Bistability driven by dichotomous noise

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We consider mean-first-passage times and transition rates in bistable systems driven by dichotomous colored noise. We carry out an asymptotic expansion for short correlation times τ_c of the colored noise and find results that differ from those reported earlier. In particular, to retain corrections to $O(\tau_c)$ we find that it is necessary to retain up to four derivatives of the potential function. We compare our asymptotic results to existing ones and also to exact ones obtained from numerical integration.

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I. INTRODUCTION

In recent years a great deal of attention has been devoted to the study of mean-first-passage times (MFPT's) and transition rates in bistable systems driven by colored noise [1–11]. A number of different techniques, at least in part designed according to the nature of the noise, have been applied to this problem. These include methods that are particularly suited to handle Gaussian noise with short correlation times (slightly colored noise) [1,6,9] and others that are especially useful for intermediate or long correlation times (highly colored noise) [1,7,8]. The comparative success of these techniques has been studied via extensive numerical simulations. Methods to deal with noise that is not Gaussian have also been developed [1–4]. Among these is the "stochastic trajectory analysis technique" (STAT), which is particularly useful when the noise can only take on a small number of values, e.g., dichotomous noise [12] and shot noise [13].

Herein we use the STAT to study the mean-first-passage time and the related transition rate problem for bistable systems driven by colored dichotomous noise. Such systems evolve according to the dynamical equation

$$\dot{X}(t) = -V'(X) + g(X)F(t) , \tag{1}$$

where $V(x)$ is a bistable "potential" function (see Fig. 1). In previous work we have obtained formal results for mean-first-passage times for this problem [12], but the explicit representation of the formal results was only considered for linear systems for which $V(x)$ is a harmonic (quadratic) potential and $g(x)=1$. In the bistable case, $-V'(x)$ has three real roots. Two of these, x_1 and x_2, are minima and therefore correspond to stable points of the potential [i.e., stable points of the system in the absence of the noise $F(t)$]. The third root, x_u, is a maximum and therefore corresponds to an unstable point of the potential. The noise $F(t)$ is a dichotomous Markov process that can take on the values $\pm a\,(a>0)$. The length of time that $F(t)$ retains either value is assumed to be governed by an exponential distribution,

$$\Psi(t) = \lambda e^{-\lambda t} , \tag{2}$$

so that λ^{-1} is the mean time between switches from one value of $F(t)$ to the other. The function $g(x)$ embodies the coupling between the deterministic system and the noise and reflects the fact that this coupling may in general depend on the state of the system. For later convenience we rewrite Eq. (1) in the form

$$\dot{X}(t) = f(X) + g(X)F(t) , \tag{3}$$

where the drift term is given by $f(x) = -V'(x)$. In general, we restrict the functions $f(x)$ and $g(x)$ to be smooth and such that the solution $X(t)$ of Eq. (3) never becomes infinite in a finite time. We further restrict $g(x)$ to be positive.

In the absence of noise ($a=0$) the system evolves according to the dynamical equation

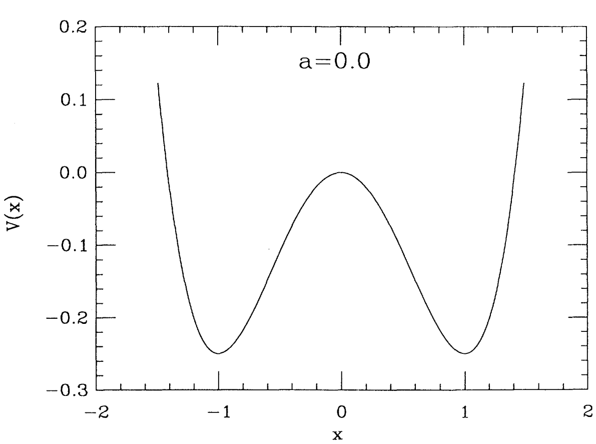

FIG. 1. The symmetric bistable potential function of Eq. (10) with $c=d=1$. Note that the theory does not require the function to be symmetric.

$$\dot{X}(t)=f(X)\ , \tag{4}$$

i.e., it evolves monotonically towards one of the minima x_1 or x_2 of the potential $V(x)$, the choice being determined by the initial condition $X(0)$. Once a minimum is reached, no transitions between minima can occur since the minima of the potential are the asymptotically fixed points of the system.

In the presence of dichotomous noise, the random process $X(t)$ alternately evolves according to one of the two deterministic dynamical equations

$$\dot{X}(t)=f(X)+ag(X)\ , \tag{5a}$$

$$\dot{X}(t)=f(X)-ag(X)\ . \tag{5b}$$

The stochastic element enters through the random lengths of time that each of these two dynamical equations alternately describes the evolution. The two effective "force" functions governing the evolution are now

$$\frac{d}{dx}V_{\pm}^{\text{eff}}(x)=V'(x)\mp ag(x)\ . \tag{6}$$

In general, one might encounter one of two situations which must be distinguished before we can proceed. If the value a of the noise is too small, then the effective potential functions still exhibit two minima and one maximum, albeit shifted from those of the original potential $V(x)$. [We assume that $g(x)$ does not introduce new minima, i.e., we assume that $V_{+}^{\text{eff}}(x)$ and $V_{-}^{\text{eff}}(x)$ have no more minima than does $V(x)$. The discussion can be generalized to situations where this is not the case.] An example is shown in Fig. 2(a). As the noise changes between its two possible values, the bistable potential alternates between V_{+}^{eff} and V_{-}^{eff} and hence the precise locations and depths of the minima also alternate. Nevertheless, once the process is in one of the wells, i.e., near one of the minima, it will always stay in that well. The entire dynamics occurs within the well and simply involves a readjustment of the process to the local changes of the well. In particular, *transitions* between minima are *not* possible since the dichotomous noise in this case is *not sufficiently strong* to cause transitions. This situation is therefore uninteresting from our point of view. The interesting situation arises when the noise is sufficiently strong to cause one of the two wells to disappear, as indicated in Fig. 2(b). Now the noise can clearly cause transitions to occur: the well in which the process finds itself suddenly disappears as the noise changes to its other value, causing the process to "roll down" the potential ramp towards the other well. It is this situation that we consider further in this paper.

Let $X_{+}(t)$ and $X_{-}(t)$ denote the solutions of Eqs. (5a) and (5b), respectively. Under our assumptions about the behavior of the functions $f(x)$ and $g(x)$, it then follows from the comparison theorem [14] that for all time

$$X_{+}(t)\geq X_{-}(t)\ . \tag{7}$$

We define $x_s^{\pm}$ to be the asymptotically fixed points of the deterministic equations (5a) and (5b) [e.g., the minima of Fig. 2(a)], so that

$$f(x_s^{\pm})\pm ag(x_s^{\pm})=0 \tag{8}$$

and

$$\lim_{t\to\infty} X_{\pm}(t)=x_s^{\pm}\ . \tag{9}$$

Note that x_s^{-} (x_s^{+}) lies below x_1 (above x_2). Also note that the process $X(t)$ always lies within the interval $[x_s^{-},x_s^{+}]$, provided that $X(0)$ does.

In general, the transitions discussed above can occur if $f(x)+ag(x)>0$ and $f(x)-ag(x)<0$ for all $x\in[x_s^{-},x_s^{+}]$ [15]. A case of particular interest is that of additive noise [$g(x)\equiv 1$] and a symmetric potential of the form

$$V(x)=-\frac{c}{2}x^2+\frac{d}{4}x^4\ . \tag{10}$$

Transitions from one well to the other of this bistable potential *will* occur if

$$a>a_c=2\frac{c^{3/2}}{3\sqrt{3d}}\ . \tag{11}$$

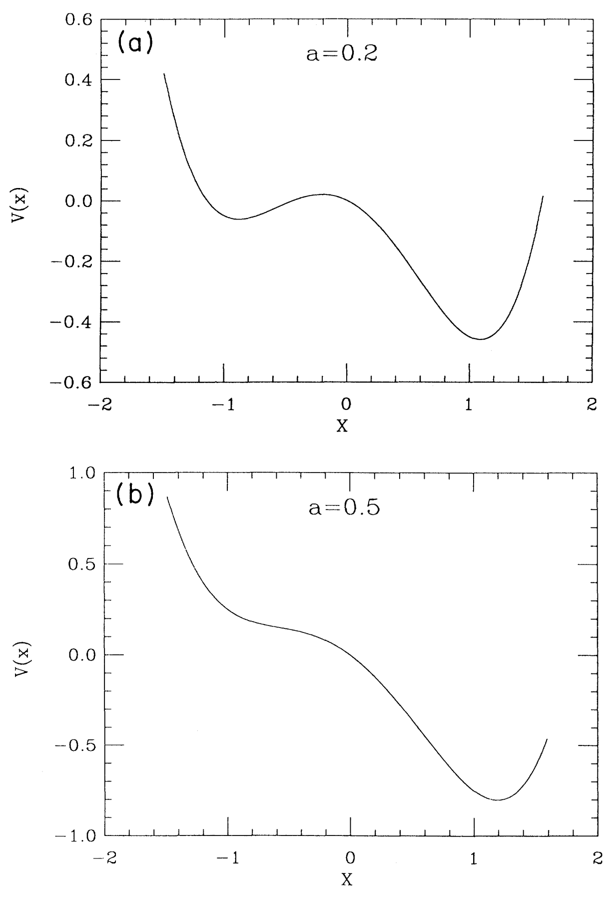

FIG. 2. The effective potential $V_{-}^{\text{eff}}=V(x)-a$ with $V(x)$ given by Eq. (10) with $c=d=1$. (a) $a=0.2$; (b) $a=0.5$.

With these preliminaries, we can now proceed to the calculation of the transition rates between the metastable states and other related quantities. The paper is organized as follows. In Sec. II we apply STAT to the system (1) to evaluate the transition rates from one stable state of the potential to the other. In Sec. III we present higher-order corrections to the asymptotic results given in Sec. II. Here we also check our results against known limits and compare them with those of other approaches. Our conclusions are presented in Sec. IV.

II. MEAN-FIRST-PASSAGE TIMES AND TRANSITION RATES

In our previous work we used STAT to obtain exact analytic expressions for the mean-first-passage time (MFPT) to either one of two critical levels, say z_1 and z_2, bounding a region, i.e., we calculated the mean time to first exit a region bounded by these values [1,4,12]. In bistability problems, on the other hand, one is interested in the MFPT to a single critical level, such as, for example, a metastable or an unstable state. The expressions that we have previously derived can be used for this latter case by careful limiting procedures that take into account that we are now interested in the MFPT to a single level. However, instead of using our previous expression, we repeat the derivation for the single-level case in Appendix A and arrive at the result

$$T_z(x_0)=2\lambda\int_{x_0}^{z}dx\frac{e^{\lambda\Phi(x)}}{ag(x)+f(x)}\int_{x_s^-}^{x}dx'\frac{e^{-\lambda\Phi(x')}}{ag(x')-f(x')}+(1-\beta)\left[\frac{1}{\lambda}+2\int_{x_s^-}^{x_0}dx'\frac{e^{-\lambda[\Phi(x')-\Phi(x_0)]}}{ag(x')+f(x')}\right], \tag{12}$$

where β is the probability that $F(0)=a$, $X(0)\equiv x_0$ is the initial value of the process, and the "effective potential function" Φ is given by the indefinite integral

$$\Phi(x)=-2\int^{x}dx'\frac{f(x')}{a^2g^2(x')-f^2(x')} . \tag{13}$$

To proceed further, let us assume that the process initially lies in the neighborhood of one of the minima of the potential, i.e., $x_0\sim x_1$ of $x_0\sim x_2$. As a final state we consider two cases that have been extensively discussed in the literature. In one, $x\sim x_u$, i.e., the final state is the maximum of the potential $V(x)$ [9]. This quantity has been denoted by $T_{\text{top}}(x_0)$ to indicate passage to the "top of the barrier," although it should be noted that in the presence of the fluctuations x_u is no longer the maximum of any instantaneous "potential" experienced by the process (this fact has elicited some discussion in the literature concerning the merits of calculating T_{top} at all when the noise driving the system is not white [16]). From Eq. (12) we have

$$T_{\text{top}}(x_0)=2\lambda\int_{x_0}^{x_u}dx\frac{e^{\lambda\Phi(x)}}{ag(x)+f(x)}\int_{x_s^-}^{x}dx'\frac{e^{-\lambda\Phi(x')}}{ag(x')-f(x')}+(1-\beta)\left[\frac{1}{\lambda}+2\int_{x_s^-}^{x_0}dx'\frac{e^{-\lambda[\Phi(x')-\Phi(x_0)]}}{ag(x')+f(x')}\right] . \tag{14}$$

In the other case $z=x_2$ and $x_0\sim x_1$ (or vice versa), i.e., we are concerned with passage from the vicinity of one minimum of the potential $V(x)$ to the other. This quantity has been denoted $T_{\text{bot}}(x_0)$ and is given by

$$T_{\text{bot}}(x_0)=2\lambda\int_{x_0}^{x_2}dx\frac{e^{\lambda\Phi(x)}}{ag(x)+f(x)}\int_{x_s^-}^{x}dx'\frac{e^{-\lambda\Phi(x')}}{ag(x')-f(x')}+(1-\beta)\left[\frac{1}{\lambda}+2\int_{x_s^-}^{x_0}dx'\frac{e^{-\lambda[\Phi(x')-\Phi(x_0)]}}{ag(x')+f(x')}\right] . \tag{15}$$

The expressions given above can easily be calculated numerically for any value of the parameters λ and a. In fact, if $f(x)$ is odd the double integral in Eq. (15) [but not the one in Eq. (14)] can be written in terms of single integrals because of symmetries when one takes $x_0=x_1$ [17] (see Appendix B). In any case, our interest here lies in obtaining analytic expressions for the mean-first-passage times and in comparing them with existing results. For this purpose we find it necessary to implement the *weak-noise approximation,* according to which [18]

$$D\equiv a^2\tau_c=\frac{a^2}{2\lambda}\ll 1 , \tag{16}$$

where $\tau_c\equiv(2\lambda)^{-1}$ is the correlation time of the noise $F(t)$ [15]. In this limit the major contributions to both T_{top} and T_{bot} come from the vicinity of $x=x_u$. The terms proportional to $(1-\beta)$ in both expressions are then negligible (and in any case vanish if $\beta=1$) and the remaining integrals can be evaluated using Laplace's method. To leading order one obtains well-known results obtained earlier by other authors [2,3]:

$$T_{\text{top}}=\frac{\pi}{(\alpha_1|\alpha_u|)^{1/2}}[1+O(\tau_c^{1/2})]e^{(a^2/2D)\Delta} \tag{17}$$

and

$$T_{\text{bot}}=\frac{2\pi}{(\alpha_1|\alpha_u|)^{1/2}}[1+O(\tau_c)]e^{(a^2/2D)\Delta} , \tag{18}$$

where

$$\Delta\equiv\Phi(x_u)-\Phi(x_1) \tag{19}$$

is the effective potential barrier and

$$\alpha_{1,u}\equiv-f'(x_{1,u}) . \tag{20}$$

As observed before, the first τ_c correction in T_{top} is of order $\tau_c^{1/2}$, whereas that of T_{bot} is of order τ_c. Moreover, up to order $\tau_c^{1/2}$ we have

$$T_{\rm bot} \sim 2T_{\rm top} \ . \tag{21}$$

We will see in the next section that this simple relation is no longer valid when one includes higher-order corrections.

With the assumptions outlined so far, the relationship between the MFPT and the transition rate from one well to the other of the bistable potential is [11,19,20]

$$r \sim \frac{1}{T_{\rm bot}} \ . \tag{22}$$

Therefore to the order of approximation considered so far and in agreement with previous results [2,3], the transition rate is given by

$$r \sim \frac{(\alpha_1|\alpha_u|)^{1/2}}{2\pi} e^{-(a^2/2D)\Delta} \ . \tag{23}$$

III. MEAN-FIRST-PASSAGE TIMES AND TRANSITION RATES: HIGHER-ORDER CORRECTIONS

Our main purpose in this paper is to consider the corrections to Eqs. (17), (18), and (23) that result from a finite correlation time τ_c in the weak-noise approximation (16). To simplify the presentation, we restrict our results to the case of additive noise [$g(x)=1$], although the general expressions obtained in Appendix B do not implement this restriction. We consider the times $T_{\rm bot}$ and $T_{\rm top}$ separately.

A. Transition rates

Consider first the higher-order corrections to $T_{\rm bot}$ and to the corresponding transition rate. Here we compare our results to those of L'Hereux and Kapral [2], who themselves reported an incorrect prefactor in the transition rate given earlier by Van den Broeck and Hänggi [3]. The asymptotic expansion detailed in Appendix B yields up to second order in the correlation time τ_c

$$T_{\rm bot} = \frac{2\pi}{(\alpha_1|\alpha_u|)^{1/2}}[1 + A\tau_c + O(\tau_c^2)]e^{(a^2/2D)\Delta} \ , \tag{24}$$

where

$$A = \frac{1}{4}\left\{\alpha_1 + |\alpha_u| + 3\frac{a^2\gamma_u}{\alpha_u^2} - 3\frac{a^2\gamma_1}{\alpha_1^2} + \frac{5}{6}\frac{a^2\beta_1^2}{\alpha_1^3} + \frac{5}{6}\frac{a^2\beta_u^2}{6|\alpha_u|^3}\right\} \tag{25}$$

and

$$\beta_{1,u} = f''(x_{1,u}), \quad \gamma_{1,u} = -\tfrac{1}{6}f'''(x_{1,u}) \ . \tag{26}$$

The effective potential Δ is defined in Eq. (19).

In the "Gaussian white-noise limit," denoted by the subscript GW,

$$\tau_c \to 0, \quad a \to \infty, \quad D = a^2\tau_c \ , \tag{27}$$

where $D = a^2\tau_c$ is finite, Eq. (24) yields the known corrections to the Kramers result [11,21]

$$T_{\rm GW} = \frac{2\pi}{(\alpha_1|\alpha_u|)^{1/2}}\left\{1 + \frac{D}{4}\left[3\left[\frac{\gamma_u}{|\alpha_u|^2} - \frac{\gamma_1}{\alpha_1^2}\right] + \frac{5}{6}\left[\frac{\beta_1^2}{\alpha_1^3} + \frac{\beta_u^2}{|\alpha_u|^3}\right]\right] + O(D^2)\right\}e^{\Delta V/D} \ , \tag{28}$$

where

$$\Delta V = \int_{x_u}^{x_1} dx\, f(x) = V(x_u) - V(x_1) \ , \tag{29}$$

The transition rate associated with the MFPT (24) is

$$r = \frac{(\alpha_1|\alpha_u|)^{1/2}}{2\pi(1 + A\tau_c)} e^{-(a^2/2D)\Delta} \ . \tag{30}$$

For the symmetric potential (10), Eq. (30) reduces to

$$r = \frac{c}{\sqrt{2}\pi[1 + (3c/4)\tau_c + (3d/2c^2)D]} e^{-(a^2/2D)\Delta} \ . \tag{31}$$

Equations (24), (30), and (31) are the principal results of this section and indeed of this paper. We compare (31) with the corresponding result of L'Hereux and Kapral [2], whose lowest-eigenvalue method yields the expression

$$r_{\rm HK} = \frac{c}{\sqrt{2}\pi(1 + 3c\tau_c)} e^{-(a^2/2D)\Delta} \ . \tag{32}$$

We argue that (31) is the correct expression: for dichotomous noise the weak-noise assumption necessarily implies a small correlation time because the intensity of the noise cannot be arbitrarily small if one is to have transitions at all [cf. Eq. (11)]. Therefore, the contributions in τ_c and D are necessarily of the same order of magnitude. Specifically, for the symmetric potential (10) with (11) we must satisfy $D > a_c^2\tau_c$ and hence $c\tau_c$ and dD/c^2 are of the same order of magnitude. Both terms must therefore be retained in the denominator of Eq. (32). In the asymptotic analysis detailed in Appendix B these effects arise from the appearance of fourth derivative terms of the potential in the expansion to first order in D. The usual procedure expands the potential $V(x)$ to second order, which leads to an incomplete calculation of the contributions of $O(D)$ to the MFPT. We find that the harmonic approximation is only valid at lowest order, i.e., in the calculation of the Kramers time, but that higher derivatives of $V(x)$ must be retained in the calculation of any corrections to the Kramers time. This conclusion is bolstered by the fact that $r_{\rm HK}$ does not go to the rate

$$r_{\rm GW} = \frac{c}{\sqrt{2}\pi[1 + (3d/2c^2)D]} e^{-\Delta V/D} \tag{33}$$

in the "Gaussian white-noise limit," whereas our result

(32) does.

A final interesting point to note is that all *asymptotic corrections* to the Kramers time in the weak-noise limit occur only in the prefactor, i.e., the Arrhenius (exponential) contribution remains unchanged.

B. MFPT to top of potential barrier

Next consider the time $T_{\rm top}$. The asymptotic evaluation of the integrals appearing in Eq. (14) when $g(x)=1$ as detailed in Appendix B leads to the result

$$T_{\rm top}=\frac{\pi}{(\alpha_1|\alpha_u|)^{1/2}}\left[1+(2/\pi)^{1/2}\left[|\alpha_u|^{1/2}+\frac{a\beta_u}{3|\alpha_u|^{3/2}}\right]\tau_c^{1/2}+A\tau_c+O(\tau_c^{3/2})\right]e^{(a^2/2D)\Delta}\ , \tag{34}$$

where A is given in Eq. (25) and Δ is defined in Eq. (19). The $\tau_c^{1/2}$ dependence has been obtained analytically [9] and numerically [22] for Gaussian colored noise. Note that the ratio

$$\frac{T_{\rm bot}}{T_{\rm top}}=2[1-\mu\tau_c^{1/2}+\mu^2\tau_c^2+O(\tau_c^{3/2})] \tag{35}$$

where

$$\mu=(2/\pi)^{1/2}\left[|\alpha_u|^{1/2}+\frac{a\beta_u}{3|\alpha_u|^{3/2}}\right] \tag{36}$$

is equivalent to order τ_c to the exponential form

$$\frac{T_{\rm bot}}{T_{\rm top}}=1+e^{-2\mu\tau_c^{1/2}}\ . \tag{37}$$

This dependence has been conjectured from computer simulation results for a bistable process driven by Gaussian colored noise [22,23]. We stress that Eq. (37) contains no more information than Eq. (35) since both are restricted to small values of τ_c and valid only to the order explicitly shown in Eq. (35). In the Gaussian white-noise limit (35) reduces to

$$\frac{T_{\rm bot}}{T_{\rm top}}=2[1-\mu_{\rm GW}D^{1/2}+\mu_{\rm GW}^2D+O(D^{3/2})]\ , \tag{38}$$

where

$$\mu_{\rm GW}=(2/\pi)^{1/2}\frac{\beta_u}{3|\alpha_u|^{3/2}}\ . \tag{39}$$

For a symmetric potential $\beta_u=0$ and one obtains the simpler result

$$\frac{T_{\rm bot}}{T_{\rm top}}=2+O(D^{3/2})\ . \tag{40}$$

Thus, even in this limit there are corrections to the frequently invoked relation $T_{\rm bot}=2T_{\rm top}$ when the diffusion coefficient is not vanishingly small.

IV. RESULTS AND CONCLUSIONS

The principal results of this paper are embodied in Eqs. (24) and (34) [see also Eq. (31)]. These are asymptotic expressions for the mean-first-passage times for transitions in a bistable potential when the correlation time of the dichotomous fluctuations driving the transition is short but nonvanishing. Equations (24) and (31) apply to transitions from one well to the other while (34) describes a transition from one well to the top of the barrier of the (noiseless) potential. Our results recover the correct white-noise limiting behavior, which is identical to that of a bistable system driven by Gaussian white noise. We find that the corrections to the zero-correlation-time limit up to $O(\tau_c)$ require the retention of derivatives of the potential function up to *fourth order,* contrary to the harmonic approximations that have been implemented in the past.

It is always difficult to assess on an analytical basis the range of validity of an asymptotic expansion of the sort that we have implemented here. Figures 3, 4, and 5 show a comparison of our analytic results for $T_{\rm bot}$ and those obtained by exact integration of Eq. (15) with $\beta=1$ for the potential function (10) with additive noise. Our result in this case reads

$$T_{\rm bot}=\sqrt{2}\pi\left[1+\frac{3c}{4}\tau_c+\frac{3d}{2c^2}D\right]e^{(a^2/2D)\Delta}\ . \tag{41}$$

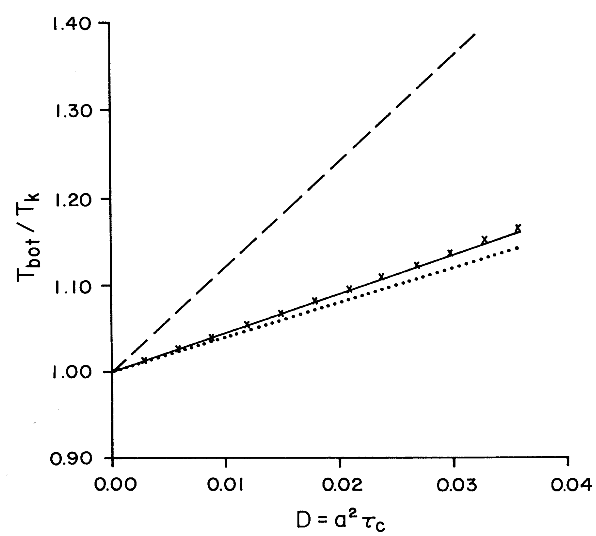

FIG. 3. Ratio of $T_{\rm bot}/T_K$ for the bistable potential of Eq. (10) with $c=d=1$ and additive noise of amplitude $a=0.5$. ×, exact results obtained by numerical integration of Eq. (15) with $\beta=1$. ——, our asymptotic expansion as given in Eq. (42); — — —, results of L'Hereux and Kapral [2]; · · · ·, results of Van den Broeck and Hänggi [3].

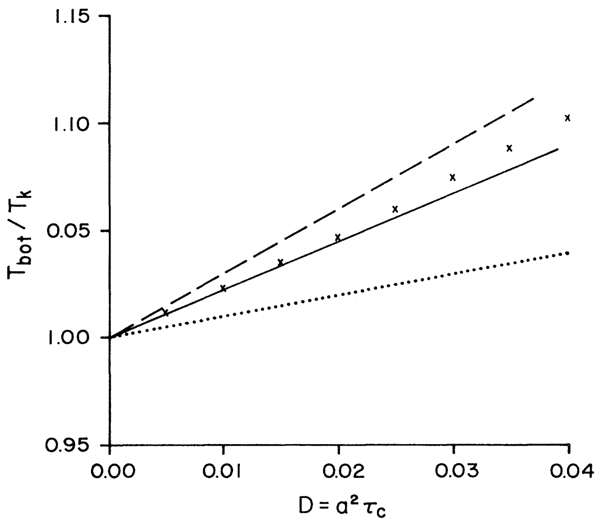

FIG. 4. Same as Fig. 3 but with $a=1.0$.

In all the figures we have taken the coefficients c and d in the potential function (10) to be unity. We have plotted the ratio of $T_{\rm bot}$ to its value when the correlation time τ_c vanishes, i.e., to the "Kramers time" $T_K \equiv \sqrt{2}\pi \exp(a^2\Delta/2D)$:

$$\frac{T_{\rm bot}}{T_K}=1+\tfrac{3}{4}\tau_c+\tfrac{3}{2}D \ . \tag{42}$$

We also show the results of L'Hereux and Kapral [2] and those of Van den Broeck and Hänggi [3]. The value of a for transitions to occur must exceed the critical value given in Eq. (11), $a_c=0.385$. The three figures correspond to different values of a: $a=0.5$, 1.0, and 5.0. The

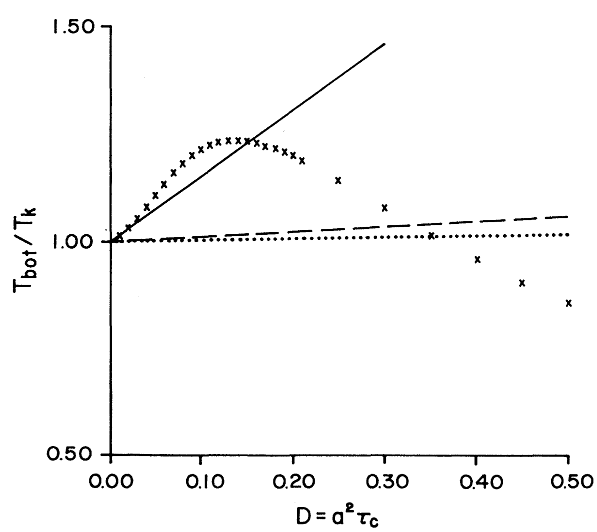

FIG. 5. Same as Fig. 3 but with $a=5.0$.

agreement of the exact results with our asymptotic ones is excellent at small values of τ_c, and is clearly better than that of other theories. As a increases, the regime of agreement is pushed to lower values of D and hence to proportionately lower values of τ_c.

We note in the exact results shown in Fig. 5 the eventual saturation of $T_{\rm bot}/T_K$ and its subsequent decrease (which is of course not predicted by the low-order asymptotic analysis since the saturation-decrease behavior is a long-correlation-time result). When $\beta=1$, the process starts at $x=-1$ from the configuration shown in Fig. 2(b) and begins to "roll down the hill" toward $x=1$. As time proceeds, a change in the value of the noise may take place and cause the ramp to slope in the opposite direction, holding the process captive near $x=-1$ if it has not rolled down the hill sufficiently far before this change occurs. After a while, another switch occurs and again the process rolls toward the desired final state $x=1$. The frequency of these switches is precisely the reciprocal of the correlation time τ_c, and the fate of the process depends on the magnitude of the correlation time compared to the time it takes for the process to roll downhill. As the correlation time increases, the transition will be completed with ever greater probability during the initial configuration of the potential. This "ballistic" time eventually reaches a limiting value $T_{\rm bal}$ dependent on the slope of the ramp:

$$T_{\rm bot}(\tau_c\to\infty)\to T_{\rm bal}=\int_{-1}^{1}dx\frac{f(x)}{f(x)+a} \ . \tag{43}$$

This time is extremely short compared to T_K (in all the cases shown in the figures $a^2\Delta/2D>10$) and hence the ratio drops precipitously for long correlation times. If we average over the initial value of the noise (e.g., $\beta=\frac{1}{2}$), then eventually the rate-limiting step in the process for long correlation times will be the time it takes for an initially unfavorable configuration to switch to the favorable one, namely, τ_c itself:

$$T_{\rm bot}\to\tau_c \ , \tag{44}$$

since $T_{\rm bal}$ is then short in comparison with τ_c. In this case the ratio $T_{\rm bot}/T_K$ also drops precipitously with increasing τ_c.

Finally, consider $T_{\rm top}$ for the potential (10) with $c=d=1$:

$$T_{\rm top}=\pi[1+(2/\pi)^{1/2}\tau_c^{1/2}+\tfrac{3}{4}\tau_c+\tfrac{3}{2}D]e^{(a^2/2D)\Delta} \ . \tag{45}$$

As said earlier, a $\tau_c^{1/2}$ term in $T_{\rm top}$ has also been found for Gaussian fluctuations, but the coefficients of the various powers of τ_c differ from those given in (45): in the Gaussian case $\tau_c^{1/2}$ is multiplied by $(2/\pi)^{1/2}\lambda_M$ where λ_M is the "Milne extrapolation length" with numerical value $\lambda_M=1.460354\ldots$ related to the Riemann ζ function [9], and the coefficient of τ_c is $\frac{3}{2}$.

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APPENDIX A: DERIVATION OF Eq. (12)

Let $p(t;x_0)$ be the first-passage time pair distribution function (PDF) from the initial ($t=0$) level x_0 to the level z, and let $p^{(\pm)}(t;x_0)$ be the conditional PDF's when the initial value of the noise is specified, i.e., when $F(0)=\pm a$. Then

$$p(t;x_0)=\beta p^{(+)}(t;x_0)+(1-\beta)p^{(-)}(t;x_0)\ , \tag{A1}$$

where β is the probability that the initial value of the noise is $F(0)=a$. Following the steps of Ref. [12] one easily establishes that for $x_0<z$ the Laplace transforms $\tilde{p}^{(\pm)}(s,x_0)$ of the conditional PDF's obey the integral equations [24]

$$\tilde{p}^{(+)}(s,x_0)=e^{-(\lambda+s)\tau}+\lambda^2\int_0^{\tau}dt_1 e^{-(\lambda+s)t_1}\int_0^{\infty}dt_2 e^{-(\lambda+s)t_2}\tilde{p}^{(+)}(s,X_2(t_2,t_1)) \tag{A2}$$

and

$$\tilde{p}^{(-)}=\lambda\int_0^{\infty}dt_1 e^{-(\lambda+s)t_1}\tilde{p}^{(+)}(s,X_1(t_1))\ , \tag{A3}$$

where

$$\tau=\int_{x_0}^{z}dx\frac{1}{f(x)+ag(x)} \tag{A4}$$

and

$$X_1(t_1)=X_+[t_1+X_+^{-1}(x_0)]\ , \tag{A5}$$

$$X_2(t_2,t_1)=X_-(t_2+X_-^{-1}\{X_+[t_1+X_+^{-1}(x_0)]\})\ . \tag{A6}$$

Here, as in the Introduction, $X_\pm(t)$ are the solutions of the deterministic dynamical equations (5a) and (5b), most easily expressed in terms of the inverse functions

$$X_\pm^{-1}(y)=\int^{y}dx\frac{1}{f(x)\pm ag(x)}\ . \tag{A7}$$

The integral equations obeyed by the mean-first-passage times from x_0 to z associated with the PDF's described above can be obtained from (A2) and (A3) via the relation

$$T_z(x_0)=-\frac{\partial}{\partial s}\tilde{p}(s,x_0)\bigg|_{s=0} \tag{A8}$$

and its analogs for the superscripted quantities. From (A2) and (A3) we obtain

$$T_z^{(+)}(x_0)=\frac{2}{\lambda}(1-e^{-\lambda\tau})+\lambda^2\int_0^{\tau}dt_1 e^{-\lambda t_1}\int_0^{\infty}dt_2 e^{-\lambda t}T_2^{(+)}(X_2(t_2,t_1)) \tag{A9}$$

and

$$T_z^{(-)}=\frac{1}{\lambda}+\lambda\int_0^{\infty}dt_1 e^{-\lambda t_1}T_z^{(+)}(X_1(t_1))\ . \tag{A10}$$

Rather than solving the integral equation (A9) directly, we find it convenient to first convert it into an integral equation for the x_0 derivative of $T_z^{(+)}(x_0)$ (together with an appropriate boundary condition). We solve this latter integral equation and then integrate once more to obtain the desired MFPT. The conversion to a useful integral equation for the derivative of $T_z^{(+)}(x_0)$ is best accomplished by introducing the change of variables

$$t_1=\int_{x_0}^{x_1}dx\frac{1}{f(x)+ag(x)}=X_+^{-1}(x_1)-X_+^{-1}(x_0)\equiv t_+(x_1,x_0)\ , \tag{A11}$$

$$t_2=\int_{x_1}^{x_2}dx\frac{1}{f(x)-ag(x)}=X_-^{-1}(x_2)-X_-^{-1}(x_1)\equiv t_-(x_2,x_1)\ . \tag{A12}$$

In terms of these new variables, we can write Eq. (A9) as

$$T_z^{(+)}(x_0)=\frac{2}{\lambda}(1-e^{-\lambda t_+(z,x_0)})+\lambda^2\int_{x_0}^{z}dx_1\frac{e^{-\lambda t_+(x_1,x_0)}}{f(x_1)+ag(x_1)}\int_{x_1}^{x_s^-}dx_2\frac{e^{-\lambda t_-(x_2,x_1)}}{f(x_2)-ag(x_2)}T_z^{(+)}(x_2)\ . \tag{A13}$$

An equation for the x_0 derivative of $T_z^{(+)}(x_0)$ is obtained by taking the derivative of (A13),

$$\frac{dT_z^{(+)}(x_0)}{dx_0}=\frac{1}{f(x_0)+ag(x_0)}\left[-2+\lambda T_z^{(+)}(x_0)-\lambda^2\int_{x_0}^{x_s^-}dx_2\frac{1}{f(x_2)-ag(x_2)}e^{-\lambda t_-(x_2,x_0)}T_z^{(+)}(x_2)\right], \tag{A14}$$

and integrating by parts noting that $t_-(x_s^-,x_0)=+\infty$:

$$\frac{dT_z^{(+)}(x_0)}{dx_0}=-\frac{2}{f(x_0)+ag(x_0)}+\frac{\lambda}{f(x_0)+ag(x_0)}\int_{x_s^-}^{x_0}dx_2 e^{-\lambda t_-(x_2,x_0)}\frac{dT_z^{(+)}(x_2)}{dx_2}. \tag{A15}$$

The information that has been lost upon taking a derivative is reinstated by introducing the boundary condition

$$T_z^{(+)}(z)=0 \tag{A16}$$

obtained directly from (A13).

The integral equation (A15) can be solved exactly using the method of the resolvent kernel [25], to obtain

$$\frac{dT_z^{(+)}(x_0)}{dx_0}=\frac{-2}{ag(x_0)+f(x_0)}-\frac{2\lambda}{ag(x_0)+f(x_0)}\int_{x_s^-}^{x_0}dx_2\frac{1}{ag(x_2)-f(x_2)}e^{-\lambda[\Phi(x_2)-\Phi(x_0)]}, \tag{A17}$$

where $\Phi(x)$ is given in Eq. (13). In turn, (A17) can be integrated directly and with the implementation of the boundary condition (A16) yields

$$T_z^{(+)}(x_0)=2\lambda\int_{x_0}^{z}dx\frac{e^{\lambda\Phi(x)}}{ag(x)+f(x)}\times\int_{x_s^-}^{x}dx'\frac{e^{-\lambda\Phi(x')}}{ag(x')-f(x')}. \tag{A18}$$

Use of (A18) in (A10) then also gives, after some algebra,

$$T_z^{(-)}(x_0)=\frac{1}{\lambda}+T_z^{(+)}(x_0)+2\int_{x_s^-}^{x_0}dx'\frac{e^{-\lambda[\Phi(x')-\Phi(x_0)]}}{ag(x')+f(x')}. \tag{A19}$$

Finally, Eq. (12) is obtained by combining these two expressions as indicated by (A1) and (A9):

$$T_z(x_0)=\beta T_z^{(+)}(x_0)+(1-\beta)T_z^{(-)}(x_0). \tag{A20}$$

APPENDIX B: ASYMPTOTIC EVALUATION OF THE MFPT

In order to evaluate the integrals that appear in the MFPT results of Sec. II we apply Laplace's method for the asymptotic expansion of integrals of the form

$$\int_a^b dx\, q(x)e^{-tp(x)}$$

when $t\gg 1$. Specifically, we make use of a theorem given in Ref. [26] and stated here in a simplified form more suitable for our purposes.

Suppose that the lower limit a is the minimum of the function $p(x)$ in the interval $a\le x\le b$ and that as $x\to a$ from the right, the following expansions hold:

$$p(x)\sim p(a)+\sum_{k=0}^{\infty}p_k(x-a)^{k+\mu},\qquad q(x)\sim\sum_{k=0}^{\infty}q_k(x-a)^{k+\nu-1}. \tag{B1}$$

Suppose also that the first of these relations is differentiable. Here μ, ν, and p_0 are *positive constants.* Then as $t\to\infty$,

$$\int_a^b dx\, q(x)e^{-tp(x)}\sim e^{-tp(a)}\sum_{k=0}^{\infty}\Gamma\left[\frac{k+\nu}{\mu}\right]\frac{a_k}{t^{(k+\nu)/\mu}}, \tag{B2}$$

where

$$a_0=\frac{p_0}{\mu p_0^{\nu/\mu}},\quad a_1=\left[\frac{q_1}{\mu}-\frac{(\nu+1)p_1q_0}{\mu^2p_0}\right]\frac{1}{p_0^{(\nu+1)/\mu}},\quad a_2=\left[\frac{q_2}{\mu}-\frac{(\nu+2)p_1q_1}{\mu^2p_0^2}+[(\mu+\nu+2)p_1^2-2\mu p_0p_2]\frac{(\nu+2)q_0}{2\mu^3p_0^2}\right]\times\frac{1}{p_0^{(\nu+2)/\mu}}, \tag{B3}$$

and we refer the reader to Ref. [26] for a general definition of the coefficients a_k.

The integrals appearing in the expressions for the mean-first-passage times can be evaluated asymptotically in the weak-noise approximation. As pointed out in Sec. II, in this limit the terms proportional to $1-\beta$ in Eqs. (14) and (15) are negligible. Thus we may write

$$T_{\rm top}\approx\frac{1}{D}\phi(x_u) \tag{B4}$$

and

$$T_{\rm bot}\approx\frac{1}{D}\phi(x_2), \tag{B5}$$

where D, a small parameter, is defined in Eq. (16). Here

$$\phi(z)=\int_{x_0}^{z}dx\, q_+(x)e^{(a^2/2D)\Phi(x)}\times\int_{x_s^-}^{x}dx'q_-(x')e^{-(a^2/2D)\Phi(x')}, \tag{B6}$$

and

$$q_{\pm}(x)=\frac{a}{ag(x)\pm f(x)} . \tag{B7}$$

We observe that for $D \ll 1$,

$$\int_{x_s^-}^{x} q_-(x')e^{-(a^2/2D)\Phi(x')}dx'$$

achieves its maximum in the neighborhood of $x=x_1$ and that it is therefore a slowly varying function of x, provided $x_s^- < x < x_1$. Therefore

$$\int_{x_s^-}^{x} dx' q_-(x')e^{-(a^2/2D)\Phi(x')} \approx \int_{x_s^-}^{x_u} dx' q_-(x')e^{-(a^2/2D)\Phi(x')} . \tag{B8}$$

Introducing Eq. (B8) into (B6) we obtain

$$\phi(z)\approx \int_{x_0}^{z} dx\, q_+(x)e^{(a^2/2D)\Phi(x)} \times \int_{x_s^-}^{x_u} dx\, q_-(x)e^{-(a^2/2D)\Phi(x)} . \tag{B9}$$

Hence the double integral in (B6) has been split into the product of two single integrals that can be evaluated using the procedure (B1)–(B4).

We note that a decoupling of the double integral (B6) into the product of two single integrals (plus another single-integral term) in fact follows exactly when $x_0=x_1$ and $z=x_2$, provided that the function $f(x)$ is odd (so that $x_1=-x_2$) [17]. To see this, let us first multiply the numerator and denominator of the inner integral in (B6) by $ag(x)+f(x)$ and note that

$$\Phi'(x)=\frac{-2f(x)}{a^2g^2(x)-f^2(x)} . \tag{B10}$$

Straightforward manipulations then yield

$$\Phi(x_2)=\int_{-x_2}^{x_2} dx\, q_+(x)e^{(a^2/2D)\Phi(x)}\int_{x_s^-}^{x} dx' g(x')q_+(x')q_-(x')e^{-(a^2/2D)\Phi(x')}+\frac{D}{a}\int_{-x_2}^{x_2} dx\, q_+(x) . \tag{B11}$$

Next, multiply the numerator and denominator of the outer integral by $ag(x)-f(x)$ and integrate by parts. Several terms cancel if $f(x)$ is odd, leaving the expression

$$\Phi(x_2)= \int_{-x_2}^{x_2} dx\, g(x)q_+(x)q_-(x)e^{(a^2/2D)\Phi(x)}\int_{x_s^-}^{x} dx' g(x')q_+(x')q_-(x')e^{-(a^2/2D)\Phi(x')} + \frac{D}{a}e^{(a^2/2D)\Phi(x_2)}\int_{-x_2}^{x_2} dx\, g(x)q_+(x)q_-(x)e^{-(a^2/2D)\Phi(x)} , \tag{B12}$$

where we have used the fact that $\Phi(x_2)=\Phi(-x_2)$. Finally, note that the double integral is the integral of an even function of two variables over half a rectangle in the (x,x') plane and is one-half the integral over the entire rectangle, which gives

$$\phi(x_2)= \tfrac{1}{2}\int_{-x_2}^{x_2} dx\, g(x)q_+(x)q_-(x)e^{(a^2/2D)\Phi(x)}\int_{x_s^-}^{x_s^+} dx\, g(x)q_+(x)q_-(x)e^{-(a^2/2D)\Phi(x)} + \frac{D}{a}e^{(a^2/2D)\Phi(x_2)}\int_{-x_2}^{x_2} dx\, g(x)q_+(x)q_-(x)e^{-(a^2/2D)\Phi(x)} . \tag{B13}$$

The exact decomposition (B13) is restricted to potentials of a particular symmetry and to particular initial and final conditions (the latter restriction makes it inapplicable to T_{top} in any case), while the approximate decomposition (B9) is more generally valid. We therefore prefer to use (B9), but we must be certain that in the appropriate special case (B9) is equivalent to (B13). It follows from straightforward manipulations similar to those that led to (B13) that (B9) can be rewritten as

$$\phi(x_2)= \tfrac{1}{2}\int_{-x_2}^{x_2} dx\, g(x)q_+(x)q_-(x)e^{(a^2/2D)\Phi(x)}\int_{x_s^-}^{x_s^+} dx\, g(x)q_+(x)q_-(x)e^{-(a^2/2D)\Phi(x)} + \frac{D}{a}e^{-(a^2/2D)\Phi(x_2)}\int_{-x_2}^{x_2} dx\, g(x)q_+(x)q_-(x)e^{(a^2/2D)\Phi(x)} . \tag{B14}$$

The difference between (B13) and (B14) occurs in the last term as a reversal in the exponential signs. Both of these terms are exponentially negligible in the limit of small D that we are considering here and therefore the two expressions are indeed equivalent.

Let us now return to Eq. (B9) and consider the application of the theorem (B1)–(B4) to each of the single integrals. Since x_1 is the minimum of $\Phi(x)$ in the interval $x_s^- < x < x_u$, we get from Eq. (B2)

$$\int_{x_s^-}^{x_u} dx\, q_-(x)e^{-(a^2/2D)\Phi(x)}\approx \left[\frac{4\pi D}{a^2\Phi''(x_1)}\right]^{1/2} e^{-(a^2/2D)\Phi(x_1)}[q_-(x_1)+QD+O(D^2)] , \tag{B15}$$

where

$$Q=\frac{2}{a^2}\left[\frac{q''_-(x_1)}{2\Phi''(x_1)}-\frac{p_-(x_1)\Phi^{iv}(x_1)}{8[\Phi''(x_1)]^2}-\frac{q'_-(x_1)\Phi'''(x_1)}{2[\Phi''(x_1)]^2}+\frac{5}{24}\frac{q_-(x_1)[\Phi'''(x_1)]^2}{[\Phi''(x_1)]^3}\right]. \tag{B16}$$

Further, recalling that x_u is the maximum of $\Phi(x)$, we find

$$\int_{x_0}^{x_2}dx\, q_+(x)e^{(a^2/2D)\Phi(x)}\approx\frac{1}{2}\left[\frac{4\pi D}{a^2|\Phi''(x_u)|}\right]^{1/2}e^{(a^2/2D)\Phi(x_u)}[q_+(x_u)-RD^{1/2}+SD+O(D^{3/4})] \tag{B17}$$

and

$$\int_{x_0}^{x_2}dx\, q_+(x)e^{(a^2/2D)\Phi(x)}\approx\left[\frac{4\pi D}{a^2|\Phi''(x_u)|}\right]^{1/2}e^{(a^2/2D)\Phi(x_u)}[q_+(x_u)+SD+O(D^2)], \tag{B18}$$

where

$$R=2/a\pi^{1/2}\left[\frac{q'_+(x_u)}{|\Phi''(x_u)|^{1/2}}+\frac{1}{3}\frac{q_+(x_u)\Phi'''(x_u)}{|\Phi''(x_u)|^{3/2}}\right] \tag{B19}$$

and

$$S=\frac{2}{a^2}\left[-\frac{q''_+(x_u)}{2\Phi''(x_u)}+\frac{q_+(x_u)\Phi^{iv}(x_u)}{8[\Phi''(x_u)]^2}+\frac{q'_+(x_u)\Phi'''(x_u)}{2[\Phi''(x_u)]^2}-\frac{5}{24}\frac{q_+(x_u)[\Phi'''(x_u)]^2}{[\Phi''(x_u)]^3}\right]. \tag{B20}$$

The appearance of the $D^{1/2}$ term in (B17) is a consequence of the fact that the upper limit of integration x_u coincides with the maximum of $\Phi(x)$, while in (B18) [as in (B15)] the maximum is within the interval of integration. Substitution of the appropriate results into (B4) and (B5) then yields the results given in Eqs. (24) and (35).

Note added. Near the time of completion of this paper there appeared in the literature a closely related paper that contains results that agree with ours: M. Kuś, E. Wajnyrb, and K. Wódkiewicz, Phys. Rev. A **43**, 4167 (1991).

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