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The Shapley-Shubik Index in the Presence of Externalities

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Abstract: In this note we characterize the restriction of the externality-free value of de Clippel and Serrano (2008) to the class of simple games with externalities introduced in Alonso-Meijide et al. (2015).

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Keywords: Externality-free value, Shapley--Shubik index, Partition function.

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1 Introduction

Ever since the seminal paper of Shapley and Shubik (1954), the *a priori* assessment of the power possessed by each agent participating in a decision making body has been an important topic in game theory. Simple coalitional games can be used to describe these situations by attaching 1 to any coalition that is strong enough to pass a proposal and 0 to the rest. If power is understood as the ability of an agent to change the outcome of a ballot, it is sensible to use the marginal contributions to develop power indices. Thus, the value proposed by Shapley (1953) to distribute the surplus generated from the cooperation of the agents in economic environments has been shown to be valuable also for evaluating the power in a legislature or committee.

In this paper, we aim at studying the distribution of power in the presence of coalitional externalities. Consider, for instance, a legislature that uses the plurality rule to elect the prime minister. There are cases where minority governments emerge just because the remaining parties do not agree on an alternative candidate. In such a situation, whether a coalition is winning or not may depend on the behavior of the rest of the parties. This shows that games in partition function form (Thrall and Lucas, 1963) are the appropiate framework in which to study situations like these. Some years ago, Bolger (1986) employed games in partition function form to study multi-candidate elections and proposed several power indices. One of the main novelties of our approach is to consider a subclass of games in partition function form that are monotonic. This class of games generalizes the simple games in characteristic function form as introduced by von Neumann and Morgenstern (1944). The aforementioned monotonicity property has been recently proposed in Alonso-Meijide et al. (2015) and makes special sense in situations with negative externalities, such as the ones outlined above.

The problem of extending the Shapley value to games in partition function form was first tackled by Myerson (1977). More recently, the topic has attracted some attention and alternative generalizations of the Shapley value have been proposed (Albizuri et al., 2005; Macho-Stadler et al., 2007; de Clippel and Serrano, 2008; Dutta et al., 2010). The existence of so many different proposals can be explained by the difficult task of generalizing marginal contributions to games in partition function form. Indeed, if we want to measure the change in the utility of a coalition when one of its members leaves it, then we should know which coalition will the defecting agent join, if any. Albizuri et al. (2005) assume that the agent can join any coalition and that any such coalition configuration is equally likely. Macho-Stadler et al. (2007) generalize the previous approach by considering a probability distribution over the different events that could take place. However, de Clippel and Serrano (2008) argue that the intrinsic marginal contribution is originated by an agent that leaves a coalition to become a singleton. In a subsequent step, the agent could join any coalition, but the effect of this move should not be considered a marginal contribution. Finally, Dutta et al. (2010) follow the potential approach and study a family of values that contains the previous proposals.

In this paper, we study the restriction of the value introduced by de Clippel and Serrano (2008) to simple games in partition function form as devised by Alonso-Meijide et al. (2015). The monotonicity property of the class of simple games considered allows us to speak about minimal winning embedded coalitions. This kind of coalition enables us to define null and symmetric players while avoiding the concept of marginal contribution. We show that this power index is the only one which is efficient, symmetric, and has both the null player property and the transfer property. These four properties are natural adaptations of the homonymous properties in frameworks without externalities. The first three are properties that any sensible power index should have (Felsenthal and Machover, 1998) and the last is the transfer property proposed by Dubey (1975). The rest of this note is organized into two sections. The preliminaries state some previous results and Section 3 presents our characterization result.

2 Preliminaries

Let N be a finite set ($|N| > 1$) of players that we keep fixed henceforth. A *characteristic function* is a mapping $v\colon 2^N = \{S : S \subseteq N\} \to \mathbb{R}$, satisfying $v(\emptyset) = 0$. The set of characteristic functions is denoted by $\mathcal{CG}$. A *value* is a mapping f that assigns a unique vector $\mathsf{f}(v) \in \mathbb{R}^N$ to every $v \in \mathcal{CG}$. The *Shapley value* (Shapley, 1953), Sh, defined for every $v \in \mathcal{CG}$ and $i \in N$ by[1]

$$\mathsf{Sh}_i(v) = \sum_{S \subseteq N \setminus i} \frac{(n-s-1)!s!}{n!} \left[v\left(S \cup i\right) - v\left(S\right) \right].$$

The set of partitions of N is denoted by $\mathcal{P}(N)$.[2] An *embedded coalition* is a pair (S, P) where $P \in \mathcal{P}(N)$ and $S \in P$. We will sometimes refer to S as the active coalition in P and we will say that a player $i \in N$ belongs to an embedded coalition (S, P) when $i \in S$. The set of embedded coalitions is denoted by $\mathcal{EC}$, i.e., $\mathcal{EC} = \{(S, P) : P \in \mathcal{P}(N) \text{ and } S \in P\}$. Given $P \in \mathcal{P}(N)$ and a nonempty coalition $S \subseteq N$, we let $P_{-S} \in \mathcal{P}(N \setminus S)$ denote the partition $P = \{T \setminus S : T \in P\}$.

[1] We abuse notation slightly and write $T \cup i$ and $T \setminus i$ instead of $T \cup \{i\}$ and $T \setminus \{i\}$, respectively, for $T \subseteq N$ and $i \in N$. We use lowercase letters to denote the cardinality of a finite set.

[2] For convenience, we assume that the empty set is an element of every partition even though we may omit writing it, i.e., for every $P \in \mathcal{P}(N)$, $\emptyset \in P$.

A *partition function* is a mapping $v : \mathcal{EC} \to \mathbb{R}$ such that $v(\emptyset, P) = 0$ for every $P \in \mathcal{P}(N)$. The set of partition functions is denoted by $\mathcal{G}$. Is is easy to see that $\mathcal{G}$ is a vector space over $\mathbb{R}$. Indeed, de Clippel and Serrano (2008) develop a basis of the vector space that we present below. Given $(S,P) \in \mathcal{EC}$, with $S \neq \emptyset$, let $e_{(S,P)} \in \mathcal{G}$ be defined for every $(T,Q) \in \mathcal{EC}$ by

$$e_{(S,P)}(T,Q) = \begin{cases} 1 & \text{if } S \subseteq T \text{ and } \forall T' \in Q \setminus T, \exists S' \in P \text{ such that } T' \subseteq S',^{3} \\ 0 & \text{otherwise.} \end{cases}$$

Then, de Clippel and Serrano (2008) show that $\{e_{(S,P)} : (S,P) \in \mathcal{EC} \text{ and } S \neq \emptyset\}$ constitutes a basis of $\mathcal{G}$.

In this paper, we focus on the so-called simple games in partition function form as introduced in Alonso-Meijide et al. (2015). This subclass of partition functions is a natural generalization of the class of simple games in characteristic function form. In order to define monotonicity in the class $\mathcal{G}$, we consider the following notion of inclusion between embedded coalitions.

Definition 2.1. *Let $(S,P),(T,Q) \in \mathcal{EC}$. We say that (S,P) is contained in (T,Q) and write $(S,P) \sqsubseteq (T,Q)$ when $S \subseteq T$ and $\forall T' \in Q \setminus T$, $\exists S' \in P$ such that $T' \subseteq S'$.*

Note that whenever $S \neq \emptyset$, $(S,P) \sqsubseteq (T,Q)$ if and only if $e_{(S,P)}(T,Q) = 1$. According to the above definition, an embedded coalition (S,P) is a subset of another embedded coalition (T,Q) if the active coalition in P is contained in the active coalitions in Q (i.e., $S \subseteq T$) and moreover, the partition P_{-T} is coarser than $Q \setminus T$. Notice that both P_{-T} and $Q \setminus T$ are partitions of $N \setminus T$.

We are now in the position to introduce the class of games under study.

Definition 2.2. *A partition function $v \in \mathcal{G}$ is said to be a* simple game (with externalities) *if it satisfies the three conditions below:*

i) For every $(S,P) \in \mathcal{EC}$, $v(S,P) \in \{0,1\}$.

ii) $v(N, \{\emptyset, N\}) = 1$.

iii) If $(S,P),(T,Q) \in \mathcal{EC}$ is such that $(S,P) \sqsubseteq (T,Q)$, then $v(S,P) \leq v(T,Q)$.

An embedded coalition, $(S,P) \in \mathcal{EC}$, is said to be winning *if $v(S,P) = 1$ and* losing *otherwise. The set of simple games is denoted by $\mathcal{SG}$.*

The monotonicity property defined in point *iii)* above allows us to properly speak about minimal winning embedded coalitions. Let $v \in \mathcal{SG}$. A winning embedded coalition, $(S,P) \in$

[3] As before, we may omit the braces and write $Q \setminus T$ instead of $Q \setminus \{T\}$ for every $T \in Q \in \mathcal{P}(N)$.

$\mathcal{EC}$, is said to be *minimal* if every proper subset of it is a losing embedded coalition, i.e., if $(T, Q) \sqsubset (S, P)$ implies that $v(T, Q) = 0$.[4] The set of all minimal winning embedded coalitions of a simple game is denoted by $\mathcal{M}(v)$ and the subset of minimal winning embedded coalitions that contain a given player $i \in N$ is denoted by $\mathcal{M}_i(v)$, i.e., $\mathcal{M}_i(v) = \{(S, P) \in \mathcal{M}(v) : i \in S\}$.

A player $i \in N$ is said to be a *null player* in $v \in \mathcal{SG}$ if i does not belong to any minimal winning embedded coalition, i.e., $\mathcal{M}_i(v) = \emptyset$.

Two players i and j are said to be *symmetric* in v if exchanging the two players in an embedded coalition in which either player participates does not change its worth. Formally, let $\pi : N \to N$ be defined by $\pi(i) = j$, $\pi(j) = i$, and for every $l \in N \setminus \{i, j\}$, $\pi(l) = l$. Then, i and j are symmetric in v if for every $(S, P) \in \mathcal{M}(v)$ such that $i \in S$ and $j \notin S$, $(\pi(S), \pi(P)) \in \mathcal{M}(v)$, where $\pi(S) = \{\pi(i) : i \in S\}$ and $\pi(P) = \{\pi(S) : S \in P\}$.

3 The Shapley–Shubik index in the presence of externalities

A *power index* is a mapping, f, that assigns a vector $\mathsf{f}(v) \in \mathbb{R}^N$ to every simple game $v \in \mathcal{SG}$, where each coordinate $\mathsf{f}_i(v)$ describes the power of player $i \in N$. Next, we present four properties that a power index may satisfy. All of them are based on well known properties in the framework of games in characteristic function form, adapted to our setting.

EFF A power index f is *efficient* if $\sum_{i \in N} \mathsf{f}_i(v) = 1$ for every $v \in \mathcal{SG}$.

NPP A power index f has the *null player property* if $\mathsf{f}_i(v) = 0$ for every $v \in \mathcal{SG}$ and every null player $i \in N$ in v.

SYM A power index f is *symmetric* property if $\mathsf{f}_i(v) = \mathsf{f}_j(v)$ for every $(N, v) \in \mathcal{SG}$ and every pair $i, j \in N$ of symmetric players in v.

TRA A power index f has the *transfer* property if $\mathsf{f}(v) + \mathsf{f}(w) = \mathsf{f}(v \vee w) + \mathsf{f}(v \wedge w)$ for every pair of simple games $v, w \in \mathcal{SG}$.

First of all, we show that these four properties single out a unique power index.

Theorem 3.1. *There is at most one power index satisfying* EFF*,* NPP*,* SYM*, and* TRA*.*

[4] A proper subset, $(T, Q) \sqsubset (S, P)$, is a subset $(T, Q) \sqsubseteq (S, P)$ satisfying $(T, Q) \neq (S, P)$.

Proof. Let f be a power index satisfying the four properties. We show, by induction on the number of minimal winning coalitions, that f is unique.

First, let $v \in \mathcal{SG}$ be such that $|\mathcal{M}(v)| = 1$. Then $\mathcal{M}(v) = \{(S,P)\}$ for some $(S,P) \in \mathcal{EC}$ and $v = e_{(S,P)}$. It is immediate to check that every $i \notin S$ is a null player in $e_{(S,P)}$. Then, by NPP, $\mathsf{f}_i\left(e_{(S,P)}\right) = 0$. Similarly, every two players in S are symmetric in $e_{(S,P)}$. Then, by SYM, they get the same payoff and by EFF we conclude that $\mathsf{f}_i\left(e_{(S,P)}\right) = \frac{1}{|S|}$ for every $i \in S$.

Second, suppose that f is uniquely determined for every $v \in \mathcal{SG}$ with $|\mathcal{M}(v)| < r$. Let $v \in \mathcal{SG}$ with $\mathcal{M}(v) = \left\{\left(S_1, P^1\right), \ldots, (S_r, P^r)\right\}$. Next, since v is monotonic, for every $(T,Q) \in \mathcal{EC}$,

$$v(T,Q) = \max_{(S,P)\in\mathcal{M}(v)} e_{(S,P)}(T,Q) = \max\left\{w(T,Q), e_{(S_r,P^r)}(T,Q)\right\},$$

where $w(T,Q) = \max_{k\in\{1,\ldots,r-1\}} e_{(S_k,P^k)}(T,Q)$. Since $v = w \vee e_{(S_r,P^r)}$, by TRA,

$$\mathsf{f}(w) + \mathsf{f}\left(e_{(S_r,P^r)}\right) = \mathsf{f}(v) + \mathsf{f}\left(w \wedge e_{(S_r,P^r)}\right).$$

Note that the two payoffs on the left-hand side of the equation above are uniquely determined by the induction hypothesis. Then it only remains to prove that the vector $\mathsf{f}\left(w \wedge e_{(S_r,P^r)}\right)$ is uniquely determined.

Third, for every $k \in \{1,\ldots,r-1\}$, we define the coalition $T_k = S_k \cap S_r$ and the partition $Q^k = \left\{U \cap V : U \in P^k \text{ and } V \in P^r\right\}$. Observe that $\left(T_k, Q^k\right) \in \mathcal{EC}$. We claim that $\mathcal{M}\left(w \wedge e_{(S_r,P^r)}\right) = \left\{\left(T_k, Q^k\right) : k \in \{1,\ldots,r-1\}\right\}$. Indeed, let $(T,Q) \in \mathcal{EC}$. Then

$$w \wedge e_{(S_r,P^r)}(T,Q) = 1 \Leftrightarrow \begin{cases} w(T,Q) = 1 \text{ and} \\ e_{(S_r,P_r)}(T,Q) = 1 \end{cases} \Leftrightarrow \begin{cases} \exists k \in \{1,\ldots,r-1\} : (S_k,P_k) \sqsubseteq (T,Q) \text{ and} \\ (S_r,P_r) \sqsubseteq (T,Q) \end{cases}$$

$$\Leftrightarrow \exists k \in \{1,\ldots,r-1\} : \begin{cases} S_k \cap S_r \subseteq T \text{ and} \\ Q \setminus T \text{ is finer than } P^k \setminus S_k \text{ and } P^r \setminus S_r \end{cases}$$

Notice that, by definition of Q^k, any partition which is finer than both $P^k \setminus S_k$ and $P^r \setminus S_r$ is necessarily finer than $Q^k \setminus T_k$. Thus, the above statement is equivalent to

$$\exists k \in \{1,\ldots,r-1\} : \begin{cases} T_k \subseteq T \text{ and} \\ Q \setminus T \text{ is finer than } Q^k \setminus T_k \end{cases} \Leftrightarrow \exists k \in \{1,\ldots,r-1\} : (T_k,Q_k) \sqsubseteq (T,Q).$$

Since all the above statements are if and only if implications, we have shown the claim.

Fourth, and last, since $\left|\mathcal{M}\left(w \wedge e_{(S_r,P^r)}\right)\right| < r$, by the induction hypothesis $\mathsf{f}\left(w \wedge e_{(S_r,P^r)}\right)$ is unique and the proof is concluded. $\square$

Finally, we show that there is indeed one power index with the four properties. As did Shapley and Shubik (1954), we consider the restriction of the externality-free value of (de Clippel and Serrano, 2008) to our class of simple games.

Definition 3.1. *The* externality-free Shapley–Shubik index*,* SS*, is the power index defined by* $\mathsf{SS}(v) = \mathsf{Sh}(v^*)$*, where* $v \in \mathcal{SG}$ *and* $v^* \in \mathcal{CG}$ *is then defined by putting* $v^*(S) = v\left(S, \left\{S, \{j\}_{j \in N \setminus S}\right\}\right)$ *for each* $S \subseteq N$*.* [5]

Theorem 3.2. SS *satisfies* EFF*,* NPP*,* SYM*, and* TRA*.*

Proof. EFF follows from the fact that Sh is efficient.

To show NPP, let $i \in N$ be a null player in $v \in \mathcal{SG}$. We will see that $v^*(S \cup i) = v^*(S)$ for every $S \subseteq N \setminus i$. Suppose, on the contrary, that there is a coalition $S \subseteq N \setminus i$ such that $v^*(S \cup i) \neq v^*(S)$. Then, by definition of v^*, $v\left(S \cup i, \left\{S \cup i, \{j\}_{j \in N \setminus (S \cup i)}\right\}\right) \neq v\left(S, \left\{S, \{j\}_{j \in N \setminus S}\right\}\right)$. Taking into account that $\left(S, \left\{S, \{j\}_{j \in N \setminus S}\right\}\right) \sqsubseteq \left(S \cup i, \left\{S \cup i, \{j\}_{j \in N \setminus (S \cup i)}\right\}\right)$ and the definition of $\mathcal{SG}$, we necessarily have that $v\left(S \cup i, \left\{S \cup i, \{j\}_{j \in N \setminus (S \cup i)}\right\}\right) = 1$ and $v\left(S, \left\{S, \{j\}_{j \in N \setminus S}\right\}\right) = 0$. Since i is a null player in v, $\left(S \cup i, \left\{S \cup i, \{j\}_{j \in N \setminus (S \cup i)}\right\}\right)$ cannot be a minimal winning embedded coalition in (N, v). Let $(T, Q) \in \mathcal{M}(v)$ be such that $(T, Q) \sqsubseteq \left(S \cup i, \left\{S \cup i, \{j\}_{j \in N \setminus (S \cup i)}\right\}\right)$. Again, since i is a null player in v, $i \notin T$ or, equivalently, $T \subseteq S$. Then

$$(T, Q) \sqsubseteq \left(S, \left\{S, \{j\}_{j \in N \setminus S}\right\}\right),$$

which contradicts the assumption that $v\left(S, \left\{S, \{j\}_{j \in N \setminus S}\right\}\right) = 0$. That is, we have shown that i is a null player in the classical sense in the characteristic function v^*. Finally, since Sh satisfies the null player property (Shapley, 1953), $\mathsf{SS}_i(N, v) = 0$.

To show SYM, let $i, j \in N$ be two symmetric players in $v \in \mathcal{SG}$ and let $S \subseteq N \setminus \{i, j\}$. Suppose that $\left(S \cup i, \left\{S \cup i, \{l\}_{l \in N \setminus (S \cup i)}\right\}\right)$ is a winning embedded coalition. We show that $\left(S \cup j, \left\{S \cup j, \{l\}_{l \in N \setminus (S \cup j)}\right\}\right)$ is also winning embedded coalition. Indeed, suppose that there is a $(T, Q) \in \mathcal{M}(v)$ such that $(T, Q) \sqsubseteq \left(S \cup i, \left\{S \cup i, \{l\}_{l \in N \setminus (S \cup i)}\right\}\right)$. On the one hand, if $i \notin T$, then $(T, Q) \subseteq \left(S \cup j, \left\{S \cup j, \{l\}_{l \in N \setminus (S \cup j)}\right\}\right)$ and we are done. On the other hand, suppose that $i \in T$. Let $\pi : N \to N$ be defined by $\pi(i) = j$, $\pi(j) = i$, and for every $l \in N \setminus \{i, j\}$, $\pi(l) = l$. Note that since i and j are symmetric players, $(\pi(T), \pi(Q)) \in \mathcal{M}(v)$. Moreover, $(\pi(T), \pi(Q)) \sqsubseteq \left(S \cup j, \left\{S \cup j, \{l\}_{l \in N \setminus (S \cup j)}\right\}\right)$ because $\pi(T) = (T \setminus i) \cup j \subseteq S \cup j$. All in all, we have shown that i and j are symmetric players (in the classical sense) in the characteristic

[5] Observe that for our class of simple games, v^* associates to every coalition its optimistic expected worth. I.e., if $v \in \mathcal{SG}$, then for every $S \subseteq N$, $v^*(S) = \max_{(S,P) \in \mathcal{EC}} \{v(S, P)\}$.

function v^*. Finally, since Sh is symmetric (Shapley, 1953), the payoffs of i and j in v according to SS coincide.

To show TRA, let $v, w \in \mathcal{SG}$. Then

$$\mathsf{SS}(v) + \mathsf{SS}(w) = \mathsf{Sh}(v^*) + \mathsf{Sh}(w^*) = \mathsf{Sh}(v^* \vee w^*) + \mathsf{Sh}(v^* \wedge w^*) = \mathsf{Sh}((v \vee w)^*) + \mathsf{Sh}((v \wedge w)^*)$$
$$= \mathsf{SS}(v \vee w) + \mathsf{SS}(v \wedge w),$$

where the first and last equalities hold by definition of SS, the second is due to the fact that Sh satisfies the classic transfer property (Dubey, 1975), and the third follows from $v^* \vee w^* = (v \vee w)^*$ and $v^* \wedge w^* = (v \wedge w)^*$. Indeed, if $S \subseteq N$, then

$$(v^* \vee w^*)(S) = \max\{v^*(S), w^*(S)\} = \max\left\{v\left(S, \{S, \{j\}_{j \in N \setminus S}\}\right), w\left(S, \{S, \{j\}_{j \in N \setminus S}\}\right)\right\}$$
$$= (v \vee w)\left(S, \{S, \{j\}_{j \in N \setminus S}\}\right) = (v \vee w)^*(S).$$

Exchanging the maximum with the minimum in the equation above shows that $v^* \wedge w^* = (v \wedge w)^*$, which concludes the proof. $\square$

To conclude, we present the characterization result and show the logical independence of the four properties.

Theorem 3.3. *The externality-free Shapley–Shubik index is the only power index satisfying* EFF, NPP, SYM, *and* TRA. *Moreover, the four properties are independent.*

Proof. The characterization is a direct consequence of Theorems 3.1 and 3.2. To show the independence, consider the following power indices.

Let f^1 be the power index defined by $\mathsf{f}^1_i(v) = 0$ for $v \in \mathcal{SG}$ and $i \in N$. Then f^1 satisfies NPP, SYM, and TRA, but not EFF.

Let f^2 be the power index defined by $\mathsf{f}^2_i(v) = \frac{1}{n}$ for $v \in \mathcal{SG}$ and $i \in N$. Then f^2 satisfies EFF, SYM, and TRA, but not NPP.

Let f^3 be the power index defined as follows: if $v \in \mathcal{SG}$ is such that $v(S, P) = 0$ for every $(S, P) \in \mathcal{EC}$ with $S \neq N$, then $\mathsf{f}^3(v) = (1, 0 \ldots, 0)$. For any other $v \in \mathcal{SG}$, put $\mathsf{f}^3(v) = \mathsf{SS}(v)$. Then f^3 satisfies EFF, NPP, and TRA, but not SYM.

The DP-externality power index defined in Alonso-Meijide et al. (2015) satisfies EFF, NPP, and SYM, but not TRA. $\square$

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