BRIEF REPORTS

Brief Reports are accounts of completed research which do not warrant regular articles or the priority handling given to Rapid Communications; however, the same standards of scientific quality apply. (Addenda are included in Brief Reports.) A Brief Report may be no longer than 3½ printed pages and must be accompanied by an abstract. The same publication schedule as for regular articles is followed, and page proofs are sent to authors.

Bistability driven by dichotomous noise: A comment

Josep M. Porrà and Jaume Masoliver
Departament de Física Fonamental, Universitat de Barcelona, Diagonal 647, 08028 Barcelona, Spain

Katja Lindenberg
*Department of Chemistry 0340, University of California, San Diego, La Jolla, California 92093
and Institute for Nonlinear Science 0402, University of California, San Diego, La Jolla, California 92093*

Ivan L'Heureux
Department of Physics, University of Ottawa, Ottawa, Ontario, Canada K1N 6N5

Raymond Kapral
Chemical Physics Theory Group, Department of Chemistry, University of Toronto, Toronto, Ontario, Canada M5S 1A1
(Received 11 December 1991)

Two recently reported treatments [J. M. Porrà *et al.*, Phys. Rev. A **44**, 4866 (1991) and I. L'Heureux and R. Kapral, J. Chem. Phys. **88**, 7468 (1988)] of the problem of bistability driven by dichotomous colored noise with a small correlation time are brought into agreement with each other and with the exact numerical results of L'Heureux and Kapral [J. Chem. Phys. **90**, 2453 (1989)].

PACS number(s): 05.40.+j, 05.60.+w, 05.20.Dd, 05.90.+m

In a recent paper, three of us (Porrá *et al.* [1]) considered the problem of bistability driven by dichotomous colored noise. Our purpose was to calculate the effect of the finite (albeit short) correlation time τ_c of the noise on mean first-passage times and associated transition rates. To obtain the transition rate from one metastable state of the system to the other, we carried out an asymptotic expansion to first order in τ_c. We reported that our results differed from earlier ones that had not retained all the necessary terms in the asymptotic expansion, including results reported by two of us (L'Heureux and Kapral [2]). Herein we reconcile this apparent discrepancy. To do this, we first note that in the earlier comparison there was an unfortunate unawareness of a more recent paper by L'Heureux and Kapral [3] where the fact that the asymptotic expansion had to be carried further than had been done previously was noted. Second, there is a mathematical error of emission in the actual expansion and in the exact numerical results in this latter reference that leads to apparent disagreement between the results of [1] and [3]. Here we note that inclusion of the missing term brings the results to complete agreement with one another and also with the numerical results.

The specific problem addressed in [1–3] is the evolution of the dynamical variable $x(t)$ according to the equation

$$\dot{x}(t)=f(x)+F(t)\ , \tag{1}$$

where the noise $F(t)$ is an exponentially correlated dichotomous Markov process, and $f(x)\equiv -V'(x)$ is the force associated with the bistable potential

$$V(x)=-\frac{c}{2}x^2+\frac{d}{4}x^4\ . \tag{2}$$

(We caution the reader about the difference in notation in the two papers, which unfortunately includes usage of the same letter for different quantities—a "translation" of notation is given in Table I.) We are interested in calculating the transition-rate coefficient for passage from the vicinity of one minimum of the bistable potential to the vicinity of the other.

In Porrá *et al* [1], the transition rate coefficient is found to be given by

$$r=\frac{c}{\sqrt{2}\pi[1+(3c/4)\tau_c+(3d/2c^2)D]}e^{-(a^2/2D)\Delta} \tag{3}$$

where a, $-a$ are the possible values of the noise, $D\equiv a^2\tau_c$ is its intensity, and Δ is the effective potential barrier $\Delta\equiv\Phi(0)-\Phi(\sqrt{-c/d})$, with

$$\Phi(x)\equiv -2\int^x dx'\frac{f(x')}{a^2-[f(x')]^2}\ . \tag{4}$$

Equation (3) agrees quantitatively with exact numerical results when D is small [1].

In [2], L'Heureux and Kapral presented the transition

TABLE I. Translation of symbols between [1] and [3].

Ref. [3]	Ref. [1]
x_A	x_1
x_B	x_2
x_0	x_u
M_A	x_s^-
M_B	x_s^+
Δ	a
γ	$1/\tau_c$
Δ^2/γ	D
a	d
b	c
c	0
$\phi(x)$	$-(a^2/2D)[\Phi(x)-\Phi(x_u)]$

rate coefficient (in the present notation)

$$r=\frac{c}{\sqrt{2}\pi(1+3c\tau_c)}e^{-(a^2/2D)\Delta}, \tag{5}$$

which was later corrected [3] by noting that Eq. (5) did not contain all the appropriate terms in the asymptotic expansion. The corrected result (in the present notation) was

$$r=\frac{c}{\sqrt{2}\pi[1+(3c/4)\tau_c+(9d/16c^2)D]}e^{-(a^2/2D)\Delta}. \tag{6}$$

The analysis involved two quantities that must be multiplied to yield r, namely, a transmission coefficient κ and the transition-state-theory (TST) value of the rate coefficient k_f^{TST}, which is itself inversely proportional to the equilibrium population η^A in the first well and proportional to the stationary probability density of the process evaluated at the barrier top, $p_s(0)$. The coefficient κ was calculated correctly and yields results that agree with exact ones. The calculations of η, however, omitted one contribution. Specifically, Eq. (4.4) of [3] for the population number η_A^{hp} around one of the minima should read (in the notation of [3])

$$\eta_A^{\mathrm{hp}}=Ze^{\phi(x_A)}\left[\frac{\pi\gamma}{b\Delta^2}\right]^{1/2}\times\left[1+\frac{b}{2\gamma}(1-3a\Delta^2/8b^3+15a\Delta^2/8b^3)\right], \tag{7}$$

where Z is a normalization factor. The last term in the parentheses, which comes from a contribution containing the square of the third derivative of the potential, had been omitted.

With this condition, the formulas in [3] lead exactly to the transition rate coefficient (3). It should also be noted that the well-known corrections [4] to the $\tau_c\to 0$ result of Kramers [5], valid in the "Gaussian white-noise limit," where $\tau_c\to 0$, $a\to\infty$, but $D=a^2\tau_c$ remains finite, are recovered from (3).

In Table II of [3], the numerical calculation of the normalization factor Z was incorrect due to a coding error. Using the correct value of Z, we recalculate the quantities η and k_f^{TST} and compare these results with the analytical estimate based on Eq. (7). The numerical results are obtained for $c=d=1$ in the potential (2). The first column in Table II (in the present notation) is the absolute value of the noise, the second is the inverse correlation time, and the third is the ratio of the effective barrier height to the intensity of the noise. The asymptotic expansion is expected to work well when this ratio is large, say ≥ 10. That this is indeed the case can be seen from the fact that the value of the equilibrium population in one well given in the fourth column is indeed equal to

TABLE II. Steepest-descent approximation to the equilibrium-population number in one of the wells surrounding one of the two minima, and numerical and approximate values of the transition-state-theory rate coefficient.

a	$1/\tau_c$	$a^2\Delta/2D$	η	k_f^{TST} (numer.)	k_f^{TST} (approx.)
0.5	10	18.0	0.5	1.267×10^{-8}	1.267×10^{-8}
	15	27.01	0.5	1.953×10^{-12}	1.954×10^{-12}
	20	36.01	0.5	2.810×10^{-16}	2.810×10^{-16}
	30	54.01	0.5	5.283×10^{-24}	5.283×10^{-24}
	40	72.01	0.5	9.311×10^{-32}	9.311×10^{-32}
	50	90.02	0.5	1.586×10^{-39}	1.586×10^{-39}
1.0	15	4.18	0.4916	1.523×10^{-2}	1.549×10^{-2}
	20	5.57	0.4952	4.489×10^{-3}	4.532×10^{-3}
	30	8.35	0.4982	3.486×10^{-4}	3.498×10^{-4}
	40	11.14	0.4991	2.516×10^{-5}	2.521×10^{-5}
	50	13.92	0.4995	1.750×10^{-6}	1.752×10^{-6}
2.0	50	3.21	0.4865	7.353×10^{-2}	7.557×10^{-2}
	60	3.85	0.4892	4.313×10^{-2}	4.408×10^{-2}
	70	4.49	0.4916	2.486×10^{-2}	2.528×10^{-2}
	80	5.13	0.4936	1.414×10^{-2}	1.432×10^{-2}

0.5, the correct value, when the ratio is large. The fifth column gives the exact numerical value of k_f^{TST}. It is seen that the agreement between the exact and approximate values (last column) of k_f^{TST} is very good. Although the numerical values of k_f^{TST} are different from those obtained in [3], the conclusion drawn from Fig. 6 of [3] regarding the critical slowing down of the transition remains unchanged.

I.L. and R.K. acknowledge financial support from the Natural Science and Engineering Research Council of Canada. K.L. gratefully acknowledges support from the U. S. Department of Energy Grant No. DE-FG03-86-ER13606. J.M.P. and J.M. acknowledge partial financial support by the Dirección General de Investigación Científica y Técnica (DGICYT) under Contract No. PB90-0012, and by the Societat Catalana de Física (Institut d'Estudis Catalans). Finally, J.M.P. is grateful for the hospitality of the University of California at San Diego and for the cordial attention of R. Pawula during this visit.

[1] J. M. Porrá, J. Masoliver, and K. Lindenberg, Phys. Rev. A **44**, 4866 (1991).

[2] I. L'Heureux and R. Kapral, J. Chem. Phys. **88**, 7468 (1988).

[3] I. L'Heureux and R. Kapral, J. Chem. Phys. **90**, 2453 (1989).

[4] R. Larson and M. D. Kostin, J. Chem. Phys. **69**, 4821 (1978).

[5] H. A. Kramers, Physica (Utrecht) **7**, 284 (1940).