INTERPOLATION SEQUENCES FOR THE BERNSTEIN ALGEBRA

XAVIER MASSANEDA AND JOAQUIM ORTEGA-CERDÀ

ABSTRACT. We give a description, in analytic and geometric terms, of the interpolation sequences for the algebra of entire functions of exponential type which are bounded on the real line.

Dedicated to Victor Petrovich Havin in his 75 birthday

1. INTRODUCTION

In this paper we describe interpolation sequences for the *Bernstein algebra* $\mathcal{B}$ of entire functions of exponential type which are bounded on the real line.

The space $\mathcal{B}$ consists of the entire functions f such that for some $A, B > 0$

$$\log |f(z)| \leq A + B |\operatorname{Im} z| \quad z \in \mathbb{C}.$$

It can also be viewed as the union of the *Bernstein spaces* B_σ , $\sigma > 0$, of entire functions f such that

$$\sup_{z \in \mathbb{C}} |f(z)| e^{-\sigma |\operatorname{Im} z|} < \infty,$$

which are precisely the bounded functions with Fourier-Laplace transforms supported in $[-\sigma, \sigma]$.

We were motivated to study the interpolation in $\mathcal{B}$ by the recent characterisation of zero sets in the same class obtained by Sergei Ju. Favorov.

Given a discrete sequence $\Lambda \subset \mathbb{C}$ and a point $z \in \mathbb{C}$, consider the counting function $n_\Lambda(z, t) = \#(\Lambda \cap \overline{D(z, t)})$, $t > 0$.

Theorem. [Fa07, Theorem 2.2] *A sequence Λ is the zero set of a function in $\mathcal{B}$ if and only if:*

- (a) $\exists \lim_{R \rightarrow \infty} \sum_{\lambda \in \Lambda \cap D(0, R) \setminus \{0\}} \lambda^{-1}$
- (b) $n_\Lambda(0, t) = O(t)$, $t \rightarrow \infty$

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- (c) $n_\Lambda(0, t+1) - n_\Lambda(0, t) = o(t), t \rightarrow \infty$
 (d) $\sup_{x \in \mathbb{R}} \int_0^\infty [n_\Lambda(b, t) - n_\Lambda(x, t)] \frac{dt}{t} < \infty$ for some $b \in \mathbb{R} \setminus \Lambda$.

Notice that a subset of a zero set Λ of $\mathcal{B}$ is not necessarily a zero set of $\mathcal{B}$, since conditions (a) and (d) imply a certain regularity in the distribution of Λ that can be destroyed when removing points.

In view of this resulted we ask ourselves whether interpolation sequences for $\mathcal{B}$ can be described in similar terms. Given the characteristic growth of functions in $\mathcal{B}$, the natural interpolation problem for this algebra is the following.

Definition. A discrete sequence Λ in $\mathbb{C}$ is called an *interpolation sequence* for $\mathcal{B}$ if for every sequence of values $\{v_\lambda\}_{\lambda \in \Lambda}$ with

$$(1) \quad \sup_{\lambda \in \Lambda} |v_\lambda| e^{-C|\operatorname{Im} \lambda|} < \infty$$

for some $C > 0$, there exists $f \in \mathcal{B}$ with

$$f(\lambda) = v_\lambda \quad \lambda \in \Lambda.$$

Letting $\mathcal{B}(\Lambda)$ denote the space of sequences $\{v_\lambda\}_{\lambda \in \Lambda}$ satisfying (1) for some $C > 0$, we can equivalently define interpolating sequences as those such that the restriction operator

$$\begin{aligned} \mathcal{R}_\Lambda : \mathcal{B} &\longrightarrow \mathcal{B}(\Lambda) \\ f &\mapsto \{f(\lambda)\}_{\lambda \in \Lambda} \end{aligned}$$

is onto.

Applying the open mapping theorem to $\mathcal{R}_\Lambda$ [BrGa95, Lemma 2.2.6] one sees that the interpolation is stable in the following sense: for each $C > 0$ there exist $M, \sigma > 0$ such that if $\{v_\lambda\}_{\lambda \in \Lambda}$ satisfies (1) then there exists $f \in \mathcal{B}$ with $f(\lambda) = v_\lambda, \lambda \in \Lambda$, and $\sup_{z \in \mathbb{C}} |f(z)| e^{-\sigma|\operatorname{Im} z|} \leq M \sup_{\lambda \in \Lambda} |v_\lambda| e^{-C|\operatorname{Im} \lambda|}$.

There is an alternative (and equivalent) definition of $\mathcal{B}$ -interpolation:

Definition. A sequence Λ is *free interpolation* (denoted $\Lambda \in \operatorname{Int}_{\ell^\infty} \mathcal{B}$) if the trace $\mathcal{R}_\Lambda(\mathcal{B})$ is *ideal*, that is, if $\ell^\infty \cdot \mathcal{R}_\Lambda(\mathcal{B}) \subset \mathcal{R}_\Lambda(\mathcal{B})$.

Thus, $\Lambda \in \operatorname{Int}_{\ell^\infty} \mathcal{B}$ if and only if whenever $f \in \mathcal{B}$ and $\{\alpha_\lambda\} \in \ell^\infty$ there exists $g \in \mathcal{B}$ such that $g(\lambda) = \alpha_\lambda f(\lambda), \lambda \in \Lambda$. Since $\mathcal{B}$ is an algebra, this happens if and only if for any sequence of values $\{\alpha_\lambda\}_{\lambda \in \Lambda} \in \ell^\infty$ there exists $f \in \mathcal{B}$ with $f(\lambda) = \alpha_\lambda, \lambda \in \Lambda$.

There exists a general theory (and in particular an analytic description of interpolating sequences) for the algebras

$$A_p = \left\{ f \in \operatorname{Hol}(\mathbb{C}) : \exists C > 0 : \sup_{z \in \mathbb{C}} |f(z)| e^{-Cp(z)} < \infty \right\},$$

where $p : \mathbb{C} \rightarrow \mathbb{R}_+$ is a subharmonic weight such that:

- (a) $\log(1 + |z|^2) = O(p(z))$,
- (b) there are constants $C, D > 0$ such that whenever $|\zeta - z| \leq 1$ then $p(\zeta) \leq Cp(z) + D$.

For this we refer to [BrGa95, Chapter 2].

Notice that $\mathcal{B}$ does not fall into this general theory, since $p(z) = |\operatorname{Im} z|$ does not satisfy (a) (in particular, $\mathcal{B}$ does not contain polynomials). However, the techniques used in the description of A_p -interpolation mentioned above (see [BrLi95, Corollary 3.5]) can be adapted to obtain an analogous result for $\mathcal{B}$.

Theorem A. *The following statements are equivalent:*

- (a) $\Lambda \in \operatorname{Int} \mathcal{B}$
- (b) $\Lambda \in \operatorname{Int}_{\ell^\infty} \mathcal{B}$
- (c) *There exist $A, B, C > 0$ and peak-functions $f_\lambda \in \mathcal{B}$ such that $f_\lambda(\lambda') = \delta_{\lambda, \lambda'}$ and*

$$\sup_{z \in \mathbb{C}} |f_\lambda(z)| e^{-C|\operatorname{Im} z|} \leq A e^{B|\operatorname{Im} \lambda|}.$$

- (d) *There exist $F \in \mathcal{B}$ and $\varepsilon, C > 0$ such that $F(\lambda) = 0$ for all $\lambda \in \Lambda$ and*

$$|F'(\lambda)| \geq \varepsilon e^{-C|\operatorname{Im} \lambda|}.$$

As soon as the analytic conditions of Theorem A hold we can proceed as in [MOO06] and obtain geometric conditions. That paper deals with A_p -interpolation for weights of the form

$$p(z) = |\operatorname{Im} z| + \omega(|z|),$$

being $\omega(t)$ a subadditive increasing continuous function, normalized with $\omega(0) = 0$ such that $\log(1 + t) \lesssim \omega(t)$ and $\int_0^\infty \frac{\omega(t)}{1+t^2} dt < \infty$. Taking $p(z) = |\operatorname{Im} z|$ (i.e. $\omega \equiv 0$) and using Theorem A we can mimic the proof of [MOO06, Theorem 1] and obtain a geometric characterisation.

In order to state this result consider the counting function $n_\Lambda(z, r)$ defined previously and the integrated version

$$N_\Lambda(z, r) = \int_0^r \frac{n_\Lambda(z, t) - n_\Lambda(z, 0)}{t} dt + n_\Lambda(z, 0) \log r.$$

Hereinafter we assume that $\Lambda \cap \mathbb{R} = \emptyset$; otherwise replace $\mathbb{R}$ by any the horizontal line that does not contain any of the points in Λ .

Theorem B. *The sequence Λ is $\mathcal{B}$ -interpolation if and only if:*

- (i) *There is $C > 0$ such that*

$$N_\Lambda(\lambda, |\operatorname{Im} \lambda|) \leq C |\operatorname{Im} \lambda| \quad \forall \lambda \in \Lambda.$$

(ii) *The following Carleson-type condition holds*

$$\sup_{x \in \mathbb{R}} \sum_{\lambda \in \Lambda} \frac{|\operatorname{Im} \lambda|}{|x - \lambda|^2} < \infty.$$

Since the Poisson kernel at λ in the corresponding half-plane (upper half-plane if $\operatorname{Im} \lambda > 0$ and lower half-plane when $\operatorname{Im} \lambda < 0$) is $P(\lambda, x) = \frac{|\operatorname{Im} \lambda|}{|x - \lambda|^2}$, a restatement of condition (ii) is that the measure $\sum_{\lambda \in \Lambda} \delta_\lambda$ has bounded Poisson balayage.

The paper is structured as follows. In Section 2 we prove Theorem A and describe other analytic properties of $\mathcal{B}$ -interpolation sequences which are relevant in the proof of Theorem B. For the sake of completeness, we briefly recall the proof of Theorem B in Section 3.

A final remark about notation. C will always denote a positive constant and its actual value may change from one occurrence to the next. $A = O(B)$ and $A \lesssim B$ mean that $A \leq cB$ for some $c > 0$, and $A \simeq B$ is $A \lesssim B \lesssim A$.

2. PROOF OF THE ANALYTIC CONDITIONS

Before the proof of Theorem A we need to spell out some of its consequences.

Lemma 1. *Let Λ be a discrete sequence in $\mathbb{C}$ for which (c) in Theorem A holds. Then*

- (i) *There exist $\alpha, \varepsilon > 0$ such that the discs $D_\lambda := D(\lambda, \delta_\lambda)$, with $\delta_\lambda = \varepsilon e^{-\alpha|\operatorname{Im} \lambda|}$, are pairwise disjoint.*
- (ii) *There exist $C > 0$ such that for all $\beta > 2$*

$$\sup_{z \in \mathbb{C}} \sum_{\lambda \in \Lambda} \frac{e^{-C|\operatorname{Im} \lambda|}}{1 + |z - \lambda|^\beta} < \infty.$$

When Λ satisfies (i) we say that Λ is *weakly separated*.

Proof. (i) Fix $\lambda \in \Lambda$ and consider any λ' such that $|\lambda - \lambda'| \leq 1$. Then

$$1 = |f_\lambda(\lambda) - f_\lambda(\lambda')| \leq \sup_{\zeta \in D(\lambda, 1)} |f'_\lambda(\zeta)| |\lambda - \lambda'|.$$

By the Cauchy estimates

$$|f'_\lambda(\zeta)| \leq \sup_{\eta: |\eta - \zeta| = 1} |f_\lambda(\eta)| \leq \sup_{z \in D(\lambda, 2)} |f_\lambda(z)| \leq \sup_{z \in D(\lambda, 2)} A e^{-B|\operatorname{Im} \lambda|} e^{-C\alpha|\operatorname{Im} z|} \lesssim e^{-(B+C)|\operatorname{Im} \lambda|},$$

which yields the desired estimate.

(ii) This is an immediate consequence of (i): if $C \geq 2\alpha$ we have

$$\sum_{\lambda \in \Lambda} \frac{e^{-C|\operatorname{Im} \lambda|}}{1 + |z - \lambda|^\beta} \lesssim \sum_{\lambda \in \Lambda} \int_{D_\lambda} \frac{e^{-(C-2\alpha)|\operatorname{Im} \zeta|}}{1 + |z - \zeta|^\beta} dm(\zeta) \lesssim \int_{\mathbb{C}} \frac{dm(\zeta)}{1 + |z - \zeta|^\beta}.$$

■

Proof of Theorem A. (a) $\Rightarrow$ (b). Obvious.

(b) $\Rightarrow$ (c). Here we use the scheme of [BrGa95, Lemma 2.2.6.]. Consider the complete metric space $S = \{v = \{v_\lambda\}_\lambda : \|v\|_\infty \leq 1\}$, and for $n \in \mathbb{N}$ let:

$$S_n = \left\{ \{v_\lambda\}_\lambda \in S : \exists f \in \mathcal{B} : f(\lambda) = v_\lambda, \lambda \in \Lambda, \text{ and } \sup_{z \in \mathbb{C}} |f(z)| e^{-n|\operatorname{Im} z|} \leq n \right\}.$$

By hypotheses $S = \bigcup_n S_n$ and by Baire's Category Theorem there exists $n \in \mathbb{N}$ such that $\overset{\circ}{S}_n \neq \emptyset$. Since $0 \in S_n$ for all n , there exists $\varepsilon \in (0, 1)$ such that

$$\|v\|_\infty \leq \varepsilon \implies v \in S_n.$$

Let $v^\lambda = \{\varepsilon \delta_{\lambda, \lambda'}\}_{\lambda'}$. There exists then $g_\lambda \in \mathcal{B}$ with $g_\lambda(\lambda') = \varepsilon \delta_{\lambda, \lambda'}$ and $\sup_z |g_\lambda(z)| e^{-n|\operatorname{Im} z|} \leq n$. Taking $f_\lambda = g_\lambda/\varepsilon$ we get the desired properties.

(c) $\Rightarrow$ (d). Define $F(z) = \sum_{\lambda \in \Lambda} w_\lambda(z) g_\lambda(z)$, where

$$w_\lambda(z) = \left(\frac{\sin(z - \lambda)}{z - \lambda} \right)^3 e^{i\sigma_\lambda M(z - \lambda)}, \quad g_\lambda(z) = f_\lambda^2(z) \sin(z - \lambda),$$

$M > 0$ will be chosen later on and

$$\sigma_\lambda = \begin{cases} 1 & \text{if } \operatorname{Im} \lambda < 0 \\ -1 & \text{if } \operatorname{Im} \lambda > 0. \end{cases}$$

Notice that w_λ is a holomorphic weight in $\mathcal{B}$ with $w_\lambda(\lambda) = 1$, while g_λ is a function in $\mathcal{B}$ vanishing at order 1 on λ and at least at order 2 on all $\lambda' \neq \lambda$.

Let's see first that $F \in \mathcal{B}$. A direct estimate shows that

$$|w_\lambda(z)| \lesssim \frac{e^{(M+3)|\operatorname{Im} z|} e^{-(M-3)|\operatorname{Im} \lambda|}}{1 + |z - \lambda|^3},$$

$$|g_\lambda(z)| \lesssim e^{(2C+1)|\operatorname{Im} z|} e^{(2B+1)|\operatorname{Im} \lambda|},$$

and therefore

$$|F(z)| \lesssim e^{(M+2C+4)|\operatorname{Im} z|} \sum_{\lambda \in \Lambda} \frac{e^{-(M-2B-4)|\operatorname{Im} \lambda|}}{1 + |z - \lambda|^3}.$$

Taking M big enough and applying Lemma 1(ii) we see that $F \in \mathcal{B}$.

It is clear that $F(\lambda) = 0$ for all $\lambda \in \Lambda$. On the other hand

$$F'(\lambda) = \sum_{\lambda' \in \Lambda} w_{\lambda'}(\lambda) g_{\lambda'}'(\lambda) = w_\lambda(\lambda) g_\lambda'(\lambda) = 1.$$

(d) $\Rightarrow$ (a). Let $\{v_\lambda\}_{\lambda \in \Lambda}$ satisfy (1) and define

$$f(z) = \sum_{\lambda \in \Lambda} v_\lambda w_\lambda(z) \frac{F(z)}{F'(\lambda)(z - \lambda)},$$

where w_λ is defined as before.

It is clear that $F(\lambda) = v_\lambda$ for all $\lambda \in \Lambda$, and a similar computation to the one in the previous implication shows that $f \in \mathcal{B}$, since $|F(z)/(F'(\lambda)(z - \lambda))| \lesssim e^{C(|\operatorname{Im} z| + |\operatorname{Im} \lambda|)}$. $\blacksquare$

We finish this section by showing that a weakly separated union of two $\mathcal{B}$ -interpolation sequences is still $\mathcal{B}$ -interpolation.

Theorem 2. *Let $\Lambda_1, \Lambda_2 \in \operatorname{Int} \mathcal{B}$ be such that $\Lambda_1 \cup \Lambda_2$ is weakly separated. Then $\Lambda_1 \cup \Lambda_2 \in \operatorname{Int} \mathcal{B}$.*

For the proof of this result we will need the following lemma.

Lemma 3. *Let $\Lambda \in \operatorname{Int} \mathcal{B}$ and let $z_0 \notin \Lambda$. There exist $A, B, C > 0$ independent of z_0 and $f \in \mathcal{B}$ such that*

- (i) $f(z_0) = 1$ and $f(\lambda) = 0$ for all $\lambda \in \Lambda$.
- (ii) $\sup_{z \in \mathbb{C}} |f(z)| e^{-C|\operatorname{Im} z|} \leq B e^{-C|\operatorname{Im} z_0|} / d(z_0, \Lambda)$.

Proof. Assume first that $z_0 = 0$. Since Λ is weakly separated there exists $\delta > 0$ be such that $|\lambda - \lambda'| \geq 10\delta$ for all $\lambda, \lambda' \in \Lambda \cap \{|\operatorname{Im} z| < 1\}$.

Lemma 4. *There exists $F \in \mathcal{B}$ with $F(0) = 0$ and such that*

$$\begin{aligned} |F(\lambda)| &\geq \varepsilon && \text{if } |\lambda| \geq \delta \\ |F(\lambda)| &\geq \varepsilon|\lambda| && \text{if } |\lambda| < \delta. \end{aligned}$$

We defer the proof of this technical result to the end of the section.

Consider the values $v_\lambda = 1/F(\lambda)$. Since $\sup_{\lambda \in \Lambda} |v_\lambda| \leq 1/\min(\varepsilon, \varepsilon d(0, \Lambda))$, there exist $A, B > 0$ and $G \in \mathcal{B}$ such that $G(\lambda) = v_\lambda = 1/F(\lambda)$ and $\sup_{z \in \mathbb{C}} |G(z)| e^{-A|\operatorname{Im} z|} \leq B/(\varepsilon d(0, \Lambda))$.

The function $f(z) = 1 - F(z)G(z)$ satisfies then the required properties.

For $z_0 \notin \Lambda$ arbitrary apply the previous case to the sequence $\Lambda - z_0$. Notice that $\Lambda - z_0 \in \operatorname{Int} \mathcal{B}$, since there exist the peak-functions required by Theorem A(c): just take $f_{\lambda - z_0}(z) = f_\lambda(z + z_0)$, where f_λ are the peak-functions associated to Λ . $\blacksquare$

Proof of Theorem 2. Let us show that there exist peak-functions as in Theorem A(c).

Fix $\lambda_1 \in \Lambda_1$. Applying the previous Lemma to Λ_2 and $z_0 = \lambda_1$ we see that there exist $A, B > 0$ and $g_{\lambda_1} \in \mathcal{B}$ such that

$$\begin{aligned} g_{\lambda_1}(\lambda_1) &= 1, \quad \text{and} \quad g_{\lambda_1}(\lambda_2) = 0 \quad \forall \lambda_2 \in \Lambda_2 \\ \sup_{z \in \mathbb{C}} |g_{\lambda_1}(z)| e^{-A|\operatorname{Im} z|} &\leq B \frac{e^{C|\operatorname{Im} \lambda_1|}}{d(\lambda_1, \Lambda_2)} \leq B e^{C'|\operatorname{Im} \lambda_1|}. \end{aligned}$$

Since $\Lambda_1 \in \operatorname{Int} \mathcal{B}$ there exist a functions $f_{\lambda_1} \in \mathcal{B}$, as given by Theorem A(c). Then $h_{\lambda_1} = f_{\lambda_1} g_{\lambda_1}$ is a peak-function for $\Lambda_1 \cup \Lambda_2$. $\blacksquare$

Proof of Lemma 4. Consider

$$\sin(\pi z) = \pi z \prod_{k \in \mathbb{Z} \setminus \{0\}} \left(1 - \frac{z}{k}\right).$$

If $|\lambda - k| \geq \delta$ for all $\lambda \in \Lambda$ and $k \in \mathbb{Z}$, we can just take $F(z) = \sin(\pi z)$.

For each $k \in \mathbb{N}$ such that there this $\lambda \in \Lambda$ with $|\lambda - k| < \delta$ or $|\lambda + k| < \delta$ we replace in $\sin(\pi z)$ the factors $1 - z/k$, $1 + z/k$ by small perturbations $1 - z/p_k$, $1 - z/p_{-k}$, where $p_k = k + i\epsilon_k$ and $\epsilon_k > 0$ is chosen so that

- (a) $|\epsilon_k| \leq 5\delta$
- (b) $\epsilon_{-k} = \epsilon_k$
- (c) $|p_k - \lambda| \geq \delta$ and $|p_{-k} - \lambda| \geq \delta$ for all $\lambda \in \Lambda$.

Denoting by $\mathcal{I}$ the indices of $\mathbb{Z} \setminus \{0\}$ affected by this modification, we define

$$F(z) = \sin(\pi z) \prod_{k \in \mathcal{I}} \frac{1 - z/p_k}{1 - z/k}.$$

Notice that the zeros of this new sine-type function are at a distance of Λ no smaller than δ .

In order to see that $F \in \mathcal{B}$ is enough to show that the latter product is bounded above and below away from the points k and p_k . To do so we group the factors corresponding to k and $-k$. Since $p_{-k} = -\overline{p_k}$ we have

$$\begin{aligned} \prod_{k \in \mathcal{I}} \frac{1 - z/p_k}{1 - z/k} &= \prod_{k \in \mathcal{I}} \frac{z - p_k}{z - k} \frac{k}{p_k} = \prod_{k \in \mathcal{I} \cap \mathbb{N}} \frac{(z - p_k)(z - p_{-k})}{(z - k)(z + k)} \frac{-k^2}{p_k p_{-k}} \\ &= \prod_{k \in \mathcal{I} \cap \mathbb{N}} \left(1 - \frac{\epsilon_k(\epsilon_k + 2iz)}{z^2 - k^2}\right) \frac{1}{1 + \epsilon_k^2/k^2}, \end{aligned}$$

from which the required properties follow easily. ■

3. PROOF OF THE GEOMETRIC CONDITIONS

In this section we briefly recall the proof of Theorem B.

3.1. Necessity. The necessity of (i) is a standard (and straightforward) application of Jensen's formula to the functions f_λ provided by Theorem A(c) on the disks $D(\lambda, |\operatorname{Im} \lambda|)$.

To see that (ii) is also necessary consider the functions $g_\lambda(z) = f_\lambda(z)e^{iCz}$, which according to Theorem A(c) are bounded in the upper half-plane $\mathbb{H}$. Applying Jensen's formula to g_λ in $\mathbb{H}$ we have

$$\sum_{\lambda' \neq \lambda: \operatorname{Im} \lambda' > 0} \log \left| \frac{\lambda - \lambda'}{\lambda - \overline{\lambda'}} \right|^{-1} \lesssim \operatorname{Im} \lambda \quad \text{for all } \lambda \in \Lambda \cap \mathbb{H}.$$

The estimate $1 - t \leq \log t^{-1}$ for $t \in (0, 1)$ yields then

$$\sup_{\lambda \in \Lambda \cap \mathbb{H}} \sum_{\lambda' \neq \lambda: \operatorname{Im} \lambda' > 0} \frac{\operatorname{Im} \lambda'}{|\lambda - \lambda'|^2} < +\infty.$$

Given $x \in \mathbb{R}$, consider the $\lambda \in \Lambda \cap \mathbb{H}$ closest to x . Then $\frac{\operatorname{Im} \lambda'}{|x - \lambda'|^2} \leq 2 \frac{\operatorname{Im} \lambda'}{|\lambda - \lambda'|^2}$. This and the corresponding computation for the lower half-plane imply (ii).

3.2. Sufficiency. According to Theorem 2 it will be enough to show that $\Lambda_+ = \Lambda \cap \{\operatorname{Im} z > 0\}$ and $\Lambda_- = \Lambda \cap \{\operatorname{Im} z < 0\}$ are $\mathcal{B}$ -interpolation.

By Theorem A(d), in order to prove that Λ_+ is $\mathcal{B}$ -interpolation it is enough to construct a function $F \in \mathcal{B}$ such that $\Lambda_+ \subset \mathcal{Z}(F)$ and

$$|F'(\lambda)| \geq \varepsilon e^{-K \operatorname{Im} \lambda} \quad \lambda \in \Lambda_+$$

for some constants $\varepsilon, k > 0$.

Start with any entire function G with zero set $\mathcal{Z}(G) = \Lambda_+$. Condition (ii) implies that Λ_+ is a Blaschke sequence in $\mathbb{H}$, i.e. the Blaschke product

$$B(z) = \prod_{\lambda \in \Lambda_+} \frac{z - \lambda}{z - \bar{\lambda}}, \quad z \in \mathbb{H}$$

converges. Define

$$\Psi(z) = N |\operatorname{Im} z| - \phi(z),$$

where

$$\phi(z) = \begin{cases} \log \left| \frac{G(z)}{B(z)} \right| & \operatorname{Im} z > 0 \\ \log |G(z)| & \operatorname{Im} z \leq 0. \end{cases}$$

A computation shows that ϕ is harmonic outside the real axis, subharmonic on $\mathbb{C}$ and $\Delta \phi(x) = \sum_{\lambda \in \Lambda_+} \frac{\operatorname{Im} \lambda}{|x - \lambda|^2} dx$ (see [MOO06, Lemma 10]). Thus,

by condition (ii), $\Delta \Psi \simeq dx$ when $N \in \mathbb{N}$ is big enough. In this situation, according to [OS99, Lemma 3], there exists a multiplier associated to Ψ , i.e., an entire function h such that:

- (a) $\mathcal{Z}(h)$ is a separated sequence contained in $\mathbb{R}$ and separated from Λ (i.e. $\inf_{z \in \mathcal{Z}(h)} d(z, \Lambda) > 0$).
- (b) Given any $\varepsilon > 0$, $|h(z)| \simeq \exp(\Psi(z))$ for all points z such that $d(z, \mathcal{Z}(h)) > \varepsilon$.

Define now $F = hG$. It is clear that $F \in \mathcal{B}$:

$$|F(z)| \lesssim e^{\Psi(z) + \log |G(z)|} \leq e^{\Psi(z) + \phi(z)} \leq e^{Np(z)} \quad z \in \mathbb{C}.$$

It is also clear that $\Lambda_+ \subset \mathcal{Z}(F)$, since $\Lambda_+ \subset \mathcal{Z}(G)$.

In order to prove that there exist $\varepsilon, C > 0$ such that

$$(2) \quad |F'(\lambda)| \geq \varepsilon e^{-C \operatorname{Im} \lambda}$$

consider the disjoint disks $D_\lambda = D(\lambda, \delta_\lambda)$, $\delta_\lambda = \delta e^{-C \frac{p(\lambda)}{m_\lambda}}$ provided by Lemma 1(i). Since Λ_+ is far from $\mathcal{Z}(h)$, the estimate

$$|G(z)| = |h(z)| e^{\phi(z)} |B(z)| \simeq e^{N |\operatorname{Im} z|} |B(z)| \quad z \in \partial D_\lambda$$

holds.

From here we finish as in [MOO06]: since the hypotheses imply that there exists $C > 0$ such that $|B(z)| \geq \epsilon e^{-C |\operatorname{Im} z|}$, $z \in \partial D_\lambda$, we have $|F(z)| \gtrsim e^{-C p(z)}$ for all $z \in \partial D_\lambda$. The function $g(z) = F(z)/(z - \lambda)$ is then holomorphic, non-vanishing in D_λ , and with $|g(z)| \gtrsim e^{-c \operatorname{Im} \lambda}$ for $z \in \partial D_\lambda$. By the minimum principle

$$|F'(\lambda)| = |g(0)| \gtrsim e^{-c \operatorname{Im} \lambda},$$

as desired.

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DEPARTAMENT DE MATEMÀTICA APLICADA I ANÀLISI, UNIVERSITAT DE BARCELONA, GRAN VIA 585, 08071-BARCELONA, SPAIN

E-mail address: xavier.massaneda@ub.edu

DEPARTAMENT DE MATEMÀTICA APLICADA I ANÀLISI, UNIVERSITAT DE BARCELONA, GRAN VIA 585, 08071-BARCELONA, SPAIN

E-mail address: jortega@ub.edu