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Preferences, Social Choice and Theories of Justice

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Abstract

This work is an introduction to Social Choice Theory and Distributive Justice. It aims to set under which conditions does exist a method to aggregate individual preferences. We first model individual behaviour in decision making taking into account preferences and utilities, and also choice behaviour. The objective is to understand how collective choice can be done. Then we formulate and prove Arrow's impossibility theorem, which states that, under certain (and reasonable) conditions, it is impossible to find an aggregation system. We also present some ways to loosen hypothesis in order to avoid this impossibility. Finally, we redefine our whole framework to allow interpersonal comparisons. This lets us characterize two main ways of social deciding: the leximin principle and the utilitarian rule.

Resum

Aquest treball és una introducció a la Teoria de l'elecció social (Social Choice Theory) i a la Justícia distributiva (Distributive Justice). Té com a objectiu determinar sota quines condicions existeixen mètodes per agrupar les opinions dels individus d'una societat. Per entendre com es pot triar col·lectivament primer de tot modelitzarem el comportament individual a l'hora de triar entre alternatives. A continuació enunciarem i demostrarem el teorema d'impossibilitat d'Arrow, que diu que, sota certes condicions (ben raonables), és impossible trobar una manera d'escollir socialment que les satisfaci. També proposarem algunes relaxacions d'hipòtesis per a evitar la impossibilitat. Finalment redefinirem el nostre marc de treball per permetre comparacions entre individus d'una mateixa societat. Això ens permetrà caracteritzar dos dels principals mètodes a l'hora de triar: el "leximin principle" i la "utilitarian rule".

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Contents

Introduction

Social Choice Theory: Origins and Evolution

Social choice theory is an analysis of collective decision making. It starts out from individual opinions, preferences, interests or welfare of the members of a society and attempts to reach a collective statement in a "satisfactory" manner, that is, in a manner compatible with the fulfillment of a variety of desirable conditions. Formal social choice theory has its origin at the end of 1940s and beginning of 1950s with the works of Black, Arrow and Guilbaud. It is considered a rebirth of what Borda and Condorcet started at the end of 18th century. However, one can find numerous precursors.

According to McLean and Urken (1995) [17] several classics discussed about the goodness of various political system, but it is not until the appearance of Pliny the Younger(AD 61 or 62-113) that we talk about precursors of social choice. McLean and London (1990) [16] research which are these precursors. Around AD 90 Pliny the Younger wrote a letter to Titius Aristo where he describes a situation of manipulation of preferences in a voting situation in the modern sense of Gibbard-Satterthwaite: a group of individuals by voting strategically forces the voting rule to generate an outcome that the members of the group prefer to the outcome that would have prevailed if they had not voted strategically.

Ramon Llull (ca. 1233-1316) recommends systems based on pair-wise (majority) voting on his novels "*Blanquerna*" and "*De Arte Eleccionis*". The procedure described in "*Blanquerna*" chose the candidate who had won the highest number of pair-wise comparisons and is identical to the one proposed by Borda in 1770. The voting rule proposed in "*De Arte Eleccionis*" ended up in a method satisfying a criteria proposed by Condorcet in 1785, but it did it in an iterative way.

In the 15th century, Nicholas of Cusa (also known as Nicholas Cusanus) in his book "*De Concordantia Catholica*" devotes some paragraphs to the description of a voting method for the election of the Emperor of the Holy Roman Empire. On an example with ten candidates he assumes that the voter ranks the candidate without ties from the least preferred to the most preferred and give marks to candidates from 1 to 10 on the basis of this ranking. This is clearly Borda's rule as it is now known.

Samuel von Pufendorf publishes in 1672 his book "De Jure Naturae and Gentium", which has caught the eye of social choice theorists and economists. He discussed weighted voting, qualified majorities and single-peaked preferences (a domain restriction that has become quite known in the middle of 20th century) amongst other subjects. Another interpretation of von Pufendorf examples would be to show how manipulate voting works as Pliny did.

Even if Nicholas of Cusa proposed Borda's rule more than three hundred years before

Borda, and Ramon Lull's description of elections are generally based on pair-wise majority voting announcing, maybe, Condorcet, their works cannot be compared with those of Borda and Condorcet. Borda was a great applied scientist of his time and Condorcet's contribution to human knowledge is still probably underestimated. What they left us on voting is not commensurate with what Cusanus or Lull left. Condorcet strongly advocated a binary notion, i.e. pairwise comparisons of candidates, whereas Borda focused on a positional approach where the positions of candidates in the individual preference orderings matter. A third author from 18th century shall be pointed out although his contributions to social choice theory are quite indirect. Jeremy Bentham in 1789 published "*An Introduction to the Principles of Morals and Legislation*" where he bets for "the greatest happiness of the greatest number". It can be somehow understood as the utilitarian way of choosing without having defined utility functions yet.

During the 19th century, in spite of the development of democratic institutions, the theory of social choice was rather dormant. Despite, it is worth naming Charles Lutwidge Dodgson, better known as Lewis Carroll, the author of "*Alice in Wonderland*", since he dealt with cyclical majorities and proposed a rule, based on pairwise comparisons, which avoids such cycles.

As said before, formal social choice theory stands out with the works of Arrow, Black and Guilbaud. Since them several social choice theorists have emerged and social choice theory has done great advances. However we shall point out two aspects towards which Social Choice Theory has focused. The first one is the preference aggregation and was started by Kenneth Arrow[2] with his theorem. It works with binary relations since its starting point is individual preference of members of society. Neither comparison between individuals nor cardinal measure of preferences are allowed. The second one was started by Amartya Sen.[3] He stated that by assigning real numbers to alternatives, welfare profiles contain a lot of information over and above the profiles of binary relations on X they induce. So one shall aggregate welfare instead of preferences. This would allow us to compare between individuals and to use cardinal information when needed. In this way we can evaluate the welfare of a society, with respect to other possible states.

It is worth noting that Gerard Debreu[4] was also at the beginning of Social Choice, with the relationship between preferences and utilities.

Recently, Social Choice Theory has developed several branches. Empirical social choice, judgment aggregation, cooperative bargaining, behavioural social choice, fair division, computational social choice theory, theories of probability aggregation, theories of general attitude aggregation, collective decision-making in non-human animals and applications to social epistemology are some issues on which experts are working at. As should be evident, social choice theory is a vast field.

[2]Kenneth J. Arrow (1921-2017) got the Nobel Prize in Economics in 1972, joint with John Hicks, "for their pioneering contributions to general economic equilibrium theory and welfare theory". he was a mathematician.

[3]Amartya K. Sen (1933) got the Nobel Prize in Economics in 1998, "for his contributions to welfare economics".

[4]Gerard Debreu (1921-2004) got the Nobel Prize in Economics in 1983, "for having incorporated new analytical methods into economic theory and for his rigorous reformulation of the theory of general equilibrium". He was a mathematician and attended École Normale Supérieure with Henri Cartan. He was influenced by the books of N. Bourbaki.

About this work

Let us briefly describe the contents of this work. Chapter 1 sets the basis to face social choice problems; it aims to model individual behaviour when making decisions. We introduce individual decision making theory in an abstract way, describe two approaches to modeling a person's decisions and give conditions to set when to move from one to another.

Chapter 2 discusses Arrow's famous impossibility theorem. We first present some possible frameworks when studying social decision making and give some examples of how social aggregation can be made. We shall present two different proofs of Arrow's theorem. The first one is Arrow's original proof, and the second one emphasises informational aspect within the Arrovian set-up. Finally, we point out some possibilities if one wants to avoid impossibilities.

Chapter 3 is the main issue of this work. Alternative theories of distributive justice are its topic. We redefine the whole set-up and widen it in order to be able to compare individuals and discuss some justice conditions. The two main ways of social deciding, utilitarianism and the Rawlsian maximin/leximin principle, are contrasted with each other and characterized from different viewpoints.

Chapter 1

Preferences and utility functions

In this chapter we analyze the preferences of one single agent over a set of different alternatives and which conditions we ask to consider these preferences as belonging to a rational agent. Moreover we give conditions to ensure when a preference relation can be represented by an utility function. We will mainly follow Mas-Colell et al. (1995) [15] and Jehle and Reny (2001) [12].

1.1 Decision problems

The starting point for any individual decision problem is a set of possible, mutually exclusive, alternatives, which will be denoted by X. There are two distinct approaches to modeling individual choice behaviours. The first one, defined in Section 1.2, focuses on decision maker's tastes and interprets them as a binary relation over the set of alternatives. The second approach is defined in Section 1.3 and defends that an individual's primitive feature is being able to choose amongst alternatives.

Let $X = \{x, y, z, \ldots\}$ denote the set of all conceivable social states, which we call alternatives. This can be a finite set, for example a group of candidates in an election, or countable as the different budgets for a project. It can be also an uncountable set such as the combinations of goods and services a consumer may choose. We denote by $N = \{1, \ldots, n\}$ a finite set of individuals, with $n \geq 2$.

1.2 Preference relations

In the preference-based approach, the objectives of a decision maker are summarized in a preference relation on X, denoted by R. Set X is usually referred as the set of alternatives. The relation R is a subset of ordered pairs of the product $X \times X$; for example we write xRy to indicate that the pair $(x, y) \in X \times X$ is in the subset given by R. Now, when we speak about individual i's preference, $i \in N$, we just write R^i. A binary relation gives just a subset of the ordered pairs of the product $X \times X$. A *preference relation* R is a binary relation on the set X. To denote this preference relation, if aRb (with $a, b \in X$) we will say that a is *at least as good as* b.

From this relation we can derive to two other important relations: the strict preference relation P and the indifference relation I, both defined as follows. Let a and b be members of a set X,

(i) we say that a is *strictly preferred* to b, and denote it by aPb if and only if $aRb \wedge \neg\, bRa$,

(ii) we say that a is *indifferent* to b, and denote it by aIb if and only if $aRb \wedge bRa$.

In much of social choice theory, individuals are assumed to be rational which means that their preference relation is complete and transitive.

Definition 1.1. *A preference relation R on X is* rational *if it is, for all $a, b, c \in X$,*

1. *reflexive, i.e. aRa,*
2. *transitive, i.e. $(aRb \wedge bRc) \rightarrow aRc$, and*
3. *complete, i.e. $\forall a, b \in X$ it holds $aRb \vee bRa$.*

When a preference relation is rational, we call it a preference ordering.

Completeness means that the individual has a well-defined preference between every two alternatives; he or she can compare any two alternatives in X. Transitivity requires the decision maker to choose in a consistent way. Sometimes, though, transitivity may be quite a hard condition to be held, and may only be needed either quasi-transitivity or acyclicity.

Definition 1.2. *R is said to be* quasi-transitive *if P is transitive.*

Definition 1.3. *R is said to be* acyclical *if for all finite sequences $x_1, x_2, \ldots, x_k \in X$ it never holds $x_1Px_2 \wedge x_2Px_3 \wedge \ldots \wedge x_{k-1}Px_k \wedge x_kPx_1$.*

All three definition aim to avoid cycles on the preference relations. We may point out some properties that follow from preference relations and orderings. The strict preference relation P is irreflexive and transitive. The indifference relation I is reflexive, symmetrical and transitive, which means it is an equivalence relation. Note that R being transitive means R is quasi transitive, and so, requires R to be acyclical. If R is an ordering, it also holds $aPb \wedge bRc \rightarrow aRc$.

Given an alternative $a \in X$, the set formed by those alternatives which are at least as good as a is defined to be its “at least as good as” set, and denote it by Ra, and the set formed by those alternatives which are no better than a id defined as its “no better than” set and denote it as aR. More formally, we define

$$Ra = \{b \in X \mid bRa\} \quad \text{and} \quad aR = \{b \in X \mid aRb\}.$$

In the same way we could also define strictly preferred sets aP and Pa and indifference sets $aI = Ia$.

We may sometimes need some topological regularity on preference orderings in order to work in an easier way. Although in social choice it is not widely needed, in some cases we may need our relations to hold continuity.

Definition 1.4. *We say that a binary relation R on X is* continuous *if the sets aR and Ra are closed sets in X.*

Continuity will be most used when considering X as a subset on $\mathbb{R}^n$. In this case it is guaranteed that sudden preference reversals do not occur. Note that continuity implies the indifference set to be also closed. Continuity requirements are closely related to utility functions, which are studied in the next paragraph.

Utility functions

Utility functions are simply a convenient device for summarizing the information contained in the consumer's preference relation. It allows to translate preferences to numbers, and then apply other tools. Sometimes it is easier to work with preference relations and its associated sets and other times it is easier to work with utility functions, especially when employing calculus methods.

Definition 1.5. *Let R be a preference relation on a set X. An utility function u representing R is a function $u : X \longrightarrow \mathbb{R}$ such that $\forall a, b \in X$, it holds*

$$aRb \iff u(a) \geq u(b).$$

If all we require of a preference relation is that ranking between alternatives be meaningful, and all we require of a utility function representing that preference relation is that it reflects the ordering of alternatives, then any other function that assigns numbers to alternatives in the same order as the utility function does will also represent that preference relation. It is clear then that uniqueness of utility functions is not met. However, in this case, any utility function representing that relation is capable of conveying to us is ordinal information. The following result characterizes which utility functions represent a preference relation.

Proposition 1.1. *Let R be a preference relation on X and suppose $u(x)$ is an utility function that represents R. Then a function $v(x)$ also represents R if and only if there exists a function f, $f : \mathbb{R} \longrightarrow \mathbb{R}$, which is strictly increasing on the set of values taken by u, such that $v = f \circ u$.*

Proof. Let us suppose there exists f satisfying such hypothesis. Let $a, b \in X$ be two alternatives. Then,

$$v(a) > v(b) \iff f(u(a)) > f(u(b)) \iff u(a) > u(b) \iff aPb.$$

In the same way $v(a) = v(b) \iff aIb$, so $v(a) \geq v(b) \iff aRb$.

Let us now suppose u represents R and v also represents R. Consider the function that assigns $v(x)$ to every $u(x)$ for any $x \in X$. This is a strictly increasing function since $u(a) > u(b) \iff aPb \iff v(a) > v(b)$. $\square$

It is clear that if an utility function represents a preference relation, this one has to be rational. However, not every rational preference relation can be described by some utility function. If X is a finite or countable set, then every rational preference can be described by an utility function, but when X is non-countable, things get more complicated. Moreover, in order to easily calculate and achieve some results, one may also guarantee this utility function to be continuous. These are our next results. The following theorem and proposition assure the existence of a continuous utility function representing a preference relation under certain restrictions. To this end we need some definitions.

A gap of a set $S \subseteq \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$ is a maximal non-degenerate interval of the complement of S which has an upper and lower bound in S. The next proposition was proved by Debreu (1954) [5], and we use this result to prove our Theorem 1.1.

Proposition 1.2 (Debreu, 1954)**.** *If S is a subset of the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$, there exists an increasing function g from S to $\overline{\mathbb{R}}$ such that all the gaps of $g(S)$ are open.*

Theorem 1.1. *Let X be a topological space satisfying the second axiom of countability*[1] *and R a continuous preference ordering defined on X. Then there exists a continuous utility function u representing R.*

Proof. We will first prove the existence of u and then its continuity.

Let $\{B_n\}_{n\in\mathbb{N}}$ denote the sets of the countable base of the topology of X. Let $a \in X$ be an alternative and let us consider the set $N(a) = \{n | aRz, \forall z \in B_n\}$. Let us define the following function:

$$u(a) = \sum_{n\in N(a)} \frac{1}{2^n}.$$

To u representing R it has to be $aRb \iff u(a) \geq u(b)$ for $a, b \in X$. If aRb, then $N(a) \supseteq N(b)$, so that $u(a) \geq u(b)$. To show the other implication, let us suppose $\neg aRb$, hence bPa. This means there exists an $n \in N(b)$ such that $a \in B_n$ but not $n \in N(a)$. Thus, $N(a) \subsetneq N(b)$ and $u(a) < u(b)$, hence not $u(a) \geq u(b)$.

To show continuity, let us now consider S an arbitrary set of $\overline{\mathbb{R}}$. Because of Proposition 1.2, there exists an increasing function g from S to $\overline{\mathbb{R}}$ such that all the gaps of $g(S)$ are open.

We then define a new utility function $v = g \circ u$. The function $v(x)$ is also an utility function representing R. All the gaps of $v(X)$ are open. To show that v is continuous it suffices to show that $\forall t \in \overline{\mathbb{R}}$ the sets $v^{-1}([t, +\infty])$ and $v^{-1}([-\infty, t])$ are closed sets.

If $t \in v(X)$, then there exists $y \in X$ such that $v(y) = t$. Given that case, $v^{-1}([t, +\infty]) = \{x \in X \mid xRy\}$ and $v^{-1}([-\infty, t]) = \{x \in X \mid yRx\}$. Both sets are closed because of R being a continuous preference ordering.

We discuss now the cases in which $t \notin v(X)$. If $t \notin v(X)$ and is not contained in a gap, then there are three possibilities:

(i) $t \leq \inf_{x\in X} v(X)$. This means $v^{-1}([t, +\infty]) = X$ and $v^{-1}([-\infty, t]) = \varnothing$. Both are closed sets.

(ii) $t \geq \sup_{x\in X} v(X)$. This means $v^{-1}([t, +\infty]) = \varnothing$ and $v^{-1}([-\infty, t]) = X$. Both are closed sets.

(iii) $[t, +\infty] = \bigcap_{\substack{\alpha<t;\\ \alpha\in v(X)}} [\alpha, +\infty]$ and $[-\infty, t] = \bigcap_{\substack{\alpha>t;\\ \alpha\in v(X)}} [-\infty, \alpha]$. In this case

$$v^{-1}([t, +\infty]) = v^{-1}\left(\bigcap_{\substack{\alpha<t;\\ \alpha\in v(X)}} [\alpha, +\infty]\right) = \bigcap_{\substack{\alpha<t;\\ \alpha\in v(X)}} v^{-1}([\alpha, +\infty])$$

and

$$v^{-1}([-\infty, t]) = v^{-1}\left(\bigcap_{\substack{\alpha>t;\\ \alpha\in v(X)}} [-\infty, \alpha]\right) = \bigcap_{\substack{\alpha>t;\\ \alpha\in v(X)}} v^{-1}([-\infty, \alpha]).$$

[1] A topological space satisfies the second axiom of countability if the topology has a countable base.

Both are intersection of closed sets, so both are closed sets.

Finally, if t belongs to a gap, it must be an open gap $]a, b[$ with $a, b \in v(X)$. Then

$$v^{-1}([t, +\infty]) = v^{-1}([b, +\infty]),$$
$$v^{-1}([-\infty, t]) = v^{-1}([-\infty, a]).$$

Both are closed sets, so v is a continuous utility function representing R. $\square$

1.3 Choice Rules

In the second approach to the theory of decision making, choice behaviour itself is taken to be the primitive object of the theory. Formally, choice behaviour is represented by means of a choice structure. This is a common approach when we deal with microeconomics (see Mas-Colell et al., 1995 [15]).

Recall that X is the set of alternatives (finite or infinite).

Definition 1.6. *A choice structure on X is a pair $(\mathscr{B}, C(\cdot))$ where*

(i) $\mathscr{B}$ is a family of non-empty subsets of X,

(ii) $C(\cdot)$ is a function $C : \mathscr{B} \longrightarrow \mathscr{B}$ that assigns a non-empty set of chosen elements of X to every set B in $\mathscr{B}$ such that $C(B) \subseteq B$.

The consideration of the family $\mathscr{B}$ allows to consider not all subset of the alternatives as acceptable. We will denote an arbitrary member of $\mathscr{B}$ by the letter B. We say that $C(B)$ contains all alternatives chosen by the individual among all possible alternatives in B.

As when working with preferences, when working with choice structures we may want to impose some restrictions of consistency regarding an individual's choice behaviour. The weak axiom of revealed preference reflects this consistency.

Definition 1.7. *Let $(\mathscr{B}, C(\cdot))$ be a choice structure on X. Let $B \in \mathscr{B}$ be a subset of X, and let $a, b \in B$ be two alternatives such that $a \in C(B)$. We say that this choice structure satisfies the* weak axiom of revealed preference *if for any $B' \in \mathscr{B}$ with $a, b \in B'$ and $b \in C(B')$, we must also have $a \in C(B')$.*

The weak axiom of revealed preference says that whenever a is chosen from a set containing a and b there cannot be another set containing a and b such that b is chosen and a not.

Now we relate choice structures on X with preference relations on X. First we associate in a natural way a preference relation to a choice structure.

Given a choice structure, we can define a preference relation from the observed choice behaviour in $C(\cdot)$. We call it the revealed preference relation.

Definition 1.8. *Let $(\mathscr{B}, C(\cdot))$ be a choice structure on X. We define the* revealed preference relation *R^* by:*

$$aR^*b \iff \exists B \in \mathscr{B} \;\; \textit{such that } a, b \in B \;\; \textit{and } a \in C(B).$$

Notice that this relation, depending on the properties of the choice structure, may be rational or not.

On the other hand, one may want to know whether a rational preference defines a choice structure satisfying the weak axiom of revealed preference or not. Also one may ask the reciprocal; whether given a choice structures satisfying the weak axiom of revealed preference, the revealed preference relation is a rational one.

To answer these questions one may need to define a choice structure with a given preference ordering. Given a preference ordering and a subset $B \subseteq X$ we define the following choice rule:

$$C^*(B, R) = \{a \in B \mid aRb \text{ for every } b \in B\}.$$

Notice that $C^*(B, R)$ may be empty for some B. However, when X is finite, or when suitable conditions are held, such as continuity on R, non-emptiness of $C^*(B, R)$ can be assured. From now on we will suppose $C^*(B, R)$ is a non-empty set.

Proposition 1.3. *Let R be a preference ordering on X. Then, the choice structure generated by R, that is $(\mathscr{B}, C^*(\cdot, R))$, satisfies the weak axiom of revealed preference.*

Proof. Let $B \in \mathscr{B}$ and let $a, b \in B$ be two alternatives such that $a \in C^*(B, R)$. This implies aRb by definition of $C^*(B, R)$. Let now $B' \in \mathscr{B}$ with $a, b \in B'$ such that $b \in C(B', R)$. This means that for every $c \in B'$ it holds bRc. But, by transitivity we have aRc and, hence, $a \in C(B', R)$ so the weak axiom of revealed preference is satisfied. $\square$

Having a choice structure satisfying the weak axiom of revealed preference, however, does no always mean to have a rational preference relation. We have to add some conditions and introduce the notion of rationalization of a choice structure to make language easier.

Definition 1.9. *Let $(\mathscr{B}, C(\cdot))$ be a choice structure on X and R a rational preference relation. We say that R rationalizes $C(\cdot)$ relative to $\mathscr{B}$ if*

$$C(B) = C^*(B, R) \text{ for all } B \in \mathscr{B}.$$

This means that if the choice structure is generated by R, then R rationalizes the choice structure. Notice that, in general, there may be more that one rationalizing relation for a given choice structure.

By Proposition 1.3 it holds that if there exists a rationalizing preference relation, then the corresponding choice structure it rationalizes must satisfy the weak axiom of revealed preference. So, only choice rules that satisfy the weak axiom of revealed preference can be rationalized, but it is not a sufficient condition. The following proposition gives sufficient conditions to a choice rule to be rationalized.

Proposition 1.4. *Let $(\mathscr{B}, C(\cdot))$ be a choice structure on X such that:*

(i) satisfies the weak axiom of revealed preference,

(ii) $\mathscr{B}$ includes all subsets of X of up to three elements.

Then, there is a rational preference relation R that rationalizes the choice structure. Furthermore, R is the only preference relation that does so.

Proof. The main candidate to rationalize a choice structure is the revealed preference relation R^*. We have to prove that it is a rational relation, that it rationalizes the choice structure and that it is the only one doing so.

Completeness is argued as follows. By assumption (ii), every pair of alternatives $\{a, b\}$ belongs to $\mathcal{B}$. This means either $a \in C(\{a, b\})$ or $b \in C(\{a, b\})$ and, by definition of revealed preference relation, it holds either aR^*b or bR^*a, or both. Thus, R^* is complete.

To prove transitivity, let a, b, c be alternatives such that aR^*b and bR^*c. Consider the set $\{a, b, c\} \in \mathcal{B}$. Proving that $a \in C(\{a, b, c\})$ is sufficient to show transitivity because of the definition of revealed preference. As $C(\{a, b, c\}) \neq \varnothing$, at least one alternative must be in $C(\{a, b, c\})$. If $a \in C(\{a, b, c\})$, we obtain aR^*c, as required. If $b \in C(\{a, b, c\})$, since aR^*b, the weak axiom of revealed preference implies $a \in C(\{a, b, c\})$. If $c \in C(\{a, b, c\})$, since bR^*c the weak axiom of revealed preference implies $b \in C(\{a, b, c\})$, so that we are in the previous case. Thus, $a \in C(\{a, b, c\})$ and aR^*c as required.

To show that it rationalizes the choice structure we have to prove $C(B) = C^*(B, R^*)$ for all $B \in \mathscr{B}$. Let $B \in \mathscr{B}$ such that $a \in C(B)$. Then $\forall b \in B$ it holds aR^*b, so this means $a \in C(B, R^*)$. Thus, $C(B) \subseteq C(B, R^*)$. Suppose now $a \in C(B, R^*)$. This means aR^*b for all $b \in B$. Then, for every b there must exist a set $B_b \in \mathcal{B}$ such that $a, b \in B_b$ and $a \in C(B_b)$. As $C(B) \neq \varnothing$ and that holds for every $b, a \in C(B)$. Thus, $C(B, R^*) \subseteq C(B)$.

It is easy to establish uniqueness. Since B includes all subsets of X formed by two elements, the choice structure determines unambiguously the relation between every pair of alternatives. $\square$

We have established when a preference relation can be interpreted as a choice structure satisfying the weak axiom of revealed preference and when a choice structure defines a rational preference relation.

Chapter 2

Arrow's impossibility Theorem

Arrow's Theorem is the cornerstone in Social Choice Theory and also in Political Science. It says, essentially, that any procedure to obtain a preference relation out of the preferences of a population or constituency such that it respects several minimal conditions, is impossible.

Kenneth J. Arrow formulated his impossibility Theorem in his book Social Choice and Individual Values (1951). Arrow got the Nobel Prize in Economic Science in 1972 (joint with John R. Hicks) "for their pioneering contributions to general economic equilibrium theory and welfare theory".

When he first showed the general impossibility of a social welfare function this result was seen with skepticism, but soon it was considered very important as it is. To set Arrow's framework and prove his theorem we will mainly follow Gaertner (2001) [9] and Gaertner (2009) [8].

2.1 Social Aggregation

Our aim in the following two chapters is to prove whether it exists or not a way to choose a social state based on what individuals of such society prefer. We will ask this way of choosing to satisfy certain properties of justice and discuss what do they mean. Let $X = \{a, b, c, \ldots\}$ be the set of all possible mutually exclusive alternatives or social states and $N = \{1, 2, \ldots, n\}$ the set of all individuals or voters. Notice that the set of alternatives may be infinite while the set of individuals will be always a finite one. We may ask the set of individuals to be infinite when treating issues in an intergenerational view for example, but this is not the case. See, for instance, Zame (2007) [28] and Lauwers (2012) [13]. Let us suppose each individual $i \in N$ has its own preference relation on X and denote it by R^i. Without any index, R refers to society's preferences and we will call it *social preference relation.*

The way to aggregate preferences will be a function, so we have to decide in which sets the domain and the image are defined. Let $\mathcal{A}$ denote the set of preference relations on X, and $\mathcal{B}$ denote the set of preference relations on X which are reflexive and complete, $\mathcal{C}$ denote the set of preference relations which are reflexive, complete and acyclical on X, $\mathcal{D}$ the set of reflexive, complete and quasi-transitive preference relations on X and $\mathcal{E}$ denote the set of preference orderings. Obviously, it holds $\mathcal{E} \subseteq \mathcal{D} \subseteq \mathcal{C} \subseteq \mathcal{B} \subseteq \mathcal{A}$. Moreover $\mathcal{A}', \mathcal{B}', \mathcal{C}', \mathcal{D}', \mathcal{E}'$ will denote subsets of $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}, \mathcal{E}$ respectively satisfying certain conditions. $\mathcal{A}'^n$ will denote the Cartesian product $\mathcal{A}' \times \ldots \times \mathcal{A}'$, n times. An element of $\mathcal{A}'^n$ is an n-tuple $(R^1, \ldots, R^n)$ and we will call it a *profile* of preference relations.

A *collective choice rule* is a mapping f from $\mathcal{A}'^n$ to $\mathcal{A}$. A *social welfare function* is a mapping f from $\mathcal{E}'^n$ to $\mathcal{E}$. A *social welfare function for quasi-transitive preferences (QT)* is a mapping from $\mathcal{D}'^n$ to $\mathcal{E}$. A *social decision function of type QT* is a mapping from $\mathcal{E}'^n$ to $\mathcal{D}$. A *social decision function* is a mapping from $\mathcal{E}'^n$ to $\mathcal{C}$.

Notice that a social decision function is a collective choice rule such that a choice function is generated over the set of alternatives. In this project only social welfare functions will be studied widely, but other types must be explained since they may be a solution for some of our problems. On Gaertner (2001) [9] can be found an extended study about which role do domains play in social aggregation.

Before going deep in this issue, some example shall be pointed out in order to understand what we mean by social aggregation. These examples are taken from Gaertner (2009) [8] and show how different ways of choosing may lead to different results. We abstain from clarifying which type of function they are as it may be quite hard for an introduction. Some of them will be deeper studied later on.

Let us suppose a society with only three individuals ($|N| = 3$) and a set of four alternatives: $X = \{a, b, c, d\}$. Let us suppose that every individual is able to give a value to every alternative. It could be, for instance, how can they distribute a cake. Suppose the alternatives are set in the following way:

$$a = \left(\frac{1}{2}, \frac{1}{2}, 0\right) \qquad b = \left(\frac{1}{2}, 0, \frac{1}{2}\right) \qquad c = \left(0, \frac{1}{2}, \frac{1}{2}\right) \qquad d = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).$$

Each individual only takes care about how much does he or she win, so the preferences of each individual are the following:

1	2	3
a, b	a, c	b, c
d	d	d
c	b	a

Alternatives on the same level are supposed to be indifferent. In this way for individual 1 we have, aI^1b-, aP^1d, dP^1c and all combinations we can get using transitivity and completeness. Now, we shall introduce various choice rules:

Example 2.1 (Simple majority rule). *The reader may be familiar with this kind of aggregation. It consists on counting votes "for" and "against" between two alternatives. Let us call $N(aPb)$ the number of individuals that prefer alternative a rather than alternative b. Formally, we can define the simple majority rule by $aRb \iff N(aPb) \geq N(bPa)$. In this case, using the simple majority rule, we would get that a, b and c are indifferent to each other and each of them is preferred to d.*

Example 2.2 (Absolute majority rule). *In this case, an alternative is preferred to another if it has more than half of the votes. This means, $aPb \iff N(aPb) > \frac{1}{2}|N|$. This would lead us aPd, bPd and cPd, and indifference between a, b and c, as in the simple majority rule.*

Example 2.3 (Borda rule). *The Borda rule ranks the alternatives and weights every rank. Thus, depending on the weight assigned to every rank, the result may be one or* *another. There is mainly a way to weight every rank, and in this case it ranks 2.5 to the first alternatives, 1 to the second and 0 to the third. To understand how the Borda rule works and its variations see Gaertner (2009)[8].*

Example 2.4 **(Utilitarian rule).** *The utilitarian rule consists in maximizing the sum of utilities (or expected utilities. We can suppose that the utility of* $\frac{1}{2}$ *cake is* $\frac{1}{2}$, $\frac{1}{3}$ *for* $\frac{1}{3}$ *cake and* 0 *for no cake. So, utilities of each alternatives are the following:*

$$\textit{(i)}\ \ u(a) = \tfrac{1}{2} + \tfrac{1}{2} + 0 = 1,$$

$$\textit{(ii)}\ \ u(b) = \tfrac{1}{2} + 0 + \tfrac{1}{2} = 1,$$

$$\textit{(iii)}\ \ u(c) = 0 + \tfrac{1}{2} + \tfrac{1}{2} = 1,$$

$$\textit{(iv)}\ \ u(d) = \tfrac{1}{3} + \tfrac{1}{3} + \tfrac{1}{3} = 1.$$

So all alternatives are indifferent in the utilitarian view.

Example 2.5 **(Rawls' approach).** *Rawls stood up for favouring the worst-off individuals in society. Note that for alternatives a, b and c, the worst-off person always gets* 0*, so they would be indifferent to each other. But in alternative d the worst-off individual gets* $\frac{1}{3}$*, so if we ranked any of the first three alternatives against d, d would be chosen.*

Example 2.6 **(Maximize the product of utilities).** *In such a case, d would be the only winner since:*

$$\textit{(i)}\ \ u(a) = \tfrac{1}{2} \cdot \tfrac{1}{2} \cdot 0 = 0,$$

$$\textit{(ii)}\ \ u(b) = \tfrac{1}{2} \cdot 0 \cdot \tfrac{1}{2} = 0,$$

$$\textit{(iii)}\ \ u(c) = 0 \cdot \tfrac{1}{2} \cdot \tfrac{1}{2} = 0,$$

$$\textit{(iv)}\ \ u(d) = \tfrac{1}{3} \cdot \tfrac{1}{3} \cdot \tfrac{1}{3} = \tfrac{1}{27}.$$

Notice that not all of them must be social welfare functions. If we had chosen three alternatives such that aP^1bP^1c, bP^2cP^2a and cP^3aP^3b, we would see a cyclical relation $aPbPcPa$.

2.2 Arrow's impossibility theorem

As said before, the aim of this chapter is to show whether it exists or not a function f giving a preference ordering $R := f(R^1, \dots, R^n)$ that meets certain conditions. We will name this kind of function f a *social welfare function.* Let $\mathcal{E}$ be the set of all possible rational relations in X, that is $\mathcal{E} = \{R \subseteq X \times X \mid R \text{ is transitive and complete}\}$. Arrow, in his book *Social Choice and Individual Values* (1951), considered four minimal conditions that every social welfare function should hold:

Condition **(*U* – Unrestricted Domain).** *The domain of* f *includes all possible combinations of individual preference relations on* X*, which means the domain of* f *is* $\mathcal{E}^n$*.*

Condition **(*WP* – Weak Pareto Principle).** *For any pair of alternatives* $a, b \in X$ *if, for every* $i \in N$ *it holds* aP^ib*, then* aPb*.*

Condition (*IIA* − Independence of Irrelevant Alternatives). *Given two profiles of individual orderings* $(R_1^1, \ldots, R_1^n)$ *and* $(R_2^1, \ldots, R_2^n)$ *if for some* $a, b \in X$ *and* $\forall i \in N$ *it holds* $aR_1^i b \longleftrightarrow aR_2^i b \ \wedge \ bR_1^i a \longleftrightarrow bR_2^i a$, *then it holds* $aR1b \longleftrightarrow aR_2 b \wedge bR_1 a \longleftrightarrow bR_2 a$.

Condition (*D* − Non-dictatorship). *There is no individual* $k \in N$ *such that for all profiles in the domain of* f *and* $\forall\ a, b \in X$, *if* $aP^k b$, *then* aPb. *If there exists such individual, we will call him a dictator.*

Condition U means that every preference ordering should be taken into consideration. Condition WP demands that if everyone strictly prefers a to b, then the same should hold also for society's preference. Condition IIA requires the social welfare function taking into consideration only individuals' preferences of a and b when comparing this alternatives and nothing more. Individuals' preferences between a and a third alternative c should not count. And finally, condition D refers to the issue that there should be no dictator. A dictator is defined as an individual whose preferences must become society's preferences for all pairs of alternatives.

Arrow stated that there was no social welfare function satisfying these conditions. They have been fully discussed and loosen some of them may change the results of the theorem. See Sen (1970) [21] for a more accurate discussion. Before formulating the impossibility theorem, let us introduce two definitions and a lemma that will be necessary for the proof of the theorem.

Definition 2.1. *A set of individuals* $V \subseteq N$ *is* almost decisive *for some* $a \in X$ *against some* $b \in X$ *when* $(\forall i \in V\ aP^i b\ \ \wedge \forall i \notin V\ bP^i a) \longrightarrow aPb$.

Definition 2.2. *A set of individuals* $V \subseteq N$ *is* decisive *for some* $a \in X$ *against some* $b \in X$ *when* $(\forall i \in V\ aP^i b) \longrightarrow aPb$.

Let us now consider that V has a single element $V = \{k\}$. We say that an individual k is almost decisive (resp. decisive) if the set formed by only this individual is almost decisive (resp. decisive). We will denote that an individual $k \in N$ is almost decisive for a against b by $D(a, b)$ and that an individual $k \in N$ is decisive for a against b by $\overline{D}(a, b)$. Notice that $\overline{D}(a, b)$ implies $D(a, b)$.

Lemma 2.1. *Let us consider a finite set of individuals* N *and let* X *be the set of alternatives with* $|X| \geq 3$. *If there is some individual* $k \in N$ *who is almost decisive for some ordered pair of alternatives* (a, b) *and there is a social welfare function that satisfies conditions U, WP and IIA, then this individual* k *is a dictator.*

Proof. Let us assume that individual $k \in N$ is almost decisive for some a against some b. Let c be another alternative. We will show that, given such hypothesis, k is decisive for every ordered pair from the triplet of alternatives (a, b, c), which means that k is a dictator for any three alternatives that contain a and b. Then we will see that this can be extended to any finite number of alternatives.

Formally written, given a set of three alternatives $X = \{a, b, c\}$ and an individual $k \in N$, we suppose $(aP^k b \wedge \forall i \neq k\ bP^i a) \longrightarrow aPb$ and $(bP^k a \wedge \forall i \neq k\ aP^i b) \longrightarrow bPa$. We want to prove the following implications:

(a) $aP^k c \longrightarrow aPc$,

(b) $bP^k c \longrightarrow bPc$,

(c) $cP^k a \longrightarrow cPa$,

(d) $cP^k b \longrightarrow cPb$,

(e) $aP^k b \longrightarrow aPb$,

(f) $bP^k a \longrightarrow bPa$.

Notice that item (a) is similar to item (b), item (c) is similar to item (d) and item (e) is similar to item (f). So we will show only items (a), (c) and (e).

Let us suppose first $aP^k c$. By condition U, we can consider any preference profile. Then consider the following profile:

$$aP^k b \ \wedge \ bP^k c$$
$$bP^i a \ \wedge \ bP^i c, \forall i \neq k.$$

Note that the preference between a and c remain unspecified for all individuals except k. As k is almost decisive for x against y, we have aPb. bPc is due to condition WP. By transitivity we obtain aPc. Condition IIA says that the social relation between two alternatives must only be consequence of the relation of individuals between these two alternatives and no other alternative has a role to play. But if we focus on the profile above, we only assumed $aP^k c$; the relation between a and c of all other individuals could have been any. This means that aPc is consequence of individual k alone whichever profile we consider. This means $aP^k c \rightarrow aPc$.
Similar arguments will be used to prove (c) and (e). Let us now suppose $cP^k a$ and consider the following profile:

$$cP^k a \wedge aP^k b$$
$$cP^i a \wedge bP^i a \ \forall i \neq k.$$

Then we have cPa and aPb, and though, cPb. As relation P^k is transitive, it is the unique that specified the relation between c and b, so we can use the same argument as above and obtain $cP^k b \rightarrow cPb$ as wanted.
Let us finally suppose $aP^k b$ and consider the following profile:

$$aP^k c \wedge cP^k b$$
$$cP^i a \wedge cP^i b \ \forall i \neq k.$$

By condition WP we have cPb. Since above we have seen $aP^k c \rightarrow aPc$, we fathom aPb by transitivity. As we only specified the relation between a and b aof individual k, we conclude $aP^k b \rightarrow aPb$ by IIA. Now let us show that this can be extended to any number of alternatives. We suppose that individual k is almost decisive for a against b. If we consider the triple (a, b, c), we have shown that k is decisive for a against c, therefore, it is also almost decisive. So we now consider the triple (a, c, d), which implies that k is decisive for c against d. This means that if k is almost decisive for a against b, then it is decisive for any pair (c, d) □

Now we are in the position to state and prove the impossibility theorem.

Theorem 2.1 (Arrow's impossibility Theorem). *For a finite number of individuals, if $|X| \geq 3$, then there is no social welfare function $f : \mathcal{E}^n \longrightarrow \mathcal{E}$ that simultaneously satisfies conditions U, WP, IIA and D.*

Proof. The procedure will be to show that there always exists an almost decisive individual for every pair of alternatives.
For every pair of alternatives (a, b) there exists at least one almost decisive set, N.
Let us choose one of the smallest sets among all almost decisive sets for some pair of alternatives and let us call it V. This means that if V' is an almost decisive for any pair of alternatives, then either $V \subseteq V'$ or V and V' are not comparable. Note that V may not be unique. Let us suppose V contains at least two individuals and is almost decisive for a against b. Let us now consider V as the disjoint union $V = V_1 \cup V_2$, where V_1 contains a single individual and V_2 the rest from V. We introduce now a third alternative c and postulate the following profile (we can do it due to condition U):

$$\text{For } k \in V_1\text{:} \quad aP^k b \wedge bP^k c$$
$$\text{For all } i \in V_2\text{:} \quad cP^i a \wedge aP^i b$$
$$\text{For all } j \in N \setminus V\text{:} \quad bP^j c \wedge cP^j a$$

As V is almost decisive for a against b, we obtain aPb. Besides, it cannot hold cPb. If it were the case, V_2 would be an almost decisive set for c against b (and though for every pair of alternatives) and it would lead us to contradiction with the fact that V is a smallest set. Thus bRc. Since R is a transitive relation, we obtain aRc. But then V_1 would be almost decisive for a against c, also in contradiction with V being smallest. Then, V must contain a single element and, because of the previous lemma, it must be a dictator. $\square$

2.3 Another approach to Arrow's theorem

In this section we will see a manner of reformulating Arrow's theorem. Although we may change some suppositions, the result is indeed the same as before. We will redefine the whole Arrowian setup in terms of utility functions, that are defined in Euclidean spaces. Let us consider alternatives as points in $\mathbb{R}^d$ for some $d \in \mathbb{N}$. Thus $X \subseteq \mathbb{R}^d$.

As before, we suppose that preference relations $R^i, i \in N$ are complete and transitive, but we now add continuity. In Chapter 1 we saw that if a binary relation defined in $\mathbb{R}^d$ is transitive, complete and continuous, then it can be represented by an utility function u^i, for all $i \in N$. So, instead of working with binary relations, we will work with utility functions and show that there does not exist a social welfare function $f(u^1, \dots, u^n)$ that follows the same previous conditions.

As before, we want to obtain an ordering, so our function will go from $\mathcal{U} \times \cdots \times \mathcal{U}$ n-times to $\mathcal{E}$. We will call this function a social welfare function although it does not adjust to the definition given in Section 2.1. This is because utility functions come from preference orderings and one may see that the results do not change. R_U is the ordering generated by f when the utility profile is $U = (u^1, \dots, u^n)$ – when needed, we will put $U(a) = (u^1(a), \dots, u^n(a))$ for $a \in X$ –. From now on we will denote individuals' utility by a lowercase u and profiles of utilities by capital letters U. Let us redefine such conditions in terms of utility:

Condition (*U* - Unrestricted Domain). *The domain of f is $\mathcal{U}^n$.*

Condition (*WP* - Weak Pareto Principle). *For any pair of alternatives $a, b \in X$ if, for every $i \in N$ it holds $u^i(a) > u^i(b)$, then $aP_U b$.*

Condition (*IIA* - Independence of Irrelevant Alternatives). *Given two profiles of individual utility functions $U_1 = (u_1^1, \dots, u_1^n)$ and $U_2 = (u_2^1, \dots, u_2^n)$ and for all $a, b \in X$,* *if* $\forall\, i \in N$ *it holds* $u_1^i(a) = u_2^i(a)$ *and* $u_1^i(b) = u_2^i(b)$ *i.e.,* $U_1(a) = U_2(a) \wedge U_1(b) = U_2(b)$*, then it holds* $aR_{U_1}b \iff aR_{U_2}b$ *and* $bR_{U_1}a \iff bR_{U_2}a$*.*

Condition (*D* - Non-dictatorship). *There is no individual* $k \in N$ *such that for all profiles in the domain of* f *and* $\forall a, b \in X$*, if* $u^k(a) > u^k(b)$*, then* aP_Ub*.*

As said before, this result differs slightly from the one on the previous chapter and another condition is needed. We now introduce a condition called Pareto Indifference, which requires that if all members are indifferent between a pair of alternatives the same should hold for society's preference over this pair.

Condition (*PI* - Pareto Indifference). $\forall\, a, b \in X$ *and* $\forall\, U \in \mathcal{U}^n$ *it holds* $U(a) = U(b) \Rightarrow aI_Ub$*.*

We also define another condition called strong neutrality. It is related to conditions *IIA* and *PI* imposed on f, since these imply that f also holds strong neutrality.

Condition (*SN* - Strong Neutrality). *Let* $a, b, c, d \in X$ *be alternatives and* U_1*,* U_2 *two profiles of utility functions. If* $U_1(a) = U_2(c)$ *and* $U_1(b) = U_2(d)$*, then* $aR_{U_1}b \iff cR_{U_2}d$*.*

Strong neutrality requires that the social evaluation functional f ignore all non-utility information with respect to the alternatives, such as names or rights or claims or procedural aspects. The only information that counts is the vector of individual utilities associated with any social alternative.

Now we give some relationships between these conditions.

Lemma 2.2. *Let* f *be a social welfare function satisfying conditions U, PI and IIA. Let* $a, b, c \in X$ *be alternatives with* $b \neq c$ *and* U_1 *and* U_2 *be two profiles of utility functions such that* $U_1(a) = U_2(a)$ *and* $U_1(b) = U_2(c)$*. Then* $aR_{U_1}b$ *implies* $aR_{U_2}c$ *and* $bR_{U_1}a$ *implies* $cR_{U_2}a$*. We will denote this as* $\{a, b\}N\{a, c\}$*.*

Proof. By condition U we can consider another profile of alternatives U_3 that satisfies $U_3(a) = U_1(a)$ and $U_3(b) = U_3(c) = U_1(b)$. By condition *PI*, we have $bI_{U_3}c$. Let us now suppose $aR_{U_1}b$. Since U_1 and U_3 give the same value to a and so they do to b, it holds $aR_{U_3}b$ by *IIA*. By transitivity, we have $aR_{U_3}c$. As before, we have $aR_{U_2}c$ by *IIA*. Thus, $aR_{U_1}b$ implies $aR_{U_2}c$. An analogous argument is used for $bR_{U_1}a$ implying $cR_{U_1}a$. □

Proposition 2.1. *Let* f *be a social welfare function satisfying condition U. Then conditions IIA and PI hold iff condition SN does.*

Proof. Let us suppose the social welfare function satisfies *SN*. Considering $a = c$ and $b = d$, *IIA* holds. Setting as well $U_1 = U_2$, *PI* holds. Let us now suppose the social welfare function satisfies *IIA* and *PI*. Let $a, b, c, d \in X$ be alternatives and U_1, U_2 two profiles of utility functions such that $U_1(a) = U_2(c)$ and $U_1(b) = U_(d)$. We can consider a third profile of alternatives U_3 such that $U_3(a) = U_3(c) = U_1(a)$ and $U_3(b) = U_3(d) = U_1(b)$. Let us suppose $aR_{U_1}b$ and we want to see $cR_{U_2}d$. By *IIA*, we have $aR_{U_3}b$. By *PI* we have $cI_{U_3}a$ and $bI_{U_3}d$. By transitivity it holds $cR_{U_3}d$ and, again by *IIA*, we have $cR_{U_2}d$ as wanted. □

Proposition 2.2. *If* f *satisfies conditions U and SN, then there is an ordering* R^* *defined on* $\mathbb{R}^n$*. Besides, if* f *fulfills conditions U, IIA and PI, the ordering* R^* *inherits these properties.*

Proof. We define R^* as follows: $\forall v, w \in \mathbb{R}^n$, vR^*w iff there exist $a, b \in X$ and $U \in \mathcal{U}^n$ such that $v = U(a)$, $w = U(b)$ and aRb. Since R is complete, R^* is complete. To establish transitivity, let us consider $v, w, z \in \mathbb{R}^n$ such that vRw and wRz. By definition, there exist $a, b, c \in X$ and $U \in \mathcal{U}^n$ such that $v = U(a)$, $w = U(b)$ and $z = U(c)$. By condition *SN*, aRb and bRc. As R is an ordering, aRc and thus vR^*z. D'Aspremont and Gevers (1977) [4] prove that R^* inherits these properties. □

Notice that utilities give a number to every alternative. There are infinite utility functions representing a preference ordering and we must, somehow, set a condition for them to be the same. We recall Proposition 1.1 and set it as a condition. Although it seems quite obvious in the framework given until now, it may change in Chapter 3.

Condition (*ON* - Ordinal Measure, Non-comparable Utilities). $\forall\, U_1, U_2 \in \mathcal{U}^n$, *let, for every* $i \in N$, ϕ^i *be a strictly increasing transformation. If* $\forall\, i \in N$ *and* $\forall\, a \in X$ *it holds* $u_2^i(a) = \phi^i(u_1^i(a))$, *then* $R_{U_1} = R_{U_2}$. *Note that* $u_1^i(\cdot)$ *and* $u_2^i(\cdot)$ *are the utility components of profiles* U_1 *and* U_2 *respectively.*

Now we are in the position to state and prove the corresponding version of Arrow's Theorem.

Theorem 2.2. *For a finite number of individuals and at least three alternatives,* $|X| \geq 3$, *then there is no social welfare function* $f : \mathcal{U}^n \longrightarrow \mathcal{E}$ *that simultaneously satisfies conditions U, WP, IIA, D, PI and ON.*

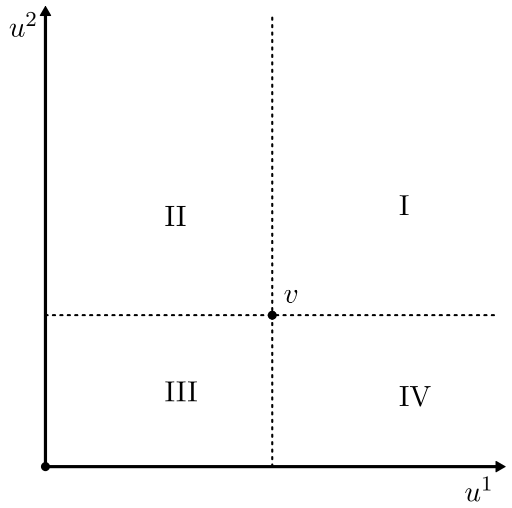
Proof. On the previous Proposition 2.2 we saw that it is the same to deal with R_U than with R^*. We will prove the theorem for two individuals, i.e. $|N| = 2$. Let us consider $v \in \mathbb{R}^2$, $v = (v^1, v^2)$ as a point of reference. Our aim is to see that $\forall w \in \mathbb{R}^2$ it happens $vR^*w \iff v^1 > w^1$ – and so individual 1 is a dictator – or $vR^*w \iff v^2 > w^2$ – and so individual 2 is a dictator–.

Since we are in $\mathbb{R}^2$, we will draw it and divide the plane into four regions as shown in Figure 2.1. For the moment, we do not consider the boundaries between the regions but only the interior of the four regions.

Let $w \in \mathbb{R}^2$ be a point of the first region (Region I). Since $w^1 > v^1$ and $w^2 > v^2$, it will be wR^*v. Similar reasoning happens with the third region (Region III), but in this case it is vR^*w. Regions II and IV require a bit more effort. If $w, \bar{w} \in \mathbb{R}^2$, w in the second region and $\bar{w}$ in the fourth, we will prove that either $wP^*v \wedge vP^*\bar{w}$ or $vP^*w \wedge \bar{w}P^*v$.

Let us show first that all points in region II must be ranked equally against v (and analogously all points in region IV must be ranked equally against v). Let $z, \bar{z}$ be points in II and let us assume zPv. As we are only using ordinal information, each of the two persons is totally free to map his or her utility scale into another one by a strictly increasing transformation without changing the rankings of the two persons. So we take one for each that map $z = (z_1, z_2)$ into $\bar{z} = (\bar{z_1}, \bar{z_2})$ and v into v. This means $\bar{z}Pv$. If it were vPz instead of zPv, the same argument would hold. In case zIv, this would lead to contradiction as, if we took $z, \bar{z}$ in region II such that $z_1 < \bar{z_1}$ and $z_2 < \bar{z_2}$, both indifferent to v, and used the argument above, it would happen $zI\bar{z}$ and lead us to contradiction. Therefore, all points in region II are ranked equally against v (obviously they are not ranked equally against each other) and they are either strictly preferred or strictly worse. Same happens with points in region IV and to prove it one must do an analogous reasoning.

Figure 2.1



We now wish to show that points in region II are ranked oppositely to points in region IV. We will use again the argument that strictly increasing transformations of utility functions do not change the informational content. Let $w = (w^1, w^2)$ be a point in region II and suppose wPv. Consider the transformation of utility scale $f(u) = u + (v^1 - w^1)$ for person 1. This means every point is moved w^1 to the right. Consider the transformation of utility scale $g(u) = u + (v^2 - w^2)$ for person 2. This means every point is moved down w^2. If we apply these transformations to the points we had, w moves to $(w^1 + v^1 - w^1, w^2 + v^2 - w^2) = (v^1, v^2) = v$ and v moves to $(2v^1 - w^1, 2v^2 - w^2) := \overline{w}$. Note that $\overline{w}$ is in the forth region, so these transformations map w into v and v into $\overline{w}$. Since, by assumption, wPv, it holds $vP\overline{w}$. If it were vPw, a similar argument would hold. This means points in regions II and IV are ranked against v contrary to each other. To recap, if a point in region II is preferred to v, then all points in region II are preferred to v and v is preferred to all points in region IV.

All that remains is dealing with the boundaries. If two regions are ranked equally against v, then its boundary is ranked the same way as them. Suppose region II is preferred to v and let w be a point on the boundary between regions I and II. There always exists a point $\overline{w}$ in region II such that $\overline{w}^1 < w^1$ and $\overline{w}^2 < w^2$. Then $wP\overline{w}$. As $\overline{w}Pv$, we have wPv. Note that we have said nothing about the other boundaries. In the following figures we will see they are not important for person 1 or 2 being a dictator.

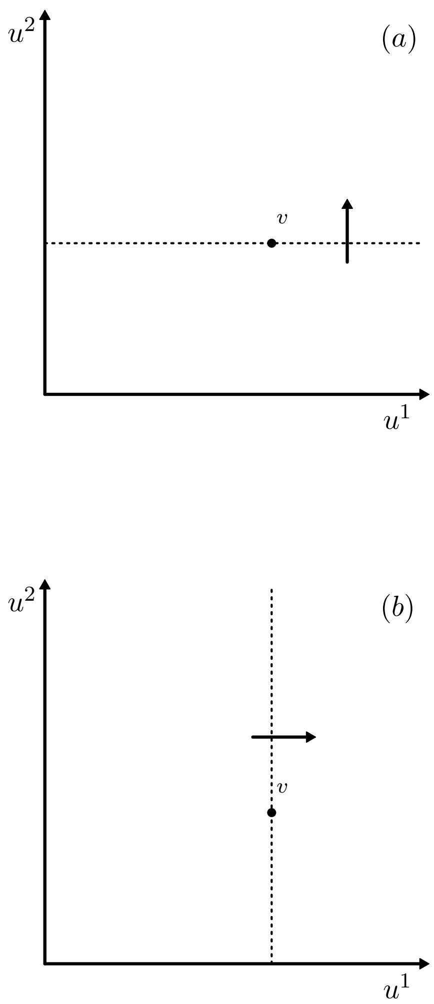
All we have seen can be summed up in two different cases (see Figure 2.2):

(a) Region II is preferred to v (and v is preferred to region IV, and thus person 2 is a dictator. In this case the boundaries between regions II and III, and regions I and IV have no matter in the condition of dictatorship as its points are indifferent to person 2.

(b) Region IV is preferred to v (and v is preferred to region II, and thus person 1 is a dictator. In this case the boundaries between regions II and I, and regions III and IV have no matter in the condition of dictatorship as its points are indifferent to person

1.

Figure 2.2



□

2.4 Escaping impossibilities

Up to now, all we have seen is that it is impossible to find a function that aggregates individuals' preferences and satisfies certain conditions. One may now ask if loosing hypothesis could be of use. Let us remember which hypothesis did we suppose. First we asked the preference relations to be rational. Then we asked f to be a social welfare function. And then, we asked f to satisfy conditions U, IIA, WP and D. Finally, condition ON was needed when we worked with utility functions. Sen (1977) [24] summarizes how can such hypothesis be dropped or lowered and what do we get in each case:

Majority rule under restricted domains

If we drop conditions related to the domain of the function f (we may either ask R not to be transitive, or drop U, or look for ways of aggregating that are not social welfare functions), the simple majority rule may be a function that satisfies some of our conditions. Sen and Pattanaik (1969) [20], Inada (1969) [11] and Gaertner (2001) [9] go deeper in this issue. Let us, for instance, restrict the preference ordering so that the domain is only shaped by single-peaked preference. Let $B(a, b, c)$ mean that alternative b is between a and c in a strong ordering S. Single peaked preferences are defined this way:

Definition 2.3. *A profile of individual preference relations* $(R^1, \ldots, R^n)$ *satisfies the condition of single-peakedness if there exists a strong ordering* S *such that for all* $i \in N$, aR^ib *and* $B(a, b, c)$ *imply* bP^ic.

Theorem 2.3. *Suppose the number of concerned voters is odd. The majority decision rule is a social welfare function for any number of alternatives if the individual orderings satisfy the property of single-peakedness over each triple of alternatives.*

Reverse dictatorship and null social welfare functions

We say that a social welfare function is a reverse dictatorship is $-f$ is a dictatorship. We say that it is null if all alternatives are indifferent to each other. Pareto conditions may seem the most obvious condition. It cannot be loosened in an intuitive way and, if we drop it, instead of getting a dictatorship, we may get either a dictatorship, a reverse dictatorship or a null social welfare function. Wilson (1972) [27] and Binmore (1976) [2] give proofs of when to get one or another. For a more detailed version of what they prove, see Ubeda (2004) [26].

The Borda Rule

The Borda Rule has been widely discussed and many interpretations and variations have come up. Sen (1977) [23] considers two variants of the Borda rule. The broad Borda Rule (*BBR*) checks the rank of all alternatives in the set X in a person's ordering and the scores are given as $(n + 1 - j)$. The narrow Borda Rule (*NBR*) is a choice function that gives the Borda scores according to rankings of the set S for a choice over S. Sen also sets a condition named α that gives certain consistency to the choice rule.

Definition 2.4. *A Choice Rule satisfies condition* α *if for all* $S \subseteq T \subseteq X$, *if an alternative* a *belongs to* S *and to* $C(T)$, *then it also belongs to* $C(S)$.

The *NBR* satisfies condition *IIA* but it does not satisfy property α. However, the *BBR* may not satisiy independence of irrelevant alternatives but it satisfies condition α, so that it generates a rational preference relation.

Other impossibility theorems

To finish, let us point out that Arrow's impossibility theorem is not the only impossibility theorem in social choice theory. There are others that have provoked disbelief like the Gibbard-Satterthwaite theorem or the Sen's impossibility theorem of a Paretian liberal (1970). To formulate the first one would go way out the aims of this work, but the second one can be formulated. To do so we have first to define condition of liberalism. Informally, we could say that a function f satisfies liberalism if for each individual i, there is at least one pair of personal alternatives $(a, b) \in X$ such that the individual is decisive both ways in the social choice process. Liberalism will be denoted by L.

Theorem 2.4. *There is no social decision functions that satisfies conditions U, WP and L.*

Chapter 3

Distributive justice

Any society has different forms to assign benefits and duties, and it is of the extreme importance to see how this is done across the members of the society. It is the result of some economic, social or political frameworks. They are changing over time and are the result of historical processes. They affect people's lives. This is the topic of what is known as distributive justice, and we will see it through the eyes of formal reasoning (mathematical in nature), which implies also a moral guidance for the assessment of political processes or structures, giving the distribution of benefits and duties.

This chapter delves into the use of utility functions and its utility information to find social welfare functions. Last chapter we saw a version of Arrow's theorem in terms of utility functions. Remember it was imperative not to compare utilities between individuals neither of utility levels nor gains or losses. All that mattered was the position an alternative had against another, so that no mattered which utility function we used as long as it represented the preference ordering of every individual. This chapter shows, though, how being able to compare utilities may change the outlook of the problem. We will mainly follow Blackorby et al.(1984) [3], D'Aspremont and Gevers (1977) [4] and Hammond (1976) [10]. We use their papers and make some adaptations to our framework.

3.1 Informational Structure and Axioms

Statements like "for individual i, a is preferred to b" were the ones we made in the previous chapter, so we supposed every individual was able to compare alternatives in twos and our aim was to find a function that gave us a preference ordering for these alternatives. However, in this chapter, we want to compare between individuals, so we want to make statements like "it is better alternative a for individual i than alternative b is for individual j". Therefore, to consider a preference ordering for every individual is not sufficient and, instead, we will consider a preference ordering on the set $X \times N$ and want to find a mapping f from the set of possible orderings on $X \times N$ to the set of orderings on alternatives X, something called $\mathcal{E}$ in the previous chapter.

To differentiate both relations, we will call $\widetilde{R}$ to the relation defined on $X \times N$ and R to the relation defined on X. Although not always needed, we will suppose $\widetilde{R}$ is continuous, so that it can be represented by an utility function. We will call $\mathcal{R}$ the set of all possible relations defined on $X \times N$. Note that this relation is a generalization from the one on the previous chapter, as $aR^ib = (a,i)\widetilde{R}(b,i)$. Also, we may want to refer to the fact that, given an alternative a, individual i gets more benefit that individual j. We will denote this by $i\widetilde{R}(a)j := (a,i)\widetilde{R}(a,j)$.

When working with utility functions function f must be redefined. Let $\mathcal{O}$ be the set of all possible utility functions representing a preference relation on $X \times N$. A social welfare function f can be interpreted both as a mapping from $\mathcal{O}$ to the set of all possible preference orderings on X, $\mathcal{E}$ and as a mapping from $\mathcal{U}^n$ to $\mathcal{E}$. Both ways $U = (u(\cdot,1),\ldots,u(\cdot,n))$ will denote a profile of utility functions.

For the next few pages our aim will be to define a whole setup that will allow us to find certain welfare social functions. As said, from now on, the preference ordering $\widetilde{R}$ will be a transitive, complete and continuous binary relation defined on $X \times N$.

Let us first define which informational requirements may be asked. Informational requirements denote whether it is possible or not to compare between individuals, how these comparisons can be made and if we can use cardinal informal or only ordinal. We will define them in terms of utility functions since some of them can only be defined this way.

Condition (*ON* - Ordinal Measure, Non-comparable Utilities). *Let U_1, U_2 be two profiles of utility functions. Let, for every i, ϕ^i be a strictly increasing transformation. If $\forall i \in N$ and $\forall a \in X$ it holds $u_2(a,i) = \phi^i(u_1(a,i))$, then $R_{U_1} = R_{U_2}$. Note that $u_1(\cdot,i)$ and $u_2(\cdot,i)$ are the utility components of profiles U_1 and U_2 respectively.*

Condition *ON* is the same it was in Chapter 2, though adapting the notation. It means that no matter which number an individual ascribes to an alternative, provided that its ordinal position in comparison to other alternatives does not change. This means we cannot compare utilities and that all that matters is the ordinal position of an alternative. When we permit comparison between utilities we may ask the following informational requirement.

Condition (*OC* - Ordinal Measure, Fully-comparable utilities). *Let U_1, U_2 be two profiles of utility functions. Let ϕ be a strictly increasing transformation. If $\forall i \in N$ and $\forall a \in X$ it holds $u_2(a,i) = \phi(u_1(a,i))$, then $R_{U_1} = R_{U_2}$.*

In this case, as we want to compare, we have to use an utility transformation such that keeps orderings in the same way as before. No matter which number one ascribes to an alternative provided that its ordinal position in comparison to other alternatives and to other individuals does not change. Note that in both cases we do not use cardinal information, only ordinal. This means it is not necessary to use utility functions, it is enough with preference orderings. Note that, when we suppose $\widetilde{R}$ to be defined in $X \times N$, we are already permitting comparisons between individuals, so all we are saying in *OC* is that utility transformations are allowed. In case *ON*, we suppose there are n preference orderings defined on X and we are allowing each of them to use utility transformations if needed.

When we want to use cardinal measure, we have to establish an origin point and a scale. In case every individual has its own origin point and scale we will not be able to compare utilities.

Condition (*CN* - Cardinal Measure, Non-comparable utilities). *Let U_1, U_2 be two profiles of utility functions. Let there be $2n$ numbers $\alpha^1,\ldots,\alpha^n$ and $\beta^1 > 0,\ldots,\beta^n > 0$. If $\forall i \in N$ and $\forall a \in X$ it holds $u_2(a,i) = \alpha^i + \beta^i \cdot u_1(a,i)$, then $R_{U_1} = R_{U_2}$.*

When we require all α^i to be the same and all β^i to be the same, we will be able to compare between individuals.

Condition (*CC* - Cardinal Measure, Fully-comparable utilities). *Let U_1, U_2 be two profiles of utility functions. Let there be α and $\beta > 0$. If $\forall i \in N$ and $\forall a \in X$ it holds $u_2(a,i) = \alpha + \beta \cdot u_1(a,i)$, then $R_{U_1} = R_{U_2}$.*

Between *CN* and *CC* we shall single out two interesting cases. One may ask only to be either all α^i the same or all β^i the same. In the first case, they all have the same origin point but different scale measure, so that we can compare utility levels. Otherwise, when β^i are all the same, they have the same scale measure but different origin point. This allows us to compare gains and loses but no utility levels.

Condition (*CUL* - Cardinal Measure, Utility-Levels-comparable utilities). *Let U_1, U_2 be two profiles of utility functions. Let there be $n+1$ numbers α and $\beta^1 > 0, \ldots, \beta^n > 0$. If $\forall i \in N$ and $\forall a \in X$ it holds $u_2(a,i) = \alpha + \beta^i \cdot u_1(a,i)$, then $R_{U_1} = R_{U_2}$.*

Condition (*CIC* - Cardinal Measure, Increment-comparable utilities). *Let U_1, U_2 be two profiles of utility functions. Let there be $n+1$ numbers $\alpha^1, \ldots, \alpha^n$ and $\beta > 0$. If $\forall i \in N$ and $\forall a \in X$ it holds $u_2(a,i) = \alpha^i + \beta \cdot u_1(a,i)$, then $R_{U_1} = R_{U_2}$.*

The following table sums up all possible informational requirements.

	Non-comparable	Half-comparable	Full-comparable
Ordinal Measure	*ON*	...	*OC*
Cardinal Measure	*CN*	*CUL* and *CIC*	*CC*

Notice that, despite *OC* only allows us to compare utility levels it has been put as full-comparable. This is due to in ordinal informational requirement we cannot compare gains and losses since this would need the use of cardinal numbers. We have six different possible informational requirements. The following theorems show that, given our framework, some of them are equivalent.

Theorem 3.1. *A social welfare function f satisfying U satisfies CN if and only if it satisfies CUL.*

Proof. Taking $\alpha^1 = \cdots = \alpha^n = \alpha$, it holds directly that *CN* imply *CUL*.

We have only to see, then, that *CUL* implies *CN*. Let $U_0 \in \mathfrak{U}^n$ be a profile of utility functions. Let there be $2n$ numbers $\alpha^1, \ldots, \alpha^n$ and $\beta^1 > 0, \ldots, \beta^n > 0$. Suppose there exists a profile of utilities U_1 such that $\forall a \in X, \forall i \in N$, $u_1(a,i) = \alpha^i + \beta^i \cdot u_0(a,i)$. We must show that if f satisfies *CUL*, then $R_{U_0} = R_{U_1}$.

Consider a profiles of utilities U_2 such that $\forall i \in N$ and $\forall a \in X$ it is $u_2(a,i) = 1 + \bar{\beta}^i \cdot u_0(a,i)$, where $\bar{\beta}^i$ are such that there exists $\theta < \min_{j \in N} \alpha^j$ with:

$$\frac{1}{\bar{\beta}^i} = (\alpha^i - \theta)\frac{1}{\beta^i}$$

Because of *CUL*, we have $R_{U_2} = R_{U_0}$. Note that we can isolate $u_0(a,i)$ and write $u_0(a,i) =$

$\frac{1}{\beta^i}(u_2(a,i)-1)$. Then,

$$\begin{aligned} u_1(a,i) &= \alpha^i + \beta^i \cdot u_0(a,i) \\ &= \alpha^i + \beta^i \cdot \frac{1}{\beta^i}(u_2(a,i)-1) \\ &= \alpha^i + (\alpha^i - \theta)(u_2(a,i)-1) \\ &= \theta + (\alpha^i - \theta) \cdot u_2(a,i) \end{aligned}$$

This, by *CUL*, means $R_{U_1} = R_{U_2}$, therefore $R_{U_0} = R_{U_1}$. $\square$

When independence of irrelevant alternatives hold, as long as comparability between individuals is not allowed, it is the same using either ordinal or cardinal information. D'Aspremont and Gevers (1977) [4] give a proof of the following theorem.

Theorem 3.2. *A social welfare function satisfying IIA satisfies CN if and only if satisfies ON.*

So, by Theorems 3.1 and 3.2, our initial six possible informational requirements have turned into four. Condition *ON* is the most restrictive and was the one used in Chapter 2 in Arrow's framework. This chapter will study deeply informational requirements *OC* and *CIC*.

First, though, let us introduce some new conditions of justice. The conditions proposed by Arrow were minimal conditions of justice that every way of choosing shall satisfy. Now that we have changed Arrow's framework, we may introduce some new ones and strengthen some of them. As we will work with both, utility functions and binary relations, conditions will be defined in terms of both utility functions and binary reltions.

Condition (*A* - Anonymity). *In terms of preference orderings: Let $\widetilde{R}$ and $\widetilde{R}'$ be two orderings on $X \times N$ and σ a permutation on N such that $R' = \sigma(R)$. Then, $R' = R$.*
In terms of utility functions: Let σ be a permutation on the set of individuals N and $a \in X$. Let $U_1, U_2 \in \mathcal{U}^n$ be two profiles of utility functions. If $\forall i \in N$ and $\forall a \in X$ it holds $u_1(a,i) = u_2(a,\sigma(i))$, then $R_{U_1} = R_{U_2}$.

Anonymity requires that, in social decision making, every individual should count the same. Renumbering the individuals or changing their name-tags should not matter or, more technically, permutations among the set of voters should play no role in the social decision procedure. Notice that, if Anonymity holds, it implies that there is no dictator, so it is a strengthening of condition *D*.

Suppes (1966) [25] proposed another condition of justice. He said that if permuting individuals leaves everybody just as well off as before, then two social states are socially indifferent.

Condition (*S* - Suppes notion of justice). *In terms of preference orderings: Let $a, b \in X$. If there exists a permutation σ on N such that $\forall i \in N$ it holds $(a,i)\widetilde{I}(b,\sigma(i))$, then aIb.*
In terms of utility functions: Let $a, b \in X$. If there exists a permutation σ on N such that $\forall i \in N$ it holds $u(a,i) = u(b,\sigma(i))$, then aIb.

Let us now introduce another Pareto condition, stronger than the others defined before.

Condition (*SP* - Strict Pareto Principle). *In terms of preference orderings: Let $a, b \in X$ and $\widetilde{R}$ be an ordering in $X \times N$ such that $\forall i \in N$ it holds $(a,i)\widetilde{R}(b,i)$. Then, aRb. If, additionally, $\exists j \in N$ such that $(a,j)P(b,j)$, then aPb.*
In terms of utility functions: Let $a, b \in X$ and U be a profile of utility functions such that $\forall i \in N$ it holds $u(a,i) \geq u(b,i)$. Then, aRb. If, moreover, $\exists j \in N$ such that $u(a,j) > u(b,j)$, then aPb.

Note that *SP* is a stronger version of the weak Pareto principle and also of the Pareto Indifference principle. This means whenever *SP* is held, *WP* and *PI* also hold.

Condition *IIA* can be defined in the same way as in Chapter 2 since $aR^i b = (a,i)\widetilde{R}(b,i)$. Unrestricted domain can is also the same as before. Strong neutrality will be needed to show some results, so it must be redefined in terms of preference relations.

Condition (*SN* - Strong Neutrality). *Let $\widetilde{R}$ and $\widetilde{R}'$ be two preference orderings in $X \times N$ and $a, b, c, d \in X$. Suppose that $\forall i \in N$ it holds $(a,i)\widetilde{R}(b,i)$ and $(c,i)\widetilde{R}'(d,i)$. Then $aRb \iff cR'd$.*

We shall now introduce the notion of concerned and unconcerned voters. Concerned voters are those who are not indifferent for at least a pair of alternatives. Unconcerned voters are those who are indifferent between every two alternatives. Separability requirement eliminates all unconcerned individuals.

Condition (*SE* - Separability of unconcerned individuals). *Let U_1, U_2 be two profiles of utility functions. Let us consider $M \subset N$. If $\forall i \in M$ and $\forall a \in X$ it holds $u_1(a,i) = u_2(a,i)$ while $\forall j \notin M$ and $\forall a, b \in X$ it holds $u_1(a,j) = u_1(b,j)$ and $u_2(a,j) = u_2(b,j)$, then $R_{U_1} = R_{U_2}$.*

The set $N \setminus M$ is the set of unconcerned individuals. As *SE* will only be used in terms of utilities, it is only defined this way. Finally, two more conditions shall be defined. They respond to the necessity of favouring an individual or another when one of them is better off than the other.

Condition (*EQ* - Equity). *In terms of preference orderings: Let $a, b \in X$ be a pair of alternatives, $i, j \in N$ two individuals and $\widetilde{R}$ an ordering on $X \times N$. If $j\widetilde{P}(a)i$, $j\widetilde{P}(b)i$, $aP^i b$, $bP^j a$ and $\forall k \in N \setminus \{i,j\}$ $aI^k b$, then aPb.*
In terms of utility functions: Let U be a profile of utility functions. Let $a, b \in X$ be two alternatives and let $i, j \in N$ be two individuals. If $\forall h \in N \setminus \{i,j\}$ it holds $u(a,h) = u(b,h)$ and $u(b,i) < u(a,i) < u(a,j) < u(b,j)$, then $aP_U b$.

Condition (*INEQ* - Inequity). *Let $U \in \mathcal{U}^n$ be a profile of utility functions. Let $a, b \in X$ be two alternatives and let $i, j \in N$ be two individuals. If $\forall h \in N \setminus \{i,j\}$ it holds $u(a,h) = u(b,h)$ and $u(b,i) < u(a,i) < u(a,j) < u(b,j)$, then $bP_U a$.*

Clearly, Equity and Inequity can only be defined when interpersonal comparison is allowed. Equity favours the worst off individual while Inequity favours the better off one.

Under conditions *U* and *IIA*, conditions *A* and *S* are equivalent. This property will be of pretty use when later characterizing the Leximin Principle. The following lemma and propositions show this result.

Lemma 3.1. *Suppose $|X| \geq 3$ and f is a social welfare function satisfying U, IIA and S. Let $a, b \in X$ be two alternatives, $\widetilde{R}$ and $\widetilde{R}'$ two orderings in $X \times N$ and σ a permutation on N such that*

$$\forall i, j \in N \quad (a,i)\widetilde{R}(b,j) \iff (a,i)\tilde{R}'(b,\sigma(j)),$$

$$\forall i, j \in N \quad (b, j)\widetilde{R}(a, i) \iff (b, \sigma(j))\widetilde{R}'(a, i).$$

Then, $aRb \iff aR'b$ and $bRa \iff bR'a$.

Proof. Let $c \in X \setminus \{a, b\}$ be an alternative. We can consider orderings $\widetilde{R}_0$ and $\widetilde{R}'_0$ such that

$$\begin{array}{ll}
\text{(i)} & \widetilde{R} = \widetilde{R}_0 \text{ in } \{a, b\}, \\
\text{(ii)} & \widetilde{R}' = \widetilde{R}'_0 \text{ in } \{a, b\}, \\
\text{(iii)} & \forall i \in N \text{ it holds } (b, i)\widetilde{I}_0(c, i), \\
\text{(iv)} & \forall i \in N \text{ it holds } (b, \sigma(i))\widetilde{I}'_0(c, i).
\end{array}$$

By U, we can consider orderings $R_0 = f(\widetilde{R}_0)$ and $R'_0 = f(\tilde{R}'_0)$. Now:

$$\begin{aligned}
(a, i)\widetilde{R}_0(c, j) &\iff (a, i)\widetilde{R}_0(b, j) \quad \text{by (iii)}, \\
&\iff (a, i)\widetilde{R}(b, j) \quad \text{by (i)}, \\
&\iff (a, i)\widetilde{R}'(b, \sigma(j)) \quad \text{by hypothesis}, \\
&\iff (a, i)\widetilde{R}'_0(b, \sigma(j)) \quad \text{by (ii)}, \\
&\iff (a, i)\widetilde{R}'_0(c, j) \quad \text{by (iv)}.
\end{aligned}$$

Similarly, $(c, j)\widetilde{R}_0(a, i) \iff (c, j)\widetilde{R}'_0(a, i)$. This means $\widetilde{R}_0 = \widetilde{R}'_0$ in $\{a, c\}$. By *IIA*, we have $R_0 = R'_0$ in $\{a, c\}$. By (i), (ii) and *IIA*, we have $R = R_0$ in $\{a, b\}$ and $R' = R'_0$ in $\{a, b\}$. By S, (iii) and (iv), we have bI_0c and bI'_0c. Finally, $aRb \iff aR_0b \iff aR_0c \iff aR'_0c \iff aR'_0b \iff aR'b$. □

Proposition 3.1. *Suppose $|X| \geq 3$ and f is a social welfare function satisfying U, IIA and S. Then f also satisfies A.*

Proof. Suppose that f satisfies S. We will first construct two ordering that satisfy hypothesis of the previous lemma and we will be able to apply it. The construction is quite similar to the proof above. Let $\tilde{R}$ and $\tilde{R}'$ be two orderings in $X \times N$ and let σ be a permutation such that $\tilde{R}' = \sigma(\tilde{R})$. This means $(a, i)\sigma(\tilde{R})(b, j)$ is $(a, \sigma(i))\tilde{R}(b, \sigma(j))$. Let us consider $a, b \in X$. Let $c \in X \setminus \{a, b\}$ be an alternative. We can consider orderings $\tilde{R}_0$ and $\tilde{R}'_0$ such that

$$\begin{array}{ll}
\text{(i)} & \tilde{R} = \tilde{R}_0 \text{ in } \{a, b\}, \\
\text{(ii)} & \tilde{R}' = \tilde{R}'_0 \text{ in } \{a, b\}, \\
\text{(iii)} & \forall i \in N \text{ it holds } (b, i)\tilde{I}_0(c, i), \\
\text{(iv)} & \forall i \in N \text{ it holds } (b, \sigma(i))\tilde{I}'_0(c, i).
\end{array}$$

By U, we can consider orderings $R_0 = f(\tilde{R}_0)$ and $R'_0 = f(\tilde{R}'_0)$. Now:

$$\begin{aligned}
(a, i)\tilde{R}_0(c, j) &\iff (a, i)\tilde{R}_0(b, j) \text{ by (iii)}, \\
&\iff (a, i)\tilde{R}(b, j) \text{ by (i)}, \\
&\iff (a, \sigma(i))\tilde{R}'(b, \sigma(j)) \text{ by hypothesis}, \\
&\iff (a, \sigma(i))\tilde{R}'_0(b, \sigma(j)) \text{ by (ii)}, \\
&\iff (a, \sigma(i))\tilde{R}'_0(c, j) \text{ by (iv)}.
\end{aligned}$$

Similarly, $(c,j)\tilde{R}_0(a,i) \iff (c,\sigma(j))\tilde{R'}_0(a,i)$. Now, $\tilde{R}_0$ and $\tilde{R'}_0$ are ordering satisfying the conditions on the previous lemma, so we have $\tilde{R}_0 = \tilde{R'}_0$ in $\{a,c\}$. But by (iii), (iv) and condition *S*, we have $b\tilde{I}_0c$ and $b\tilde{I'}_0c$, so that $R_0 = R'_0$ in $\{a,b\}$. But by (i), (ii) and condition *IIA*, it follows $R = R'$ in $\{a,b\}$. As a,b can be any alternatives, condition *A* is satisfied. □

Proposition 3.2. *Suppose $|X| \geq 3$ and f is a social welfare function satisfying U, IIA, SP and A. Then f also satisfies S.*

Proof. Let f be a social welfare function satisfying *U*, *IIA*, *SP* and *A*. Let $\tilde{R}$ be an ordering on $X \times N$ and σ a permutation of N of only two individuals i and j, i.e., $\sigma(i) = j$, $\sigma(j) = i$ and $\forall k \neq i,j$, $\sigma(k) = k$. Suppose that for all $l \in N$ it holds $(a,l)\tilde{I}(b,\sigma(l))$. Our aim is to show aIb.

Suppose it does not hold aIb and, instead, we have aPb. Let us construct two orderings $\tilde{R'}$ and $\tilde{R''}$ satisfying the following conditions:

(i) $(a,i)\tilde{P'}(b,i)$ and $(a,i)\tilde{P'}(a,j)$,

(ii) $(b,j)\tilde{P'}(a,j)$ and $(b,j)\tilde{P'}(b,i)$,

(iii) $(a,i)\tilde{I'}(b,j)$ and $(b,i)\tilde{I'}(a,j)$,

(iv) $(a,j)\tilde{P''}(b,j)$ and $(a,j)\tilde{P''}(a,i)$,

(v) $(b,i)\tilde{P''}(a,i)$ and $(b,i)\tilde{P''}(b,j)$,

(vi) $(a,i)\tilde{I''}(b,j)$, and $(b,i)\tilde{I''}(a,j)$.

Due to aPb, condition (iii) and *IIA* it holds $aP'b$. Note that $R'' = \sigma(R')$. From Anonymity it follows $aP''b$. But, as *IIA* and *SP* hold, *SN* follows. And taking $a = c$ and $b = d$ we fathom $bP'a$, which leads us to contradiction. If it were bPa an analogous argument would be followed.

Now we move to general permutations. As *SN* is satisfied, we can consider the ordering R^* on $\mathbb{R}^n$ associated with f shown on Proposition 2.2. Thus, we can consider alternatives as vectors in $\mathbb{R}^n$. Let $u,v \in \mathbb{R}^n$ be the same except for an exchange of elements between two columns. We have just shown uIv. Since N is a finite set, any permutations of elements of a vector may be a product of, at most, n permutations involving only 2 columns. Indifference is preserved along the sequence, so a and b must be socially indifferent. □

D'Aspremont and Gevers (1977) show that with the ordering on $\mathbb{R}^n$ associated to a social welfare function f, which was defined in Proposition 2.2, the following results are achieved.

Proposition 3.3. *The ordering R^* satisfies SP if f satisfies IIA and SP. If f also satisfies OC (respectively CIC, A), then R^* satisfies OC (respectively (CIC, A).*

Proposition 3.4. *If f satisfies IIA and SP, then the ordering R^* satisfies SE (respectively EQ, INEQ) if, and only if, f satisfies SE (respectively EQ,INEQ).*

3.2 The Rawlsian form

Rawls (1971) developed his concept of *justice as fairness* proposing two principles of justice which are meant to be guidelines for how the basic structure of society is to realize the values of liberty and equality. It is probably fair to say that Rawls's work has become a powerful contestant of utilitarianism over the last few decades. Rawls's second principle, the *difference principle,* on which economists have focused in particular and on which we will focus, requires that social and economic inequalities are to be arranged so that they are both to the greatest benefit of the least advantaged members of society and attaches to offices and positions open to all under conditions of fair equality of opportunity.

In terms of binary relations, *Rawls's difference principle, or maximin rule,* denoted by P^D can be defined as follows. Given an individual $i \in N$ and two alternatives $a, b \in X$:

$$aP^Db \text{ iff } \big(\forall j \in N\ (b,j)\tilde{R}(b,i)\big) \Longrightarrow \big(\forall j \in N\ (a,j)\tilde{P}(b,i)\big).$$

Here, individual i is worst off in state b; aP^Db if, and only if, everybody is better off in state a than i is in state b. Later on, we will define Rawls's difference principle in terms of utility functions.

It is easy to see that P^D is irreflexive and asymmetric. If R^D the preference relation associated, one may prove it is an ordering. Given any ordering $\tilde{R}$, we define the ordering $R^D = f^D(\tilde{R})$, so that f^D is a social welfare functions. It can be proved easily by its definition that conditions *U*, *IIA* and *S* are satisfied. Then, f^D also satisfies *A*. *WP* is also satisfied, but not *SP*. We prove next that f^D also satisfies *EQ*.

Proposition 3.5. *f^D satisfies condition EQ.*

Proof. Suppose $iP(a)j$, $iP(b)j$, aP^ib, bP^ja and that $\forall k \in N \setminus \{i,j\}$ we have aI^kb. Let $l \in N$ be an individual so that $\forall k \in N$ it holds $(a,k)\tilde{R}(a,l)$. Obviously individual l can not be individual j. Then, $(a,l)\tilde{R}(b,l)$. It is false then, that $\forall k \in N\ (b,k)\tilde{P}(a,l)$. As the premise of the maximin rule is true and the consequent is false, it is false that bP^Da, from what follows aR^Db and condition *EQ* is satisfied. □

We will now strengthen the difference principle and replace it by Sen's lexical difference principle. The idea is to rank individuals from more advantaged to less. We will apply the maximin rule from the less advantaged to the most until it is decided whether aPb or bPa.

Let us, then, give each individual $i \in N$ a rank for each social state, $r(i,a)$, so that for every alternative a every person is assigned an integer between 1 and n. The individuals who are less advantaged have smaller numbers than those who are more advantaged. The individuals with smaller numbers will receive greater precedence than those who are more advantaged. Formally, for every alternative $a \in X$ and every person $i \in N$, we define en integer between 1 and n such that $(a,i)\tilde{P}(a,j) \Longleftrightarrow r(i,a) > r(j,a)$. If there are ties, i.e. $(a,i)\tilde{I}(a,j)$, we will see that they can be broken without affecting the final social ordering. In any social state $a \in X$ for each integer r there exists a unique individual $i \in N$ whose rank is r, and it will be denoted by $i(r,a)$. We will write a_r to the pair $(a, i(r,a))$. With this notation we have, for $1 \le r \le n-1$, $a_{r+1}\tilde{R}a_r$. With this notation we can easily redefine the maximin rule in this way: $aP^Db \Longleftrightarrow a_1\tilde{P}b_1$.

We can now define the *lexical difference principle or leximin, P^L*. Let $a, b \in X$ be alternatives:

$$aP^L b \iff \exists m \geq 1 \mid \forall r \in \{1, \ldots, m-1\};\ a_r \tilde{I} b_r \wedge a_m \tilde{P} b_m.$$

Then, if R^L is the corresponding preference relation and I^L the corresponding indifference relation, we have $aI^L b \iff a_r \tilde{I} b_r\ \forall r \in N$. R^L is transitive and complete and, if f^L is the corresponding social welfare function, $R^L = f^L(\tilde{R})$ satisfies conditions *U*, *IIA*, *S* (and so, *A*) by definition. One may see it is a strengthening of the difference principle. f^L also satisfies *SP* and *EQ*. To prove this we need some auxiliary lemmas. Let $N(k) = \{1, \ldots, k\}$ denote the set of integers from 1 to k.

Lemma 3.2. *Let σ be a permutation on N and $r \in N$. If $k \in N$ is such that $r > k \geq \sigma(r)$ then there exists $s \leq k$ such that $\sigma(s) > k$.*

Proof. Suppose $r \in N$, $r > k \geq \sigma(r)$ and $\sigma(s) \leq k$ for all s between 1 and k. Then $\sigma\big(N(k) \cup \{r\}\big) \subseteq N(k)$. But $N(k) \cup \{r\}$ has $k+1$ members while $N(k)$ has k members. Therefore, we reach a contradiction since σ cannot be a permutation. □

Lemma 3.3. *Let σ be a permutation on N and $a, b \in X$. If, for any $m \in \{1, \ldots, n\}$ we have $a_r \widetilde{R} b_{\sigma(r)}$ with $1 \leq r \leq m$, then $a_r \widetilde{R} b_r$ (with $1 \leq r \leq m$).*

Proof. For any $r \in \{1, \ldots, m\}$ if can be either (i) $\sigma(r) \geq r$ or (ii) $\sigma(r) < r$. In the first case (i) it holds $y_{\sigma(r)} \widetilde{R} y_r$ for all $y \in X$ and, by transitivity with the hypothesis we have $a_r \widetilde{R} b_r$. In case (ii), we take $k = r - 1$. Because of Lemma 3.2 there exists $s \leq k$ such that $\sigma(s) > k$. Then, we have $a_r \widetilde{R} a_s$, $a_s \widetilde{R} b_{\sigma(s)}$, $b_{\sigma(s)} \widetilde{R} b_{\sigma(r)}$ and, by transitivity, $a_r \widetilde{R} b_{\sigma(r)}$ □

Lemma 3.4. *Let σ be a permutation on N and $a, b \in X$. If, for any $m \in \{1, \ldots, n\}$ it holds $b_r \tilde{I} a_r$ and $a_r \tilde{R} b_{\sigma(r)}$ with $1 \leq r \leq m$, then $a_r \tilde{I} b_{\sigma(r)}$.*

Proof. It is enough for us to show $b_{\sigma(r)} \tilde{R} b_r$. Suppose it does not hold, i.e. $b_r \tilde{P} b_{\sigma(r)}$. Then there exists an integer k such that $r > k \geq \sigma(r)$ and $b_{k+1} \tilde{P} b_k$. Using Lemma 3.2 there exists $s \leq k$ such that $\sigma(s) > k$. It follows, then, $\sigma(s) \geq k+1$ and we arrive to contradiction using transitivity in $b_{k+1} \tilde{P} b_k$, $b_k \tilde{R} b_s$, $b_s \tilde{R} b_{\sigma(s)}$ and $b_{\sigma(s)} \tilde{R} b_{k+1}$. □

Lemma 3.5. *Let $a, b \in X$ and $j \in N$. If $(a, j) \tilde{P} (b, j)$ and it holds*

$$\forall k \in N\ (b, k) \tilde{P} (a, k) \Rightarrow (a, k) \tilde{P} (a, j).$$

Then, $aP^L b$.

Proof. Let us consider a permutation σ in N such that $\forall i \in N$ it holds $\sigma(r(i, a)) = r(i, b)$. Let $s = r(j, a)$. Then $\sigma(s) = r(j, b)$ and $a_s \tilde{P} b_{\sigma(s)}$. Let also $k \in N$ be an alternative and name $t = r(k, a)$, so that $\sigma(t) = r(k, b)$. Then $b_{\sigma(t)} \tilde{P} a_t \Rightarrow a_t \tilde{P} a_s$. This means $b_{\sigma(t)} \tilde{P} a_t \Rightarrow t > s$. So, for $1 \leq t \leq s$ we have $a_t \tilde{R} b_{\sigma(t)}$. Because of Lemma 3.3 we have $a_t \tilde{R} b_t$ for $1 \leq t \leq s$.

Suppose now that for those t it holds $a_t \tilde{I} b_t$. Since it also holds $a_t \tilde{R} b_{\sigma(t)}$, by Lemma 3.4 we have $a_t \tilde{I} b_{\sigma(t)}$ $(1 \leq t \leq s)$. This contradicts $a_s \tilde{P} b_{\sigma(s)}$. Therefore, there exists a t, $1 \leq m \leq s$ such that $a_m \tilde{P} b_m$ while for other t, $1 \leq t \leq s$ it is $a_t \tilde{R} b_t$, and by definition of leximin, it follows $aP^L b$. □

Proposition 3.6. *f^L satisfies conditions SP and EQ.*

Proof. Let $a, b \in X$ be alternatives. To prove *SP* suppose that $\forall i \in N$ it holds $(a,i)\tilde{R}(b,i)$ and there exists at least one $j \in N$ such that $(a,j)\tilde{P}(b,j)$. In this case, hypothesis of Lemma 3.5 are satisfied, so we have aP^Lb.
To show condition *EQ* suppose now $(a,j)\tilde{P}(a,i)$, $(b,j)\tilde{P}(b,i)$, $(a,i)\tilde{P}(b,i)$, $(b,j)\tilde{P}(a,j)$ and $\forall k \in N \setminus \{i,j\}$ $(a,k)\tilde{I}(b,k)$. In this case hypothesis of Lemma 3.5 are also satisfied, so aP^Lb. □

Conversely, f^L is the only social welfare function satisfying *U*, *IIA*, *A*, *SP* and *EQ*. This leads to a characterization of the leximin principle. To prove it, we also need a previous lemma. In it, let $a, b \in X$ and we use the notation $aP(J)b$, where:

$$\begin{array}{rl} \text{(i)} & J = \{j \in N \mid (b,j)\tilde{P}(a,j)\}, \\ \text{(ii)} & \exists i \in N \text{ such that } (a,i)\tilde{P}(b,i), \\ \text{(iii)} & \forall j \in J \text{ it holds } (a,j)\tilde{P}(b,i). \end{array}$$

Lemma 3.6. *Suppose that X has at least three alternatives, $|X| \geq 3$, and that f is a social welfare function satisfying U, IIA, EQ and SP. Then, for any $a, b \in X$ and for any set $J \subseteq N$, $aP(J)b$ implies aPb.*

Proof. As N is a finite set, we may prove it by induction on the number of individuals on J. If J is empty, then everyone satisfies prefers a to b and there exists at least one individual who strictly prefers a to b. This means aPb by *SP*.

Let us fix two alternatives $a, b \in X$. Suppose J_{m-1} is a non-empty set of $m-1$ individuals for which, for any ordering $\tilde{R}_0$ in $X \times N$, $aP(J_{m-1})b$ implies aPb. Let now $J_m = J_{m-1} \cup \{k\}$ be a non-empty set of m individuals for which it holds $aP(J_m)b$. We want to show aPb. Let c be an alternative in $X \setminus \{a,b\}$. We want to construct the ordering $\tilde{R}_0$ so that aPb is followed from $aP(J_m)b$. Let $\tilde{R}_0$ be an ordering that satisfies:

$$\text{(iv)}\ \ a\tilde{R}_0 b \Longleftrightarrow a\tilde{R}b,$$

$$\text{(v)}\ \ \forall j \in J,\ \ (a,j)\tilde{P}_0(c,i),$$

$$\text{(vi)}\ \ (a,i)\tilde{P}_0(c,i)\ \wedge\ (c,i)\tilde{P}_0(b,i),$$

$$\text{(vii)}\ \ (a,k)\tilde{I}_0(c,k),$$

$$\text{(viii)}\ \ \forall j \in N \setminus \{i,k\},\ \ (b,j)\tilde{I}_0(c,j).$$

This ordering exists and by U we can consider the social welfare function $f(\tilde{R}_0) = R_0$. The proof will proceed in two steps:

Step 1 aP_0c. This will be proved by showing that $aP(J_{m-1})c$ and using the induction hypothesis. Note that by (vi) we have $(a,i)\tilde{P}_0(c,i)$. From (v) it follows $\forall j \in J_{m-1}$ $(a,j)\tilde{P}_0(c,i)$. Also, $J_{m-1} = \{j \in N \mid (c,j)\tilde{P}_0(a,j)\}$. This is because

$$\begin{aligned} (c,j)\tilde{P}_0(a,j) &\Longleftrightarrow (b,j)\tilde{P}_0(a,j) \text{ and } j \in N \setminus \{i,k\} \quad \text{by (vi), (vii) and (viii)}, \\ &\Longleftrightarrow (b,j)\tilde{P}(a,j) \text{ and } j \in N \setminus \{i,k\} \quad \text{by (iv)}, \\ &\Longleftrightarrow j \in J_{m-1} \quad \text{by (i), (ii) and the definition of } J_{m-1}. \end{aligned}$$

As (i), (ii) and (iii) are satisfied, we have $aP(J_{m-1})c$, and thus aP_0c.

Step 2 cR_0b. This will be proved using *EQ*. By (vi) we have $(c,i)\tilde{P}_0(b,i)$ (α). By (viii) it holds $\forall j \in N \setminus \{i,k\}$, $(b,j)\tilde{I}_0(c,j)$ (β). By definition, we have $k \in J_m$, so by (i) we have $(b,k)\tilde{P}(a,k)$. Using (iv), $(b,k)\tilde{P}_0(a,k)$, and because of (vii) it holds $(b,k)\tilde{P}_0(c,k)$ (γ). Also, using (v) and that $k \in J_m$, $(a,k)\tilde{P}_0(c,i)$ holds. From (vii) it follows $(c,k)\tilde{P}_0(c,i)$ (δ). Finally, using again $k \in J_m$, (i) and (iii), it follows $(b,k)\tilde{P}_0(b,i)$ (ϵ). Note that α, β, γ, δ, ϵ are the premises of condition E, so we can conclude cR_0b.

As R_0 is transitive, we have aP_0b. Since condition *IIA* holds and (iv), we obtain aPb, as required. □

Theorem 3.3 (Leximin's characterization)**.** *If $|X| \geq 3$, f^L is the only social welfare function satisfying U, IIA, S, SP and EQ.*

Proof. We have seen above that if the social welfare function is the leximin principle, then it satisfies conditions *U*, *IIA*, *SP*, *EQ* and *S*. It only lasts, then, to show the reciprocal. Let f be a social welfare function satisfying *U*, *IIA*, *SP*, *E* and *S*. We want to see that f is f^L, i.e. given two alternatives $a, b \in X$, we want to see R^L and R rank them in the same way.

Suppose aI^Lb.
This means $a_r\tilde{I}b_r$ for $1 \leq r \leq n$. Let us consider a permutation σ on N such that $\forall i \in N$ it holds $r(i,a) = r(\sigma(i),b)$. This means $(a,i)\tilde{I}(b,\sigma(i))$ and, from condition *S*, it follows aIb.

Suppose now aP^Lb.
This means there exists an $m \in N$ such that $a_m\tilde{P}b_m$ and $a_r\tilde{I}b_r$ $\forall r \in \{1,\dots,m-1\}$. Let $c \in X \setminus \{a,b\}$. Let us construct an ordering $\tilde{R}_0$ on $X \times N$ such that $b_r\tilde{I}_0(c,i(r,a))$ for $1 \leq r \leq n$ and also it ranks a against b the same way as $\tilde{R}$ does i.e. $a\tilde{R}b \iff a\tilde{R}b \ \wedge \ b\tilde{R}a \iff b\tilde{R}_a$. It is clear that this ordering exists. By U we can define $R_0 = f(\tilde{R}_0)$. Note that $r(i,a) = r(\sigma(i),b)$ is the same as $i(r,a) = \sigma(i(r,b))$, so we can write $(b,i(r,b))\tilde{I}_0(c,\sigma(i(r,b)))$ and, because of S, we have bI_0c. See that if it were aP_0c, we would have aP_0b and by condition IIA, aPb would follow as required.

Let us then prove that aP_0c holds. We will prove it using Lemma 3.6. Let $J = \{j \in N \mid (c,j)\tilde{P}_0(a,j)\}$. As $a_m\tilde{P}b_m$ and by definition of the ordering $\tilde{R}_0$, we have $a_m\tilde{P}_0b_m$. Also, by definition of $\tilde{R}_0$ we can write $a_m\tilde{P}_0(c,i(m,a))$. So taking $k = i(m,a)$, there exists $k \in N$ such that $(a,k)\tilde{P}_0(c,k)$. It is only left to show $(c,j)\tilde{P}_0(a,j)$ implies $(a,j)\tilde{P}_0(c,k)$. Suppose $(c,j)\tilde{P}_0(a,j)$. Let $r = i(r,a)$. By construction of the ordering $\tilde{R}_0$, we have $b_r\tilde{I}_0(c,j)$. This implies $b_r\tilde{P}_0(a,j)$ and so $b_r\tilde{P}_0a_r$. As $\tilde{R}$ and $\tilde{R}_0$ are identically defined for a against b, it holds $b_r\tilde{P}a_r$. It has to be, then, $r > m$. In this case, $a_r\tilde{R}_0a_m$. But, as $k = i(m,a)$, $a_m\tilde{P}_0(c,k)$. Then, by definition of r, we finally have $(a,j)\tilde{P}_0(c,k)$.

This means the hypothesis of the lemma are satisfied and aP_0c holds, so aPb follows. □

We have characterized the leximin principle with conditions *U*, *IIA*, *SP*, *S* and *EQ*. As conditions *IIA* and *SP* hold together, *S* and *A* are equivalent. This means we can also characterize the leximin principle with condition *A* instead of *S*. Notice that the leximin principle can be understood as a dictatorship of the worst-off individuals and this may seem to contradict Anonymity (or non-dictatorship). There is, though, a slight difference between this dictatorship and the one defined in Chapter 2. This one is called positional dictatorship. In Chapter 2, given a profile of preference orderings $(R^1, \ldots, R^n)$. there was an individual k who was a dictator. If we changed the profile, the new individual k would be the dictator. However, with the positional dictatorship, it is the individual with rank k who is the dictator (and is does not matter if it is individual 1, k or n). If we changed the profile of preference relations, the dictator would still be the person with rank k but it may have changed his or her position (if it were individual k, now it may be individual l).

We can also define a symmetric social welfare function but, instead of favouring the worst off individuals, we may favour the better off ones. We will call it the leximax principle. Symmetrically, the leximax principle is characterized by *U*, *IIA*, *SP*, *A* and *INEQ*.

D'Aspremont and Gevers (1977) give another characterization of the leximin principle. Equity may seem quite a strong condition, mostly when interests of the worst off individual become on conflict with all the more favoured members of society. Moreover, in Theorem 3.3, the characterization of leximin, condition *SE* is absent and one may want it to appear. They proved that setting few conditions of justice, the social welfare function leads us either to Equity or Inequity. To prove it we will have to set-up all conditions and social welfare functions in terms of utilities. All needed conditions have been defined in terms of utility functions above. We will first need a lemma essentially technical to prove the result. Recall that, if condition *SN* is satisfied (which is due to *IIA* and *SP*), we can consider every alternative as a vector in an n-dimensional Euclidean space. We also define the rank function r in terms of utility as follows: given a vector $u \in \mathbb{R}^n$, $u^{r(i)} < u^{r(j)} \iff r(i) < r(j)$.

Lemma 3.7. *If f satisfies U, IIA, SP, A, SE and the informational requirement OC, then $\forall i, j \in N$ with $r(i) < r(j)$ and $\forall u, v \in \mathbb{R}^n$ such that $v^{r(i)} < u^{r(i)} < u^{r(j)} < v^{r(j)}$ and $v^{r(k)} = u^{r(k)}$ for every $k \in N \setminus \{i, j\}$ it holds either uP^*v or vP^*u.*

Proof. Let $u_0, u_1, v_0, v_1 \in \mathbb{R}^n$ be vectors satisfying the conditions of the lemma. As R^* is an ordering, it happens either $u_0P^*v_0$, or $v_0P^*u_0$, or $u_0I^*v_0$. Suppose $u_0P^*v_0$. We know, by Propositions 2.2, 3.3 and 3.4 that the ordering R^* inherits the properties of f. Because of the informational requirement, there exists an strictly increasing function ϕ such that

$$u_0^{r(i)} = \phi(u_1^{r(i)}),\ v_0^{r(i)} = \phi(v_1^{r(i)}),\ u_0^{r(j)} = \phi(u_1^{r(j)}) \text{ and } v_0^{r(j)} = \phi(v_1^{r(j)}).$$

We choose now $u_2, v_2 \in \mathbb{R}^n$ such that $\forall k \in N$

$$u_0^{r(k)} = \phi(u_2^{r(k)}) \text{ and } v_0^{r(k)} = \phi(v_2^{r(k)}).$$

Since conditions *A* and *OC* hold, $u_2P^*v_2$ holds. By conditions *A* and *SE*, $u_2P^*v_2$ implies $u_1P^*v_1$. This means $u_0P^*v_0$ implies $u_1P^*v_1$.

One may also prove on the same way that $u_0I^*v_0$ implies $u_1I^*v_1$ and $v_0P^*u_0$ implies $v_1P^*u_1$. Because of Anonymity and Separability of unconcerned individuals, if this result holds for some $i, j \in N$ with $r(i) < r(j)$, it is also true for all $k, h \in N$ with $r(k) < r(n)$.

It remains to prove, however that the case $u_0I^*v_0$ is impossible. Let us suppose it is possible and select $c \in \mathbb{R}^n$ such that $v_0^{r(i)} < c^{r(i)} < u_0^{r(i)}$ and $c^{r(k)} = u_0^{r(k)}$ $\forall k \in N \setminus \{k\}$. Following the same argument as before, we get cI^*v_0. By transitivity, cI^*u_0, but this contradicts condition *SP*. $\square$

Theorem 3.4. *If f satisfies U, IIA, SP, A, the informational requirement OC and SE, then it either satisfies EQ or INEQ.*

Proof. Consider any $i, j \in N$ and let $u, v \in \mathbb{R}^n$ such that

$$v^{r(i)} < u^{r(i)} < u^{r(j)} < v^{r(j)} \text{ and } u^{r(k)} = v^{r(k)} \;\; \forall k \in N \setminus \{i, j\}.$$

Consider $u_0, v_0 \in \mathbb{R}^n$ such that

$$u_0^{r(i)} = u^{r(i)};\; u_0^{r(j)} = u^{r(j)};\; v_0^{r(i)} = v^{r(i)};\; v_0^{r(j)} = v^{r(j)} \;\text{ and}$$
$$u_0^{r(k)} = v_0^{r(k)} = \beta < v^{r(i)} \;\text{ for all }\; k \in N \setminus \{i, j\}.$$

Note that u_0 and v_0 satisfy the conditions of the previous lemma. As *SE* holds, we have uR^*v if, and only if, $u_0R^*v_0$, and vR^*u iff $v_0R^*u_0$, and we can conclude that either *EQ* or *INEQ* hold. By Proposition 3.4, the social welfare function also satisfies either *EQ* or *INEQ*. □

So, a social welfare function satisfying *U*, *IIA*, *SP*, *A*, *OC* and *SE* must lead to a dictatorship either of the better off members of society or the worst off ones. One may ask the social welfare function not to be a dictatorship of the better off members of society.

Condition (MEQ - Minimal Equity). *The social welfare function is not the leximax principle.*

Together these conditions and theorems lead us to the last characterization of the leximin principle.

Theorem 3.5. *The leximin principle is characterized by conditions U, IIA, SP, the informational requirement OC, SE and MEQ.*

Note that this is a direct consequence of Theorems 3.3, 3.4 and the leximax's characterization. In this characterization it does not appear Equity, which has been highly questioned and appears Separability of unconcerned individuals, which was pretty claimed amongst experts.

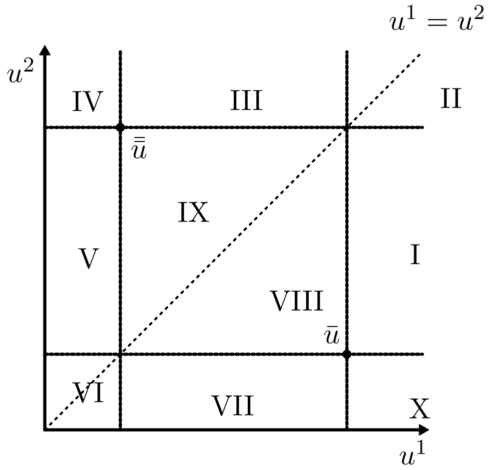
A diagrammatical approach

Though we have already given all results related to the leximin's characterization, we may stop a moment to remake those theorems we have just done in a diagrammatical way in order to have a more intuitive idea of the issue and understand how proofs above come to mind. We will prove Theorem 3.4 for just two individuals. The procedure will be quite the same but emphasizing points of the plane instead of relation between vectors.

Theorem 3.6. *Suppose $|N| = 2$. If f satisfies U, IIA, SP, A and the informational requirement OC, then it satisfies either EQ or INEQ.*

Proof. Recall that in Theorem 3.4 we used *SE* to go from two people who were in conflict to the general case. In this proof, as only two individuals are considered, condition *SE* will not be needed.

Figure 3.1



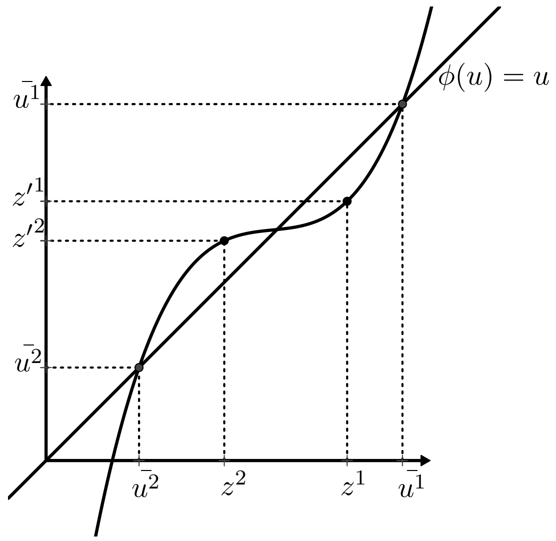
Our aim is to show that either the better off or the worst off of the two individuals who are in conflict will be decisive socially. As conditions *IIA* and *SP* hold, then *SN* holds, which means we can consider alternatives as points in $\mathbb{R}^2$. We will divide the plane in ten different regions, as shown in Figure 3.1.

We start from reference point $\bar{u}$. Due to Anonymity, $\bar{\bar{u}}$ is indifferent to $\bar{u}$. Note that all points $x = (x^1, x^2)$ in region I and II are such that $x^1 > \bar{u}^1$ and $x^2 > \bar{u}^2$. This, by *SP* means $xP\bar{u}$ for all x in regions I and II. Same happens with points in regions II and III regarding point $\bar{\bar{u}}$; if x is in regions II or III, then, by *SP*, $xP\bar{\bar{u}}$. Since $\bar{u}$ and $\bar{\bar{u}}$ are indifferent to one another, every point x in regions I, II or III is preferred to $\bar{u}$ and to $\bar{\bar{u}}$. In a symmetric way, $\bar{u}$ and $\bar{\bar{u}}$ are preferred to points y in regions V, VI and VII.

As we did in the diagrammatic proof of Arrow's theorem, we want to prove that all points in VIII are ranked equally against $\bar{u}$. By A, this would mean that all points in IX are ranked equally against $\bar{u}$(and so, against $\bar{\bar{u}}$). Let $z = (z^1, z^2)$ be a point on region VIII. Note that the following statements hold: (i) individual 1 is better off than individual 2 since $z^1 > z^2$ (ii) individual 1 is worse off in c than in $\bar{u}$ since $z^1 < \bar{u}^1$ (iii) individual 2 is better off in c than in $\bar{u}$ since $z^2 > \bar{u}^2$ (iv) individual 1 is better off in c than individual 2 in $\bar{u}$ since $z^1 > \bar{u}^2$ (v) individual 2 is worse off in c than individual 1 in $\bar{u}$ since $z^2 < \bar{u}^1$. This means $\bar{u}^2 < z^2 < z^1 < \bar{u}^1$. Note that these are the premises of conditions *EQ* and *INEQ*. To prove that all points in region VII are ranked equally against $\bar{u}$, let us consider two points $z = (z^1, z^2)$ and $z' = (z'^1, z'^2)$ in this region. It holds $\bar{u}^2 < z^2 < z^1 < \bar{u}^1$ and $\bar{u}^2 < z'^2 < z'^1 < \bar{u}^1$. As the informational requirement is *OC*, we can use an strictly increasing transformation ϕ such that is maps $\bar{u}$ and $\bar{\bar{u}}$ into themselves and z into z'. Figure 3.2 may help to understand that this functions exists. This means z and z' are ranked equally against $\bar{u}$.

It must be, then, that all points in region VII ans IX are either preferred to $\bar{u}$ and $\bar{\bar{u}}$, or indifferent to $\bar{u}$ and $\bar{\bar{u}}$, or $\bar{u}$ and $\bar{\bar{u}}$ are preferred to them. If it were indifference, we could consider two points z, z' in region VII such that $z^1 < z'^1$ and $z^2 < z'^2$. By *SP* it would hold $z'Pz$ and, by transitivity, $\bar{u}Pz$, which is a contradiction. So, it can only be that points in regions VIII and IX are preferred to $\bar{u}$ or vice versa.

Figure 3.2: Transformation of utility functions in OC

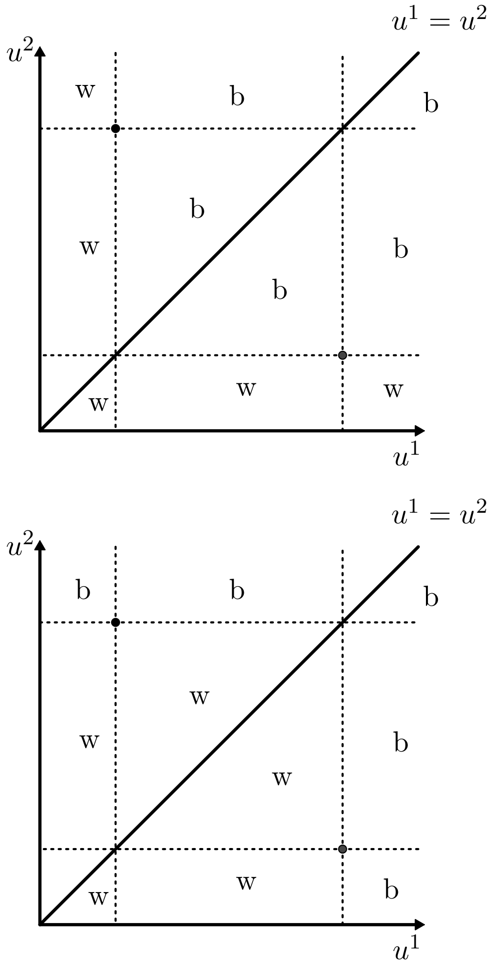


An analogous reasoning can be done with points on region X. By Anonymity, the same would hold for region IV,so that all points in regions X and IV are either preferred to $\bar{u}$ or $\bar{u}$ is preferred to them. Note that the situation of $\bar{u}$ and $\bar{\bar{u}}$ is quite similar to the one we have on Section 2.3. Let us use the same argument without explaining it step by step. All we need is to find an strictly increasing transformation ϕ such that it maps points in region VIII into $\bar{u}$ and $\bar{u}$ into a point in X This leads us to state that points on sections VII and IX are ranked oppositely to points on sections IV and X. We have already analysed all ten region in which we divided the plane. Only boundaries are left. As in Arrow's theorem, is two adjacent regions are ranked equally against $\bar{u}$, then points in the boundary between these two regions are ranked in the same way as the regions. Boundaries between regions that are not ranked in the same way will be taken care later.

What we have shown is that there are two basic possibilities: the ones shown on Figure 3.3. In this figure, regions marked with a "b" are preferred to $\bar{u}$ and $\bar{\bar{u}}$ and regions marked with a "w" "dispreferred" against them. If regions VII and IX are preferred to $\bar{u}$ we obtain the first picture while if regions IV and X are the ones preferred to $\bar{u}$ we obtain the second one. We pointed out before that points in region VII are such that $\bar{u}^2 < z^2 < z^1 < \bar{u}^1$. In a symmetric way, points in region X are such that $t^2 < \bar{u}^2 < \bar{u}^1 < t^1$. Let us fix points z and t in regions VIII and X respectively and suppose they represent alternatives a and b respectively. Let us suppose also, $\bar{u}$ represents alternative c. So, we can write $u(c,2) < u(a,2) < u(a,1) < u(c,1)$ and $u(b,2) < u(c,2) < u(c,1) < u(b,1)$. Note that these are the premises for *EQ* and *INEQ*. Individual 1 is better of than individual 2 for a against c, and for b against c. In conditions of Equity, individual 2 would be the dictator, so that it would be aPc and cPb. Note that this is the case of the first picture of Figure 3.3. In the second case it holds bPc and cPa, so that individual 1 is the dictator and condition of Inequity holds. Remind that this dictatorship is not the one defined by Arrow (1951), but the positional dictatorship discussed in Section 3.1.

□

Figure 3.3: Ranking possibilities of points against $\bar{u}$



3.3 The Utilitarian form

The utilitarian form is by far the most common and widely applied social welfare function in economics. Under a utilitarian rule social states are ranked according to he linear sum of utilities. Obviously, to perform the summation utilities must be cardinally measurable. Furthermore, statements of the form "in the move from a to b p, person i gains more than person j loses" must be meaningful. Together, these considerations imply that increments in utility must be both meaningful. Thus, the utilitarian form will only be possible when working with utility functions (not with preference orderings) and the informational requirement CIC will be needed. Formally, the utilitarian form is defined as follows. Let R^U denote the utilitarian ordering.

$$aR^U b \iff \sum_{i=1}^{n} u(a,i) \geq \sum_{i=1}^{n} u(b,i)$$

Let f^U be the corresponding social welfare function. It satisfies conditions *U*, *IIA*, *SP*, and *A*.

Proposition 3.7. *f^U satisfies conditions U, IIA, SP and A.*

Proof. It is obvious that every profile of utility functions can be considered in the utilitarian rule, so it satisfies *U*. To show *IIA*, let us consider two profiles of utility functions U_1 and U_2 and two alternatives $a, b \in X$ such that $U_1(a) = U_2(a)$ and $U_1(b) = U_2(b)$. Then, $\sum_i u_1(a,i) = \sum_i u_2(a,i)$ and $\sum_i u_1(b,i) = \sum_i u_2(b,i)$, so they rank the same way a against b.

Let us now prove condition *SP*. To do so, let U be a profile of utility functions and $a, b \in X$ alternatives such that $\forall i \in N$ it holds $u(a,i) \geq u(b,i)$. Then it follows $\sum_i u(a,i) = \sum_i u(b,i)$ and so $aR^U b$. To show A let U_1 and U_2 be two profiles of alternatives and σ a permutation on N such that $\forall i \in N$ and $\forall a \in X$ it holds $u_1(a,i) = u_2(a,\sigma(i))$. In this case, $\sum_i u_1(a,i) = \sum_i u_2(a,\sigma(i))$ and so they rank the same way all alternatives. □

In this case we can also characterize this rule, as f^U is the only social welfare function satisfying *U*, *IIA*, *SP*, *A* and the informational requirement *CIC*.

Theorem 3.7 (Utilitarianism's characterization)**.** *In the informational requirement CIC, if* $|X| \geq 3$*,* f^U *is the only social welfare function satisfying U, IIA, SP and A.*

Proof. We have already proved that the utilitarian form satisfies *U*, *IIA*, *SP* and *A*. It misses to show that whenever a social welfare function satisfies these conditions, then it is an utilitarian rule. Let us suppose, then, that f is a social welfare function satisfying *U*, *IIA*, *SP* and *A*. Let $a, b \in X$ be alternatives. We have to show that

$$\textstyle\sum_{i\in N} u(a,i) \geq \sum_{i\in N} u(b,i) \implies aRb.$$

Suppose $\sum_{i\in N} u(a,i) = \sum_{i\in N} u(b,i)$. Let us consider the ordering in $\mathbb{R}^n$ defined in Proposition 2.2. Note that, due to Proposition 3.3 it inherits all properties demanded. Let u_0 (respectively v_0)$\in \mathbb{R}^n$ be the vectors obtained by ordering in increasing size the members of $U(a) = (u(a,1), \ldots, u(a,n))$ (respectively $U(b)$). As Anonymity holds, we have $U(a)I^*u_0$ and $U(b)I^*v_0$. Let us now consider $u_1, v_1 \in \mathbb{R}^n$ constructed by subtracting $\min_{i\in N} u_0^i, v_0^i$ to u_0 and v_0. Note that one of these vectors now has a 0 as its first number. Now, from conditions *IIA* and *SP*, it follows $u_1 I^* v_1 \implies u_0 I^* v_0$.

Note that we can repeat this process a finite number of steps $m \leq 2n$ until we get $u_m = v_m = (0, \ldots, 0)$ and, at each step k we get $u_k I^* v_k \implies u_{k-1} I^* v_{k-1}$. So, $u_m I^* v_m \implies u_0$. But, as $u_m = v_m = (0, \ldots, 0)$, $u_m I^* v_m$ holds, and so $u_0 I^* v_0$ does. We saw above that u_0 and v_0 are indifferent to $U(a)$ and $U(b)$ respectively, what means $U(a)I^*U(b)$. We conclude, then, aIb when $\sum_{i\in N} u(a,i) = \sum_{i\in N} u(b,i)$.

Suppose now $\sum_{i\in N} u(a,i) \geq \sum_{i\in N} u(b,i)$ or $\sum_{i\in N} u(a,i) \leq \sum_{i\in N} u(b,i)$. Having an analogous reasoning, we would get either $u_m = (0, \ldots, 0)$ or $v_m = (0, \ldots, 0)$ (not both), while the other would have at least one strictly positive component. This would lead us to aPb in the first case and bPa in the second as required. □

If we drop Anonymity, we are able to discriminate among members of society. This leads us to the so called generalized utilitarianism, which is defined as follows:

$$aRb \iff \textstyle\sum_{i\in N} \alpha_i \cdot u(a,i) \geq \sum_{i\in N} \alpha_i \cdot u(b,i)$$

with $\alpha_i \geq 0$ for all $i \in N$ and $\alpha_j > 0$ for at least one $j \in N$. Blackorby, Donaldson and Weymark (1984) give a characterization of this generalized utilitarian rule.

Conclusions

With this project I had the opportunity of learning the fundamentals of Social Choice Theory and delve into Distributive Justice. Before starting the work I had no idea even that Social Choice Theory even existed and little by little it has given me a deeper understanding of how do politics and economics work and how can they be modeled in a mathematical way. I found quite shocking that, with just basic knowledge of formal logic and analysis, one can set up a whole framework that allows us to prove some important and interesting results such as characterizations of the leximin principle and the utilitarian principle.

I really enjoyed initiating myself in such a little known field. Although they had their own difficulties, none, modeling individual decision making and studying the Arrovian framework, presented a big deal. However, I found quite a challenge to understand and explain notions related to the third chapter. Changing the whole framework and discern when to use preference orderings and when utility functions required most of my attention. Besides, information was divided into some different articles and it was not easy to assemble them into one only chapter. My work is only a small part of what I found and what has been written, so I am very curious of what comes next.

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