COMPOSITION OPERATORS ACTING ON WEIGHTED DIRICHLET SPACES

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ABSTRACT. We study composition operators acting on weighted Dirichlet spaces. We obtain estimates for the essential norm, describe the membership in Schatten-Von Neumann ideals and characterize the composition operators with closed range.

1. INTRODUCTION

Let $\mathbb{D}$ denote the open unit disk in the complex plane, and let $\varphi : \mathbb{D} \to \mathbb{D}$ analytic. The function φ induces a composition operator C_φ acting on $H(\mathbb{D})$, the space of all analytic functions on $\mathbb{D}$, by the formula

$$C_\varphi f(z) = f(\varphi(z)).$$

It is an interesting problem to describe the operator properties of C_φ in terms of the function properties of the symbol φ when the operator C_φ acts on several spaces of analytic functions in $\mathbb{D}$. In this paper, we are going to study the composition operator C_φ acting on weighted Dirichlet spaces $\mathcal{D}_\alpha$, $\alpha > 0$, so let's proceed to introduce these spaces.

For $\alpha \geq 0$, the Dirichlet type space $\mathcal{D}_\alpha$ consists of those analytic functions f on $\mathbb{D}$ with

$$\|f\|_{\mathcal{D}_\alpha} = \left(|f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \, dA_\alpha(z) \right)^{1/2} < \infty,$$

where

$$dA_\alpha(z) = (1+\alpha)\,(1-|z|^2)^\alpha \, dA(z),$$

and $dA(z) = \frac{1}{\pi}\, dx\, dy$ is the normalized area measure on $\mathbb{D}$. Observe that $f \in \mathcal{D}_\alpha$ if and only if f' belongs to the Bergman space A^2_α. Let $\alpha > -1$, the weighted Bergman space A^2_α consists of those functions $f \in H(\mathbb{D})$ with

$$\|f\|_{A^2_\alpha} = \left(\int_{\mathbb{D}} |f(z)|^2 \, dA_\alpha(z) \right)^{1/2} < \infty.$$

It is well known that $\mathcal{D}_1 = H^2$, the classical Hardy space, and $\mathcal{D}_\alpha = A^2_{\alpha-2}$ if $\alpha > 1$. The results we are going to obtain about composition operators on the spaces $\mathcal{D}_\alpha$ are well known for the Hardy and Bergman spaces. Therefore we will focus on the case $0 < \alpha < 1$ (the reason why we exclude the case $\alpha = 0$ will be explained later). We are going to estimate the essential

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norm of the composition operator acting on $\mathcal{D}_\alpha$, obtain a description of the membership in the Schatten-Von Neumann ideal S_p of the composition operator on $\mathcal{D}_\alpha$, and finally, we characterize the composition operators with closed range. All of this will be accomplished with the aid of the generalized Nevanlinna counting function, which we study in the next section.

2. The generalized Nevanlinna counting function

Suppose that $\varphi : \mathbb{D} \to \mathbb{D}$ is analytic. The (classical) *Nevanlinna counting function* of φ is

$$N_\varphi(z) = \sum_{w:\varphi(w)=z} \log \frac{1}{|w|}, \qquad z \in \mathbb{D} \setminus \{\varphi(0)\},$$

where the sum is interpreted as being zero if $z \notin \varphi(\mathbb{D})$. An important property of the counting function N_φ is that, being not necessarily subharmonic, it satisfies the submean value property (see [22] or [25]).

Submean value property for N_φ. Suppose that $\varphi : \mathbb{D} \to \mathbb{D}$ is analytic. Then

$$N_\varphi(z) \leq \frac{1}{|B|} \int_B N_\varphi(w)\, dA(w),$$

where $z \in \mathbb{D} \setminus \{\varphi(0)\}$ and B is any euclidian disk centered at z contained in $\mathbb{D} \setminus \{\varphi(0)\}$. Here $|B|$ denotes the area of the disk B.

Let $\alpha > 0$ and $\varphi : \mathbb{D} \to \mathbb{D}$ analytic. The *generalized Nevanlinna counting function* of φ is

$$N_{\varphi,\alpha}(z) = \sum_{w:\varphi(w)=z} (1-|w|^2)^\alpha, \qquad z \in \mathbb{D} \setminus \{\varphi(0)\},$$

where, as before, the last sum is interpreted as being zero if $z \notin \varphi(\mathbb{D})$.

The following change of variables formula can be found, for example, in [1, Proposition 2.1] or [6, Theorem 2.32].

Change of variables formula. If f is nonnegative on $\mathbb{D}$ and $\varphi : \mathbb{D} \to \mathbb{D}$ is analytic, then

$$\int_{\mathbb{D}} f\big(\varphi(z)\big)\, |\varphi'(z)|^2\, (1-|z|^2)^\alpha\, dA(z) = \int_{\mathbb{D}} f(z)\, N_{\varphi,\alpha}(z)\, dA(z).$$

For $a \in \mathbb{D}$, denote by σ_a the disk automorphism defined by

$$\sigma_a(z) = \frac{a-z}{1-\bar{a}z}, \qquad z \in \mathbb{D}.$$

A key tool for our results is the submean value property for the generalized counting function $N_{\varphi,\alpha}$. This fact is deduced from the corresponding property for the classical Nevanlinna counting function together with the following formula due to A. Aleman (see Lemma 2.3 in [1]) that relates the function $N_{\varphi,\alpha}$ with the classical Nevanlinna counting function N_φ.

Aleman's formula. Let $0 < \alpha < 1$ and $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic and nonconstant. Then for every $\zeta \in \varphi(\mathbb{D})$,

$$N_{\varphi,\alpha}(\zeta) = -\frac{1}{2}\int_{\mathbb{D}} \Delta\,\omega_\alpha(z)\, N_{\varphi\circ\sigma_z}(\zeta)\, dA(z), \tag{2.1}$$

where $\omega_\alpha(z) = (1-|z|^2)^\alpha$, and Δ denotes the standard Laplace operator.

Proposition 2.1. *Let* $0 < \alpha < 1$ *and* $0 < p < \infty$*. Suppose that* $\varphi : \mathbb{D} \to \mathbb{D}$ *is analytic. Then there is a positive constant* $C = C_p < \infty$ *such that*

$$N_{\varphi,\alpha}(\zeta)^p \le \frac{C}{|B|}\int_B N_{\varphi,\alpha}(w)^p\, dA(w),$$

where $\zeta \in \mathbb{D} \setminus \{\varphi(0)\}$ *and* B *is any euclidian disk centered at* ζ *contained in* $\mathbb{D} \setminus \{\varphi(0)\}$*. Moreover, one can take* $C = 1$ *if* $p \ge 1$*.*

Proof. We begin with the case $p = 1$. This case has been proved recently in [10]. Let $\omega_\alpha(z) = (1-|z|^2)^\alpha$. An application of Aleman's formula together with the submean value property for the classical Nevanlinna counting function, and taking into account that $\Delta\omega_\alpha(z) \le 0$, gives

$$N_{\varphi,\alpha}(\zeta) \le -\frac{1}{2}\int_{\mathbb{D}} \Delta\,\omega_\alpha(z)\left(\frac{1}{|B|}\int_B N_{\varphi\circ\sigma_z}(w)\, dA(w)\right) dA(z),$$

for any euclidian disk B centered at ζ contained in $\mathbb{D} \setminus \{\varphi(0)\}$. Now, we continue by using Fubini's theorem together with another application of Aleman's formula to obtain

$$\begin{aligned} N_{\varphi,\alpha}(\zeta) &\le \frac{1}{|B|}\int_B \left(-\frac{1}{2}\int_{\mathbb{D}} \Delta\,\omega_\alpha(z)\, N_{\varphi\circ\sigma_z}(w)\, dA(z)\right) dA(w) \\ &= \frac{1}{|B|}\int_B N_{\varphi,\alpha}(w)\, dA(w), \end{aligned}$$

which proves the desired result for $p = 1$ with $C = 1$. This special case together with Hölder's inequality then implies that the desired inequality holds with $C = 1$ for all $p \ge 1$.

To deal with the case $p \in (0,1)$ we use an idea from [19]. Since the result holds for $p = 1$ with $C = 1$, a standard regularization argument from potential theory (see p. 51 of [21]) produces a nonnegative subharmonic function u_α on $\mathbb{D} \setminus \{\varphi(0)\}$ such that $N_{\varphi,\alpha} \le u_\alpha$ everywhere in $\mathbb{D} \setminus \{\varphi(0)\}$ and $N_{\varphi,\alpha} = u_\alpha$ almost everywhere on $\mathbb{D} \setminus \{\varphi(0)\}$ (the function u_α is just the upper semicontinuous regularization of $N_{\varphi,\alpha}$). Now, by an inequality due to Hardy and Littlewood (see [7, Lemma 2] or Lemma 3.7 in Chapter III of [8]), there is a positive constant $C = C_p$ such that

$$u_\alpha(\zeta)^p \le \frac{C}{|B|}\int_B u_\alpha(w)^p\, dA(w),$$

for any euclidian disk B centered at ζ contained in $\mathbb{D} \setminus \{\varphi(0)\}$. Observe that the inequality in [7] or [8] is stated for the absolute value of a harmonic function, but its proof actually works for nonnegative subharmonic functions. Thus

$$N_{\varphi,\alpha}(\zeta)^p \le u_\alpha(\zeta)^p \le \frac{C}{|B|}\int_B u_\alpha(w)^p\, dA(w) = \frac{C}{|B|}\int_B N_{\varphi,\alpha}(w)^p\, dA(w)$$

completing the proof of the Proposition. $\square$

Remark: When considering the composition operator acting on the classical Dirichlet space $\mathcal{D}$, one should replace the function $N_{\varphi,\alpha}(z)$ by the counting function $n_\varphi(z) := \#\{w : \varphi(w) = z\}$. However, the function n_φ does not satisfy the analogue of Proposition 2.1, that is, the counting function $n_\varphi(z)$ does not necessarily satisfies the generalized submean value property (see [27] for an example). For that reason, and due to the fact that Proposition 2.1 plays a prominent role in the proof of our results, we exclude the case $\alpha = 0$ of our study.

3. The essential norm of the composition operator

The *essential norm* of a bounded linear operator T is defined to be the distance to the compact operators, that is,

$$\|T\|_e = \inf\{\|T - K\| : K \text{ is compact}\}.$$

It is our aim in this section to obtain estimates for the essential norm of the composition operator C_φ acting on $\mathcal{D}_\alpha$. Before doing that, we describe the symbols φ inducing bounded composition operators on $\mathcal{D}_\alpha$. This result can be found in [10]. For completeness we offer here the proof. A different description in terms of Carleson measure properties of $N_{\varphi,\alpha}\, dA$ can be found in [27].

Theorem 3.1. *Let $0 < \alpha < 1$ and $\varphi : \mathbb{D} \to \mathbb{D}$ be analytic. The composition operator C_φ is bounded on $\mathcal{D}_\alpha$ if and only if*

$$\sup_{a\in\mathbb{D}} \frac{N_{\varphi,\alpha}(a)}{(1-|a|^2)^\alpha} < \infty. \tag{3.1}$$

Proof. Suppose first that the composition operator C_φ is bounded on $\mathcal{D}_\alpha$, that is, $\|C_\varphi f\|_{\mathcal{D}_\alpha} \le C\|f\|_{\mathcal{D}_\alpha}$ for any $f \in \mathcal{D}_\alpha$. For $a \in \mathbb{D}$, apply the previous inequality to the functions

$$f_a(z) = (1-|a|^2)^{1+\frac{\alpha}{2}} \int_0^z \frac{d\zeta}{(1-\bar{a}\zeta)^{2+\alpha}}, \quad z \in \mathbb{D}.$$

The use of Lemma 3.10 in [25] gives $\sup_{a\in\mathbb{D}} \|f_a\|^2_{\mathcal{D}_\alpha} \le C$. Also, the change of variables formula yields

$$\|C_\varphi f_a\|^2_{\mathcal{D}_\alpha} = |f_a(\varphi(0))|^2 + (1+\alpha)\int_{\mathbb{D}} |(f_a)'(z)|^2 N_{\varphi,\alpha}(z)\, dA(z),$$

and therefore, we have

$$\int_{\mathbb{D}} \frac{(1-|a|^2)^{2+\alpha}}{|1-\bar{a}z|^{4+2\alpha}} N_{\varphi,\alpha}(z)\, dA(z) \le C, \tag{3.2}$$

with the constant C not depending on a. Now, for $a \in \mathbb{D}$ with $|a| > \frac{1}{2}(1+|\varphi(0)|)$, let $D(a) = \big\{z \in \mathbb{D} : |z-a| < \frac{1}{2}(1-|a|)\big\}$. Observe that $\varphi(0) \notin D(a)$. Then, Proposition 2.1, the well known fact that $|1-\bar{a}z| \asymp (1-|a|^2)$ for $z \in D(a)$, and (3.2), finally gives

$$\begin{aligned} N_{\varphi,\alpha}(a) &\le \frac{4}{(1-|a|)^2}\int_{D(a)} N_{\varphi,\alpha}(z)\, dA(z) \le C \int_{D(a)} \frac{(1-|a|^2)^{2+2\alpha}}{|1-\bar{a}z|^{4+2\alpha}}\, N_{\varphi,\alpha}(z)\, dA(z) \\ &\le C\int_{\mathbb{D}} \frac{(1-|a|^2)^{2+2\alpha}}{|1-\bar{a}z|^{4+2\alpha}}\, N_{\varphi,\alpha}(z)\, dA(z) \le C(1-|a|^2)^\alpha, \end{aligned}$$

where C does not depend on the point a. This gives

$$\sup_{|a|>\frac{1+|\varphi(0)|}{2}} \frac{N_{\varphi,\alpha}(a)}{(1-|a|^2)^\alpha} < \infty. \tag{3.3}$$

On the other hand, we know from the proof of Proposition 2.1 that there is a subharmonic function u_α such that $N_{\varphi,\alpha} \leq u_\alpha$. Since any upper semi-continuous function is bounded above on compact sets, we get

$$\sup_{|a|\leq\frac{1+|\varphi(0)|}{2}} \frac{N_{\varphi,\alpha}(a)}{(1-|a|^2)^\alpha} \leq \frac{2^\alpha}{(1-|\varphi(0)|)^\alpha} \sup_{|a|\leq\frac{1+|\varphi(0)|}{2}} u_\alpha(a) < \infty.$$

This, together with (3.3) proves (3.1).

Conversely, assume that (3.1) holds. Then

$$\int_{\mathbb{D}} |f'(z)|^2\, N_{\varphi,\alpha}(z)\, dA(z) \leq C\|f\|^2_{\mathcal{D}_\alpha}.$$

Also, the well known pointwise estimate for functions in $\mathcal{D}_\alpha$ gives

$$|f(\varphi(0))|^2 \leq \frac{C}{(1-|\varphi(0)|^2)^\alpha}\|f\|^2_{\mathcal{D}_\alpha}.$$

Since, by the change of variables formula, we have

$$\|C_\varphi f\|^2_{\mathcal{D}_\alpha} = |f(\varphi(0))|^2 + (1+\alpha)\int_{\mathbb{D}} |f'(z)|^2\, N_{\varphi,\alpha}(z)\, dA(z),$$

then, by the previous inequalities, the composition operator C_φ is bounded on $\mathcal{D}_\alpha$. □

It is now a classical result of J.H. Shapiro (see [22]) that the essential norm of the composition operator C_φ acting on the Hardy space H^2 equals

$$\limsup_{|z|\to1^-} \sqrt{\frac{N_\varphi(z)}{\log\frac{1}{|z|}}}.$$

In particular, C_φ is compact on H^2 if and only if $\lim_{|z|\to1^-} \frac{N_\varphi(z)}{\log\frac{1}{|z|}} = 0$ (see [22] or [6, Theorem 3.20]). Concerning the essential norm of C_φ acting on $\mathcal{D}_\alpha$ we have the following result (see also [24]).

Theorem 3.2. *Let $0 < \alpha < 1$, and let $\varphi : \mathbb{D} \to \mathbb{D}$ analytic inducing a bounded composition operator on $\mathcal{D}_\alpha$. Then, the following estimate for the essential norm of the composition operator C_φ acting on $\mathcal{D}_\alpha$ holds:*

$$\|C_\varphi\|^2_e \asymp \limsup_{|a|\to1^-} \frac{N_{\varphi,\alpha}(a)}{(1-|a|^2)^\alpha}.$$

Proof. The upper estimate

$$\|C_\varphi\|_e^2 \le C_1 \limsup_{|a|\to 1^-} \frac{N_{\varphi,\alpha}(a)}{(1-|a|^2)^\alpha},$$

for some positive constant C_1 can be found in p.136 of Cowen-MacCluer's book [6]. So let's proceed to show that the lower estimate

$$\|C_\varphi\|_e^2 \ge C_2 \limsup_{|a|\to 1^-} \frac{N_{\varphi,\alpha}(a)}{(1-|a|^2)^\alpha} \tag{3.4}$$

holds for some positive constant C_2. To this end, for each $a \in \mathbb{D}$ consider the functions

$$f_a(z) = \frac{(1-|a|^2)^{1-\frac{\alpha}{2}}}{1-\bar{a}z}, \quad z \in \mathbb{D}.$$

Using Lemma 3.10 in [25] it is easy to see that $\sup_{a\in\mathbb{D}} \|f_a\|_{\mathcal{D}_\alpha} \le C$. It is also clear that the functions f_a converges to zero as $|a| \to 1^-$ uniformly on compact subsets of $\mathbb{D}$. Therefore, $\lim_{|a|\to 1^-} \|K(f_a)\|_{\mathcal{D}_\alpha} = 0$ for every compact operator K on $\mathcal{D}_\alpha$. This implies that

$$\begin{aligned} \|C_\varphi - K\| &\ge C \limsup_{|a|\to 1^-} \|C_\varphi f_a - K(f_a)\|_{\mathcal{D}_\alpha} \\ &\ge C \limsup_{|a|\to 1^-} \|C_\varphi f_a\|_{\mathcal{D}_\alpha}. \end{aligned}$$

Since this holds for any compact operator K, and

$$\|C_\varphi\|_e = \inf\{\|C_\varphi - K\| : K \text{ is compact}\},$$

we have

$$\|C_\varphi\|_e \ge C \limsup_{|a|\to 1^-} \|C_\varphi f_a\|_{\mathcal{D}_\alpha}. \tag{3.5}$$

Now, by the change of variables formula,

$$\begin{aligned} \|C_\varphi f_a\|_{\mathcal{D}_\alpha}^2 &= |f_a(\varphi(0))|^2 + (1+\alpha)\int_{\mathbb{D}} |(f_a)'(z)|^2\, N_{\varphi,\alpha}(z)\, dA(z) \\ &= \frac{(1-|a|^2)^{2-\alpha}}{|1-\bar{a}\varphi(0)|^2} + (1+\alpha)|a|^2(1-|a|^2)^{2-\alpha}\int_{\mathbb{D}} \frac{N_{\varphi,\alpha}(z)}{|1-\bar{a}z|^4}\, dA(z). \end{aligned}$$

It is clear that the first term tends to zero as $|a| \to 1^-$. Thus

$$\limsup_{|a|\to 1^-} \|C_\varphi f_a\|_{\mathcal{D}_\alpha}^2 \ge (1+\alpha) \limsup_{|a|\to 1^-} |a|^2(1-|a|^2)^{2-\alpha}\int_{D(a)} \frac{N_{\varphi,\alpha}(z)}{|1-\bar{a}z|^4}\, dA(z),$$

where, as before,

$$D(a) = \Big\{z \in \mathbb{D} : |z-a| < \frac{1}{2}(1-|a|)\Big\}.$$

Observe that $\varphi(0) \notin D(a)$ if $|a|$ is close enough to 1, and so, after using that $|1-\bar{a}z| \asymp (1-|a|)$ for $z \in D(a)$, we can apply Proposition 2.1 in order to obtain

$$\limsup_{|a|\to 1^-} \|C_\varphi f_a\|_{\mathcal{D}_\alpha}^2 \ge C \limsup_{|a|\to 1^-} \frac{N_{\varphi,\alpha}(a)}{(1-|a|^2)^\alpha}.$$

This together with (3.5) proves (3.4) finishing the proof of the theorem. □

Since $\|C_\varphi\|_e = 0$ if and only if C_φ is compact, as an immediate consequence of the previous theorem, we obtain the following description of compact composition operators on Dirichlet type spaces, result that has been obtained recently by Kellay and Lefevre in [10].

Corollary 3.3. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ analytic and $0 < \alpha < 1$. Then C_φ is compact on $\mathcal{D}_\alpha$ if and only if*

$$\frac{N_{\varphi,\alpha}(a)}{(1-|a|)^\alpha} \to 0 \qquad \textit{as} \quad |a| \to 1^-.$$

Corollary 3.4. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ univalent and $0 < \alpha < 1$. Then C_φ is compact on $\mathcal{D}_\alpha$ if and only if*

$$\lim_{|a|\to 1^-} \frac{1-|\varphi(a)|^2}{1-|a|^2} = \infty. \tag{3.6}$$

Proof. Since φ is univalent, then $N_{\varphi,\alpha}(a) = (1-|\varphi^{-1}(a)|^2)^\alpha$, and the result is an immediate consequence of the previous Corollary. □

We remark here that, by the Julia-Caratheodory Theorem, the condition (3.6) is equivalent to the fact that φ does not have finite angular derivatives at any point of the circle.

4. Schatten class composition operators

Schatten classes. If H and $\mathcal{K}$ are separable Hilbert spaces, a compact operator T from H to $\mathcal{K}$ is said to belong to the Schatten class $S_p = S_p(H, \mathcal{K})$ if its sequence of singular numbers is in the sequence space ℓ^p. Recall that the singular numbers of a compact operator T are the square root of the eigenvalues of the positive operator T^*T, where T^* denotes the adjoint of T. Also, the compact operator T admits a decomposition of the form

$$T = \sum_n \lambda_n \langle \cdot, e_n \rangle_H f_n,$$

where $\{\lambda_n\}$ are the singular numbers of T, $\{e_n\}$ is an orthonormal set in H, and $\{f_n\}$ is an orthonormal set in $\mathcal{K}$. Moreover, if the singular values $\{\lambda_n\}$ are ordered in a decreasing order, then

$$\lambda_n = \lambda_n(T) = \inf\{\|T-K\| : \text{ rank}\, K \le n\}.$$

For $p \ge 1$, the class S_p is a Banach space with the norm

$$\|T\|_p = \left(\sum_n |\lambda_n|^p \right)^{1/p},$$

while for $0 < p < 1$ one has the inequality

$$\|S+T\|_p^p \le \|S\|_p^p + \|T\|_p^p.$$

We refer to [25, Chapter 1] for a brief account on the theory of Schatten p-classes.

Schatten class Toeplitz operators on Bergman spaces. Let $\alpha > -1$ and let ψ be a positive function in $L^1(\mathbb{D}, dA_\alpha)$. The Toeplitz operator T_ψ with symbol ψ acting on the Bergman space A^2_α is

$$T_\psi f(z) = \int_{\mathbb{D}} \frac{\psi(w)\, f(w)}{(1-\bar{w}z)^{2+\alpha}}\, dA_\alpha(w), \qquad f \in A^2_\alpha.$$

Note that the operator T_ψ is well defined on H^∞, the algebra of bounded analytic functions on $\mathbb{D}$, that is dense on A^2_α, and therefore the Toeplitz operator T_ψ is densely defined on A^2_α. A description of those positive symbols ψ for which T_ψ is bounded or compact on A^2_α can be found, for example, in Chapter 7 of K. Zhu's book [25], where one can also found the following description, essentially due to D. Luecking [17], of Toeplitz operators with positive symbols belonging to the Schatten class S_p.

Theorem A. *Let $\psi \in L^1(\mathbb{D}, dA_\alpha)$ be a nonnegative function on $\mathbb{D}$, and let $\alpha > -1$ and $0 < p < \infty$. Then $T_\psi \in S_p(A^2_\alpha)$ if and only if the function*

$$\widehat{\psi}_r(z) = \frac{1}{(1-|z|^2)^{2+\alpha}} \int_{\Delta(z,r)} \psi(w)\, dA_\alpha(w), \quad 0 < r < 1,$$

is in $L^p(\mathbb{D}, d\lambda)$.

Here, $d\lambda(z) = (1-|z|^2)^{-2} dA(z)$ is the hyperbolic measure in $\mathbb{D}$, and $\Delta(z,r) = \{w : |\sigma_z(w)| < r\}$ is the pseudo-hyperbolic disk with center z and radius r. We are going to use this result for our study of the Schatten class composition operators on Dirichlet type spaces.

Composition operators belonging to the Schatten ideals. A characterization of the membership in the Schatten class S_p for $0 < p < \infty$ of the composition operator in the Hardy space H^2 was obtained by D. Luecking and K. Zhu [19]. His result reads as follows: let $0 < p < \infty$ and let $\varphi : \mathbb{D} \to \mathbb{D}$ analytic. Then $C_\varphi \in S_p(H^2)$ if and only if

$$\frac{N_\varphi(z)}{\log \frac{1}{|z|}} \in L^{p/2}(\mathbb{D}, d\lambda).$$

Next, we are going to extend this result to all Dirichlet type spaces $\mathcal{D}_\alpha$ with $0 < \alpha < 1$.

Theorem 4.1. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ analytic, $0 < \alpha < 1$ and $0 < p < \infty$. Then $C_\varphi \in S_p(\mathcal{D}_\alpha)$ if and only if*

$$\frac{N_{\varphi,\alpha}(z)}{(1-|z|^2)^\alpha} \in L^{p/2}(\mathbb{D}, d\lambda).$$

In order to prove this result we use a well known relationship of composition operators and Toeplitz operators in order to deduce the result from Theorem A and Proposition 2.1. Before going to the proof, we need some preliminaries in order to make clear the connection mentioned above.

Let $\dot{\mathcal{D}}_\alpha$ be the closed subspace of $\mathcal{D}_\alpha$ consisting of functions f with $f(0) = 0$. If $\varphi : \mathbb{D} \to \mathbb{D}$ is analytic and $\varphi(0) = 0$, then the composition operator C_φ leaves $\dot{\mathcal{D}}_\alpha$ invariant, and we can think of C_φ as an operator on $\dot{\mathcal{D}}_\alpha$. In the case that $\varphi(0) = 0$, it turns out that it is more convenient to work with $C_\varphi : \dot{\mathcal{D}}_\alpha \to \dot{\mathcal{D}}_\alpha$ instead of $C_\varphi : \mathcal{D}_\alpha \to \mathcal{D}_\alpha$. Also, when discussing the membership in

Schatten classes of the composition operator C_φ, without loss of any generality, we can always assume $\varphi(0) = 0$ and consider the operator acting on $\dot{\mathcal{D}}_\alpha$. Indeed, if $\varphi : \mathbb{D} \to \mathbb{D}$ is analytic with $a = \varphi(0)$, then the function $\psi = \sigma_a \circ \varphi$ satisfies $\psi(0) = 0$, and $\varphi = \sigma_a \circ \psi$, so that $C_\varphi = C_\psi C_{\sigma_a}$. Since C_{σ_a} is invertible (the inverse being itself), it follows that C_φ is in the Schatten class $S_p(\mathcal{D}_\alpha)$ if and only if C_ψ is in the same Schatten class $S_p(\mathcal{D}_\alpha)$. Since $\dot{\mathcal{D}}_\alpha$ has codimension 1 in $\mathcal{D}_\alpha$, the membership in S_p of $C_\psi : \mathcal{D}_\alpha \to \mathcal{D}_\alpha$ is the same as that of $C_\psi : \dot{\mathcal{D}}_\alpha \to \dot{\mathcal{D}}_\alpha$.

Now, consider the linear mapping $U_\alpha : \dot{\mathcal{D}}_\alpha \to A^2_\alpha$ defined by $U_\alpha f(z) = f'(z)$. It is obvious that U_α is a unitary operator from $\dot{\mathcal{D}}_\alpha$ onto A^2_α. We now consider the action of U_α on a composition operator. If $\varphi : \mathbb{D} \to \mathbb{D}$ is analytic with $\varphi(0) = 0$, and C_φ is the composition operator acting on $\dot{\mathcal{D}}_\alpha$, then it follows easily from elementary calculations that $U_\alpha C_\varphi U^*_\alpha = D_\varphi$ on A^2_α, where D_φ is the weighted composition operator on A^2_α given by

$$D_\varphi f(z) = f\big(\varphi(z)\big)\varphi'(z).$$

Therefore, the study of the membership in S_p of $C_\varphi : \dot{\mathcal{D}}_\alpha \to \dot{\mathcal{D}}_\alpha$ is the same as that of $D_\varphi : A^2_\alpha \to A^2_\alpha$. The following result exhibits the relationship between D_φ and a certain Toeplitz operator acting on the Bergman space A^2_α.

Lemma 4.2. *Assume that $\varphi : \mathbb{D} \to \mathbb{D}$ is analytic with $\varphi(0) = 0$, and let*

$$\psi(z) = \frac{N_{\varphi,\alpha}(z)}{(1 - |z|^2)^\alpha}, \qquad z \in \mathbb{D}. \tag{4.1}$$

Then $D^*_\varphi D_\varphi = T_\psi$ on A^2_α.

Proof. Let $\langle\, ,\, \rangle_\alpha$ be the inner product in A^2_α. Using the properties of the reproducing kernel of A^2_α it is easy to see that, for any f and g in A^2_α, we have

$$\langle T_\psi f, g\rangle_\alpha = (1+\alpha) \int_{\mathbb{D}} f(w)\, \overline{g(w)}\, N_{\varphi,\alpha}(w)\, dA(w).$$

On the other hand, by the change of variables formula,

$$\begin{aligned} \langle D^*_\varphi D_\varphi f, g\rangle_\alpha &= \langle D_\varphi f, D_\varphi g\rangle_\alpha \\ &= \int_{\mathbb{D}} f\big(\varphi(z)\big)\, \overline{g\big(\varphi(z)\big)}\, |\varphi'(z)|^2\, dA_\alpha(z) \\ &= (1+\alpha) \int_{\mathbb{D}} f(w)\, \overline{g(w)}\, N_{\varphi,\alpha}(w)\, dA(w). \end{aligned}$$

It follows that $\langle T_\psi f, g\rangle_\alpha = \langle D^*_\varphi D_\varphi f, g\rangle_\alpha$ for all f and g in A^2_α and hence $D^*_\varphi D_\varphi = T_\psi$ on A^2_α. □

Proof of Theorem 4.1. By the previous observations we may assume that $\varphi(0) = 0$. Also, by the remarks made and Lemma 4.2, we have $C_\varphi \in S_p(\mathcal{D}_\alpha)$ if and only if $D_\varphi \in S_p(A^2_\alpha)$ if and only if the Toeplitz operator $T_\psi \in S_{p/2}(A^2_\alpha)$, where ψ is the function defined in (4.1). By Theorem A, $T_\psi \in S_{p/2}(A^2_\alpha)$ if and only if the function $\widehat{\psi}_r$ is in $L^{p/2}(\mathbb{D}, d\lambda)$. Now, it is clear that the pseudo-hyperbolic disk $\Delta(z, r)$ contains an euclidian disk centered at z of radius $\eta(1 - |z|)$ with

η depending only on r. Therefore, since $(1-|w|) \asymp (1-|z|)$ for $w \in \Delta(z,r)$, using Proposition 2.1 we see that, for $0 < q < \infty$, there is a positive constant C such that

$$\psi(z)^q \leq \frac{C}{(1-|z|^2)^{2+\alpha}} \int_{\Delta(z,r)} \psi(w)^q \, dA_\alpha(w), \quad |z| > r. \tag{4.2}$$

In particular, the case $q = 1$ gives $\psi(z) \leq C\widehat{\psi}_r(z)$. Hence $\psi \in L^{p/2}(\mathbb{D}, d\lambda)$ if $\widehat{\psi}_r$ is in $L^{p/2}(\mathbb{D}, d\lambda)$ proving the necessity part of the Theorem.

Conversely, if $\psi \in L^{p/2}(\mathbb{D}, d\lambda)$ we will show that $\widehat{\psi}_r$ is in $L^{p/2}(\mathbb{D}, d\lambda)$. From the case $q = p/2$ of (4.2) and the well known estimates $(1-|w|^2) \asymp (1-|z|^2) \asymp |1-\bar{w}z|$ for $w \in \Delta(z,r)$, we deduce

$$\begin{aligned}
\widehat{\psi}_r(z)^{p/2} &\leq C \sup \big\{ \psi(w)^p : w \in \Delta(z,r) \big\} \\
&\leq C \sup_{w \in \Delta(z,r)} \frac{1}{(1-|w|^2)^{2+\alpha}} \int_{\Delta(w,r)} \psi(u)^{p/2} \, dA_\alpha(u) \\
&\leq C(1-|z|^2)^{2+\alpha} \int_{\mathbb{D}} \frac{\psi(u)^{p/2}}{|1-\bar{u}z|^{4+2\alpha}} \, dA_\alpha(u).
\end{aligned}$$

Therefore, by Fubini's theorem and Lemma 3.10 of [25],

$$\begin{aligned}
\int_{\mathbb{D}} \widehat{\psi}_r(z)^{p/2} d\lambda(z) &\leq C \int_{\mathbb{D}} \psi(u)^p \left(\int_{\mathbb{D}} \frac{(1-|z|^2)^\alpha}{|1-\bar{u}z|^{4+2\alpha}} dA(z) \right) dA_\alpha(u) \\
&\leq C \int_{\mathbb{D}} \psi(u)^{p/2} d\lambda(u).
\end{aligned}$$

This completes the proof of the theorem. □

Some consequences. It is known that the results on compactness and membership in Schatten classes for composition operators on Hardy spaces can be reformulated in terms of the function

$$\tau_\varphi^*(z) = \frac{N_\varphi^*(z)}{1-|z|^2},$$

where

$$N_\varphi^*(z) := \sum_{\varphi(w)=z} (1-|w|^2).$$

Then Shapiro's compactness theorem can be restated by saying that C_φ is compact in H^2 if and only if $\lim_{|z|\to 1} \tau_\varphi^*(z) = 0$, and the necessary and sufficient condition for the membership in C_φ in S_p in Luecking-Zhu's theorem is then $\tau_\varphi^* \in L^{p/2}(\mathbb{D}, d\lambda)$. Also, since τ_φ^* is a bounded function, one has

$$\tau_\varphi^*(z) \leq C\tau_\varphi^*(z)^\alpha \leq C\frac{N_{\varphi,\alpha}(z)}{(1-|z|^2)^\alpha}, \qquad 0 < \alpha < 1.$$

Taking all of this into account and bearing in mind Theorems 3.1 and 4.1 it follows that, for $0 < \alpha < 1$, the compactness of the composition operator C_φ in $\mathcal{D}_\alpha$ implies the compactness of C_φ in H^2, and the membership of C_φ in the Schatten class S_p of $\mathcal{D}_\alpha$ implies that C_φ belongs to $S_p(H^2)$. Also, when φ is univalent (or even with bounded valence) C_φ is compact on the Hardy space H^2 if and only if C_φ is compact on $\mathcal{D}_\alpha$, $0 < \alpha < 1$. Therefore, several existing examples of composition operators on the Hardy space may be used to produce relevant examples of composition operators on $\mathcal{D}_\alpha$. In particular, Lotto's simpler example [15] of a compact composition operator on H^2 not being Hilbert-Schmidt in H^2 is actually an example of a compact composition operator on $\mathcal{D}_\alpha$ that is not Hilbert-Schmidt in $\mathcal{D}_\alpha$. Also, Akeroyd's example [2] provides a compact composition operator on $\mathcal{D}_\alpha$ that does not belong to any Schatten ideal S_p of $\mathcal{D}_\alpha$.

5. Composition operators between different Dirichlet type spaces

One can also consider the composition operator C_φ acting from one Dirichlet type space $\mathcal{D}_\alpha$ to another Dirichlet type space $\mathcal{D}_\beta$ with $\alpha \neq \beta$. The proof of the result stated below is the same as the case $\alpha = \beta$ considered above, and therefore it is omitted.

Theorem 5.1. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ analytic. Let $\alpha \geq 0$ and $\beta > 0$. Then*

(a) *$C_\varphi : \mathcal{D}_\alpha \to \mathcal{D}_\beta$ is bounded if and only if* $\sup_{z\in\mathbb{D}} \frac{N_{\varphi,\beta}(z)}{(1-|z|^2)^\alpha} < \infty$.

(b) *The essential norm of $C_\varphi : \mathcal{D}_\alpha \to \mathcal{D}_\beta$ is comparable to the quantity*

$$\limsup_{|z|\to 1^-} \frac{N_{\varphi,\beta}(z)}{(1-|z|^2)^\alpha}.$$

(c) *$C_\varphi : \mathcal{D}_\alpha \to \mathcal{D}_\beta$ is compact if and only if* $\lim_{|z|\to 1^-} \frac{N_{\varphi,\beta}(z)}{(1-|z|^2)^\alpha} = 0$.

(d) *Let $0 < p < \infty$. Then $C_\varphi \in S_p(\mathcal{D}_\alpha, \mathcal{D}_\beta)$ if and only if*

$$\frac{N_{\varphi,\beta}(z)}{(1-|z|^2)^\alpha} \in L^{p/2}(\mathbb{D}, d\lambda).$$

We want to remark here that in the previous result the case $\alpha = 0$ is included, that is, when the starting space is the classical Dirichlet space.

6. Composition operators with closed range

In this section we are going to describe those bounded composition operators acting on $\mathcal{D}_\alpha$ with closed range. The corresponding result for the Hardy space H^2 was obtained by N. Zorboska [26] (see [5] for a different description): a composition operator C_φ has closed range in H^2 if and only if there are positive constants ε, δ and $0 < r < 1$ so that if $G_\varepsilon = \{w : \tau_\varphi(w) > \varepsilon\}$, then

$$Area(G_\varepsilon \cap \Delta(z,r)) \geq \delta(1-|z|^2)^2$$

for all pseudo-hyperbolic disks $\Delta(z,r)$ with $z \in \mathbb{D}$. Here $\tau_\varphi(z) = N_\varphi(z)/(1-|z|)$. A similar result for Bergman spaces was also obtained there [26] (see also [3] and [4] for some variants). A key ingredient for the proof of that result (that we also will use in order to characterize the closed range composition operators on $\mathcal{D}_\alpha$) is the following result due to D. Luecking [16].

Theorem B. *Let G be a measurable subset of $\mathbb{D}$. The following conditions are equivalent:*

(a) *There is a positive constant C such that for every g in A^2_α, $\alpha > -1$,*

$$\int_{\mathbb{D}} |g(z)|^2 \, dA_\alpha(z) \leq C \int_G |g(z)|^2 \, dA_\alpha(z).$$

(b) *There exists* $\delta > 0$ *and* $0 < r < 1$ *such that*

$$Area(G \cap \Delta(z,r)) \geq \delta(1-|z|^2)^2$$

for every $z \in \mathbb{D}$.

We need a modified version of this theorem whose proof can be deduced from Theorem B in the same way as in Corollary 3.34 of [6], so that the proof will be omitted.

Proposition 6.1. *Let* $\tau(z)$ *be a bounded, measurable non-negative function on* $\mathbb{D}$. *The following conditions are equivalent:*

(a) *There exists* $M > 0$ *such that for every* $f \in \mathcal{D}_\alpha$, $\alpha \geq 0$,

$$\int_{\mathbb{D}} |f'(z)|^2 \, dA_\alpha(z) \leq M \int_{\mathbb{D}} |f'(z)|^2 \, \tau(z) \, dA_\alpha(z).$$

(b) *There exists* $\varepsilon > 0$, $\delta > 0$ *and* $0 < r < 1$ *such that*

$$Area(G_\varepsilon \cap \Delta(z,r)) \geq \delta(1-|z|^2)^2$$

for every $z \in \mathbb{D}$, *where* $G_\varepsilon = \{w \in \mathbb{D} : \tau(w) > \varepsilon\}$.

We also need the following well known result (see [6, Proposition 3.30] for example) saying that the closed range composition operators are precisely those operators that are bounded below.

Lemma C. *A bounded composition operator* C_φ *acting on a Hilbert space* H *of analytic functions in the disk has closed range if and only if there exists* $m > 0$ *such that*

$$\|C_\varphi f\|_H \geq m\|f\|_H \qquad \text{for all } f \in H.$$

For $\varphi : \mathbb{D} \to \mathbb{D}$ analytic and $\alpha > 0$, consider the function

$$\tau_{\varphi,\alpha}(z) = \frac{N_{\varphi,\alpha}(z)}{(1-|z|^2)^\alpha}.$$

Note that this is the same function that appears in (4.1). Now we are prepared to state and prove our description of the closed range composition operators acting on $\mathcal{D}_\alpha$.

Theorem 6.2. *Let* $0 < \alpha < 1$ *and let* $\varphi : \mathbb{D} \to \mathbb{D}$ *analytic inducing a bounded composition operator* C_φ *on* $\mathcal{D}_\alpha$. *Then the composition operator* C_φ *has closed range on* $\mathcal{D}_\alpha$ *if and only if there are positive constants* ε, δ *and* $0 < r < 1$ *so that if* $G_\varepsilon^\alpha = \{w : \tau_{\varphi,\alpha}(w) > \varepsilon\}$, *then*

$$Area(G_\varepsilon^\alpha \cap \Delta(z,r)) \geq \delta(1-|z|^2)^2$$

for all pseudo-hyperbolic disks $\Delta(z,r)$ *with* $z \in \mathbb{D}$.

Proof. Since, by Theorem 3.1, $\tau_{\varphi,\alpha}(z)$ is a bounded function, the result follows from a direct application of Proposition 6.1 and the following claim: a bounded composition operator C_φ has closed range in $\mathcal{D}_\alpha$ if and only if there exists a constant $M > 0$ so that

$$\int_{\mathbb{D}} |f'(z)|^2 \, \tau_{\varphi,\alpha}(z) \, dA_\alpha(z) \geq M \int_{\mathbb{D}} |f'(z)|^2 \, dA_\alpha(z) \tag{6.1}$$

whenever f is in $\mathcal{D}_\alpha$. In order to prove that claim, we first assume that $\varphi(0) = 0$. In this case, we can think as C_φ acting on $\dot{\mathcal{D}}_\alpha$, the closed subspace of $\mathcal{D}_\alpha$ consisting of functions f with $f(0) = 0$.

By Lemma C the composition operator C_φ has closed range if and only if there exists a constant $m > 0$ so that

$$\|f\|^2_{\mathcal{D}_\alpha} \le m\|C_\varphi f\|^2_{\mathcal{D}_\alpha}$$

for all $f \in \dot{\mathcal{D}}_\alpha$. By the change of variables formula, we have

$$\|C_\varphi f\|^2_{\mathcal{D}_\alpha} = \int_{\mathbb{D}} |f'(z)|^2 \tau_{\varphi,\alpha}(z)\, dA_\alpha(z),$$

that proves the claim in that case.

Next, assume that $\varphi(0) = a \neq 0$. Consider $\psi = \sigma_a \circ \varphi$, so that $\psi(0) = 0$. Since $C_\varphi = C_\psi C_{\sigma_a}$ and C_{σ_a} is invertible on $\mathcal{D}_\alpha$, then C_φ has closed range if and only if C_ψ does. By the case already considered, C_ψ has closed range if and only if there exists $m > 0$ so that

$$\int_{\mathbb{D}} |f'(z)|^2\, \tau_{\psi,\alpha}(z)\, dA_\alpha(z) \ge m \int_{\mathbb{D}} |f'(z)|^2\, dA_\alpha(z). \tag{6.2}$$

If (6.1) holds, by the change of variables formula,

$$\begin{aligned}\int_{\mathbb{D}} |f'(z)|^2\, \tau_{\psi,\alpha}(z)\, dA_\alpha(z) &= \int_{\mathbb{D}} |(f\circ\psi)'(z)|^2\, dA_\alpha(z)\\ &= \int_{\mathbb{D}} |(f\circ\sigma_a)'(z)|^2\, \tau_{\varphi,\alpha}(z)\, dA_\alpha(z)\\ &\ge M \int_{\mathbb{D}} |(f\circ\sigma_a)'(z)|^2\, dA_\alpha(z)\\ &= (1+\alpha)M \int_{\mathbb{D}} |f'(z)|^2\, (1-|\sigma_a(z)|^2)^\alpha\, dA(z)\\ &\ge M2^{-2\alpha}\,(1-|a|^2)^\alpha \int_{\mathbb{D}} |f'(z)|^2\, dA_\alpha(z).\end{aligned}$$

Hence (6.2) holds with $m = M2^{-2\alpha}\,(1-|\varphi(0)|^2)^\alpha$, and therefore C_φ has closed range. Conversely, if C_φ has closed range in $\mathcal{D}_\alpha$, then (6.2) holds, and since $\varphi = \sigma_a \circ \psi$ we can use the same argument to show that (6.1) is satisfied. This completes the proof. $\square$

As an immediate consequence, we obtain the following alternate description of closed range composition operators in $\mathcal{D}_\alpha$ similar to the one obtained recently in [13, Theorem 5.1] for the Hardy space.

Corollary 6.3. *Let $0 < \alpha < 1$ and let $\varphi : \mathbb{D} \to \mathbb{D}$ analytic inducing a bounded composition operator C_φ on $\mathcal{D}_\alpha$. Then C_φ has closed range in $\mathcal{D}_\alpha$ if and only if there is $0 < r < 1$ and a positive constant C such that*

$$\frac{1}{Area(\Delta(z,r))} \int_{\Delta(z,r)} \tau_{\varphi,\alpha}(w)\, dA(w) \ge C \tag{6.3}$$

for any pseudo-hyperbolic disk $\Delta(z,r)$, $z \in \mathbb{D}$.

Proof. If C_φ has closed range in $\mathcal{D}_\alpha$, by Theorem 6.2, there are constants $\varepsilon > 0$ and $\delta > 0$ such that

$$Area(G_\varepsilon^\alpha \cap \Delta(z,r)) \geq \delta \, Area(\Delta(z,r))$$

for any pseudo-hyperbolic disk $\Delta(z,r)$. Therefore,

$$\begin{aligned} \frac{1}{Area(\Delta(z,r))} \int_{\Delta(z,r)} \tau_{\varphi,\alpha}(w)\, dA(w) &\geq \frac{1}{Area(\Delta(z,r))} \int_{G_\varepsilon^\alpha \cap \Delta(z,r)} \tau_{\varphi,\alpha}(w)\, dA(w) \\ &\geq \varepsilon \frac{Area(G_\varepsilon^\alpha \cap \Delta(z,r))}{Area(\Delta(z,r))} \geq \varepsilon\delta. \end{aligned}$$

Conversely, assume that (6.3) holds. Choose $0 < \varepsilon < \frac{C}{2}$. Then

$$\begin{aligned} C &\leq \frac{1}{Area(\Delta(z,r))} \int_{G_\varepsilon^\alpha \cap \Delta(z,r)} \tau_{\varphi,\alpha}(w)\, dA(w) + \frac{1}{Area(\Delta(z,r))} \int_{\Delta(z,r) \setminus G_\varepsilon^\alpha} \tau_{\varphi,\alpha}(w)\, dA(w) \\ &\leq \frac{1}{Area(\Delta(z,r))} \int_{G_\varepsilon^\alpha \cap \Delta(z,r)} \tau_{\varphi,\alpha}(w)\, dA(w) + \varepsilon, \end{aligned}$$

so that

$$\frac{C}{2} \leq \frac{1}{Area(\Delta(z,r))} \int_{G_\varepsilon^\alpha \cap \Delta(z,r)} \tau_{\varphi,\alpha}(w)\, dA(w).$$

Since, by Theorem 3.1, $M := \sup_{w \in \mathbb{D}} \tau_{\varphi,\alpha}(w) < \infty$, we deduce that

$$Area(G_\varepsilon^\alpha \cap \Delta(z,r)) \geq \delta \, Area(\Delta(z,r))$$

with $\delta = C/(2M)$. Thus, by Theorem 6.2, the composition operator C_φ has closed range in $\mathcal{D}_\alpha$. □

Corollary 6.4. *Let $0 < \alpha < \beta$ and $\varphi : \mathbb{D} \to \mathbb{D}$ analytic. Assume that C_φ is bounded on $\mathcal{D}_\alpha$. If C_φ has closed range on $\mathcal{D}_\beta$, then C_φ has closed range on $\mathcal{D}_\alpha$.*

Proof. Since, for $\beta > \alpha$, one has $\tau_{\varphi,\alpha}(z) \geq C\tau_{\varphi,\beta}(z)$ for some positive constant C and, clearly, if C_φ is bounded on $\mathcal{D}_\alpha$ then C_φ is bounded on $\mathcal{D}_\beta$ (see Theorem 3.1), the result is an immediate consequence of Corollary 6.3. □

Remark: It is standard to see that the results of Theorem 6.2 and Corollary 6.3 can be reformulated in terms of Carleson sectors. Recall that, for a a given arc I of the unit circle with normalized arclength $|I|$, the Carleson sector $S(I)$ based on I is

$$S(I) = \{z = re^{it} \in \mathbb{D} : 1 - r < |I|;\, e^{it} \in I\}.$$

Then, in the condition in Theorem 6.2 one can replace the pseudo-hyperbolic disks by Carleson sectors, so that C_φ has closed range in $\mathcal{D}_\alpha$ if and only if

$$Area(G_\varepsilon^\alpha \cap S(I)) \geq \delta |I|^2$$

for all arcs I. Also, the condition (6.3) in Corollary 6.3 can be replaced by the condition

$$\frac{1}{|I|^2} \int_{S(I)} \tau_{\varphi,\alpha}(w)\, dA(w) \geq C$$

for all arcs I in the unit circle.

Some examples. Some consequences of Theorem 6.2 are listed below. All the assertions can be proved in a similar manner as in [26], so that the proofs will be omitted here. Let $\alpha > 0$ and $\varphi : \mathbb{D} \to \mathbb{D}$ analytic. Then:

• If the range of φ on $\mathbb{D}$ misses a neighborhood of a point on the unit circle then C_φ does not have a closed range on $\mathcal{D}_\alpha$.

• If the range of φ on $\mathbb{D}$ has a hole that includes a disc internally tangent to the unit disc, then C_φ does not have a closed range on $\mathcal{D}_\alpha$.

• Assume that φ is univalent, and let $\psi : \mathbb{D} \to \mathbb{D}$ analytic with $\psi(\mathbb{D}) \subset \varphi(\mathbb{D})$. If C_φ does not have a closed range in $\mathcal{D}_\alpha$, then C_ψ does not have a closed range in $\mathcal{D}_\alpha$.

• If $\varphi(\mathbb{D}) \subset \mathbb{D} \setminus [0, 1)$, then C_φ does not have a closed range on $\mathcal{D}_\alpha$.

Closed range composition operators on the Dirichlet space. The counting function $n_\varphi(z)$ does not necessarily satisfies the submean value property, and the analogues of Theorem 6.2 and Corollary 6.3 for the Dirichlet space (replacing the function $\tau_{\varphi,\alpha}$ by the counting function n_φ) can fail. As far as we know, it is still an open problem to describe the composition operators with closed range in the Dirichlet space $\mathcal{D}$. We refer to [9] and [18] for more information on closed range composition operators on the Dirichlet space.

7. Pullback measures vs counting functions

It is well known that the results on compactness or membership in Schatten classes of composition operators on Hardy spaces can be restated in terms of pullback measures. More concretely, given $\varphi : \mathbb{D} \to \mathbb{D}$ analytic, consider the measure m_φ defined on subsets E of the closed unit disk by

$$m_\varphi(E) = \sigma\Big((\varphi^*)^{-1}(E)\Big)$$

where σ denotes the arclength measure on the unit circle and φ^* is the radial limit function of φ. Then, see [20], the composition operator C_φ is compact on H^2 if and only if m_φ is a vanishing Carleson measure, that is, if

$$\lim_{|I|\to 0} \frac{m_\varphi(S(I))}{|I|} = 0,$$

or, equivalently, if

$$\lim_{|z|\to 1^-} \frac{m_\varphi(S(I_z))}{1 - |z|} = 0,$$

where, for $z = re^{it} \in \mathbb{D}$, I_z denotes the arc centered at e^{it} with $|I_z| = 1 - |z|$. In view of Shapiro's compactness theorem [22], it is natural to expect some relationship between the quantities $N_\varphi(z)$ and $m_\varphi(S(I_z))$. More evidences on that is the fact that the membership in the Schatten classes of the composition operator on H^2 can also be described in terms of the pullback measure m_φ

(see [17] and [11]). Essentially, the result is that C_φ belongs to $S_p(H^2)$ if and only if the function $m_\varphi(S(I_z))/(1-|z|)$ is in $L^{p/2}(\mathbb{D}, d\lambda)$ (compare this with Luecking-Zhu's condition mentioned in Section 4). Recently, in [12], the relation between $N_\varphi(z)$ and $m_\varphi(S(I_z))$ has been quantified in the following form: there exists positive constants C_1, c_1, C_2, c_2 such that, for $z \in \mathbb{D}$ with $|z|$ close enough to 1, one has

$$N_\varphi(z) \le C_1 m_\varphi(S(c_1 I_z)), \tag{7.1}$$

and

$$m_\varphi(S(I_z)) \le \frac{C_2}{(1-|z|)^2} \int_{S(c_2 I_z)} N_\varphi(w)\, dA(w). \tag{7.2}$$

Here, given a positive constant c, we denote by cI the arc with the same center as I and length $c|I|$. Of course, we can replace $N_\varphi(z)$ by $N_\varphi^*(z) := N_{\varphi,1}(z)$ in the previous inequalities.

A similar phenomenon occurs in the setting of the weighted Bergman spaces A^2_α with $\alpha > -1$. The pullback measure that enters in action now is the measure $m_{\varphi,\alpha}$ defined on subsets E of the open unit disk $\mathbb{D}$ by

$$m_{\varphi,\alpha}(E) = \int_{\varphi^{-1}(E)} dA_\alpha(z).$$

The boundedness, compactness and membership in Schatten classes of the composition operator on A^2_α can be described in terms of the properties of the Nevanlinna counting function $N_{\varphi,2+\alpha}$ or in terms of properties of the function $m_{\varphi,\alpha}(S(I_z))$ (see [20] and [22] for boundedness and compactness, and [17], [19] for the membership in Schatten classes). As before, the relationship between $N_{\varphi,2+\alpha}(z)$ and $m_{\varphi,\alpha}(S(I_z))$ can be quantified. The following result has been proved in [14] for $\alpha = 0$, but the same proof works for all $\alpha > -1$. For the sake of completeness we provide the proof here.

Proposition 7.1. *Let $\alpha > -1$ and $\varphi : \mathbb{D} \to \mathbb{D}$ analytic. There exists positive constants C_1, c_1, C_2, c_2 such that, for $z \in \mathbb{D}$ with $|z|$ close enough to 1, one has*

(a) $N_{\varphi,2+\alpha}(z) \le C_1 m_{\varphi,\alpha}(S(c_1 I_z))$,

(b) $m_{\varphi,\alpha}(S(I_z)) \le \frac{C_2}{(1-|z|)^2} \int_{S(c_2 I_z)} N_{\varphi,2+\alpha}(w)\, dA(w)$.

Proof. As in [22, Proposition 6.6] one has

$$N_{\varphi,2+\alpha}(z) = 4(2+\alpha)(1+\alpha) \int_0^1 N_{\varphi,1}(r,z)\,(1-r^2)^\alpha\, r\, dr, \tag{7.3}$$

where

$$N_{\varphi,1}(r,z) := \sum_{\varphi(w)=z, |w|<r} (1-|w|^2/r^2) = N_{\varphi_r,1}(z), \quad 0 < r \le 1,$$

with $\varphi_r(z) = \varphi(rz)$. Then, from (7.3) and (7.1), we obtain

$$\begin{aligned} N_{\varphi,2+\alpha}(z) &= 4(2+\alpha)(1+\alpha) \int_0^1 N_{\varphi_r,1}(z)\,(1-r^2)^\alpha\, r\, dr \\ &\le C_\alpha \int_0^1 m_{\varphi_r}(S(c_1 I_z))\,(1-r^2)^\alpha\, r\, dr \\ &= C_\alpha \int_0^1 \sigma\big\{e^{it} : \varphi(re^{it}) \in S(c_1 I_z)\big\}\,(1-r^2)^\alpha\, r\, dr \\ &= C_\alpha m_{\varphi,\alpha}(S(c_1 I_z)) \end{aligned}$$

which proves (a). Part (b) is proved in a similar manner bearing in mind that

$$m_{\varphi,\alpha}(S(I_z)) = \int_0^1 m_{\varphi_r}(S(I_z))\,(1-r^2)^\alpha\, r\, dr$$

and using (7.2) and (7.3). □

In the setting of Dirichlet type spaces, one considers the pullback measure $\mu_{\varphi,\alpha}$ defined on subsets E of the open unit disk $\mathbb{D}$ by

$$\mu_{\varphi,\alpha}(E) = \int_{\varphi^{-1}(E)} |\varphi'(z)|^2\, dA_\alpha(z).$$

The boundedness and compactness of the composition operator C_φ on $\mathcal{D}_\alpha$ can also be described in terms of the pullback measure $\mu_{\varphi,\alpha}$ [20]. Indeed, a change of variables gives

$$\int_{\mathbb{D}} |(C_\varphi f)'(z)|^2\, dA_\alpha(z) = \int_{\mathbb{D}} |f'(w)|^2\, d\mu_{\varphi,\alpha}(w).$$

It follows that C_φ is bounded on $\mathcal{D}_\alpha$ if and only if $\mu_{\varphi,\alpha}$ is a Carleson measure for A^2_α, that is (see [25, Theorem 7.4] for example), if

$$\sup_{z\in\mathbb{D}} \frac{\mu_{\varphi,\alpha}(D(z))}{(1-|z|^2)^{2+\alpha}} < \infty,$$

where $D(z) = \{w \in \mathbb{D} : |w - z| < \frac{1}{2}(1-|z|)\}$; or equivalently (using Carleson sectors) if

$$\sup_{z\in\mathbb{D}} \frac{\mu_{\varphi,\alpha}(S(I_z))}{(1-|z|^2)^{2+\alpha}} < \infty.$$

Also C_φ is compact on $\mathcal{D}_\alpha$ if and only if $\mu_{\varphi,\alpha}$ is a vanishing Carleson measure for A^2_α, that is (see [25, Theorem 7.7]), if

$$\lim_{|z|\to 1^-} \frac{\mu_{\varphi,\alpha}(D(z))}{(1-|z|^2)^{2+\alpha}} = 0 \Leftrightarrow \lim_{|z|\to 1^-} \frac{\mu_{\varphi,\alpha}(S(I_z))}{(1-|z|^2)^{2+\alpha}} = 0.$$

Also, the membership of the composition operator C_φ in the Schatten class $S_p(\mathcal{D}_\alpha)$ can be described in terms of properties of the function $\mu_{\varphi,\alpha}(D(z))$ (see [17]). Thus, in view of Theorem 3.1, Corollary 3.3 and Theorem 4.1, it is natural to expect some relationship between the quantities $N_{\varphi,\alpha}(z)$ and $\mu_{\varphi,\alpha}(D(z))/(1-|z|)^2$. This is quantified in the next result.

Proposition 7.2. *Let $\alpha > 0$ and $\varphi : \mathbb{D} \to \mathbb{D}$ analytic. There is a constant $C > 1$ such that*

$$\frac{1}{C} N_{\varphi,\alpha}(z) \leq \frac{\mu_{\varphi,\alpha}(D(z))}{(1-|z|)^2} \leq \frac{C}{(1-|z|)^2} \int_{D(z)} N_{\varphi,\alpha}(w)\, dA(w).$$

Also, the same result holds for Carleson sectors, that is,

$$\frac{1}{C} N_{\varphi,\alpha}(z) \leq \frac{\mu_{\varphi,\alpha}(S(2I_z))}{(1-|z|)^2} \leq \frac{C}{(1-|z|)^2} \int_{S(2I_z)} N_{\varphi,\alpha}(w)\, dA(w).$$

Proof. Denote by χ_E the characteristic function of the set E. The definition of the measure $\mu_{\varphi,\alpha}$ together with the change of variables formula yields

$$\begin{aligned}\mu_{\varphi,\alpha}(D(z)) &= \int_{\mathbb{D}} \chi_{\varphi^{-1}(D(z))}(w)\, |\varphi'(w)|^2\, dA_\alpha(w) = \int_{\mathbb{D}} \chi_{D(z)}(\varphi(w))\, |\varphi'(w)|^2\, dA_\alpha(w) \\ &= (1+\alpha) \int_{\mathbb{D}} \chi_{D(z)}(\zeta) N_{\varphi,\alpha}(\zeta)\, dA(\zeta).\end{aligned}$$

From here the upper inequality is trivial, and the lower inequality is a consequence of the sub-mean value property for $N_{\varphi,\alpha}$ (see Proposition 2.1). The result in terms of Carleson sectors follows exactly the same proof. □

REFERENCES

[1] A. Aleman, *Hilbert spaces of analytic functions between the Hardy and the Dirichlet space*, Proc. Amer. Math. Soc. 115 (1992), 97–104.

[2] J. Akeroyd, *On Shapiro's compactness criterion for composition operators*, J. Math. Anal. Appl. 379 (2011), 1–7.

[3] J. Akeroyd and P. Ghatage, *Closed-range composition operators on $\mathbb{A}^2$*, Illinois J. Math. 52 (2008), 533–549.

[4] J. Akeroyd, P. Ghatage and M. Tjani, *Closed-range composition operators on $\mathbb{A}^2$ and the Bloch space*, Integr. Equations Operator Theory 68 (2010), 503–517.

[5] J. Cima, J. Thomson and W. Wogen, *On some properties of composition operators*, Indiana Univ. Math. J. 24 (1974), 215–220.

[6] C. Cowen and B. MacCluer, 'Composition Operators on Spaces of Analytic Functions', CRC Press, Boca Raton, Florida, 1995.

[7] C. Fefferman and E. Stein, *H^p spaces of several variables*, Acta Math. 129 (1972), 137–193.

[8] J.B. Garnett, 'Bounded Analytic Functions', Revised First Edition, Springer-Verlag, New York,

[9] M. Jovovic and B. MacCluer, *Composition operators on Dirichlet spaces*, Acta Sci. Math. (Szeged) 83 (1997), 229–247.

[10] K. Kellay and P. Lefèvre, *Compact composition operators on weighted Hilbert spaces of analytic functions*, J. Math. Anal. Appl. 386 (2012), 718–727.

[11] P. Lefèvre, D. Li, H. Queffelec and L. Rodríguez-Piazza, *Some examples of compact composition operators on H^2*, J. Funct. Anal. 255 (2008), 3098–3124.

[12] P. Lefèvre, D. Li, H. Queffelec and L. Rodríguez-Piazza, *Nevanlinna counting function and Carleson function of analytic maps*, Math. Ann. 351 (2011), 305–326.

[13] P. Lefèvre, D. Li, H. Queffelec and L. Rodríguez-Piazza, *Some revisited results about composition operators on Hardy spaces*, Revista Mat. Iberoamericana 28 (2012), 57–76.

[14] P. Lefèvre, D. Li, H. Queffelec and L. Rodríguez-Piazza, *Compact composition operators on Bergman-Orlicz spaces*, preprint

[15] B.A. Lotto, *A compact composition operator that is not Hilbert-Schmidt*, Contemp. Math. 213 (1998), 93–97.

[16] D. Luecking, *Inequalities on Bergman spaces*, Illinois J. Math. 25 (1981), 1–11.

[17] D. Luecking, *Trace ideal criteria for Toeplitz operators,* J. Funct. Anal. 73 (1987), 345–368.

[18] D. Luecking, *Bounded composition operators with closed range on the Dirichlet space*, Proc. Amer. Math. Soc. 128 (2000),1109–1116.

[19] D. Luecking and K. Zhu, *Composition operators belonging to the Schatten ideals*, Amer. J. Math. 114 (1992), 1127–1145.

[20] B. MacCluer and J.H. Shapiro, *Angular derivatives and compact composition operators on the Hardy and Bergman spaces*, Canadian J. Math. 38 (1986), 878–906.

[21] T. Ransford, 'Potential Theory in the Complex Plane', Cambridge University Press, Cambridge, 1995.

[22] J.H. Shapiro, *The essential norm of a composition operator*, Ann. of Math. 125 (1987), 375–404.

[23] J.H. Shapiro, 'Composition Operators and Classical Function Theory', Springer-Verlag, New York, 1993.

[24] S. Stevic and A.K. Sharma, *Essential norm of composition operators between weighted Hardy spaces*, Appl. Math. Comput. 217 (2011), 6192–6197.

[25] K. Zhu, 'Operator Theory in Function Spaces', Second Edition, Math. Surveys and Monographs, Vol. 138, American Mathematical Society: Providence, Rhode Island, 2007.

[26] N. Zorboska, *Composition operators with closed range*, Trans. Amer. Math. Soc. 344 (1994), 791–801.

[27] N. Zorboska, *Composition operators on weighted Dirichlet spaces*, Proc. Amer. Math. Soc. 126 (1998), 2013–2023.

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