Fractional Quantum Hall Effect with Cold Atoms

Author: Juan Miguel Escalante Charlez
Facultat de Física, Universitat de Barcelona, Diagonal 645, 08028 Barcelona, Spain.

Advisor: Nuria Barberán

Abstract: This work presents an introduction to the simulation of the Fractional Quantum Hall Effect with cold neutral atoms. In order to do this we perform the study of the energy spectrum of a neutral particle moving in a two dimensional square lattice. First, we analyse the Hamiltonian in absence of magnetic field. Then, we observe how the Hamiltonian changes undergoing artificial and uniform magnetic field, the so-called Hofstadter model. Finally, we reach the famous energy spectrum given by Hofstadter butterfly[1].

I. INTRODUCTION

The quantum hall effects are observed in two dimensional systems subjected to low temperatures and high magnetics fields, where the Hall resistivity is quantized. The Integer Quantum Hall Effect was discovered in 1982 by G. Dorda and M. Pepper which is quite well understood. Two years later in 1982, the Fractional Quantum Hall Effect was discovered by D.C Tsui, H.L. Stomer and A.C. Gossard.

Nowadays for electrons, large amount of information exist in both regimes Integer Quantum Hall and Fractional Quantum Hall but not so much for atoms. Besides, there is no performed experiment for atoms in the strong magnetic field regime.

The protocol to reach the simulation follows the next steps. First, one needs to solve the periodic Hamiltonian without magnetic field. Once solved, an uniform magnetic field has to be simulated in an effective way, perpendicular to the motion plane. Furthermore, one needs to introduce an artificial electric field and obtain the evolution of the solutions which will lead to the expected values of the Hall current and finally deduce the expression of the conductivity. Within this root, we perform the starting part in which we obtain the spectrum of an atom in a square lattice under an artificial magnetic field.

We will start by introducing a fundamental concept of magnetism, the Magnetic Translation. Straightaway, we will talk about the importance of Artificial Gauge Field owing to the charge neutrality of the cold atoms and how to overcome this limitation designing systems which describe the same properties of charged particle in a magnetic field. Then, we will introduce the concept known as gauge invariance. At the end, we will study the energy spectrums for a particle moving in a two dimensional square lattice in the tight-binding approximation. With and without magnetic field.

Understanding the behaviour of atoms in optical lattices is one of the most powerful ways to reproduce the Fractional Quantum Hall Effect.

II. THEORETICAL BACKGROUND

For a free particle with mass M and charge q in a presence of a magnetic field we can write the Hamiltonian

$$\hat{H} = \frac{(\hat{\mathbf{p}} - q\mathbf{A}(\hat{\mathbf{r}}))^2}{2M}, \quad q < 0 \tag{1}$$

where $\hat{\mathbf{p}}$ is the canonical momentum, $\mathbf{A}(\hat{\mathbf{r}})$ the vector potential associated to magnetic field $\mathbf{B}$ and the whole expression in the numerator is the square of the kinetic momentum of the particle $\hat{\Pi} = (\hat{\mathbf{p}} - q\mathbf{A}(\hat{\mathbf{r}}))$.

Formally, the same Hamiltonian can be obtained inducing an artificial magnetic field and appropriately defining a vector potential $\mathbf{A}$ over neutral atoms.

A. Magnetic Translation

In the absence of magnetic field the operator $\hat{\mathbf{p}}$ acts as generator of translations. But if we add an uniform magnetic field along $\hat{z}$, the system must remain translationally invariant and the operator that plays the role of being the new generator of translations[2] which commute with the Hamiltonian is given by

$$\mathbf{K} = \mathbf{p} - e\mathbf{A} + e\mathbf{B} \times \mathbf{r} = \mathbf{p} + e\mathbf{A} \tag{2}$$

considering the symmetric gauge, $\mathbf{A} = B/2(-y, x, 0)$ and $\mathbf{B} = B\mathbf{u}_z$. The translational operator of distance $\mathbf{x}$ can be written as

$$\hat{t}(\mathbf{x}) = e^{-i\mathbf{x}\cdot\mathbf{K}/\hbar} = e^{-i\mathbf{p}\cdot\mathbf{x}/\hbar} \cdot e^{-i\mathbf{x}\cdot\frac{e}{\hbar}\mathbf{A}} \tag{3}$$

bearing in mind that $[\mathbf{p}, \mathbf{A}]$=0. One can notice, by applying equation (3) over the wave function, that the displacement involves a phase shift

$$e^{i\mathbf{x}\cdot\mathbf{K}/\hbar} f(\mathbf{r}) = e^{ie\mathbf{x}\cdot\mathbf{A}(\mathbf{r})/\hbar} f(\mathbf{r} + \mathbf{x}). \tag{4}$$

If translation is in straight line the argument in the exponential is a scalar product. Instead in bended trajectories it becomes an integral[3]

$$\phi = \frac{e}{\hbar} \oint \mathbf{A}(\mathbf{r})\, d\mathbf{r} = \frac{e}{\hbar} \int \mathbf{B}(\mathbf{r})\, dS = 2\pi \frac{\Phi}{\Phi_0} \tag{5}$$

where $\Phi_0 = h/e$ is the flux quanta.

This means that $\mathbf{K}$ translates and produces a phase shift. This phase change can be interpreted as Aharonov-Bohm phase that an electron acquires when it surround magnetic flux even though $\mathbf{B} = 0$ along the path.

B. Artificial magnetic field

The goal of generating artificial magnetic fields is to simulate the behaviour of a charged particle under influence of real magnetics fields in neutral atoms[4].

Many different methods have been proposed for generating such artificial magnetic fields but generally fall into two categories: By introducing a rotating parabolic trap that confines the system or by designing suitable laser-atom coupling to introduce artificial gauge potentials.

In order to prove this phenomenon we will compare the Hamiltonian for a free charged particle undergone a magnetic field and the Hamiltonian for a neutral particle in a trapping potential. First, we assume $\mathbf{B} = B\mathbf{u}_z$ so the electron experiences a Lorentz force $\mathbf{F} = -e\,(\mathbf{v} \times \mathbf{B})$. Taking $\mathbf{A} = \frac{B}{2}\,(-y, x, 0)$, the so-called symmetric gauge, and substituting in equation (1) the single-electron system is described by

$$H_e = \frac{1}{2M} \left[\left(p_x - \frac{eBy}{2} \right)^2 + \left(p_y + \frac{eBx}{2} \right)^2 \right]. \tag{6}$$

Now, by applying a rotation we wish to create an artificial magnetic field to simulate the Lorentz force. The rotating frame leads to a Coriolis force $\mathbf{F}_c = 2M\,(\mathbf{v} \times \mathbf{\Omega})$. To see how rotating the system yields an artificial magnetic field we must switch the Hamiltonian in Lab. frame to Hamiltonian in Rotation frame

$$H_{\text{Rot}} = H_L - \mathbf{\Omega} \cdot \mathbf{L} \tag{7}$$

where $\mathbf{L}$ is the total angular momentum and assuming the rotating axis is perpendicular to the motion plane $(x - y)$. This leads to

$$\begin{aligned} H_{\text{Rot}} =& \frac{1}{2M} \left[(p_x + M\Omega y)^2 + (p_y + M\Omega x)^2 \right] + \\ & \frac{M}{2} \left(\omega_\perp^2 - \Omega^2 \right) \left(x^2 + y^2 \right) \end{aligned} \tag{8}$$

where the vector potential has been defined as $\mathbf{A} = M\Omega\,(x\mathbf{u}_y - y\mathbf{u}_x)$. Thus, the artificial magnetic field is given by $B = 2M\Omega$. Finally, is easy to observe by comparing (6) and (8) that the electron has the same behaviour in a magnetic field as in a rotating system for atoms when the centrifugal term is well-balanced by the harmonic trap potential.

We are not going to use either particle under magnetic field nor rotating the system in order to confine the atoms. We will proceed on a different manner reproducing the magnetic field perpendicular to the lattice in an effective way by introducing a building-up phase in each translation of the particle.

C. Gauge Invariance

Let us consider a static magnetic field and from Maxwell's equations we know $\nabla \mathbf{B}(\mathbf{r}) = 0$ which means the flux across any closed surface is zero. Knowing this, we can deduce that $\mathbf{B}(\mathbf{r})$ can be written as

$$\mathbf{B}(\mathbf{r}) = \nabla \times \mathbf{A}(\mathbf{r}) \tag{9}$$

where $\mathbf{A}(\mathbf{r})$ is the vector potential. One can easily observe that it is possible to choose different $\mathbf{A}(\mathbf{r})$ for a given $\mathbf{B}(\mathbf{r})$, this is called a freedom degree in the gauge. The addition of any scalar function to the vector potential: [3]

$$\mathbf{A}'(\mathbf{r}) = \mathbf{A}(\mathbf{r}) + \nabla\chi \tag{10}$$

where χ has to be sufficiently regular, reproduces the same magnetic field.

The same observables must be reproduced, so we need to know how to change the wave function. In order to do this we define the transformed wave function as

$$\psi'(r,t) = \psi(r,t)e^{-i\chi} \tag{11}$$

and substituting it into the time independent Schrödinger equation we check that this fulfill the equation

$$\frac{1}{2M} \left[\left(-i\hbar\nabla - q\mathbf{A}(\mathbf{r}) + i\hbar\,(i\nabla\chi) \right)^2 \right] \psi e^{i\chi} = E\psi e^{i\chi} \tag{12}$$

$$H'\psi' = E\psi' \tag{13}$$

in general $\chi = \chi(r)$.

Thus, the gauge freedom does not affect to the expected values of the operators. This freedom to choose the vector potential is called gauge invariance.

III. OPTICAL LATTICE

Optical lattices are artificial crystals of light created by interfering optical laser beams producing a spatially periodic potential which can be used to trap neutral atoms. One of the major advantages of optical lattices are that the lattice geometry can be controlled by the configuration of laser beams used and the depth is tunable by the laser intensity.

In the following sections we will perform the study over a two dimensional square optical lattice with the characteristic parameter a which is the period of the potential.

A. Absence of magnetic field

We use the tight binding approximation. Within this model we consider as a single-particle base of functions, the states $|j,l\rangle$ localized at the sites $\mathbf{r} = a\left(j\mathbf{u}_x + l\mathbf{u}_y\right)$[3].

Then, the single-particle Hamiltonian can be written as

$$\hat{H} = -J\sum_{j,l}\left(|j+1,l\rangle\langle j,l| + |j,l+1\rangle\langle j,l|\right) + h.c. \quad (14)$$

where we bear in mind that the particle in the lattice moves via tunneling from a given site $|j,l\rangle$ to the nearest neighbour places $|j\pm 1,l\rangle$ and $|j,l\pm 1\rangle$. Constant J is the tunneling energy charecterizing the hopping between neighbouring sites. In our approximation the hopping to further places is disregarded.

According to Bloch's theorem the solution for a periodic potential that solves the Schrödinger equation has the following form

$$|\Psi(\mathbf{q})\rangle = \sum_{j,l} e^{ia(jq_x+lq_y)}|j,l\rangle \quad (15)$$

where $\mathbf{q}$ is the wave vector. This eigenfunctions are characterized by the well defined quasi-momentum $\hbar\mathbf{q}$. Due to the periodicity of the reciprocal lattice, the analysis can be restricted to the first Brillouin zone, $-\pi/a < q_j < \pi/a$, $j = x, y$. The spectrum is given by

$$E(\mathbf{q}) = -2J\left(\cos(aq_x) + \cos(aq_y)\right) \quad (16)$$

and looks like FIG.1

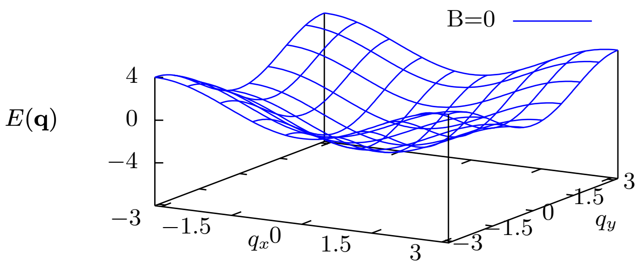

FIG. 1: Energy spectrum for a particle moving on an optical square lattice in tight-binding approximation without magnetic field versus the two components of the Bloch's vector. It is centered on E=0 with a full width 8J.

B. Presence of magnetic field

In order to mimic the effect of a magnetic field acting perpendicular to the square lattice ($\mathbf{B} = B\mathbf{u}_z$). The unique needed modification is given by

$$J \rightarrow Je^{i\phi(\mathbf{r}\rightarrow\mathbf{r}')} \quad (17)$$

to give trace of the accumulation of phase when the particle moves under the magnetic field. This artificial magnetic field is associated with $\mathbf{A}$.

Accordingly, the Hamiltonian (14) turns into the following form,

$$\begin{aligned} H = -J\sum_{j,l}\Big(&e^{i\phi(j,l\rightarrow j+1,l)}|j+1,l\rangle\langle j,l| \\ &+e^{i\phi(j,l\rightarrow j,l+1)}|j,l+1\rangle\langle j,l|\Big) + h.c., \end{aligned} \quad (18)$$

from now on we choose $\mathbf{A} = -By\mathbf{u}_x$.

The Aharonov-Bohm phase must be calculated using the relation (5)

$$\begin{aligned} &\phi(j,l\rightarrow j,l+1) = 0 \\ &\phi(j,l\rightarrow j+1,l) = \frac{q}{\hbar}\int_{ja}^{(j+1)a}\mathbf{A}d\mathbf{r} = -2\pi l\alpha \end{aligned} \quad (19)$$

where $\alpha = \Phi/\Phi_0$.

Once knowing the phase and introducing in equation (18) yields

$$H = -J\sum_{j,l}\left(e^{i2\pi l\alpha}|j+1,l\rangle\langle j,l| + |j,l+1\rangle\langle j,l|\right) + h.c. \quad (20)$$

This new Hamiltonian, owing to the presence of vector potential, is no longer invariant under translations since the symmetry has been broken along y. We need to recover the periodicity in y to be able to work out the Schrödinger equation using the Bloch solutions. To this end we consider the case given by $\alpha = p'/p$ where p and p' are coprime positive numbers. One can rapidly check that the periodicity throughout y has been recovered.

$$\begin{aligned} &\phi(j,l+p\rightarrow j+1,l+p) = -2\pi\alpha(l+p) = \\ &-2\pi\alpha l - 2\pi p' = -2\pi\alpha l, \quad \text{modulo } 2\pi. \end{aligned} \quad (21)$$

Namely, within this hypothesis ($\alpha = p'/p$) we recover the periodicity of the lattice, but the cell size has increased up to $a \times pa$ and because the new cell contains "p" sites, the initial band splits into "p" subbands.

We will solve the problem for $\alpha = 1/3$ in order to see how the energy spectrum gets fragmented in three subbands. The $\mathbf{q}$ vector is still well-defined being aware of the domain of q_y is reduced to $-\pi/3a < q_y < \pi/3a$. Then, the Bloch function becomes

$$|\Psi(\mathbf{q})\rangle = \sum_{j,l'} e^{ia(jq_x+3l'q_y)} \left(\beta_1|A_{j,l'}\rangle \right. \\ \left. + e^{iaq_y}\beta_2|B_{j,l'}\rangle + e^{2iaq_y}\beta_3|C_{j,l'}\rangle\right) \tag{22}$$

where we use the following notation, we place the A site at $\mathbf{r} = a\,(j\mathbf{u}_x + 3l'\mathbf{u}_y)$. Thus

$$|A\rangle = |j, 3l'\rangle, \;\; |B\rangle = |j, 3l'+1\rangle, \;\; |C\rangle = |j, 3l'+2\rangle \tag{23}$$

and the domain of $\mathbf{q}$

$$\frac{-\pi}{a} < q_x < \frac{\pi}{a}, \quad \frac{-\pi}{3a} < q_y < \frac{\pi}{3a} \tag{24}$$

Having reached this point, using the eigenvalue equation

$$\hat{H}|\Psi\rangle = E|\Psi\rangle \tag{25}$$

we will be able to find the coefficient β_i that is amounted to look for the eigenvalues of the Hamiltonian

$$\hat{\aleph}(\mathbf{q}) = -J \begin{pmatrix} 2\cos(aq_x) & e^{iaq_y} & e^{-iaq_y} \\ e^{-iaq_y} & 2\cos(aq_x + \frac{2\pi}{3}) & e^{iaq_y} \\ e^{iaq_y} & e^{-iaq_y} & 2\cos(aq_x + \frac{4\pi}{3}) \end{pmatrix} \tag{26}$$

in reciprocal space.

The eigenvalues obtained from the diagonalization of this matrix gives three eigenvalues $E^{(i)}, i = 1, 2, 3$ which depend on q_x and q_y. Assigning values to q_x and q_y within the intervals given by equation (24) we will generate the three subbands that set up the energy spectrum, see FIG. 2,3,4 and 5.

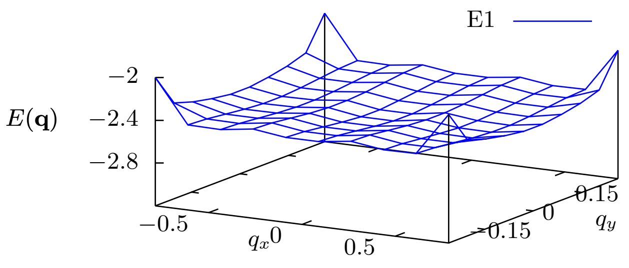

FIG. 2: Lowest energy band, it has the minimum in $E = -2.73J$ and a maximum $E = -2J$ with a band width $E = 0.73$.

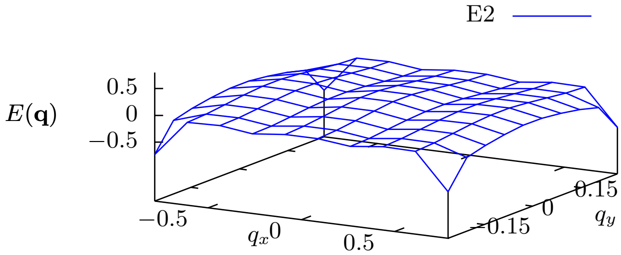

FIG. 3: Intermediate energy band, it has the minimum in $E = -0.73J$ and the maximum in $E = 0.73J$. It is symmetric centered in $E = 0J$.

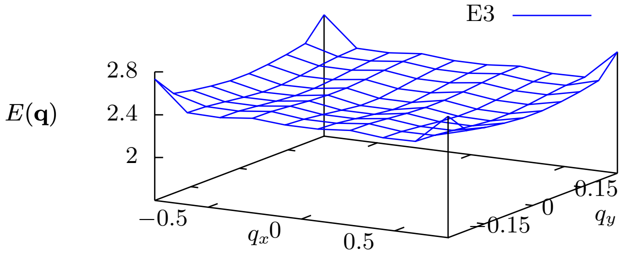

FIG. 4: Highest energy band, the maximum is $E = 2.73J$ and the minimum is $E = 2J$, it has a width $E = 0.73J$.

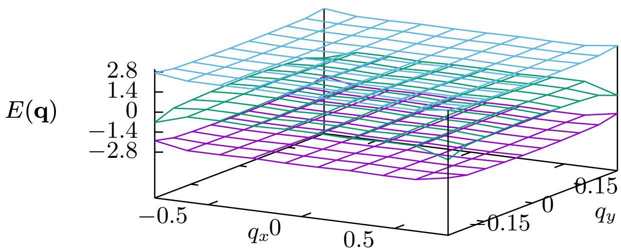

FIG. 5: Representation of the three subbands. The lowest and the highest are equals, having the same maximums and minimums with opposite sign. It is appreciable the gaps between lowest-intermediate and intermediate-highest bands have the same width $E = 1.27J$.

Now, computing the different matrixes for distinct values of α one can get the famous energy spectrum called *Hosftadter's Butterfly*[1] showed in Fig. 6

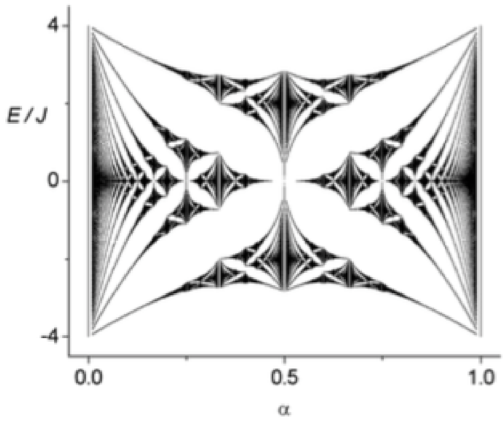

FIG. 6: The famous Hofstadter butterfly[1]:Energy spectrum for a single particle moving on a 2D periodic system in the tight-binding approximation under an external magnetic field. The x-axis describes the magnetic flux per cell $\alpha = p'/p = \Phi/\Phi_0 \in [0,1]$ with $q < 100$ and y-axis describes the energy. The plot has been performed for rational values of α.

IV. CONCLUSIONS

In this brief introduction to Fractional Quantum Hall Effect, we have exposed the energy spectrum of a particle moving in a 2D lattice under an external magnetic field. First, the study of the system in absence of magnetic field has allowed us to set out the explicit Hamiltonian which describes the particle states. Furthermore, the principle of the Aharonov-Bohm effect allows us to interpret the magnetic field as a build-up phase. Combining both we can reach the Hamiltonian for a particle in a 2D lattice undergone magnetic field and get the famous Hofstadter butterfly energy spetrum.

Acknowledgments

I would like to express my great gratitude to my supervisor Nuria Barberán for her guidance and help during the fulfillment of this work. I would also thank to my parents and colleagues for the support and trust shown in me throughout my study.

[1] Hofstadter, D.R., 1976, Phys. Rev. B14,2239
[2] Yoshioka, D., *The Quantum Hall Effect*, Springer 1998
[3] Dalibard, J., Introduction to the physics of artificial gauge fields, 2015, Phys. Rev. 83, 1504
[4] Dalibard, J. and Gerber, F., Colloquium: Artificial gauge potentials for neutral atoms, 2011, Phys. Rev. 83, 1523