Landau damping of transverse quadrupole oscillations of an elongated Bose-Einstein condensate

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We have studied the interaction between the low-lying transverse collective oscillations and the thermal excitations of an elongated Bose-Einstein condensate by means of perturbation theory. We consider a cylindrical trapped condensate and calculate the transverse elementary excitations at zero temperature by solving the linearized Gross-Pitaevskii equations in two dimensions (2D). We use them to calculate the matrix elements between the thermal excited states and the quasi-2D collective modes. The Landau damping of transverse collective modes is studied as a function of temperature. At low temperatures, the corresponding damping rate is in agreement with the experimental data for the decay of the transverse quadrupole mode, but it is too small to explain the observed slow decay of the transverse breathing mode. The reason for this discrepancy is discussed.

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I. INTRODUCTION

Transverse collective oscillations have recently been observed in elongated condensates confined in axially symmetric harmonic traps [1]. At low temperatures, the frequency of the both monopole and quadrupole transverse modes turns out to be in very good agreement with the predictions of the Gross-Pitaevskii theory for two-dimensional (2D) condensates, thus revealing their quasi-2D character. The experiments also show some peculiar features of the transverse breathing oscillation of these elongated condensates. First, its excitation frequency is close to twice the radial trapping frequency and is nearly independent of the strength of the two-body interaction, the number of atoms, and the temperature. Second, its damping rate is much smaller than that of the other modes. For instance, the measured damping rate of the transverse breathing mode is approximately ten times smaller than that of the quadrupole mode. This is rather different from the 3D case, where both the monopole and the quadrupole oscillations have comparable decay rates. Conversely, the observations are in agreement with the expected behavior of the breathing oscillation of 2D isotropic condensates [2–4] for which theory predicts no damping and a frequency fixed by a universal quantity. This means that the experiments of Ref. [1] do indeed show the two-dimensional nature of transverse oscillations of elongated condensates and test the effects of reduced dimensionality.

In the present paper we consider a cylindrical condensate, subject to harmonic confinement in the radial (transverse) direction. The excitation spectrum at zero temperature is calculated by solving the linearized Gross-Pitaevskii equation. The main purpose is to investigate the coupling between the transverse collective oscillations and the thermal excitations in the collisionless regime. This coupling causes a Landau damping of the collective oscillations, i.e., the decay of the collective mode through the exchange of energy with thermally activated excitations.

The paper is organized as follows. In Sec. II, we calculate the excitation spectrum using the Gross-Pitaevskii theory. In Sec. III, we summarize the perturbation theory for the interaction of collective modes with thermal excitations (as in Ref. [5]). We then apply this theory to calculate the decay rate due to Landau damping of transverse oscillations. In Sec. IV, we present the main results for the transverse quadrupole mode. The situation for the monopole mode is also discussed.

II. ELEMENTARY EXCITATIONS OF A CYLINDRICAL TRAPPED CONDENSATE

At low temperatures the elementary excitations of a trapped weakly interacting degenerate Bose gas are described by the well-known time-dependent Gross-Pitaevskii (GP) equation for the order parameter in an external confining potential. The trapping potential is usually axially symmetric and it is given by $V_{\rm ext}(r_\perp,z)=M(\omega_\perp^2 r_\perp^2+\omega_z^2 z^2)/2$, with $r_\perp^2=x^2+y^2$. The ratio between axial (ω_z) and radial ($\omega_\perp$) trapping frequencies defines the anisotropy parameter of the trap $\lambda=\omega_z/\omega_\perp$.

In the experiment of Ref. [1], the transverse modes have been excited in the radial plane of a highly anisotropic cigar-shaped trap, with $\lambda\sim 0.0646$. Thus, as a reasonable approximation, we consider an idealized cylindrical trap that is uniform in the z direction ($\omega_z=0$) and has an isotropic trapping potential in the radial plane,

$$V_{ext}(r_\perp)=\frac{1}{2}M\omega_\perp^2 r_\perp^2 . \tag{1}$$

The harmonic trap frequency $\omega_\perp$ provides a typical length scale for the system, $a_\perp=(\hbar/M\omega_\perp)^{1/2}$. We assume a cylindrical condensate with large longitudinal size L, compared with the radial size R and with a number of atoms per unit length, $N_\perp=N/L$.

At low temperatures, the dynamics of the condensate is described by the linearized time-dependent GP equation. The excited states can be found by assuming small oscillations of the order parameter around the ground-state value,

$$\Psi(\mathbf{r},t)=e^{-i\mu t/\hbar}[\Psi_0(r_\perp)+\delta\Psi(\mathbf{r},t)], \tag{2}$$

where μ is the chemical potential.

In a cylindrical trap, the modes of the condensate are labeled by $i=(n,m,k)$, where n is the number of nodes in the radial solution, m is the z projection of the orbital angular momentum, and k is the longitudinal wave vector. The normal modes of the condensate can be obtained by seeking fluctuations that correspond to propagating waves of the form [6,7]

$$\delta\Psi(\mathbf{r},t)=U_k(r_\perp)e^{i(m\theta+kz)}e^{-i\omega t}+V_k^*(r_\perp)e^{-i(m\theta+kz)}e^{i\omega t}, \tag{3}$$

where we have used cylindrical coordinates $(r_\perp,\theta,z)$. The "particle" and "hole" components characterizing the Bogoliubov transformations in a cylindrical symmetric trap are then $u_{nmk}(\mathbf{r})=U_{nmk}(r_\perp)e^{im\theta}e^{ikz}$ and $v_{nmk}(\mathbf{r})=V_{nmk}(r_\perp)e^{im\theta}e^{ikz}$, respectively. Quasi-2D collective modes are described by the $k=0$ case, since in this case fluctuations are restricted to the transverse plane.

By inserting Eqs. (2) and (3) in the GP equation and retaining terms up to first order in U_k and V_k, one obtains three equations. The first is the nonlinear equation for the order parameter of the ground state,

$$[H_0+g\Psi_0^2(r_\perp)]\Psi_0(r_\perp)=\mu\Psi_0(r_\perp), \tag{4}$$

where $H_0=-(\hbar^2/2M)\nabla^2_{r_\perp}+V_{\text{ext}}(r_\perp)$ with $\nabla^2_{r_\perp}=d^2/dr_\perp^2+r_\perp^{-1}d/dr_\perp$. The interaction coupling constant is $g=4\pi\hbar^2a/M$ and is fixed by the s-wave scattering length a. The other two equations contain the unknown functions $U_k(r_\perp)$ and $V_k(r_\perp)$ [8]:

$$\hbar\omega U_k(r_\perp)=\left[H_0+\frac{\hbar^2}{2M}\left(\frac{m^2}{r_\perp^2}+k^2\right)-\mu+2g\Psi_0^2\right]U_k(r_\perp)+g\Psi_0^2V_k(r_\perp), \tag{5}$$

$$-\hbar\omega V_k(r_\perp)=\left[H_0+\frac{\hbar^2}{2M}\left(\frac{m^2}{r_\perp^2}+k^2\right)-\mu+2g\Psi_0^2\right]V_k(r_\perp)+g\Psi_0^2U_k(r_\perp). \tag{6}$$

We have numerically checked from Eqs. (5) and (6) that the state with $(m=0,n=1)$ and $k=0$ corresponds to the transverse breathing mode with excitation frequency $2\omega_\perp$. This frequency does neither depend on the strength of the two-body interaction potential nor on the number of particles. Of course, according to the universal nature of this mode, this general result is also valid in the Thomas-Fermi (TF) limit, where the dispersion relation can be found analytically [6,9]. However, the eigenfunctions for this mode, in general, do not coincide with their Thomas-Fermi limit.

The state with $(m=2,n=0)$ and $k=0$ is a transverse quadrupole mode. Its frequency depends on the two-body interaction, ranging from the TF value $\sqrt{2}\omega_\perp$ [valid for large condensates $(aN_\perp\gg1$ or equivalently $\mu\gg\hbar\omega_\perp)$], to the noninteracting value $2\omega_\perp$. Therefore, the transverse quadrupole has to be obtained numerically by solving Eqs. (5) and (6) for each particular value of the dimensionless parameter $aN_\perp$.

It is well known that the dipole mode $(m=1,n=0)$ is unaffected by the two-body interactions due to the translational invariance of the interatomic force, which cannot affect the motion of the center of mass [8] and its excitation frequency is $1\omega_\perp$. We have numerically checked that the transverse dipole mode satisfies these conditions. The fact that the frequency of the monopole mode is independent of the interaction is worth stressing. It is a unique property of 2D systems, related to the presence of a hidden symmetry of the problem described by the two-dimensional Lorentz group SO(2,1) [4].

III. LANDAU DAMPING

Let us consider a collective mode with frequency Ω_{osc} and the excited states i,j available by thermal activation, with energies E_i and E_j, respectively. Suppose that this collective mode has been excited and, therefore, the condensate oscillates with the corresponding frequency Ω_{osc}. Due to interaction effects, the thermal cloud of excitations can either absorb or emit quanta of this mode thus producing damping of the collective motion. We want to study the decay process in which a quantum of oscillation $\hbar\Omega_{\text{osc}}$ is annihilated (created) and the ith excitation is transformed into the jth one (or vice-versa). This mechanism is known as Landau damping. Another possible decay mechanism, also due to the coupling between collective and thermal excitations, is Beliaev damping [10], which is based on the decay of an elementary excitation into a pair of excitations.

The Landau damping rate within perturbation theory is given by [5,11]

$$\frac{\gamma}{\Omega_{\text{osc}}}=\sum_{ij}\gamma_{ij}\delta(\omega_{ij}-\Omega_{\text{osc}}), \tag{7}$$

where the transition frequencies are $\omega_{ij}=(E_j-E_i)/\hbar$ and the "damping strength" is

$$\gamma_{ij}=\frac{\pi}{\hbar^2\Omega_{\text{osc}}}|A_{ij}|^2(f_i-f_j). \tag{8}$$

We have assumed that the thermal cloud is at thermodynamic equilibrium, and the states i,j are thermally occupied with the usual Bose factor $f_i=[\exp(E_i/k_BT)-1]^{-1}$. The matrix element that couples the low-energy collective mode $(u_{\text{osc}},v_{\text{osc}})$ with the higher energy single-particle excitations (for which we use the indices i,j) is [5]

$$A_{ij}=2g\int d\mathbf{r}\Psi_0[(u_j^*v_i+v_j^*v_i+u_j^*u_i)u_{\text{osc}}+(v_j^*u_i+v_j^*v_i+u_j^*u_i)v_{\text{osc}}]. \tag{9}$$

In this work, we calculate the quantities γ_{ij} by using the numerical solutions u and v, avoiding the use of further approximations in the spectrum of excitations.

We are interested in the decay of transverse collective excitations $(k=0)$, $u_{\rm osc}(r_\perp,\theta)=U_{\rm osc}(r_\perp)e^{im_{\rm osc}\theta}$, due to the coupling with thermally excited states (i,j). From Eq. (9), it follows that $k\equiv k_i=k_j$ $(\Delta k=0)$. Since k is a continuous quantum number, the sum over k becomes an integral and Eq. (7) yields

$$\frac{\gamma}{\Omega_{\rm osc}}=\sum_{n_in_jm_im_j}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\gamma_{ij}\delta(\omega_{ij}-\Omega_{\rm osc}); \tag{10}$$

the sum is restricted to pairs that satisfy $\Delta m=|m_j-m_i|=m_{\rm osc}$. Then, in Eq. (10), the integration over k yields

$$\frac{\gamma}{\Omega_{\rm osc}}=\frac{1}{2\pi}\sum_{\tilde{k}}\sum_{n_in_jm_im_j}\gamma_{ij}\left[\left|\frac{\partial\omega_i}{\partial k}-\frac{\partial\omega_j}{\partial k}\right|_{\tilde{k}}\right]^{-1}, \tag{11}$$

where $\tilde{k}$ is the wave vector in which the conservation of energy is verified, i.e., $|\omega_i(\tilde{k})-\omega_j(\tilde{k})|=\Omega_{\rm osc}$.

IV. RESULTS

In order to present numerical results, we choose a gas of ^{87}Rb atoms (scattering length $a=5.82\times10^{-7}$ cm) confined in a cylindrical trap with radial frequency $\omega_\perp=219\times2\pi$ Hz, which corresponds to an oscillator length $a_\perp=0.729\times10^{-4}$ cm. The number of condensate atoms per unit length is taken to be $N_\perp=N/L=2800\,a_\perp^{-1}$. These conditions are close to the experimental conditions of Ref. [1].

We have solved the linearized GP equations (4)–(6) at zero temperature to obtain an exact description of the ground state $\Psi_0(r_\perp)$ and the normal modes of the condensate within Bogoliubov theory without using the Thomas-Fermi or Hartree-Fock approximations. The following numerical results were obtained: $\mu=9.526\hbar\omega_\perp$, and the excitation frequencies of the transverse monopole $\Omega_M=2.0\,\omega_\perp$ and of the transverse quadrupole mode $\Omega_Q=1.436\,\omega_\perp$.

For a fixed number of trapped atoms, the number of atoms in the condensate depends on the temperature. At zero temperature, quantum depletion is negligible and all the atoms can be assumed to be in the condensate [12]. At a finite temperature, the condensate coexists with the thermal cloud. The temperature dependence of the condensate atoms $N_0(T)$ is important at $k_BT>\mu$. However, if the temperature is low enough in comparison with the transition temperature, the size of the thermal cloud is large in comparison with that of the condensate, and the number of thermal atoms weakly depends on the interaction (see Ref. [13]). Thus, the number of condensate atoms can be approximated by the ideal-gas expression $N_0(T)=N[1-(T/T_c^0)^3]$, where N is the total number of atoms and T_c^0 is the transition temperature for Bose-Einstein condensation in the ideal Bose gas. Moreover, at low enough temperatures the excitation spectrum can be safely calculated by neglecting the interaction between the condensate and the thermal excitations [8]. This means that the excitation energies at a given T can be obtained within the Bogoliubov theory at $T=0$ by normalizing the number of condensate atoms to the corresponding condensate fraction $N_0(T)$.

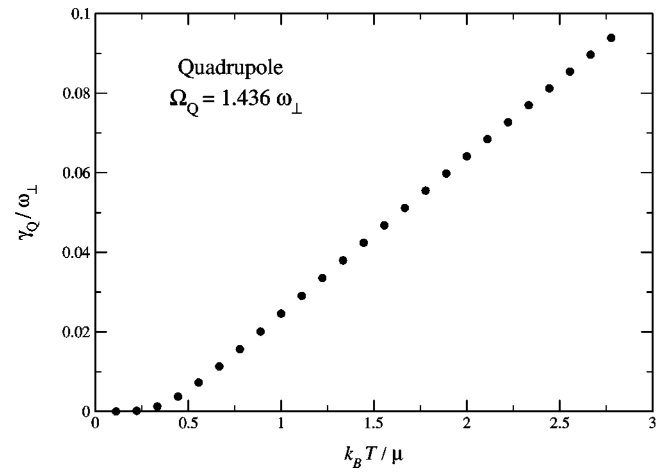

FIG. 1. Dimensionless damping rate of the transverse quadrupole mode $\gamma_Q/\omega_\perp$ as a function of k_BT/μ for a cylindrical condensate with $N_\perp=2800a_\perp^{-1}$ atoms of ^{87}Rb per unit length.

We have restricted our calculations to levels with energy $E\leqslant7\mu$. The contribution of higher excited levels can be neglected since their occupation becomes negligible in the range of temperatures we have considered.

It is worth recalling that the dipole mode is undamped [8] since it is not affected by the two-body interactions. We have checked that within our formalism the Landau damping of the transverse dipole mode is zero, as expected.

First of all, we have calculated the Landau damping of the transverse quadrupole mode as a function of T for a fixed number of condensate atoms, $\gamma_Q(T)$. It was found that there are 34 pairs of levels that verify the transverse quadrupole selection rules $(\Delta m=2,\ \Delta k=0)$ and the energy conservation at finite wave vectors, i.e., at $\tilde{k}\neq0$.

In Fig. 1, we plot the damping rate versus k_BT/μ for the transverse quadrupole mode for a fixed number of condensate atoms per unit length in a cylindrical trap. As expected, Landau damping increases with temperature since the number of excitations available at thermal equilibrium is larger when T increases. One can distinguish the two different regimes, one at very low T $(k_BT\ll\mu)$ and the other at higher T, where damping is linear with temperature. However, this linear behavior appears at relatively small temperature $(k_BT\sim\mu)$ in comparison with the homogeneous system [5], where this regime occurs at $k_BT\gg\mu$. For the highly elongated condensate of Ref. [1], at $k_BT\simeq0.7\mu$, a quality factor [14] for the transverse quadrupole mode, $Q=\Omega_Q/(2\gamma_Q^{\rm expt})\sim200$ has been measured, which gives a damping rate $\gamma_Q^{\rm expt}\sim10^{-2}\omega_\perp$. This value is in agreement with our calculations of the Landau damping for this mode (see Fig. 1). A more quantitative comparison is not possible at present due to the absence of the detailed measurements of damping of this mode. However, it is reasonable to conclude that the transverse quadrupole mode in elongated condensates decays via Landau damping. It is worth noting that the order of magnitude of the damping rate of the transverse quadrupole mode is the same as that previously estimated for a uniform gas [5,15–17], for spherical traps [11], and for anisotropic traps [18,19].

The Landau damping for other modes can be calculated analogously. However, the physical situation for the decay of the singular transverse monopole mode turns out to be much more complicated.

From the spectrum of elementary excitations, we have found that there are only 16 pairs of levels that verify the transverse monopole selection rules ($\Delta m=0$, $\Delta k=0$) and the energy conservation at $\tilde{k}\neq 0$. The damping strengths associated with the allowed transitions are smaller than the γ_{ij} values associated with the quadrupolelike transitions. As a result from our calculations, quite small amount of damping occurs. At $T\simeq 1.5\mu$, we obtain Landau damping $\gamma_M\simeq 7\times 10^{-5}\omega_\perp$, which is one order of magnitude smaller than the experimental decay [14] $\gamma_M^{\rm expt}\simeq 6\times 10^{-4}\omega_\perp$ measured in Ref. [1]. Thus, Landau damping cannot explain the experimentally observed decay of the transverse breathing mode.

Actually, there are reasons to believe that even this small value of Landau damping obtained in the present calculations could be larger than the true value. The point is that in our calculations, based on perturbation theory, the thermal cloud is assumed to be in the state of thermodynamic equilibrium and, thus, one can neglect its motion. This is correct for the case of quadrupole oscillations, where the frequencies of the collective motion of the condensate and the cloud are different. However, due to the unique peculiar nature of the 2D breathing mode, the breathing oscillation of the thermal cloud has the same frequency $2\omega_\perp$ as the condensate. Then, if the oscillation is excited by the deformation of the trap, as takes place in the present experiment [1], the cloud will oscillate in phase with the condensate, which will further decrease the Landau damping [20]. Therefore, this quasi-2D monopole mode in highly elongated condensates may decay via another damping mechanism, different from the Landau damping.

After finishing this work, a paper by Jackson and Zaremba has appeared [22] in which the effect of the collisions between elementary excitations has also been included. This effect is usually small in comparison with Landau damping. Nevertheless, for this peculiar mode the collisions seem to be the dominant damping mechanism, yielding a decay rate that it is in agreement with the experimentally measured value [1]. Note, however, that the experiments are produced at quite a high oscillation amplitude and nonlinear effects can play an important role. For example, as was noted in Ref. [23], the transverse breathing mode exhibits a "parametric" instability due to the decay into two or more excitations with nonzero momentum along the axis. In Ref. [24], the process in which the transverse breathing oscillation decays into two excitations propagating along z with momentum k and $-k$ was considered. This damping mechanism is active also at zero temperature. However, in this case the damping rate depends on the amplitude of the breathing oscillation.

V. SUMMARY

We have investigated the decay of the low-lying transverse oscillations of a large cylindrical condensate. First of all, the transverse normal modes of the condensate were calculated, as well as the excitations with finite k, by solving the linearized time-dependent Gross-Pitaevskii equation. Then, within the formalism of Ref. [11], the matrix elements associated with the transitions between the excited states, allowed by the selection rules of the transverse collective modes, were numerically calculated. The Landau damping of the transverse collective modes due to the coupling with thermal excited levels, as a function of temperature were studied within a first-order perturbation theory assuming the thermal cloud to be in thermodynamic equilibrium. For the damping rate of the transverse quadrupole mode, we have found good agreement with the experimentally measured value of Ref. [1]; whereas our result for the transverse breathing mode is one order of magnitude smaller than the experimental decay. One can conclude that in a highly elongated cylindrical symmetric condensate, transverse quadrupole oscillations decay via Landau damping mechanism, but the transverse breathing mode, which has an anomalously small measured damping rate, may decay via other damping mechanisms.

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[1] F. Chevy, V. Bretin, P. Rosenbusch, K.W. Madison, and J. Dalibard, Phys. Rev. Lett. **88**, 250402 (2002).
[2] L.P. Pitaevskii, Phys. Lett. A **221**, 14 (1996).
[3] Yu. Kagan, E.L. Surkov, and G.V. Shlyapnikov, Phys. Rev. A **54**, R1753 (1996).
[4] L.P. Pitaevskii and A. Rosch, Phys. Rev. A **55**, R853 (1997).
[5] L.P. Pitaevskii and S. Stringari, Phys. Lett. A **235**, 398 (1997).
[6] S. Stringari, Phys. Rev. A **58**, 2385 (1998).
[7] E. Zaremba, Phys. Rev. A **57**, 518 (1998).
[8] F. Dalfovo, S. Giorgini, L.P. Pitaevskii, and S. Stringari, Rev. Mod. Phys. **71**, 463 (1999).
[9] T.-L. Ho and M. Ma, J. Low Temp. Phys. **115**, 61 (1999).
[10] S.T. Beliaev, Zk. Eksp. Teor. Fiz. **7**, 469 (1958) [Sov. Phys. JETP **34**, 323 (1958)].
[11] M. Guilleumas and L.P. Pitaevskii, Phys. Rev. A **61**, 013602 (2000).
[12] F. Dalfovo, S. Giorgini, M. Guilleumas, L.P. Pitaevskii, and S. Stringari, Phys. Rev. A **56**, 3840 (1997).
[13] S. Giorgini, L.P. Pitaevskii, and S. Stringari, J. Low Temp. Phys. **109**, 309 (1997).
[14] In Ref. [1], the definition of the damping rate contains a factor 2 with respect to our definition.
[15] W.V. Liu and W.C. Schieve, e-print cond-mat/9702122.
[16] W.V. Liu, Phys. Rev. Lett. **79**, 4056 (1997).

[17] S. Giorgini, Phys. Rev. A **57**, 2949 (1998).

[18] P.O. Fedichev, G.V. Shlyapnikov, and J.T.M. Walraven, Phys. Rev. Lett. **80**, 2269 (1998).

[19] P.O. Fedichev and G.V. Shlyapnikov, Phys. Rev. A **58**, 3146 (1998).

[20] Note that the ''perturbative'' calculations of Landau damping can correspond to a different type of experiment, in which the oscillations of the condensate are excited by the action of a thin laser beam on the condensate, but not on the thermal cloud. In this case, the ''perturbative'' calculations will describe the initial stage of the process, when the cloud is still at rest. As it was presented in the lecture by E. Zaremba [21], the Landau damping corresponding to this type of experiment was calculated using the method developed in Ref. [22] in an elongated condensate with the same experimental parameters of Ref. [1]. However, they have obtained larger values for the Landau damping of the transverse breathing mode in comparison with our results. This discrepancy could be related to the semiclassical nature of their method [22] that does not take into account the discrete nature of the energy spectrum, and also to the fact that they consider a cigar-shaped condensate instead of an infinite cylinder as in the present calculations. The finite value of the longitudinal frequency ω_z could be important for the decay of this transverse breathing mode. Therefore, this is still an interesting open problem.

[21] E. Zaremba, lecture delivered at the BEC Summer Programme, Trento, 2002 (unpublished).

[22] B. Jackson and E. Zaremba, Phys. Rev. Lett. **89**, 150402 (2002).

[23] L.P. Pitaevskii and S. Stringari, Phys. Rev. Lett. **81**, 4541 (1998).

[24] Yu. Kagan and L.A. Maksimov, Phys. Rev. A **64**, 053610 (2001).