Intensity correlation function of dye lasers: Short-time behavior

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We propose an equation to calculate the intensity correlation function of a dye-laser model with a pump parameter subject to finite-bandwidth fluctuations. The equation is valid, in the weak-noise limit, for all times. It incorporates novel non-Markovian features. Results are given for the short-time behavior of the correlation function. It exhibits a characteristic initial plateau. Our findings are supported by a numerical simulation of the model.

Experimental results of Kaminishi *et al.*[1] show that the statistical properties of light from a dye laser close to its instability point cannot be explained in terms of the usual single-mode laser on resonance. It has been suggested that such anomalous fluctuation phenomenon is due to the existence of stochastic fluctuations of the pump parameter.[1-7] From this point of view dye lasers constitute an interesting example of the effects of external noise.[8] A model with Gaussian white-noise fluctuations of the pump parameter was used to describe the statistical properties very close to threshold.[2] Later experiments[3] have shown that this model needs some modification when moving away from the immediate vicinity of the instability point. Short *et al.*[3] attributed the discrepancy to the white-noise assumption for the pump parameter fluctuations. Following this suggestion, Dixit and Sahni[4] have introduced a model in which the correlation time of the pump parameter fluctuations is taken into account replacing the Gaussian white noise by an Ornstein-Uhlenbeck noise. A numerical simulation of this model[4] seems to give a fair agreement with the experimental results. A more microscopic approach to the problem of a single-mode laser pumped by a stochastic field has been presented recently by Fox *et al.*[5] Here we are concerned with the phenomenological model introduced by Dixit and Sahni.[4] The intensity stationary moments of this model have been calculated analytically[6,7] from a Fokker-Planck equation for the probability density derived under a weak-noise assumption.[9] The intensity stationary distribution has been calculated for an arbitrary strength of the noise by a matrix continued-fraction technique.[10] An analytical calculation of the long-time behavior of the intensity steady-state correlation function has also been reported.[7] The purpose of this paper is to present a calculation of the short-time behavior of such a correlation function. It is precisely in the initial time decay where a white-noise model fails to give acceptable results.[4,6,7] Our calculation is based in the equation satisfied by the joint probability density at different times of a non-Markovian process.[11] This equation is valid for all times. The dynamical non-Markovian features that it incorporates become important at initial times.[11-13] The equation for the correlation function is solved for early times by a decoupling ansatz. The correlation function shows a peculiar slow initial decay characteristic of a non-white-noise model. We have carried out a numerical simulation of the model which substantiates the existence of an initial plateau in the correlation function. Our results are shown to be more consistent with the experimental ones than those obtained under white-noise assumptions. However, the validity of the model remains an open question because the available experimental data[3] do not seem to exhibit the initial plateau implied by the model. A more careful experimental analysis would be necessary to check the detailed features of this model.

The deterministic model which describes a single-mode laser operating on resonance is[14]

$$\frac{d}{d\bar{t}}E=\bar{\alpha}E-\beta\,|E\,|^2E\ , \tag{1}$$

where $E(\bar{t})$ is the complex field amplitude, and $\bar{\alpha}$ is the pump parameter. We assume that $\bar{\alpha}$ takes random values around a real mean value $\bar{\alpha}_0$: $\bar{\alpha}=\bar{\alpha}_0+\eta(\bar{t})$. The complex fluctuations $\eta(\bar{t})=\eta_1(\bar{t})+i\eta_2(\bar{t})$ are modeled by an Ornstein-Uhlenbeck process[4,6,7,9,10] with zero mean value and correlation function

$$\langle\eta^*(\bar{t})\eta(\bar{t}\,')\rangle=(D/\bar{\tau})\exp(-\,|\,\bar{t}-\bar{t}\,'\,|\,/\bar{\tau})\ . \tag{2}$$

D is the noise intensity and $\bar{\tau}$ its correlation time. The white-noise limit is obtained taking the limit $\bar{\tau}\to 0$ with D fixed. The variable of interest in the problem is the intensity of the field $|E\,|^2$. Defining new variables by $I=(\beta/D)\,|E\,|^2$, $t=2D\bar{t}$, $\alpha=\bar{\alpha}_0/D$, $\xi=(1/D)\eta_1$, $\tau=2D\bar{\tau}$, we obtain the following stochastic differential equation for the intensity of the field I:

$$\frac{d}{dt}I=\alpha I-I^2+I\xi\ , \tag{3}$$

where $\xi(t)$ is a real Ornstein-Uhlenbeck process with correlation time τ and unit noise intensity. The two dimensionless parameters left in the theory are α and τ.

We wish to calculate the intensity stationary correlation function defined by

$$\langle I(s)I(0)\rangle_{st}=\lim_{t\to\infty}\langle I(t+s)I(t)\rangle \;. \tag{4}$$

Owing to the non-Markovian character of the process $I(t)$ defined by (3), the correlation function $\langle I(s)I(0)\rangle_{st}$ cannot be calculated from the long-time limit of the equation satisfied by the probability density of the process $P(I,t)$: In order to calculate non-Markovian correlations at different times it is also necessary to know the joint probability distribution $P(I,t+s;I',t)$.[11,12] The equations satisfied by $P(I,t)$ and $P(I,t+s;I',t)$ in the steady state, reached as $t\to\infty$, have been derived in a general case in Refs. 9 and 11, respectively, under the weak-noise assumption ($D \ll \bar{\alpha}_0$) and for arbitrary values of τ. In the case of the process defined by (3) we have, neglecting transients,

$$\frac{\partial}{\partial t}P(I,t)=L(\tau)P(I,t) \;, \tag{5}$$

$$\frac{\partial}{\partial s}P_{st}(I,t+s;I',t) = \left[L(\tau)+\exp(-s/\tau)\frac{\partial}{\partial I}I\frac{\partial}{\partial I'}(I'-\tau_R I'^2)\right] \times P_{st}(I,t+s;I',t) \;, \tag{6}$$

where the Fokker-Planck operator $L(\tau)$ is defined by

$$L(\tau)=-\frac{\partial}{\partial I}(\alpha I-I^2)+\frac{\partial}{\partial I}I\frac{\partial}{\partial I}(I-\tau_R I^2) \tag{7}$$

and

$$\tau_R=\frac{\tau}{1+\tau\alpha} \;. \tag{8}$$

Equation (6) has to be solved with the initial conditions at $s=0$,

$$P_{st}(I,t;I',t)=\delta(I-I')P_{st}(I) \;, \tag{9}$$

where $P_{st}(I)$ is the stationary solution of (5).

The time-independent properties of the steady state like the moments $\langle I^n\rangle_{st}$ are modified with respect to their values in the white-noise limit due to the τ dependence of $L(\tau)$ in (5).

The basic feature of the non-Markovian dynamics is the fact that the time evolution of $P_{st}(I,t+s,I',t)$ is not completely determined by the operator $L(\tau)$: There exists a second term in the right-hand side (rhs) of (6) which is particularly important for small values of s. The τ dependence of the Fokker-Planck operator $L(\tau)$ and the existence of a second term in the rhs of (6) are also the two sources of modifications of the stationary correlation function with respect to the white-noise limit. From Eq. (6) the equation satisfied by $\langle I(s)I(0)\rangle_{st}$ is immediately obtained:

$$\frac{d}{ds}\langle I(s)I(0)\rangle_{st}=(\alpha+1)\langle I(s)I(0)\rangle_{st}-(1+\tau_R)\langle I^2(s)I(0)\rangle_{st}+\exp(-s/\tau)[\langle I(s)I(0)\rangle_{st}-\tau_R\langle I(s)I^2(0)\rangle_{st}] \;. \tag{10}$$

The last term in (10) comes from the last term in (6). Equation (10) has to be solved with initial conditions at $s=0$ given by $P_{st}(I)$.

Before facing the problem of solving (10) we wish to comment on other attempts of calculating $\langle I(s)I(0)\rangle_{st}$. We first note that the last term in (10) is negligible for $s\to\infty$. Nevertheless, the long-time behavior of $\langle I(s)I(0)\rangle_{st}$ depends on the existence of this term through the short-time behavior of $\langle I(s)I(0)\rangle_{st}$. This is a clear non-Markovian effect. It is physically obvious that such an effect can be taken into account in the calculation of the long-time behavior of $\langle I(s)I(0)\rangle_{st}$ neglecting the term proportional to $e^{-s/\tau}$ in (10) and solving it with an effective initial condition at a time $s=s_0\gg\tau$. The effective initial condition replaces the effect of the short-time processes in the long-time behavior. This attitude has been adopted by Haake and Lewenstein[15] in the context of the adiabatic elimination of variables. In this context usually one is only interested in a description in a coarse-grained time scale. These authors have coined the term "initial slip" to describe the change in initial conditions. From a similar point of view Schenzle and Graham[7] have calculated the long-time behavior of $\langle I(s)I(0)\rangle_{st}$. In fact, their equation for the conditional probability density is our Eq. (5) supplemented with effective initial conditions.

Equation (10) is an adequate starting point to study the evolution of $\langle I(s)I(0)\rangle_{st}$ for all times. The time domain $s\ll s_0$ is also of interest in an external noise problem with moderate values of τ. It is not trivial to obtain a solution of (10) due to the nonlinearities of the problem.[20] A solution which takes fully into account the nonlinearities exists in the white-noise limit.[2] The nonlinearities are also crucial in the understanding of the long-time behavior in the non-Markovian case.[7] Here we argue that the initial decay can be described using a linearizing decoupling ansatz due to Stratonovich.[16] Assuming only weak fluctuations around the macroscopic stationary state, the simplest linearization scheme consists in approximating the solution of Eq. (3) by $I=\alpha+I_1$, where I_1 satisfies the linear equation

$$\frac{\partial}{\partial t}I_1=-\alpha I_1+\alpha\xi(t) \;.$$

This approximation is obtained in the lowest order of a systematic perturbation expansion in powers of α^{-1} around $I=\alpha$. Linear processes driven by an Ornstein-Uhlenbeck noise $\xi(t)$ can be solved exactly.[12] In this linear approximation the normalized correlation function $\lambda(s)$ defined by

$$\lambda(s)=\frac{\langle I(s)I(0)\rangle_{st}-\langle I\rangle_{st}^2}{\langle I\rangle_{st}^2} \tag{11}$$

becomes

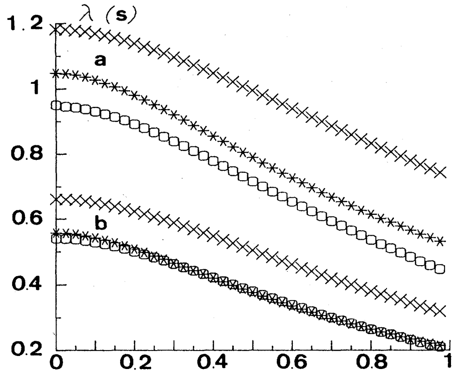

FIG. 1. Intensity correlation function $\lambda(s)$. (a) $\alpha=0.7$, (b) $\alpha=1.13$. □, integral of (14); $*$, numerical simulations; $\times$, linear approximation Eq. (12).

$$\lambda(s)=\frac{1}{(1+\alpha\tau)(1-\alpha\tau)}[\alpha^{-1}\exp(-\alpha s)-\tau\exp(-s/\tau)] . \tag{12}$$

Stratonovich's ansatz[16] leads to a linearization of Eq. (10) that goes beyond the simple linear model that we have just discussed. The basic idea is to express the nonlinear correlations $\langle I^n(s)I^m(0)\rangle_{st}$ by linear functions of $\langle I(s)I(0)\rangle_{st}$. The decoupling ansatz is written as

$$\frac{\langle I^n(s)I^m(0)\rangle_{st}-\langle I^n\rangle_{st}\langle I^m\rangle_{st}}{\langle I^{n+m}\rangle_{st}-\langle I^n\rangle_{st}\langle I^m\rangle_{st}}=\frac{\langle I(s)I(0)\rangle_{st}-\langle I\rangle_{st}^2}{\langle I^2\rangle_{st}-\langle I\rangle_{st}^2} . \tag{13}$$

With this approximation, (10) becomes a linear equation for $\langle I(s)I(0)\rangle_{st}$:

$$\frac{d}{ds}\langle I(s)I(0)\rangle_{st}=\left[-\frac{(\alpha+1)}{(1+2\tau_R)}+\frac{(1-2\tau_R\alpha)}{(1+2\tau_R)}\exp(-s/\tau)\right]\langle I(s)I(0)\rangle_{st}+\frac{\langle I^2\rangle_{st}\langle I\rangle_{st}^2}{(1+2\tau_R)}\{1+\tau_R[1+\exp(-s/\tau)]\} . \tag{14}$$

The ansatz (13) is an identity for $s=0$. The ansatz corresponds to linearize the dynamics but making no approximation on the values of the time-independent moments $\langle I^n\rangle_{st}$. It has been shown in general[13] that this decoupling ansatz corresponds, in the white-noise limit, to the zeroth-order approximation of a continued-fraction expansion method[17] in which the exact values of the stationary moments are used. This method has been successfully used to calculate the linewidth of the standard model of a single-mode laser.[17] Such an approximation is known to give good results for the initial decay and sufficiently large values of α (weak noise). In a simple perturbation scheme in powers of α^{-1}, statics and dynamics are approximated simultaneously in a given order in α^{-1}. Using

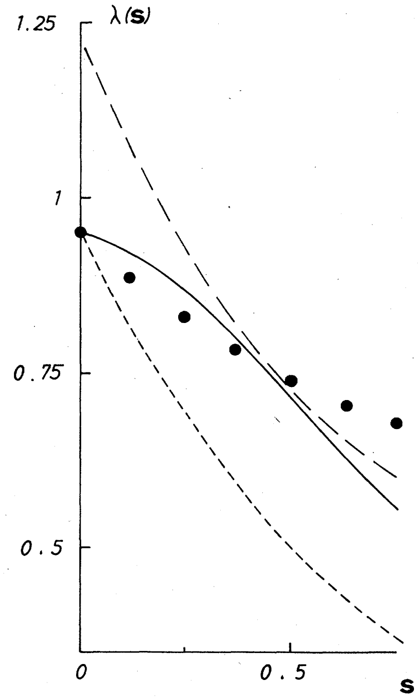

FIG. 2. Intensity correlation function for $\lambda(0)=0.95$. Solid line, integral of (14) with $\alpha=0.7$; circles, experimental data from Ref. 3; -----, white-noise theory with $\alpha=\lambda(0)^{-1}=1.05$; — — —, white-noise theory with $\alpha=0.82$.

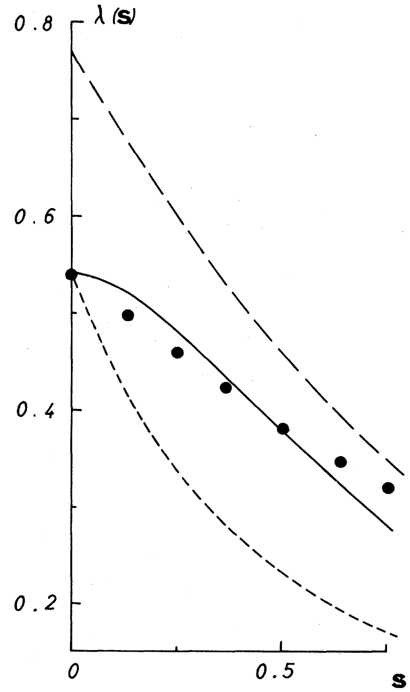

FIG. 3. Intensity correlation function for $\lambda(0)=0.54$. Solid line, integral of (14) with $\alpha=1.13$; circles, experimental data from Ref. 3; -----, white-noise theory with $\alpha=\lambda(0)^{-1}=1.84$; — — —, white-noise theory with $\alpha=1.3$.

the ansatz (13), nonlinearities are in principle fully taken into account in the static quantities that enter in (14). As a consequence, the solution of (14) is expected to be reliable for moderate values of α for which a simple linear model cannot be used. The values of $\langle I\rangle_{st}$ and $\langle I^2\rangle_{st}$ which appear in (11), (14), and the initial condition of (14) are calculated from the stationary solution of (5). We have[6,7]

$$\langle I^n\rangle_{st}=\frac{\Gamma(n+\alpha)\Gamma(\tau_R^{-1})}{\Gamma(\alpha)\tau_R^n\Gamma(n+\tau_R^{-1})} . \tag{15}$$

In particular

$$\frac{\langle I^2\rangle_{st}-\langle I\rangle_{st}^2}{\langle I\rangle_{st}^2}=\lambda(0)=\alpha^{-1}[1+\tau(1+\alpha)]^{-1} \tag{16}$$

so that

$$\alpha=[\lambda(0)]^{-1}\left[\frac{2}{1+\tau}\right] \times\left[1+\left[1+\frac{4\tau}{(1+\tau)^2}[\lambda(0)]^{-1}\right]^{1/2}\right]^{-1} . \tag{17}$$

This equation relates the two parameters of the theory with the experimental value $\lambda(0)$.

In Fig. 1 we compare the normalized correlation function $\lambda(s)$ obtained from (14) with the result (12) of the linear model and also with a numerical simulation of Eq. (3).[18] We compare the three results for the same values of the two parameters of the model, α and τ. We take $\tau=0.3$ (Ref. 7) and α given by (17) with experimental values of $\lambda(0)$ from Short *et al.*[3,19] As expected, the linear model and our approximation become better for larger α and our approximation is still reasonable for moderate α. The main discrepancies between the simulation and our approximation can be traced back to the different values of $\lambda(0)$ for the same α and τ. This is not a deficiency of the ansatz (13). It rather measures the accuracy of the weak-noise approximate stationary solution (15), used to determine $\lambda(0)$ through (16). Correcting for the shift in the initial value $\lambda(0)$, our approximation gives good results during a few correlation times of the noise. The most remarkable feature of $\lambda(s)$ is the existence of a slow initial decay in which $\lambda(s)$ shows a peculiar plateau.

Once we have discussed the validity of our calculation of $\lambda(s)$, we compare our results with experimental data in Figs. 2 and 3. Our results are given in dimensionless time and the comparison is made choosing $D=0.2$.[2-4] Our results are consistent with the experimental data in the initial decay. However, the data extracted from Ref. 3 do not seem to exhibit an initial plateau. In the same time interval of a few correlation times, there exist important discrepancies of the experimental data with results given by white-noise models: in Figs. 2 and 3 we also include the results of two white-noise models with the same $D=0.2$. In the first model $\alpha=\lambda^{-1}(0)$ as given by a consistent white-noise theory.[2,3] In the second model α is chosen to fit the long-time behavior.[4,7] Our results for the Ornstein-Uhlenbeck noise model start at the correct value of $\lambda(0)$ and interpolate between the two white-noise models.

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[18]The numerical simulation has been done following the algorithm explained in Ref. 9. The correlation function is obtained averaging over 1000 trajectories and over 15 000 integration steps in each trajectory. To assure that the stationary state has been reached we eliminate the first 2500 integration steps. We used an integration step $\Delta=0.005$.
[19]We do not examine the value $\lambda(0)=1.92$ of Ref. 3. For this value of $\lambda(0)$, α is quite small ($\alpha=0.37$), and due to the critical slowing down, the white-noise approximation is reasonably good.
[20]See J. M. Sancho and M. San Miguel, in *Fluctuations and Sensitivity in Nonequilibrium Systems,* edited by W. Horsthemke and D. K. Kondepudi (Springer, Berlin, 1984).