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Decay of unstable states in the presence of colored noise and random initial conditions. I. Theory of nonlinear relaxation times

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The general theory of nonlinear relaxation times is developed for the case of Gaussian colored noise. General expressions are obtained and applied to the study of the characteristic decay time of unstable states in different situations, including white and colored noise, with emphasis on the distributed initial conditions. Universal effects of the coupling between colored noise and random initial conditions are predicted.

I. INTRODUCTION

The dynamics of stochastic processes driven by colored noise has received an increasing interest in the last years.[1] One of the most studied aspects has been the dynamics of the steady state,[2,3] addressing, for instance, the study of correlation functions and relaxation times. However, some systematic work is still to be done in the more varied context of transient dynamics, that is, on the study of the relaxation of initial conditions. Most of the efforts in that direction have been concerned with the problem of time scales. The success of the first-passage-time (FPT) techniques in the context of white noise has motivated the study of the mean-first-passage-time (MFPT) problem in the presence of colored noise. However, the difficulties involved in the mathematical treatment of this problem for non-Markovian processes have been an important handicap.[4] A possible alternative to these techniques has been proposed very recently[5] regarding other definitions of characteristic times which in some cases are expected to circumvent some of the difficulties of the non-Markovian FPT theory. These are the so-called nonlinear relaxation times (NLRT), which are defined in terms of time integrals of the transient moments. For a process $x(t)$ defined by a Langevin-like equation of the general form

$$\dot{x}=v(x)+g(x)\xi(t)\ , \tag{1.1}$$

the NLRT associated with the average [over realizations of the noise $\xi(t)$ and over initial conditions] of any arbitrary quantity $\phi(x)$ is defined by[6]

$$T_\phi=\int_0^\infty \frac{\langle\phi(t)\rangle-\langle\phi\rangle_{\rm st}}{\langle\phi\rangle_i-\langle\phi\rangle_{\rm st}}dt\ . \tag{1.2}$$

Depending on the particular problem and for appropriate choices of the quantity $\phi(x)$, this may be a good definition of a global time scale for the relaxation of initial conditions towards the steady state. The interest of that definition lies in the fact that the NLRT's can be usually calculated via very general techniques based on the knowledge of the evolution operator of the probability densities, and with no use of any explicit form for the time dependence of the function $\langle\phi(x(t))\rangle$, which for nonlinear problems is usually unknown. The formalism can be systematized in a way that parallels the calculation of the FPT moments and has turned out to have some practical and theoretical advantages over them. Its usefulness has been checked in the context of white noise,[7,8] but the advantages may be particularly relevant in colored-noise problems.[5]

The fundamental difficulty in the study of the transient dynamics of processes driven by colored noise is that, due to the non-Markovian character of the process $x(t)$, the evolution of $P(x,t)$ not only depends on its initial condition $P_i(x)$ but also on the initial state of the noise and its coupling to the variable x. The usual point of view is to assume that the noise variable is in its steady state and is also initially decoupled from the system variable (statistical independence). This means that their joint probability density factorizes. One of the points we want to emphasize here is that this assumption, made for the sake of mathematical simplicity, is not necessarily the most interesting from the physical point of view. A natural way to prepare the variable x with a given probability distribution could be, for instance, as the steady state of an appropriate auxilliary model. A typical example would be the instantaneous change of a control parameter at $t=0$ which leaves the system in an unstable situation but with a probability distribution which corresponds to a previously stable state. The point is that for any such steady state the joint probability density of the system variable and the noise variable is never factorized. In this paper we will discuss the dynamical consequences of this initial coupling on the transient dynamics.

The NLRT approach provides a particularly appropriate framework for this discussion, since it permits a natural treatment of the case of random initial conditions (initially distributed states), contrary to what happens with the MFPT. Other advantages come from the absence of the absorbing boundary conditions which cause most of the troubles in the MFPT calculations. This will enable us to obtain very general expressions, which are relatively simple and provide a clear interpretation of the different transient and preparation effects, particularly on the distinction between the purely non-Markovian ones and

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those which can be implemented in a quasi-Markovian description of the problem. These general aspects of the theory of NLRT's with colored noise is the topic of Sec. II.

In this paper we also apply the theoretical approach of Sec. II to study the decay of unstable states in the presence of colored noise and random initial conditions. For at least the last two decades, the study of the decay of unstable states has called attention to the theoretical[9-15] and the experimental[16-19] points of view. Its relevance is remarkable, for instance, in the study of laser systems.[13,16-19] The decay of an unstable state is governed by the initial fluctuations which are amplified through the deterministic evolution. Hence this type of dynamical process is very sensitive to the statistics of the initial conditions and of the noise. This makes it very appropriate for our discussion on the effects associated with an eventual coupling between the noise and the system variable due to different types of preparation of the system. The application of the NLRT approach to this study is presented in Sec. III. The existence of some results in the literature, mainly based on quasideterministic theory (QDT),[11-15,20] will also be useful in our discussion. Exact calculations on some mathematically simple models will allow us to extract general conclusions on the basis of universality. Conclusions are summarized in Sec. IV.

II. GENERAL THEORY OF NLRT WITH COLORED NOISE

The standard approximations for the time evolution of the probability density $P(x,t)$ of the non-Markovian process (1.1), when the noise is a Gaussian colored process with zero mean and correlation

$$\langle \xi(t)\xi(t')\rangle = \frac{D}{\tau}\exp(-|t-t'|/\tau) \tag{2.1}$$

for small τ, usually assume, implicitly or explicitly, that the noise variable and the system variable are uncoupled at $t=0$. This leads to effective Fokker-Planck equations with time-dependent diffusion. In our case, in order to admit completely general initial conditions and preserve all possible non-Markovian effects of the preparation of the system, we will address the two-variable Markovian formulation of the problem (1.1) with (2.1) defined by

$$\dot{x} = v(x) + \frac{g(x)}{\epsilon}\mu , \tag{2.2a}$$

$$\dot{\mu} = -\frac{\mu}{\epsilon^2} + \frac{\sqrt{D}}{\epsilon}\eta(t) , \tag{2.2b}$$

with $\langle \eta(t)\eta(t')\rangle = 2\delta(t-t')$, $\epsilon^2=\tau$, and where the noise variable has been scaled as $\xi(t)=\mu(t)/\epsilon$. The Fokker-Planck operator associated to (2.2) is

$$L(x,\mu) = -\frac{\partial}{\partial x}v(x) + \frac{1}{\epsilon^2}\frac{\partial}{\partial \mu} - \frac{1}{\epsilon}\mu\frac{\partial}{\partial x}g(x) + \frac{D}{\epsilon^2}\frac{\partial^2}{\partial \mu^2} . \tag{2.3}$$

The starting point for the calculation of the NLRT associated with $\phi(x,\mu)$ is obtained by commuting the average and the time integral in the definition (1.2) and reads

$$T_\phi = \frac{1}{\langle \phi\rangle_i - \langle \phi\rangle_{st}} \int d\mu \int dx\, \phi(x,\mu) R(x,\mu) , \tag{2.4}$$

where the quantity $R(x,\mu)$, defined by

$$R(x,\mu) = \int_0^\infty dt [P(x,\mu;t) - P_{st}(x,\mu)] , \tag{2.5}$$

obeys the equation

$$L(x,\mu)R(x,\mu) = P_{st}(x,\mu) - P_i(x,\mu) . \tag{2.6}$$

If we knew $P_{st}(x,\mu)$ and we could obtain an explicit expression for $R(x,\mu)$ from (2.6) we would have reduced the problem to quadrature, inserting it into (2.4). In our case, despite the fact that (2.6) is not exactly solvable, standard approximate techniques can be applied. In particular, a singular perturbation approach for small correlation time of the noise is suitable for our purposes. The application of this approach to NLRT's was already outlined in Ref. 21. Here we will give some more details of the essential steps necessary to follow the calculation.

According to (2.3), the exact Fokker-Planck operator for our system (2.2) can be written as

$$L(x,\mu) = \frac{F_1}{\epsilon^2} + \frac{F_2}{\epsilon} + F_3 , \tag{2.7}$$

where

$$F_1 = \frac{\partial}{\partial \mu}\mu + D\frac{\partial^2}{\partial \mu^2} , \tag{2.8a}$$

$$F_2 = -\mu\frac{\partial}{\partial x}g(x) , \tag{2.8b}$$

$$F_3 = -\frac{\partial}{\partial x}v(x) . \tag{2.8c}$$

Equation (2.6) to solve then reads

$$P_{st}(x,\mu) - P_i(x,\mu) = \left[\frac{F_1}{\epsilon^2} + \frac{F_2}{\epsilon} + F_3\right] R(x,\mu) . \tag{2.6$'$}$$

The starting point of the method is then the ansatz of an expansion in powers of ϵ of the quantities involved in (2.6′)

$$P_{st}(x,\mu) = P_0(x)P_s(\mu) + \epsilon P_1(x,\mu) + \epsilon^2 P_2(x,\mu) + \cdots , \tag{2.9a}$$

$$P_i(x,\mu) = P_0^i(x)P_s(\mu) + \epsilon P_1^i(x,\mu) + \epsilon^2 P_2^i(x,\mu) + \cdots , \tag{2.9b}$$

$$R(x,\mu) = R_0(x,\mu) + \epsilon R_1(x,\mu) + \epsilon^2 R_2(x,\mu) + \cdots , \tag{2.9c}$$

where $P_s(\mu)$ is the stationary probability of μ, so that inserting (2.9) into (2.6) and collecting the different orders in ϵ one gets the infinite set of equations

$$\begin{aligned}
&F_1R_0 = 0 \quad \text{for } \epsilon^{-2} , \\
&F_2R_0 + F_1R_1 = 0 \quad \text{for } \epsilon^{-1} , \\
&F_3R_0 + F_2R_1 + F_1R_2 = [P_0(x) - P_0^i(x)]P_s(\mu) \quad \text{for } \epsilon^0 , \\
&F_3R_1 + F_2R_2 + F_1R_3 = P_1(x,\mu) - P_1^i(x,\mu) \quad \text{for } \epsilon^1 , \\
&F_3R_n + F_2R_{n+1} + F_1R_{n+2} = P_n(x,\mu) - P_n^i(x,\mu) \quad \text{for } \epsilon^n .
\end{aligned} \tag{2.10}$$

The ansatz (2.9a) leads automatically to an expansion in powers of ϵ of the NLRT itself. According to (2.4) the non-normalized NLRT will read

$$\overline{T}_\phi = \int dx\, d\mu\, \phi(x,\mu) R(x,\mu) = \overline{T}_0 + \epsilon \overline{T}_1 + \epsilon^2 \overline{T}_2 + \cdots , \tag{2.11a}$$

where

$$\overline{T}_n = \int dx\, d\mu\, \phi(x,\mu) R_n(x,\mu) . \tag{2.11b}$$

Since we are mainly interested in the NLRT for quantities $\phi(x)$ independent of the noise variable, we will write Eq. (2.11b) as

$$\overline{T}_\phi = \int dx\, \phi(x) R_0(x) + \epsilon \int dx\, \phi(x) R_1(x) + \cdots , \tag{2.11c}$$

which defines the quantities $R_k(x)$ as

$$R_k(x) = \int d\mu\, R_k(x,\mu) . \tag{2.12}$$

Therefore, to obtain the NLRT up to a given order ϵ^n we have to solve (2.10) up to that order. Equations (2.10) are formally similar to those set up in Ref. 21 for the steady-state probability density $P_{\rm st}(x,\mu)$ of this problem, except for the nonhomogeneous terms in the right-hand side (rhs) of (2.10) for $k \geq 0$, so we will follow a parallel procedure. While the quantities of (2.9b) are an arbitrary input, the expansion (2.9a) has to be obtained consistently, solving the homogeneous part of (2.10), which is what was done in Ref. 21; so those quantities are known (up to order ϵ^2).

The analog of the normalization condition for the probability density will be in our case

$$\int dx\, d\mu\, R(x,\mu) = \int dx\, d\mu \int_0^\infty dt [P(x,\mu;t) - P(x,\mu;\infty)] = 0 , \tag{2.13a}$$

which has to hold for all ϵ, so that

$$\int dx\, d\mu\, R_k(x,\mu) = \int dx\, R_k(x) = 0 . \tag{2.13b}$$

We will also assume that the noise is always in its steady state, that is,

$$P(\mu,t) = \int dx\, P(x,\mu;t) = P_s(\mu) , \tag{2.13c}$$

so that

$$\int dx\, P_k(x,\mu;t) = 0, \quad k = 1,2,\ldots \tag{2.14}$$

and condition (2.13b) reduces simply to

$$\int dx\, R_k(x,\mu) = 0, \quad k = 0,1,2,\ldots . \tag{2.15}$$

We define now the quantities $r_k(x,\mu)$ and $q_k(x,\mu)$ as

$$R_k(x,\mu) = r_k(x,\mu) P_s(\mu) ,$$
$$P_k(x,\mu) = q_k(x,\mu) P_s(\mu) , \tag{2.16}$$
$$P_k^i(x,\mu) = q_k^i(x,\mu) P_s(\mu) .$$

From the equation to order ϵ^{-2} in (2.10) we obtain immediately the factorization

$$R_0(x,\mu) = R_0(x) P_s(\mu) , \tag{2.17}$$

with $r_0(x,\mu) = r_0(x) \equiv R_0(x)$ and from (2.15) $\int R_0(x) dx = 0$. As we have explicitly assumed in (2.9a), we also have, from the corresponding system (2.10) for the $P_s(x,\mu)$, the equation $F_1 P_s = 0$, which also implies the factorization $P_0(x,\mu) = P_0(x) P_s(\mu)$, so that $q_0(x,\mu) = q_0(x) \equiv P_0(x)$, with $\int P_0(x) dx = 1$. The same conditions have to be assumed for the lowest order of the initial probability density.

Now, using the operator relation

$$F_1 P_s(\mu) = P_s(\mu) F_1^\dagger , \tag{2.18}$$

where $F_1^\dagger$ is the backward operator of the Ornstein-Uhlenbeck process, the equations to solve take the form

$$F_1^\dagger r_k(x,\mu) = I_k(x,\mu), \quad k = 0,1,2,\ldots \tag{2.19}$$

with

$$I_0(x,\mu) = 0 , \tag{2.20a}$$

$$I_1(x,\mu) = \mu \frac{\partial}{\partial x} g(x) r_0(x) , \tag{2.20b}$$

$$I_2(x,\mu) = P_0(x) - P_0^i(x) + \mu \frac{\partial}{\partial x} g(x) r_1(x,\mu) + \frac{\partial}{\partial x} v(x) r_0(x) , \tag{2.20c}$$

$$\vdots$$

$$I_k(x,\mu) = q_{k-2} - q_{k-2}^i + \mu \frac{\partial}{\partial x} g r_{k-1} + \frac{\partial}{\partial x} v r_{k-2} . \tag{2.20d}$$

The system (2.19) is formally the same as that encountered for the $P_s(x,\mu)$. The differences are on the functions $I_k(x,\mu)$, which include now a dependence on the steady state and initial probability densities, and on the supplementary boundary conditions to fulfill the r_k.

The system (2.19) has to be solved consistently with the solvability condition

$$\int P_s(\mu) I_k(x,\mu) d\mu = 0, \quad k = 1,2,\ldots \tag{2.21}$$

which comes from the relation

$$\int d\mu\, P_s(\mu) F_1^\dagger r_k(x,\mu) = \int d\mu\, r_k(x,\mu) F_1 P_s(\mu) = 0 \tag{2.22}$$

and expresses that the inhomogeneous parts $I_k(x,\mu)$ of (2.19) have to be orthogonal to the null space of F_1.

The solvability condition (2.21) required to a given order will allow us to obtain the explicit solution of $r_k(x,\mu)$ of lower orders in a systematic way. In order to get the contributions to order $\epsilon^2 = \tau$, one has to perform the calculations up to $k = 4$. In Appendix A we give more details of the calculations up to this order. Despite being solvable in general, for the sake of simplicity and for its physical relevance, we have assumed a particular form for the quantities $q_k^i(x,\mu)$ which corresponds to a steady-state-like initial condition. This means that the q_k^i have the same functional form as that of the $q_k(x,\mu)$ in $v(x)$ and $g(x)$ (see Appendix A), but now on the arbitrary quantities $\tilde{v}(x)$ and $\tilde{g}(x)$. These functions define the preparation model which is assumed to be coupled to the same noise source ($\tilde{D}=D$, $\tilde{\tau}=\tau$).

From the results (A12) of Appendix A, the nonnormalized NLRT of a function $\phi(x)$ can be written as ($\tau=\epsilon^2$)

$$\overline{T}=\int_a^b[\phi(x)-\langle\phi\rangle_0]P_0(x)C_1(x)+\tau\left[\langle\phi\rangle_i-\langle\phi\rangle_0+\int_a^b[\phi(x)-\langle\phi\rangle_0]P_0(x)C_2(x)dx\right]+O(\tau^2)\ . \tag{2.23}$$

The first term of (2.23) after an integration by parts is identified with the white-noise solution.[7,8] However, Eq. (2.23) admits a more interesting form both from a theoretical and a practical point of view. Taking into account that the steady-state probability density for the variable x in the same approximation is of the form

$$P_\tau(x)=P_0(x)\left\{1-\tau\left[g\left[\frac{v}{g}\right]'+\frac{1}{2D}\left[\frac{v}{g}\right]^2-\left\langle g\left[\frac{v}{g}\right]'\right\rangle_0-\frac{1}{2D}\left\langle\left[\frac{v}{g}\right]^2\right\rangle_0\right]\right\}+O(\tau^2) \tag{2.24a}$$

and

$$\left\langle g\left[\frac{v}{g}\right]'\right\rangle_0+\frac{1}{2D}\left\langle\frac{v^2}{g^2}\right\rangle_0=-\frac{1}{2D}\left\langle\frac{v^2}{g^2}\right\rangle_0\ , \tag{2.24b}$$

after some manipulations it is possible to show that (2.23) is equivalent, except for corrections of order τ^2, to a compact expression for the normalized NLRT of the form

$$T=T_0(\tau)+\tau(1-T_1)+O(\tau^2)\ , \tag{2.25}$$

where now we know how to interpret the different terms.

The first term on the rhs of (2.25) is identified as the NLRT solution of the problem defined by the Fokker-Planck operator

$$L(x)=-\frac{\partial}{\partial x}v(x)+D\frac{\partial}{\partial x}g(x)\frac{\partial}{\partial x}H(x)\ , \tag{2.26a}$$

where $H(x)$ defines any possible effective diffusion function[20,22] whose first-order form agrees with

$$H(x)=g(x)\left[1+\tau g(x)\left[\frac{v(x)}{g(x)}\right]'\right]+O(\tau^2)\ , \tag{2.26b}$$

so that $T_0(\tau)$ includes the contributions coming from the steady-state effective Fokker-Planck approximation, that is, the one with a time-independent diffusion function, obtained in the limit $t\rightarrow\infty$. If we denote the steady-state probability density of (2.26) by $P_\tau(x)$ and the corresponding to the preparation model as $\tilde{P}_\tau(x)$, the first term of (2.25) reads

$$T_0(\tau)=\frac{1}{\langle\tilde{\phi}\rangle_\tau-\langle\phi\rangle_\tau}\int_a^b\frac{F(x)I(x)}{Dg(x)H(x)P_\tau(x)}dx\ , \tag{2.27a}$$

where

$$F(x)=-\int_a^x[\phi(x')-\langle\phi\rangle_\tau]P_\tau(x')dx'\ , \tag{2.27b}$$

$$I(x)=-\int_a^x[\tilde{P}_\tau(x')-P_\tau(x')]dx'\ . \tag{2.27c}$$

The averages are taken with the corresponding steady-state probability densities and a,b is the domain of definition of the process $x(t)$.

The second term on the rhs of (2.25) has two parts. First there is a systematic positive amount of τ completely general for any model, any initial condition, and any relaxing function $\phi(x)$. This is a typical non-Markovian effect analogous to that found in Ref. 3 for the LRT, which accounts for the expected slowing down in the presence of colored noise. Finally, we have the coefficient T_1, which depends on the preparation of the system and is given by

$$T_1=\frac{1}{\langle\tilde{\phi}\rangle_0-\langle\phi\rangle_0}\int_a^b\frac{F_0(x)}{Dg(x)}\left[\frac{v(x)}{g(x)}-\frac{\tilde{v}(x)}{\tilde{g}(x)}\right]\frac{\tilde{P}_0(x)}{P_0(x)}dx\ , \tag{2.28}$$

where the zero subscript indicates Gaussian white noise. This coefficient contains additional transient information, particularly about the initial coupling of the variables x and μ, since it contains the dependence on $\tilde{v}(x)$ and $\tilde{g}(x)$. This term, which could not have been obtained from the standard one-variable effective Fokker-Planck description, provides the distinction between the coupled and the uncoupled initial conditions. In fact, for an uncoupled initial condition, T_1 would read

$$T_1=\frac{1}{\langle\tilde{\phi}\rangle_0-\langle\phi\rangle_0}\int_a^b\frac{F_0(x)P_i(x)}{Dg^2(x)P_0(x)}v(x)dx \tag{2.29}$$

[the term $T_0(\tau)$ would be the same identifying $\tilde{P}_\tau(x)$ with $P_i(x)$]. The most remarkable particular case of (2.29) is $P_i(x)=\delta(x-x_0)$. If x_0 is a deterministically stationary point [$v(x_0)=0$], T_1 will vanish.

For the coupled case, an explicit evaluation of (2.28) can be performed with great generality for arbitrary $v(x)$, $g(x)$, and $\tilde{v}(x)$, with the only restriction of $\tilde{g}(x)\equiv g(x)$ and yields the simple result

$$T_1=1\ . \tag{2.30}$$

This is a quite general and remarkable result which applies to most interesting situations. It implies that only the term $T_0(\tau)$ survives in (2.25), so that the problem is reduced to the effective steady-state Fokker-Planck description (2.26), with a time-independent effective diffusion function.[22]

It is to be remarked that the dynamics of our system for almost any steady-state-like preparation has thus been reduced, having included all the transient effects (to first order in τ), to an effective Gaussian white-noise one, where the τ dependence enters only parametrically into the equations, so that the usual machinery of Markovian processes is applicable.

III. EXPLICIT RESULTS FOR UNSTABLE STATES

A. Definitions and models

The usual definition of an unstable state x_0 for a general model (1.1) is

$$v(x_0)=0, \quad v'(x_0)>0 . \tag{3.1}$$

If $v'(x_0)=0$, then we have a marginal state which is not the subject of this work. Nevertheless, our approach can be extended to that situation.[23]

A very common model one can find in the literature as a prototype for the study of unstable states is the Landau model defined by the equation

$$\dot{x}=ax-bx^3+\xi(t), \quad a,b>0 \tag{3.2}$$

where $x_0=0$ is the unstable state. This model has two stable states at $|x|=\sqrt{a/b}$. However, even for such a simple model, few results can be obtained exactly. In order to find analytical and explicit results and on the basis of the universal properties of the unstable states, we define a "representative model" which is mathematically simpler than (3.2) but captures its essential features. This model is defined by a linear equation

$$\dot{x}=ax+\xi(t), \quad x\in[-R,R] \tag{3.3}$$

subject to two reflecting boundaries at $|x|=R$. The effect of these boundaries is essentially that of the nonlinearities in the model (3.2), that is, to provide a saturation regime which stabilizes the system at a typical distance R from the unstable state. The model (3.3) has thus the same parameters as the original model (3.2), with the equivalence $R=\sqrt{a/b}$ (a comparative study of these models can be found in Ref. 8).

The models (3.2) and (3.3) define the dynamics of the system for $t>0$. Since we are interested in the preparation of steady-state-like initial conditions at $t=0$, we have to fix now the preparation models which define the evolution of the system for $t<0$, from an arbitrary state at $t=-\infty$. In our case the preparation of the system at $t=0$ for the models (3.2) and (3.3) will be given, respectively, as the steady state of

$$\dot{x}=-a_0x-bx^3+\xi(t), \quad a_0,b>0, \quad -\infty<t<0 \tag{3.4}$$

and

$$\dot{x}=-a_0x+\xi(t), \quad -\infty<t<0 . \tag{3.5}$$

Equation (3.5) is the linear approximation of (3.4). This approximation is justified when the intensity of the noise is very small, which is the limit in which we are interested. In the case of a large intensity of the noise, the dynamics of the relaxation is completely different and does not admit the usual picture of unstable states.[8]

The steady-state probabilities for the Gaussian white-noise case are, respectively,

$$P_{\rm st}(x)\sim\exp\left[-\frac{a_0x^2}{2D'}-\frac{bx^4}{4D'}\right], \tag{3.6}$$

$$P_{\rm st}(x)\sim\exp\left[-\frac{a_0x^2}{2D'}\right], \tag{3.7}$$

where we admit in general a different noise intensity D' for $t<0$. In the case of Gaussian colored noise the steady distribution is exactly known for (3.7). The solution is the same as for (3.7) but with the substitution

$$D'\to\frac{D'}{1+a_0\tau} . \tag{3.8}$$

The preparation models (3.4) and (3.5) describe the typical situation in which the system suffers an instantaneous "quench" of the coefficient of the linear term $-a_0\to a$ which leaves the system initially located around an unstable state. Up to very recently the usual initial condition considered was

$$P_i(x)=\delta(x) . \tag{3.9}$$

From the experimental point of view, this means that the spread or uncertainty of the initial condition is much smaller than the actual intensity of the noise that will trigger the decay. However, in many situations both quantities will essentially originate from a unique noise source, so that they will be of comparable size. If the noise is colored, a particular attention has to be paid to the coupling of the system variable and the noise variable. When the system is prepared as a real steady state, the joint probability density of the variables x and μ never factorizes. This means that due to the history of the system during the preparation at $t<0$ there is a correlation between the noise and the system variable. Only if the noise source is not the same for $t<0$ and $t>0$ (or there is not any and the system starts from a point distributed at random) can the two variables be considered as statistically independent. Some recent theoretical results have considered the study of unstable states with the latter assumption.[12,14] However, here we claim that this approach is not justified for simulations of quenched unstable states like (3.2) with (3.4) or those of Ref. 19, since, as we will show, the net effects of the initial coupling are not negligible.

For the sake of a better comprehension of the colored-noise effects and for further reference we will start analyzing the white-noise case with distributed initial conditions. In the rest of the paper we will always refer to the NLRT associated to the second moment $[\phi(x)=x^2]$.

B. White noise

The NLRT problem for the decay of an unstable state with fixed initial conditions (3.9) and white noise was solved very recently.[7,8] The conclusion was that for a (symmetric) unstable-state model, characterized by the parameter a and the length R separating the stable states from the unstable one, the NLRT, for a small intensity of the noise, is given by the general law

$$T^F(D,\tau=0)=\frac{1}{a}\left[\frac{1}{2}\ln\left[\frac{R^2a}{2D}\right]+C\right]+O(D) , \tag{3.10}$$

where the logarithmic term is universal and the numeric constant C is characteristic of each model and accounts for the details of the deterministic relaxation in the nonlinear and saturation regimes. In Appendix B we give more details about the meaning and the calculation of that constant for fixed initial conditions and white noise. For the linear model (3.3) we have[8]

$$C_{\rm L}=\tfrac{1}{2}(\gamma+2\ln 2-1)=0.482\ldots\,, \qquad (3.11)$$

where γ is the Euler constant. The value for the nonlinear (NL) model (3.3), from (B9), is just

$$C_{\rm NL}=C_{\rm L}+\tfrac{1}{2}=0.982\ldots\,. \qquad (3.12)$$

In this section we are interested in the corrections to that constant due to nonfixed initial conditions and to colored noise. Despite the fact that the value of the constant is model dependent, those corrections are expected to be universal.

Let us see first the effects of the distributed initial conditions (3.7) in the model (3.3). The NLRT expression for white noise is given by (2.39) with $\tau=0$. The explicit calculation for the model (3.3) is explained in Appendix C. The final result in the limits of both $D,D'\to 0$ with D/D' finite reads

$$T_{\rm L}^{D}(\tau=0)=\frac{1}{a}\left[\frac{1}{2}\ln\left[\frac{R^2}{2(D/a+D'/a_0)}\right]+C_{\rm L}\right]+O(D,D')\,. \qquad (3.13)$$

We see that the effect of the distributed initial condition defines a larger effective diffusion coefficient. The difference between (3.13) and (3.10) is

$$T^{D}(\tau=0)-T^{F}(\tau=0)=-\frac{1}{2a}\ln\left[1+\frac{D'a}{Da_0}\right]+O(D,D')\,, \qquad (3.14)$$

so that the presence of distributed initial conditions speeds the decay of the unstable state. Its effect is important when the width of the initial distribution D' and the diffusion coefficient D are of comparable size, even when both are very small. Although the formula (3.14) was calculated for the representative case (3.3), (3.5), and (3.7), that result has to be general [$R^2=a/b$ in model (3.2)], since in the limits $D,D'\to 0$ only the linear part of the dynamics is involved. This result could also be used in real experiments to estimate the actual uncertainty on the initial condition.

C. Colored noise

Now we will consider the case $\tau\neq 0$ so we have to evaluate our general result (3.37) in the different cases. The simplest situation is that corresponding to coupled initial conditions given that $T_1=1$, so only the evaluation of $T_0(\tau)$ is necessary. As we established in Sec. II, this contribution is exactly that of the effective white-noise problem defined by (2.26), so we can use the result (3.13) with the substitutions

$$D\to D/(1-a\tau)\,, \qquad (3.15a)$$

$$D'\to D/(1+a_0\tau)\,, \qquad (3.15b)$$

and we get finally the result

$$T^{D}(\tau)=T^{D}(\tau=0)+O(\tau^2,D)\,, \qquad (3.16)$$

which has to hold also for the nonlinear model (3.2) and (3.4).

For decoupled initial conditions we have that for the case (3.9) $T_1=0$, so that, using (3.13) with (3.15a) and $D'=0$, the result can be written as

$$T^{F}(\tau)=T^{F}(\tau=0)+\frac{\tau}{2}+O(D,\tau^2)$$
$$=\frac{1}{2a}\left[\ln\frac{aR^2}{2D}+a\tau\right]+C+O(D,\tau^2)\,. \qquad (3.17)$$

This result coincides to first order in τ with the prediction of the QDT,[12,14] up to an additive constant.

Assuming a distributed initial condition for the variable x given by (3.7), but decoupled from the steady-state distribution for the noise, one gets

$$T_1=\frac{a}{a+a_0}+O(D) \qquad (3.18)$$

and

$$\hat{T}^{D}(\tau)=T^{D}(\tau)+\frac{\tau}{1+a/a_0}+O(\tau^2,D,D')\,, \qquad (3.19)$$

where the hat means uncoupled initial conditions. As in the previous cases these results are claimed to be universal, so they have to hold also for the Landau model. Equations (3.14), (3.16), (3.17), and (3.19) are the main predictions of this work.

D. Comparison with the MFPT and QDT approaches

The comparison between the NLRT and the MFPT approaches has been widely discussed in Ref. 8 for the case of white noise and fixed initial conditions. In the limit of small intensity of the noise both results coincide up to a systematic constant $\frac{1}{2}$, whose relevance is manifest in the limit of large D.[8] In the case of distributed initial conditions, the MFPT approach also makes sense from an experimental point of view.[19] Assuming that the initial distribution is very peaked with respect to the escape region studied, the MFPT can be evaluated by a simple averaging over the initial distribution. For the model (3.3) with $\tau=0$, this leads to the same result (3.13), except for the additive $\frac{1}{2}$, characteristic of the NLRT.

In the context of colored noise, the advantages of the NLRT approach may be more relevant. The NLRT provides a simple characterization of the whole relaxation process and avoids some technical difficulties of the boundary conditions in the MFPT theory. On the other hand, it is also naturally adapted to the treatment of arbitrarily distributed initial conditions.

In the particular case of unstable states, the difficulties of the MFPT with colored noise have been avoided in the literature by means of the QDT approach. The explicit results for (3.3) and (3.5) were obtained in Ref. 15 and are ($D=D'$)

$$\langle t \rangle = -\frac{1}{2a}\ln\left[\frac{D}{a(1+a\tau)}+\frac{D}{a_0(1+a_0\tau)}+\frac{2D\tau}{(1+a_0\tau)(1+a\tau)}\right]+\frac{1}{2a}[\gamma+\ln(2x_F^2)] . \tag{3.20}$$

Taking $x_F^2=R^2$, we recover, if $\tau=0$, the result (3.13) up to the constant $\frac{1}{2}$ in (3.11). To first order in τ we see that

$$\langle t \rangle = \frac{1}{a}\left[\frac{1}{2}\ln\left[\frac{R^2 a}{2D}\right]+C_L-\frac{1}{2}\right]+O(D,\tau^2) . \tag{3.21}$$

This is a clear manifestation of the universality of the process. The decay of an unstable state in each case has a well-defined characteristic time scale which can be obtained using different approaches. Particularly, the corrections to the model-dependent constant due to distributed initial conditions and coupling effects are claimed to be universal and should be encountered also for other characteristic times like the MFPT.

IV. SUMMARY AND CONCLUSIONS

We have developed the general theory of NLRT's for colored-noise driven systems and applied it to the study of the problem of the decay of unstable states in different situations, with special stress on the non-Markovian effects associated with the preparation of the system in the presence of colored noise. We have obtained different predictions with theoretical and practical interest, which are also in qualitative agreement with those of the QDT analysis. The advantage of our formalism is that, contrary to what happens with the QDT, it is completely general, so that it can be applied to other relaxation processes such as those which involve metastable or marginal states. It is to be remarked that, despite the fact that our general results are valid for small τ, they include all non-Markovian effects. Its application to particular cases is straightforward: there is a contribution which comes from a quasi-Markovian effective problem, for which standard Markovian techniques can be applied, and then a simple expression which supplies the purely non-Markovian effects. The formalism can be seen as a complementary approach to that of the MFPT that circumvents some of the its difficulties, and can be a useful tool, both from a theoretical and a practical point of view, in the study of non-Markovian transient dynamics.

Analog experiments and digital simulations will be presented in the second part of this work[24] in order to check the range of validity of the theoretical predictions presented here.

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APPENDIX A

The solvability condition (SC) (2.21) required to a given order will allow us to obtain the explicit solution of $r_k(x,\mu)$ of lower orders in a systematic way, which we will describe in the following. This method parallels step by step that described in Ref. 21. We include here some details in order to be more explicit and emphasize the main differences with that calculation.

That condition is trivially satisfied to order ϵ^{-1}. The solution of (2.19) for $k=1$ then reads

$$r_1(x,\mu)=H_1(x)-I_1(x,\mu) , \tag{A1}$$

where $H_1(x)$ is an arbitrary function compatible with the condition (2.15). Now Eq. (2.19) for $k=2$ takes the form

$$F_1^\dagger r_2=P_0-P_i+\mu\frac{\partial}{\partial x}g\left[H_1-\mu\frac{\partial}{\partial x}gr_0\right]+\frac{\partial}{\partial x}vr_0 , \tag{A2}$$

so the SC leads to

$$\left[-\frac{\partial}{\partial x}v+D\frac{\partial}{\partial x}g\frac{\partial}{\partial x}g\right]r_0=P_0-P_i . \tag{A3}$$

Equation (A3) is the usual one encountered for the NLRT for the case (1.1) with a Gaussian white noise.[7] Its solution, eliminating one arbitrary constant from the condition of vanishing probability current at the natural boundaries a,b and the other from the condition (2.15), is

$$r_0(x)=P_0(x)\left[\int_a^x\frac{I(x')}{Dg^2(x')P_0(x')}dx'-\left\langle\int_a^x\frac{I}{Dg^2P_0}\right\rangle_0\right] , \tag{A4a}$$

where

$$I(x)=\int_a^x[P_0(x')-P_i(x')]dx' \tag{A4b}$$

and $\langle\ \rangle_0$ means the average with $P_0(x)$. After a formal integration in both members, Eq. (A3) permits us to write (A1) as

$$r_1(x,\mu)=H_1(x)-\frac{1}{D}\mu\frac{1}{g(x)}[I(x)+v(x)r_0(x)] . \tag{A1$'$}$$

On the other hand, using also (A3), Eq. (A2) can be written as

$$F_1^\dagger r_2=P_0-P_i-\frac{\mu^2}{D}\left[P_0-P_i+\frac{\partial}{\partial x}vr_0\right]+\mu\frac{\partial}{\partial x}gH_1+\frac{\partial}{\partial x}vr_0 \tag{A2$'$}$$

and its general solution is of the form

$$r_2(x,\mu)=\frac{\mu^2}{2D}\left[P_0-P_i+\frac{\partial}{\partial x}v(x)r_0(x)\right]-\mu\frac{\partial}{\partial x}g(x)H_1(x)+H_2(x)\ , \tag{A5}$$

where $H_2(x)$ is another arbitrary function compatible with (2.15). Notice that it has been necessary to go to order $k=2$ to determine $r_0(x)$, and that we have two arbitrary functions $H_1(x)$ and $H_2(x)$. These functions will be determined by imposing the SC, respectively, to the orders $k=3$ and 4.

Equation (2.19) for $k=3$ reads

$$F_1^\dagger r_3=q_1-q_1^i+\mu\frac{\partial}{\partial x}gr_2+\frac{\partial}{\partial x}vr_1\ . \tag{A6a}$$

From Ref. 21 we know that

$$q_1(x,\mu)=-\frac{1}{D}\frac{v(x)}{g(x)}P_0(x)\mu\ , \tag{A6b}$$

but $q_1^i(x,\mu)$ is to be determined from the initial probability density for the variable x. At this point we make the assumption of a steady-state-like initial condition, that is, a probability density which comes from a true steady state of the two-variable problem of the type (1.1) associated with a preparation model defined by $\tilde{v}(x),\tilde{g}(x)$ and driven by the same noise. We also assume that the intensity of the noise D for the preparation model is the same, in order to ensure that the noise is always in its steady state. This case is the simplest one after the completely decoupled case [$q_k^i(x)=0$ for $k>0$] and is also the more interesting from a physical point of view. In this case we will have

$$q_1-q_1^i\equiv\Delta q_1(x,\mu)=\mu\left[\frac{\tilde{v}}{D\tilde{g}}\tilde{P}_0-\frac{v}{Dg}P_0\right]\ , \tag{A6c}$$

so that the SC to the order $k=3$ given by (A6) and after the substitution of (A1′) and (A5) gives

$$-D\frac{\partial}{\partial x}g\frac{\partial}{\partial x}gH_1+\frac{\partial}{\partial x}vH_1=0\ . \tag{A7}$$

For natural boundary conditions Eq. (A7) implies that $H_1(x)$ vanishes identically.[21]

Finally, to determine $H_2(x)$ let us consider the order $k=4$,

$$F_1^\dagger r_4=\Delta q_2(x,\mu)+\mu\frac{\partial}{\partial x}gr_3+\frac{\partial}{\partial x}vr_2\ , \tag{A8}$$

where we define $\Delta q_2(x,\mu)=q_2(x,\mu)-q_2^i(x,\mu)$, $q_2^i(x,\mu)$ corresponding to the preparation system defined above. From Ref. 21 we also know that

$$q_2(x,\mu)=\frac{1}{2D}\mu^2\frac{\partial}{\partial x}vP_0+P_0\left[\frac{1}{2D}\frac{v^2}{g^2}-\frac{1}{2D}\left\langle\frac{v^2}{g^2}\right\rangle-\frac{3}{2}\frac{1}{P_0}\frac{\partial}{\partial x}vP_0\right]\ , \tag{A9}$$

and similarly for $q_2^i(x,\mu)$. Now we have to proceed with the SC for $k=4$, which gives rise to

$$\begin{aligned}&-\frac{1}{2}\frac{\partial}{\partial x}v\frac{\partial}{\partial x}vr_0+\frac{\partial}{\partial x}vH_2-\frac{D}{2}\frac{\partial}{\partial x}g\frac{\partial}{\partial x}g\frac{\partial}{\partial x}vr_0\\&+D\frac{\partial}{\partial x}g\frac{\partial}{\partial x}\left[\frac{v^2}{Dg}r_0-g\frac{\partial}{\partial x}vr_0-gH_2\right]\\&+D\frac{\partial}{\partial x}g\frac{\partial}{\partial x}\frac{vI}{Dg}\\&-\frac{3}{2}D\frac{\partial}{\partial x}g\frac{\partial}{\partial x}g\Delta P_0+\frac{1}{2}\frac{\partial}{\partial x}v\Delta P_0\\&+\Delta\bar{q}_2(x)\\&-D\frac{\partial}{\partial x}g(x)\Delta\bar{q}_1(x)=0\ ,\end{aligned} \tag{A10}$$

where $\Delta P_0(x)=P_0(x)-P_i(x), \Delta\bar{q}_1(x)=q_1(x,\mu)/\mu$ and

$$\Delta\bar{q}_2(x)=\int P_s(\mu)\Delta q_2(x,\mu)d\mu\ .$$

After some more algebra, using the explicit form of $P_0(x)=Ng^{-1}\exp(\int^x v/Dg^2)$ and $r_0(x)$ given by (A4), integrating and rearranging, this yields

$$\begin{aligned}\tilde{H}_2(x)=\frac{H_2(x)}{P_0(x)}=&\int_a^x\frac{1}{Dg^2P_0}\left[\frac{v^2}{Dg^2}I-v\Delta P_0-Dg\Delta\bar{q}_1+\int_a^x\Delta\bar{q}_2\right]-\frac{1}{2D}\int_a^x\frac{I}{Dg^2P_0}\frac{v^2}{g^2}-\frac{1}{2D}\frac{v}{g^2}\frac{I}{P_0}-\frac{3}{2}\frac{\Delta P_0}{P_0}\\&-\left[\frac{1}{D}\frac{v^2}{g^2}+\frac{3}{2}g\left[\frac{v}{g}\right]'\right]\left[\int_a^x\frac{I}{Dg^2P_0}-\left\langle\int_a^x\frac{I}{Dg^2P_0}\right\rangle\right]+K\ ,\end{aligned} \tag{A11}$$

where the constant K is to be determined by (2.15). Now, in order to get the $R_k(x)$ defined by (2.12), we have to use the explicit $r_k(x,\mu)$ obtained and integrate over the domain of μ using (2.16). We then get, after some rearranging,

$$R(x)=R_0(x)+\epsilon R_1(x)+\epsilon^2R_2(x)+\cdots\ , \tag{A12a}$$

where

$$R_0(x)=P_0(x)[C_1(x)-\langle C_1(x)\rangle_0]\ , \tag{A12b}$$

$$R_1(x)=0\ , \tag{A12c}$$

$$R_2(x)=-\Delta P_0(x)+P_0(x)[C_2(x)-\langle C_2(x)\rangle_0]\ , \tag{A12d}$$

with

$$C_1(x)=\int_a^x\frac{I}{Dg^2P_0}\ , \tag{A12e}$$

$$C_2(x)=\int_a^x \frac{1}{Dg^2P_0}\left[-v\Delta P_0-Dg\Delta\bar{q}_1+\int_a^x\Delta\bar{q}_2\right] -\left[\frac{1}{2D}\frac{v^2}{g^2}+g\left[\frac{v}{g}\right]'\right][C_1(x)-\langle C_1(x)\rangle_0] . \tag{A12f}$$

APPENDIX B

Here we will give more details about the constant C defined by (3.10). We can write it as

$$C=C(n)+C(m) , \tag{B1}$$

where $C(n)$ is conjectured to be a universal part which depends only on the dimension n of the space (assuming an isotropic unstable state in n variables[8]) and which was also found for the MFPT (Ref. 8) and reads in terms of the ψ function[25]

$$C(n)=-\tfrac{1}{2}\psi\left[\frac{n}{2}\right] . \tag{B2}$$

The other term in (B1) accounts for the deterministic relaxation in the nonlinear and saturation regimes and is model dependent. For the linear model and the relaxation of the mth power $[\phi(x)=x^m]$, it reduces to

$$C_L(m)=-\frac{1}{m} . \tag{B3}$$

Since the value of the constant C_L for the linear model is known, what we have to evaluate for the other cases is simply the difference with respect to that one which will be given by

$$\Delta C(m)=\lim_{\substack{D\to 0\\ x_0\to 0}}[T^{(m)}(D,x_0)-T_L^{(m)}(D,x_0)] , \tag{B4}$$

where $T^{(m)}$ stands for the NLRT corresponding to the mth moment, and x_0 is the initial condition. Both models are assumed to be scaled in such a way that $v(0)=v(1)=v(-1)=0$ and $v'(0)=1$. The divergence of each term is then exactly the same and the finite difference is just that of the respective constants. Assuming $n=1$ for simplicity and taking first the deterministic limit $D\to 0$, we can write

$$T^{(m)}(0,x_0)=\frac{1}{x_0^m-1}\int_0^\infty[x^m(t)-1]dt , \tag{B5}$$

and with the change of variables given by the deterministic equation $dx=v(x)dt$ this reduces to

$$T^{(m)}(0,x_0)=\frac{1}{x_0^m-1}\int_{x_0}^1\frac{x^m-1}{v(x)}dx , \tag{B6}$$

so that $C(m)$ is just

$$C(m)=-\frac{1}{m}+\int_0^1(x^m-1)\left[\frac{1}{x}-\frac{1}{v(x)}\right]dx . \tag{B7}$$

For the Landau model (B4) will be

$$\Delta C_{NL}(m)=\int_0^1\frac{x^m-1}{x^2-1}x dx , \tag{B8}$$

so that for arbitrary n and m

$$C_{NL}=-\tfrac{1}{2}\gamma-\frac{1}{2}\left[\psi\left[\frac{n}{2}\right]-\psi\left[\frac{m}{2}\right]\right] . \tag{B9}$$

APPENDIX C

From the general result (2.25) and (2.27) the NLRT for the linear model (3.3) ($a=R=1$) with white noise ($\tau=0$) and prepared as (3.5) ($a_0=1$) is given by

$$T=4\alpha e^{-\alpha^2}\int_0^\alpha\left[\frac{z}{2\alpha^2}-F(z)\right]e^{z^2}F(z)dz -\frac{1}{\alpha^2}\int_0^\alpha z\Phi(\beta z)dz+2\int_0^\alpha F(z)\Phi(\beta z)dz , \tag{C1}$$

where $\alpha=\sqrt{1/2D}$, $\alpha'=\sqrt{1/2D'}$, and $\beta=\alpha'/\alpha$, and where $F(x)$ is the Dawson integral and $\Phi(x)$ is the error function.[25] For simplicity, in (C1) we have written the leading terms when $\alpha\gg 1$ for each of the four terms. Here we are interested in the correction $\Delta C(\beta)$ to the numeric constant in (3.10) as a function of β. This quantity will correspond to the difference between the NLRT for initially distributed initial conditions ($\beta<\infty$) with respect to the NLRT with a fixed initial condition (3.10) ($\beta=\infty$), in the limits $\alpha,\alpha'\to\infty$, that is,

$$\Delta C(\beta)=\lim_{\alpha\to\infty}\left[\frac{1}{\alpha^2}\int_0^\alpha z[1-\Phi(\beta z)]dz -2\int_0^\alpha F(z)[1-\Phi(\beta z)]dz\right] , \tag{C2}$$

since only the terms depending on β remain. The first term of (C2) vanishes as $1/\alpha^2$, so that we only have to calculate

$$\Delta C(\beta)=-2\int_0^\infty F(z)(1-\Phi(\beta z))dz . \tag{C3}$$

An integration by parts gives

$$\Delta C(\beta)=-\frac{4\beta}{\sqrt{\pi}}\int_0^\infty e^{-\beta^2z^2}G(z)dz-2G(z)[1-\Phi(\beta z)]|_0^\infty . \tag{C4}$$

where

$$G(z)=\int_0^z F(x)dx . \tag{C5}$$

This function $G(z)$ diverges as $\frac{1}{2}\ln z$ when $z\gg 1$, so the surface term vanishes, taking into account that $1-\Phi(z)$ goes as $\exp(-z^2)/(\sqrt{\pi}z)$. The remaining integral can be evaluated by expanding the function $G(z)$ in powers of z,

$$G(z)=\sum_{k=1}^\infty\frac{z^{2k}}{(2k)!}(-1)^{k-1}2^{2(k-1)}(k-1)! . \tag{C6}$$

Substituting (C6) into (C4) and making the Gaussian integrals, one obtains a power series which can be summed for $\beta>1$ and, by analytic prolongation, defines the function $\Delta C(\beta)$ for all β as

$$\Delta C(\beta) = -\tfrac{1}{2}\ln\left[1+\frac{1}{\beta^2}\right] = -\tfrac{1}{2}\ln\left[1+\frac{D'}{D}\right] . \quad \text{(C7)}$$

This result, which has been obtained for simplicity for the relaxation of x^2 in dimension $n=1$, is actually general for any moment x^m and for arbitrary n. It has a simple interpretation as the time needed for the process to broaden up to a width D' from a fixed initial condition, in the linear regime (D and D' small).

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