Noise and dynamics of self-organized critical phenomena

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Different microscopic models exhibiting self-organized criticality are studied numerically and analytically. Numerical simulations are performed to compute critical exponents, mainly the dynamical exponent, and to check universality classes. We find that various models lead to the same exponent, but this universality class is sensitive to disorder. From the dynamic microscopic rules we obtain continuum equations with different sources of noise, which we call internal and external. Different correlations of the noise give rise to different critical behavior. A model for external noise is proposed that makes the upper critical dimensionality equal to 4 and leads to the possible existence of a phase transition above $d = 4$. Limitations of the approach of these models by a simple nonlinear equation are discussed.

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I. INTRODUCTION

In the past few years a lot of attention has been paid to the phenomenon known as self-organized criticality. Bak, Tang, and Wiesenfeld [1] studied a cellular automaton model as a paradigm for the explanation of two widely occurring phenomena in nature: $1/f$ noise and fractal structures. $1/f$ noise has its origin in the superposition of a wide range of time scales, whereas fractals are linked to spatial self-similarity. Both have in common a lack of characteristic scales. This scale invariance suggests that the system is critical in analogy with classical equilibrium critical phenomena, but in self-organized criticality one deals with dynamical nonequilibrium statistical properties. On the other hand, the system evolves naturally to the critical state without any tuning of external parameters.

Different aspects of self-organized criticality have been studied: extensive numerical simulations for discrete and continuous models, scaling laws satisfied by the critical exponents, mean-field theory, coarse-grained versions satisfying symmetry rules to obtain critical exponents by means of the dynamic renormalization group, results for exactly solvable models, experimental values for a rich variety of systems, and applications to earthquakes (for a complete list, see, for instance, Ref. [2]).

In this paper we are interested in two of the aspects outlined above: numerical simulations and the analytical description by a nonlinear equation. Numerical simulations are performed on lattices with a continuously distributed variable; this model was introduced by Zhang [3] and has been used by other authors [4–6]. This model is expected to be in the same universality class as the original sandpile model [1]. We want to check the universality classes of the Zhang model by changing the microscopic rules to see under which circumstances the macroscopic behavior (dynamical exponent) is modified, including the effect of disorder.

From the dynamical rules one can obtain a coarse-grained version where the microscopic parameters enter the transport coefficients. Different models then give rise to different macroscopic equations. From these nonlinear equations containing the threshold condition one usually builds up simple nonlinear equations that retain the underlying symmetries and conservation laws as well as the characteristics of the noise sources. Our aim here is to analyze the simplest nonlinear equations obtained from different microscopic rules and to compare these results with the numerical simulations. The models we have studied have different symmetries than the models treated in other similar approaches [7–11] and, hence, different long-time and large-scale behavior is expected.

The outline of the papers is as follows. In Sec. II we describe the original model and introduce a model with an additional symmetry. In Sec. III we report the results of numerical simulations with different microscopic rules and show that disorder can be responsible for the appearance of a phase transition. In Sec. IV we analyze the nonlinear diffusion equation and discuss the effect of different sources of noise; the dynamic renormalization group is used to get the critical exponents for the simplest nonlinear equation satisfying the symmetry properties; and, finally, in Sec. V we summarize the results and present the conclusions.

II. DESCRIPTION OF THE MODEL

The model originally proposed by Zhang [3] consists of a lattice in which any site can store some energy E continuously distributed between 0 and E_c. This variable, which we call energy, can have different physical interpretations [5]. The system is perturbed by adding at a randomly chosen site an amount of energy δE, which is also a random variable. Once a site reaches a value of the energy greater than some critical value E_c this site becomes active and transfers isotropically the full amount of energy to its nearest neighbors. At this point the input of energy from the outside is turned off and the energy

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transferred to the neighboring sites can make them active, leading to new transfers of energy, giving rise to an activation cluster or avalanche that ends when all the sites have reached a value of the energy smaller than E_c. It is only when the avalanche has stopped that energy is added again; otherwise the system remains quiescent. After this procedure is repeated a large enough number of times there exists a well-defined distribution of energies characterizing the dynamical equilibrium state, which is homogeneous and isotropic, on the average [5]. This distribution is equivalent to the original discrete sandpile model of Bak, Tang, and Wiesenfeld [1], in which the variable describing the system state is the slope. Actually both models share some of the critical exponents and seem to belong to the same universality class [3, 5].

The microscopic rules for this model can be written in the form of a set of algebraic equations, one for each site,

$$E(i,t+1) = [1 - \Theta(E(i,t) - E_c)]E(i,t) + \sum_{\rm NN} \Theta(E(j,t) - E_c)E(j,t)/q + \eta_e(i,t), \tag{1}$$

where $\Theta(x)$ is the Heaviside step function, q is the lattice coordination number, and $\eta_e(i,t)$ is the external noise that generates the dynamics of the system. The sum runs over the nearest neighbors (NN)j to the lattice site i. Energy flows out freely through the boundaries to preserve overall energy conservation.

In Sec. III we will discuss in detail some of the variations one can introduce in this model in order to check the universality classes. However, one of them deserves some remarks. A simple change in the microscopic rules can be done in such a way that an amount of energy equal to E_c is transferred to the set of neighboring sites when a site becomes active instead of its total energy $E > E_c$. This new model is closer in spirit to the original sandpile model of Bak, Tang, and Wiesenfeld [1], where this is necessary due to the discrete nature of the critical variable, the slope. In this case the set of algebraic equations read

$$E(i,t+1) = E(i,t) - \Theta(E(i,t) - E_c)E_c + \sum_{\rm NN} \Theta(E(j,t) - E_c)E_c/q + \eta_e(i,t). \tag{2}$$

The importance of this model lies in the fact that it introduces a new symmetry: the deterministic equations are invariant under the transformation $E - E_c \rightarrow -(E - E_c)$. There are other symmetries that are common to the original Zhang model discussed previously; both are invariant under translation, rotation, and reflection.

It is well known from the theory of critical phenomena [12, 13] that symmetries play a crucial role in the establishment of universality classes. This and other aspects of the different models will be treated in detail in the following sections.

III. COMPUTER SIMULATIONS AND UNIVERSALITY CLASSES

We have performed computer simulations of the model sketched in the previous section together with some variants in order to determine the universality classes to which they belong. Our analysis does not pretend to be exhaustive, since our goal is to obtain mainly one of the exponents, the dynamic exponent, which can be analytically calculated from a stochastic partial differential equation by means of the dynamic renormalization-group (DRG) procedure [14–16], which will be the subject of Sec. IV.

The study is not complete since it is the whole set of critical exponents [17] that determines the universality classes and not a subset of them. Moreover, we do not try to obtain very good estimates of the dynamic exponent by means of a large number of runs in a large lattice but to get enough qualitative results to enable one to differentiate the macroscopic properties of different microscopic models.

In our simulations we define the size of an avalanche as the number of sites that have become critical irrespective of the number of times it has happened for a given site, while we define the time as the number of steps the avalanche takes and finally the characteristic length of the avalanche as the radius of gyration with respect to its center of mass. There are other choices in the literature; some of them are equivalent but other choices can lead to wrong conclusions, as we think happens in Ref. [2], where the authors take the maximum distance to a perimeter site from the seed as a characteristic length of the avalanche. In our opinion this makes the dynamic exponent (the exponent relating the duration of the avalanche to its characteristic length) closer to unity, since avalanches in a given direction make this choice of the length grow linearly with time. It is worth noting that this can modify the critical dimension of this class of systems. Following the results reported in Ref. [2] the exponents change when going from dimension 4 to 5 but the only exponent that does not involve the characteristic length of the avalanche is not changed within numerical accuracy.

It has been shown that Zhang's model in a hypercubic lattice gives rise to a distribution of energies at the dynamical equilibrium state that has a number $2D$ of pronounced peaks, whose width is a finite-size effect [3–5]. We have done simulations on a hexagonal lattice to generalize this statement and one can conclude that for any type of lattice the number of peaks equals the lattice coordination number.

Concerning the critical behavior, we have measured the dynamic exponent for a 128×128 square lattice with a starting configuration in which all sites are critical, taking $E_c = 1$ without loss of generality. We make 1000 runs for the system to get the dynamical equilibrium state and 10 000 runs to get the dynamical statistical properties. Averages are taken over the avalanches that are induced instead of doing an ensemble average, since it is assumed that the system is ergodic [2]. In Fig. 1 we plot the duration of the avalanches as a function of their characteristic length for the different definitions mentioned above, each extracted from the same set of simulations. The simulation data have been coarse grained in order to get smoother curves. We consider time intervals of exponentially increasing amplitude 2,3–4,5–8,9–16,..., and

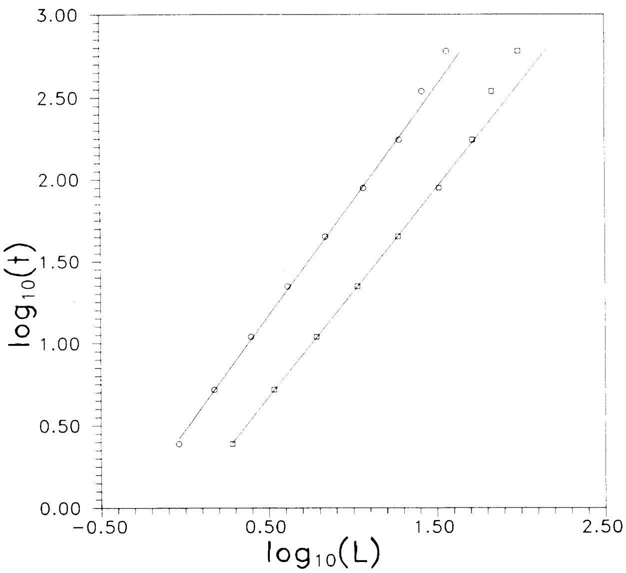

FIG. 1. Duration of an avalanche vs its mean characteristic length in logarithmic scales, for two different definitions of the length: o, length is defined as the radius of gyration with respect to the center of mass; □, the length is the maximum distance to a perimeter site from the seed. Straight lines are best linear fits to the sets of data with slopes $z = 1.40$ and 1.28, respectively.

we associate this interval with the averaged lifetime and the averaged length of the avalanches. For the definition of the characteristic length we have chosen, we get a dynamic exponent $z = 1.40 \pm 0.03$, whereas for the definition given in Ref. [2] one gets a smaller value $z = 1.28 \pm 0.03$, in agreement with our previous comments about this choice. In order to minimize the finite-size effects, we have performed an extrapolation to infinite lattice size from which we get an estimate for the dynamic exponent $z = 1.36 \pm 0.03$, as can be seen in Fig. 2. By studying the spatial characteristics of the avalanche clusters we have found that they are compact objects, in agreement with previous simulations and with the theoretical approach of Ref. [18].

We have also checked universality by changing some of the parameters of the dynamical rule given by Eq. (1). Instead of transferring the full amount of energy we have considered that only a fixed fraction is isotropically transferred from a critical site to its four nearest neighbors. This pushes the system towards criticality, i.e., all sites are closer to the critical value of the energy E_c, and avalanches become smaller. In Fig. 3 we plot averaged lifetimes against characteristic lengths for different values of the energy fraction released. From there we get a set of dynamic exponents that do not change compared to the original model (simulations are performed on a 128×128 lattice and no extrapolation to infinite length has been made) within numerical precision. Although some of the characteristic features of the model are changed (the distribution of energies still has four peaks but now they are closer to E_c and the average energy per site is different), the macroscopic behavior given by the dynamic exponent is unaltered. However, the avalanche lifetime and size distributions are modified for a value 0.5 of the fraction of energy released. They are still power-law distributed, but the exponents are different. The same can be concluded when modifying the intensity of the noise, i.e., the amount of energy we add to perturb the system when it is at equilibrium. In Fig. 4 we plot how the avalanche lifetime grows with its linear size for a random value δE between 0 and different $\delta E_{\max}$. The fractal dimension of the avalanches is not changed when introducing these

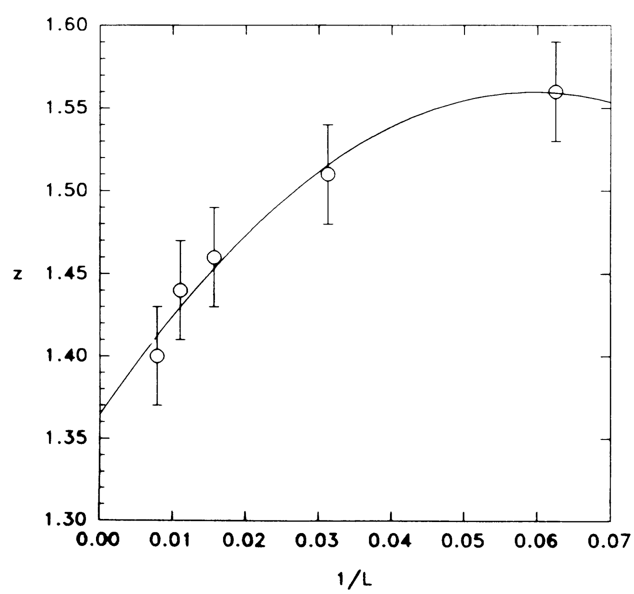

FIG. 2. Infinite-size extrapolation of the calculated dynamic exponent. This gives an estimation of $z = 1.36 \pm 0.03$.

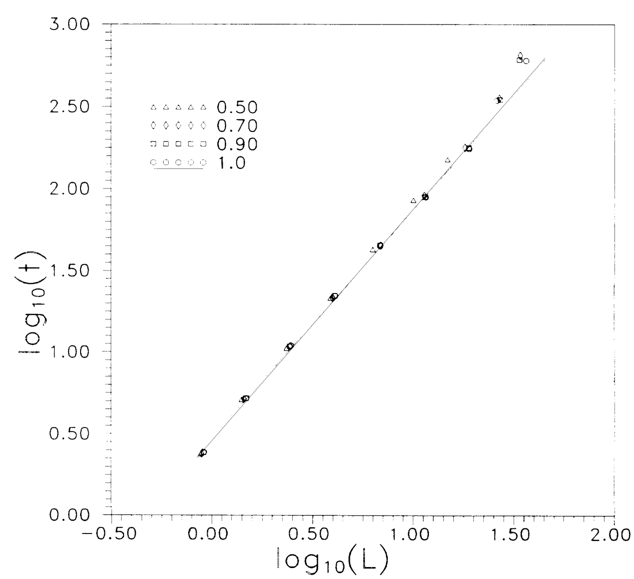

FIG. 3. Same as Fig. 1 for different fractions of the energy released when a site becomes critical. The solid line corresponds to the best fit for the original model in order to be taken as reference.

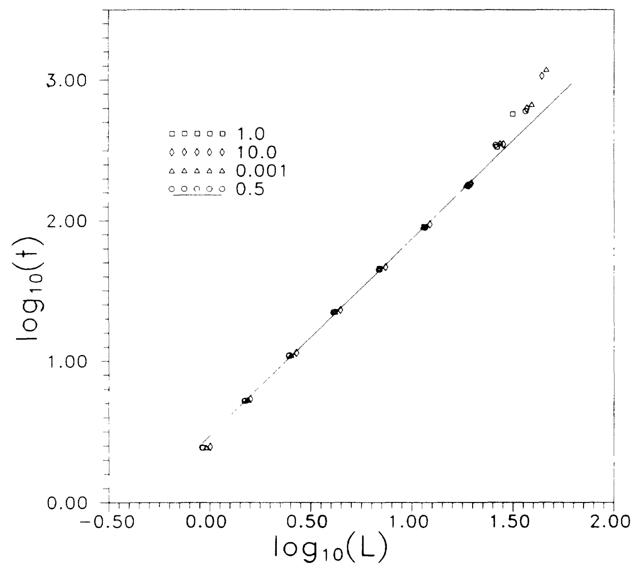

FIG. 4. Same as Fig. 3 for different values of $\delta E_{\max}$.

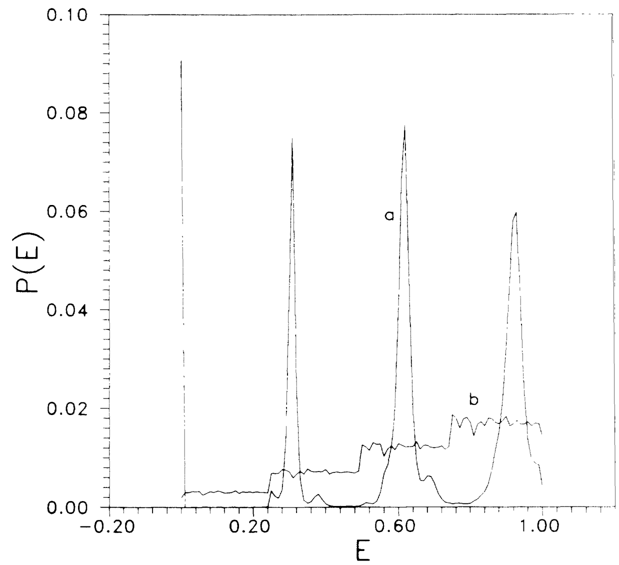

FIG. 5. Distribution of energies for the microscopic dynamical rules described in the text: (a) original Zhang model, (b) introducing a new symmetry.

modifications of the microscopic rules.

We have also checked how randomness can modify the behavior of this system. In Ref. [5] it has been assumed that energy is divided in a random way between the 2D neighbors. This local and instantaneous anisotropy rule breaks the peaks distribution, but the average energy per site is unchanged. In this paper we introduce randomness in a different way: a random fraction of the energy is released when a site becomes critical. When this fraction of the energy is close to 1 (for example, if we choose it between 0.9 and 1), the peaks structure is clearly preserved, but for smaller values the peaks are broadened until they finally disappear. For two different kinds of disorder, annealed (the fraction of energy released is chosen at random every time the site becomes critical) and quenched (there exists a random fraction of energy associated with each site), the transition happens at different values. In Table I we list the set of dynamic exponents we get from these simulations. From these results, however, it is not clear whether there is a phase transition. The differences could be due either to a finite-size effect or to corrections to scaling.

In the previous section we presented an alternative to the original Zhang model that introduces an additional symmetry. This model seems to be closer to the discrete sandpile models that coined the term self-organized criticality. Although the energy has again a continuous distribution in each site, the distribution of energies is changed completely. In Fig. 5 we plot this distribution and the distribution obtained with the Zhang model in order to compare them, and in Fig. 6 the avalanche lifetime is plotted against the characteristic length for both models as well. It is difficult to distinguish both sets of points, and so one can conclude that both models belong to the same universality class unless the right scaling regime has not been reached. This is confirmed by inspection of the distribution of lifetimes and sizes of the avalanches, and although the agreement is not so good, within numerical accuracy the curves have the same behavior. In this case

TABLE I. Numerical values of the dynamical exponents for a random fraction of the energy released when a site becomes critical

	0.90–1.0	0.75–1.0	0.50–1.0
Annealed	1.42	1.47	1.49
Quenched	1.42	1.47	1.47

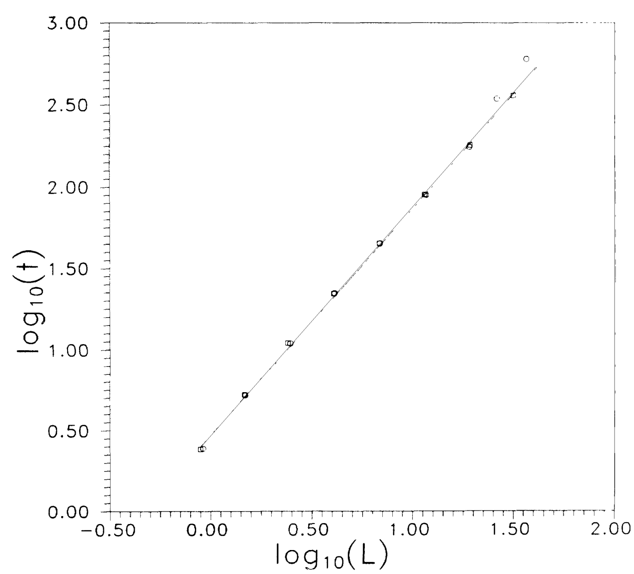

FIG. 6. Same as Figs. 3 and 4 for the models with different symmetries described in the text: o, original Zhang model; □, modified model.

the avalanche clusters appear to be compact objects as well.

IV. ANALYTICAL APPROACH

From the microscopic rules (1) one can construct an effective-medium equation in terms of a rescaled energy $E - E_c \rightarrow E$ in which the microscopic scales enter the transport coefficients, as follows:

$$\frac{\partial E(\mathbf{r},t)}{\partial t} = \alpha\nabla^2\{\Theta(E(\mathbf{r},t))[E(\mathbf{r},t) + E_c]\} + \eta_e(\mathbf{r},t) + \eta_i(\mathbf{r},t), \tag{3}$$

where α is related to the lattice spacing, to the unit time step, and to the fraction of energy released, and plays the role of a diffusion constant. We have introduced $\eta_i(\mathbf{r},t)$ as an internal noise, which accounts for the removed microscopic degrees of freedom, the type of lattice, for instance [19]. This internal noise would obey a fluctuation-dissipation theorem linked to a conserved magnitude; in this case, the energy [20]. Therefore it has a zero mean and a correlation function given by

$$\langle\eta_i(\mathbf{r},t)\eta_i(\mathbf{r}',t')\rangle \propto 2\alpha\nabla\nabla'\delta(\mathbf{r}-\mathbf{r}')\delta(t-t'). \tag{4}$$

Since the external noise is fundamental for the dynamical properties of the model, this point deserves some discussion. The input of energy to the system is a random number between 0 and $\delta E_{\max}$ and the noise acts only between avalanches. In the previous section we have reported numerical simulations, and the critical behavior (dynamic exponent) is not changed when $\delta E_{\max}$ is varied by orders of magnitude. For very small $\delta E_{\max}$ the evolution is slowed down since many more inputs are needed to make a site critical and to start the avalanche. Moreover, for such small intensity of the noise we have ascertained that, if it is not turned off between avalanches, i.e., the noise is constant with time, the dynamic exponent is not modified and interactions between avalanches remain negligible if the intensity of the noise is decreased as the system size grows. Therefore we conclude that the external noise can be modeled by a time-independent stochastic process.

The noise character can also be analyzed from a different point of view. One can assume a Gaussian process with zero mean [21] and correlation function given by

$$\langle\eta(\mathbf{r},t)\eta(\mathbf{r}',t')\rangle = \frac{2D}{\tau}e^{-|t-t'|/\tau}\delta^d(\mathbf{r}-\mathbf{r}'). \tag{5}$$

This is an Ornstein-Uhlenbeck process, with τ the correlation time. This general case can describe two very different limits. When dealing with internal noise, which involves a microscopic time scale, τ should be very small and in this limit we recover the usual white noise in space and time

$$\langle\eta(\mathbf{r},t)\eta(\mathbf{r}',t')\rangle = 2D\delta(t-t')\delta^d(\mathbf{r}-\mathbf{r}'). \tag{6}$$

But in our problem the external noise has no characteristic time scale. The only thing we know is that the noise acts once the avalanche is over, so it means that the only characteristic time would be a macroscopic time that scales with some power of the system size [22]. Then the limit $\tau \rightarrow \infty$ is appropriate and we can write

$$\langle\eta(\mathbf{r},t)\eta(\mathbf{r}',t')\rangle = \frac{2D}{T}\delta^d(\mathbf{r}-\mathbf{r}'), \tag{7}$$

where T is a macroscopic time much larger than the unit time step. Therefore, both points of view are linked by the fact that there are only short-range spatial correlations and an intensity of the noise that scales with $1/T$ or $1/L^\mu$ ($\mu > 0$). Both types of noise (6) and (7) are nonconservative due to the way energy is added from the outside. This nonconservative nature breaks the detailed balance. This is believed to be one of the main ingredients of self-organized criticality [9].

Clearly, Eq. (3) is a stochastic nonlinear differential equation from which one wants to obtain the hydrodynamic, long-time, and large-scale behavior of the system. In order to make it tractable we choose one of the representations of the step function

$$\Theta(x) = \lim_{\beta\rightarrow\infty}\frac{1}{2}(1+\tanh\beta x) \tag{8}$$

and make a series expansion in powers of the argument. This can be performed by keeping β finite instead of $\beta \rightarrow \infty$ [23]. We can then write

$$\frac{\partial E(\mathbf{r},t)}{\partial t} = \alpha\nabla^2 E(\mathbf{r},t) + \sum_{n=2}^{\infty}\lambda_n\nabla^2 E^n(\mathbf{r},t) + \eta_e(\mathbf{r},t) + \eta_i(\mathbf{r},t). \tag{9}$$

By simple dimensional analysis one can show that the internal noise with correlation function (4) makes the contribution of the nonlinearities to be irrelevant for any spatial dimensionality [24]. This makes the internal noise itself irrelevant, so from now on we will ignore this noise and discuss the effects of external noise as given by (7). Moreover, one realizes that all coupling constants λ_n are relevant when $d < 4$ and all are irrelevant when $d > 4$. Thus we can conclude that the upper critical dimension is 4 [25, 26], and that below it one needs the full set of nonlinearities to study the hydrodynamic behavior of this model, contrarily to what happens, for instance, in surface growth, where the Kardar-Parisi-Zhang equation [27] has only one relevant nonlinearity, allowing the critical exponents to be computed from this equation. Nevertheless let us assume for the moment that some qualitative aspects of models exhibiting self-organized criticality can be obtained from the simplest nonlinear equation

$$\frac{\partial E(\mathbf{r},t)}{\partial t} = \alpha\nabla^2 E(\mathbf{r},t) + \lambda\nabla^2 E^2(\mathbf{r},t) + \eta_e(\mathbf{r},t), \tag{10}$$

which is consistent with conservation laws and with the symmetries of the problem: reflection, rotation, translation, and lack of any characteristic time or length scale. Note that some of the symmetries obeyed by other continuum models such as Galilean invariance [8, 11] or discrete lattice structure [7] are lost.

We follow a DRG [15, 16] procedure to analyze the hydrodynamic behavior of the system given by Eq. (10). The infrared divergencies of momentum integration are avoided by integrating out the fast modes with momenta in the range $\Lambda e^{-l} \leq k \leq \Lambda$, where l is the shell thickness and Λ is the short-distance cutoff. In order to recover the original Brillouin zone one has to perform the following scaling transformation for the remaining short-wavelength modes:

$$E(k,\omega) \longrightarrow e^{(\chi+d+z)l} E(ke^{l}, \omega e^{zl}), \tag{11}$$

where z and χ are, respectively, the dynamic and the roughening exponent. These modes obey Eqs. (7) and (10) with renormalized parameters that satisfy, under an infinitesimal renormalization-group (RG) transformation and in the hydrodynamic limit ($k \to 0$, $\omega \to 0$), the following recursion relations up to one-loop order:

$$\frac{d\alpha(l)}{dl} = \alpha\left((z-2) - \frac{D\lambda^2}{T\alpha^4}16A_d\Lambda^{d-4}\right), \tag{12}$$

$$\frac{d(D/T)(l)}{dl} = \frac{D}{T}(2z - d - 2\chi), \tag{13}$$

$$\frac{d\lambda(l)}{dl} = \lambda\left((\chi + z - 2) + \frac{D\lambda^2}{T\alpha^4}48A_d\Lambda^{d-4}\right), \tag{14}$$

where $A_d = S_d/2(2\pi)^d$, with S_d being the surface area of a unit d-dimensional sphere. One can notice that $\bar{\lambda} = (16\Lambda^{d-4}A_dD\lambda^2/T\alpha^4)^{1/2}$ is the effective dimensionless coupling constant for which we can write the following RG recursion relation:

$$\frac{d\bar{\lambda}(l)}{dl} = \bar{\lambda}\left(\frac{4-d}{2} + 5\bar{\lambda}^2\right), \tag{15}$$

which enables us to evaluate fixed points and critical exponents up to the above-mentioned one-loop order. One notices that the upper critical dimension is $d_{uc} = 4$, since for $d < 4$ there exists a fixed point at $\bar{\lambda} = 0$ that is unstable and for $d > 4$ there are three fixed points, $\bar{\lambda} = 0$ and $\bar{\lambda} = \pm\sqrt{(d-4)/10}$. The first one is stable, whereas the other two are unstable. Under RG transformations the flow in parameter space has the following behavior. For $d < 4$ a small nonlinearity flows away from the mean-field fixed point ($\bar{\lambda} = 0$), and a dynamic exponent z different from 2 is expected. On the other hand, for $d > 4$ there is a basin of attraction for the stable mean-field fixed point, and purely diffusive behavior should be observed. However, for values of the effective coupling constant such that $|\bar{\lambda}| > \sqrt{(d-4)/10}$ the behavior is dominated by the strong-coupling unstable fixed point that gives rise to superdiffusive behavior ($z < 2$).

We conclude that it should be possible to observe a phase transition at $d \geq 4$ between a logarithmically rough phase with mean-field exponents and a smooth phase with a nondiffusive behavior [7, 8, 10, 11]. At this point it is worth noting that with a correlation function for the noise as given by (6) the upper critical dimension is lowered down to 2 [28]. In the previous section we have reported extensive numerical simulations on the Zhang model for $d = 2$ and different effective coupling constants (α is related to the fraction of energy released and D/T is linked to $\delta E_{\max}$ as the intensity of the noise) and no phase transition has been observed, thus providing some support for the assumptions about the external noise that we have made in the present analysis.

The same line of reasoning can be applied to the model described by microscopic rules (2). In this case, a continuum equation for a rescaled energy is written

$$\frac{\partial E(\mathbf{r},t)}{\partial t} = \alpha\nabla^2\left[\Theta(E(\mathbf{r},t))E_c\right] + \eta_e(\mathbf{r},t), \tag{16}$$

neglecting the effect of internal noise. Equation (6) clearly shows the reflection invariance of the energy variable. Now the simplest nonlinear equation in agreement with symmetry rules is

$$\frac{\partial E(\mathbf{r},t)}{\partial t} = \alpha\nabla^2 E(\mathbf{r},t) + \lambda\nabla^2 E^3(\mathbf{r},t) + \eta_e(\mathbf{r},t), \tag{17}$$

giving rise to a different hydrodynamic behavior. This model has the same upper critical dimensionality $d_{uc} = 4$, but some qualitative differences appear. The RG recursion relation for the effective coupling constant $\bar{\lambda} = 12\Lambda^{d-4}A_dD\lambda/T\alpha^3$ is

$$\frac{d\bar{\lambda}(l)}{dl} = \bar{\lambda}(4 - d - 9\bar{\lambda}). \tag{18}$$

Now there are two fixed points above and below $d = 4$. For $d < 4$ the mean-field fixed point ($\bar{\lambda} = 0$) is unstable and the fixed point corresponding to $\bar{\lambda} = (4-d)/9$ is stable, whereas for $d > 4$ stability is exchanged. This enables us to obtain a dynamic exponent $z = (14+d)/9$ below $d = 4$, which is, however, far from the results obtained in the numerical simulations. Thus this analytical approach would lead us to conclude that both models do not belong to the same universality class, whereas from the simulations the conclusion is the opposite. All this leads us to believe that the approach based on simple nonlinear equations is incomplete and that one should study the full nonlinear equations (3) and (16), keeping the threshold condition, as will be the subject of a subsequent paper.

In a recent paper Hentschel and Family [29] have proposed an approach to obtain the scaling behavior of stochastic differential equations of type (9). This method has been proven to give values of the critical exponents for a variety of problems that are in agreement with numerical simulations and renormalization-group calculations. Following this idea one can get a set of dynamical exponents, depending on the term that is dominant in the power-series expansion (9), $z = [4 + (m-1)d]/2m$. This assumption gives the exponent one gets in the numerical simulations for $m = 3$. It is not clear to us why this term gives the correct value. Moreover, one should take into account that the value obtained in this way differs from the one we computed from (17) showing the limitations of Hentschel-Family procedure when dealing with an equation where Galilean invariance is broken [27] and hence the coupling constant is renormalized.

V. CONCLUSIONS

In this paper we have performed numerical simulations in some models showing self-organized criticality in order to compute one of the exponents characterizing their critical behavior, the dynamical exponent. Starting with the dynamical microscopic rules proposed by Zhang [3] we have checked related models with different rules. We have emphasized that one must carefully define the avalanche characteristic length, since this can be relevant for the determination of the upper critical dimension.

When either the fraction of energy released at a critical site or the intensity of the noise are changed, the same dynamical exponent is obtained in a 2D square lattice. The fact that the dynamical exponent is not modified when varying the effective coupling constant is important for the appropriate choice of the noise correlations. However, the critical behavior is modified when disorder is introduced in the lattice, either quenched or annealed. A phase transition seems to happen in such a situation. We have also checked a different model that introduces a new symmetry, but the dynamic exponent is not changed in this case, belonging then, in principle, to the same universality class.

We have analytically studied continuum models derived from the above microscopic dynamical rules. The stochastic differential equations satisfied by these models have two sources of noise: internal and external. Internal noise comes from the removed microscopic degrees of freedom and hence it is described by a fluctuation-dissipation theorem. On the other hand, external noise is specified by the model. We show that the internal noise turns out to be irrelevant and external noise with the appropriate correlation makes the critical dimension equal to 4, in agreement with numerical simulations and other analytical approaches, and suggests that a phase transition, as a function of the bare coupling constant, above $d = 4$ should be observed. However, in the analytical approach two models with different symmetries belong to different universality classes, in contrast with our numerical simulations. This suggests the need for a more complete analysis involving the full nonlinear equations describing these models.

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