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Volterra type operators on Bergman spaces with exponential weights

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Abstract. In this paper we characterize the boundedness, compactness and membership in Schatten p-classes of Volterra type operators on Bergman spaces with exponential weights.

1. Introduction and main results

Let $\mathbb{D}$ be the unit disc in the complex plane, $dm(z) = \frac{dx\,dy}{\pi}$ be the normalized area measure on $\mathbb{D}$, and denote by $H(\mathbb{D})$ the space of all analytic functions in $\mathbb{D}$. If $g \in H(\mathbb{D})$, we consider the linear operator J_g defined by

$$(J_g f)(z) = \int_0^z f(\zeta)\, g'(\zeta)\, d\zeta, \qquad f \in H(\mathbb{D}).$$

This operator was introduced by C. Pommerenke in [**9**] as a tool in his study of $BMOA$ functions. The operator J_g has received many names in the literature: the Pommerenke operator, a Volterra type operator (since the choice $g(z) = z$ gives the usual Volterra operator), the generalized Cesàro operator (since the usual Cesàro operator appears with the choice $g(z) = -\log(1-z)$), a Riemann-Stieltjes type operator, or simply called an integration operator. It not was until the works of Aleman and Siskakis in [**2**] and [**3**] that the operator J_g began to be extensively studied. The operator J_g is related to the multiplication operator $M_g(f) = gf$ by the formula $M_g(f) = f(0)g(0) + J_g(f) + I_g(f)$, where I_g is another integration operator defined by

$$(I_g f)(z) = \int_0^z f'(\zeta)\, g(\zeta)\, d\zeta, \qquad f \in H(\mathbb{D}).$$

We refer to [**1**] and [**11**] for surveys on the operator J_g acting in several spaces of analytic functions. We are mainly interested on the operator J_g acting on weighted

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Bergman spaces, so let's recall the definition.

A *weight function* is a positive function $w(r)$, $0 \leq r < 1$, which is integrable in $(0,1)$. We extend w to $\mathbb{D}$ setting $w(z) = w(|z|)$, $z \in \mathbb{D}$. For $0 < p < \infty$, the weighted Bergman space $A^p(w)$ is the space of functions $f \in H(\mathbb{D})$ such that

$$\|f\|^p_{A^p(w)} = \int_{\mathbb{D}} |f(z)|^p w(z)\, dm(z) < \infty.$$

A characterization of those symbols $g \in H(\mathbb{D})$ such that J_g is bounded on $A^p(w)$, where w belongs to a large class of radial weights including the standard weights $w(r) = (1-r)^\alpha$, $\alpha > -1$, but excluding the exponential ones

$$w_{\gamma,\alpha}(r) = (1-r)^\gamma \exp\left(\frac{-c}{(1-r)^\alpha}\right), \quad \gamma \geq 0,\ \alpha > 0,\ c > 0, \tag{1.1}$$

is offered in [**3**].

The following result describes the boundedness and compactness of the operator J_g on $A^p(w_{\gamma,\alpha})$ in terms of the growth of the maximum modulus of g', for the exponential type weights $w_{\gamma,\alpha}$.

THEOREM 1.1. *Let $0 < p < \infty$, $g \in H(\mathbb{D})$, and consider the weights $w_{\gamma,\alpha}$ defined by (1.1). Then*

(i) $J_g : A^p(w_{\gamma,\alpha}) \to A^p(w_{\gamma,\alpha})$ *is bounded if and only if*

$$\sup_{z\in\mathbb{D}} (1-|z|)^{1+\alpha} |g'(z)| < \infty.$$

(ii) $J_g : A^p(w_{\gamma,\alpha}) \to A^p(w_{\gamma,\alpha})$ *is compact if and only if*

$$\lim_{|z|\to 1^-} (1-|z|)^{1+\alpha} |g'(z)| = 0.$$

We note that Theorem 1.1 answers the question which appears in [**3**, p. 353]. The case $p = 2$, $c > 0$ and $\alpha \in (0,1]$ was proved by Dostanić in [**4**], while the general case is proved by the authors in [**7**], where a characterization is also obtained for a general class of radial rapidly decreasing weights. It is our aim in the first part of this note to provide a different proof of Theorem 1.1 using the test functions considered by Dostanic when $\alpha \in (0,1]$, and Oleinik's description [**6**] of the Carleson measures for $A^p(w_\alpha)$ when $\alpha > 1$, where w_α are the exponential weights

$$w_\alpha(r) = \exp\left(\frac{-c}{(1-r)^\alpha}\right), \quad c > 0,\ \alpha > 0. \tag{1.2}$$

One of the main tools in order to prove Theorem 1.1 is a description of the weighted Bergman spaces in terms of derivatives obtained in [**8**]. The version proved in [**8**] is much more general than the one we state next, and uses a suitable distorsion function.

THEOREM A. *Let $0 < p < \infty$, and $g \in H(\mathbb{D})$. Then*

$$\|g\|^p_{A^p(w_{\gamma,\alpha})} \asymp |g(0)|^p + \int_{\mathbb{D}} |g'(z)|^p \,(1-|z|)^{(1+\alpha)p}\, w_{\gamma,\alpha}(z)\, dm(z).$$

Let H be a separable Hilbert space. Given $0 < p < \infty$, let $\mathcal{S}_p(H)$ denote the Schatten p-class of operators on H. $\mathcal{S}_p(H)$ contains those compact operators T

on H whose sequence of characteristic (or singular) numbers λ_n belongs to ℓ^p, the p-summable sequence space. The singular numbers of an operator T are defined by

$$\lambda_n = \lambda_n(T) = \inf\{\|T - K\| : \operatorname{rank} K \leq n\}.$$

Thus finite rank operators belong to every $\mathcal{S}_p(H)$, and the membership of an operator in $\mathcal{S}_p(H)$ measures in some sense the size of the operator. If $1 \leq p < \infty$, $\mathcal{S}_p(H)$ is a Banach space with the norm $\|T\|_p = \|\{\lambda_n\}\|_{\ell^p}$. We refer to [**12**, Chapter 1] for more information about $\mathcal{S}_p(H)$.

Our next result will be a characterization, in terms of the symbol g, of the membership of the operator J_g in the Schatten p-classes of $A^2(w_{\gamma,\alpha})$. In order to state our result, we recall the definition of another class of analytic function spaces, the so called Besov type spaces B^p_σ. Let $0 < p < \infty$, and $\sigma \geq 0$. The space B^p_σ consists of those analytic functions on $\mathbb{D}$ with

$$\|f\|^p_{B^p_\sigma} = \int_{\mathbb{D}} |f'(z)|^p \,(1 - |z|^2)^{p-2+\sigma}\, dm(z) < \infty.$$

THEOREM 1.2. *Let $1 < p < \infty$, $g \in H(\mathbb{D})$, and consider the weights $w_{\gamma,\alpha}$ defined by (1.1). Then $J_g \in \mathcal{S}_p(A^2(w_{\gamma,\alpha}))$ if and only if $g \in B^p_{\alpha(p-1)}$.*

This result was also proved by the authors in [**7**] for more general weights. However, here we will present a different proof.

The paper is organized as follows: Section 2 is devoted to some preliminaries needed for the proofs of the main results. We prove Theorem 1.1 in Section 3 and Theorem 1.2 in Section 4.

Throughout the paper, the letter C will denote an absolute constant whose value may change at different occurrences. We also use the notation $a \lesssim b$ to indicate that there is a constant $C > 0$ with $a \leq Cb$, and the notation $a \asymp b$ means that $a \lesssim b$ and $b \lesssim a$.

2. Preliminary results

In this section we shall prove a few preliminary results which are used for the proofs of the main results of the paper.

From now on, we will always use the following notations: $D(z, r)$ is the Euclidean disc centered at z with radius $r > 0$; For fixed $\alpha > 0$, the function τ_α is defined by

$$\tau_\alpha(z) = (1 - |z|^2)^{1+\frac{\alpha}{2}}.$$

If there is no confusion and for easy of notation, we shall write $\tau_\alpha = \tau$ and for any $\delta > 0$, $D(\delta\tau(z))$ for the disc $D(z, \delta\tau(z))$.

The following result (see [**7**] or [**6**]) says that $|f(z)|^p\, w_{\gamma,\alpha}(z)$ verifies a certain sub-mean-value property.

LEMMA 2.1. *Let $\gamma \geq 0$ and $0 < p, \alpha < \infty$. Then there exist constants $M = M(\alpha, \gamma) \geq 1$ and $m = m(\alpha, \gamma) > 0$ such that*

$$|f(a)|^p\, w_{\gamma,\alpha}(a) \leq \frac{M}{\delta^2\, \tau(a)^2} \int_{D(\delta\tau(a))} |f(z)|^p\, w_{\gamma,\alpha}(z)\, dm(z),$$

for all $0 < \delta \leq m$ and $f \in H(\mathbb{D})$.

An immediate consequence of Lemma 2.1 is that the point evaluations are bounded linear functionals on $A^p(w_{\gamma,\alpha})$. In particular, $A^2(w_{\gamma,\alpha})$ is a reproducing kernel Hilbert space: there are reproducing kernels $K_z \in A^2(w_{\gamma,\alpha})$ with

$$f(z) = \langle f, K_z \rangle = \int_{\mathbb{D}} f(\zeta)\overline{K_z(\zeta)}\, w_{\gamma,\alpha}(\zeta) dm(\zeta).$$

It also follows from Lemma 2.1 that $\|K_z\|^2_{A^2(w_{\gamma,\alpha})}\, w_{\gamma,\alpha}(z) \lesssim (1-|z|^2)^{-2-\alpha}$. In fact, it is proved in [**5**, Lemma 3.5] (see also [**7**, Corollary 1]) that this is the corresponding growth of the reproducing kernel, that is,

$$\|K_z\|^2_{A^2(w_{\gamma,\alpha})}\, w_{\gamma,\alpha}(z) \asymp (1-|z|^2)^{-2-\alpha}, \quad z \in \mathbb{D}. \tag{2.1}$$

Next, bearing in mind Lemma 4 of Dostanic's paper [**4**], the following "test functions" are constructed in order to prove Theorem 1.1 for $0 < \alpha \le 1$.

Lemma 2.2. *Let $0 < \alpha \le 1$, $\gamma \ge 0$ and $c > 0$. For each $a \in \mathbb{D}$, consider the functions*

$$F_a(z) = \frac{1}{(1-\bar{a}z)^{\gamma/2}} \exp\left(\frac{2^\alpha c}{(1-\bar{a}z)^\alpha}\right).$$

Then $F_a \in A^2(w_{\gamma,\alpha})$ with $\|F_a\|^2_{A^2(w_{\gamma,\alpha})} \le C\,(1-|a|^2)^{2+\alpha+\gamma/2}\,|F_a(a)|$.

Proof. Since $|1-\bar{a}z|^\gamma \ge (1-|a|)^\gamma$, it follows from [**4**, Lemma 4] that

$$\begin{aligned}
\|F_a\|^2_{A^2(w_{\gamma,\alpha})} &\le C \int_{\mathbb{D}} \left|\exp\left(\frac{2^\alpha c}{(1-\bar{a}z)^\alpha}\right)\right|^2 w_\alpha(z)\, dm(z) \\
&\le C(1-|a|^2)^{2+\alpha} \exp\left(\frac{2^\alpha c}{(1-|a|^2)^\alpha}\right) \\
&= C\,(1-|a|^2)^{2+\alpha+\gamma/2}\,|F_a(a)|.
\end{aligned}$$

□

Finally, we remind the reader a description of Carleson measures for $A^p(w_\alpha)$ due to Oleinik (see [**6**, Theorem 3.3]), for $\alpha > 1$.

Theorem B. *Suppose that μ is a finite positive Borel measure on $\mathbb{D}$, $\alpha > 1$ and $0 < p \le q < \infty$. The following are equivalent:*

(i) $I_d : A^p(w_\alpha) \to L^q(\mu)$ is a bounded operator.
(ii) If $\delta > 0$ is sufficiently small then

$$K_{\mu,\alpha} = \sup_{a\in\mathbb{D}} \frac{1}{\tau(a)^{2q/p}} \int_{D(\delta\tau(a))} w_\alpha(z)^{-q/p}\, d\mu(z) < \infty.$$

Moreover, if (i) or (ii) holds, then $K_{\mu,\alpha} \asymp ||I_d||^q_{A^p(w_\alpha)\to L^q(\mu)}$.

Theorem C. *Suppose that μ is a finite positive Borel measure on $\mathbb{D}$, $\alpha > 1$ and $0 < p \le q < \infty$. The following are equivalent:*

(i) *$I_d : A^p(w_\alpha) \to L^q(\mu)$ is a compact operator.*
(ii) *If $\delta > 0$ is sufficiently small then*

$$\lim_{r\to 1^-} \sup_{|a|>r} \frac{1}{\tau(a)^{2q/p}} \int_{D(\delta\tau(a))} w_\alpha(z)^{-q/p}\, d\mu(z) = 0.$$

3. Proof of Theorem 1.1.

Throughout this section, for each $z \in \mathbb{D}$ and $g \in H(\mathbb{D})$, we will use the notation:

$$B_g(z) \stackrel{\text{def}}{=} (1-|z|)^{1+\alpha}|g'(z)|.$$

Proof of (i). Suppose first that $\sup_{z\in\mathbb{D}} B_g(z) < \infty$, and let $f \in A^p(w_{\gamma,\alpha})$. Since $(J_g f)'(z) = f(z)g'(z)$, from Theorem A we obtain

$$\begin{aligned}
\|J_g f\|^p_{A^p(w_{\gamma,\alpha})} &\asymp |(J_g f)(0)|^p + \int_{\mathbb{D}} |(J_g f)'(z)|^p \,(1-|z|)^{(1+\alpha)p}\, w_{\gamma,\alpha}(z)\, dm(z) \\
&= \int_{\mathbb{D}} |f(z)|^p \,|g'(z)|^p \,(1-|z|)^{(1+\alpha)p}\, w_{\gamma,\alpha}(z)\, dm(z) \\
&\leq \big(\sup_{z\in\mathbb{D}} B_g(z)\big)^p \,\|f\|^p_{A^p(w_{\gamma,\alpha})},
\end{aligned}$$

and it follows that $J_g : A^p(w_{\gamma,\alpha}) \to A^p(w_{\gamma,\alpha})$ is bounded.

Suppose now that J_g is bounded on $A^p(w_{\gamma,\alpha})$ and choose $\delta > 0$ sufficiently small. We shall split the proof of this implication in two cases.

Case $\mathbf{0} < \alpha \leq \mathbf{1}$. If $f \in A^p(w_{\gamma,\alpha})$ and a is any point of $\mathbb{D}$, then by Lemma 2.1 we have

$$\begin{aligned}
|(J_g f)'(a)|^p &\lesssim \frac{1}{w_{\gamma,\alpha}(a)\tau(a)^2} \int_{D(\delta\tau(a))} |(J_g f)'(z)|^p \, w_{\gamma,\alpha}(z)\, dm(z) \\
&\lesssim \frac{(1-|a|)^{-(1+\alpha)p}}{w_{\gamma,\alpha}(a)\tau(a)^2} \int_{D(\delta\tau(a))} |(J_g f)'(z)|^p \,(1-|z|)^{(1+\alpha)p}\, w_{\gamma,\alpha}(z)\, dm(z).
\end{aligned}$$

In the last inequality we have used the fact that $(1-|a|) \asymp (1-|z|)$ for $z \in D(\delta\tau(a))$. Since $(J_g f)'(a) = f(a)g'(a)$, then it follows from Theorem A and the boundedness of J_g that

$$|f(a)|^p \,(1-|a|)^{(1+\alpha)p}\, |g'(a)|^p \lesssim \frac{\|J_g f\|^p_{A^p(w_{\gamma,\alpha})}}{w_{\gamma,\alpha}(a)\,\tau(a)^2} \lesssim \frac{\|J_g\|^p\,\|f\|^p_{A^p(w_{\gamma,\alpha})}}{w_{\gamma,\alpha}(a)\,\tau(a)^2}. \tag{3.1}$$

Now, consider the test function $F_a(z)$ defined in Lemma 2.2. Since $F_a(z)$ has no zeros on $\mathbb{D}$, then the function $H_a(z) = (F_a(z))^{2/p}$ belongs to $A^p(w_{\gamma,\alpha})$ with

$$\|H_a\|^p_{A^p(w_{\gamma,\alpha})} = \|F_a\|^2_{A^2(w_{\gamma,\alpha})}. \tag{3.2}$$

Since $2^{-\gamma} \leq |F_a(a)|\, w_{\gamma,\alpha}(a)\,(1-|a|^2)^{-\gamma/2}$, it follows from Lemma 2.2 that

$$\|F_a\|^2_{A^2(w_{\gamma,\alpha})} \leq C\,(1-|a|)^{2+\alpha}\, w_{\gamma,\alpha}(a)\,|F_a(a)|^2. \tag{3.3}$$

Therefore, taking the function $f = H_a$ in (3.1), using (3.2), (3.3) and recalling that $\tau(a)^2 = (1-|a|)^{2+\alpha}$, we get

$$\begin{aligned}
\big((1-|a|)^{1+\alpha}\,|g'(a)|\big)^p &\lesssim \left(\frac{\|F_a\|_{A^2(w_{\gamma,\alpha})}}{|F_a(a)|}\right)^2 \frac{\|J_g\|^p}{w_{\gamma,\alpha}(a)\,\tau(a)^2} \\
&\lesssim \|J_g\|^p,
\end{aligned}$$

and then, bearing in mind that a is arbitrary, we have

$$\sup_{a\in\mathbb{D}} B_g(a) \lesssim \|J_g\|.$$

This finishes the proof for the case $0 < \alpha \leq 1$.

Case $\alpha > 1$. It follows from Theorem A and the boundedness of J_g that

$$\int_{\mathbb{D}} |f(z)|^p \, |g'(z)|^p \, (1-|z|)^{(1+\alpha)p} \, w_{\gamma,\alpha}(z) \, dm(z) \asymp \|J_g f\|^p_{A^p(w_{\gamma,\alpha})} \le C\|J_g\|^p \, \|f\|^p_{A^p(w_{\gamma,\alpha})}, \tag{3.4}$$

whenever f belongs to $A^p(w_{\gamma,\alpha})$. Next, note that if h is a function in $A^p(w_\alpha)$, then

$$f_\zeta(z) \stackrel{\text{def}}{=} \frac{h(z)}{(1-\bar{\zeta}z)^{\gamma/p}} \in A^p(w_{\gamma,\alpha}), \qquad \text{for any } \zeta \in \mathbb{D},$$

and moreover $\sup_{\zeta\in\mathbb{D}} \|f_\zeta\|^p_{A^p(w_{\gamma,\alpha})} \le C\|h\|^p_{A^p(w_\alpha)}$. Consequently, if we write

$$d\mu_\zeta(z) = |g'(z)|^p \, (1-|z|^2)^{(1+\alpha)p} \, \frac{(1-|z|^2)^\gamma}{|1-\bar{\zeta}z|^\gamma} \, w_\alpha(z) \, dm(z),$$

bearing in mind (3.4), we deduce that

$$\sup_{\zeta\in\mathbb{D}} \int_{\mathbb{D}} |h(z)|^p \, d\mu_\zeta(z) \le C \, \|J_g\|^p \sup_{\zeta\in\mathbb{D}} \|f_\zeta\|^p_{A^p(w_{\gamma,\alpha})} \le C\|J_g\|^p \, \|h\|^p_{A^p(w_\alpha)},$$

where C is a constant independent of ζ. That is, the operators $I_d : A^p(w_\alpha) \to L^p(\mu_\zeta)$, $\zeta \in \mathbb{D}$, have norm uniformly bounded by $C\|J_g\|$. Then, if $\delta > 0$ is sufficiently small, by Oleinik's theorem (see Theorem B) one has

$$\sup_{\zeta\in\mathbb{D}} \sup_{a\in\mathbb{D}} \frac{1}{\tau(a)^2} \int_{D(\delta\tau(a))} \frac{d\mu_\zeta(z)}{w_\alpha(z)} \le C \, \|J_g\|^p.$$

So, taking $\zeta = a$, we get

$$\sup_{a\in\mathbb{D}} \frac{1}{\tau(a)^2} \int_{D(\delta\tau(a))} \frac{d\mu_a(z)}{w_\alpha(z)} \le C\|J_g\|^p. \tag{3.5}$$

On the other hand, for any $a \in \mathbb{D}$, the subharmonicity of $|g'|^p$ yields

$$\big(B_g(a)\big)^p = (1-|a|)^{(1+\alpha)p} \, |g'(a)|^p \lesssim \frac{(1-|a|)^{(1+\alpha)p}}{\tau(a)^2} \int_{D(\delta\tau(a))} |g'(z)|^p \, dm(z).$$

This together with the fact that $(1-|a|) \asymp (1-|z|) \asymp |1-\bar{a}z|$ for $z \in D(a, \delta\tau(a))$ gives

$$\begin{aligned} B_g(a)^q \quad &\lesssim \quad \frac{1}{\tau(a)^2} \int_{D(\delta\tau(a))} \frac{|g'(z)|^p(1-|z|^2)^{(1+\alpha)p}}{w_\alpha(z)} \, w_\alpha(z) \, \frac{(1-|z|^2)^\gamma}{|1-\bar{a}z|^\gamma} \, dm(z) \\ &= \quad \frac{1}{\tau(a)^2} \int_{D(\delta\tau(a))} \frac{d\mu_a(z)}{w_\alpha(z)}. \end{aligned} \tag{3.6}$$

Finally, bearing in mind (3.5), this gives

$$\sup_{a\in\mathbb{D}} B_g(a) \lesssim \|J_g\|.$$

Thus, the proof is complete. □

Before going into the proof of the compactness part, some previous results will be needed. Using the fact that the point evaluation functionals are bounded on $A^p(w_{\gamma,\alpha})$, the proof of the following result is standard, and we omit it here.

LEMMA 3.1. *Let $0 < p < \infty$ and $g \in H(\mathbb{D})$. Then J_g is compact on $A^p(w_{\gamma,\alpha})$ if and only if whenever $\{f_n\}$ is a bounded sequence in $A^p(w_{\gamma,\alpha})$ such that $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, then $J_g f_n \to 0$ in $A^p(w_{\gamma,\alpha})$.*

Now we choose the appropriate test functions to study the compactness.

LEMMA 3.2. *Let $0 < \alpha \leq 1$, $\gamma \geq 0$, and let $\lambda = 1 + \alpha/2 + \gamma/4$. For each $a \in \mathbb{D}$, consider the functions*

$$f_a(z) = (1 - |a|^2)^{-\lambda} \, \frac{F_a(z)}{\sqrt{F_a(a)}},$$

where F_a is the function defined in Lemma 2.2. Then $\|f_a\|_{A^2(w_{\gamma,\alpha})} \leq C$, where $C > 0$ does not depend on the point a, and

$$\lim_{|a| \to 1^-} |f_a(z)| = 0$$

uniformly on compact subsets of $\mathbb{D}$.

PROOF. The fact that $\|f_a\|_{A^2(w_{\gamma,\alpha})} \leq C$ is a consequence of Lemma 2.2. Now, for $z \in \mathbb{D}$ we have

$$|F_a(z)| \leq \frac{1}{(1 - |z|)^{\gamma/2}} \exp\left(\frac{2^\alpha c}{(1 - |z|)^\alpha}\right), \qquad a \in \mathbb{D}.$$

Therefore the result follows from the fact that

$$\lim_{|a| \to 1} \frac{(1 - |a|^2)^{-\lambda}}{\sqrt{F_a(a)}} = 0. \qquad \square$$

PROOF OF (ii). Suppose first that $g \in H(\mathbb{D})$ satisfies

$$\lim_{|z| \to 1^-} B_g(z) = 0, \tag{3.7}$$

and let $\{f_n\}$ be a bounded sequence of functions in $A^p(w_{\gamma,\alpha})$ such that $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$. Fixed $\varepsilon > 0$, by (3.7) there is $r \in (0,1)$ such that $B_g(z)^p < \varepsilon$, for all $z \in \{\xi \in \mathbb{D} : r \leq |\xi| < 1\}$. Moreover, since $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$, there is $n_0 \in \mathbb{N}$ such that

$$|f_n(z)|^p < \varepsilon, \quad \text{for all } n \geq n_0 \text{ and } z \in \{\xi : |\xi| < r\}.$$

Since (3.7) gives that $\sup_{z \in \mathbb{D}} (1 - |z|)^{(1+\alpha)} |g'(z)| < \infty$, by Theorem A the function g belongs to $A^p(w_{\gamma,\alpha})$. Using again Theorem A, for $n \geq n_0$ we obtain

$$\begin{aligned}
\|J_g(f_n)\|^p_{A^p(w_{\gamma,\alpha})} &\leq C \int_{\mathbb{D}} |g'(z)|^p |f_n(z)|^p \, (1 - |z|)^{(1+\alpha)p} \, w_{\gamma,\alpha}(z) \, dm(z) \\
&\leq C\varepsilon \int_{|z| < r} |g'(z)|^p \, (1 - |z|)^{(1+\alpha)p} \, w_{\gamma,\alpha}(z) \, dm(z) \\
&\quad + C \int_{r \leq |z| < 1} |g'(z)|^p |f_n(z)|^p \, (1 - |z|)^{(1+\alpha)p} \, w_{\gamma,\alpha}(z) \, dm(z). \\
&\leq C\varepsilon \, \|g\|^p_{A^p(w_{\gamma,\alpha})} + C \int_{r \leq |z| < 1} |f_n(z)|^p \, B_g(z)^p \, w_{\gamma,\alpha}(z) \, dm(z) \\
&\leq C\varepsilon \left(\|g\|^p_{A^p(w_{\gamma,\alpha})} + \sup_n \|f_n\|^p_{A^p(w_{\gamma,\alpha})} \right) \\
&\leq C\varepsilon, \quad \text{for all } n \geq n_0,
\end{aligned}$$

that is, $\lim_{n\to\infty} \|J_g(f_n)\|^p_{A^p(w_{\gamma,\alpha})} = 0$. So by Lemma 3.1, J_g is compact.

Conversely, suppose that J_g is compact. We shall split the proof of this implication in two cases.

Case $0 < \alpha \le 1$. Consider the function f_a from Lemma 3.2. Since $f_a(z)$ never vanishes on $\mathbb{D}$, then, by Lemma 3.2, the function $h_a(z) = (f_a(z))^{2/p}$ belongs to $A^p(w_{\gamma,\alpha})$ with $\|h_a\|^p_{A^p(w_{\gamma,\alpha})} = \|f_a\|^2_{A^2(w_{\gamma,\alpha})} \le C$, and $h_a \to 0$ as $|a| \to 1$ uniformly on compact subsets of $\mathbb{D}$. Since J_g is compact, Lemma 3.1 implies that

$$\lim_{|a|\to 1^-} \|J_g(h_a)\|^p_{A^p(w_{\gamma,\alpha})} = 0. \tag{3.8}$$

For $f \in A^p(w_{\gamma,\alpha})$, proceeding as in the proof of the boundedness part (see equation (3.1)), we obtain

$$|f(a)|^p\,(1-|a|^2)^{(1+\alpha)p}\,|g'(a)|^p \le C\,\frac{\|J_g f\|^p_{A^p(w_{\gamma,\alpha})}}{w_{\gamma,\alpha}(a)\,\tau(a)^2}. \tag{3.9}$$

Recall that $\tau(a)^2 = (1-|a|^2)^{2+\alpha}$. On the other hand,

$$w_{\gamma,\alpha}(a)|f_a(a)|^2 = w_{\gamma,\alpha}(a)(1-|a|^2)^{-2\lambda}|F_a(a)| \ge C(1-|a|^2)^{-2-\alpha},$$

so

$$\frac{1}{w_{\gamma,\alpha}(a)(1-|a|^2)^{2+\alpha}} \le |f_a(a)|^2,$$

which together with (3.9) (with $f = h_a$) and (3.8) gives that

$$\begin{aligned}\lim_{|a|\to 1^-} B_g(a)^p &\lesssim \lim_{|a|\to 1^-} \Big((1-|a|)^{(1+\alpha)p}|g'(a)|^p|f_a(a)|^2 w_{\gamma,\alpha}(a)\,\tau^2(a)\Big)\\ &\lesssim \lim_{|a|\to 1^-} \|J_g(h_a)\|^p_{A^p(w_{\gamma,\alpha})} = 0.\end{aligned}$$

This finishes the proof of this case.

Case $\alpha > 1$. This can be proved with similar arguments as in the boundedness part using Theorem C. We left the details to the interested reader. □

4. Schatten p-classes

In this section we will prove Theorem 1.2. For easy of notation, throughout this section we denote $\mathcal{S}_p := \mathcal{S}_p(A^2(w_{\gamma,\alpha}))$, the norm $\|\cdot\|$ is the norm in $A^2(w_{\gamma,\alpha})$, and $\langle\cdot,\cdot\rangle$ is the inner product in $A^2(w_{\gamma,\alpha})$. First, we need several definitions and preparatory results that can be of independent interest. Let $F = \{f_n\}$ be a sequence of analytic functions on $\mathbb{D}$. We denote

$$\|F(z)\|_{\ell^2} = \left(\sum_n |f_n(z)|^2\right)^{1/2}, \qquad z \in \mathbb{D},$$

and for $0 < p < \infty$, consider the p-integral means

$$M_p^p(r,F) = \int_0^{2\pi} \big\|F(re^{i\theta})\big\|^p_{\ell^2}\,\frac{d\theta}{2\pi}, \qquad 0 \le r < 1.$$

If ω is a weight function, following Siskakis [**10**], we define the distorsion function of ω as

$$\psi_\omega(z) = \frac{1}{\omega(z)}\int_{|z|}^1 \omega(s)\,ds.$$

Now, the proof of the following lemma is analogue to the case of one function (see [**10**, Lemma 2.1]. We shall give an sketch of the proof for the sake of completeness.

LEMMA 4.1. *Let $1 \le p < \infty$, $F = \{f_n\} \subset H(\mathbb{D})$ with $F(z) \in \ell^2$ for each $z \in \mathbb{D}$, and let $F' = \{f_n'\}$. Then for any weight function ω one has*

$$\int_{\mathbb{D}} \|F(z)\|_{\ell^2}^p \,\omega(z)\,dm(z) \le C\left(\|F(0)\|_{\ell^2}^p + \int_{\mathbb{D}} \|F'(z)\|_{\ell^2}^p \,\psi_\omega(z)^p\,\omega(z)\,dm(z)\right),$$

where the constant C depends only on p and the weight ω.

PROOF. First, we will show that

$$\frac{d}{dr}M_p^p(r,F) \le p\,M_p^{p-1}(r,F)\,M_p(r,F'), \quad 0<r<1, \qquad \text{for } p \ge 1. \tag{4.1}$$

If $F = 0$, (4.1) is clear. If $F \ne 0$, at points $z \in \mathbb{D}$ where F is not zero, by Cauchy-Schwarz inequality

$$\begin{aligned} r\frac{\partial\,\|F(z)\|_{\ell^2}^p}{\partial r} &= \frac{rp}{2}\,\|F(z)\|_{\ell^2}^{p-2}\,\frac{\partial\,\|F(z)\|_{\ell^2}^2}{\partial r} \\ &= \frac{p}{2}\,\|F(z)\|_{\ell^2}^{p-2}\sum_n\left(r\frac{\partial\,|f_n(z)|^2}{\partial r}\right) \\ &= p\,\|F(z)\|_{\ell^2}^{p-2}\sum_n \operatorname{Re}\left(zf_n(z)\overline{f_n'(z)}\right) \\ &\le rp\,\|F(z)\|_{\ell^2}^{p-1}\,\|F'(z)\|_{\ell^2}\,, \end{aligned}$$

and consequently

$$\frac{d}{dr}M_p^p(r,F) \le rp\int_0^{2\pi} \left\|F(re^{i\theta})\right\|_{\ell^2}^{p-1}\left\|F'(re^{i\theta})\right\|_{\ell^2}\frac{d\theta}{2\pi}.$$

Thus (4.1) holds for $p = 1$. If $p > 1$ apply Hölder's inequality to obtain (4.1). From now, the proof can be mimicked from that of [**10**, Lemma 2.1]. □

We also need the fact that for any orthonormal set $\{e_n\}$ of $A^2(w_{\gamma,\alpha})$, one has

$$\sum_n |e_n(z)|^2 \le \|K_z\|^2, \quad z \in \mathbb{D}, \tag{4.2}$$

with equality if $\{e_n\}$ is also an orthonormal basis.

The following Proposition gives the sufficiency in Theorem 1.2.

PROPOSITION 4.2. *Let $1 < p < \infty$. If $g \in B^p_{\alpha(p-1)}$, then $J_g \in \mathcal{S}_p$*

PROOF. If $p \ge 2$, then $J_g \in \mathcal{S}_p$ if and only if

$$\sum_n \|J_g e_n\|^p < \infty$$

for all orthonormal sets $\{e_n\}$ of $A^2(w_{\gamma,\alpha})$ (see [**12**, Theorem 1.33]). But, by Theorem B we get

$$\begin{aligned} \|J_g e_n\|^p &= \left(\int_{\mathbb{D}} |J_g e_n(z)|^2\,w_{\gamma,\alpha}(z)\,dm(z)\right)^{p/2} \\ &\asymp \left(\int_{\mathbb{D}} |e_n(z)|^2\,|g'(z)|^2\,(1-|z|^2)^{2(1+\alpha)}\,w_{\gamma,\alpha}(z)\,dm(z)\right)^{p/2}. \end{aligned}$$

Therefore, since $p/2 \geq 1$, and $\|e_n\| = 1$, Hölder's inequality, (4.2) and (2.1) give

$$\begin{aligned}
\sum_n \|J_g e_n\|^p &\asymp \sum_n \left(\int_{\mathbb{D}} |e_n(z)|^2 \, |g'(z)|^2 \, (1-|z|^2)^{2(1+\alpha)} \, w_{\gamma,\alpha}(z) \, dm(z) \right)^{p/2} \\
&\leq \sum_n \int_{\mathbb{D}} |e_n(z)|^2 \, |g'(z)|^p \, (1-|z|^2)^{(1+\alpha)p} \, w_{\gamma,\alpha}(z) \, dm(z) \\
&= \int_{\mathbb{D}} \left(\sum_n |e_n(z)|^2 \right) |g'(z)|^p \, (1-|z|^2)^{(1+\alpha)p} \, w_{\gamma,\alpha}(z) dm(z) \\
&\leq \int_{\mathbb{D}} \|K_z\|^2 \, |g'(z)|^p \, (1-|z|^2)^{(1+\alpha)p} \, w_{\gamma,\alpha}(z) dm(z) \\
&\asymp \int_{\mathbb{D}} |g'(z)|^p \, (1-|z|^2)^{(1+\alpha)p-2-\alpha} \, dm(z) \\
&\leq \|g\|^p_{B^p_{\alpha(p-1)}}.
\end{aligned}$$

This finishes the proof for $p \geq 2$.

If $1 < p < 2$, then $J_g \in \mathcal{S}_p$ if and only if

$$\sum_n |\langle J_g e_n, e_n \rangle|^p < \infty$$

for all orthonormal sets $\{e_n\}$ of $A^2(w_{\gamma,\alpha})$ (see [**12**, Theorem 1.27]). We begin the proof of this case by establishing the inequality

$$\sum_n |\langle J_g e_n, e_n \rangle|^p \leq \int_{\mathbb{D}} \Big(\sum_n |J_g e_n(z)|^2 \Big)^{p/2} \|K_z\|^{2-p} \, w_{\gamma,\alpha}(z) dm(z). \tag{4.3}$$

Since $p > 1$ and $\|e_n\| = 1$, we can use Hölder's inequality to obtain

$$\begin{aligned}
\sum_n |\langle J_g e_n, e_n \rangle|^p &\leq \sum_n \left(\int_{\mathbb{D}} |J_g e_n(z)| \, |e_n(z)| \, w_{\gamma,\alpha}(z) dm(z) \right)^p \\
&\leq \sum_n \int_{\mathbb{D}} |J_g e_n(z)|^p \, |e_n(z)|^{2-p} \, w_{\gamma,\alpha}(z) dm(z) \\
&= \int_{\mathbb{D}} \Big(\sum_n |J_g e_n(z)|^p \, |e_n(z)|^{2-p} \Big) \, w_{\gamma,\alpha}(z) dm(z)
\end{aligned}$$

Next, since $p < 2$, we can use Hölder's inequality with exponent $2/p > 1$

$$\begin{aligned}
\sum_n |\langle J_g e_n, e_n \rangle|^p &\leq \int_{\mathbb{D}} \Big(\sum_n |J_g e_n(z)|^2 \Big)^{\frac{p}{2}} \Big(\sum_n |e_n(z)|^2 \Big)^{\frac{2-p}{2}} w_{\gamma,\alpha}(z) dm(z) \\
&\leq \int_{\mathbb{D}} \Big(\sum_n |J_g e_n(z)|^2 \Big)^{\frac{p}{2}} \|K_z\|^{2-p} \, w_{\gamma,\alpha}(z) dm(z),
\end{aligned}$$

and this proves (4.3).

Now, (4.3) and the fact that $\|K_z\|^2 \, w_{\gamma,\alpha}(z) \lesssim (1-|z|^2)^{-2-\alpha}$ gives

$$\sum_n |\langle J_g e_n, e_n \rangle|^p \lesssim \int_{\mathbb{D}} \big\|\{J_g e_n(z)\}\big\|^p_{\ell^2} \, \omega^*(z) \, dm(z), \tag{4.4}$$

where

$$\omega^*(z) = (1-|z|^2)^{-\frac{(2+\alpha)(2-p)}{2}}\, w_{\gamma,\alpha}(z)^{p/2} = (1-|z|^2)^{\gamma^*} \exp\left(\frac{-cp/2}{(1-r)^\alpha}\right),$$

with $\gamma^* = \frac{p}{2}\gamma - \frac{(2+\alpha)(2-p)}{2}$. Since the distorsion function of the weight ω^* is comparable to $(1-|z|^2)^{1+\alpha}$ (see [**10**, Example 3.2]), then Lemma 4.1 together with (4.4) gives

$$\begin{aligned}
\sum_n |\langle J_g e_n, e_n\rangle|^p &\lesssim \int_{\mathbb{D}} \left\|\{(J_g e_n)'(z)\}\right\|_{\ell^2}^p (1-|z|^2)^{(1+\alpha)p}\omega^*(z)\, dm(z)\\
&= \int_{\mathbb{D}} \Big(\sum_n |(J_g e_n)'(z)|^2\Big)^{p/2} (1-|z|^2)^{(1+\alpha)p}\omega^*(z)\, dm(z)\\
&= \int_{\mathbb{D}} |g'(z)|^p \Big(\sum_n |e_n(z)|^2\Big)^{p/2} (1-|z|^2)^{(1+\alpha)p}\omega^*(z)\, dm(z)\\
&\le \int_{\mathbb{D}} |g'(z)|^p\, \|K_z\|^p\, (1-|z|^2)^{(1+\alpha)p}\omega^*(z)\, dm(z)\\
&\lesssim \int_{\mathbb{D}} |g'(z)|^p\, w_{\gamma,\alpha}(z)^{-p/2}\, (1-|z|^2)^{\frac{\alpha p}{2}}\, \omega^*(z)\, dm(z)\\
&= \int_{\mathbb{D}} |g'(z)|^p\, (1-|z|^2)^{p-2+\alpha(p-1)}\, dm(z) = \|g\|^p_{B^p_{\alpha(p-1)}}.
\end{aligned}$$

This completes the proof of the Proposition. □

For the necessity we need first some lemmas.

Lemma A (Oleinik [**6**]). *Let $\tau(z) = (1-|z|^2)^{1+\frac{\alpha}{2}}$. There is a number δ_0 and a sequence of points $\{z_j\} \subset \mathbb{D}$, such that for each $\delta \in (0,\delta_0)$ one has:*

- *(i)* $z_j \notin D(\delta\tau(z_k))$, $j \neq k$.
- *(ii)* $\bigcup_j D(\delta\tau(z_j)) = \mathbb{D}$.
- *(iii)* $\tilde{D}(\delta\tau(z_j)) \subset D(3\delta\tau(z_j))$, *where* $\tilde{D}(\delta\tau(z_j)) = \bigcup_{z\in D(\delta\tau(z_j))} D(\delta\tau(z))$, $j = 1,2,\dots$
- *(iv)* $\{D(3\delta\tau(z_j))\}$ *is a covering of* $\mathbb{D}$ *of finite multiplicity* N.

Let $k_z = K_z/\|K_z\|$ be the normalized reproducing kernels of $A^2(w_{\gamma,\alpha})$.

Lemma 4.3. *Let $\{z_j\}$ be the sequence given in Lemma A. Then for every ortonormal sequence $\{e_j\}$ in $A^2(w_{\gamma,\alpha})$, the operator B taking e_j to k_{z_j} is bounded.*

Proof. It is required to show

$$\left\| B\Big(\sum_j a_j e_j\Big)\right\| \le C\left(\sum_j |a_j|^2\right)^{1/2}.$$

For any $g \in A^2(w_{\gamma,\alpha})$, we have

$$\left|\left\langle B\Big(\sum_j a_j e_j\Big), g\right\rangle\right| = \left|\left\langle \sum_j a_j k_{z_j}, g\right\rangle\right| = \left|\sum_j a_j \langle k_{z_j}, g\rangle\right| \le \sum_j |a_j| \frac{|g(z_j)|}{\|K_{z_j}\|}$$
$$\le \left(\sum_j |a_j|^2\right)^{1/2} \left(\sum_j |g(z_j)|^2 \, \|K_{z_j}\|^{-2}\right)^{1/2}.$$

Now the result follows from the fact that, by (2.1), Lemma 2.1 and Lemma A

$$\begin{aligned}\sum_j |g(z_j)|^2 \, \|K_{z_j}\|^{-2} &\asymp \sum_j |g(z_j)|^2 w_{\gamma,\alpha}(z_j)\tau(z_j)^2 \\ &\lesssim \sum_j \int_{D(\tau(z_j))} |g(z)|^2 \, w_{\gamma,\alpha}(z)\, dm(z) \\ &\le C\|g\|^2.\end{aligned}$$

□

The next result gives the necessity in Theorem 1.2 completing the proof of that Theorem.

Proposition 4.4. *Let $0 < p < \infty$. If $J_g \in \mathcal{S}_p(A^2(w_{\gamma,\alpha}))$, then $g \in B^p_{\alpha(p-1)}$.*

Proof. We consider first the case $p \ge 2$. Suppose that J_g is in $\mathcal{S}_p$, and let $\{e_k\}$ be an orthonormal set in $A^2(w_{\gamma,\alpha})$. By Lemma 4.3, the operator B taking e_j to the normalized reproducing kernels k_{z_j} is bounded on $A^2(w_{\gamma,\alpha})$, where $\{z_j\}$ is the sequence from Lemma A. Since $\mathcal{S}_p$ is a two-sided ideal in the space of bounded linear operators on $A^2(w_{\gamma,\alpha})$, then $J_g B$ belongs to $\mathcal{S}_p$ (see [**12**, p.27]). Thus, by [**12**, Theorem 1.33]

$$\sum_j \|J_g(k_{z_j})\|^p = \sum_k \|J_g B e_j\|^p < \infty.$$

Now, using the subharmonicity of $|g'|^2$ and Lemma A we obtain

$$\begin{aligned}\|g\|^p_{B^p_{\alpha(p-1)}} &\lesssim \int_{\mathbb{D}} \left(\frac{1}{\tau(\zeta)^2}\int_{D(\delta\tau(\zeta))} |g'(z)|^2\, dm(z)\right)^{p/2} (1-|\zeta|^2)^{p-2+\alpha(p-1)} dm(\zeta) \\ &= \sum_j \int_{D(\delta\tau(z_j))} \left(\int_{D(\delta\tau(\zeta))} |g'(z)|^2\, dm(z)\right)^{p/2} (1-|\zeta|^2)^{\frac{\alpha p}{2}}\, \tau(\zeta)^{-2}\, dm(\zeta) \\ &\lesssim \sum_j \left(\int_{D(3\delta\tau(z_j))} |g'(z)|^2\, (1-|z|^2)^\alpha\, dm(z)\right)^{p/2}.\end{aligned}$$

This together with (2.1), the fact that (see [**5**, Lemma 3.6])

$$|k_{z_j}(z)| \asymp \|K_z\| \qquad \text{for } z \in D(\delta\tau(z_j)),$$

and Theorem A gives

$$\begin{aligned}
\|g\|^p_{B^p_{\alpha(p-1)}} &\asymp \sum_j \left(\int_{D(3\delta\tau(z_j))} \|K_z\|^2 \, |g'(z)|^2 \, (1-|z|^2)^{2(1+\alpha)} \, w_{\gamma,\alpha}(z) \, dm(z) \right)^{p/2} \\
&\asymp \sum_j \left(\int_{D(3\delta\tau(z_j))} |k_{z_j}(z)|^2 \, |g'(z)|^2 \, (1-|z|^2)^{2(1+\alpha)} \, w_{\gamma,\alpha}(z) \, dm(z) \right)^{p/2} \\
&\leq \sum_j \left(\int_{\mathbb{D}} |k_{z_j}(z)|^2 \, |g'(z)|^2 \, (1-|z|^2)^{2(1+\alpha)} \, w_{\gamma,\alpha}(z) dm(z) \right)^{p/2} \\
&\asymp \sum_j \|J_g(k_{z_j})\|^p < \infty.
\end{aligned}$$

This completes the proof for the case $p \geq 2$.

If $0 < p < 2$ we follow the argument in [**12**, Proposition 7.15]. If $J_g \in \mathcal{S}_p$ then the positive operator $J_g^* J_g$ belongs to $\mathcal{S}_{p/2}$. Without loss of generality we may assume that $g' \neq 0$. Suppose $J_g^* J_g f = \sum_n \lambda_n \langle f, e_n \rangle \, e_n$ is the canonical decomposition of $J_g^* J_g$. Then $\{e_n\}$ is also an orthonormal basis. Indeed, if there is an unit vector $e \in A^2(w_{\gamma,\alpha})$ such that $e \perp e_n$ for all $n \geq 1$, then by Theorem A,

$$\int_{\mathbb{D}} |g'(z)|^2 |e(z)|^2 (1-|z|^2)^{2(1+\alpha)} \, w_{\gamma,\alpha}(z) \, dm(z) \asymp \|J_g e\|^2 = \langle J_g^* J_g e, e \rangle = 0$$

because $J_g^* J_g$ is a linear combination of the vectors e_n. This would give $g' \equiv 0$.

Now (2.1), the fact that equality holds in (4.2) (since $\{e_n\}$ is an orthonormal basis), and Hölder's inequality yields

$$\begin{aligned}
\|g\|^p_{B^p_{\alpha(p-1)}} &\asymp \int_{\mathbb{D}} |g'(z)|^p \, (1-|z|^2)^{(1+\alpha)p} \, \|K_z\|^2 \, w_{\gamma,\alpha}(z) \, dm(z) \\
&= \sum_n \int_{\mathbb{D}} |g'(z)|^p \, |e_n(z)|^2 \, (1-|z|^2)^{(1+\alpha)p} \, w_{\gamma,\alpha}(z) \, dm(z) \\
&\leq \sum_n \left(\int_{\mathbb{D}} |g'(z)|^2 \, |e_n(z)|^2 \, (1-|z|^2)^{2(1+\alpha)} \, w_{\gamma,\alpha}(z) \, dm(z) \right)^{p/2} \\
&\lesssim \sum_n \langle J_g^* J_g e_n, e_n \rangle^{p/2} = \sum_n \lambda_n^{p/2} = \|J_g^* J_g\|^{p/2}_{\mathcal{S}_{p/2}}.
\end{aligned}$$

The last inequality is due to Theorem A. This completes the proof. □

Corollary 4.5. *Let $0 < p \leq 1$. Then $J_g \in \mathcal{S}_p$ if and only if g is constant.*

Proof. The sufficiency is obvious, and the necessity follows from Proposition 4.4, since $B^p_{\alpha(p-1)}$ contains only constant functions for $0 < p \leq 1$. □

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