ON THE ZEROS OF FUNCTIONS IN DIRICHLET-TYPE SPACES

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Abstract. We study the sequences of zeros for functions in the Dirichlet spaces $D_s$. Using Carleson-Newman sequences we prove that there are great similarities for this problem in the case $0 < s < 1$ with that for the classical Dirichlet space.

1. Introduction and main results

The problem of describing the zero sets for the Dirichlet-type spaces $D_s$ is an old one, and to the best of our knowledge, is still an open problem whose best results are the ones given by Carleson in [8], [11], and by Shapiro and Shields in [39]. The purpose of this paper is to give some light on this difficult problem. Since the Dirichlet-type spaces are subclasses of the Hardy space $H^2$, any zero sequence $\{z_n\}$ satisfies the Blaschke condition $\sum (1 - |z_n|^2) < \infty$ ([18, p. 18]). However, this condition is far from being sufficient. Many examples of Blaschke sequences that are not $D_s$- zero sets can be found in the literature (see [12], [29] and [39]). When $0 < s < 1$, Carleson proved in [8] that the condition

$$\sum (1 - |z_n|^2)^s < \infty$$

implies that the Blaschke product $B$ with zeros $\{z_n\}$ belongs to the space $D_s$, and therefore, it is a sufficient condition for the sequence $\{z_n\}$ to be a $D_s$-zero set. Concerning the Dirichlet space $D$ (the case $s = 0$), since it does not contain infinite Blaschke products, one must go in a different way. In [11], by constructing a function $g \in D$ with $gB \in D$, Carleson found the sufficient condition $\sum \left( \log \frac{1}{1 - |z_n|^2} \right)^{-1 + \epsilon} < \infty$, for a sequence $\{z_n\}$ to be a zero set for the Dirichlet space. Using Hilbert space techniques, this was improved in [39] by Shapiro and Shields, who proved that the condition

$$\sum_n \left( \log \frac{1}{1 - |z_n|^2} \right)^{-1} < \infty$$

is sufficient for $\{z_n\}$ to be a Dirichlet zero set.

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Note that the spaces $D_s$ are Hilbert function spaces with the norm of the corresponding reproducing kernels $k_z$ comparable to $(\log(1/|z|))^{1/2}$ if $s = 0$, and to $(1 - |z|^2)^{-s/2}$ if $s > 0$. So, the corresponding sufficient conditions stated before can be restated as $\sum ||k_z||_{D_s}^2 < \infty$. On the other hand, if $\{r_n\} \subset (0,1)$ and $\sum ||k_{r_n}||_{D_s}^2 = \infty$, with $0 \leq s < 1$, in \[29\], Nagel, Rudin, and Shapiro constructed a sequence of angles $\{\theta_n\}$ such that $\{r_ne^{i\theta_n}\}$ is not the zero set of any function in $D_s$. Together with the previous sufficient condition, this implies that given $\{r_n\} \subset (0,1)$, then $\{r_ne^{i\theta_n}\}$ is a zero set for $D_s$ for any choice of angles $\{\theta_n\}$ if and only if
\[
\sum_n ||k_{r_n}||_{D_s}^2 < \infty. 
\]

We also note that, in \[7\], Bogdan described the regions $\Omega \subset \mathbb{D}$ for which any Blaschke sequence of points in $\Omega$ must be a Dirichlet zero set. For example, it follows that any Blaschke sequence that lies in a region with finite order of contact with the unit circle must be a Dirichlet zero set.

What about conditions on the angles? Here we touch the notion of a Carleson set. Given a sequence of points $\{e^{i\theta_n}\}$, the sequence $\{r_ne^{i\theta_n}\}$ is a zero sequence of $D$ for any choice of radius $\{r_n\}, 0 < r_n < 1$ with $\sum(1 - r_n) < \infty$ if and only if the closure of $\{e^{i\theta_n}\}$ in the unit circle is a Carleson set. Indeed, if the closure of $\{e^{i\theta_n}\}$ in the unit circle is a Carleson set, Caughran proved in \[13\] that there is a function $f$ with all derivatives bounded in the unit disk vanishing at the points $\{r_ne^{i\theta_n}\}$. Conversely, if $\{e^{i\theta_n}\}$ is not a Carleson set, by modifying the construction in \[12\] Theorem 1, he obtained in \[13\] a sequence $\{r_n\}$ for which $\{r_ne^{i\theta_n}\}$ is not contained in the zero set of any function with finite Dirichlet integral. We will see that the same holds for the spaces $D_s$ when $0 < s < 1$.

In \[26\] Corollary 13, Marshall and Sundberg proved that the zero sets of the Dirichlet-type spaces $D_s, 0 \leq s \leq 1$, coincide with the zero sets of its multiplier algebra (see also \[2\] Corollary 9.39)). From this follows the remarkable result that the union of two zero sets is also a zero set for $D_s$. Note that the corresponding result for the weighted Bergman spaces (the case $s > 1$) is not true; the first example was given by Horowitz in \[22\]. A complete description of the zeros of functions in Bergman spaces is still open, but the gap between the necessary and sufficient known conditions is small. We refer to \[19\] Chapter 4, \[21\] Chapter 4, \[22\], \[25\], \[31\] and \[33\] for more information on this interesting problem.

1.1. Main results. Let $\mathbb{D}$ denote the open unit disk of the complex plane, let $T$ denote the unit circle and let $H(\mathbb{D})$ be the class of all analytic functions on $\mathbb{D}$. For $s \geq 0$, the weighted Dirichlet-type space $D_s$ consists of those functions $f \in H(\mathbb{D})$ for which
\[
||f||^2_{D_s} \overset{\text{def}}{=} |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 (1 - |z|^2)^s \, dA(z) < \infty,
\]
where $dA(z) = \frac{1}{2} \, dx \, dy$ is the normalized area measure on $\mathbb{D}$. As usual, $D_0$ will be simply denoted by $D$.

Given a space $X$ of analytic functions in $\mathbb{D}$, a sequence $Z = \{z_n\} \subset \mathbb{D}$ is said to be an $X$-zero set if there exists a function in $X$ that vanishes on $Z$ and nowhere else.

A sequence $\{z_n\} \subset \mathbb{D}$ is said to be separated if $\inf_{j \neq k} \varrho(z_j, z_k) > 0$, where $\varrho(z, w) = \frac{|z - w|}{1 - \overline{z}w}$ denotes the pseudohyperbolic metric in $\mathbb{D}$. This condition is
equivalent to the fact that there is a positive constant $\delta < 1$ such that the pseudo-
hyperbolic discs $\Delta(z_j, \delta) = \{ z : g(z, z_j) < \delta \}$ are pairwise disjoint.

We denote by $H^p$ ($0 < p \leq \infty$) the classical Hardy spaces of analytic functions
on $\mathbb{D}$ (see [18]). We remind the reader that $\{ z_k \} \subset \mathbb{D}$ is an interpolating sequence
if for each bounded sequence $\{ w_k \}$ of complex numbers there exists $f \in H^\infty$ such that $f(z_k) = w_k$ for all $k$. It is a classical result of Carleson (see e.g. [18]) that $\{ z_k \} \subset \mathbb{D}$ is an interpolating sequence if and only if
\begin{equation}
\inf_k \prod_{j \neq k} \rho(z_j, z_k) > 0.
\end{equation}

Clearly a sequence satisfying (1.2) is separated. A finite union of interpolating
sequences is usually called a Carleson-Newman sequence.

In this research on $\mathcal{D}_s$-zero sets, $0 < s < 1$, the additional hypothesis of being
a Carleson-Newman sequence enables us to obtain better results. The key is the
following one which moves the problem to a new situation on the boundary.

**Theorem 1.** Suppose that $0 < s < 1$ and $\{ z_k \}$ is a Carleson-Newman sequence. Then the following conditions are equivalent:

(i) $\{ z_k \}$ is a $\mathcal{D}_s$-zero set.

(ii) There exists an outer function $g \in \mathcal{D}_s$ such that
\begin{equation}
\sum_{k=1}^{\infty} |g(z_k)|^2 (1 - |z_k|^2)^s < \infty.
\end{equation}

(iii) There exists an outer function $g \in \mathcal{D}_s$ such that
\begin{equation}
\sum_{k=1}^{\infty} (1 - |z_k|^2)^{1+s} \int_T |g(e^{it})|^2 \frac{dt}{|e^{it} - z_k|^2} < \infty.
\end{equation}

We recall that a function $g \in H(\mathbb{D})$ is called an outer function if $\log |g|$ belongs
to $L^1(\mathbb{T})$ and
\begin{equation}
g(z) = \exp \left( \frac{1}{2\pi} \int_\mathbb{T} \log |g(e^{it})| \frac{e^{it} + z}{e^{it} - z} \, dt \right).
\end{equation}

Although obviously there are $\mathcal{D}_s$-zero sets that are not Carleson-Newman se-
quencies, this additional assumption is not an obstacle in order to construct relevant
examples, and to get analogous results for $\mathcal{D}_s$ to those known for $\mathcal{D}$. Combining
ideas from [10], [12] and Theorem 1, the next result follows.

**Corollary 1.** Suppose that $0 < s < 1$ and $\{ z_k \}$ is a Carleson-Newman sequence.
If $\{ z_k \}$ is a $\mathcal{D}_s$-zero set, then
\begin{equation}
\int_\mathbb{T} \log \left( \sum_{k=1}^{\infty} \frac{(1 - |z_k|^2)^{1+s}}{|e^{it} - z_k|^2} \right) \, dt < \infty.
\end{equation}

We note that this result remains true for $s = 0$ without assuming that the
sequence is Carleson-Newman (see [12]); that is, if $\{ z_k \}$ is a $\mathcal{D}$-zero set, then
\begin{equation}
\int_\mathbb{T} \log \left( \sum_{k=1}^{\infty} \frac{1 - |z_k|^2}{|e^{it} - z_k|^2} \right) \, dt < \infty.
\end{equation}

Corollary 1 allows us to extend Theorem 1 of [12] to the case $0 < s < 1$. 

Theorem 2. Let $0 < s < 1$. Then there exists a Blaschke sequence $\{z_n\}$ which is not a $D_s$-zero set and with 1 as a unique accumulation point.

Denote by $|E|$ the normalized Lebesgue measure of a subset $E$ of the unit circle $\mathbb{T}$. A Carleson set is a closed subset $E \subset \mathbb{T}$ of Lebesgue measure zero for which, if the intervals $\{I_k\}$ complementary to $E$ have lengths $|I_k|$, then $\sum_k |I_k| \log |I_k| > -\infty$. This notion was introduced in [3], and in [9] Carleson used it to describe the sets of uniqueness of some function spaces. Corollary [1] is also useful to obtain results on the angular distribution of the $D_s$-zero sets.

Theorem 3. Let $0 < s < 1$, and $\{e^{i\theta_n}\} \subset \mathbb{T}$. The following are equivalent:

(i) the sequence $\{r_n e^{i\theta_n}\}$ is a $D_s$-zero set for any choice of $\{r_n\} \subset (0,1)$ with $\sum (1-r_n) < \infty$;

(ii) the closure of $\{e^{i\theta_n}\}$ in the unit circle is a Carleson set.

As noted before, if $0 \leq s < 1$ and $\{r_n\} \subset (0,1)$ is a Blaschke sequence that does not satisfy (1.1), then there is a sequence of angles $\{\theta_n\}$ such that $Z = \{r_n e^{i\theta_n}\}$ is not a $D_s$-zero set. The sequences doing that which have been constructed in [24] (and also the examples in [39]) satisfy that every $\xi \in \mathbb{T}$ is an accumulation point of $Z$. Ross, Richter and Sundberg proved in [35] that this can be done in $D$ with a sequence $Z$ which accumulates to a single point in $\mathbb{T}$. We shall extend this result to the range $0 < s < 1$, which improves our Theorem 2 but whose proof is much more technical.

Theorem 4. Let $0 < s < 1$. Suppose that $\{r_n\} \subset (0,1)$ satisfies

$$\sum_{n=0}^{\infty} (1-r_n)^s = \infty.$$  

Then there exists a sequence $\{\theta_n\}$ such that $\{r_n e^{i\theta_n}\} \cap \mathbb{T} = \{1\}$ and $\{r_n e^{i\theta_n}\}$ is not a $D_s$-zero set.

Let $X$ be a space of analytic functions in $\mathbb{D}$ contained in the Nevanlinna class (see [18]), so every function $f \in X$ has nontangential limits a.e. on $\mathbb{T}$. Denote also by $f$ the function of boundary values of $f$ (taken as a nontangential limit). A closed set $E \subset \mathbb{T}$ is called a set of uniqueness for $X$ if it has the property that $f \equiv 0$ if $f \in X$ vanishes at all points $\xi \in E$. It is well known that $E \subset \mathbb{T}$ is a set of uniqueness for a Lipschitz class $\Lambda_\alpha$ if and only if $E$ is not a Carleson set. We remind the reader that $f \in H(\mathbb{D})$ belongs to $\Lambda_\alpha$, $0 < \alpha \leq 1$, if there is $C > 0$ such that

$$|f(z) - f(w)| \leq C|z-w|^\alpha, \quad \text{for all } z, w \in \overline{\mathbb{D}}.$$

In [9] Theorem 5, under a very weak additional assumption, the sets of uniqueness for the classical Dirichlet space are described.

If $\alpha > 0$, we denote by $C_\alpha(E)$ the $\alpha$-capacity of a subset of $\mathbb{T}$ (see Section 4 for a definition). The following result is an extension of Theorem 5 in [9].

Theorem 5. Let $0 \leq s < \alpha < 1$ and $E \subset \mathbb{T}$ with null Lebesgue measure. Suppose that there exists $m > 0$ such that for each interval $I \subset \mathbb{T}$ centered at a point of $E$,

$$C_\alpha(E \cap I) \geq m|I|.$$  

Then $E$ is a set of uniqueness for $D_s$ if and only if $E$ is not a Carleson set.
The paper is organized as follows. Section 2 is devoted to the study of Carleson-Newman sequences as $D_s$-zero sets proving Theorem 1, Corollary 1, Theorem 2, and Theorem 3. Theorem 3 is proved in Section 4, and Theorem 5 is proved in Section 6. In Section 5, we shall give a new proof of a result of Bogdan [7] on the description of Blaschke sets for $D$. Finally, in Section 6, between other results, we prove that $D_s$-zero sets and the zero sets of their generated Möbius invariant spaces coincide.

In the sequel, the notation $A \asymp B$ will mean that there exist two positive constants $C_1$ and $C_2$ which only depend on some parameters $p, \alpha, s, \ldots$ such that $C_1 A \leq B \leq C_2 A$. Also, we remark that throughout the paper we shall be using the convention that the letter $C$ will denote a positive constant whose value may depend on some parameters $p, \alpha, s, \ldots$, not necessarily the same at different occurrences.

2. Carleson-Newman $D_s$-zero sets

We first recall some useful concepts and results. The Carleson square $S(I)$ of an interval $I \subset \mathbb{T}$ is defined as

$$S(I) = \{r e^{i\theta} : e^{i\theta} \in I, \quad 1 - |I| \leq r < 1\}.$$ 

Given $s > 0$ and a positive Borel measure $\mu$ on $\mathbb{D}$, we say that $\mu$ is an $s$-Carleson measure if there exists a positive constant $C$ such that

$$\mu(S(I)) \leq C |I|^s, \quad \text{for every interval } I \subset \mathbb{T}. $$

If $s = 1$ we simply say that $\mu$ is a Carleson measure. We recall that a sequence $\{z_n\} \subset \mathbb{D}$ is Carleson-Newman if and only if the measure $d\mu_{z_n} = \sum (1 - |z_n|) \delta_{z_n}$ is a Carleson measure (see [27] and [28]). Here, as usual, $\delta_{z_n}$ denotes the point mass at $z_n$. A Blaschke product whose zero sequence is Carleson-Newman is called a Carleson-Newman Blaschke product (a CN-Blaschke product, for short).

Let $P_z(e^{it})$ denote the Poisson kernel at a point $z \in \mathbb{D}$, so that

$$P_z(e^{it}) = \frac{1 - |z|^2}{|e^{it} - z|^2}, \quad e^{it} \in \mathbb{T},$$

and let

$$\Psi(z, \phi) = \frac{1}{2\pi} \int_{\mathbb{T}} \phi(e^{it}) P_z(e^{it}) dt - \exp \left( \frac{1}{2\pi} \int_{\mathbb{T}} \log \phi(e^{it}) P_z(e^{it}) dt \right), \quad z \in \mathbb{D},$$

where $\phi$ is a positive function which belongs to $L^1(\mathbb{T})$. Observe that the arithmetic-geometric inequality implies that $\Psi(z, \phi) \geq 0$. If $\phi \in L^2(\mathbb{T})$, $\phi \geq 0$, we set

$$\Phi(z, \phi) = \Psi(z, \phi^2).$$

We observe that for an outer function $g \in H^2$,

$$\Phi(z, |g|) = P(|g|^2)(z) - |g(z)|^2,$$

where $P(|g|^2)$ is the Poisson integral of $|g|^2$.

The following result, Theorem 3.1 of [12] (see [6] for related results), characterizes the membership in $D_s$ of an outer function in terms of its modulus on the boundary.

**Theorem A.** Suppose that $0 < s < 1$ and $f$ is an outer function. Then the following are equivalent:

(i) $f \in D_s$,

(ii) $\int_{\mathbb{D}} \Phi(z, |f|) \frac{dA(z)}{(1 - |z|^2)^{s+1}} < \infty$. 


In order to prove Theorem 1 we need some lemmas. The following result is implicit in some places (see e.g. [33, Theorem 5] or [15, Theorem 8]). For completeness we sketch a proof here.

**Lemma 1.** Suppose that $0 < s < 1$, $f \in \mathcal{D}_s$ and let $B$ be a Carleson-Newman Blaschke product with zeros $\{z_k\} \subset \mathbb{D}$. Then $fB \in \mathcal{D}_s$ if and only if

$$\sum_{k=1}^{\infty} |f(z_k)|^2 (1 - |z_k|^2)^s < \infty.$$  

Moreover,

$$\|fB\|_{\mathcal{D}_s}^2 \leq \|f\|_{\mathcal{D}_s}^2 + \sum_{k=1}^{\infty} |f(z_k)|^2 (1 - |z_k|^2)^s.$$  

**Proof.** Suppose first that $fB \in \mathcal{D}_s$. By Theorem 4 of [16],

$$\|fB\|_{\mathcal{D}_s}^2 \leq \|f\|_{\mathcal{D}_s}^2 + \int_{\mathbb{D}} |f(z)|^2 (1 - |B(z)|^2) (1 - |z|^2)^{s-2} \, dA(z).$$

Since $B$ is a CN-Blaschke product, there is a positive constant $C$ such that (see e.g. [16] p. 15)

$$1 - |B(z)|^2 \geq C \sum_{n} \frac{(1 - |z_n|^2)(1 - |z|^2)}{|1 - z_n z|^2}.$$  

Therefore, if $\Delta_n = \{ \varrho(z, z_n) < 1/2 \}$, the subharmonicity of $|f|^2$ gives

$$\sum_{n} |f(z_n)|^2 (1 - |z_n|^2)^s \leq C \sum_{n} \int_{\Delta_n} |f(z)|^2 \frac{(1 - |z|^2)^s}{|1 - z_n z|^2} \, dA(z)$$

$$\leq C \sum_{n} (1 - |z_n|^2) \int_{\Delta_n} |f(z)|^2 \frac{(1 - |z|^2)^{s-1}}{|1 - z_n z|^2} \, dA(z)$$

$$\leq C \sum_{n} (1 - |z_n|^2) \int_{\mathbb{D}} |f(z)|^2 \frac{(1 - |z|^2)^{s-1}}{|1 - z_n z|^2} \, dA(z)$$

$$\leq C \int_{\mathbb{D}} |f(z)|^2 (1 - |B(z)|^2) (1 - |z|^2)^{s-2} \, dA(z).$$

For the converse we refer to [4] Proposition 3.2, where an elementary proof is given. \[\square\]

Next, if $g \in H^2$ we shall see that the function $\Phi(z, |g|)$, although it is superharmonic, verifies a certain sub-mean-value property.

**Lemma 2.** Suppose that $g$ is an outer function which belongs to $H^2$. Then there is a constant $M > 1$ such that

$$\Phi(z, |g|) \leq \frac{M}{A(D(z, r))} \int_{D(z, r)} \Phi(w, |g|) \, dA(w), \quad \text{for all } r \in \left(0, \frac{1 - |z|}{2}\right),$$

where $D(z, r)$ is the Euclidean disk of center $z$ and radius $r$.  

Lemma 1. Suppose that there is an outer function \(g\) such that
\[
\sum_{k} \|g(z_k)\| (1 - |z_k|^2)^s < \infty.
\]

(iii) \(\Rightarrow\) (ii). Without loss of generality we may assume that \(\{z_k\}\) is separated. Therefore, there is a positive constant \(\varepsilon < 1\) such that the pseudohyperbolic disks \(\Delta(z_k, \varepsilon)\) are pairwise disjoint.

Suppose that there is an outer function \(g\) which satisfies (1.3). It is observed that
\[
\sum_{k} (1 - |z_k|^2)^{1+s} \int_{\gamma} |g(e^{it})|^2 \frac{dt}{|e^{it} - z_k|^2}
\]
(2.5)
\[
\leq \sum_{k} \Phi(z_k, |g|) (1 - |z_k|^2)^s + \sum_{k=1}^{\infty} |g(z_k)|^2 (1 - |z_k|^2)^s.
\]
Next, bearing in mind Lemma 2 the separation of \( \{z_k\} \) and Theorem A we deduce that
\[
\sum_k \Phi(z_k, |g|) (1 - |z_k|^2)^s \leq C \sum_k (1 - |z_k|^2)^{s-2} \int_{\Delta(z_k, r)} \Phi(z, |g|) \, dA(z)
\]
so bearing in mind that \( \log |g| \in L^1(\mathbb{T}) \) and the geometric-arithmetic inequality, the result follows.

Finally, \( (iii) \) follows from (1.3), (2.6) and (2.5).

**Proof of Corollary 1** By Theorem 1 there is an outer function \( g \in \mathcal{D}_s \) such that
\[
\int_{\mathbb{T}} |g(e^{it})|^2 \left( \sum_k \frac{(1 - |z_k|^2)^{1+s}}{|e^{it} - z_k|^2} \right) \, dt < \infty,
\]
so bearing in mind that \( \log |g| \) is a Carleson-Newman sequence which accumulates only at \( \{z_n\} \), the result follows.

**Proof of Theorem 2** The same sequence given in the proof of Theorem 1 works. Choose a sequence \( \{\varepsilon_n\} \) such that \( 0 < \varepsilon_n < 1, \sum_n \varepsilon_n \leq 1 \) and \( \sum_n \varepsilon_n \log \varepsilon_n = -\infty \). Next, take disjoint open arcs of \( \mathbb{T} \) with \( |I_n| = \varepsilon_n \) converging to 1. Let \( r_n = 1 - \varepsilon_n \) and \( z_n = r_n e^{i\theta_n} \), where \( \theta_n \) is the center of \( I_n \). If \( I \) is an arc of \( \mathbb{T} \), then
\[
\sum_{\varepsilon_n \in S(I)} (1 - |z_n|) \leq \sum_{|I_n| \leq 2|I|} |I_n| \leq 2|I|,
\]
proving that the measure \( \mu = \sum (1 - |z_n|) \delta_{z_n} \) is a Carleson measure. So, \( \{z_n\} \) is a Carleson-Newman sequence which accumulates only at \( \{1\} \). Moreover, since
\[
\int_{\mathbb{T}} \log \left( \sum_{k=1}^\infty \frac{(1 - |z_k|^2)^{1+s}}{|e^{it} - z_k|^2} \right) \, dt \geq \sum_{j=1}^{\infty} \int_{I_j} \log \left( \sum_{k=1}^\infty \frac{(1 - |z_k|^2)^{1+s}}{|e^{it} - z_k|^2} \right) \, dt \geq \sum_{j=1}^{\infty} \int_{I_j} \log \left( \frac{(1 - |z_j|^2)^{1+s}}{|e^{it} - z_j|^2} \right) \, dt \geq \sum_{j=1}^{\infty} |I_j| \log (4|I_j|^s - 1) = \infty,
\]
it follows from Corollary 1 that \( \{z_n\} \) is not a \( \mathcal{D}_s \)-zero set. The proof is complete.

**Proof of Theorem 3** If \( \{e^{i\theta_n}\} \) is a Carleson set and \( \sum (1 - r_n) < \infty \), then it follows from Theorem 2 that there is a function \( f \) with all derivatives bounded that vanishes only at \( \{r_n e^{i\theta_n}\} \).

Suppose now that \( E = \{e^{i\theta_n}\} \) is not a Carleson set. Let \( \{I_n\} \) be the complementary intervals of \( E \), with \( I_n = (e^{i\theta_n}, e^{i(\theta_n + |I_n|)}) \). Set \( r_n = (1 - |I_n|) e^{i\theta_n} \),
which satisfies \( \sum (1 - r_n) < \infty \). Clearly, the sequence \( \{z_n\} = \{r_n e^{i\theta_n}\} \) is Carleson-Newman, and arguing as in the proof of Theorem 2 we have
\[
\int_T \log \left( \sum_n \frac{1 - |z_n|^2}{|e^{it} - z_n|^2} \right) \, dt \geq C \sum_n |I_n| \log (4|I_n|^{-s-1}) = \infty.
\]
Hence, by Corollary 1, the sequence \( \{r_n e^{i\theta_n}\} \) is not a \( D_s \)-zero set.

**Proof.**

Let \( \Lambda \) be an arc with center \( z_0 \). Arguing as in the proof of Theorem 2, we have
\[
\Lambda \subset \bigcup_{j \in \mathbb{Z}} I_j.
\]

This finishes the proof. \( \square \)

### 3. Proof of Theorem 4

Some new concepts and preliminary results will be needed in the proof of Theorem 4. For \( 0 < s \leq 1 \), the \( s \)-dimensional Hausdorff capacity of \( E \subset \mathbb{T} \) is determined by
\[
\Lambda^\infty_s(E) = \inf \left\{ \sum_j |I_j|^s : E \subset \bigcup_j I_j \right\},
\]
where the infimum is taken over all coverings of \( E \) by countable families of open arcs \( I \subset \mathbb{T} \).

Although we think that the next result is known, a proof is included here since we were not able to find any clear reference.

**Lemma 3.** Let \( 0 < s \leq 1 \). Then there exists a universal constant \( C \) such that \( \Lambda^\infty_s(E) \geq C|E|^s \) for all \( E \subset \mathbb{T} \).

**Proof.** Let \( E \subset \mathbb{T} \). If \( |E| = 0 \), the result is clear. Suppose that \( |E| > 0 \) and take \( \varepsilon \in \left(0, \frac{|E|^s}{2}\right) \). Then there exists a covering \( \{I_j\}_j \) of \( E \), such that
\[
\Lambda^\infty_s(E) \geq \sum_j |I_j|^s - \varepsilon \geq \left( \sum_j |I_j| \right)^s - \varepsilon \geq |E|^s - \frac{|E|^s}{2} = \frac{|E|^s}{2}.
\]
This finishes the proof. \( \square \)

The homogeneous \( D_s \)-capacity of a set \( E \subset \mathbb{T} \) is defined by
\[
\text{cap} (E, D_s) = \inf \left\{ ||f||^2_{C_s} : f \in L^2(\mathbb{T}) \text{ and } f \geq 1 \text{ a.e. on } E \right\}.
\]

**Lemma 4.** Let \( J \subset \mathbb{T} \) be an open arc with center \( e^{i\theta_0} \). Suppose that \( F \in D_s \) with \( E = \{e^{it} \in J : |F(e^{it})| \geq 1\} \).

If \( |E| \geq \frac{|J|}{2} \), then there exists a universal constant \( C \) such that
\[
\int_{S(J)} |F'(z)|^2 (1 - |z|^2)^s \, dA(z) \geq C|J|^s.
\]

**Proof.** Let \( z_0 = (1 - \frac{|J|}{2}) e^{i\theta_0} \). Arguing as in the proof of [36] Lemma 3, we deduce that there is a universal constant \( C \) such that the harmonic measure of \( E \) with respect to \( Q := S(J) \) at \( z_0, \mu_{z_0}^Q(E) \), satisfies
\[
\mu_{z_0}^Q(E) \geq C.
\]

Consider a conformal map \( \varphi : \mathbb{D} \to Q \) with \( \varphi(0) = z_0 \) and take \( g = F \circ \varphi \). Then \( g \geq 1 \) on \( \varphi^{-1}(E) \) and \( |\varphi^{-1}(E)| = \mu_{z_0}^Q(E) \geq C \). Thus, putting together (5.1.3) of [1]
and Lemma 3 we have

\[ \|g\|_{L^2(D_s)} \geq \text{cap} (\varphi^{-1}(E), D_s) \geq C \left( A_{s'}^\varphi (\varphi^{-1}(E)) \right)^\gamma \geq C \mu^Q_{s_0}(E)^{s'_\gamma} \geq C, \]

where \( s' \in (s, 1) \) and \( \gamma \in (0, 1) \).

Next, since \( \varphi \) is a conformal map (see [34, Chapter 1]),

\[ |\varphi'(z)| \geq \frac{1}{4} |\varphi'(0)| \geq C d(z_0, \partial Q) \geq C |J|, \]

Moreover, since \( Q \) is convex, reasoning as in [20, Proposition 5] and bearing in mind (3.2) we obtain that

\[ |\varphi'(z)| \geq \frac{1}{4} |\varphi'(0)| \geq C d(z_0, \partial Q) \geq C |J|, \]

where \( d(z_0, \partial Q) \) is the Euclidean distance from \( z_0 \) to \( \partial Q \).

Taking into account (3.1), (3.2) and (3.3) we deduce that

\[ \int_Q |F'(z)|^2 \left( 1 - |z|^2 \right)^s dA(z) \geq \int_Q |F'(z)|^2 d(z, \partial Q)^s dA(z) \]

\[ \geq \int_{\mathbb{D}} |g'(z)|^2 d(\varphi(z), \partial Q)^s dA(z) \]

\[ \geq C \int_{\mathbb{D}} |g'(z)|^2 \left( (1 - |z|^2)|\varphi'(z)| \right)^s dA(z) \]

\[ \geq C |J|^s \int_{\mathbb{D}} |g'(z)|^2 (1 - |z|^2)^s dA(z) \]

\[ \geq C |J|^s. \]

This finishes the proof.  \( \Box \)

**Proof of Theorem 4.** Let \( \{r_n\} \subset (0, 1) \) be an increasing sequence such that

\[ \sum_n (1 - r_n)^s = \infty. \]

We can find

\[ 1 \leq n_1 < m_1 < n_2 < m_2 < \cdots < n_k < m_k < \cdots \]

such that

\[ (1 - r_n)^{1-s} < k^{-2} e^{-2k^2} \quad \text{if} \quad n \geq n_k, \quad k = 1, 2, \ldots \]

and

\[ ke^{2k^2} \leq \sum_{n=n_k}^{m_k} (1 - r_n)^{s} < ke^{2k^2} + 1, \quad k = 1, 2, \ldots. \]

For each \( k \), lay out arcs \( J_{n_k}, J_{n_k+1}, \ldots, J_{m_k} \) on the unit circle end-to-end starting at \( e^{i\theta} = 1 \) and such that

\[ |J_n| = (1 - r_n)^{s} k^{-2} e^{-2k^2}, \quad n_k \leq n \leq m_k. \]

Observe that (3.4) together with (3.5) implies that

\[ |J_n| > (1 - r_n). \]
Let $e^{i\theta_n}$ be the center of $J_n$ and set $\lambda_n = (1 - r_n)e^{i\theta_n}$. Suppose that there is $F \in \mathcal{D}_s$ with $F(\lambda_n) = 0$ for all $n_k \leq n \leq m_k$. By [6, Theorem 3.4] we may assume that $||F||_{H^\infty} \leq 1$. Set

$$A_k = \left\{ n : n_k \leq n \leq m_k \text{ and } |F| \geq e^{-k^2} \text{ on a set } E_n \subset J_n \text{ with } |E_n| \geq \frac{|J_n|}{2} \right\},$$

$$B_k = \left\{ n : n_k \leq n \leq m_k, \ n \notin A_k \right\}.$$

Using Lemma 4 and (3.6) with $S(J_n)$, $n \in A_k$, we deduce that

$$\int_{S(J_n)} |F'(z)|^2 (1 - |z|^2)^{s} \ dA(z) \geq Ce^{-2k^2} |J_n|^s \geq Ce^{-2k^2} (1 - r_n)^s.$$

Moreover if $n \in B_k$,

$$\int_{J_n} \log \frac{1}{|F(\xi)|} \ d\xi \geq \frac{1}{2} k^2 |J_n| = \frac{1}{2} (1 - r_n)^s e^{-2k^2}.$$

So, bearing in mind (3),

$$\sum_{n \in A_k} \int_{S(J_n)} |F'(z)|^2 (1 - |z|^2)^{s} \ dA(z) + \sum_{n \in B_k} \int_{J_n} \log \frac{1}{|F(\xi)|} \ d\xi$$

$$\geq Ce^{-2k^2} \sum_{n = n_k}^{m_k} (1 - r_n)^s \geq Ck,$$

which together with the integrability of $\log |F|$ on the boundary (see Theorem 2.2 of [18]), implies that $F$ must be the zero function. Finally, arguing as in the proof of Theorem 2 of [36], the proof can be finished.

**4. Zeros on the boundary. Sets of uniqueness**

In order to prove Theorem 5 the notion of $\alpha$-capacity must be introduced. We shall recall some definitions (see [41] and [8]). Given $E \subset [0, 2\pi)$, let $\mathcal{P}(E)$ be the set of all probability measures supported on $E$. If $\alpha > 0$ and $\sigma \in \mathcal{P}(E)$, the $\alpha$-potential associated to $\sigma$ is

$$U_{\alpha}\sigma(\tau) = \int_{E} \frac{d\sigma(\theta)}{|	heta - \tau|^\alpha}.$$

Let

$$V_{E, \alpha} = \inf \int_{E} U_{\alpha}\sigma(\tau) \ d\sigma(\tau),$$

where the infimum is taken over all $\sigma \in \mathcal{P}(E)$. If $V_{E, \alpha} < \infty$, there is $\mu \in \mathcal{P}(E)$ where the value $V_{E, \alpha}$ is attained, and that measure $\mu$ is called the equilibrium distribution for the $\alpha$-potentials of $E$. It is known that $U_{\alpha}\mu(\tau) = V_{E, \alpha}$ for a.e. ($\mu$).

The $\alpha$-capacity of $E$ is determined by

$$C_{\alpha}(E) = (V_{E, \alpha})^{-1}.$$

**Proof of Theorem 5.** Suppose that $E$ is a set of uniqueness for $\mathcal{D}_s$. Then $E$ is also a set of uniqueness for any Lipschitz class $\Lambda_\beta$ with $\beta > \frac{1}{2}$, due to $\Lambda_\beta \subset \mathcal{D}_s$. So, by Theorem 1 of [9], $E$ is not a Carleson set.

For the converse, we shall follow the argument in the proof of Theorem 5 in [9]. Let $\mu$ be the equilibrium distribution for the $\alpha$-potentials of $E$. Then, if $\{\gamma_n\}$ are
the Fourier-Stieltjes coefficients of $\mu$, there is a constant $C$ which only depends on $\alpha$ such that
\begin{equation}
\sum_n n^{\alpha - 1} |\gamma_n|^2 \leq CV_{E, \alpha}.
\end{equation}

Suppose that there is a bounded function $f \in D$, $f \neq 0$, that vanishes on $E$. We shall see that this leads to a contradiction. The function $h(\theta) = |f(e^{i\theta})|$ can be written as
\[ h(\theta) = \sum_n c_n e^{in\theta}, \]
where
\begin{equation}
\sum_n n^{1-s} |c_n|^2 < \infty.
\end{equation}

For each $t \in (0, \pi)$, let us consider $h_t(\theta) = \frac{1}{2t} \int_{\theta - t}^{\theta + t} h(s) \, ds$. Integrating the Fourier series of $h$, it follows that the Fourier coefficients of $h_t$ are $\frac{\sin(nt)}{nt} c_n$. Then by (4.1) and Schwarz’s inequality,
\begin{equation}
\left| \int_E h_{t}(\theta) \, d\mu(\theta) \right| = \left| \int_E (h_{t}(\theta) - h(\theta)) \, d\mu(\theta) \right|
= \left| \sum_n \left(1 - \frac{\sin(nt)}{nt}\right) c_n \int_E e^{in\theta} \, d\mu(\theta) \right|
\leq C \sum_n \left(1 - \frac{\sin(nt)}{nt}\right) |c_n||\gamma_n|
\leq C \left(\sum_n \left(1 - \frac{\sin(nt)}{nt}\right)^2 |c_n|^2 n^{1-\alpha}\right)^{\frac{1}{2}} \left(\sum_n n^{\alpha - 1} |\gamma_n|^2\right)^{\frac{1}{2}}.
\end{equation}

We claim that there is $C > 0$ such that
\begin{equation}
n^{s-\alpha} \left(1 - \frac{\sin(nt)}{nt}\right)^2 \leq Ct^{\alpha - s}, \quad t > 0, \quad n = 1, 2, \ldots.
\end{equation}
If $nt \leq 1$, there is a positive constant $C$ which does not depend on $n$ or $t$, such that $1 - \frac{\sin(nt)}{nt} \leq C(nt)^2$, so
\begin{equation}
n^{s-\alpha} \left(1 - \frac{\sin(nt)}{nt}\right)^2 \leq C^2 n^{s-\alpha}(nt)^4 \leq C^2 n^{s-\alpha}(nt)^{\alpha-s} \leq C^2 t^{\alpha-s}.
\end{equation}

On the other hand, if $nt \geq 1$, bearing in mind that $1 - \frac{\sin(\theta)}{\theta}$ is a bounded function of $\theta$, we deduce that
\[ n^{s-\alpha} \left(1 - \frac{\sin(nt)}{nt}\right)^2 \leq Cn^{s-\alpha} \leq Ct^{\alpha-s}, \]
which together with (4.4) gives (4.4).

Therefore, using (4.3), (4.5), (4.1) and (4.2), it follows that
\begin{equation}
\int_E h_{t}(\theta) \, d\mu(\theta) \leq Ct^{\frac{s}{\alpha-1}} \left(\sum_n n^{s-\alpha} |c_n|^2\right)^{\frac{1}{2}} \left(\sum_n n^{\alpha - 1} |\gamma_n|^2\right)^{\frac{1}{2}}
\leq Ct^{\frac{s}{\alpha-1}} \|f\|_{D_2} \|V^{1/2}_{E, \alpha}}.
\end{equation}
Now, let $k_n$ be the number of complementary intervals of $E$ whose lengths are in $[2^{-n}, 2^{-n+1})$. Since $E$ is not a Carleson set,

$$(4.7) \quad \sum \frac{n k_n}{2^n} = \infty.$$ 

Let $\{\omega_i\}_{i=1}^{k_n}$ be those intervals, and let $\{\theta_i\}_{i=1}^{2 k_n}$ be the endpoints of $\{\omega_i\}_{i=1}^{k_n}$. We consider the open intervals $\\{\delta_i\}_{i=1}^{2 k_n}$ of length $2^{-n}$ with midpoints $\{\theta_i\}_{i=1}^{2 k_n}$. Take $\gamma \in \left(0, \frac{\alpha}{2}\right)$ and let $S$ be the set of those $\delta_i$ such that

$$(4.8) \quad h_\tau(\theta_i) > 2^{-\gamma n}, \quad \tau = 2^{-n}.$$ 

Observe that (4.8) implies that $h_\tau(x) > 2^{-\gamma n-1}$ holds for $\theta \in \delta_i$ whenever $\delta_i \in S$, which, together with the general relation (4.6), gives that for $\mu^*$ the equilibrium distribution for the $\alpha$-potentials of $E \cap S$,

$$2^{-\gamma n-1} \leq \int_{E \cap S} h_\tau(\theta) \, d\mu^*(\theta) \leq CV^{1/2}_{E \cap S} 2^{-\gamma n} (\alpha - s),$$

so

$$(4.9) \quad C_\alpha(E \cap S) \leq C2^{(2\gamma-(\alpha-s))n}.$$ 

Let $N$ be the number of intervals $\delta_i$ which belong to $S$. We shall estimate $N$ using condition (1.6). Take $\mu_i$ to be the equilibrium distribution for the $\alpha$-potentials of $E \cap \delta_i$. Let us consider $\sigma = N^{-1} \sum_{\delta_j \subset S} \mu_i$ and $u$ the corresponding $\alpha$-potential. Suppose that $\tau \in \delta_k$, where $\delta_k \in S$, and let $\delta_{k-1}$ and $\delta_{k+1}$ be the intervals in $S$ which are on the left and on the right of $\delta_k$. We shall define $F = \{k-1, k, k+1\}$. Then bearing in mind that the intervals $\{\delta_j\}$ are disjoint, the distance between the intervals $\{\delta_j\}$, and condition (1.6) we deduce that

$$u(\tau) = \int_{E \cap S} \frac{d\sigma(\theta)}{\theta - \tau} \leq \sum_{j \in F} \int_{\delta_j \cap S} \frac{d\sigma(\theta)}{\theta - \tau} + \sum_{j=1, j \notin F}^{N} \int_{\delta_j \cap S} \frac{d\sigma(\theta)}{\theta - \tau} \leq N^{-1} \left( \sum_{j \in F} \int_{\delta_j \cap S} \frac{d\mu_j(\theta)}{\theta - \tau} + \sum_{j=1, j \notin F}^{N} \int_{\delta_j \cap S} \frac{d\mu_j(\theta)}{\theta - \tau} \right) \leq CN^{-1} \left( 2^n + \sum_{j=1}^{N} \frac{1}{(j2^{-n})^\alpha} \right) \leq CN^{-1} 2^n,$$

which together with (4.3) gives

$$N^{-1} 2^n \geq C \frac{u}{C_\alpha(E \cap S)} \geq C2^{(2\gamma+(\alpha-s))n},$$

so due to $\gamma < \frac{\alpha-s}{2}$, one obtains

$$(4.10) \quad N \leq C 2^{pn}, \quad \text{for some } p \in (0, 1).$$
If \( \omega_\nu = (\theta_{2\nu-1}, \theta_{2\nu}) \) and (4.8) does not hold for \( \theta_{2\nu-1} \) and \( \theta_{2\nu} \), then by the arithmetic-geometric inequality,

\[
\frac{1}{|\omega_\nu|} \int_{\omega_\nu} \log h(\theta) \, d\theta \leq \log \left( \frac{1}{|\omega_\nu|} \int_{\omega_\nu} h(\theta) \, d\theta \right) \\
\leq \log \left[ \frac{1}{|\omega_\nu|} \left( \int_{\theta_{2\nu-1}+2^{-n}}^{\theta_{2\nu}+2^{-n}} h(\theta) \, d\theta + \int_{\theta_{2\nu-1}-2^{-n}}^{\theta_{2\nu}-2^{-n}} h(\theta) \, d\theta \right) \right] \\
= \log \left[ \frac{2^{-n+1}}{|\omega_\nu|} (h_+(\theta_{2\nu-1}) + h_-(\theta_{2\nu})) \right] \\
\leq -\gamma n + C.
\]

By (4.10), the number of indices \( n \) for which the above inequality is true is greater than \( k_n - 2N \geq k_n - C2^p n \). Hence

\[
\sum_{\nu=1}^{k_n} \int_{\omega_\nu} \log h(\theta) \, d\theta \leq -\gamma n 2^{-n} (k_n - C2^p n) + C \sum_{\nu=1}^{k_n} |\omega_\nu|,
\]

which, joined to the fact that \( p < 1 \), gives

\[
\int_0^{2\pi} \log h(\theta) \, d\theta \leq -\gamma \sum_n n 2^{-n} k_n + C.
\]

Consequently, bearing in mind that \( \gamma > 0 \) and (4.7), this implies a contradiction. \( \square \)

5. Blaschke sets

A subset \( A \) of the unit disc \( \mathbb{D} \) is called a Blaschke set for \( \mathcal{D} \) if any Blaschke sequence with elements in \( A \) is a zero set of \( \mathcal{D} \). These sets were characterized by Bogdan in [2]. Here we shall give a new proof of that result.

Theorem 6. \( A \subset \mathbb{D} \) is a Blaschke set for \( \mathcal{D} \) if and only if

\[
(5.1) \quad \int_\mathbb{T} \log \text{dist}(e^{it}, A) \, dt > -\infty.
\]

Some definitions and results will be introduced. A tent is an open subset \( T \) of \( \mathbb{D} \) bounded by an arc \( I \subset \mathbb{T} \) with \( |I| < \frac{1}{4} \) and two straight lines through the endpoints of \( I \) forming with \( I \) an angle of \( \frac{\pi}{2} \). The closed arc \( \overline{I} \) will be called the base of the tent \( T = T_I \). A tent \( T \) is said to support \( A \) if \( T \cap A = \emptyset \) but \( \overline{T} \cap \overline{A} \neq \emptyset \). A finite or countable collection of tents \( \{T_n\} \) is an A-belt if \( \{T_n\} \) are pairwise disjoint, A-supporting and \( \mathbb{T} \setminus \overline{A} \subset \bigcup_n \overline{T_n} \). The following result can be found in [24, Lemma 1].

Lemma B. Let \( A \subset \mathbb{D} \) such that \( \overline{T} \setminus \overline{A} \neq \emptyset \). Let \( \{T_n\} \) be an A-belt. Then (5.1) holds if and only if \( \overline{A} \cap \mathbb{T} \) has zero Lebesgue measure, and

\[
\sum_n |I_n| \log \left( \frac{e}{|I_n|} \right) < \infty.
\]

Lemma 5. Let \( \{z_n\} \) be a \( \mathcal{D} \)-zero set. If \( \{\lambda_n\} \subset \mathbb{D} \) satisfies that \( g(z_n, \lambda_n) < \delta < 1 \) for each \( n \), then \( \{\lambda_n\} \) is a \( \mathcal{D} \)-zero set.
Proof. Since $Z = \{z_n\}$ is a $D$-zero set, there is a function $g$ in $D$ such that $gB_Z \in D,$ where $B_Z$ is the Blaschke product with zeros $\{z_n\}$. By Carleson’s formula for the Dirichlet integral (see [11] and also [35]), we have

$$\|gB_\Lambda\|_2^2 = \|g\|_2^2 + \int_T \sum_n P_{\lambda_n}(e^{it}) |g(e^{it})|^2 \, dt$$

$$\leq \|g\|_2^2 + C \int_T \sum_n P_{z_n}(e^{it}) |g(e^{it})|^2 \, dt$$

$$\leq C \|gB_Z\|_2^2 < \infty.$$ 

Hence, $\{\alpha_n\}$ is a $D$-zero set, and the proof is complete. \hfill $\square$

Remark 1. Note that this result implies that, if $A$ is a Blaschke set for $D$ and $\{w_k\}$ is a sequence such that $\varrho(\{w_k\}, A) \leq C < 1,$ then $A \cup \{w_k\}$ is also a Blaschke set for $D$.

Proof of Theorem 6. Suppose that (5.1) holds, and let $Z$ be a Blaschke sequence of points in $A$. Then

$$\int_T \log \text{dist}(e^{it}, Z) \, dt > -\infty,$$

and by a result of Taylor and Williams in [40], $Z$ is a $\Lambda_\alpha$-zero set for any $\alpha$. Since $\Lambda_\alpha \subset D$ for $\alpha > \frac{1}{2},$ it follows that $A$ is a Blaschke set for $D$.

Suppose that $A$ is a Blaschke set for $D$. We shall use Lemma B to see that (5.1) holds. Suppose that $|A \cap T| > 0.$ Then we can choose a sequence $\{\varepsilon_n\}$ of positive numbers satisfying

$$\sum_n \varepsilon_n \leq |A \cap T|, \quad \sum_n \varepsilon_n \log \frac{1}{\varepsilon_n} = \infty,$$

and a collection of disjoint arcs $\{I_n\}$ in $T$ such that

$$|I_n| = \varepsilon_n, \quad I_n \cap \overline{A} \neq \emptyset, \quad n \geq 1.$$ 

In order to construct this sequence of subsets $\{I_n\}$, take $I_1$ with $|I_1| = \varepsilon_1$ and $I_1 \cap \overline{A} \neq \emptyset$, and once $I_n$ has been taken, choose $I_{n+1}$ such that $I_{n+1} \cap \left(\overline{A} \setminus \bigcup_{j=1}^n I_j\right) \neq \emptyset$ with $|I_{n+1}| = \varepsilon_{n+1}.$

Next, take a sequence $\{w_n\} \subset A$ such that $\text{dist}(w_n, I_n \cap \overline{A}) \leq \varepsilon_n$ and let $p_n$ be the integer part of $\varepsilon_n/(1 - |w_n|).$ Let $Z$ be the sequence of points in $A$ that consists of $p_n$ repetitions of each point $w_n.$ Observe that $Z$ is a Blaschke sequence,

$$\sum_{z \in Z} (1 - |z|) = \sum_n p_n (1 - |w_n|) \leq \sum_n \varepsilon_n < \infty,$$
so that \( Z \) must be a sequence of zeros of \( D \). We also have
\[
\int_T \log \left( \sum_{z \in Z} \frac{1 - |z|^2}{|e^{it} - z|^2} \right) \, dt = \int_T \log \left( \sum_n p_n \frac{1 - |w_n|^2}{|e^{it} - w_n|^2} \right) \, dt
\]
\[
\geq \sum_k \int_{I_k} \log \left( \frac{1 - |w_k|^2}{4\pi^2} \right) \, dt
\]
\[
\geq \sum_k |I_k| \log \left( \frac{1}{8\varepsilon_k} \right) = \infty,
\]
which gives a contradiction with condition (1.5). Therefore, \( \overline{A} \cap T \) has zero Lebesgue measure.

Next, let \( \{T_n\} \) be an \( A \)-belt. Then for each \( n \) there is \( w_n \in \overline{A} \cap \partial T_n \). We may assume that \( w_n \) belongs to \( A \). Indeed, if \( w_n \) is an endpoint of the arc \( I_n \), there is a point \( \alpha_n \in A \) which is in the Stolz angle with vertex \( w_n \) and aperture \( \varepsilon/2 \). Consequently, if \( \tilde{\alpha}_n \) is the closest point in \( \partial T_n \) with the same modulus as \( \alpha_n \), then \( g(\alpha_n, \tilde{\alpha}_n) \leq C < 1 \), where \( C \) is independent of \( n \), and now we can use the remark after Lemma 5.

Let \( v_n \) be the vertex of the tent \( T_n \). Since \( \{I_n\} \) is a sequence of disjoint arcs, \( \{v_n\} \) is a Blaschke sequence. We denote by \( q_n \) the integer part of \( (1 - |v_n|)/(1 - |w_n|) \) and we consider \( Z \) to be the sequence of points in \( A \) that consists of \( q_n \) repetitions of each point \( w_n \). Arguing as before, it follows that \( Z \) is a Blaschke sequence, and moreover there is \( C > 0 \) such that
\[
|w_n - e^{it}|^2 \leq C|v_n - e^{it}|^2, \quad \text{for each } n \text{ and } e^{it} \in T.
\]
So, bearing in mind that \( A \) is a Blaschke set for \( D \), (1.5) and (5.2), we have that
\[
\infty > \int_T \log \left( \sum_{z \in Z} \frac{1 - |z|^2}{|e^{it} - z|^2} \right) \, dt = \int_T \log \left( \sum_n q_n \frac{1 - |w_n|^2}{|e^{it} - w_n|^2} \right) \, dt
\]
\[
\geq \int_T \log \left( C \sum_n q_n \frac{1 - |w_n|^2}{1 - |v_n|^2} \frac{1 - |v_n|^2}{|e^{it} - v_n|^2} \right) \, dt
\]
\[
\geq \int_T \log \left( \sum_n C \frac{1 - |v_n|^2}{|e^{it} - v_n|^2} \right) dt
\]
\[
\geq \sum_k \int_{I_k} \log \left( \frac{C}{|I_k|} \right) \, dt
\]
This finishes the proof. \( \square \)

6. Other results

6.1. Other necessary angular conditions on \( D_\alpha \)-zero sets. First we shall prove the following result of its own interest.
Lemma 6. Suppose that $0 < s < 1$, $B$ is a Blaschke product with ordered sequence of zeros $\{z_k\}_{k=1}^\infty$ and $f \in \mathcal{D}_s$. Then

$$\|fB\|_{\mathcal{D}_s}^2 \asymp \|f\|_{\mathcal{D}_s}^2 + \sum_{k=1}^\infty (1 - |z_k|^2) \int_{\mathbb{D}} \frac{|f(z)|^2 |B_k(z)|^2}{|1 - z_k z|^2} \frac{dA(z)}{(1 - |z|^2)^{1-s}},$$

where $B_k(z)$ is the Blaschke product of the first $k - 1$ zeros.

Proof. Bearing in mind (2.2), the result follows from the identity (see [3, p. 191])

$$\frac{1 - |B(z)|^2}{1 - |z|^2} = \sum_k |B_k(z)|^2 \frac{1 - |z_k|^2}{|1 - z_k z|^2}, \quad z \in \mathbb{D}.$$

We also obtain different conditions from (1.4) (which can work for any Blaschke sequence) on the angular distribution of a Blaschke sequence $\{z_k\}$ to be a $\mathcal{D}_s$-zero set, $0 < s < 1$.

Proposition 1. Suppose that $0 < s < 1$ and $\{z_k\} \subset \mathbb{D}$. If there exists $r_0 \in (0, 1)$ such that

$$(6.1) \quad M(\{z_k\}) \overset{\text{def}}{=} \inf_{r_0 \leq |z| < 1} \sum_k (1 - |z_k|^2)(1 - |z|^2)^s \frac{(1 - |z|^2)}{|1 - z_k z|^2} > 0,$$

then $\{z_k\}$ is not a $\mathcal{D}_s$-zero set.

Proof. Suppose that $\{z_k\}$ is a $\mathcal{D}_s$-zero set and satisfies (6.1). Then, there exists $F \in \mathcal{D}_s$ which vanishes uniquely on $\{z_k\}$, so $F = f \cdot B$, where $f \in \mathcal{D}_s$ and $B$ is the Blaschke product with zeros $\{z_k\}$. Thus, Lemma 6 and (6.1) imply that

$$\lim_{k \to \infty} \sum_k (1 - |z_k|^2) \int_{\mathbb{D}} \frac{|f(z)|^2 |B_k(z)|^2}{|1 - z_k z|^2} \frac{dA(z)}{(1 - |z|^2)^{1-s}} \geq \int_{\mathbb{D}} |f(z)|^2 |B(z)|^2 \left( \sum_k \frac{(1 - |z_k|^2)(1 - |z|^2)^s}{|1 - z_k z|^2} \right) \frac{dA(z)}{(1 - |z|^2)}$$

$$\geq M(\{z_k\}) \int_{\mathbb{D}} |F(z)|^2 \frac{dA(z)}{(1 - |z|^2)};$$

consequently $F \equiv 0$. This finishes the proof.

This result allows us to make constructions of Blaschke sequences which are not $\mathcal{D}_s$-zero sets.

Corollary 2. For $0 < s < 1$, set

$$z_{k,j}^{(s)} \overset{\text{def}}{=} \left(1 - 2^{-\frac{s}{1+s}}\right) \exp\left(\frac{2\pi j}{2^k - 1}\right), \quad k = 0, 1, 2, \ldots,$$

$$j = 0, 1, \ldots, 2^k - 1.$$

The sequence $\{z_{k,j}^{(s)}\}$ is not a $\mathcal{D}_s$-zero set.

Proof. There is $\beta = \beta(s) > 0$ such that for each $z \in \mathbb{D}$ we can find a pair $(k(z), j(z))$ with $1 - |z| \asymp 1 - |z_{k(z), j(z)}^{(s)}|$, and

$$|1 - \overline{z_{k(z), j(z)}^{(s)}}z|^2 \leq \beta(1 - |z|^2)^{1+s}.$$
Therefore
\[
\sum_{k=0}^{\infty} \sum_{j=0}^{2^k-1} \frac{(1 - |z_{k,j}|^2)(1 - |z|^2)^s}{|1 - \overline{z_{k,j}}z|^2} \geq \frac{(1 - |z(\tau)|^2)(1 - |z|^2)^s}{|1 - \overline{z(\tau)}z|^2} \geq C\beta^{-1},
\]
so, by Proposition 2 \(\{z_{k,j}\}\) is not a \(D_s\)-zero set.

6.2. Möbius invariant spaces generated by \(D_s\). The space \(Q_s\), \(0 \leq s < \infty\), is the Möbius invariant space generated by \(D_s\), that is, \(f \in Q_s\) if
\[
\sup_{a \in \mathbb{D}} \|f \circ \varphi_a - f(a)\|_{D_s}^2 < \infty.
\]

It is known that \(Q_1\) coincides with \(BMOA\). However, if \(0 < s < 1\), \(Q_s\) is a proper subspace of \(BMOA\) and has many interesting properties (see the detailed monograph \[42\]).

As usual, for a space of analytic functions \(X\), we shall write \(M(X)\) for the algebra of (pointwise) multipliers of \(X\), that is,
\[
M(X) \overset{\text{def}}{=} \{g \in H(\mathbb{D}) : gf \in X \text{ for all } f \in X\}.
\]

**Theorem 7.** Suppose that \(0 < s \leq 1\). Then \(D_s\), \(Q_s\), \(Q_s \cap H^\infty\) and \(M(D_s)\) have the same zero sets.

**Proof.** If \(s = 1\), the result is well known because \(D_1 = H^2\), \(M(H^2) = H^\infty\) and \(Q_1 = BMOA\). If \(0 < s < 1\), by \[26\] Corollary 13 the zeros sets of \(D_s\) and \(M(D)\) coincide, so the result follows from the chain of embeddings (see \[4\] Lemma 5.1)
\[
M(D_s) \subset Q_s \cap H^\infty \subset Q_s \subset D_s.
\]

This finishes the proof. \(\square\)

Since from different values of \(s \in (0, 1)\), the \(D_s\)-zero sets are not the same, we obtain directly the following result.

**Corollary 3.** Suppose that \(0 \leq s < p < 1\). Then there exists \(Z \subset \mathbb{D}\), which is a \(Q_p\)-zero but not a \(Q_s\)-zero set.

A stronger result, in the following sense, can be proved. A sequence \(\{z_n\}\) is interpolating for \(Q_p \cap H^\infty\), \(0 < p < 1\), if for each bounded sequence \(\{w_k\}\) of complex numbers, there exists \(f \in Q_p \cap H^\infty\) such that \(f(z_k) = w_k\) for all \(k\). A characterization of these sequences in terms of \(p\)-Carleson measures is given in \[30\]. It is clear that each interpolating sequence for \(Q_p \cap H^\infty\) is a \(D_p\)-zero set.

**Theorem 8.** Suppose that \(0 < s < p < 1\). Then, there exists \(Z = \{z_n\}_{n=0}^\infty \subset \mathbb{D}\) which is an interpolating sequence for \(Q_p \cap H^\infty\) and such that it is not a \(D_s\)-zero set.

**Proof.** Set
\[
z_n = \left(1 - \frac{1}{n^{1/s}}\right) e^{i\theta_n}, \quad n = 2, 3, \ldots,
\]
where
\[
\theta_n = \sum_{k=1}^{n-1} \frac{1}{k + \frac{1}{2n}}, \quad n = 2, 3, \ldots.
\]

The proof of \[29\] Theorem 5.10 \[\|\] gives that \(\{z_n\}\) is not a \(D_s\)-zero set. Moreover, borrowing the argument of the proof of \[32\] Theorem 2, we have that \(\{z_n\}\) is
Suppose that measures. Using Corollary 3 as a main tool we shall prove the following result.

Finally, we note that in a recent paper [31], the algebra of (pointwise) multipliers of $Q_s$, $0 < s < 1$, has been characterized in terms of $\alpha$-logarithmic $s$-Carleson measures. Using Corollary 3 as a main tool we shall prove the following result.

**Corollary 4.** Suppose that $0 < s < p < 1$. Then

$$M(Q_p, Q_s) \equiv \{ g \in H(D) : gf \in Q_s \text{ for all } f \in Q_p \} = \{ 0 \}.$$  

**Proof.** Suppose that $M(Q_p, Q_s) \neq \{ 0 \}$. Let $g \in M(Q_p, Q_s)$, $g \neq 0$ and denote by $W$ its zero set. By Corollary 3 there exists $f \in Q_p$, $f \neq 0$, whose sequence of zeros $Z$ is not a $Q_s$-zero set. It is clear that $Z \cup W$ is the zero set of $fg \in Q_s$, and since $g \in Q_s$, $W$ satisfies the Blaschke condition. Now, taking $B$ to be the Blaschke product with zeros $W$ and bearing in mind that $Q_s$ has the $f$-property (see Corollary 1 of [14] or Corollary 5.4.1 of [42]), we obtain that $\frac{f}{g} \in Q_s$, whose zero set is $Z$. This finishes the proof. □

7. FURTHER REMARKS

We would like to emphasize that conditions $(ii)$ and $(iii)$ of Theorem 1 are equivalent when $\{z_n\}$ is a finite union of separated Blaschke sequences. So, it seems natural to ask whether or not for finite unions of separated Blaschke sequences, condition $(ii)$ implies that $\{z_n\}$ is a $D_s$-zero set. Although we are not able to answer this question, if the function $g$ has some additional regularity properties, one can prove that condition $(ii)$ implies that $\{z_n\}$ is a $D_s$-zero set, as the following result shows.

**Proposition 2.** Let $\{z_n\} \subset D$ be a Blaschke sequence, $0 < s < 1$ and $\alpha > \frac{1-s}{2}$. If there exists a function $g \in \Lambda_\alpha$ such that

$$\sum_n |g(z_n)|^2 (1 - |z_n|^2)^s < \infty,$$

then $\{z_n\}$ is a $D_s$-zero set.

**Proof:** Let $B$ be the Blaschke product with zeros $\{z_n\}$. We shall prove that $gB \in D_s$. Using the fact that $g \in \Lambda_\alpha$, and [43] Lemma 4.2.2, one has

$$\sum_n (1 - |z_n|^2) \int_D |g(z) - g(z_n)|^2 \frac{(1 - |z|^2)^{s-1}}{|1 - \bar{z}_n z|^2} dA(z)$$

$$\leq C \sum_n (1 - |z_n|^2) \int_D (1 - |z|^2)^{s-1} \frac{1}{|1 - \bar{z}_n z|^2} dA(z)$$

$$\leq C \sum_n (1 - |z_n|^2) < \infty.$$  

(7.1)

Also, by our assumption and [43] Lemma 4.2.2,

$$\sum_n (1 - |z_n|^2) |g(z_n)|^2 \int_D \frac{(1 - |z|^2)^{s-1}}{|1 - \bar{z}_n z|^2} dA(z)$$

$$\leq C \sum_n |g(z_n)|^2 (1 - |z_n|^2)^s < \infty.$$  

(7.2)
Now, since $\Lambda_{\alpha} \subset D_s$ for $\alpha > \frac{1-s}{2}$, it follows easily from (7.1) and (7.2) that
\[
\|gB\|^2_{D_s} \leq C\|g\|^2_{D_s} + C \int_D |(gB')(z)|^2 (1-|z|^2)^s \, dA(z) < \infty.
\]
□

In view of all this, we state the following related problem.

**Problem.** For $0 < s < 1$, describe those separated Blaschke sequences $\{z_n\} \subset \mathbb{D}$ such that there is $g \in D_s$, $g \neq 0$, with
\[
\sum_n |g(z_n)|^2 (1-|z_n|^2)^s < \infty.
\]

Another interesting problem is to find sufficient conditions in order for a sequence $\{z_n\}$ to be a zero set for the analytic Besov space $B_p$, $1 < p < \infty$ (see [33, Chapter 5]). Since the point evaluations are bounded linear functionals in $B_p$, there are reproducing kernels $k_z \in B_p'$, where $p'$ is the conjugate exponent of $p$. Also, it is well known that
\[
\|k_z\|^{-p}_{B_p'} \simeq \left( \log \frac{1}{1-|z|} \right)^{-(p-1)}.
\]
So, bearing in mind (1.1), it seems natural to ask the following.

**Question.** Let $1 < p < \infty$, and let $\{z_n\} \subset \mathbb{D}$ such that
\[
\sum_n \left( \log \frac{1}{1-|z_n|^2} \right)^{-(p-1)} < \infty.
\]
Is the sequence $\{z_n\}$ a $B_p$-zero set?

In order to answer that question, it seems that a more constructive proof of the case $p = 2$ (the Shapiro-Shields result [39]) must be given, not relying so heavily on Hilbert space techniques.

**References**


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