Second-order processes driven by dichotomous noise

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We study free second-order processes driven by dichotomous noise. We obtain an exact differential equation for the marginal density $p(x,t)$ of the position. It is also found that both the velocity $\dot{X}(t)$ and the position $X(t)$ are Gaussian random variables for large t.

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I. INTRODUCTION

First-order processes driven by dichotomous noise are useful in modeling a variety of physical phenomena and interesting from a mathematical point of view because dichotomous noise is known to be non-Markovian when the distribution between switches is not exponential. In recent years these processes have been extensively studied. Thus when the distribution between switches is exponential (in this case dichotomous noise is Markovian and is sometimes referred to as the random telegraph signal) the probability density function obeys an integro-differential equation which reduces to the telegrapher's equation in the driftless case [1,2]. Also exact analytical expressions for the extreme events of such processes have been recently found [3,4].

On the other hand, second-order processes driven by dichotomous noise are of practical interest in many branches of physics and in engineering [5]. One example of this is provided by the evaluation of the probability density function for the output of second-order filters driven by the random telegraph signal [6]. The only results available for this problem are Monte Carlo simulations and other approximate results for the distribution function of second-order Butterworth filters (i.e., dichotomous-noise-driven oscillators) [7]. Unfortunately, second-order processes are much more difficult to address (even in the simplest cases) than first-order processes.

Our aim in this paper is to study second-order processes driven by dichotomous noise starting from the simplest one, thus we will deal with random processes $X(t)$ whose dynamical evolution is governed by the equation

$$\frac{d^2X(t)}{dt^2}=F(t) \tag{1.1}$$

where $F(t)$ is dichotomous noise alternately taking on values $\pm a$. The times that the variable retains the values $+a$ or $-a$ are governed by a switch probability density function $\psi(t)$, i.e., $\psi(t)dt$ is the probability of a switch first occurring in the interval $(t,t+dt)$. If $F(t)$ is a dichotomous Markov process then this density is exponential,

$$\psi(t)=\lambda e^{-\lambda t} , \tag{1.2}$$

where λ^{-1} is the average time between switches. In this case the correlation function of $F(t)$ reads [8]

$$\langle F(t)F(t')\rangle=a^2e^{-2\lambda|t-t'|} . \tag{1.3}$$

Equation (1.3) shows that the correlation time of the random telegraph signal is

$$\tau_c=\frac{1}{2\lambda} . \tag{1.4}$$

The second-order process (1.1) is equivalent to the bidimensional first-order process $(X(t),Y(t))$ specified by

$$\dot{X}(t)=Y , \quad \dot{Y}(t)=F(t) . \tag{1.5}$$

The random process $(X(t),Y(t))$ can take all real values $-\infty<X(t)<\infty$ and $-\infty<Y(t)<\infty$ and the properties of the process of principal interest will be contained in the joint probability density function defined by

$$p(x,y,t)dxdy=\Pr(x<X(t)<x+dx,y<Y(t)<y+dy) . \tag{1.6}$$

We will show that for a general switch density, $\psi(t)$, the joint density obeys a rather complicated integral equation. When $\psi(t)$ is exponential the integral equation reduces to a third-order partial differential equation. In this case we recover the Fokker-Planck equation in the Gaussian white-noise limit and telegrapher's equation for the marginal density of the velocity:

$$p(y,t)=\int_{-\infty}^{\infty}p(x,y;t)dx . \tag{1.7}$$

We will also obtain a quite simple partial differential equation for the marginal density of the position:

$$p(x,t)=\int_{-\infty}^{\infty}p(x,y;t)dy . \tag{1.8}$$

We will show that this marginal density goes to a Gaussian density as t goes to infinity. Specifically we will show that, for times much greater than the correlation time (1.4), $p(x,t)$ becomes the marginal density of a free second-order process driven by Gaussian white noise. We finally obtain a hierarchy of equations that is a convenient way of evaluating the moments of the process.

II. ANALYSIS

In this section we set the general formalism in order to evaluate the joint density $p(x,y;t)$ of the process. Two intermediate functions denoted by $\Omega_+(x,y;t)$ and $\Omega_-(x,y;t)$ will be required in our analysis and they are defined as follows:

$$\Omega_\pm(x,y;t)dxdydt = \Pr(\text{a sojourn with } F(t)=+a\,(-a) \textit{ ends} \text{ with the process } (X(t),Y(t)) \text{ in the volume } dxdy \text{ during the time interval } (t,t+dt)) \,. \tag{2.1}$$

We observe that these functions describe the state of the process at switching times. The evolution of the process between switches is deterministic and is given by the equation

$$\ddot{x}(t)=\pm a \,. \tag{2.2}$$

We also assume that the initial state of the process is known, i.e., $x_0=X(t=0)$ and $y_0=\dot{X}(t=0)$ are given quantities. Therefore the functions $\Omega_\pm(x,y;t)$ obey the following set of coupled integral equations:

$$\Omega_+(x,y;t)=\beta_+\delta(x-x_0-y_0t-\tfrac{1}{2}at^2)\delta(y-y_0-at)\psi(t) + \int_0^t d\tau\int_{-\infty}^{\infty} du\int_{-\infty}^{\infty} dv\, \Omega_-(u,v;\tau)\delta(x-u-v(t-\tau)-\tfrac{1}{2}a(t-\tau)^2)\delta(y-v-a(t-\tau))\psi(t-\tau) \,, \tag{2.3}$$

$$\Omega_-(x,y;t)=\beta_-\delta(x-x_0-y_0t+\tfrac{1}{2}at^2)\delta(y-y_0+at)\psi(t) + \int_0^t d\tau\int_{-\infty}^{\infty} du\int_{-\infty}^{\infty} dv\, \Omega_+(u,v;\tau)\delta(x-u-v(t-\tau)+\tfrac{1}{2}a(t-\tau)^2)\delta(y-v+a(t-\tau))\psi(t-\tau) \tag{2.4}$$

where $\beta_\pm$ are the probabilities of having $F(0)=\pm a$ $(\beta_+ + \beta_- = 1)$. These equations are derived from the consideration that when a sojourn in an occurrence of the plus (minus) state ends at time t, it is either the end of the very first sojourn [accounting for the factor β_+ (β_-) in the first term] or else a sojourn in the minus (plus) state ended at time $\tau<t$, at which time the process was in the point (u,v) of the phase space, and the subsequent sojourn in the plus (minus) state lasted for a time $t-\tau$.

We decompose the joint probability density into two components

$$p(x,y;t)=p_+(x,y;t)+p_-(x,y;t) \,, \tag{2.5}$$

where, for example, $p_+(x,y;t)$ is the probability density for $(X(t),Y(t))$ to be equal to (x,y) at time t while in the plus state, with an analogous definition for $p_-(x,y;t)$.

It is not difficult to convince oneself that $p_\pm(x,y;t)$ obey similar equations to that of $\Omega_\pm(x,y;t)$ just by replacing the ψ's by Ψ's, where $\Psi(t)$ is the probability that the time between switches is greater than t, i.e.,

$$\Psi(t)=\int_t^\infty dt'\psi(t') \,. \tag{2.6}$$

Therefore

$$p_\pm(x.y;t)=\beta_\pm\delta(x-x_0-y_0t\mp\tfrac{1}{2}at^2)\delta(y-y_0\mp at)\Psi(t) + \int_0^t d\tau\int_{-\infty}^{\infty} du\int_{-\infty}^{\infty} dv\, \Omega_\mp(u,v;\tau)\delta(x-u-v(t-\tau)\mp\tfrac{1}{2}a(t-\tau)^2) \times\delta(y-v\mp a(t-\tau))\Psi(t-\tau) \,. \tag{2.7}$$

Equations (2.3), (2.4), and (2.7) furnish a formal solution to the problem and can be a convenient starting point for numerical analysis when no further analytical treatment can be made.

We finally note that for an exponential switch density we have

$$\Psi(t)=e^{-\lambda t} \tag{2.8}$$

and

$$p(x,y;t)=\frac{1}{\lambda}[\Omega_+(x,y;t)+\Omega_-(x,y;t)] \,. \tag{2.9}$$

III. THE JOINT DENSITY

In what follows we will assume that the switch density is exponential, i.e., $\psi(t)=\lambda e^{-\lambda t}$, and without loss of generality we may also assume that $x_0=y_0=0$. With these assumptions Eqs. (2.3) and (2.4) read

$$\Omega_{\pm}(x,y;t)=\lambda\beta_{\pm}\delta(x\mp\tfrac{1}{2}at^2)\delta(y\mp at)e^{-\lambda t}+\lambda\int_0^t d\tau\int_{-\infty}^{\infty}du\int_{-\infty}^{\infty}dv\,\Omega_{\mp}(u,v;t-\tau)\delta(x-u-v\tau\mp\tfrac{1}{2}a\tau^2)\delta(y-v\mp a\tau)e^{-\lambda\tau}\,. \tag{3.1}$$

The Fourier transform of the position and the Laplace transform of time, defined by

$$\tilde{\Omega}_{\pm}(\omega,y;s)=\int_0^{\infty}dt\,e^{-st}\int_{-\infty}^{\infty}dx\,e^{-i\omega x}\Omega_{\pm}(x,y;t)$$

turns Eq. (3.1) into the following set of coupled integral equations:

$$\tilde{\Omega}_{+}(\omega,y;s)=\tilde{\Gamma}_{+}(\omega,y;s)+\frac{\lambda}{a}\int_{-\infty}^{y}dv\,\tilde{\Omega}_{-}(\omega,v;s)\times\exp\left[-\frac{\lambda+s}{a}(y-v)-i\frac{\omega}{2a}(y^2-v^2)\right], \tag{3.2}$$

$$\tilde{\Omega}_{-}(\omega,y;s)=\tilde{\Gamma}_{-}(\omega,y;s)+\frac{\lambda}{a}\int_{y}^{\infty}dv\,\tilde{\Omega}_{+}(\omega,v;s)\times\exp\left[\frac{\lambda+s}{a}(y-v)+i\frac{\omega}{2a}(y^2-v^2)\right] \tag{3.3}$$

where

$$\tilde{\Gamma}_{\pm}(\omega,y;s)=\frac{\lambda}{a}\beta_{\pm}\Theta(\pm y)\exp\left[\mp\frac{\lambda}{a}y\mp i\frac{\omega}{2a}y^2\right] \tag{3.4}$$

and $\Theta(\pm y)$ is the Heaviside step function. After differentiating Eqs. (3.2) and (3.3) with respect to y we find that the set of coupled integral equations is equivalent to the following system of first-order differential equations:

$$a\frac{\partial\tilde{\Omega}_{+}}{\partial y}+(\lambda+s+i\omega y)\tilde{\Omega}_{+}-\lambda\tilde{\Omega}_{-}=\lambda\beta_{+}\delta(y)\,, \tag{3.5}$$

$$a\frac{\partial\tilde{\Omega}_{-}}{\partial y}-(\lambda+s+i\omega y)\tilde{\Omega}_{-}+\lambda\tilde{\Omega}_{+}=-\lambda\beta_{-}\delta(y)\,. \tag{3.6}$$

It follows from Eq. (3.1) that the functions $\Omega_{\pm}(x,y;t)$ satisfy the initial conditions

$$\Omega_{\pm}(x,y;t)=\lambda\beta_{\pm}\delta(x)\delta(y)\,. \tag{3.7}$$

Hence the Fourier-Laplace inversion of Eqs. (3.5) and (3.6) leads to the following system of first-order partial differential equations:

$$\frac{\partial\Omega_{+}}{\partial t}+y\frac{\partial\Omega_{+}}{\partial x}+a\frac{\partial\Omega_{+}}{\partial y}+\lambda\Omega_{+}-\lambda\Omega_{-}=0\,, \tag{3.8}$$

$$\frac{\partial\Omega_{-}}{\partial t}+y\frac{\partial\Omega_{-}}{\partial x}-a\frac{\partial\Omega_{-}}{\partial y}+\lambda\Omega_{-}-\lambda\Omega_{+}=0\,. \tag{3.9}$$

As we have shown in Sec. II, when $\psi(t)$ is exponential the joint density is given by

$$p(x,y;t)=\frac{1}{\lambda}[\Omega_{+}(x,y;t)+\Omega_{-}(x,y;t)]\,, \tag{3.10}$$

thus, if we define the auxiliary function

$$q(x,y;t)=\frac{1}{\lambda}[\Omega_{+}(x,y;t)-\Omega_{-}(x,y;t)] \tag{3.11}$$

we obtain the following system:

$$\frac{\partial p}{\partial t}+y\frac{\partial p}{\partial x}+a\frac{\partial q}{\partial y}=0\,, \tag{3.12}$$

$$\frac{\partial q}{\partial t}+y\frac{\partial q}{\partial x}+a\frac{\partial p}{\partial y}+2\lambda q=0\,. \tag{3.13}$$

The differentiation of Eq. (3.13) with respect to y and the use of Eq. (3.12) yields

$$\frac{\partial^2 p}{\partial t^2}+2y\frac{\partial^2 p}{\partial t\partial x}+y^2\frac{\partial^2 p}{\partial x^2}-a^2\frac{\partial^2 p}{\partial y^2}+2\lambda\frac{\partial p}{\partial t}+2\lambda y\frac{\partial p}{\partial x}-a\frac{\partial q}{\partial x}=0\,,$$

another differentation with respect to y yields

$$\frac{\partial}{\partial y}\left[\frac{\partial^2 p}{\partial t^2}+2y\frac{\partial^2 p}{\partial t\partial x}+y^2\frac{\partial^2 p}{\partial x^2}-a^2\frac{\partial^2 p}{\partial y^2}+2\lambda\frac{\partial p}{\partial t}+2\lambda y\frac{\partial p}{\partial x}\right]+\frac{\partial^2 p}{\partial t\partial x}+y\frac{\partial^2 p}{\partial x^2}=0\,. \tag{3.14}$$

Equation (3.14) is a third-order partial differential equation with three independent variables. In order to reduce the number of independent variables in the equation we make the double Fourier transform:

$$\tilde{p}(\omega,\rho;t)=\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dy\,e^{-i(\omega x+\rho y)}p(x,y;t)$$

which turns Eq. (3.14) into

$$\frac{\partial^2\tilde{p}}{\partial t^2}-2\omega\frac{\partial^2\tilde{p}}{\partial t\partial\rho}+\omega^2\frac{\partial^2\tilde{p}}{\partial\rho^2}+\left[2\lambda+\frac{\omega}{\rho}\right]\frac{\partial\tilde{p}}{\partial t}-\omega\left[2\lambda+\frac{\omega}{\rho}\right]\frac{\partial\tilde{p}}{\partial\rho}+a^2\rho^2\tilde{p}=0\,. \tag{3.15}$$

Equation (3.15) is a second-order parabolic equation with two independent variables ρ and t and characteristic curves given by $\rho+\omega t=$const. If we now define new independent variables ξ and τ suggested by the characteristics (we rescale variables to get dimensionless units)

$$\xi=\frac{\lambda\rho}{\omega}\,,\quad \tau=\frac{\lambda\rho}{\omega}+\lambda t \tag{3.16}$$

we may further reduce Eq. (3.15) to an ordinary differential equation

$$\frac{\partial^2 \tilde{p}}{\partial \xi^2} - \left[2 + \frac{1}{\xi}\right] \frac{\partial \tilde{p}}{\partial \xi} + \mu^2 \xi^2 \tilde{p} = 0 \tag{3.17}$$

where

$$\mu \equiv a\omega/\lambda^2 \ . \tag{3.18}$$

We see from Eq. (3.7) that the initial conditions satisfied by the joint density $p(x,y;t)$ and the auxiliary function $q(x,y;t)$ are

$$p(x,y;t) = \delta(x)\delta(y) \ , \tag{3.19}$$

$$q(x,y;t) = \beta\delta(x)\delta(y) \ , \tag{3.20}$$

where

$$\beta \equiv \beta_+ - \beta_- \ . \tag{3.21}$$

On the other hand, the joint Fourier transform of Eq. (3.12) followed by the change of variables given by Eq. (3.16) yields

$$\frac{\partial}{\partial \xi} \tilde{p}(\omega,\xi;\tau) = -i\mu\xi\tilde{q}(\omega,\xi;\tau) \ . \tag{3.22}$$

Finally, taking into account that $t=0$ implies $\xi = \tau$ we obtain the following conditions to be satisfied by $\tilde{p}(\omega,\xi;\tau)$;

$$\tilde{p}(\omega,\xi=\tau;\tau) = 1 \ , \tag{3.23}$$

$$\frac{\partial}{\partial \xi} \tilde{p}(\omega,\xi=\tau;\tau) = -i\mu\beta\tau \ . \tag{3.24}$$

Therefore we have reduced the problem of finding the joint density of the process to solving a linear second-order differential equation with initial conditions. In the following sections we will obtain two special and relevant solutions to the problem.

Before closing this section we will show that in the Gaussian white-noise limit Eq. (3.15) turns into a Fokker-Planck equation. As is well known in the limit

$$a \to \infty, \quad \lambda \to \infty, \quad a^2/\lambda = D < \infty \tag{3.25}$$

the dichotomous noise $F(t)$ becomes Gaussian white noise [1]. In other words, in this limit the process given by Eq. (1.1) goes to the process

$$\frac{d^2X(t)}{dt^2} = \eta(t) \tag{3.26}$$

where $\eta(t)$ is Gaussian and δ-correlated noise. The joint density $p(x,y;t)$ $(y \equiv \dot{x})$ of the process (3.26) obeys the following Fokker-Planck equation:

$$\frac{\partial p}{\partial t} = y\frac{\partial p}{\partial x} + \frac{1}{2} D \frac{\partial^2 p}{\partial^2 y} \ . \tag{3.27}$$

Taking the limit (3.25) in Eq. (3.15) we obtain

$$\frac{\partial \tilde{p}}{\partial t} = \omega \frac{\partial \tilde{p}}{\partial \rho} - \frac{1}{2} D\rho^2 \tilde{p} \ , \tag{3.28}$$

which is the Fourier transform of Eq. (3.27).

IV. MARGINAL DENSITY OF THE VELOCITY

The marginal density of the velocity is defined by

$$p(y,t) = \int_{-\infty}^{\infty} p(x,y;t)dx \tag{4.1}$$

where $p(x,y;t)$ is the joint density of the second-order process. This marginal density is the probability density function of the first-order process [cf. Eq. (1.5)]

$$\dot{Y}(t) = F(t) \tag{4.2}$$

hence, $p(y,t)$ obeys the telegrapher's equation [1]

$$\frac{\partial^2 p}{\partial t^2} + 2\lambda \frac{\partial p}{\partial t} = a^2 \frac{\partial^2 p}{\partial y^2} \ . \tag{4.3}$$

In this section we will readily derive Eq. (4.3) and solve it from the above formalism. From the Fourier transform, $\tilde{p}(\omega,\rho;t)$, of the joint density we can easily obtain the transformed marginal density (i.e., the characteristic function), $\tilde{p}(\rho;t)$, just by setting $\omega=0$ in Eq. (3.15):

$$\frac{\partial^2 \tilde{p}}{\partial t^2} + 2\lambda \frac{\partial \tilde{p}}{\partial t} + a^2\rho^2\tilde{p} = 0 \ . \tag{4.4}$$

The Fourier inversion of Eq. (4.4) immediately leads to Eq. (4.3). Let us now solve this equation. We know that $\tilde{p}(\rho;t)$ satisfies the initial conditions

$$\tilde{p}(\rho;0) = 1 \ , \tag{4.5}$$

$$\left.\frac{\partial \tilde{p}}{\partial t}\right|_{t=0} = -ia\beta\rho \tag{4.6}$$

and the solution to the problem (4.4)–(4.6), in the Laplace domain, reads

$$\hat{\tilde{p}}(\rho,s) = \frac{2\lambda + s - ia\beta\rho}{s^2 + 2\lambda s + a^2\rho^2} \tag{4.7}$$

where

$$\hat{\tilde{p}}(\rho,s) = \int_0^{\infty} e^{-st}\tilde{p}(\rho,t)dt \ .$$

The Fourier inversion of Eq. (4.7) yields

$$\hat{p}(y,s) = \frac{1}{2a}\left[\frac{2\lambda+s}{(s^2+2\lambda s)^{1/2}} + \beta\,\mathrm{sgn}(y)\right] \times e^{-(s^2+2\lambda s)^{1/2}|y|/a} \tag{4.8}$$

where $\mathrm{sgn}(y) = 1(-1)$ if $y > 0(<0)$. Finally inverting the Laplace transform we obtain after some algebra [9]

$$p(y,t) = \tfrac{1}{2}e^{-\lambda t}\left\{[1+\beta\,\mathrm{sgn}(y)]\delta(at-|y|) + \lambda\left[(1/a)I_0\left[\frac{\lambda}{a}(a^2t^2-y^2)^{1/2}\right] + \frac{t+\beta y/a}{(a^2t^2-y^2)^{1/2}} I_1\left[\frac{\lambda}{a}(at^2-y^2)^{1/2}\right]\right]\Theta(at-|y|)\right\} \tag{4.9}$$

where $I_{0,1}(x)$ are modified Bessel functions. When $\beta=0$ Eq. (4.9) agrees with previous solutions of the telegrapher's equation [10]. We note that if $\beta=0$ then $p(y,t)$ is symmetric with respect to the plane $y=0$ (i.e., the plane $y=y_0$ if $y_0\neq 0$) which confirms intuition [we recall that $\beta=0$ implies $F(0)=a$ or $F(0)=-a$ with equal probability]. We will see below [cf. Eq. (4.10)] that this asymmetry vanishes as time increases.

Asymptotic behavior

Let us now show that the process $Y(t)$ goes to a diffusion process as t goes to infinity. In fact, this is a known result and is simply a restatement of the central limit theorem [11]. From the asymptotic expansions of Bessel functions it is readily seen that

$$p(y,t)\sim\left[\frac{\lambda}{2\pi a^2 t}\right]^{1/2}\left[1+\frac{\beta y}{2at}+O\left[\frac{1}{t}\right]\right] \times e^{-\lambda y^2/2a^2 t} . \tag{4.10}$$

We thus see that the term containing β is of the same order of magnitude than terms already neglected. We may conclude that for large t

$$p(y,t)\sim\left[\frac{1}{2\pi Dt}\right]^{1/2} e^{-y^2/2Dt} \tag{4.11}$$

where $D\equiv a^2/\lambda$ is the diffusion coefficient. Therefore the velocity of the process becomes, in the asymptotic (in time) limit, a Wiener process. It is instructive to obtain this result following a different reasoning based on Tauberian theorems [11,12]. Thus the symptotic (in time) value of $p(y,t)$ may be obtained by passing to the limit $s\to 0$ in Eq. (4.7). If we assume $\beta=0$ then the small s approximation is found in this way to be

$$2\lambda(s\hat{\tilde{p}}-1)=-a^2\rho^2\hat{\tilde{p}} \tag{4.12}$$

which is the Fourier-Laplace transform of the diffusion equation

$$\frac{\partial p}{\partial t}=(a^2/2\lambda)\frac{\partial^2 p}{\partial y^2} \tag{4.13}$$

with initial condition $p(y,0)=\delta(y)$ and whose solution is precisely given by Eq. (4.11).

V. THE MARGINAL DENSITY OF $X(t)$

In terms of the joint density $p(x,y;t)$ the marginal density of $X(t)$ is given by

$$p(x,t)=\int_{-\infty}^{\infty} p(x,y;t)dy \tag{5.1}$$

and its Fourier transform is given by

$$\tilde{p}(\omega,t)=\hat{p}(\omega,\rho=0;t) \tag{5.2}$$

where $\tilde{p}(\omega,\rho;t)$ is the joint Fourier transform of $p(x,y;t)$.

The evaluation of $p(x,t)$ for second-order processes is usually a very difficult task, and this is so because the position at a given time is strongly dependent on the velocity and to get rid of this dependence becomes quite involved. One example of the difficulties in dealing with marginal distributions is provided by the derivation of the Smoluchowski equation for second-order processes driven by Gaussian white noise where an approximate equation for $p(x,t)$, valid only for large damping constants, is derived from Kramers's equation [13]. Obviously such a procedure is unapplicable when no damping is present (as is the present case).

A. Equation for $p(x,t)$

Let us now derive an exact equation for the marginal density $p(x,t)$. The starting point of our derivation is Eq. (3.17) that we write in the form

$$\tilde{p}''-\left[2+\frac{1}{\xi}\right]\tilde{p}'+\mu^2\xi^2\tilde{p}=0 \tag{5.3}$$

where $\tilde{p}=\tilde{p}(\omega,\xi;\tau)$ is the joint characteristic function, ξ and τ are given by Eq. (3.16), and the primes denote derivatives with respect to ξ. Equation (5.3) has to be solved under the conditions

$$\tilde{p}(\omega,\xi=\tau;\tau)=1 , \tag{5.4}$$

$$\tilde{p}'(\omega,\xi=\tau;\tau)=-i\mu\beta\tau \tag{5.5}$$

where μ is given by Eq. (3.18). In the Appendix we show that the solution to the problem (5.3)–(5.5) is given by

$$\tilde{p}(\omega,\xi;\tau)=\frac{e^{-2\tau}}{2\tau}\{[v'(\tau)+i\beta\mu\tau v(\tau)]u(\xi) -[u'(\tau)+i\beta\mu\tau u(\tau)]v(\xi)\} \tag{5.6}$$

where $u(\xi)$ and $v(\xi)$ are two linearly independent solutions of Eq. (5.3) such that

$$u(\xi)=1+O(\xi^4),\quad v(\xi)=\xi^2+O(\xi^3) . \tag{5.7}$$

In terms of the original variables ρ and t we have

$$\tilde{p}(\omega,\rho;t)=\frac{e^{-2(\lambda\rho/\omega+\lambda t)}}{2(\lambda\rho/\omega+\lambda t)}\left\{\left[v'\left[\frac{\lambda\rho}{\omega}+\lambda t\right]+i\frac{a\beta}{\lambda}(\rho+\omega t)v\left[\frac{\lambda\rho}{\omega}+\lambda t\right]\right]u\left[\frac{\lambda\rho}{\omega}\right] -\left[u'\left[\frac{\lambda\rho}{\omega}+\lambda t\right]+i\frac{a\beta}{\lambda}(\rho+\omega t)u\left[\frac{\lambda\rho}{\omega}+\lambda t\right]\right]v\left[\frac{\lambda\rho}{\omega}\right]\right\} . \tag{5.8}$$

Now setting $\rho=0$ and taking into account Eq. (5.7) we obtain the marginal characteristic function of $X(t)$

$$\tilde{p}(\omega;t)=\frac{e^{-2\lambda t}}{2\lambda t}\left[v'(\lambda t)+i\frac{a\beta}{\lambda}\omega t v(\lambda t)\right] . \tag{5.9}$$

We will get a closed equation for $p(x,t)$ when $\beta=0$. Indeed, in this case we have

$$\tilde{p}(\omega;t)=\frac{e^{-2\lambda t}}{2\lambda t}v'(\lambda t) , \tag{5.10}$$

but $v(\zeta)$ $(\zeta \equiv \lambda t)$ is a solution to Eq. (5.3), that is,

$$\tilde{v}'' - \left[2+\frac{1}{\zeta}\right]\tilde{v}' + \mu^2\zeta^2\tilde{v} = 0 \ . \tag{5.11}$$

If we differentiate Eq. (5.11) with respect to ζ and use again Eq. (5.11) to write v in terms of v' and v'' we get

$$\frac{d^2v'}{d\zeta^2} - \left[2+\frac{3}{\zeta}\right]\frac{dv'}{d\zeta} + \left[\mu^2\zeta^2 + \frac{4}{\zeta} + \frac{3}{\zeta^2}\right]v' = 0 \ . \tag{5.12}$$

The substitution of Eq (5.10) into Eq. (5.12) gives the following equation for $\tilde{p}(\omega,t)$:

$$\frac{\partial^2\tilde{p}}{\partial t^2} + \left[2\lambda - \frac{1}{t}\right]\frac{\partial\tilde{p}}{\partial t} + a^2\omega^2 t^2\tilde{p} = 0 \ . \tag{5.13}$$

Finally, the Fourier inversion of Eq. (5.13) yields

$$\frac{\partial^2 p}{\partial t^2} + \left[2\lambda - \frac{1}{t}\right]\frac{\partial p}{\partial t} = a^2 t^2 \frac{\partial^2 p}{\partial x^2} \ . \tag{5.14}$$

Equation (5.14) is an exact equation for $p(x,t)$ and is the key result of the paper. We thus see that while the probability density of the velocity, $\dot{X}(t)$, obeys the telegrapher's equation with constant coefficients [cf. Eq. (4.3)] the density of the process itself obeys a telegrapher's equation with time-varying coefficients.

B. Diffusion process

From Eq. (5.14) it is possible to obtain an exact equation for $p(x,t)$ for free second-order processes driven by Gaussian white noise [cf. Eq. (3.26)]. Indeed, if in Eq. (5.14) we take the limit given by Eq. (3.25) we get

$$\frac{\partial p}{\partial t} = \frac{1}{2} D t^2 \frac{\partial^2 p}{\partial x^2} \tag{5.15}$$

where $D = a^2/\lambda$. Equation (5.15) is a Fokker-Planck equation with a time-dependent coefficient and represents an *exact* equation for free Brownian motion without damping effects. The solution to Eq. (5.15) with initial condition $p(x,0)=\delta(x)$ is

$$p(x,t) = \left[\frac{3}{2\pi D t^3}\right]^{1/2} \exp\left[-\frac{3x^2}{2Dt^3}\right] . \tag{5.16}$$

In this case the mean-squared displacement is of the form

$$\langle X^2(t)\rangle \sim t^3$$

which corresponds to the so-called anomalous diffusion [14].

C. Asymptotic behavior

We will now show that, in the dichotomous case, the asymptotic (in time) behavior of $p(x,t)$ is precisely given by Eq. (5.16). In order to show it we use the method, outlined at the end of Sec. IV, based on Tauberian theorems which assert that the large time behavior of $p(x,t)$ may be obtained by the small s behavior of its Laplace transform. We first note that $p(x,t)$ satisfies the initial conditions

$$p(x,0) = \delta(x) \ , \tag{5.17}$$

$$\frac{\partial}{\partial t} p(x,0) = 0 \ . \tag{5.18}$$

Thus the Laplace transform of Eq. (5.13) reads

$$(2\lambda + s)\frac{\partial}{\partial s}(s\hat{\tilde{p}} - 1) + 2(s\hat{\tilde{p}} - 1) + a^2\omega^2\frac{\partial^3\hat{\tilde{p}}}{\partial^3 s} = 0 \tag{5.19}$$

where $\hat{\tilde{p}} = \hat{\tilde{p}}(\omega,s)$ is the joint Fourier-Laplace transform of $p(x,t)$. Now the small s approximation (i.e., $|s| \ll \lambda$) turns Eq. (5.19) into

$$2\lambda\frac{\partial}{\partial s}(s\hat{\tilde{p}} - 1) + 2(s\hat{\tilde{p}} - 1) + a^2\omega^2\frac{\partial^3\hat{\tilde{p}}}{\partial^3 s} = 0 \ . \tag{5.20}$$

The Fourier-Laplace inversion of Eq. (5.20) immediately leads to the following diffusion equation:

$$\frac{\partial p}{\partial t} = \frac{a^2 t^3}{2(\lambda t - 1)}\frac{\partial^2 p}{\partial x^2} \tag{5.21}$$

whose solution, in the Fourier domain, reads

$$\tilde{p}(\omega,t) = e^{-a^2\omega^2 t^3/6\lambda}\left[1 + O\left[\frac{1}{\lambda t}\right]\right] , \tag{5.22}$$

whence

$$p(x,t) \sim \left[\frac{3\lambda}{2\pi a^2 t^3}\right]^{1/2} \exp\left[-\frac{3\lambda x^2}{2a^2t^3}\right] . \tag{5.23}$$

which agrees with Eq. (5.16). We observe that the asymptotic behavior given by Eq. (5.23) is valid for times much larger than the correlation time, $\tau_c = 1/2\lambda$, of $F(t)$. This is clearly seen from the above derivation since the small s approximation, $|s| \ll \lambda$, corresponds to $t \gg 2\tau_c$.

We finally note that the approximation given by Eq. (5.23) is consistent with the so-called dominant balance method [15]. Indeed, Eq. (5.23) suggests that in Eq. (5.14) one should have

$$2\lambda\left|\frac{\partial p}{\partial t}\right| \gg \left|\frac{\partial^2 p}{\partial t^2}\right| \quad \text{and} \quad 2\lambda\left|\frac{\partial p}{\partial t}\right| \gg \frac{1}{t}\left|\frac{\partial p}{\partial t}\right| \tag{5.24}$$

(for large t) and these conditions are certainly satisfied by Eq. (5.23).

D. Moments

Although to get analytical exact solutions to Eq. (5.14) seems to be quite involved it is less difficult to obtain exact expressions for the moments

$$m_n(t) \equiv \langle X^n(t)\rangle \tag{5.25}$$

$(n = 1,2,3,\ldots)$ of the process. In terms of the characteristic function $\tilde{p}(\omega,t)$ moments are given by

$$m_n(t) = (-i)^n \left.\frac{\partial^n\tilde{p}(\omega,t)}{\partial\omega^n}\right|_{\omega=0} . \tag{5.26}$$

From Eqs. (5.13) and (5.26) we get the following hierarchy of equations:

$$m_n''(t)+\left[2\lambda-\frac{1}{t}\right]m_n'(t)=n(n-1)t^2m_{n-2}(t) \qquad (5.27)$$

($n=0,1,2,3,\ldots$) with initial conditions

$$m_n(0)=\delta_{n,0} \qquad (5.28)$$

and

$$m_n'(0)=0 \qquad (5.29)$$

where $\delta_{n,0}$ is the Kronecker symbol. The problem (5.27)–(5.29) can be exactly solved. Thus, for example, the second moment is

$$m_2(t)=\frac{t^3}{3\lambda}-\frac{t^2}{4\lambda^2}+\frac{1}{8\lambda^4}-\frac{1}{8\lambda^4}(1+2\lambda t)e^{-2\lambda t} \qquad (5.30)$$

and, as $t\to\infty$, we have

$$m_2(t)\sim\frac{t^3}{3\lambda} \qquad (5.31)$$

which corresponds to the second moment of the Gaussian probability density (5.23). We also observe from Eqs. (5.27)–(5.29) that all odd moments are zero. We finally note that moments can also be evaluated from the dynamical equation (1.1). Thus, for instance, the second moment is given by

$$\langle X^2(t)\rangle=\int_0^t dt_1\int_0^{t_1}dt_2\int_0^t dt_1'\int_0^{t_1'}dt_2'\langle F(t_2)F(t_2')\rangle \qquad (5.32)$$

and the substitution of Eq. (1.3) into Eq. (5.32) yields Eq. (5.30). Apparently this way of evaluating moments seems to be more straightforward but it turns out to be messier than solving Eq. (5.27), especially when $n>2$.

VI. CONCLUSIONS

We have studied free second-order processes driven by dichotomous noise. For a general dichotomous noise the joint probability density $p(x,y;t)$ of the process may be obtained through a set of coupled integral equations. When the switch density is exponential the joint density obeys a third-order partial linear differential equation which, in the Fourier domain and with a convenient choice of variables, reduces to a second-order ordinary differential equation. The marginal density $p(y,t)$ of the velocity of process obeys the telegrapher's equation. In this case a complete solution has been provided. The marginal density $p(x,t)$ of the position obeys the following telegrapher's equation with variable coefficients:

$$\frac{\partial^2 p}{\partial t^2}+\left[2\lambda-\frac{1}{t}\right]\frac{\partial p}{\partial t}=a^2t^2\frac{\partial^2 p}{\partial x^2} .$$

The Gaussian white-noise limit turns this equation into the following exact equation for the position of a free Brownian particle without damping effects:

$$\frac{\partial p}{\partial t}=\frac{1}{2}Dt^2\frac{\partial^2 p}{\partial x^2} .$$

In this limit the mean-squared displacement shows an anomalous diffusion behavior of the form $\langle X^2(t)\rangle\sim t^3$.

The asymptotic (in time) behavior of both $p(y,t)$ and $p(x,t)$ has also been investigated. Thus, while the velocity asymptotically behaves as an ordinary Wiener process with a constant diffusion given by $D=a^2/\lambda$, the position asymptotically behaves as a one-dimensional free diffusion process with time-varying diffusion given by $D(t)=(a^2t^2/\lambda)[1+O(1/2\lambda t)]$ [cf. Eq. (5.21)]. In other words, as t becomes much larger than the correlation time of the driving noise, $X(t)$ behaves as the position of free Brownian particles without damping.

We have finally presented a hierarchy of equations that is a convenient way of calculating moments of any degree.

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APPENDIX: DERIVATION OF EQ. (5.6)

We see from Eq. (5.3) that $\xi=0$ is a regular singular point of the differential equation. This allows us to seek a series solution of the form [15]

$$\tilde{p}(\omega,\xi,\tau)=\sum_{n=0}^{\infty}a_n(\omega,\tau)\xi^{n+r} . \qquad (A1)$$

The substitution of Eq. (A1) into Eq. (5.3) yields

$$\tilde{p}(\omega,\xi,\tau)=A(\omega,\tau)u(\xi)+B(\omega,\tau)v(\xi) \qquad (A2)$$

where A and B are to be determined from initial conditions and

$$u(\xi)=1-\tfrac{1}{8}\mu^2\xi^4-\tfrac{1}{15}\mu^2\xi^5+O(\xi^6) , \qquad (A3)$$

$$v(\xi)=\xi^2+\tfrac{4}{3}\xi^3+\tfrac{8}{15}\xi^5+O(\xi^6) . \qquad (A4)$$

From Eq. (5.3) and Eqs. (A3) and (A4) we see that the Wronskian of $u(\xi)$ and $v(\xi)$ reads

$$W(u(\xi),v(\xi))=2\xi e^{2\xi} . \qquad (A5)$$

Finally combining Eqs. (5.4) and (5.5) with Eqs. (A2)–(A5) we obtain Eq. (5.6).

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