Internal and external fluctuations around nonequilibrium steady states in one-dimensional heat-conduction problems

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Thermal fluctuations around inhomogeneous nonequilibrium steady states of one-dimensional rigid heat conductors are analyzed in the framework of generalized fluctuating hydrodynamics. The effect of an external source of noise is also considered. External fluctuations come from temperature and position fluctuations of the source. Contributions of each kind of noise to the temperature correlation function are computed and compared through the study of its asymptotic behavior.

I. INTRODUCTION

In recent years a great deal of attention has been paid to the study of fluctuations about nonequilibrium steady states.[1] This interest lies in the fact that away from equilibrium the study of fluctuations introduces a great variety of differences with respect to fluctuations about equilibrium states.[2] For example, in steady states, the equal-time correlation functions are long ranged in contrast to correlations at equilibrium which are δ-correlated. Moreover, such correlations contain the external gradients, imposed to generate a nonequilibrium steady state, which sometimes appear in the form of expansion parameters. The stationary states have been normally assumed to be simple functions as linear temperature or velocity profiles.

Nonequilibrium steady states introduce an essential feature: To maintain the steady state, energy must be supplied through the boundaries of the system. The consequence is that sometimes the nature of boundaries should be taken into account in the analysis of fluctuations. This is what happens, for example, when absorbing walls are considered;[3] then one shows that the presence of fluctuating boundaries modifies the spectrum of light scattering. Another example is the displacement of the onset of Bénard instability due to a random surface temperature.[4] A review of results concerning the role of external fluctuations around homogeneous systems has been given in Ref. 5.

Our aim in this paper is precisely to incorporate the effect of external fluctuations to the study of fluctuations about nonequilibrium steady states. To this purpose and for the sake of simplicity we will treat a problem of one-dimensional heat conduction. Then our system consists of a thin solid rod in which heat may flow by conduction and radiation to a thermal bath. By considering the problem as one dimensional we can obtain analytical expressions for the temperature correlation function coming from different sources of noise, which for higher dimensionality could not be achieved.

On the other hand, it is well known that, in general, far away from critical points, fluctuations are quite small. However, our attention is not focused on the size of such fluctuations but on the form in which nonlinear profiles affect the temperature correlation function or on comparing the different contributions to the temperature correlation function due to external and internal sources of noise. Our analysis could be, of course, extended to more general situations.

The distribution of the paper is as follows. In Sec. II we introduce our physical system obtaining the stationary solution and we study internal fluctuations around it. Section III is devoted to external noise. Finally, in Sec. IV we compare temperature correlation functions obtained with both kinds of noise.

II. THERMAL FLUCTUATIONS AROUND NONEQUILIBRIUM STEADY STATES

Our system consists of a thin solid rod surrounded by a heat bath kept at constant temperature. In such a situation the temperature along the rod varies according to[6]

$$c\frac{\partial T(x,t)}{\partial t}=-\frac{\partial}{\partial x}J(x,t)-Q(x,t)\ , \tag{2.1}$$

where c is the heat capacity per unit length of the solid, $J(x,t)$ is the heat flux along the rod, and $Q(x,t)$ is the heat flux interchanged by radiation at the surface. These fluxes obey the following linear phenomenological laws:

$$J(x,t)=-\lambda\frac{\partial}{\partial x}T(x,t)\ , \tag{2.2}$$

$$Q(x,t)=q[T(x,t)-T_B]\ , \tag{2.3}$$

λ being the heat conductivity of the rod that, in principle, depends on the local temperature. Equation (2.3) stands for Newton's law of heat interchange between the system and a heat bath kept at a temperature T_B. Without loss of generality this temperature can be taken to be zero in such a way that in what follows all the temperatures will be measured taking the bath temperature as reference. The coefficient q is related to the surface conductance[7] and depends not only on the nature of the rod and of the surrounding medium but also on the geometry of the rod.

In this paper, for the sake of simplicity, the heat conductivity λ and the coefficient q will be assumed to be constant along the temperature range to be considered.

Inserting (2.2) and (2.3) into (2.1) one obtains

$$\frac{\partial T(x,t)}{\partial t}=\alpha\frac{\partial^2 T(x,t)}{\partial x^2}-\beta T(x,t)\ , \tag{2.4}$$

where the thermal diffusivity α and the heat-transfer coefficient β are defined by

$$\alpha=\lambda/c\ ,\qquad \beta=q/c\ . \tag{2.5}$$

It is easily seen from Eq. (2.4) that the stationary solution depends on $\mu=(\beta/\alpha)^{1/2}$. The inverse of this quantity can be interpreted as a decay length which compares the importance of diffusion along the rod and radiation into the heat bath.

Obviously, the stationary solution depends on both the geometry and the boundary conditions. In heat conduction problems, boundary conditions are related to a heat source supplying a constant flux or maintaining a point at a fixed temperature. In the present paper we are going to consider a heat source located at $x=0$ maintaining this point at a fixed temperature T_0. Such a position of the source allows us to consider long rods without losing the effect of boundary conditions.

The stationary solution of (2.4) with the above-mentioned condition is not physically acceptable since the temperature would grow when moving away from the source. However, this problem is avoided by obtaining solutions in each side of the rod. Assuming that the temperature of the end points is equal to that of the bath, which will be a good approximation for long rods,[8] one obtains a general solution

$$T_s(x)=T_0\frac{\sinh[\mu(L-|x|)]}{\sinh(\mu L)} \tag{2.6}$$

which is not a stationary solution of (2.4). This is solved by adding a new term to the equation satisfied by the stationary temperature

$$0=\alpha\frac{d^2T_s(x)}{dx^2}-\beta T_s(x)+2\mu\alpha T_0[\coth(\mu L)]\delta(x)\ , \tag{2.7}$$

which is a common feature when a located source is present such as, for example, in crystal growth,[9] in nuclear reactors,[10] or when a fluid is in contact with absorbing walls.[3]

The stability of the system can be studied through the stability of a small perturbation from the steady state. On this perturbation one imposes the same kind of boundary conditions as those for the stationary solutions, concluding that any perturbation, and the stationary solution as well, is stable.

Once the stability is demonstrated, it is possible to study fluctuations around stationary states. Internal fluctuations are incorporated by adding stochastic currents J^R and Q^R to the linear phenomenological laws (2.2) and (2.3) in the form

$$J(x,t)=-\frac{\partial}{\partial x}T(x,t)+J^R(x,t)\ , \tag{2.8}$$

$$Q(x,t)=qT(x,t)+Q^R(x,t)\ . \tag{2.9}$$

It is known that fluctuations of macroscopic variables are small far from critical points. For this reason the local temperature in Eq. (2.4) can be linearized around the stationary temperature. Taking into account (2.8) and (2.9) one arrives at

$$\frac{\partial\delta T(x,t)}{\partial t}=\alpha\frac{\partial^2\delta T(x,t)}{\partial x^2}-\beta\delta T(x,t)-\frac{1}{c}\frac{\partial J^R(x,t)}{\partial x}-\frac{1}{c}Q^R(x,t)\ , \tag{2.10}$$

where temperature fluctuations are defined as $\delta T(x,t)=T(x,t)-T_s(x)$. According to the extension of Landau-Lifshitz fluctuating hydrodynamics,[11] the stochastic currents have zero mean and are not cross correlated. Moreover, their autocorrelation functions are

$$\langle J^R(x,t)J^R(x',t')\rangle_{\rm NE}=2\lambda k_B T_s^2(x)\delta(x-x')\delta(t-t')\ , \tag{2.11}$$

$$\langle Q^R(x,t)Q^R(x',t')\rangle_{\rm NE}=2qk_B T_s^2(x)\delta(x-x')\delta(t-t')\ , \tag{2.12}$$

where k_B is the Boltzmann constant. In Eqs. (2.11) and (2.12), $\langle\cdots\rangle_{\rm NE}$ stands for nonequilibrium averages. Henceforth, the symbol NE will be omitted.

Notice that, in principle, the temperature appearing in the fluctuation-dissipation theorems (2.11) and (2.12) is the absolute temperature, whereas the stationary temperature coming from Eq. (2.6) is taken in a scale whose origin is the heat-bath temperature. This fact does not introduce any problem since, in our temperature scale, the temperature correlation function measures how temperature fluctuations around an inhomogeneous steady state differ from fluctuations around a state which is at thermal equilibrium with the heat bath.

By introducing Fourier transforms, as usual, Eq. (2.10) becomes

$$\delta T(k,w)=\frac{F^R(k,w)}{\alpha k^2+\beta-iw}\equiv G(k,w)F^R(k,w)\ , \tag{2.13}$$

where F^R is the total stochastic current

$$F^R(k,w)=-c^{-1}[ikJ^R(k,w)+Q^R(k,w)] \tag{2.14}$$

and $G(k,w)$ is the Green function describing the propagation of fluctuations.

Notice that the Fourier integrals defined above are applicable to a finite system when the length of the system is much greater than a characteristic length.[11] This fact introduces a restriction in the value of wave numbers in the sense that our expressions are not valid for small wave numbers. However, integrals over all the values of k can be performed since the leading contribution to the integral containing the Green propagator introduced in (2.13) occurs for values of k close to the pole of such a propagator.

The temperature correlation function in k-w representation can be obtained from (2.13):

$$\langle\delta T(k,w)\delta T(k',w')\rangle=\frac{4\pi v_B}{c^2}\delta(w+w')(q-\lambda kk')T_s^2(k+k')\times G(k,w)G(k',w')\ , \tag{2.15}$$

$T_s^2(k)$ being the Fourier transform of the square of the stationary temperature.

As stated above, the rod with a source located at the origin permits one to take the limit $L\to\infty$ without losing the effect of the source. In this case the stationary solution (2.6) becomes

$$T_s(x)=T_0 e^{-\mu|x|} \ . \tag{2.16}$$

Fourier transforming the square of (2.16), one obtains

$$T_s^2(k)=\frac{4\mu T_0^2}{4\mu^2+k^2} \ , \tag{2.17}$$

and the equal-time temperature correlation function can be written

$$\langle \delta T(x,t)\delta T(x',t)\rangle=\frac{k_B}{c}T_s^2(x)\delta(x-x')-\frac{\mu k_B T_0^2}{\pi c}\int_{-\infty}^{+\infty}dp\ p^2\frac{e^{ip(x+x')}e^{-(p^2+\mu^2)^{1/2}|x-x'|}}{(p^2+\mu^2)^{3/2}} \ . \tag{2.18}$$

The integral appearing in this last expression can be evaluated for small separation between x and x'.[12] Defining $y=x-x'$, one obtains up to second order in y/x

$$\begin{aligned}\langle \delta T(x,t)\delta T(x-y,t)\rangle &=\frac{k_B}{c}T_s^2(x)\delta(y)+\frac{2\mu k_B T_0^2}{\pi c}\Bigg[-K_0(2\mu|x|)(1+2\mu^2xy+\tfrac{1}{2}\mu^2y^2)\\ &\quad+2\mu|x|K_1(2\mu|x|)\left[1-\frac{1}{2}\frac{y}{x}+\frac{1}{2}\mu^2y^2\right]-\frac{\pi}{2}\mu|y|e^{-2\mu|x|}\left[1-\mu y\frac{|x|}{x}\right]\Bigg] \ ,\end{aligned} \tag{2.19}$$

where K_0 and K_1 are modified Bessel functions of zeroth and first order, respectively. Equation (2.19) shows the long-range behavior and no translational invariance of temperature fluctuations. The violation of translational invariance is an expected result coming from the presence of a point heat source. In general, this fact occurs for temperature correlation functions due to any source of noise.

III. INFLUENCE OF AN EXTERNAL SOURCE OF NOISE

As the external heat flow is supplied by a source that maintains the origin at temperature T_0, external noise must be related to this source. In this case, two origins of external noise can be considered, namely, a temperature fluctuating source and a fluctuating position of this source.

As shown by Horsthemke and Lefever[13] the effect of internal and external sources of noise are in general coupled. This is the case when both are represented by Ornstein-Uhlenbeck processes. However, these authors also show that in the white-noise limit, that is, the case we are going to analyze, this coupling disappears. As a consequence, temperature correlation functions coming from external noise do not contain the steady-state temperature profile. Moreover, both sources of external noise are assumed to be independent.

A. Effect of a fluctuating temperature of the source

First of all, we will compute the equal-time correlation function when the temperature of the source is no longer T_0 but may fluctuate around this value, such that

$$T_0(t)=T_0+T_0^R(t) \ . \tag{3.1}$$

The random temperature is described by means of a Gaussian white-noise process in the form

$$\begin{aligned}&\langle T_0^R(t)\rangle=0 \ ,\\ &\langle T_0^R(t)T_0^R(t')\rangle=2\gamma\delta(t-t') \ ,\end{aligned} \tag{3.2}$$

where γ is the intensity of the noise. Bearing in mind that the source temperature can be maintained by Joule heating of an electric device, a fluctuating temperature of the source could be easily produced by an electric noise generator.

The effect of a time-dependent source temperature is included in the local temperature by the generalization of the effect of a point source as we did in Eq. (2.7). Thus Eq. (2.4) becomes, in the case of an infinite rod,

$$\frac{\partial T}{\partial T}=\alpha\frac{\partial^2 T}{\partial x^2}-\beta T+2\mu\alpha T_0(t)\delta(x) \ , \tag{3.3}$$

in such a way that the random part of the source temperature leads to temperature fluctuations around the stationary state,

$$\delta T(x,t)=2\mu\alpha\int_0^\infty d\tau\, G(x,\tau)T_0^R(t-\tau) \ , \tag{3.4}$$

where $G(x,t)$ is the Green function which is nothing but the Fourier transform of $G(k,w)$ defined by (2.13),

$$G(x,t)=\frac{e^{-\beta t}e^{-x^2/4\alpha t}}{(4\pi\alpha t)^{1/2}} \ . \tag{3.5}$$

From (3.2) and (3.4), the equal-time temperature correlation function reads

$$\langle \delta T(x,t)\delta T(x',t)\rangle=\frac{4\alpha\gamma\mu^2}{\pi}K_0(\mu\{2[x^2+(x')^2]\}^{1/2}) \ , \tag{3.6}$$

where one can see that not only the intensity of the noise but also the inverse of the correlation length μ and the thermal diffusivity α determine the intensity of temperature fluctuations.

B. Effect of a fluctuating position of the source

As we have considered a point source of heat kept at a fixed temperature, we can also study the effect on correlation functions of temperature fluctuations due to a random position of the external source. As we did above, the effect of a point source in local temperature is performed by adding a new term in Eq. (2.4). In the present case we consider a source whose position fluctuates in such a way that Eq. (2.4) can be rewritten

$$\frac{\partial T}{\partial t}=\alpha\frac{\partial^2 T}{\partial x^2}-\beta T+2\mu\alpha T_0\delta(x-\epsilon(t)) , \tag{3.7}$$

where $\epsilon(t)$ is the random position of the source. Up to linear order in the fluctuating position one arrives at

$$\delta T(x,t)=2\mu\alpha T_0\frac{\partial}{\partial x}\int_0^\infty d\tau\, G(x,\tau)\epsilon(t-\tau) , \tag{3.8}$$

where $G(x,t)$ was defined in (3.5).

As in the previous case we assume that the random position of the source is described by a Gaussian white-noise process with zero mean, and variance

$$\langle \epsilon(t)\epsilon(t')\rangle=2\nu\delta(t-t') , \tag{3.9}$$

where ν is the intensity of the noise. Taking into account (3.9), the equal-time temperature correlation function coming from the fluctuating position of the source, (3.8), can be written

$$\langle \delta T(x,t)\delta T(x',t)\rangle = \frac{8\nu\alpha\mu^4 T_0^2}{\pi}\frac{xx'}{x^2+(x')^2}K_2(\mu\{2[x^2+(x')^2]\}^{1/2}) , \tag{3.10}$$

which shows the long-range behavior proper of fluctuations about nonequilibrium states.

IV. COMPARISON BETWEEN INTERNAL AND EXTERNAL NOISE

Our aim in the present section is to confront the results obtained in the previous sections. First of all, it is important to note that the parameter μ, the inverse of a characteristic length of the system, which accounts for the relative importance of both effects taking place, plays a crucial role in the behavior of any kind of fluctuation. It is not only this parameter but also the thermal diffusivity α which appear in the temperature correlation function of both external sources of noise. It implies that also α and β will have to be considered.

As the asymptotic behavior of Bessel functions is well known[14] there is no problem in comparing temperature correlation functions in different limiting cases. Previously, to compare analytically the relative importance of the different kinds of noise it was interesting to study their behavior when one of the processes taking place (diffusion and heat interchange) dominated over the other one.

Thus, when $\mu\to 0$ the rod is at thermal equilibrium with the source and diffusion dominates over radiation. In this case, internal fluctuations are δ-correlated and fluctuations coming from a fluctuating position vanish since the rod is unaffected by the position of the source. However, fluctuations due to a fluctuating intensity depend on the form the limit $\mu\to 0$ reaches. If there is no radiation to the heat bath, the temperature correlation function behaves as $\delta(x)\delta(x')$ since they have a located origin. In contrast, if thermal diffusivity is very large, then fluctuations are immediately propagated along the rod and make the correlation function diverge.

On the other hand, if $\mu\to\infty$, the rod is at thermal equilibrium with the heat bath and, since this is the contribution we have eliminated, internal fluctuations vanish. Although external fluctuations vanish as well, they do it because the effect of a fluctuating point source is rapidly lost.

In order to simplify the comparison, we are going to consider that both points whose correlation we study are very close but are not the same, to avoid divergencies in the internal fluctuations [cf. Eq. (2.19)]. Moreover, the respective intensities of the external noises are not considered since they are particular properties of a kind of noise and not of the system itself. This is also what has been plotted in Figs. 1—3, i.e., correlation functions in an arbitrary scale on which we must only take into account the relative speed of either growing or decaying.

If both points are very close, the limits which have to be investigated will be the case in which they are near the origin compared with the characteristic length of the system μ^{-1} and the opposite case where their distance to the origin is large enough confronted with μ^{-1}. But, as stated

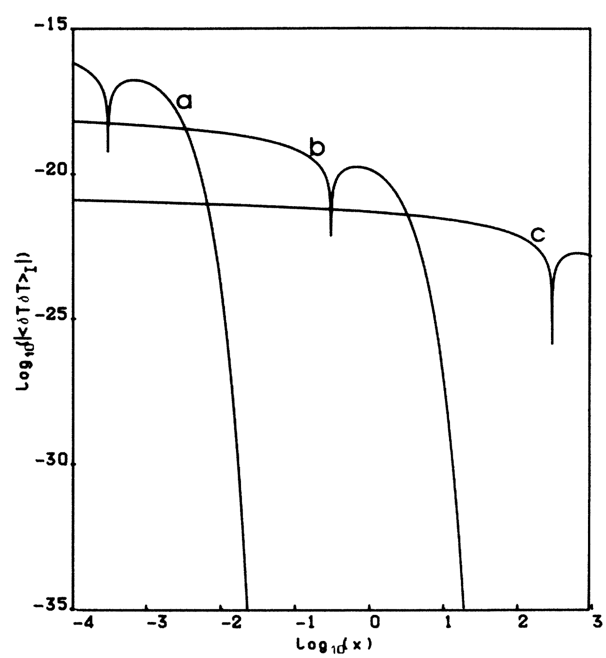

FIG. 1. Absolute value of correlation function of internal temperature fluctuations, with $x\sim x'$ but not strictly equal, vs the logarithm of the position (both in arbitrary units). The peaks are due to the fact that at long distance the correlation function is positive whereas at short distance it is negative, and in the intermediate region, temperature correlation function is small. The different curves correspond to correlation lengths: a, $\mu=10^3$; b, $\mu=1$; and c, $\mu=10^{-3}$.

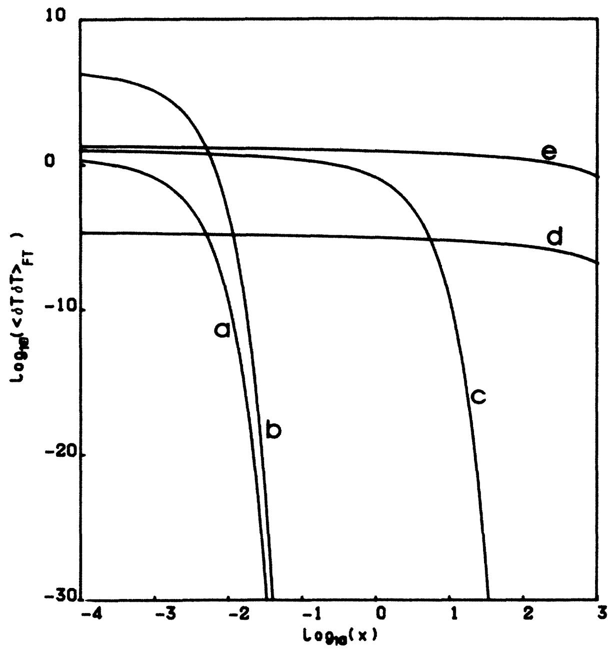

FIG. 2. Equal-position temperature correlation function due to a fluctuating temperature of the source with the following values of the parameters: a, $\mu=10^3$, $\alpha=10^{-6}$; b, $\mu=10^3$, $\alpha=1$; c, $\mu=1$, $\alpha=1$; d, $\mu=10^{-3}$, $\alpha=1$; and e, $\mu=10^{-3}$, $\alpha=10^6$.

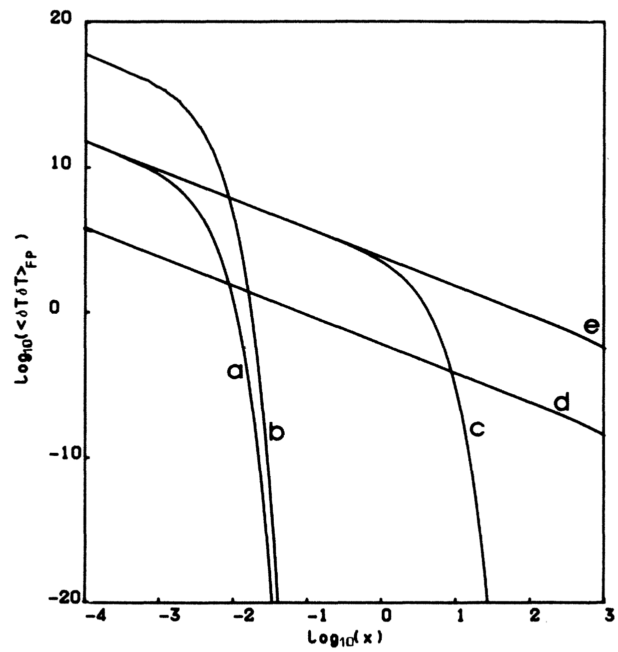

FIG. 3. Same as in Fig. 2 for the equal-position temperature correlation function due to a fluctuating position of the source.

above, the asymptotic behavior will also depend on both the thermal diffusivity α and the transfer coefficient β.

The internal noise is found to be the largest one in two cases. The first case is when the decay length μ^{-1} tends to infinity due to a zero transfer coefficient, provided that the distance to the origin is not too short. This is due to the fact that the strength of external fluctuations is appreciably reduced, as we note in curves c in Figs. 2 and 3. The other event in which the internal noise dominates is, obviously, when the distance to the origin is large, since internal fluctuations do not have a located source as external fluctuations do (compare the way in which the curves in Figs. 1—3 decay at long distances).

About the fluctuating intensity of the source, it only dominates when the thermal diffusivity is very large, so that any temperature fluctuation propagates quickly along the rod.

Finally, the fluctuating position of the source is the dominant source of noise at short distances (observe in Fig. 3 the growing for small x) because these positions are easily reached by the fluctuating source. It also dominates when the decay length μ^{-1} is small whenever the distance is not too large. At this state one must distinguish between the case where any fluctuation escapes immediately to the bath ($\beta \rightarrow \infty$) and the case where there is no diffusion along the rod ($\alpha \rightarrow 0$), although in both cases the dominant noise is the fluctuating position, the internal noise begins to be comparable at a shorter distance in the $\alpha \rightarrow 0$ case than in the $\beta \rightarrow \infty$ case. This is due to the fact that with immediate radiation, fluctuations decay rapidly, but if there is no diffusion, this decay is even faster.

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