RUBIO DE FRANCIA’S EXTRAPOLATION THEOREM FOR $B_p$ WEIGHTS

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Abstract. In this paper, we prove some of Rubio de Francia’s extrapolation results for the class $B_p$ of weights for which the Hardy operator is bounded on $L^p(w)$ restricted to decreasing functions. Applications to the boundedness of operators on $L^p_{\text{dec}}(w)$ are given. We also present an extension to the $B_\infty$ case and some connections with classical $A_p$ theory.

1. Introduction

In 1984, J.L. Rubio de Francia [10] proved that if $T$ is a sublinear operator that is bounded on $L^r(w)$ for every $w$ in the Muckenhoupt class $A_r$ ($r > 1$) with constant depending only on $\|w\|_{A_r}$, then for every $1 < p < \infty$, $T$ is bounded on $L^p(w)$ for every $w \in A_p$ with constant depending only on $\|w\|_{A_p}$. Since then, many results concerning this topic have been published (see [8], [6], [7]). From these results, it is now known that, in fact, the operator $T$ plays no role; that is, if $(f, g)$ are a pair of functions such that for some $1 \leq p_0 < \infty$,

$$\int_{\mathbb{R}^n} f^{p_0}(x)w(x)dx \leq C \int_{\mathbb{R}^n} g^{p_0}(x)w(x)dx$$

for every $w \in A_{p_0}$ with $C$ depending on $\|w\|_{A_{p_0}}$, then for every $1 < p < \infty$,

$$\int_{\mathbb{R}^n} f^p(x)w(x)dx \leq C \int_{\mathbb{R}^n} g^p(x)w(x)dx$$

for every $w \in A_p$ with $C$ depending on $\|w\|_{A_p}$. The theory has also been generalized to the case of $A_\infty$ weights and many interesting consequences have been derived from it.

The purpose of this paper is to develop a completely parallel theory in the setting of $B_p$ weights. The techniques are different as usually happens with these...
two theories and things are, in some sense, clearer and more natural. We think that the results in this paper should help to clarify what is happening in the $A_p$ context and we hope to solve that case in a forthcoming paper.

Before presenting the main results of this paper, let us just recall some important facts concerning $B_p$ weights which will be fundamental for our purposes. First of all, let us recall that a positive and locally integrable function $w$ on $(0, \infty)$ is called a $B_p$ weight if the following condition holds:

$$||w||_{B_p} = \inf \left\{ C > 0; \int_0^r w(t) dt + r^p \int_r^\infty \frac{w(t)}{t^p} dt \leq C \int_0^r w(t) dt, \forall r > 0 \right\} < \infty.$$ 

It is known (II) that $w \in B_p$ with $p > 0$ if and only if, for every decreasing function $f$,

$$\int_0^\infty \left( \frac{1}{t} \int_0^t f(s) ds \right)^p w(t) dt \leq C \int_0^\infty f^p(s) w(s) ds$$

with $C$ depending on $||w||_{B_p}$. Observe also that $||w||_{B_p} > 1$ if $w$ is not identically zero.

An important property that these classes of weights satisfy (see [4], Chapter 3, Section 3.3) is that, for every $p > 0$ and every $w \in B_p$, there exists $\varepsilon > 0$ such that $w \in B_{p-\varepsilon}$; moreover,

(1.1) $$||w||_{B_{p-\varepsilon}} \leq \frac{C||w||_{B_p}}{1 - \varepsilon \alpha^p||w||_{B_p}},$$

where $C$ and $0 < \alpha < 1$ are universal constants and $\varepsilon$ is such that $1 - \varepsilon \alpha^p||w||_{B_p} > 0$.

Since $B_p \subset B_q$ for every $0 < p \leq q < \infty$, we can define (similarly to $A_p$ theory) the class $B_\infty$ as the collection of weights belonging to some $B_p$; that is,

$$B_\infty = \bigcup_{p>0} B_p.$$ 

Let us also define

$$||w||_{B_\infty} = \inf\{||w||_{B_p}; w \in B_p\}.$$ 

We shall denote by $C$ a universal constant depending possibly on $p$ but independent of the weight $w$. Also $C$ might not be the same in all instances. We write $A \lesssim B$ if there exists a universal constant $C$ such that $A \leq CB$ and $A \approx B$ if $A \lesssim B$ and $B \lesssim A$.

2. Main results

Our first result is the counterpart in this setting of the new version of Rubio de Francia’s extrapolation result:

**Theorem 2.1.** Let $\varphi$ be an increasing function on $(0, \infty)$, let $(f, g)$ be a pair of positive decreasing functions defined on $(0, \infty)$ and let $0 < p_0 < \infty$. Suppose that for every $w \in B_{p_0}$,

$$\int_0^\infty f^{p_0} w \leq \varphi(||w||_{B_{p_0}}) \int_0^\infty g^{p_0} w.$$ 

Then, for every $p > 0$ and $w \in B_p$,

$$\int_0^\infty f^p w \leq \tilde{\varphi}(||w||_{B_p}) \int_0^\infty g^p w,$$
Let $\Phi(x) = \int_0^x \phi(t)dt$. The generalized Hardy operator associated to $\phi$ is defined, for $f$ decreasing, by

$$S_\phi f(x) = \frac{1}{\Phi(x)} \int_0^x f(t)\phi(t)dt.$$
Lemma 2.4. Let $0 < p < \infty$. Then, $S_\phi$ is bounded on $L^p_{\text{dec}}(w)$ with constant $A$ if and only if

\[ \int_0^r w(x)dx + \Phi(r)^p \int_r^\infty \frac{w(x)}{\Phi(x)^p} dx \leq A^p \int_0^r w(x)dx, \quad \text{for all } r > 0. \tag{2.1} \]

Proof. This result has been proved in [5] (Theorem 4.1) for the case $p > 1$. The proof also works (and is easier) for $p = 1$.

Let us now prove the case $0 < p < 1$. The necessary condition follows as in [5] by taking $f = \chi_{(0,r)}$. Conversely, let $f$ be decreasing. Then, $f(s) \leq \frac{1}{\Phi(s)} \int_0^s f(t)\phi(t)dt$ for every $s > 0$ and therefore

\[ \left( \int_0^s f(t)\phi(t)dt \right)^{p-1} \leq f(s)^{p-1} \Phi(s)^{p-1}. \]

Taking this into account,

\[ \begin{align*}
\int_0^\infty (S_\phi f(x))^p w(x)dx &= \int_0^\infty \left( \frac{1}{\Phi(x)} \int_0^x f(s)\phi(s)ds \right)^p w(x)dx \\
&= p \int_0^\infty \int_0^x \left( \int_0^s f(t)\phi(t)dt \right)^{p-1} f(s)\phi(s)ds \frac{w(x)}{\Phi(x)^p} dx \\
&\leq p \int_0^\infty \int_0^x f(s)^p\phi(s) \Phi(s)^{p-1} ds \frac{w(x)}{\Phi(x)^p} dx.
\end{align*} \tag{2.2} \]

Since $f$ is decreasing, Corollary 2.2 in [5] gives that the chain of inequalities in (2.2) can be continued as follows:

\[ \begin{align*}
&\leq p \int_0^\infty \int_0^\infty \int_0^{\lambda_{f}(y)} \chi_{(0,x)}(s)\phi(s)\Phi(s)^{p-1} ds dy \frac{w(x)}{\Phi(x)^p} dx \\
&\leq p \int_0^\infty \Phi(\min\{\lambda_{f}(y), x\})^p dy \frac{w(x)}{\Phi(x)^p} dx \\
&= \int_0^\infty \Phi(\min\{\lambda_{f}(y), x\})^p dy \frac{w(x)}{\Phi(x)^p} dx \\
&= \int_0^\infty \left( \int_0^{\lambda_{f}(y)} w(x)dx + \Phi(\lambda_{f}(y))^p \int_0^\infty \frac{w(x)}{\Phi(x)^p} dx \right) dy \\
&\leq A^p \int_0^\infty \lambda_{f}(y) w(x)dy = A^p \int_0^\infty f(y)^p w(y)dy,
\end{align*} \]

where the last inequality is obtained from the hypothesis. \qed
Proof of Theorem 2.1. Let $p > 0$, $w \in B_p$ and $0 < \varepsilon < p_0$. Using the fact that $f$ is decreasing and Lemma 2.3, we get

$$
\int_0^\infty f(t)^p w(t) dt \leq \int_0^\infty \left( \frac{p_0 - \varepsilon}{tp_0 - \varepsilon} \int_0^t f(s)^{p_0} s^{p_0-1} ds \right)^{p/p_0} w(t) dt
$$

(2.3)

$$
\leq \varphi \left( \frac{p_0}{\varepsilon} \right)^{p/p_0} \int_0^\infty \left( \frac{p_0 - \varepsilon}{tp_0 - \varepsilon} \int_0^t g(s)^{p_0} s^{p_0-1} ds \right)^{p/p_0} w(t) dt
$$

$$
= \varphi \left( \frac{p_0}{\varepsilon} \right)^{p/p_0} \int_0^\infty (S_\phi g^{p_0}(t))^{p/p_0} w(t) dt,
$$

where $\phi(t) = t^{p_0-1-\varepsilon}$. The proof will be finished once we compute $A$ such that

$$
\int_0^\infty (S_\phi g^{p_0}(t))^{p/p_0} w(t) dt \leq A \int_0^\infty g(t)^p w(t) dt,
$$

and by Lemma 2.4, we only have to compute $A$ such that

$$
\int_0^r w(x) dx + r \int_0^\infty \frac{w(x)}{x^{(p_0-1)p_0}} dx \leq A \int_0^r w(x) dx,
$$

which is equivalent to saying that $w \in B_{\frac{(p_0-1)p_0}{p_0}}$ with $A = \|w\|_{B_{\frac{(p_0-1)p_0}{p_0}}}.$

Now, since $w \in B_p$, there exists $\bar{\varepsilon} > 0$ so that $w \in B_{p-\bar{\varepsilon}}$. Then, it suffices to take $\varepsilon$ small enough so that $p - \varepsilon = \frac{(p_0-1)p_0}{p_0}$ to get the result. Moreover, by (1.1), we have that

$$
A = \|w\|_{B_{\frac{(p_0-1)p_0}{p_0}}} = \|w\|_{B_{p-\bar{\varepsilon}}} \leq \frac{C||w||_{B_p}}{1 - \varepsilon^\alpha P_0 ||w||_{B_p}}.
$$

Consequently, for every $0 < \varepsilon < \frac{p_0}{\alpha P_0 ||w||_{B_p}}$,

$$
\int_0^\infty f(t)^p w(t) dt \leq \varphi \left( \frac{p_0}{\varepsilon} \right)^{p/p_0} \frac{C||w||_{B_p}}{1 - \varepsilon^\alpha P_0 ||w||_{B_p}} \int_0^\infty g(t)^p w(t) dt,
$$

and the result follows by taking the infimum of such $\varepsilon$’s.

\[ \square \]

Proof of Theorem 2.2. By hypothesis we have that

$$
\int_0^\infty f^{p_0} w \leq \varphi(||w||_\infty) \int_0^\infty g^{p_0} w,
$$

for every $w \in B_\infty$. Then, taking $w(t) = \chi_{(0,s)}(t)t^\beta$ with $s > 0$ and $\beta > -1$, we have that $w \in B_\infty$ and $||w||_{B_\infty} = 1$. Hence

$$
\int_0^s f^{p_0}(t)t^\beta dt \leq \varphi(1) \int_0^s g^{p_0}(t)t^\beta dt, \quad \text{for all } t > 0, \beta > -1.
$$

(2.4)

Now let $p > 0$ and let $w \in B_\infty$ be arbitrary. Then, by definition of $B_\infty$, there exists $q > 0$ such that $w \in B_q$. Using again that $f$ is decreasing and inequality (2.4), we
obtain that for every $\beta > -1$,
\[
\int_0^\infty f(t)^p w(t) dt \leq \int_0^\infty \left( \frac{1 + \beta}{t^{1+\beta}} \int_0^t f(s)^{p_0} s^\beta ds \right)^{p/p_0} w(t) dt \\
\leq \varphi(1)^{p/p_0} \int_0^\infty \left( \frac{1 + \beta}{t^{1+\beta}} \int_0^t g(s)^{p_0} s^\beta ds \right)^{p/p_0} w(t) dt \\
= \varphi(1)^{p/p_0} \int_0^\infty (S_\phi g^{p_0}(t))^{p/p_0} w(t) dt,
\]
where $\varphi(t) = t^\beta$. To finish the proof we only have to check that $S_\phi$ is bounded in $L^{p/p_0}(w)$ and this is equivalent to showing that $w \in B^{(1+\beta)p_0}_{p_0}$. Therefore, it suffices to choose $\beta > -1$ such that $(1+\beta)p_0 = q$, i.e., $\beta = \frac{q-1}{p_0} - 1$, to get that
\[
\int_0^\infty f(t)^p w(t) dt \leq \varphi(1)^{p/p_0} ||w||_{B_\beta} \int_0^\infty g(t)^p w(t) dt.
\]
Taking the infimum of such $q$’s we are done. \hfill \Box

3. Application and Examples

In this section, we shall present mainly two applications which have interesting consequences. Both of them are consequences of the following observation:

Remark 3.1. It has been implicitly proved that, given $0 < p < \infty$ fixed and a pair of decreasing functions $(f,g)$,
\[
\int_0^\infty f(t)w(t) dt \leq C_w \int_0^\infty g(t)w(t) dt
\]
holds for every $w \in B_p$ with constant $C_w$ depending only on $||w||_{B_p}$ if and only if, for every $s > 0$ and every $-1 < \beta < p - 1$,
\[
\int_0^s f(t)t^\beta dt \lesssim C_\beta \int_0^s g(t)t^\beta dt,
\]
with $C_\beta$ independent of $s$.

Application I. The above observation is especially useful for characterizing the boundedness on $L^{p/p_0}_{\text{dec}}(w)$ of certain operators.

Theorem 3.2. Let $T$ be an operator such that
\begin{itemize}
  \item[i)] for every decreasing function $f$, $Tf$ is also a decreasing function whenever it is well defined;
  \item[ii)] for every decreasing function $g$, a function $T^*g$ is well defined by
\end{itemize}
\[
\int_0^\infty Tf(t)g(t) dt = \int_0^\infty f(t)T^*g(t) dt, \quad \forall f \downarrow.
\]

Let $0 < p < \infty$ be fixed. Then,
\[
T : L^{p/p_0}_{\text{dec}}(w) \longrightarrow L^p(w)
\]
is bounded for every $w \in B_p$ with constant depending only on $||w||_{B_p}$ if and only if, for every $r, s > 0$ and every $-1 < \alpha < 0$,
\[
\int_0^s T\chi_{(0,r)}(t)t^\alpha dt \lesssim C_\alpha \min(r, s)^{\alpha+1},
\]
with $C_\alpha$ independent of $r$ and $s$. 

Proof. If $T$ satisfies (3.1), then taking $f$ to be a decreasing function, we can apply Theorem 2.1 to the pair $(Tf, f)$ to deduce that

$$T : L^1_{\text{dec}}(w) \rightarrow L^1(w)$$

for every $w \in B_1$, and by the previous remark this is equivalent to having that, for every $s > 0$ and every $-1 < \alpha < 0$,

$$\int_0^\infty f(t) T^*(u^\alpha \chi_{(0,s)}(u))(t) dt = \int_0^s Tf(t)t^\alpha dt \lesssim C_\alpha \int_0^s f(t)t^\alpha dt.$$

Now, it is known (see [5]) that the above inequality holds for every decreasing $f$ if and only if, for every $r > 0$,

$$\int_0^s T\chi_{(0,r)}(t)t^\alpha dt = \int_0^r T^*(u^\alpha \chi_{(0,s)}(u))(t) dt \lesssim C_\alpha \int_0^{\min(s,r)} t^\alpha dt \approx C_\alpha \min(r,s)^{\alpha+1},$$

as we wanted to show. \hfill □

In particular, we can consider integral operators with positive kernel, which have been intensively studied in [9].

Corollary 3.3. Let

$$Tf(x) = \int_0^\infty f(t)k(x,t)dt$$

with $k$ a positive kernel such that, for every decreasing function $f$, $Tf$ is also a decreasing function whenever it is well defined. Then,

$$T : L^p_{\text{dec}}(w) \rightarrow L^p(w)$$

is bounded for every $w \in B_p$ with constant $C_w$ depending only on $||w||_{B_p}$ if and only if, for every $r, s > 0$ and every $-1 < \alpha < 0$,

$$(3.3) \quad \int_0^s \int_0^r k(x,t)x^\alpha dt dx \lesssim C_\alpha \min(r,s)^{\alpha+1},$$

with $C_\alpha$ independent of $r$ and $s$.

Similarly, in the case of two linear operators:

Corollary 3.4. If $T_1$ and $T_2$ are two linear operators satisfying the hypothesis of Theorem 3.2 we have

a) $$(3.4) \quad \int_0^\infty (T_1f)^p(t)w(t)dt \lesssim C_w \int_0^\infty (T_2f)^p(t)w(t)dt$$

for every $w \in B_p$ and every decreasing function $f$ with $C_w$ depending only on $||w||_{B_p}$ if and only if, for every $r, s > 0$ and every $-1 < \alpha < 0$,

$$\int_0^s T_1\chi_{(0,r)}(t)t^\alpha dt \lesssim C_\alpha \int_0^s T_2\chi_{(0,r)}(t)t^\alpha dt,$$

with $C_\alpha$ independent of $r$ and $s$. 

\vspace{1cm}
b) If \( T_j \) are integral operators with positive kernels \( k_j \) satisfying the hypothesis of Corollary 3.3, then (3.4) holds for every \( w \in B_p \) if and only if, for every \( r, s > 0 \) and every \(-1 < \alpha < 0\),

\[
\int_0^s \int_0^r k_1(x, t)x^\alpha dt \, dx \lesssim C_\alpha \int_0^s \int_0^r k_2(x, t)x^\alpha dt \, dx,
\]

with \( C_\alpha \) independent of \( r \) and \( s \).

**Examples**

Let us now give some examples of well known operators for which boundedness on \( L^p_{\text{dec}}(w) \) is true for every \( w \in B_p \) and examples in which this condition fails.

**Example I.** The Calderón operator.

Let \( \lambda, \beta, \gamma > 0 \) with \( \lambda \geq \beta \gamma \) and let us consider the operator

\[
Tf(x) = x^{-\lambda} \int_0^x t^{\gamma-1} f(t) dt.
\]

Then, \( T \) is an integral operator with kernel

\[
k(x, t) = x^{-\lambda} \chi_{(0,x)}(t) t^{\gamma-1}
\]

and hence using Corollary 3.3 it is immediate to see the following result:

**Theorem 3.5.** Let \( T \) be the Calderón operator defined above. Then, the following conditions are equivalent:

(i) There exists \( 0 < p < \infty \) such that

\[
T : L^p_{\text{dec}}(w) \rightarrow L^p(w)
\]

is bounded for every \( w \in B_p \).

(ii) For every \( 0 < p < \infty \),

\[
T : L^p_{\text{dec}}(w) \rightarrow L^p(w)
\]

is bounded for every \( w \in B_p \).

(iii) \( \beta = 1 \) and \( \gamma = \lambda \geq 1 \).

**Example II.** The Riemann-Liouville fractional operator is defined by

\[
R_\lambda f(x) = x^{-\lambda} \int_0^x (x-t)^{\lambda-1} f(t) dt,
\]

with \( 0 < \lambda \leq 1 \).

**Theorem 3.6.** For every \( 0 < p < \infty \), the operator

\[
R_\lambda : L^p_{\text{dec}}(w) \rightarrow L^p(w)
\]

is bounded for every \( w \in B_p \).

**Proof.** In this case \( k(x, t) = x^{-\lambda} \chi_{(0,x)}(t) (x-t)^{\lambda-1} \). We already know that, in order to prove the result, it is enough to show that for all \(-1 < \alpha < 0 \) and all \( r, s > 0 \) we have

\[
\int_0^s \int_0^r k(x, t)x^\alpha dt \, dx \lesssim C_\alpha \min(r, s)^{\alpha+1}.
\]

To see this, suppose first that \( s \leq r \). Then, for \( x \in (0, s) \),

\[
\int_0^r k(x, t) dt = \int_0^x x^{-\lambda}(x-t)^{\lambda-1} dt = \frac{1}{\lambda}.
\]
Therefore,

\[ \int_0^s \int_0^r k(x,t)x^\alpha dx\,dt = \frac{1}{\lambda} \int_0^s x^\alpha dx = C s^{\alpha+1} = C \min(r,s)^{\alpha+1}. \]

Suppose now that \( r < s \). Then there are two possible cases: \( s \leq 2r \) and \( 2r < s \). In the case where \( s \leq 2r \) we have

\[ \int_0^s \int_0^r k(x,t)x^\alpha dx\,dt = \frac{1}{\lambda} \int_0^s x^\alpha dx = C (2r)^{\alpha+1} = C \min(r,s)^{\alpha+1}. \]

If \( 2r < s \), then

\[ \int_0^s \int_0^r k(x,t)x^\alpha dx\,dt = \frac{1}{\lambda} \int_0^s x^\alpha dx = C (2r)^{\alpha+1} = C \min(r,s)^{\alpha+1}. \]

For the first summand we proceed as in the previous case:

\[ \int_0^{2r} \int_0^r k(x,t)x^\alpha dx\,dt \leq \int_0^{2r} \int_0^r k(x,t)x^\alpha dx\,dt \leq C (2r)^{\alpha+1} = C \min(r,s)^{\alpha+1}. \]

Let us estimate the second one. By the mean value theorem applied to the function \( f(u) = (x - u)^\lambda \) on the interval \([0, r]\), we have that there exists \( c \in (0, r) \) such that

\[ (x - r)^\lambda - x^\lambda = -\lambda r(x - c)^{\lambda-1}. \]

Then

\[ \int_0^r k(x,t)dt = x^{-\lambda} \left( \frac{x^\lambda - (x - r)^\lambda}{\lambda} \right) = x^{-\lambda} r (x - c)^{\lambda-1} \leq x^{-\lambda} r \frac{x^\lambda}{x - r} = \frac{r}{x - r}. \]

Therefore,

\[ \int_0^s \int_0^r k(x,t)x^\alpha dx\,dt = \int_2^r \int_0^r k(x,t)x^\alpha dx\,dt = r \int_2^r x^{\alpha-1} \frac{x}{x - r} dx. \]

Since the function \( g : [2r, s] \to \mathbb{R} \) given by \( g(x) = \frac{x}{x - r} \) is decreasing and \( \alpha < 0 \), we have that

\[ \int_2^r \int_0^r k(x,t)x^\alpha dx\,dt \leq 2r \left( \frac{s^\alpha - (2r)^\alpha}{\alpha} \right) = 2r \left( \frac{(2r)^\alpha - s^\alpha}{-\alpha} \right) \leq C (2r)^{\alpha+1} = C \min(r,s)^{\alpha+1}, \]

and (3.6) is proved.

\[ \square \]

**Remark 3.7.** With the same technique, we can also prove that neither the adjoint Calderón operator defined by

\[ Tf(x) = x^{-\lambda} \int_{x^\beta}^1 t^{\gamma-1} f(t)dt \]

with \( \lambda, \beta, \gamma > 0 \) nor the Laplace operator

\[ Lf(x) = \int_0^\infty e^{-xt} f(t)dt \]

satisfy the condition of boundedness on \( L^p_{\text{dec}}(w) \) for every \( w \in B_p \).

In the first case the kernel is

\[ k(x,t) = x^{-\lambda} \chi_{(x^\beta,1)}(t) t^{\gamma-1} \]

and it is enough to show that it is not true that for each \( -1 < \alpha < 0 \) and \( r, s > 0 \),

\[ \int_0^s \int_0^r k(x,t)x^\alpha dx\,dt \lesssim \min(r,s)^{\alpha+1}. \]
Let $0 < s < 1 < r$. Then
\[
\int_0^r k(x, t) dt = \int_0^r x^{-\lambda} \chi_{(x^{\beta}, 1)}(t) t^{\gamma-1} dt = x^{-\lambda} \int_0^1 t^{\gamma-1} dt = \frac{1}{\gamma} x^{-\lambda} (1 - x^{\beta \gamma}).
\]
Hence,
\[
\int_0^s \int_0^r k(x, t) x^\alpha dt dx = \frac{1}{\gamma} \int_0^s x^{\alpha-\lambda} (1 - x^{\beta \gamma}) dx \geq \frac{1}{\gamma} \int_0^s x^{\alpha-\lambda} (1 - s^{\beta \gamma}) dx
\]
for any $\alpha$ such that $-1 < \alpha < -1 + \lambda$.

In the second case the kernel is $k(x, t) = e^{-xt}$. Let us take $0 < s < r$ and observe that
\[
\int_0^r e^{-xt} dt = rf(xr),
\]
where $H$ denotes the Hardy operator and $f(t) = e^{-t}$. Then, making the substitution $xr = u$, we get
\[
\int_0^s x^\alpha \int_0^r k(x, t) dt dx = r \int_0^s x^\alpha H f(xr) dx = \frac{1}{r^\alpha} \int_0^{sr} u^\alpha H f(u) du
\]
\[
= \frac{1}{r^\alpha} \int_0^{sr} u^\alpha \frac{1 - e^{-u}}{u} du.
\]
If we keep $sr = 1$ and let $r$ tend to infinity, then $\int_0^{sr} u^\alpha \frac{1 - e^{-u}}{u} du = \int_0^1 u^\alpha \frac{1 - e^{-u}}{u} du$ is a positive constant and, as $-1 < \alpha < 0$, $\frac{1}{r^\alpha} \to \infty$ while $\min(r, s)^{\alpha + 1} = s^{\alpha + 1} \to 0$.

**Application II.** Let $g^*(t) = \inf \{s > 0 : \lambda_0(s) \leq t \}$ be the decreasing rearrangement of $g$, where $\lambda_0(y) = | \{ x \in \mathbb{R}^n : |g(x)| > y \} |$ is the distribution function of $g$ with respect to Lebesgue measure, and let $f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds$.

In [3] the space $S_p(w)$ defined by
\[
||f||_{S_p(w)} = \left( \int_0^\infty (f^{**}(t) - f^*(t))^p w(t) dt \right)^{1/p}
\]
was studied and it was proved that it coincides with the Lorentz space $\Gamma_p(w)$ defined by
\[
||f||_{\Gamma_p(w)} = \left( \int_0^\infty (f^{**}(t))^p w(t) dt \right)^{1/p}
\]
if $w \in RB_p$; that is, for every $r > 0$,
\[
\int_0^r w(s) ds \lesssim r^p \int_r^\infty \frac{w(s)}{s^p} ds.
\]
To see this, it was proved that if $w \in RB_p$, the following inequality holds:
\[
\int_0^\infty (f^{**}(t))^p w(t) dt \lesssim \int_0^\infty (f^{**}(t) - f^*(t))^p w(t) dt.
\]
Now, making the change of variable $u = 1/t$ the previous inequality is the same as (3.7)
\[
\int_0^\infty \left( \int_0^1 f^*(s) ds \right)^p u^{p-2} w \left( \frac{1}{u} \right) du \lesssim \int_0^\infty \left( \frac{1}{u} (f^{**} \left( \frac{1}{u} \right) - f^* \left( \frac{1}{u} \right)) \right)^p u^{p-2} w \left( \frac{1}{u} \right) du.
\]
On the other hand, the following hold:
i) \( w \in RB_p \) if and only if \( u^{p-2}w\left(\frac{1}{u}\right) \in B_p \).

ii) \( g(u) = \int_0^1 f^*(s)ds \) is clearly a decreasing function.

iii) \( h(u) = \frac{1}{u}\left(f^{**}(\frac{1}{u}) - f^*(\frac{1}{u})\right) \) is also a decreasing function (see [3]).

Therefore, inequality (3.7) can be read as
\[
\int_0^\infty g(u)^pv(u)du \lesssim \int_0^\infty h(u)^pv(u)du
\]
for every \( v \in B_p \) with \( g \) and \( h \) being decreasing functions, and thus it is equivalent to proving that for every \( s > 0 \),
\[
\int_0^s g(u)u^\alpha du \lesssim \int_0^s h(u)u^\alpha du
\]
for every \(-1 < \alpha < 0\), which can be seen with an easy computation.

4. Final comments

1) In the context of \( A_p \) weights developed in [10], [6], [7] and [3], we have a pair of positive functions \((f,g)\) not necessarily decreasing such that, for some \( 1 < p_0 < \infty \) and every \( w \in A_{p_0} \), there exists a constant \( C > 0 \) depending only on \(||w||_{A_{p_0}}\) satisfying
\[
\int_0^\infty f^{p_0}w \leq C \int_0^\infty g^{p_0}w.
\]
Then, it is natural to ask whether it is true that there exists an operator \( T \) satisfying
\[
f \leq Tf, \quad Tf \leq Tg,
\]
and
\[
T : L^p(w) \rightarrow L^p(w)
\]
for every \( w \in A_p \).

Observe that if this were the case, then for every \( w \in A_p \),
\[
\int_0^\infty f^pw \leq \int_0^\infty (Tf)^pw \leq \int_0^\infty (Tg)^pw \leq C \int_0^\infty g^pw,
\]
and we get the extrapolation result in the aforementioned papers.

Also observe that this is what happens in the \( B_p \) context since upon taking
\[
Tf(t) = \left(\frac{1}{t^{p_0-\varepsilon}} \int_0^t f^{p_0}(s)s^{p_0-1-\varepsilon} ds\right)^{1/p_0}
\]
we have that \( T \) satisfies the three conditions mentioned above for \( f \) a decreasing function.

2) In the context of the interpolation theory of Banach spaces, we also have a similar result to the ones developed in [2] (Theorems 3.8 and 5.2): Given two compatible Banach spaces \( \bar{A} \) and \( \bar{B} \) and a linear operator \( T \) such that, for some \( 0 < p < \infty \),
\[
T : \bar{A}_{p,w;K} \rightarrow \bar{B}_{p,w;K}
\]
is bounded for every \( w \in B_p \) with constant depending only on \(||w||_{B_p}||\), we have that
for every $w \in B_q$ and every $0 < q < \infty$,

$$T : \tilde{A}_{q,w;K} \rightarrow \tilde{B}_{q,w;K}$$

is bounded with constant depending only on $\|w\|_{B_q}$.

To see this, observe that by hypothesis,

$$\int_0^\infty \left( \frac{K(t, T f; \tilde{B})}{t} \right)^p w(t) dt \lesssim C_w \int_0^\infty \left( \frac{K(t, f; \tilde{A})}{t} \right)^p w(t) dt,$$

and since $\frac{K(t, T f; \tilde{B})}{t}$ is a decreasing function, we can apply our results directly.

Moreover, we have that (4.1) holds for some $p$ and every $w \in B_p$ (or equivalently, for every $0 < p < \infty$ and every $w \in B_p$) if and only if, for every $r > 0$ and every $-1 < \alpha < 0$,

$$\int_0^r K(t, T f; \tilde{B}) t^{\alpha - 1} dt \lesssim C_\alpha \int_0^r K(t, f; \tilde{A}) t^{\alpha - 1} dt,$$

with $C_\alpha$ independent of $r > 0$.

REFERENCES


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