Hankel operators on standard Bergman spaces

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Abstract. We study Hankel operators on the standard Bergman spaces A^2_α, $\alpha > -1$. A description of the boundedness and compactness of the (big) Hankel operator H_f with general symbols $f \in L^2(\mathbb{D}, dA_\alpha)$ is obtained. Also, we provide a new proof of a result of Arazy-Fisher-Peetre on the membership in Schatten p-classes of Hankel operators with conjugate analytic symbols.

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1. Introduction

If T is an operator induced (in some way) by a symbol f going from some Hilbert space to another Hilbert space, one is going to hope that individual properties of the symbol (such as smoothness or growth conditions) will give information on the properties of the operator (boundedness, compactness, or membership in Schatten-Von Neumann ideals). In the present paper, we will study this when dealing with Hankel operators on standard weighted Bergman spaces. For $\alpha > -1$, the weighted Bergman space A^2_α consists of those functions f analytic on the unit disk $\mathbb{D}$ such that

$$\|f\|_\alpha = \left(\int_{\mathbb{D}} |f(z)|^2 \, dA_\alpha(z) \right)^{1/2} < \infty,$$

where $dA_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha \, dA(z)$ and dA is the normalized area measure on $\mathbb{D}$. The space A^2_α is a Hilbert space with reproducing kernel given by $K_z(w) = (1-\bar{z}w)^{-2-\alpha}$; it is also a closed subspace of $L^2(\mathbb{D}, dA_\alpha)$, and the orthogonal projection from $L^2(\mathbb{D}, dA_\alpha)$ to A^2_α is given by

$$P_\alpha f(z) = \langle f, K_z \rangle_\alpha = \int_{\mathbb{D}} \frac{f(w)}{(1-\bar{w}z)^{2+\alpha}} \, dA_\alpha(w), \qquad f \in L^2(\mathbb{D}, dA_\alpha).$$

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Given a function $f \in L^2(\mathbb{D}, dA_\alpha)$, the Hankel operator with symbol f is the linear operator $H_f : A^2_\alpha \to L^2(\mathbb{D}, dA_\alpha)$ defined by

$$H_f g = (I - P_\alpha)(fg), \qquad g \in A^2_\alpha.$$

The operator H_f is densely defined on A^2_α. For example, it is well defined in H^∞, the algebra of all bounded analytic functions on $\mathbb{D}$. The following integral formula is very useful when one is going to estimate the norm of a Hankel operator:

$$H_f g(z) = \int_{\mathbb{D}} \frac{f(z) - f(w)}{(1 - \bar{w} z)^{2+\alpha}} \, g(w) \, dA_\alpha(w), \qquad g \in A^2_\alpha.$$

It has been a lot of activity in the theory of Hankel operators on Bergman spaces in recent years, and this topic has become a classical theme in complex analysis and operator theory (see for example [1], [3], [4], [10], [11], [13], and [17]). For Hankel operators with conjugate analytic symbols, that is $H_{\bar{f}}$ with $f \in A^2_\alpha$, one has that $H_{\bar{f}}$ is bounded on A^2_α if and only if the symbol f belongs to the Bloch space; $H_{\bar{f}}$ is compact if and only if f belongs to the little Bloch space (see [1], [2]); and the membership in Schatten p-classes of the Hankel operator $H_{\bar{f}}$ is equivalent to the function f being in the analytic Besov space B_p for $1 < p < \infty$, and to f being constant when $0 < p \leq 1$. Therefore, for conjugate analytic symbols, the picture on the boundedness, compactness and Schatten p-classes is complete. However, the proof of the necessity and sufficiency of the condition $f \in B_p$ for $H_{\bar{f}}$ being in the Schatten class S_p when $1 < p < 2$ given in [1] is rather difficult and technical, and it is our aim to provide a more "elementary" proof of that result.

Theorem 1. *Let $1 < p < 2$, $\alpha > -1$ and $f \in A^2_\alpha$. The Hankel operator $H_{\bar{f}}$ belongs to S_p if and only if $f \in B_p$.*

Recall that, for $1 < p < \infty$, the analytic Besov space B_p consists of those functions f analytic on $\mathbb{D}$ for which

$$\int_{\mathbb{D}} |f'(z)|^p \, (1 - |z|^2)^{p-2} \, dA(z) < \infty.$$

Also, if H and $\mathcal{K}$ are separable Hilbert spaces, a compact operator T from H to $\mathcal{K}$ is said to belong to the Schatten class S_p if its sequence of singular numbers is in the sequence space ℓ^p. Recall that the singular numbers of a compact operator T are the square root of the eigenvalues of the positive operator T^*T, where T^* denotes the adjoint of T. Also, the compact operator T admits a decomposition of the form

$$T = \sum_n \lambda_n \langle \cdot, e_n \rangle_H f_n,$$

where $\{\lambda_n\}$ are the singular numbers of T, $\{e_n\}$ is an orthonormal set in H, and $\{f_n\}$ is an orthonormal set in $\mathcal{K}$. For $p \geq 1$, the class S_p is a Banach space with the norm $\|T\|_p = \left(\sum_n |\lambda_n|^p\right)^{1/p}$, while for $0 < p < 1$ one has the inequality $\|S + T\|_p^p \leq \|S\|_p^p + \|T\|_p^p$. We refer to [17, Chapter 1] for a brief account on the theory of Schatten p-classes.

We consider also the study of the boundedness, compactness and membership of Schatten p-classes of the Hankel operator H_f for general symbols $f \in L^2(\mathbb{D}, dA_\alpha)$. In order to state the next result we need to introduce some notation. For $z \in \mathbb{D}$ and $r > 0$, let

$$D(z,r) = \{w \in \mathbb{D} : \beta(z,w) < r\}$$

denote the hyperbolic disk with center z and radius r. Here $\beta(z,w)$ is the Bergman or hyperbolic metric on $\mathbb{D}$. Also, for any Lebesgue measurable set E in $\mathbb{D}$, we use the notation $|E|_\alpha := \int_E dA_\alpha$ for the dA_α-measure of E.

Theorem 2. *Let $\alpha > -1$ and $f \in L^2(\mathbb{D}, dA_\alpha)$. The following conditions are equivalent:*

(a) *H_f is bounded on A^2_α.*
(b) $\sup_{z\in\mathbb{D}} \|H_f k_z\|_\alpha < \infty$.
(c) *For any (or some) $r > 0$, the function F_r defined by*

$$F_r(z)^2 = \inf \left\{ \frac{1}{|D(z,r)|_\alpha} \int_{D(z,r)} |f-h|^2 \, dA_\alpha : h \in A^2_\alpha \right\}$$

is bounded on $\mathbb{D}$.

(d) *f admits a decomposition $f = f_1 + f_2$, where $f_1 \in C^1(\mathbb{D})$ satisfies $(1-|z|^2)\,\overline{\partial} f_1(z) \in L^\infty(\mathbb{D})$, and f_2 has the property that for any (or some) $r > 0$ the function G_r defined by*

$$G_r(z)^2 = \frac{1}{|D(z,r)|_\alpha} \int_{D(z,r)} |f_2(w)|^2 \, dA_\alpha(w)$$

is bounded on $\mathbb{D}$.

Note that we can put $\alpha = 0$ in the statements of parts (c) and (d) since the weight factor $(1-|z|^2)^\alpha$ in $dA_\alpha(z)$ is essentially cancelled out by the extra factor $(1-|z|^2)^\alpha$ in $|D(z,r)|_\alpha \asymp (1-|z|^2)^{2+\alpha}$.

The case $\alpha = 0$ of Theorem 2 was proved by D. Luecking in [10], who also noticed that the same proof also applies to the case $-1 < \alpha < 1$, and that the only missing part for the weighted case is a proof of the implication (d) implies (a), and this will be our contribution. The proof uses $\overline{\partial}$-techniques, and the main ideas for the proof were essentially given in [7] (see also the related papers [6] and [8] of the same authors), where the corresponding result for a class of weighted Bergman spaces is obtained. However, the standard weights $(1-|z|^2)^\alpha$ are in the class considered in [7] only for $\alpha > 2$, and also the statement that they give is for symbols in $L^2(\mathbb{D})$. We remark that K. Zhu in p.233 of his book [17] considers the question of describing the boundedness of Hankel operators with general symbols on weighted Bergman spaces as an open problem. Here we will use the appropriate modifications in order to obtain a unified proof for all $\alpha > -1$. We also obtain the corresponding analogues for compactness and membership in Schatten-Von Neumann ideals.

Throughout the paper, the letter C will denote an absolute constant whose value may change at different occurrences. The paper is organized as follows: Section 2 is devoted to some preliminaries needed for the proofs of the main results. A proof of Theorem 1 is given in Section 3, and we study the boundedness, compactness and membership in Schatten classes of Hankel operators with general symbols $f \in L^2(\mathbb{D}, dA_\alpha)$ in Section 4. Finally, we look at little Hankel operators in Section 5.

2. Preliminaries

We will use the fact that for any orthonormal set $\{e_n\}$ of A^2_α, one has

$$\sum_n |e_n(z)|^2 \leq \|K_z\|^2_\alpha, \quad z \in \mathbb{D}, \tag{2.1}$$

with equality if $\{e_n\}$ is also an orthonormal basis.

The following integral estimate (see [17]) has become indispensable in this area of analysis, and will be used repeatedly throughout the paper.

Lemma 2.1. *Suppose $z \in \mathbb{D}$, $c > 0$ and $t > -1$. The integral*

$$I_{c,t}(z) = \int_{\mathbb{D}} \frac{(1-|w|^2)^t}{|1-\bar{w}z|^{2+t+c}}\, dA(w)$$

is comparable to $(1-|z|^2)^{-c}$.

We also need some well known variants of the previous lemma. First recall that a sequence $\{z_n\}$ of points in the unit disk $\mathbb{D}$ is said to be separated in the Bergman metric if there is a constant $\delta > 0$ such that $\beta(z_j, z_k) \geq \delta$ for all j and k with $j \neq k$. In particular, there is a constant $r > 0$ such that the hyperbolic disks $D(z_k, r)$ are pairwise disjoint.

Lemma 2.2. *Let $\{z_k\}$ be a separated sequence in $\mathbb{D}$, and let $1 < t < s$. Then*

$$\sum_k \frac{(1-|z_k|^2)^t}{|1-\bar{z}_k z|^s} \leq C\,(1-|z|^2)^{t-s}, \qquad z \in \mathbb{D}.$$

Lemma 2.3. *Let $c > 0$ and $t > -1$. Then*

$$\int_{\mathbb{D}} \frac{(1-|w|^2)^t\, dA(w)}{|z-w|\,|1-\bar{w}z|^{1+t+c}}$$

is comparable to $(1-|z|^2)^{-c}$.

The following solution of the $\bar{\partial}$-equation will be a key ingredient in the proof of Theorem 2.

Proposition 2.4. *Let $1 < p < \infty$ and $\alpha > -1$. Then the function*

$$u(z) = \int_{\mathbb{D}} \frac{f(w)\,(1-|w|^2)^{1+\alpha}}{(z-w)(1-\bar{w}z)^{1+\alpha}}\, dA(w)$$

solves the equation $\overline{\partial} u = f$ *in* $\mathbb{D}$ *with*

$$\int_{\mathbb{D}} |u(z)|^p \, dA_\alpha(z) \le C \int_{\mathbb{D}} |f(z)|^p \, (1-|z|^2)^{p+\alpha} \, dA(z),$$

provided the right hand-side integral is finite.

Proof. The function u clearly satisfies the equation $\overline{\partial} u = f$ in $\mathbb{D}$. The corresponding estimate will follow from Hölder's inequality and Lemma 2.3. Indeed, let $\varepsilon > 0$ with $\alpha - \varepsilon > -1$ and $\alpha - \frac{\varepsilon}{p-1} > -1$. Then

$$\begin{aligned} |u(z)|^p &\le \left(\int_{\mathbb{D}} \frac{|f(w)|^p \, (1-|w|^2)^{p+\alpha+\varepsilon} dA(w)}{|z-w| \, |1-\bar{w}z|^{1+\alpha}} \right) \left(\int_{\mathbb{D}} \frac{(1-|w|^2)^{\alpha - \frac{\varepsilon}{p-1}} \, dA(w)}{|z-w| \, |1-\bar{w}z|^{1+\alpha}} \right)^{p-1} \\ &\le C \, (1-|z|^2)^{-\varepsilon} \left(\int_{\mathbb{D}} \frac{|f(w)|^p \, (1-|w|^2)^{p+\alpha+\varepsilon} \, dA(w)}{|z-w| \, |1-\bar{w}z|^{1+\alpha}} \right). \end{aligned}$$

Thus, Fubini's theorem and Lemma 2.3 gives

$$\begin{aligned} &\int_{\mathbb{D}} |u(z)|^p \, dA_\alpha(z) \\ &\quad \le C \int_{\mathbb{D}} |f(w)|^p \, (1-|w|^2)^{p+\alpha+\varepsilon} \left(\int_{\mathbb{D}} \frac{(1-|z|^2)^{\alpha-\varepsilon}}{|z-w| \, |1-\bar{w}z|^{1+\alpha}} \, dA(z) \right) dA(w) \\ &\quad \le C \int_{\mathbb{D}} |f(w)|^p \, (1-|w|^2)^{p+\alpha} \, dA(w). \end{aligned}$$

□

We also need the concept of an r-lattice in the Bergman metric. Let $r > 0$. A sequence $\{a_k\}$ of points in $\mathbb{D}$ is called an r-lattice if the unit disk is covered by the Bergman metric disks $\{D(a_k, r)\}$, and $\beta(a_i, a_j) \ge r/2$ for all i and j with $i \ne j$. If $\{a_k\}$ is an r-lattice in $\mathbb{D}$, then it also has the following property: for any $R > 0$ there exists a positive integer N (depending on r and R) such that every point in $\mathbb{D}$ belongs to at most N sets in $\{D(a_k, R)\}$. There are elementary constructions of r-lattices in $\mathbb{D}$. See [17, Chapter 4] for example.

A positive Borel measure μ in the unit disk is a Carleson measure for A^2_α if there exists a finite positive constant C such that

$$\int_{\mathbb{D}} |f(z)|^2 \, d\mu(z) \le C \|f\|^2_\alpha$$

for all $f \in A^2_\alpha$. Also, μ is said to be a vanishing Carleson measure for A^2_α if the inclusion map $i : A^2_\alpha \to L^2(\mathbb{D}, d\mu)$ is compact. It is well known (see [9], or Theorems 7.4 and 7.7 in [17] for example) that the Carleson measures for A^2_α are characterized by the condition

$$\sup_{a \in \mathbb{D}} \frac{\mu\big(D(a,r)\big)}{(1-|a|^2)^{2+\alpha}} < \infty.$$

Also, the condition

$$\lim_{|a| \to 1^-} \frac{\mu\big(D(a,r)\big)}{(1-|a|^2)^{2+\alpha}} = 0$$

describes the vanishing Carleson measures for A^2_α.

3. Proof of Theorem 1

Let $f \in A^2_\alpha$ and $1 < p < 2$. We prove the necessity first. So, suppose that the Hankel operator $H_{\bar f}$ belongs to S_p. We must show that $f \in B_p$. Let $\beta = (2+\alpha)p$. By Theorem 5.21 of [17], $f \in B_p$ if and only if

$$I_p(f) := \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|f(z)-f(w)|^p}{|1-\bar{w}z|^{4+2\beta}}\, dA_\beta(z)\, dA_\beta(w) < \infty.$$

If K_z is the reproducing kernel of A^2_α, since $fK_w \in A^2_\alpha$ for each $w \in \mathbb{D}$, an easy computation gives

$$\begin{aligned} H_{\bar f} K_z(w) &= \overline{f(w)} K_z(w) - P_\alpha(\bar f K_z)(w) = \overline{f(w)} K_z(w) - \langle \bar f K_z, K_w\rangle_\alpha \\ &= \overline{f(w)} K_z(w) - \langle K_z, f K_w\rangle_\alpha = \left(\overline{f(w)-f(z)}\right) K_z(w). \end{aligned}$$

On the other hand, let $H_{\bar f} = \sum_n \lambda_n \langle \cdot, e_n\rangle_\alpha\, f_n$ be a decomposition of the operator $H_{\bar f}$, with $\{\lambda_n\}$ being the singular numbers of $H_{\bar f}$, and $\{e_n\}$, $\{f_n\}$ are orthonormal sets in A^2_α and $L^2(\mathbb{D}, dA_\alpha)$ respectively. Then

$$H_{\bar f} K_z(w) = \sum_n \lambda_n \overline{e_n(z)}\, f_n(w).$$

Hence, by Hölder's inequality,

$$\begin{aligned} |H_{\bar f} K_z(w)|^p &\le \left(\sum_n |\lambda_n|^p\, |f_n(w)|^p\, |e_n(z)|^{2-p}\right) \left(\sum_n |e_n(z)|^2\right)^{p-1} \\ &\le \left(\sum_n |\lambda_n|^p\, |f_n(w)|^p\, |e_n(z)|^{2-p}\right) \|K_z\|_\alpha^{2(p-1)}. \end{aligned}$$

This, together with the fact that $\|K_z\|^2_\alpha = (1-|z|^2)^{-2-\alpha}$ and $\beta = (2+\alpha)p$, gives

$$\begin{aligned} I_p(f) &= \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|H_{\bar f} K_z(w)|^p}{|1-\bar w z|^{4+(2+\alpha)p}}\, dA_\beta(z)\, dA_\beta(w) \\ &\le \sum_n |\lambda_n|^p \int_{\mathbb{D}} |f_n(w)|^p \int_{\mathbb{D}} \frac{|e_n(z)|^{2-p}}{|1-\bar w z|^{4+(2+\alpha)p}}\, \|K_z\|_\alpha^{2(p-1)}\, dA_\beta(z) dA_\beta(w) \\ &\asymp \sum_n |\lambda_n|^p \int_{\mathbb{D}} |f_n(w)|^p \int_{\mathbb{D}} \frac{|e_n(z)|^{2-p}}{|1-\bar w z|^{4+(2+\alpha)p}}\, dA_{2+\alpha}(z)\, dA_\beta(w). \end{aligned}$$

Since $\|H_{\bar f}\|^p_{S_p} = \sum_n |\lambda_n|^p$, it is enough to show that

$$J_n := \int_{\mathbb{D}} |f_n(w)|^p \int_{\mathbb{D}} \frac{|e_n(z)|^{2-p}}{|1-\bar w z|^{4+(2+\alpha)p}}\, dA_{2+\alpha}(z)\, dA_\beta(w) \le C,$$

for some positive constant C independent of n. Now, since $p < 2$, we can use Hölder's inequality with exponent $2/p > 1$ to obtain

$$J_n \leq C\,\|f_n\|^p_{L^2(\mathbb{D},dA_\alpha)} \left(\int_{\mathbb{D}} \left(\int_{\mathbb{D}} \frac{|e_n(z)|^{2-p}}{|1-\bar{w}z|^{4+(2+\alpha)p}}\, dA_{2+\alpha}(z)\right)^{\frac{2}{2-p}} dA_\gamma(w)\right)^{\frac{2-p}{2}},$$

where $\gamma = \frac{p(4+\alpha)}{2-p}$. Now we use Hölder's inequality once again and Lemma 2.1 to obtain

$$\begin{aligned}
&\left(\int_{\mathbb{D}} \frac{|e_n(z)|^{2-p}}{|1-\bar{w}z|^{4+(2+\alpha)p}}\, dA_{2+\alpha}(z)\right)^{\frac{2}{2-p}} \\
&\qquad \leq \left(\int_{\mathbb{D}} \frac{|e_n(z)|^{2}}{|1-\bar{w}z|^{4+(2+\alpha)p}}\, dA_{2+\alpha}(z)\right)\left(\int_{\mathbb{D}} \frac{dA_{2+\alpha}(z)}{|1-\bar{w}z|^{4+(2+\alpha)p}}\right)^{\frac{p}{2-p}} \\
&\qquad \leq C\int_{\mathbb{D}} \frac{|e_n(z)|^{2}}{|1-\bar{w}z|^{4+(2+\alpha)p}}\, dA_{2+\alpha}(z)\left((1-|w|^2)^{\alpha-(2+\alpha)p}\right)^{\frac{p}{2-p}}.
\end{aligned}$$

Note that

$$\gamma + \frac{\alpha p-(2+\alpha)p^2}{2-p} = \frac{4p+2\alpha p-2p^2-\alpha p^2}{2-p} = (2+\alpha)p.$$

Therefore, since $\|f_n\|_{L^2(\mathbb{D},dA_\alpha)} = 1$, an application of Fubini's theorem and Lemma 2.1 yields

$$\begin{aligned}
J_n &\leq C\left(\int_{\mathbb{D}} \left(\int_{\mathbb{D}} \frac{|e_n(z)|^{2}}{|1-\bar{w}z|^{4+(2+\alpha)p}} dA_{2+\alpha}(z)\right)(1-|w|^2)^{(2+\alpha)p}\, dA(w)\right)^{\frac{2-p}{2}} \\
&\leq C\left(\int_{\mathbb{D}} |e_n(z)|^2 \left(\int_{\mathbb{D}} \frac{(1-|w|^2)^{(2+\alpha)p}}{|1-\bar{w}z|^{4+(2+\alpha)p}}\, dA(w)\right) dA_{2+\alpha}(z)\right)^{\frac{2-p}{2}} \\
&\leq C\|e_n\|_\alpha^{2-p} = C.
\end{aligned}$$

This finishes the proof of the necessity part.

Now we proceed to prove the sufficiency of the condition $f \in B_p$. Since B_p is included in the little Bloch space, it follows that $H_{\bar{f}}$ is compact. We want to show that the Hankel operator $H_{\bar{f}}$ is in the Schatten class S_p or, equivalently, that the positive operator $S = H^*_{\bar{f}} H_{\bar{f}}$ belongs to $S_{p/2}$. To this end, we fix a sufficiently large number b and use the atomic decomposition of A^2_α (see [17, Theorem 4.33]) to find a separated sequence $\{z_n\}$ in $\mathbb{D}$ such that A^2_α consists exactly of functions of the form $g(z) = \sum_n \lambda_n h_n(z)$, where $\lambda = \{\lambda_n\} \in \ell^2$,

$$h_n(z) = \frac{(1-|z_n|^2)^{b-\frac{2+\alpha}{2}}}{(1-\bar{z}_n z)^b},$$

and $\|g\|_\alpha \leq C\|\lambda\|_{\ell^2}$ for some positive constant C independent of $\{\lambda_n\}$.

Fix an orthonormal basis $\{e_n\}$ for A^2_α and define a linear operator B on A^2_α by

$$B\left(\sum_n \lambda_n e_n\right) = \sum_n \lambda_n h_n.$$

Since B is a bounded surjective linear operator on A^2_α, by [17, Proposition 1.30] we will have $S \in S_{p/2}$ if we can prove that the operator $H = B^* S B$ is in $S_{p/2}$. Moreover, since H is a positive operator and $p/2 < 1$, according to [17, Corollary 1.32], we will have $H \in S_{p/2}$ if

$$\sum_n \langle He_n, e_n \rangle_\alpha^{p/2} < \infty.$$

Since

$$\begin{aligned} \langle He_n, e_n \rangle_\alpha^{p/2} &= \langle B^* S B e_n, e_n \rangle_\alpha^{p/2} = \langle SBe_n, Be_n \rangle_\alpha^{p/2} = \langle Sh_n, h_n \rangle_\alpha^{p/2} \\ &= \langle H^*_{\bar f} H_{\bar f} h_n, h_n \rangle_\alpha^{p/2} = \langle H_{\bar f} h_n, H_{\bar f} h_n \rangle_\alpha^{p/2} = \|H_{\bar f} h_n\|_\alpha^p, \end{aligned}$$

we need to show that

$$\sum_n \|H_{\bar f} h_n\|_\alpha^p < \infty.$$

Since $H_{\bar f} h_n = \bar f h_n - P_\alpha(\bar f h_n)$ is the solution of the equation $\bar\partial u = h_n \, \overline{f'}$ with minimal $L^2(\mathbb{D}, dA_\alpha)$-norm, it follows from Proposition 2.4 that $\|H_{\bar f} h_n\|_\alpha \le C \|M_h h_n\|_\alpha$, where M_h is the operator of multiplication by the function $h(z) = (1-|z|^2)^2 f'(z)$. Therefore, it is enough to show that

$$M := \sum_n \|M_h h_n\|_\alpha^p < \infty.$$

Let $\{a_n\} \subset \mathbb{D}$ be an r-lattice in the Bergman metric. Since $|h_n(z)|$ is comparable to $|h_n(a_k)|$ for $z \in D(a_k, r)$ we obtain

$$\begin{aligned} M &= \sum_n \left(\int_{\mathbb{D}} (1-|z|^2)^2 \, |f'(z)|^2 \, |h_n(z)|^2 \, dA_\alpha(z) \right)^{p/2} \\ &\le \sum_n \left(\sum_k \int_{D(a_k,r)} (1-|z|^2)^2 \, |f'(z)|^2 \, |h_n(z)|^2 \, dA_\alpha(z) \right)^{p/2} \\ &\le C \sum_n \left(\sum_k |h_n(a_k)|^2 \int_{D(a_k,r)} (1-|z|^2)^2 \, |f'(z)|^2 \, dA_\alpha(z) \right)^{p/2}. \end{aligned}$$

Also, the subharmonicity of $|f'|^p$ and the fact that $D(z,r) \subset D(a_k, 2r)$ for $z \in D(a_k, r)$ gives

$$\begin{aligned} |f'(z)|^2 &\le C \left(\frac{1}{(1-|z|^2)^2} \int_{D(z,r)} |f'(w)|^p \, dA(w) \right)^{2/p} \\ &\le C \left(\int_{D(a_k,2r)} |f'(w)|^p \, d\lambda(w) \right)^{2/p} \end{aligned}$$

for each $z \in D(a_k, r)$ (see [17, Proposition 4.13]). Recall that $d\lambda(w) = (1 - |w|^2)^{-2} \, dA(w)$ is the hyperbolic metric in $\mathbb{D}$. This together with the fact that

$p/2 \le 1$ yields

$$M \le C \sum_n \left(\sum_k |h_n(a_k)|^2 (1-|a_k|^2)^2 \left(\int_{D(a_k,2r)} |f'(w)|^p d\lambda(w) \right)^{\frac{2}{p}} \big|D(a_k,r)\big|_\alpha \right)^{p/2}$$

$$\le C \sum_n \sum_k |h_n(a_k)|^p \, (1-|a_k|^2)^p \int_{D(a_k,2r)} |f'(w)|^p \, d\lambda(w) \, \big|D(a_k,r)\big|_\alpha^{p/2}.$$

Finally, since $\{z_n\}$ is a separated sequence, it follows from Lemma 2.2 that

$$\sum_n |h_n(a_k)|^p = \sum_n \frac{(1-|z_n|^2)^{bp-(2+\alpha)\frac{p}{2}}}{|1-\bar{z}_n a_k|^{bp}} \le C(1-|a_k|^2)^{-(2+\alpha)\frac{p}{2}}.$$

This, together with the fact that $|D(a_k,r)|_\alpha \asymp (1-|a_k|^2)^{2+\alpha}$ gives

$$M \le C \sum_k (1-|a_k|^2)^p \int_{D(a_k,2r)} |f'(w)|^p \, d\lambda(w)$$

$$\le C \sum_k \int_{D(a_k,2r)} |f'(w)|^p \, (1-|w|^2)^p \, d\lambda(w) \le C\|f\|_{B_p}^p.$$

This completes the proof of the theorem.

4. Hankel operators with general symbols

Proof of Theorem 2. The implications (a) ⇒ (b); (b) ⇒ (c); and (c) ⇒ (d) follows the same arguments of the proof given in [10] (see also the proof of Theorem 8.34 in K. Zhu's monograph [17]). So, suppose that (d) holds. The condition on f_2 says that $|f_2|^2 \, dA_\alpha$ is a Carleson measure for A^2_α. Therefore

$$\|H_{f_2} g\|_\alpha^2 \le \|f_2 g\|_\alpha^2 = \int_{\mathbb{D}} |g(z)|^2 \, |f_2(z)|^2 \, dA_\alpha(z) \le C\|g\|_\alpha^2,$$

and the Hankel operator H_{f_2} is bounded on A^2_α. It remains to show that the Hankel operator H_{f_1} is bounded on A^2_α. Let $g \in H^\infty$. Since $(1-|z|^2)\overline{\partial} f_1(z)$ is bounded in $\mathbb{D}$, using Lemma 2.4 we see that there exists a solution u of the equation $\overline{\partial} u = g \, \overline{\partial} f_1$ in $\mathbb{D}$ with

$$\|u\|_\alpha^2 = \int_{\mathbb{D}} |u(z)|^2 \, dA_\alpha(z) \le C \int_{\mathbb{D}} \Big((1-|z|^2) |\overline{\partial} f_1(z)| \Big)^2 \, |g(z)|^2 \, dA_\alpha(z)$$

$$\le C \int_{\mathbb{D}} |g(z)|^2 \, dA_\alpha(z) = C\|g\|_\alpha^2.$$

Since $H_{f_1} g = f_1 g - P_\alpha(f_1 g)$ is the solution of the equation $\overline{\partial} u = g \, \overline{\partial} f_1$ with minimal $L^2(\mathbb{D}, dA_\alpha)$-norm, we obtain

$$\|H_{f_1} g\|_\alpha^2 \le C\|g\|_\alpha^2,$$

completing the proof of the boundedness of H_f. □

The corresponding result for compactness is the following one.

Theorem 3. *Let $\alpha > -1$ and $f \in L^2(\mathbb{D}, dA_\alpha)$. The following conditions are equivalent:*

(a) *H_f is compact on A^2_α.*

(b) $\lim_{|z|\to 1^-} \|H_f k_z\|_\alpha = 0$.

(c) *For any (or some) $r > 0$, the function F_r defined by*

$$F_r(z)^2 = \inf\left\{ \frac{1}{|D(z,r)|_\alpha} \int_{D(z,r)} |f-h|^2 \, dA_\alpha : h \in A^2_\alpha \right\}$$

is in $C_0(\mathbb{D})$.

(d) *f admits a decomposition $f = f_1 + f_2$, where $f_1 \in C^1(\mathbb{D})$ satisfies $(1-|z|^2)\,\overline{\partial} f_1(z) \in C_0(\mathbb{D})$, and f_2 has the property that for any (or some) $r > 0$ the function G_r defined by*

$$G_r(z)^2 = \frac{1}{|D(z,r)|_\alpha} \int_{D(z,r)} |f_2(w)|^2 \, dA_\alpha(w)$$

is in $C_0(\mathbb{D})$.

Proof. Since every function in $C_0(\mathbb{D})$ is bounded in $\mathbb{D}$, each of the conditions implies the boundedness of the Hankel operator H_f on A^2_α. We will prove only the implication (d) $\Rightarrow$ (a), since the other implications are well known. Thus, suppose that $f = f_1 + f_2$ with $(1-|z|^2)\,\overline{\partial} f_1(z) \in C_0(\mathbb{D})$, and $|f_2|^2\, dA_\alpha$ being a vanishing Carleson measure for A^2_α. We will show that both H_{f_1} and H_{f_2} are compact. The condition on f_2 implies that the multiplication operator M_{f_2} is compact from A^2_α into $L^2(\mathbb{D}, dA_\alpha)$ (see Theorem 7.8 in [17]), and therefore the Hankel operator $H_{f_2} = (I - P_\alpha) M_{f_2}$ is compact. To show that H_{f_1} is compact, let $\{g_n\}$ be a bounded sequence in A^2_α that converges to 0 uniformly on compact subsets of $\mathbb{D}$. Given any $\varepsilon > 0$, there exists $0 < r < 1$ such that

$$(1-|z|^2)|\overline{\partial} f_1(z)| < \varepsilon, \qquad |z| > r,$$

and we can choose a positive integer n_0 with

$$|g_n(z)| < \varepsilon, \qquad |z| \le r, \quad n \ge n_0.$$

From the proof of Theorem 2 it follows that

$$\begin{aligned} \|H_{f_1} g_n\|^2_\alpha &\le C \int_{\mathbb{D}} \Big((1-|z|^2)|\overline{\partial} f_1(z)| \Big)^2 |g_n(z)|^2 \, dA_\alpha(z) \\ &\le C\,\varepsilon \Big\| (1-|z|^2)|\overline{\partial} f_1(z)| \,\Big\|^2_\infty + C\varepsilon \|g_n\|^2_\alpha \\ &\le C\varepsilon \end{aligned}$$

for all $n \ge n_0$. This proves that $\lim_n \|H_{f_1} g_n\|_\alpha = 0$, so H_{f_1} is compact from A^2_α into $L^2(\mathbb{D}, dA_\alpha)$ finishing the proof of the theorem. □

The following result characterizes the membership of the Hankel operator H_f in the Schatten classes S_p for $p \ge 1$. Recall that $d\lambda(z) = (1-|z|^2)^{-2}$ is the hyperbolic measure on $\mathbb{D}$.

Theorem 4. *Let $p \ge 1$ and $f \in L^2(\mathbb{D}, dA_\alpha)$ such that H_f is bounded on A^2_α. The following conditions are equivalent:*

(a) *H_f belongs to S_p.*
(b) *For any (or some) $r > 0$, the function*

$$z \mapsto \left(\int_{D(z,r)} |H_f k_z(w)|^2 \, dA_\alpha(w) \right)^{1/2}$$

belongs to $L^p(\mathbb{D}, d\lambda)$.
(c) *For any (or some) $r > 0$, the function F_r defined in Theorem 2 belongs to $L^p(\mathbb{D}, d\lambda)$.*
(d) *f admits a decomposition $f = f_1 + f_2$, where for any (or some) $r > 0$ the function*

$$H_r(z) = \left(\frac{1}{|D(z,r)|_\alpha} \int_{D(z,r)} \left|(1-|z|^2)\,\overline{\partial} f_1(z)\right|^2 dA_\alpha(z) \right)^{1/2}$$

belongs to $L^p(\mathbb{D}, d\lambda)$, and the function G_r defined in Theorem 2 also belongs to $L^p(\mathbb{D}, d\lambda)$.

Remark: For $p \geq 2$, condition (b) can be replaced by the condition

(b2) The function $z \mapsto \|H_f k_z\|_\alpha$ belongs to $L^p(\mathbb{D}, d\lambda)$,

and (d) can also be replaced by the condition

(d2) f admits a decomposition $f = f_1 + f_2$, where the function $(1-|z|^2)\,\overline{\partial} f_1(z)$ belongs to $L^p(\mathbb{D}, d\lambda)$ and for any (or some) $r > 0$ the function G_r defined in Theorem 2 also belongs to $L^p(\mathbb{D}, d\lambda)$.

Proof. Apart from the implication (d) implies (a), all the other implications are known (see [10] or [17, Theorem 8.36]). So, suppose that (d) holds, that is, $f = f_1 + f_2$, where the functions H_r and G_r both belong to $L^p(\mathbb{D}, d\lambda)$. Since for any $g \in H^\infty$ one has $\|H_{f_2} g\|_\alpha \leq \|f_2 g\|_\alpha$ and $\|H_{f_1} g\|_\alpha \leq C\|hg\|_\alpha$ with $h(z) = (1-|z|^2)\,\overline{\partial} f_1(z)$, it suffices to show that the multiplication operator $M_\psi : A^2_\alpha \to L^2(\mathbb{D}, dA_\alpha)$ belongs to S_p for $\psi = f_2$ or $\psi = h$. This is equivalent to $M_\psi^* M_\psi$ being in $S_{p/2}$, and since $M_\psi^* M_\psi = T_{|\psi|^2}$ where T_φ denotes the Toeplitz operator with symbol φ, by Theorem 7.18 of [17] the conditions in (d) are exactly what is needed to have both $T_{|f_2|^2}$ and $T_{|h|^2}$ belong to $S_{p/2}$. Thus the corresponding multiplication operators M_{f_2} and M_h are in S_p finishing the proof of the theorem.

□

5. Little Hankel operators

Let $\overline{A^2_\alpha}$ be the space of conjugate analytic functions in A^2_α. For $f \in L^2(\mathbb{D}, dA_\alpha)$, the little Hankel operator $h_f : A^2_\alpha \to \overline{A^2_\alpha}$ is defined by the formula

$$h_f g(z) = \int_{\mathbb{D}} \frac{f(w)\,g(w)}{(1-\bar{z}w)^{2+\alpha}} \, dA_\alpha(w).$$

The operator h_f is unbounded in general. However, h_f is bounded if f is bounded, and we clearly have $\|h_f\| \leq \|f\|_\infty$. In the study of little Hankel operators, it turns out that it is more convenient to study $h_{\bar{f}}$ instead of h_f. Throughout this section, let V_α be the integral operator defined by

$$V_\alpha f(z) = \langle \bar{k}_z, h_{\overline{f}}\, k_z \rangle_\alpha = (1-|z|^2)^{2+\alpha} \int_{\mathbb{D}} \frac{f(w)}{(1-\bar{w}z)^{4+2\alpha}}\, dA_\alpha(w).$$

Recall that k_z are the normalized reproducing kernels of A^2_α. For a given function $f \in L^2(\mathbb{D}, dA_\alpha)$, one has the identity $h_{\bar{f}} = h_{c_\alpha \overline{V_\alpha f}}$ for some positive constant c_α depending only on α, in the sense that $h_{\bar{f}} g = h_{c_\alpha \overline{V_\alpha f}} g$ for all $g \in H^\infty$, which is dense in A^2_α (see [17, Chapter 8]). The properties of $V_\alpha f$ can be used in order to obtain descriptions of the boundedness, compactness and membership in Schatten classes of the little Hankel operator $h_{\bar{f}}$. In fact, it is proved in [5] and [14] that $h_{\bar{f}}$ is bounded on A^2_α if and only if $V_\alpha f \in L^\infty(\mathbb{D})$, and the compactness is characterized by the condition $V_\alpha f \in C_0(\mathbb{D})$. The corresponding description for the membership in the Schatten classes S_p with $p \geq 1$ is also obtained, and it is our aim to give a "more elementary" proof of that result, especially for the case $1 \leq p < 2$. Note that the proof we give below works also in the setting of the unit ball of $\mathbb{C}^N$, or other domains Ω in $\mathbb{C}^N$.

Theorem 5. *Let $f \in L^2(\mathbb{D}, dA_\alpha)$ and $1 \leq p < \infty$. Then $h_{\overline{f}}$ belongs to S_p if and only if $V_\alpha f$ is in $L^p(\mathbb{D}, d\lambda)$.*

Proof. Suppose that $h_{\overline{f}}$ is in the Schatten class S_p. Since $|V_\alpha f(z)| \leq \|h_{\overline{f}} k_z\|_\alpha$, the case $p \geq 2$ follows from the well known fact that $\|h_{\overline{f}} k_z\|_\alpha \in L^p(\mathbb{D}, d\lambda)$ is a necessary condition for $h_{\overline{f}}$ being in S_p if $p \geq 2$. What is curious is that the case $1 \leq p < 2$ can be proved in a similar way. Indeed, one has

$$h_{\overline{f}}\, K_z(w) = \sum_n \lambda_n\, \overline{e_n(z)}\, f_n(w),$$

where $\{\lambda_n\}$ are the singular values of $h_{\overline{f}}$, and $\{e_n\}$, $\{f_n\}$ are orthonormal sets of A^2_α and $\overline{A^2_\alpha}$ respectively. Thus

$$\begin{aligned}
\langle h_{\bar{f}} k_z, \overline{k_z} \rangle_\alpha &= \|K_z\|_\alpha^{-2} \langle h_{\bar{f}} K_z, \overline{K_z} \rangle_\alpha \\
&= \|K_z\|_\alpha^{-2} \sum_n \lambda_n\, \overline{e_n(z)}\, \langle f_n, \overline{K_z} \rangle_\alpha \\
&= \|K_z\|_\alpha^{-2} \sum_n \lambda_n\, \overline{e_n(z)}\, f_n(z).
\end{aligned}$$

Since $1 \leq p < 2$, using Hölder's inequality and (2.1), we obtain

$$\begin{aligned}
|V_\alpha f(z)|^p &\leq \|K_z\|_\alpha^{-2p} \left(\sum_n |\lambda_n|^p\, |e_n(z)|^{2-p}\, |f_n(z)|^p \right) \left(\sum_n |e_n(z)|^2 \right)^{p-1} \\
&\leq \left(\sum_n |\lambda_n|^p\, |e_n(z)|^{2-p}\, |f_n(z)|^p \right) \|K_z\|_\alpha^{-2},
\end{aligned}$$

and this, together with another use of Hölder's inequality, gives

$$\begin{aligned}\int_{\mathbb{D}} |V_\alpha f(z)|^p \, d\lambda(z) &\le \sum_n |\lambda_n|^p \int_{\mathbb{D}} |e_n(z)|^{2-p} \, |f_n(z)|^p \, dA_\alpha(z) \\ &\le \sum_n |\lambda_n|^p \, \|e_n\|_\alpha^{2-p} \, \|f_n\|_\alpha^p \\ &= \sum_n |\lambda_n|^p = \|h_{\bar f}\|_{S_p}^p.\end{aligned}$$

This proves the necessity in the case $1 \le p < 2$.

Suppose now that $V_\alpha f$ is in $L^p(\mathbb{D}, d\lambda)$. Since $V_\alpha f(z) = (1 - |z|^2)^{2+\alpha} h(z)$ with h analytic on $\mathbb{D}$, this implies that $h \in A^p_\beta$ with $\beta = (2+\alpha)p - 2$. Since any function in A^p_β satisfies the growth condition

$$\lim_{|z|\to 1^-} (1 - |z|^2)^{(2+\beta)/p} \, h(z) = 0,$$

it follows that $V_\alpha f \in C_0(\mathbb{D})$ proving that $h_{\bar f}$ is compact. Next we proceed to show that $h_{\bar f}$ is in S_p. Since $h_{\bar f} = h_{\overline{V_\alpha f}}$, it suffices to prove that h_φ belongs to S_p whenever $\varphi \in L^p(\mathbb{D}, d\lambda)$. To see this, it is enough to prove that

$$\sum_n |\langle h_\varphi e_n, \overline{f_n} \rangle_\alpha|^p < \infty$$

for all orthonormal sets $\{e_n\}$ and $\{f_n\}$ of A^2_α. But notice that, by Fubini's theorem

$$\begin{aligned}\langle h_\varphi e_n, \overline{f_n} \rangle_\alpha &= \int_{\mathbb{D}} (h_\varphi e_n)(z) \, f_n(z) \, dA_\alpha(z) \\ &= \int_{\mathbb{D}} \left(\int_{\mathbb{D}} \frac{\varphi(w) \, e_n(w)}{(1 - \bar z w)^{2+\alpha}} \, dA_\alpha(w) \right) f_n(z) \, dA_\alpha(z) \\ &= \int_{\mathbb{D}} \varphi(w) \, e_n(w) \left(\int_{\mathbb{D}} \frac{f_n(z)}{(1 - \bar z w)^{2+\alpha}} dA_\alpha(z) \right) dA_\alpha(w) \\ &= \int_{\mathbb{D}} \varphi(w) \, e_n(w) \, f_n(w) \, dA_\alpha(w).\end{aligned}$$

Therefore, since

$$\sum_n |e_n(w)| \, |f_n(w)| \le \left(\sum_n |e_n(w)|^2 \right)^{1/2} \left(\sum_n |f_n(w)|^2 \right)^{1/2} \le \|K_w\|_\alpha^2$$

we finally obtain

$$\begin{aligned}\sum_n |\langle h_\varphi e_n, \overline{f_n}\rangle_\alpha|^p &\le \sum_n \Big(\int_{\mathbb{D}} |\varphi(w)|\, |e_n(w)|\, |f_n(w)|\, dA_\alpha(w)\Big)^p \\ &\le \sum_n \int_{\mathbb{D}} |\varphi(w)|^p\, |e_n(w)|\, |f_n(w)|\, dA_\alpha(w) \\ &\le \int_{\mathbb{D}} |\varphi(w)|^p\, \|K_w\|_\alpha^2\, dA_\alpha(w) \\ &\le C \int_{\mathbb{D}} |\varphi(w)|^p\, d\lambda(w).\end{aligned}$$

This completes the proof of the theorem. □

6. Further remarks and questions

6.1. Hankel operators on A^p_α with $p > 1$

For $p > 0$ and $\alpha > -1$, let A^p_α be the space of all analytic functions f on $\mathbb{D}$ with

$$\|f\|_{p,\alpha} = \Big(\int_{\mathbb{D}} |f(z)|^p\, dA_\alpha(z)\Big)^{1/p} < \infty.$$

Since for $p > 1$, the Bergman projection P_α is bounded from $L^p(\mathbb{D}, dA_\alpha)$ to A^p_α, using the density of H^∞ in A^p_α, for symbols $f \in L^p(\mathbb{D}, dA_\alpha)$, we can define the (big) Hankel operator on A^p_α as

$$H_f g(z) = (I - P_\alpha)(fg) = \int_{\mathbb{D}} \frac{f(z) - f(w)}{(1-\bar{w}z)^{2+\alpha}}\, g(w)\, dA_\alpha(w), \qquad g \in H^\infty.$$

For conjugate analytic symbols, it is known (see, for example, [16]) that $H_{\bar f}$ is bounded on A^p_α if and only if f belongs to the Bloch space, and the compactness of $H_{\bar f}$ is characterized by f being in the little Bloch space. In [16], it is also obtained a characterization of the simultaneous boundedness (and compactness) of the operators H_f and $H_{\bar f}$ in A^p_α in the setting of the unit ball. For general symbols $f \in L^p(\mathbb{D}, dA_\alpha)$, the boundedness of the Hankel operator H_f on A^p_α can be characterized as follows.

Theorem 6. *Let $1 < p < \infty$, $\alpha > -1$ and $f \in L^p(\mathbb{D}, dA_\alpha)$. The following conditions are equivalent:*

(a) *H_f is bounded on A^p_α.*

(b) $\sup_{z\in\mathbb{D}} \; dist_{L^p(\mathbb{D},dA_\alpha)}(f \circ \varphi_z, A^p_\alpha) < \infty$.

(c) *For any (or some) $r > 0$, the function F_r defined by*

$$F_r(z)^p = \inf \left\{ \frac{1}{|D(z,r)|_\alpha} \int_{D(z,r)} |f-h|^p\, dA_\alpha : h \in A^p_\alpha \right\}$$

is bounded on $\mathbb{D}$.

(d) *f admits a decomposition $f = f_1 + f_2$, where $f_1 \in C^1(\mathbb{D})$ satisfies $(1-|z|^2)\,\overline{\partial} f_1(z) \in L^\infty(\mathbb{D})$, and f_2 has the property that for any (or some) $r > 0$ the function G_r defined by*

$$G_r(z)^p = \frac{1}{|D(z,r)|_\alpha} \int_{D(z,r)} |f_2(w)|^p \, dA_\alpha(w)$$

is bounded on $\mathbb{D}$.

The case $-1 < \alpha < 1/(p-1)$ was proved by Luecking in [10]. Also observe that for $p = 2$, condition (b) in the previous theorem coincides with condition (b) in Theorem 2, since $\|H_f k_z\|_\alpha = \|f \circ \varphi_z - P_\alpha(f \circ \varphi_z)\|_\alpha$, where $\varphi_z(w) = \frac{z-w}{1-\bar{z}w}$. The proof of Theorem 6 follows the same argument as in Theorem 2. As before, only the implication (d) implies (a) must be proved, since the others implications are well known. If (d) holds and $g \in H^\infty$, then using Proposition 2.4 there is a solution u of the equation $\overline{\partial} u = g \overline{\partial} f_1$ with $\|u\|_{p,\alpha} \le C \|g\|_{p,\alpha}$. Since any such solution must be of the form $u = f_1 g + h$ for some $h \in A^p_\alpha$, the boundedness of the Bergman projection $P_\alpha : L^p(\mathbb{D}, dA_\alpha) \to A^p_\alpha$ for $p > 1$ gives

$$\begin{aligned} \|H_{f_1} g\|_{p,\alpha} &\le \|H_{f_1} g - u\|_{p,\alpha} + \|u\|_{p,\alpha} = \|P_\alpha(f_1 g + h)\|_{p,\alpha} + \|u\|_{p,\alpha} \\ &\le (1 + \|P_\alpha\|)\, \|u\|_{p,\alpha}. \end{aligned}$$

This shows that H_{f_1} is bounded, and the proof of the boundedness of H_{f_2} follows the same lines as in Theorem 2. Similarly, one can obtain the corresponding result for compactness of the Hankel operator.

6.2. The two-sided ideal problem

One can also consider a two-sided ideal problem, namely, for a function $f \in L^2(\mathbb{D}, dA_\alpha)$ describe, in terms of properties of f, the simultaneous membership of H_f and $H_{\bar{f}}$ in S_p (this is equivalent to the membership of H_f in S_p when f is real valued). In the case that $p \ge 2$, K. Zhu obtained in [15] the following description: H_f and $H_{\bar{f}}$ are in S_p if and only if $MO_\alpha(f) \in L^p(\mathbb{D}, d\lambda)$, where $d\lambda(z) = (1-|z|^2)^{-2}\, dA(z)$ is the hyperbolic measure, and

$$MO_\alpha(f)(z) = \left(B_\alpha(|f|^2)(z) - \left|B_\alpha f(z)\right|^2\right)^{1/2}.$$

For the unweighted case $\alpha = 0$, in [12] J. Xia proved that the same condition $MO(f) \in L^p(\mathbb{D}, d\lambda)$ still describes the simultaneous membership of H_f and $H_{\bar{f}}$ in S_p when $1 < p < 2$. Note that for $0 < p \le 2/(2+\alpha)$, the condition $MO_\alpha(f) \in L^p(d\lambda)$ implies f being constant, so the natural conjecture for the weighted case is that the condition $MO_\alpha(f) \in L^p(d\lambda)$ will also be the correct condition for the case $2/(2+\alpha) < p < 2$. For this case one can use the method of J. Xia, but since it uses the duality between the S_p classes, it can only be used to obtain results for $p \ge 1$, and this will still give the gap $2/(2+\alpha) < p < 1$ for the case $\alpha > 0$. Therefore, it seems that new techniques are needed here.

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