BISTABILITY DRIVEN BY DICHOTOMOUS NOISE: A COMMENT

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In a recent paper, three of us (Porrà et al. [1]) considered the problem of bistability driven by dichotomous colored noise. Our purpose was to calculate the effect of the finite (albeit short) correlation time \( \tau_c \) of the noise on mean first-passage times and associated transition rates. To obtain the transition rate from one metastable state of the system to the other, we carried out an asymptotic expansion to first order in \( \tau_c \). We reported that our results differed from earlier ones that had not retained all the necessary terms in the asymptotic expansion, including results reported by two of us (L’Heureux and Kapral [2]). Herein we reconcile this apparent discrepancy. To do this, we first note that in the earlier comparison there was an unfortunate unawareness of a more recent paper by L’Heureux and Kapral [3] where the fact that the asymptotic expansion had to be carried further than had been done previously was noted. Second, there is a mathematical error of omission in the actual expansion and in the exact numerical results in this latter reference that leads to apparent disagreement between the results of [1] and [3]. Here we note that inclusion of the missing term brings the results to complete agreement with one another and also with the numerical results.

The specific problem addressed in [1–3] is the evolution of the dynamical variable \( x(t) \) according to the equation

\[
\dot{x}(t) = f(x) + F(t) ,
\]

where the noise \( F(t) \) is an exponentially correlated dichotomous Markov process, and \( f(x) \equiv -V'(x) \) is the force associated with the bistable potential

\[
V(x) = \frac{c}{2} x^2 + \frac{d}{4} x^4 .
\]

(We caution the reader about the difference in notation in the two papers, which unfortunately includes usage of the same letter for different quantities—a “translation” of notation is given in Table I.) We are interested in calculating the transition-rate coefficient for passage from the vicinity of one minimum of the bistable potential to the vicinity of the other.

In Porrà et al. [1], the transition rate coefficient is found to be given by

\[
r = \frac{c}{\sqrt{2\pi} [1 + (3c/4) \tau_c + (3d/2c^2) D]} e^{-\left(a^2/2D\right)\Delta} ,
\]

where \( a, -a \) are the possible values of the noise, \( D \equiv a^2 \tau_c \) is its intensity, and \( \Delta \) is the effective potential barrier

\[
\Phi(x) \equiv -2 \int x' dx' \frac{f(x')}{a^2 - [f(x')]^2} .
\]

Equation (3) agrees quantitatively with exact numerical results when \( D \) is small [1].

In [2], L’Heureux and Kapral presented the transition
rate coefficient (in the present notation)

\[ r = \frac{c}{\sqrt{2\pi(1 + 3c\tau_c)}} e^{-\left(a^2/2D\right)\Delta}, \]  

which was later corrected [3] by noting that Eq. (5) did not contain all the appropriate terms in the asymptotic expansion. The corrected result (in the present notation) was

\[ r = \frac{c}{\sqrt{2\pi\left[1 + (3c/4)\tau_c + (9d/16c^2)D\right]}} e^{-\left(a^2/2D\right)\Delta}. \]  

The analysis involved two quantities that must be multiplied to yield \( r \), namely, a transmission coefficient \( \kappa \) and the transition-state-theory (TST) value of the rate coefficient \( k_{\text{TST}}^2 \), which is itself inversely proportional to the equilibrium population \( \eta^d \) in the first well and proportional to the stationary probability density of the process evaluated at the barrier top, \( \rho^d(0) \). The coefficient \( \kappa \) was calculated correctly and yields results that agree with exact ones. The calculations of \( \eta \), however, omitted one contribution. Specifically, Eq. (4.4) of [3] for the population number \( \eta^h \) around one of the minima should read (in the notation of [3])

\[ \eta^h = Z e^{\Phi(x^i) \left( \frac{\pi Y}{b\Delta^2} \right)^{1/2}} \times \left[ 1 + \frac{b}{2Y}(1 - 3a\Delta^2/8b^3 + 15a\Delta^2/8b^3) \right], \]  

where \( Z \) is a normalization factor. The last term in the parentheses, which comes from a contribution containing the square of the third derivative of the potential, had been omitted.

With this condition, the formulas in [3] lead exactly to the transition rate coefficient (3). It should also be noted that the well-known corrections [4] to the \( \tau_c \to 0 \) result of Kramers [5], valid in the “Gaussian white-noise limit,” where \( \tau_c \to 0, \ a \to \infty \), but \( D = a^2/2\tau_c \) remains finite, are recovered from (3).

In Table II of [3], the numerical calculation of the normalization factor \( Z \) was incorrect due to a coding error. Using the correct value of \( Z \), we recalculate the quantities \( \eta \) and \( k_{\text{TST}}^2 \) and compare these results with the analytical estimate based on Eq. (7). The numerical results are obtained for \( c = d = 1 \) in the potential (2). The first column in Table II (in the present notation) is the absolute value of the noise, the second is the inverse correlation time, and the third is the ratio of the effective barrier height to the intensity of the noise. The asymptotic expansion is expected to work well when this ratio is large, say \( \leq 10 \). That this is indeed the case can be seen from the fact that the value of the equilibrium population in one well given in the fourth column is indeed equal to

<table>
<thead>
<tr>
<th>( a )</th>
<th>( 1/\tau_c )</th>
<th>( a^2\Delta/2D )</th>
<th>( \eta )</th>
<th>( k_{\text{TST}}^2 ) (numer.)</th>
<th>( k_{\text{TST}}^2 ) (approx.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>10</td>
<td>18.0</td>
<td>0.5</td>
<td>( 1.267 \times 10^{-8} )</td>
<td>( 1.267 \times 10^{-8} )</td>
</tr>
<tr>
<td>15</td>
<td>27.01</td>
<td>0.5</td>
<td>( 1.953 \times 10^{-12} )</td>
<td>( 1.954 \times 10^{-12} )</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>36.01</td>
<td>0.5</td>
<td>( 2.810 \times 10^{-16} )</td>
<td>( 2.810 \times 10^{-16} )</td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>54.01</td>
<td>0.5</td>
<td>( 5.283 \times 10^{-24} )</td>
<td>( 5.283 \times 10^{-24} )</td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>72.01</td>
<td>0.5</td>
<td>( 9.311 \times 10^{-32} )</td>
<td>( 9.311 \times 10^{-32} )</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>90.02</td>
<td>0.5</td>
<td>( 1.586 \times 10^{-39} )</td>
<td>( 1.586 \times 10^{-39} )</td>
<td></td>
</tr>
<tr>
<td>1.0</td>
<td>15</td>
<td>4.18</td>
<td>0.4916</td>
<td>( 1.523 \times 10^{-2} )</td>
<td>( 1.549 \times 10^{-2} )</td>
</tr>
<tr>
<td>20</td>
<td>5.57</td>
<td>0.4952</td>
<td>( 4.489 \times 10^{-3} )</td>
<td>( 4.532 \times 10^{-3} )</td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>8.35</td>
<td>0.4982</td>
<td>( 3.486 \times 10^{-4} )</td>
<td>( 3.498 \times 10^{-4} )</td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>11.14</td>
<td>0.4991</td>
<td>( 2.516 \times 10^{-5} )</td>
<td>( 2.521 \times 10^{-5} )</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>13.92</td>
<td>0.4995</td>
<td>( 1.750 \times 10^{-6} )</td>
<td>( 1.752 \times 10^{-6} )</td>
<td></td>
</tr>
<tr>
<td>2.0</td>
<td>50</td>
<td>3.21</td>
<td>0.4865</td>
<td>( 7.353 \times 10^{-2} )</td>
<td>( 7.557 \times 10^{-2} )</td>
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<tr>
<td>60</td>
<td>3.85</td>
<td>0.4892</td>
<td>( 4.313 \times 10^{-2} )</td>
<td>( 4.408 \times 10^{-2} )</td>
<td></td>
</tr>
<tr>
<td>70</td>
<td>4.49</td>
<td>0.4916</td>
<td>( 2.486 \times 10^{-2} )</td>
<td>( 2.528 \times 10^{-2} )</td>
<td></td>
</tr>
<tr>
<td>80</td>
<td>5.13</td>
<td>0.4936</td>
<td>( 1.414 \times 10^{-2} )</td>
<td>( 1.432 \times 10^{-2} )</td>
<td></td>
</tr>
</tbody>
</table>
0.5, the correct value, when the ratio is large. The fifth column gives the exact numerical value of $k_{TST}^I$. It is seen that the agreement between the exact and approximate values (last column) of $k_{TST}^I$ is very good. Although the numerical values of $k_{TST}^I$ are different from those obtained in [3], the conclusion drawn from Fig. 6 of [3] regarding the critical slowing down of the transition remains unchanged.

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