Passage times for the decay of an unstable state triggered by colored noise

J. M. Sancho
Departament d'Estructura i Constituents de la Matèria, Universitat de Barcelona, Diagonal 647, E-08028 Barcelona, Spain

M. San Miguel
Departament de Física, Universitat de les Illes Balears, E-07071 Palma de Mallorca, Spain

(Received 20 October 1988)

We study the decay of an unstable state in the presence of colored noise by calculating the moment generating function of the passage-time distribution. The problems of the independence of the initial condition in this non-Markovian process and that of nonlinear effects are addressed. Our results are compared with recent analog simulations.

An interesting analog simulation of the decay of an unstable state triggered by a colored noise has been recently reported.[1] In this simulation, data for the probability density of the relaxation time are presented. Results for the mean relaxation time have been explained in terms of a formula by Suzuki,[2] which determines a characteristic time up to an undetermined constant. This formula was derived in the context of the scaling theory with independent initial conditions for the decay of an unstable state and it appears as a matching time between two dynamical regimes. The first regime corresponds to the early stages of the decay of the unstable state and is dominated by noise and linear terms. The second regime is dominated by nonlinearities, and noise plays no essential role. In this sense the characteristic time of Suzuki is a measure of the lifetime of the unstable state and a limit of validity of the linear theory. It is important to note that in the derivation of Suzuki the initial condition is assumed to be independent of the noise acting during the decay process.

Our aim in this Brief Report is to show that Suzuki's formula for the characteristic time can be derived within a mean first-passage-time framework and also generalized to account for the fact that the initial condition is physically determined by the same noise that triggers the decay process, so that initial condition and noise cannot be considered to be statistically independent when the noise has a finite correlation time. In our derivation the undetermined constant is evaluated and in addition it permits us to calculate any moment of the passage-time distribution. In particular, results for the variance of the first-passage time are given. The derivation is based on the calculation of the generating function associated with the passage-time distribution. Such probability distribution considers simultaneously different noise realizations and different initial conditions. We follow the same method already used to describe the switch-on statistics of a dye laser.[3,4] In this last problem, multiplicative colored pump noise plays an important role. The same method has been used to study the case in which a control parameter is changed with a finite velocity.[5]

A general simple model for the decay of an unstable state triggered by additive colored noise is given by

$$d_t x = \alpha x + N(x) + \mu(t) , \tag{1}$$

where $\alpha > 0$, $N(x)$ contains nonlinear terms, and $\mu(t)$ is the noise term assumed here to be Gaussian with zero mean and correlation

$$\langle \mu(t)\mu(t')\rangle = (D/\tau)\exp(-|t-t'|/\tau) . \tag{2}$$

D is the noise intensity and τ its correlation time. As mentioned above, the characteristic lifetime is determined by the linear and noise terms. Thus neglecting nonlinear terms the integration of (1) gives

$$x(t) = h(t)\exp(\alpha t) , \tag{3}$$

where

$$h(t) = x_0 + \int_0^t \mu(t')\exp(-\alpha t')dt' , \tag{4}$$

and x_0 is the initial value of x. The system is assumed to be at $t=0$ in a stable state associated with a value of the control parameter $\alpha = -\alpha_0$. In other words, for $t<0$ the system is described in the linear approximation by

$$d_t x = -\alpha_0 x + \mu(t) . \tag{5}$$

Hence x_0 is not arbitrary, and it is given by the solution of (5)

$$x_0 = x(0) = \int_{-\infty}^0 \mu(t') e^{\alpha_0 t'} dt' . \tag{6}$$

At $t=0$ the control parameter is instantaneously changed from $-\alpha_0$ to a value $\alpha > 0$ so that the system becomes unstable. The exact result for the probability distribution $P(x,t)$ associated with (5) is known.[6] In particular, the stationary distribution is Gaussian with a variance

$$\langle x_0^2 \rangle = \frac{D}{\alpha_0(1+\alpha_0\tau)} . \tag{7}$$

The effects of the colored noise with respect to the limit $\tau \to 0$ in (7) is seen to be just the replacement of D by $D/(1+\alpha_0\tau)$. However, an additional important consequence of having $\tau \neq 0$ is that x_0 given by (6) is a random variable determined by $\mu(t)$ so that $\langle x_0 h(t)\rangle \neq 0$. With these properties for x_0, $h(t)$ is a Gaussian process of zero mean and second moment

$$\langle h^2(t)\rangle=\langle x_0^2\rangle+\frac{D}{\alpha(1+\alpha\tau)}\left[1-e^{-2\alpha t}+\frac{2\alpha\tau}{(1-\alpha\tau)}(e^{-(\alpha+1/\tau)t}-e^{-2\alpha t})\right]+\frac{2D\tau}{(1+\alpha_0\tau)(1+\alpha\tau)}(1-e^{-(\alpha+1/\tau)t}) , \quad (8)$$

where the last term corresponds to the coupling between $\mu(t)$ and x_0, $t>0$. This is the correct way to take into account the preparation procedure (5) which was not considered in the theoretical analysis of Ref. 1.

For times $\alpha t \gg 1$, $h(t)$ can be approximated by a time-independent Gaussian random variable with a second moment

$$\langle h^2(\infty)\rangle=\langle x_0^2\rangle+\frac{D}{\alpha(1+\alpha\tau)}+\frac{2D\tau}{(1+\alpha_0\tau)(1+\alpha\tau)} . \quad (9)$$

In this time regime Eq. (3) can be inverted as

$$t=(1/2\alpha)\ln(x_F^2/h^2) . \quad (10)$$

This gives the time t to reach a prescribed value x_F^2 as a random quantity whose statistics are determined by those of $h(\infty)$. This random time is identified with a passage time to the value x_F.

The generating function for the passage-time distribution is

$$W(\beta)=\langle \exp(-\beta t)\rangle . \quad (11)$$

From $W(\beta)$ the statistical averages are evaluated as

$$\langle t\rangle=(1/2\alpha)\ln(x_F^2/2\langle h^2\rangle)-\psi(\tfrac{1}{2})/2\alpha , \quad (12)$$

$$\langle (t-\langle t\rangle)^2\rangle=\psi'(\tfrac{1}{2})/4\alpha^2 , \quad (13)$$

where $\psi(\frac{1}{2})$ and $\psi'(\frac{1}{2})$ are, respectively, the digamma function and its derivative.

At this point it is worth to comment Suzuki's formula[2]

$$\langle t\rangle=-(1/2\alpha)\left[\ln\left[\langle x_0^2\rangle+\frac{D}{\alpha(1+\alpha\tau)}\right]+\ln C\right] , \quad (14)$$

which was derived for the case of independent initial conditions but with an undetermined constant C which is here calculated using our formalism as

$$-\ln C=\gamma+\ln(2x_F^2) , \quad (15)$$

where $\gamma=0.577\,21\ldots$ is the Euler's constant. Formula (14) appeared also in Ref. 7 [Eq. (4.6)] and in Ref. 8 [Eq. (3.15)]. When $\langle h^2(\infty)\rangle$ of (9) is substituted in (12), we recover the formula (14) used in Ref. 1 plus the extra term coming from the initial preparation and the explicit expression for C given by (15). We remark that this result is completely independent of any nonlinear contribution, since it has been here obtained neglecting all nonlinear terms. The usefulness of the calculation above is that it leads simultaneously to higher-order moments of the passage-time distribution. In particular, the variance (13) is found to be independent of the noise intensity D and also of the correlation time τ. This seems to be at least in qualitative agreement with the passage time distributions displayed in Ref. 1.

The role of the noise parameters D and τ is seen more transparently rewriting (12) as

$$\langle t\rangle=-(1/2\alpha)\ln D+B(\tau)+B_0 , \quad (16)$$

where

$$B(\tau)=(1/2\alpha)\ln\left[\frac{(1+\tau\alpha)(1+\tau\alpha_0)}{1+\tau(\alpha+\alpha_0)}\right] , \quad (17)$$

$$B_0=(1/2\alpha)\ln\left[\frac{2x_F^2e^{\gamma}}{\alpha_0^{-1}+\alpha^{-1}}\right] . \quad (18)$$

The first term on the right-hand side (rhs) of (16) gives the dominant contribution for small D and it is independent of τ. The term $B(\tau)$ gives the contribution to $\langle t\rangle$ due to the finite correlation time τ. It vanishes in the white noise limit $B(\tau=0)$. Finally, B_0 gives a constant contribution (noise independent) which already appears in the white-noise limit.

We wish now to compare our theoretical predictions with the analog simulation results of Ref. 1. The simulation was made for a particular model defined by the Langevin equation

$$d_t x=x\left[-1+\frac{A(t)}{1+x^2}\right]+\mu(t) , \quad (19)$$

where $A(t)$ is the control parameter. For $A<1$, the model has a stable steady state and for $A>1$ it has bistability.[1] In particular, $A(t)$ behaves as

$$A(t)=\begin{cases}A_0<1, & t<0\\ A>1, & t>0 .\end{cases} \quad (20)$$

In order to clarify the linear assumption made in our calculation for this model we rescale Eq. (19) by changing the variable

$$x=\sqrt{D}q . \quad (21)$$

Then Eq. (19) becomes

$$d_t q=q\left[-1+\frac{A}{1+Dq^2}\right]+\Theta(t) , \quad (22)$$

where $\Theta(t)$ has a correlation

$$\langle\Theta(t)\Theta(t')\rangle=(1/\tau)\exp(-|t-t'|/\tau) . \quad (23)$$

If $q^2D \ll 1$, Eq. (22) reduces to its linear form,

$$d_t q=\alpha q+\Theta(t), \quad \alpha=A-1 . \quad (24)$$

We may conclude that the linear approximation is reliable while

$$Dq^2(t) \ll 1 . \quad (25)$$

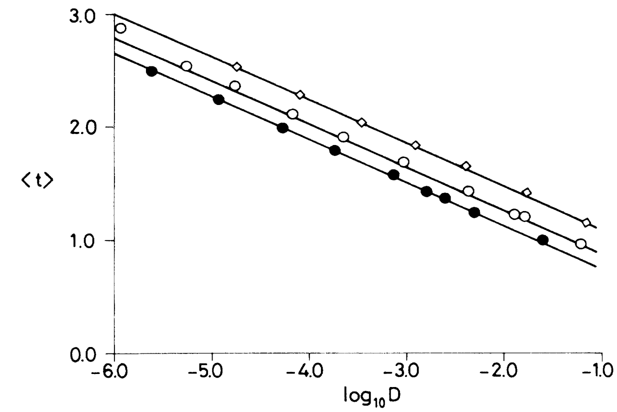

FIG. 1. MFPT vs decimal logarithm of D. The values of τ are the following: 0.1, closed circles; 1.0, open circles; and 5.0, diamonds. The solid lines correspond to Eqs. (16), (17), and (28). The points correspond to the analog data of the Fig. 7 of Ref. 1.

In fact, using that $q^2(t) \approx \exp(2\alpha t)$, Eq. (25) gives as an upper limit of validity of the linear theory a time $T \approx (2\alpha)^{-1} \ln(1/D)$ which is the dominant contribution in (16). Hence for small enough D a linear calculation of $\langle t \rangle$ makes sense.

We now compare our predictions [Eqs. (16)–(18)] taking the explicit expression for the parameters

$$\alpha = A - 1, \quad \alpha_0 = 1 + |A_0| \tag{26}$$

with the analog results of Ref. 1. We choose the case which the authors of Ref. 1 consider to be the worse to fit the value of the constant term B_0. This is the set of data closest to the white-noise limit. The value of the parameters (20) are $A=4$ and $A_0=-2.4$. Our formula (18) gives

$$B_0 = 0.24 , \tag{27}$$

which underestimates the value that fits better the analogic data

$$B_0 \approx 0.34 . \tag{28}$$

Our interpretation of this fact is the following. It is clear from the theory presented here (see Refs. 3 and 4 for details and also the classic work by Haake, Haus, and Glauber[9]) that nonlinear terms give a small positive contribution to the constant B_0. This is the whole effect of the nonlinear terms. For small enough D, this contribution to B_0 can be neglected. However, for the values of D used in the analog simulation, B_0 becomes noticeable and it was here calculated within a linear approximation. To clarify this point we will divide B_0 in two parts:

$$B_0 = B_L + B_{NL} . \tag{29}$$

B_L includes the linear contributions and it is given by Eq. (18). B_{NL} incorporates the nonlinear contributions which are independent of the noise parameters. These contributions are necessary to have a precise estimation of the mean first-passage time (MFPT), when D is not very small. An explicit calculation of this quantity is very complicated due to the special nonlinearities of the model (19). However, if our argument is correct an estimation of B_0 from analog results for a given value of τ, in particular $\tau=0$, should be enough to fit all the data for any other value of $\tau \neq 0$. From (27) and (28) we see that the nonlinear contribution is

$$B_{NL} = 0.10 , \tag{30}$$

which is a small correction to the total result for $\langle t \rangle$, Eq. (16), and it justifies the use of linear approximations in the study of the relaxation of unstable states at least as an asymptotic theory.

In Fig. 1 we present our theoretical results, using (28), compared with the analog data of Ref. 1. One can see that the agreement is good even for τ large. Our interpretation is different of that of Ref. 1. We consider the case $\tau \neq 0$ as an extension of the white-noise case while in Ref. 1. B_0 was estimated for the largest value of τ.

A point to be noted is that (17) implies that there does not exist a first-order correction in τ to the white-noise result, the first correction being of order τ^2. This does not conflict with the analog data since the small range of τ values for which (17) predicts a τ-independent $\langle t \rangle$ is beyond the experimental accuracy.

Financial support from Dirección General de Investigación Científica y Técnica, Spain (Project Nos. PB87-0014 and PB86-0534) is acknowledged. We thank Frank Moss for communication of the analog simulation results before publication. We also acknowledge helpful discussions with C. Van den Broeck and J. Casademunt.

[1] M. James, F. Moss, P. Hanggi, and C. Van den Broeck, Phys. Rev. A **38**, 4690 (1988).
[2] M. Suzuki, Y. Liu, and T. Tsuno, Physica A **138**, 433 (1986).
[3] F. De Pasquale, J. M. Sancho, M. San Miguel, and P. Tartaglia, Phys. Rev. Lett. **56**, 2473 (1986).
[4] A pedagogical introduction to the method is given in J. M. Sancho and M. San Miguel, in *Noise in Dynamical Physical Systems*, edited by F. Moss and P. V. E. McClintock (Cambridge University Press, Cambridge, in press), Vol. I.
[5] M. C. Torrent and M. San Miguel, Phys. Rev. A **38**, 245 (1988).
[6] J. M. Sancho and M. San Miguel, Z. Phys. B **36**, 357 (1980).
[7] M. C. Valsakumar, J. Stat. Phys. **39**, 347 (1985).
[8] A. K. Dhara and S. V. G. Menon, J. Stat. Phys. **46**, 743 (1987).
[9] F. Haake, J. Haus, and R. Glauber, Phys. Rev. A **23**, 3255 (1981).