Direct observation of field quantization

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Abstract: We have studied a recent experiment reported in [1] which manifests the discrete nature of the electromagnetic field inside a cavity. There, it is observed the coupling of atoms to the e.m. field. We have reproduced the calculations leading to the observation of discrete transition frequencies between two states of the atom. These frequencies are proportional to the square root of successive integers, which is a direct evidence of field quantization. Also, we have discussed the validity of the approximations that we have made.

I. INTRODUCTION

We have studied the probability transitions of a two level atom placed in a cavity enclosed between two mirrors separated by a distance such that the fundamental cavity mode is resonant with the atom energy gap.

The two atom states $|a\rangle$ and $|b\rangle$ are adjacent circular Rydberg states with principal quantum numbers $n = 50$ and $n = 51$ of Rubidium atoms corresponding to a transition frequency $\omega_{ba}/2\pi = 51.099$GHz. In these atoms, the valence electron is confined near the classical Bohr orbit which, due to the high principal quantum number, is far distant from the nucleus ($\langle r\rangle \sim 10^3$Å). Moreover, in such states the atoms have a long radiative lifetimes ($32ms$ and $30ms$ for $|b\rangle$ and $|a\rangle$ respectively), which make atomic relaxation negligible [5].

Based on a quantum treatment of both the atom and the electromagnetic field in the cavity, we have reproduced these transitions which have been observed in the experiment reported in [1].

In the experiment, the Rubidium atoms cross the cavity made of two niobium superconducting mirrors whose lower frequency mode is tuned into resonance with the $|b\rangle$ to $|a\rangle$ transition frequency. Thus, the joint atom-cavity system undergoes Rabi oscillations as well as revival patterns depending on the initial state of the field: With a given number of photons or with a small coherent field [4].

The oscillation frequencies, which are directly related with the number of photons of the field, not only uncovers the discrete nature of the radiation stored in the cavity, but also gives us a method to discern the mean number of photons of an electromagnetic field.

Thus, the article is organized as follows: at first, we have given a quantum description of the contributing mode of the electromagnetic field in the cavity. Afterwards, we have analyzed the Hamiltonian of the atom-cavity system by introducing further approximations. Finally, we have obtained the transition probabilities between the states $|b\rangle$ and $|a\rangle$ as a function of time with different states of the electromagnetic field, contrasting our calculations with the experimental results obtained in [1].

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II. QUANTIZED RADIATION FIELD

We will consider a cavity with a height L and area S, with mirrors in the planes $z = 0$ and $z = L$ (FIG:1) in which we can contain different modes of a electromagnetic field.

In the Coulomb transverse gauge, and in c.g.s. units, the expression of the contributing modes of the field can be written as

$$\vec{A}(z) = \sum_k \hat{\epsilon}\sqrt{\frac{4\pi c^2\hbar}{2\omega V}}(a_k + a_k^\dagger)\sin(kz) \qquad (1)$$

$$\text{with} \qquad k = \frac{2\pi}{L}m \quad ; \quad m = 0, 1, 2\ldots$$

where $\omega = ck$; a, $a^\dagger$ are the annihilation and creation operators for each mode that satisfy the relation $[a, a^\dagger] = 1$; and $\hat{\epsilon}$ is the polarization vector perpendicular to the vector $\vec{k} = (0, 0, k)$ [2]. Also, we have chosen $\hat{\epsilon}$ as the vector $\hat{i} = (1, 0, 0)$.

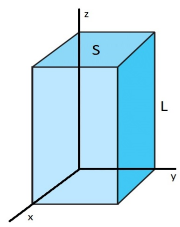

FIG. 1: Simplified scheme of the cavity; the mirrors are contained in $z = 0$ and $z = L$ planes.

Moreover, in our case, there is only one contributing mode of the field $k = \omega_{ba}/c$.

The expression of the radiation Hamiltonian, henceforth H_{rad}, once subtracted the zero point energy, can be written as

$$H_{rad} = \hbar\omega\ a^{\dagger}a = \hbar\omega\hat{N}. \qquad (2)$$

Notice that the n photon states $|n\rangle$ are eigenstates of H_{rad} with eigenvalues $n\hbar\omega$

III. COUPLING OF THE FIELD WITH AN ATOM

Following with the experiment mentioned in [1] we will consider Rubidium atoms. These atoms are Alkali atoms with only one valence electron. Thus, the Hamiltonian that describes the complete atom radiation system with one valence electron is

$$H_T = H_{rad} + \frac{1}{2m_e}(\hat{P} - \frac{q}{c}\hat{A})^2 + \hat{V}_{eff}(r) + H_{Pauli} \qquad (3)$$

where $\hat{H}_{Pauli} = g\mu_B\frac{\hat{S}\cdot\hat{B}}{\hbar}$ is the Spin interaction with the magnetic field where μ_B is the Bohr magneton and $g \approx 2$ is the gyromagnetic constant; and $V_{eff}(r)$ is the potential felt by the valence electron.

At this point, we make two further approximations: On the one hand, we only keep linear terms of the electric charge in the atom radiation interaction; on the other hand the Pauli term, which is proportional to the photon momentum ($\sim \hbar/\lambda$), is much smaller than $\frac{q}{mc}\hat{A}\cdot\hat{P}$ because the electron momentum in the atom is proportional to $\hbar/\langle r\rangle$ where $\langle r\rangle$ is the mean radius of the electron orbits. For the transitions that we shall consider $\langle r\rangle/\lambda \sim 10^{-5}$.

Thus, the Hamiltonian reads as

$$H_T = H_{rad} + H_{at} - \frac{q}{mc}\hat{A}\cdot\hat{P} \qquad (4)$$

where $H_{at} = \frac{\hat{P}^2}{2m} + V_{eff}(r)$.

The experiment is performed with Rubidium atoms and deals with transitions between adjacent Rydberg states with principal quantum numbers $n = 50$ and $n = 51$, $|a\rangle$ and $|b\rangle$ respectively. The cavity is such that its lower frequency mode is tuned into resonance with these two states, which selects the contributing mode of the electromagnetic field. Thus, the set of orthonormal states $\{|a,n\rangle,|b,n\rangle\}$ where $|a,n\rangle \equiv |a\rangle\otimes|n\rangle$ and $|b,n\rangle \equiv |b\rangle\otimes|n\rangle$ forms a basis of the Hilbert space of the entire system. Here $|n\rangle$ denotes the field state with n photons.

As (4) suggests, the non-vanishing terms of the Hamiltonian matrix differ by one photon. This is because the expression of $\hat{A}$ in (1) allows transitions from $|n\rangle$ to $|n+1\rangle$ and $|n-1\rangle$ corresponding to photon emission and absorption processes respectively. In order to evaluate them, we will introduce the electric dipole approximation which consists of freezing the value of z to the position of the atomic nucleus z_0. The corrections to this are powers of $k\langle r\rangle/ \sim 10^{-5}$ that we neglect. Finally, with the help of

$$\hat{P} = \frac{im}{\hbar}[H_{at}, \hat{X}]$$

we find:

Emission:

$$\langle i, n+1| - \frac{q}{mc}\hat{P}\cdot\hat{A}|j,n\rangle = iC(\omega)\sqrt{n+1}\,\omega_{ij}X_{ij}$$

Absorption:

$$\langle i, n-1| - \frac{q}{mc}\hat{P}\cdot\hat{A}|j,n\rangle = iC(\omega)\sqrt{n}\,\omega_{ij}X_{ij}$$

Where $C(\omega) = -q\sin(kz_0)\sqrt{\frac{4\pi\hbar}{2\omega V}}$, $X_{ij} = \langle i|\hat{X}|j\rangle$ and $\omega_{ij} = \frac{E_i - E_j}{\hbar}$. The indices i,j run in $\{a,b\}$

Finally, using the above expressions and setting the zero point energy in $E_a = 0$ the Hamiltonian matrix is

$$\begin{pmatrix} 0 & 0 & 0 & i\gamma X_{ab} & 0 & \cdots \\ 0 & \hbar\omega & i\gamma X_{ba} & 0 & 0 & \cdots \\ 0 & -i\gamma X_{ab} & \hbar\omega & 0 & 0 & \cdots \\ -i\gamma X_{ba} & 0 & 0 & 2\hbar\omega & i\sqrt{2}\gamma X_{ba} & \cdots \\ 0 & 0 & 0 & -i\sqrt{2}\gamma X_{ab} & 2\hbar\omega & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

In this expression, the states of the basis are ordered as follows: $\{|a,0\rangle; |b,0\rangle, |a,1\rangle; |b,1\rangle, |a,2\rangle; \ldots\}$. Also, $\gamma = C(\omega)\omega_{ba}$. Notice that $X_{ab} = X_{ba}^*$ and $|X_{ab}| \sim \langle r\rangle \sim 10^3$Å for our Rydberg states.

As we can see, H_T is block-diagonal except for a group of terms corresponding to the photon absorption from $|b\rangle$ to $|a\rangle$ and the photon emission from $|a\rangle$ to $|b\rangle$. These terms, which violate energy conservation, and the processes they give rise to are very much suppressed ($|\gamma X_{ab}|/\hbar\omega \sim 10^{-6}$). Thus, we will neglect them (This approximation is explicitly justified in the appendix).

By doing this, the matrix is completely 2×2 block-diagonal which involve degenerate state doublets $\{|b,n\rangle, |a,n+1\rangle\}$ with energy $(n+1)\hbar\omega_{ba}$. Notice that, within the resonant approximation, $\omega_{ba} = \omega$. Thus, our system is a set of 2-dimensional subspaces spanned by the state doublets such that

$$\begin{pmatrix} \hbar\omega(n+1) & \sqrt{n+1}\gamma X_{ab} \\ \sqrt{n+1}\gamma X_{ba} & \hbar\omega(n+1) \end{pmatrix} \qquad (5)$$

which can be readily diagonalized with the diagonal basis

$$|\phi_n^+\rangle = \tfrac{1}{\sqrt{2}}(|b,n\rangle + i|a,n+1\rangle)$$

$$|\phi_n^-\rangle = \tfrac{1}{\sqrt{2}}(|b,n\rangle - i|a,n+1\rangle)$$

with energies $E_{n\pm} = (n+1)\hbar\omega \pm \gamma|X_{ab}|\sqrt{n+1}$. Moreover, the degeneracy is lifted by the interaction.(FIG:2)

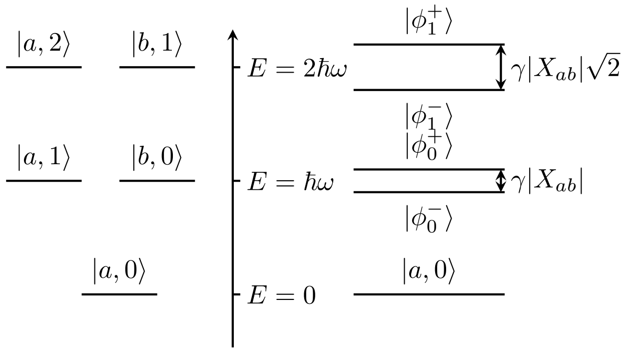

FIG. 2: Energy levels of (4). On the left hand side, the degenerate doublets are shown; On the right hand side, we represent the effect of the interaction which splits the energy of these doublets.

IV. TRANSITION PROBABILITIES

Now we are going to find the transition probabilities $|b\rangle \to |a\rangle$ as a function of time in the following cases: starting from $|b,0\rangle$ (the electromagnetic field in the vacuum); $|b,n\rangle$ (with a n photons state) and $|b,\alpha\rangle$ (a small quantum coherent field)

A. Interaction with the vacuum $|0\rangle$

The initial wave function is $|\psi(0)\rangle = |b,0\rangle$, which, in terms of the diagonal basis is

$$|b,0\rangle = \frac{1}{\sqrt{2}}(|\phi_0^+\rangle + |\phi_0^-\rangle)$$

And its time evolution is

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}e^{-i\omega t}\left(\, e^{-i\Omega t}|\phi_0^+\rangle + e^{i\Omega t}|\phi_0^-\rangle \,\right)$$

Where we have defined $\Omega \equiv |X_{ab}|\gamma/\hbar$. Rewritten in the original basis it reads

$$|\psi(t)\rangle = e^{-i\omega t}\left(\, -i\sin(\Omega t)|a,1\rangle + \cos(\Omega t)|b,0\rangle \,\right)$$

Finally, the transition probability to the state $|a,1\rangle$ is

$$\begin{aligned} P(t)_{|b,0\rangle\to|a,1\rangle} &= |\langle a,1|\psi(t)\rangle|^2 = \sin^2(\Omega t) = \\ &= \tfrac{1}{2} - \tfrac{1}{2}\cos(2\Omega t) \end{aligned} \tag{6}$$

This is a typical oscillatory behavior, characteristic of transitions in a two level system, with frequency 2Ω.

B. Interaction with an eigenstate of the field $|n\rangle$

In this case the initial wave function is $|b,n\rangle$, and proceding as before, we obtain

$$\begin{aligned} |\psi(t)\rangle = e^{-i\omega t}(-i\sin(\Omega\sqrt{n+1}\;t)|a,n+1\rangle + \\ + \cos(\Omega\sqrt{n+1}\;t)|b,n\rangle) \end{aligned}$$

as well as the transition probability $P_{|b,n\rangle\to|a,n+1\rangle} =$

$$\begin{aligned} &= |\langle a,n+1|\psi(t)\rangle|^2 = \sin^2(\Omega\sqrt{n+1}\;t) = \\ &= \tfrac{1}{2} - \tfrac{1}{2}\cos(2\Omega\sqrt{n+1}\;t). \end{aligned} \tag{7}$$

Again, the same oscillatory behaviour is observed as in (6) which manifests the two level nature of the coupled doublet in (5). These are the so called Rabi oscillations of the joint atom-cavity system between $|b,n\rangle$ and $|a,n+1\rangle$.

It is interesting to note that the state of the full system $|\psi(t)\rangle$ is a superposition of states with different number of photons, which has no classical analogue whatsoever. Moreover, notice that the oscillation frequencies are discrete and depend on the number of photons, proportionally to $\sqrt{n+1}$.

It is worth pointing out that the symmetry of the matrix in (5) implies that the transition from $|a,n+1\rangle$ to $|b,n\rangle$ occurs with the same frequency.

C. Interaction with a coherent state

Finally let us consider the transitions from a coherent state $|b,\alpha\rangle$ which is the closest approximation of a classical monochromatic plane wave, which can be reproduced in the laboratory. We recall that $|\alpha\rangle$ satisfies $a|\alpha\rangle = \alpha|\alpha\rangle$ from which

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

where α is a complex number, the modulus squared of which is the photon number expectation value $|\alpha|^2 = \langle\hat{N}\rangle \equiv n_0$. Furthermore the uncertainty of the photon number turns out to be $\sigma_N = |\alpha|$

Thus, proceeding as before we find

$$|\psi(t)\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n e^{-i\omega t}}{\sqrt{n!}}\times$$

$$\times\big[\, -i\sin(\Omega\sqrt{n+1}\;t)|a,n+1\rangle + \cos(|\Omega\sqrt{n+1}\;t)|b,n\rangle \,\big]$$

Since we are looking for the probability of ending in state $|a\rangle$ irrespectively of the number of photon, we sum up the probabilities for any possible final number of photons.

$$P_\alpha \equiv P_{b,\alpha\to a} = \sum_{k=1}^{\infty} |\langle a,k|\psi(t)\rangle|^2 = \\ = \sum_{k=1}^{\infty} \frac{e^{-|\alpha^2|}|\alpha^k|^2}{2k!}(1-\cos(2\Omega\sqrt{k+1}t)) \tag{8}$$

Unfortunately, the analytical summation is not available. Even so, we have found numerically the result shown in FIG:3. We have extended the summation up to the contribution which amounts the 0.01% of the dominant term $k \approx |\alpha|$

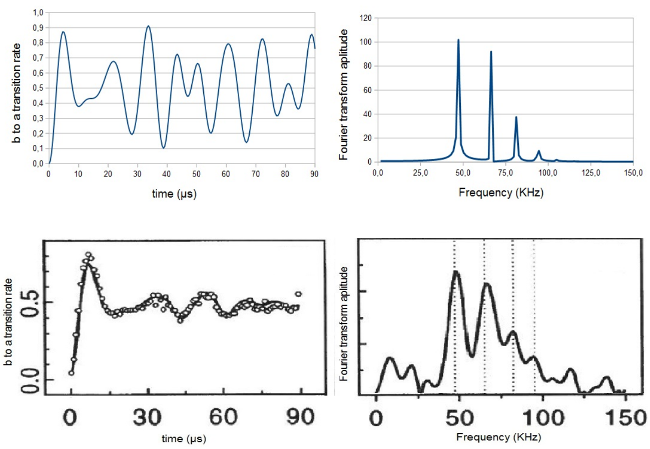

FIG. 3: Transition probability and its Fourier transform for a coherent field calculated in (8)(*above*) and obtained experimentally in [1](*below*) with $|\alpha|^2 = 0.85$. The Frequency peaks are observed in 47.3KHz, 66.4KHz, 88.2KHz and 94.5KHz, corresponding to the first four photon states.

If $|\alpha|^2 \geq O(1)$ the pattern observed is that of oscillations that die off and resurrect after a certain time interval during which the transition probability remains essentially flat. Moreover, this revival phenomenon evidences the quantum nature of the field in a cavity [6]. Also, the revival time grows with the mean number of photons n_0, disappearing in the $n_0 >> 1$ limit as we shall show in **D**.

Notice that the Fourier transform of the probability evidences the discreteness of the oscillation frequencies which uncovers the selection of electromagnetic modes obtained in the cavity. These allowed frequencies are $\Omega\sqrt{n+1}/\pi$. And only the ones centered around $n = |\alpha|^2$ have large Fourier components and significant contribution. In this case $\Omega/\pi = 47$KHz ($\omega_{ab}/2\pi = 5\cdot 10^7$KHz; $|X_{ab}| = 10^{-7}$m and $V_{cavity} = 1.87\cdot 10^{-6}\text{m}^3$ [1][4]).

Moreover, a damping of the oscillation is observed experimentally, which is due to experimental imperfections [1].

D. Large number of photons approximation

When $|\alpha|^2 >> 1$ the series in (8) can be approximated by an integral. To this end, one can use the Gaussian limit of the Poisson distribution in (8) by using the Stirling formula and expanding in the vicinity of $n = n_0$, the average photon mumber. That is

$$e^{-|\alpha|^2}\frac{|\alpha|^2 n}{n!} \approx \frac{1}{\sqrt{2\pi n_0}}\exp\left(-\frac{(n-n_0)^2}{2n_0}\right)$$

Also, by linearizing the dependence of $\Omega_n \equiv \Omega\sqrt{n+1}$ in the vicinity of n_0 as

$$\Omega_n = \Omega_{n_0} + \Omega_0\frac{n-n_0}{2\sqrt{n_0+1}}$$

one finds

$$P_\alpha = \frac{1}{2} - \frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi n_0}}\exp\left(-\frac{(n-n_0)^2}{2n_0}\right)\cos(2\beta t)dn$$

$$\beta = \Omega_{n_0} + \Omega_0\frac{n-n_0}{2\sqrt{n_0+1}}$$

where the lower limit of the integral has been extended to negative values of n with negligible error, given that $|\alpha|^2 >> 1$. Thus, the result of the integral is:

$$P_\alpha = \frac{1}{2} - \frac{1}{2}\cos(\Omega_{n_0}t)\exp\left(-\frac{\Omega_0^2 t^2 n_0}{8(n_0+1)}\right) \tag{9}$$

In this case, P_α does not exhibit the revivals we found in the previous section, FIG:3, as can be seen in FIG:4.

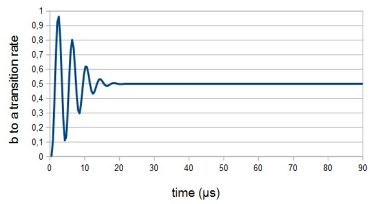

FIG. 4: Transition probability in the approximation $|\alpha|^2 >> 1$; in this case $|\alpha|^2 = 10$.

V. CONCLUSIONS

We have studied the transition probability between Rydberg states of rubidium atoms that travel across a cavity with an electromagnetic field in the context of quantum field theory. The theoretical results, contrasted with the experimental results obtained in [1], show Rabi oscillations of the transition probability the frequencies of which are directly related to the number of photons of the field; this fact gives us a method to evaluate the number of photons for extremely small fields, a highly difficult task.

Moreover, the transition probability patterns show revival features when the atom interacts with a coherent electromagnetic field with a small mean number of photons $|\alpha|^2 \sim O(1)$. These revivals are observed experimentally, and cannot be explained by a classical treatment of the field. This fact not only gives validity to the studied model, but also compels to a quantum interpretation of the atom-cavity system.

VI. APPENDIX

A. 3-state toy model

The analytical solutions we have obtained have been found under the approximation of retaining the Hamiltonian matrix elements between states of equal energy $(H_{rad} + H_{at})$ only. This provides the main contribution to these results. In order to justify this, we propose the following toy model with three levels, two of which are energy degenerate. In a basis $\{|a\rangle, |b_1\rangle, |b_2\rangle\}$:

$$H = \begin{pmatrix} E_a & & \\ & E_b & \\ & & E_b \end{pmatrix} + \hbar \begin{pmatrix} 0 & \gamma & \gamma \\ \gamma & 0 & \gamma' \\ \gamma & \gamma' & 0 \end{pmatrix}$$

According to the approximation above mentioned, we neglect the matrix elements $\hbar\gamma$ between $|a\rangle$ and $|b_1\rangle$,$|b_2\rangle$

$$H_{approx} = \begin{pmatrix} E_a & 0 & 0 \\ 0 & E_b & \hbar\gamma' \\ 0 & \hbar\gamma' & E_b \end{pmatrix}$$

Thus, under this approximation we find:

$$P^{(approx)}_{b_1\to b_2}(t) = \sin^2(\gamma' t); \quad P^{(approx)}_{b_1\to a}(t) = 0 \tag{10}$$

The *exact* results are:

$$P_{b_1\to a}(t) = \frac{\gamma^2}{\Delta^2 + 2\gamma^2} \sin^2(\sqrt{\delta^2 + 2\gamma^2}\; t) \tag{11}$$

where $\Delta = (\omega_{ba} + \gamma')/2$. If $\gamma << \omega_{ba}$ this probability remains very small at any time giving validity to our approximation.

Also

$$\begin{aligned} P_{b_1\to b_2}(t) = \sin^2\left(\tfrac{\Omega-\Omega_A}{2}t\right) - \\ -2\left(1-\tfrac{\Delta}{\Omega}\right)\sin(\Omega t)\cos\left(\tfrac{\Omega+\Omega_A}{2}t\right)\sin\left(\tfrac{\Omega-\Omega_A}{2}t\right) + \\ + \left(1-\tfrac{\Delta}{\Omega}\right)^2 \sin^2(\Omega t) \end{aligned} \tag{12}$$

where $\Omega = \sqrt{\Delta^2 + 2\gamma^2}$ and $\Omega_A = \Delta - 2\gamma'$.

In the limit $\gamma = 0$, (12) and (11) reduce to (10). Notice that $\dfrac{\Omega - \Omega_A}{2} = \gamma' + O(\dfrac{\gamma^2,\ \gamma'^2}{\omega^2_{ba}})$ and $\left(1 - \frac{\Delta}{\Omega}\right) = O\left(\frac{\gamma}{\omega_{ba}}\right)^2$ so that the (10) expressions are excellent approximations especially for not too large times (FIG:5).

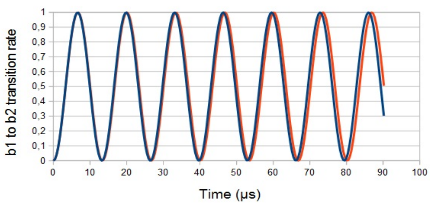

FIG. 5: Transition probability $P_{b_1\to b_2}$ calculated via (12)(*red*) and via the approximation (10)(*blue*). In this case, to illustrate the validity of the approximation in (10) we have chosen $\gamma/2\pi = 23.5$KHz and $\gamma/\omega_{ab} = 10^{-2} >> |\gamma X_{ab}|/\hbar\omega_{ba}$.

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