)\gamma \le x^s \le z_1-m\gamma . \quad (5.47)$$

For the case $f(x)=-x(x^s=0)$ with $\lambda=1$, $z_1>0$, $\gamma<z_1$, and $z_2=z_1-2\gamma$, we have

$$T(x_0)=\begin{cases} 1+\dfrac{C}{x_0}, & z_1-\gamma \le x_0 \le z_1 \\ 2+\dfrac{C}{x_0}\ln\left|1+\dfrac{x_0}{\gamma}\right|, & \tau_2-2\gamma \le x_0 \le z_1-\gamma, \end{cases} \quad (5.48)$$

where

$$C=\frac{z_1-\gamma}{1-\ln(z_1/\gamma)} \quad (5.49)$$

(Fig. 5).

VI. CONCLUSION

We have studied the problem of the first-passage-time statistics for general one-dimensional processes driven by external shot noise. The problem has been reduced to the solution of integral equations for the Laplace transform of the first-passage-time distribution function. These integral equations are valid for the shot noise defined in Eq. (1.2) which has arbitrary correlation properties and arbitrary distribution of pulse heights. The method used for constructing such integral equations has been the one applied before to the dichotomous noise.[1] This method relies on the explicit construction of trajectories which allows us to keep track of each trajectory and to select those that reach the critical levels at a given time.

For free processes the above integral equations can be solved in a completely general way. For the case of bound processes we have obtained exact expressions of the mean first-passage time when the time-interval distribution is exponential (i.e., white shot noise) and the jump distribution is either exponential or a δ function. Therefore, the problem of the extrema statistics has been completely solved for large classes of processes driven by external shot noise.

We now summarize the principal results that can be deduced from the examples considered in Secs. IV and V. From all these examples we clearly see that the statistical properties of the distribution of jump sizes clearly affect the smoothness of the mean first-passage time $T(x_0)$ and the trapping efficiency of the driven process. Thus, when the jump sizes are δ distributed, $T(x_0)$ is not a smooth function having either discontinuities of the first kind (free process) or discontinuities in the first derivative (linearly bound process). On the other hand, we see from Figs. 3, 4, and 5 that $T(x_0)$ is larger for exponentially distributed jumps, except when both critical levels are greater than the fixed point (cf. Fig. 4).

There are several generalizations that can be investigated. To name a few, we can consider nowhite shot noise (either non-δ pulses or time-interval distributions that are not exponential) or to allow pulse heights γ_i with variable sign. These cases are presently under investigation.

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APPENDIX A: DIFFERENTIAL EVOLUTION EQUATIONS FOR EXPONENTIALLY DISTRIBUTED JUMPS

1. Critical levels at one side of the fixed point

It is convenient to introduce new variables of integration into Eq. (5.4). We make the change of variables suggested by the dynamics of the system during the first time interval

$$x_1=\phi(t_1+\phi^{-1}(x_0)) \tag{A1}$$

or, equivalently, by

$$t_1=\int_{x_0}^{x_1}dX\frac{1}{f(X)}\equiv t(x_1,x_0)\ . \tag{A2}$$

In terms of these new variable we can write Eq. (5.4) as

$$T(x_0)=\frac{1}{\lambda}\left[1-e^{-\lambda t(y,x_0)}\right]+\frac{\lambda}{\gamma}\int_{x_0}^{y}dx_1\frac{1}{f(x_1)}e^{-\lambda t(x_1,x_0)}\times\int_0^{z_1-x_1}d\gamma_1 e^{-\gamma_1/\gamma}T(x_1+\gamma_1)\ , \tag{A3}$$

where $y=z_2$ (z_1) if the critical values are greater (lower) then the fixed point x^s. The x_0 derivative of (A3) is

$$\frac{dT(x_0)}{dx_0}=\frac{1}{f(x_0)}[\lambda T(x_0)-1]-\frac{\lambda/\gamma}{f(x_0)}\int_0^{z_1-x_0}d\gamma_1 e^{-\gamma_1/\gamma}T(x_0+\gamma_1)\ . \tag{A4}$$

Another x_0 derivative, and integration by parts and reorganization of terms, lead to Eq. (5.5):

$$\frac{d^2T(x_0)}{dx_0^2}+\left[\frac{f'(x_0)}{f(x_0)}-\frac{\lambda}{f(x_0)}-\frac{1}{\gamma}\right]\frac{dT(x_0)}{dx_0}=\frac{1/\gamma}{f(x_0)}\ . \tag{A5}$$

The boundary conditions are obtained by setting $x_0=y$ and $x_0=z_1$ in Eqs. (A3) and (A4), respectively,

$$\text{(i) } T(y)=0\ , \tag{A6}$$

$$\text{(ii) } \left.\frac{dT(x_0)}{dx_0}\right|_{x_0=z_1}=\frac{1}{f(z_1)}[\lambda T(z_1)-1]\ . \tag{A7}$$

2. Critical values at distinct sides of the fixed point

The change of variables (A2) when applied to Eq. (5.34) yields

$$T(x_0)=\frac{1}{\lambda}+\frac{\lambda}{\gamma}\int_{x_0}^{x^s}dx\frac{1}{f(x_1)}e^{-\lambda(x_1,x_0)}\times\int_0^{z_1-x_1}d\gamma_1 e^{-\gamma_1/\gamma}T(x_1+\gamma_1)\ . \tag{A8}$$

Taking one x_0 derivative, we have Eq. (A4) and a second x_0 derivative gives the differential equation (A5). One boundary condition is obtained by setting $x_0=z_1$ in Eq. (A4) which gives Eq. (A7). The other boundary condition is given by Eq. (5.30).

APPENDIX B: DIFFERENTIAL EVOLUTION EQUATIONS FOR JUMPS OF THE SAME SIZE

1. Critical levels at one side of the fixed point

Introducing the δ distribution (5.16) into Eq. (5.2) and performing the change of variables given by Eq. (A2), we get the following integral equation:

$$T(x_0)=\frac{1}{\lambda}\left[1-e^{-\lambda t(y,x_0)}\right]+\lambda\int_{x_0}^{y}dx\frac{1}{f(x_1)}\Theta(z_1-x_1-\gamma)\times e^{-\lambda t(x_1,x_e)}T(x_1+\gamma)\;, \tag{B1}$$

where $\Theta(x)$ is the Heaviside function. Setting $x_0=y$ in Eq. (B1), we have the boundary condition

$$T(y)=0\;, \tag{B2}$$

where $y=z_2$ (z_1) if $z_1\geq z_2>x^s$ $(z_2\leq z_1<x^s)$. The x_0 derivative of Eq. (B1) yields

$$\frac{dT(x_0)}{dx_0}+\frac{1}{f(x_0)}[\Theta(z_1-x_0-\gamma)T(x_0+\gamma)-T(x_0)]=\frac{-1}{f(x_0)}\;. \tag{B3}$$

We have two cases.

(i) $z_1-\gamma<x_0\leq z_1$. Then $\Theta(z_1-x_0-\gamma)=0$ and Eq. (B3) turns into an ordinary differential equation whose solution is

$$T(x_0)=\frac{1}{\lambda}+C^{(0)}e^{\lambda\rho(x_0)}\;, \tag{B4}$$

where $C^{(0)}$ is a constant and

$$\rho(x_0)\equiv\int^{x_0}dx\frac{1}{f(x)}\;. \tag{B5}$$

(ii) $z_2\leq x_0<z_1-\gamma$. In this case Eq. (B3) becomes

$$\frac{dT(x_0)}{dx_0}+\frac{\lambda}{f(x_0)}[T(x_0+\gamma)-T(x_0)]=\frac{-1}{f(x_0)}\;. \tag{B6}$$

Equation (B6) has to be solved along with the "initial" condition (B4) and the boundary condition (B2).

2. Critical values at distinct sides of the fixed point

When

$$z_1>x^s>z_2\;,$$

instead of (B1) we have the equation

$$T(x_0)=\frac{1}{\lambda}+\lambda\int_{x_0}^{x^s}dx\frac{1}{f(x_1)}\Theta(z_1-x_1-\gamma)\times e^{-\lambda t(x_1,x_0)}T(x_1+\gamma)\;. \tag{B7}$$

(i) When $\gamma>z_1-x^s$ and $x_0>z_1-\gamma$, Eq. (B7) yields

$$T(x_0)=\frac{1}{\lambda}\;, \tag{B8}$$

i.e., the system crosses the z_1 level at the first jump. When $x_0<z_1-\gamma$, Eq. (B7) becomes

$$T(x_0)=\frac{1}{\lambda}+\lambda\int_0^{z_1-\gamma}dx_1\frac{1}{f(x_1)}e^{-\lambda t(x_1,x_0)}T(x_1+\gamma)\;. \tag{B9}$$

The x_0 derivative of (B9) shows that it is equivalent to the differential equation (B6).

(ii) When $\gamma<z_1-x^s$ and $x_0>z_1-\gamma$, Eq. (B7) becomes

$$T(x_0)=\frac{1}{\lambda}+\lambda\int_{z_1-\gamma}^{x^s}dx_1\frac{1}{f(x_1)}e^{-\lambda t(x_1,x_0)}T(x_1+\gamma)\;, \tag{B10}$$

which is equivalent to the differential equation

$$\frac{dT(x_0)}{dx_0}-\frac{\lambda}{f(x_0)}T(x_0)=\frac{-1}{f(x_0)}\;, \tag{B11}$$

whose solution reads

$$T(x_0)=\frac{1}{\lambda}+C^{(0)}e^{\lambda\rho(x_0)}\;. \tag{B12}$$

When $x_0<z_1-\gamma$, Eq. (B7) becomes

$$T(x_0)=\frac{1}{\lambda}+\lambda\int_{x_0}^{x^s}dx_1\frac{1}{f(x_1)}e^{-\lambda t(x_1,x_0)}T(x_1+\gamma)\;, \tag{B13}$$

which is equivalent to the differential equation with deviating arguments, Eq. (B6). In this case the system can begin its evolution at $x_0=x^s$ and, therefore, from Eq. (5.30) we have the boundary condition

$$T(x^s)=\frac{1}{\lambda}+T(x^s+\gamma)\;. \tag{B14}$$

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