DOCUMENTS DE TREBALL
DE LA FACULTAT D'ECONOMIA I EMPRESA

Col.lecció d'Economia

E09/212

The induced generalized OWA operator

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Abstract: We present the induced generalized ordered weighted averaging (IGOWA) operator. It is a new aggregation operator that generalizes the OWA operator by using the main characteristics of two well known aggregation operators: the generalized OWA and the induced OWA operator. Then, this operator uses generalized means and order inducing variables in the reordering process. With this formulation, we get a wide range of aggregation operators that include all the particular cases of the IOWA and the GOWA operator, and a lot of other cases such as the induced ordered weighted geometric (IOWG) operator. We further generalize the IGOWA operator by using quasi-arithmetic means. The result is the Quasi-IOWA operator. Finally, we also develop a numerical example of the new approach in a financial decision making problem.

Keywords: Aggregation operator; OWA operator; Generalized mean; Quasi-arithmetic mean; Decision making.

JEL Classification: C44, C49, D81, D89.

Resumen: Se presenta el operador OWA generalizado inducido (IGOWA). Es un nuevo operador de agregación que generaliza al operador OWA a través de utilizar las principales características de dos operadores muy conocidos como son el operador OWA generalizado y el operador OWA inducido. Entonces, este operador utiliza medias generalizadas y variables de ordenación inducidas en el proceso de reordenación. Con esta formulación, se obtiene una amplia gama de operadores de agregación que incluye a todos los casos particulares de los operadores IOWA y GOWA, y otros casos particulares. A continuación, se realiza una generalización mayor al operador IGOWA a través de utilizar medias cuasi-aritméticas. Finalmente, también se desarrolla un ejemplo numérico del nuevo modelo en un problema de toma de decisiones financieras.

Palabras clave: Operadores de agregación; operador OWA; media generalizada; media cuasi-aritmética; toma de decisiones.

1. Introduction

In the literature, we find a wide range of aggregation operators for aggregating the information. A very common aggregation method is the ordered weighted averaging (OWA) operator (Yager, 1988). It provides a parameterized family of aggregation operators that include the maximum, the minimum and the average, as special cases. Since its appearance, the OWA operator has been used in a wide range of applications (Amin, 2007; Ahn, 2008; Beliakov, 2005; Beliakov et al., 2007; Calvo et al., 2002; Chiclana et al., 2000; Chiclana et al., 2004; 2007; Fodor et al., 1995; Herrera-Viedma et al., 2007a; 2007b; Karayiannis, 2000; Liu, 2007; Liu and Han, 2008; Merigó and Gil-Lafuente, 2007; Mitchell and Estrakh, 1997; Wang and Hao, 2006; Wang et al., 2007; Wang and Parkan, 2007; Wu et al., 2007; Xu, 2005; Xu and Da, 2002; 2003; Yager, 1988; 1992; 1993; 1994; 1996; 2002; 2003a; 2003b; 2004a; 2004b; 2007a; 2007b; Yager and Filev, 1994; 1999; Yager and Kacprzyk, 1997).

In 1999, Yager and Filev, motivated by the work of Mitchell and Estrakh (1997), developed an extension of the OWA operator called induced ordered weighted averaging (IOWA) operator. The difference is that the reordering step is not developed with the values of the arguments. In this case, the reordering step is induced by another mechanism such that the ordered position of the arguments depends upon the values of their associated order inducing variables. In the last years, the IOWA operator has been receiving increasing attention as it is seen in the different works developed about it such as (Chiclana et al., 2004; 2007; Herrera-Viedma, 2007a; 2007b; Xu and Da, 2003; Yager, 2002; 2003a; 2004a).

Another interesting extension is the generalized OWA (GOWA) operator (Karayiannis, 2000; Yager, 2004b) that generalizes the OWA operator by using generalized means. The generalized mean (Dujmovic, 1974; Dyckhoff and Pedrycz, 1984) generalizes a wide range of mean operators such as the arithmetic mean, the geometric mean and the quadratic mean. Then, with the GOWA operator, it is possible to generalize a wide range of OWA operators such as the OWA itself, the ordered weighted geometric (OWG) operator and the ordered weighted quadratic averaging (OWQA) operator. In 2005, Beliakov developed a further extension of the GOWA operator by using quasi-arithmetic means. Then, he obtained the Quasi-OWA operator developed by (Fodor et al., 1995). Further studies on these generalizations are found in (Beliakov et al., 2007; Calvo et al., 2002).

The objective of this paper is to introduce the induced generalized OWA (IGOWA) operator. It is an extension of the OWA operator that uses the main characteristics of the IOWA and the GOWA operator. That is to say, it uses order inducing variables in the reordering process and generalized means. Then, we can obtain a generalization that includes the IOWA operator and its particular cases, and a lot of other situations such as the induced OWG (IOWG) operator (Chiclana et al., 2004; Xu and Da, 2003), the induced OWQA (IOWQA) operator and the induced OWHA (IOWHA) operator. Note that this generalization also includes the GOWA operator and its special cases such as the OWA, the generalized mean (GM), the weighted generalized mean (WGM), etc. We will study different properties and families of this operator such as the olympic-IGOWA, the median-IGOWA, the S-IGOWA, etc.

We will further generalize the IGOWA operator by using quasi-arithmetic means. Then, we will get the Quasi-IOWA operator. Note that the Quasi-IOWA can be seen as an extension of the Quasi-OWA operator that uses order inducing variables in the reordering process. With this generalization, we will get as special cases, the IGOWA operator and a lot of other situations such as the exponential IOWA, the trigonometric IOWA, the radical IOWA, etc.

We will also develop an application of the new approach. We will focus on a financial decision making problem about selection of investments. Note that the main advantage of the IGOWA operator in decision making is that it includes a lot of particular cases that can be used for taking the decision. Then, it is possible to consider different types of aggregations that may lead to different decisions. Note that this situation is also found with the OWA operator but with the IGOWA, we have more possibilities. Note also that other decision making applications could be developed such as the selection of financial products, human resource management, strategic decision making, product management, etc.

In order to do so, this paper is organized as follows. In Section 2, we briefly review some basic concepts such as the OWA, the IOWA and the GOWA operator. In Section 3, we present the IGOWA operator. Section 4 analyzes different families of IGOWA operators. In Section 5 we present the Quasi-IOWA operator. In Section 6 we develop an application of the new approach. Finally, Section 7 summarizes the main conclusions of the paper.

2. Preliminaries

In this Section, we will briefly describe the OWA operator, the IOWA operator and the GOWA operator.

2.1. OWA operator

The OWA operator was introduced by Yager in (1988) and it provides a parameterized family of aggregation operators that include the arithmetic mean, the maximum and the minimum. It can be defined as follows.

Definition 1. An OWA operator of dimension n is a mapping $OWA:R^n \rightarrow R$ that has an associated weighting vector W of dimension n such that the sum of the weights is 1 and $w_j \in [0,1]$, then:

$$OWA(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} w_j b_j \tag{1}$$

where b_j is the jth largest of the a_i.

From a generalized perspective of the reordering step, we can distinguish between the descending OWA (DOWA) operator and the ascending OWA (AOWA) operator (Yager, 1992). The OWA operator is commutative, monotonic, bounded and idempotent (Yager, 1988).

2.2. Induced OWA operator

The IOWA operator was introduced by Yager and Filev (1999) and it represents an extension of the OWA operator. Its main difference is that the reordering step is not developed with the values of the arguments a_i. In this case, the reordering step is developed with order inducing variables. The IOWA operator also includes as particular cases the maximum, the minimum and the average criteria. It can be defined as follows.

Definition 2. An IOWA operator of dimension n is a mapping $IOWA$: $R^n \rightarrow R$ that has an associated weighting vector W of dimension n such that the sum of the weights is 1 and $w_j \in [0,1]$, then:

$$IOWA(\langle u_1,a_1\rangle, \langle u_2,a_2\rangle \ldots, \langle u_n,a_n\rangle) = \sum_{j=1}^{n} w_j b_j \tag{2}$$

where b_j is the a_i value of the IOWA pair $\langle u_i,a_i\rangle$ having the jth largest u_i, u_i is the order inducing variable and a_i is the argument variable.

Note that it is possible to distinguish between the Descending IOWA (DIOWA) operator and the Ascending IOWA (AIOWA) operator. The IOWA operator is also monotonic, bounded, idempotent and commutative (Yager and Filev, 1999).

2.3. Generalized OWA operator

The generalized OWA (GOWA) operator was introduced in (Karayiannis, 2000; Yager, 2004b). It generalizes the OWA operator by using generalized means. It can be defined as follows.

Definition 3. A GOWA operator of dimension n is a mapping $GOWA$:$R^n \rightarrow R$ that has an associated weighting vector W of dimension n such that the sum of the weights is 1 and $w_j \in [0,1]$, then:

$$GOWA(a_1, a_2,\ldots, a_n) = \left(\sum_{j=1}^{n} w_j b_j^{\lambda} \right)^{1/\lambda} \tag{3}$$

where b_j is the jth largest of the a_i, and λ is a parameter such that $\lambda \in (-\infty, \infty)$.

In this case, it is also possible to distinguish between the descending generalized OWA (DGOWA) operator and the ascending generalized OWA (AGOWA) operator. The weights of these operators are related by $w_j = w^*_{n+1-j}$, where w_j is the jth weight of the DGOWA (or GOWA) operator and w^*_{n+1-j} the jth weight of the AGOWA operator.

As it is demonstrated in (Karayiannis, 2000; Yager, 2004b), the GOWA operator is a mean operator. This is a reflection of the fact that the operator is commutative, monotonic, bounded and idempotent both for the DGOWA and the AGOWA operator. It can also be demonstrated that the GOWA operator has as special cases the maximum, the minimum, the generalized mean and the weighted generalized mean. Note that the weighted generalized mean is obtained when $j = i$, for all i and j, where j is the jth argument of the b_j and i is the ith argument of the a_i.

If we look to different values of the parameter λ, we can also obtain other special cases as the usual OWA operator (Yager, 1988), the ordered weighted geometric (OWG) operator (Chiclana et al., 2000; Xu and Da, 2002), the ordered weighted harmonic averaging (OWHA) operator (Yager, 2004b) and the ordered weighted quadratic averaging (OWQA) operator (Yager, 2004b). When $\lambda = 1$, we obtain the usual OWA operator. When $\lambda = 0$, the OWG operator. When $\lambda = -1$, the OWHA operator. And when $\lambda = 2$, the OWQA operator.

Another interesting issue to consider is the attitudinal character of the GOWA operator. In (2004b), Yager defined it as:

$$\alpha(W)=\left(\sum_{j=1}^{n} w_j\left(\frac{n-j}{n-1}\right)^{\lambda}\right)^{1/\lambda} \tag{4}$$

It can be shown that $\alpha \in [0, 1]$. The more of the weight located near the top of W, the closer α to 1 and the more of the weight located toward the bottom of W, the closer α to 0. Note that for the optimistic criteria $\alpha(W) = 1$ and for the pessimistic criteria $\alpha(W) = 0$.

If we replace b^{λ} with a general continuous strictly monotone function $g(b)$ (Beliakov, 2005), then, the GOWA operator becomes the Quasi-OWA operator (Fodor et al., 1995). It can be formulated as follows.

Definition 4. A Quasi-OWA operator of dimension n is a mapping QOWA: $R^n \to$ R that has an associated weighting vector W of dimension n such that the sum of the weights is 1 and $w_j \in [0, 1]$, then:

$$\text{QOWA}(a_1, a_2, \ldots, a_n) = g^{-1}\left(\sum_{j=1}^{n} w_j g\left(b_{(j)}\right)\right) \tag{5}$$

where b_j is the jth largest of the a_i.

3. The induced generalized OWA operator

The D-S theory of evidence The induced generalized OWA (IGOWA) operator represents an extension of the GOWA operator. The main difference between them is that the reordering step of the IGOWA operator is not developed with the values of the arguments a_i. In this case, the reordering step is induced by another mechanism represented as u_i, where the ordered position of the arguments a_i depends upon the values of the order inducing variable u_i.

Definition 5. An IGOWA operator of dimension n is a mapping $IGOWA$: $R^n \rightarrow R$ that has an associated weighting vector W of dimension n such that the sum of the weights is 1 and $w_j \in [0,1]$, then:

$$IGOWA(\langle u_1,a_1\rangle, \langle u_2,a_2\rangle, \ldots, \langle u_n,a_n\rangle) = \left(\sum_{j=1}^{n} w_j b_j^{\lambda} \right)^{1/\lambda} \tag{6}$$

where b_j is the a_i value of the IGOWA pair $\langle u_i,a_i\rangle$ having the jth largest u_i, u_i is the order inducing variable, a_i is the argument variable and λ is a parameter such that $\lambda \in (-\infty, \infty)$.

From a generalized perspective of the reordering step, we can distinguish between the descending induced generalized OWA (DIGOWA) operator and the ascending induced generalized OWA (AIGOWA) operator. The weights of these operators are related by $w_j = w^*_{n+1-j}$, where w_j is the jth weight of the DGOWA (or GOWA) operator and w^*_{n+1-j} the jth weight of the AGOWA operator.

If B is a vector corresponding to the ordered arguments b_j^{λ}, we shall call this the ordered argument vector and W^T is the transpose of the weighting vector, then, the IGOWA operator can be expressed as:

$$IGOWA(\langle u_1,a_1\rangle, \langle u_2,a_2\rangle, \ldots, \langle u_n,a_n\rangle) = \left(W^T B\right)^{1/\lambda} \tag{7}$$

Note that if the weighting vector is not normalized, i.e., $W = \sum_{j=1}^{n} w_j \neq 1$, then, the IGOWA operator can be expressed as:

$$IGOWA(\langle u_1,a_1\rangle, \langle u_2,a_2\rangle, \ldots, \langle u_n,a_n\rangle) = \left(\frac{1}{W} \sum_{j=1}^{n} w_j b_j^{\lambda} \right)^{1/\lambda} \tag{8}$$

The IGOWA operator is a mean or averaging operator. This is a reflection of the fact that the operator is commutative, monotonic, bounded and idempotent. These properties can be proved with the following theorems.

Theorem 1 (Monotonicity). *Assume f is the IGOWA operator, if* $a_i \geq e_i$, *for all* a_i, *then*:

$$f(\langle u_1,a_1\rangle, \ldots, \langle u_n,a_n\rangle) \geq f(\langle u_1,e_1\rangle, \ldots, \langle u_n,e_n\rangle) \tag{9}$$

Proof. Let

$$f(\langle u_1,a_1\rangle, \ldots, \langle u_n,a_n\rangle) = \left(\sum_{j=1}^{n} w_j b_j^{\lambda} \right)^{1/\lambda} \tag{10}$$

$$f(\langle u_1,e_1\rangle, \ldots, \langle u_n,e_n\rangle) = \left(\sum_{j=1}^{n} w_j d_j^{\lambda} \right)^{1/\lambda} \tag{11}$$

Since $a_i \geq e_i$, for all a_i, it follows that, $a_i \geq e_i$, and then

$$f(\langle u_1,a_1\rangle, \ldots, \langle u_n,a_n\rangle) \geq f(\langle u_1,e_1\rangle, \ldots, \langle u_n,e_n\rangle) \qquad \blacksquare$$

Theorem 2 (Commutativity). *Assume f is the IGOWA operator, then*:

$$f\,(\langle u_1,a_1\rangle, \ldots, \langle u_n,a_n\rangle) = f\,(\langle u_1,e_1\rangle, \ldots, \langle u_n,e_n\rangle) \tag{12}$$

where $(\langle u_1,a_1\rangle, \ldots, \langle u_n,a_n\rangle)$ *is any permutation of the arguments* $(\langle u_1,e_1\rangle, \ldots, \langle u_n,e_n\rangle)$.

Proof. Let

$$f\,(\langle u_1,a_1\rangle, \ldots, \langle u_n,a_n\rangle) = \left(\sum_{j=1}^{n} w_j b_j^{\lambda}\right)^{1/\lambda} \tag{13}$$

$$f\,(\langle u_1,e_1\rangle, \ldots, \langle u_n,e_n\rangle) = \left(\sum_{j=1}^{n} w_j d_j^{\lambda}\right)^{1/\lambda} \tag{14}$$

Since $(\langle u_1,a_1\rangle, \ldots, \langle u_n,a_n\rangle)$ is a permutation of $(\langle u_1,e_1\rangle, \ldots, \langle u_n,e_n\rangle)$, we have $a_j = e_j$, for all j, and then

$$f\,(\langle u_1,a_1\rangle, \ldots, \langle u_n,a_n\rangle) = f\,(\langle u_1,e_1\rangle, \ldots, \langle u_n,e_n\rangle) \qquad \blacksquare$$

Theorem 3 (Idempotency). *Assume f is the IGOWA operator, if* $a_i = a$, *for all* a_i, *then*:

$$f\,(\langle u_1,a_1\rangle, \ldots, \langle u_n,a_n\rangle) = a \tag{15}$$

Proof. Since $a_i = a$, for all a_i, we have

$$f\,(\langle u_1,a_1\rangle, \ldots, \langle u_n,a_n\rangle) = \left(\sum_{j=1}^{n} w_j b_j^{\lambda}\right)^{1/\lambda} = \left(\sum_{j=1}^{n} w_j a^{\lambda}\right)^{1/\lambda} = \left(a^{\lambda} \sum_{j=1}^{n} w_j\right)^{1/\lambda} \tag{16}$$

Since $\sum_{j=1}^{n} w_j = 1$, we get

$$f(\langle u_1,a_1\rangle, \ldots, \langle u_n,a_n\rangle) = a \qquad \blacksquare$$

Theorem 4 (Bounded). *Assume f is the IGOWA operator, then*:

$$\mathrm{Min}\{a_i\} \le f(\langle u_1,a_1\rangle, \ldots, \langle u_n,a_n\rangle) \le \mathrm{Max}\{a_i\} \tag{17}$$

Proof. Let max{a_i} = c, and min{a_i} = d, then

$$f(\langle u_1,a_1\rangle, \ldots, \langle u_n,a_n\rangle) = \left(\sum_{j=1}^{n} w_j b_j^{\lambda}\right)^{1/\lambda} \le \left(\sum_{j=1}^{n} w_j c^{\lambda}\right)^{1/\lambda} = \left(c^{\lambda} \sum_{j=1}^{n} w_j\right)^{1/\lambda} \tag{18}$$

$$f(\langle u_1,a_1\rangle, \ldots, \langle u_n,a_n\rangle) = \left(\sum_{j=1}^{n} w_j b_j^{\lambda}\right)^{1/\lambda} \ge \left(\sum_{j=1}^{n} w_j d^{\lambda}\right)^{1/\lambda} = \left(d^{\lambda} \sum_{j=1}^{n} w_j\right)^{1/\lambda} \tag{19}$$

Since $\sum_{j=1}^{n} w_j = 1$, we get

$$f(\langle u_1,a_1\rangle, \ldots, \langle u_n,a_n\rangle) \le c \tag{20}$$

$$f(\langle u_1,a_1\rangle, \ldots, \langle u_n,a_n\rangle) \ge d \tag{21}$$

Therefore,

$$\mathrm{Min}\{a_i\} \le f(\langle u_1,a_1\rangle, \ldots, \langle u_n,a_n\rangle) \le \mathrm{Max}\{a_i\} \qquad \blacksquare$$

An interesting issue when analysing induced aggregation operators is the problem of ties in the reordering step. In order to solve this problem, we recommend to follow the policy developed by Yager and Filev (1999) where they replace each argument of the tied IOWA pair by their average. For the GOWA operator, instead of using the arithmetic mean, we will replace each argument of the tied IGOWA pair by its generalized mean. Then, depending on the parameter of λ, we will use a different type of mean to replace the tied arguments.

As it is explained in (Yager and Filev, 1999) for the IOWA operator, when studying the order inducing variables of the IGOWA operator, we should note that the values used can be drawn from a space such that the only requirement is to have a linear ordering. Then, it is possible to use different kinds of attributes for the order inducing variables that permit us, for example, to mix numbers with words in the aggregations (Yager and Filev, 1999). For the IGOWA operator, this would mean that we have numerical arguments to be ordered by linguistic order inducing variables. Note that in some situations it is possible to use the implicit lexicographic ordering associated with words such as the ordering of words in dictionaries (Yager and Filev, 1999).

The IGOWA operator is a generalization of the IOWA operator. Therefore, the IGOWA operator is applicable to different situations already discussed for the IOWA operator. For example, we could use it for modeling nearest neighbour rule (Yager and Filev, 1999), for model building (Yager and Filev, 1999) and for the aggregation of complex objects (Yager, 2003). Other potential applications could be developed for decision making, group decision making, business decisions, etc. Note that in this paper we will develop an application in financial decision making.

4. Families of IGOWA operators

In this Section, we will consider different types of IGOWA operators. We will distinguish between two main classes. The first class will focus on the weighting vector W and the second class on the parameter λ.

4.1. Analysing the weighting vector W

By choosing a different manifestation of the weighting vector in the IGOWA operator, we are able to obtain different types of aggregation operators. For example, we can obtain the maximum, the minimum, the generalized mean, the weighted generalized mean and the GOWA operator. Note that these results can be obtained both for the DIGOWA and the AIGOWA operators.

The maximum is obtained if $w_p = 1$ and $w_j = 0$, for all $j \neq p$, and u_p = Max$\{a_i\}$, then, $IGOWA(\langle u_1,a_1\rangle, \langle u_2,a_2\rangle\ldots, \langle u_n,a_n\rangle)$ = Max$\{a_i\}$. The minimum is obtained if $w_p = 1$ and $w_j = 0$, for all $j \neq p$, and u_p = Min$\{a_i\}$, then, $IGOWA(\langle u_1,a_1\rangle, \langle u_2,a_2\rangle\ldots, \langle u_n,a_n\rangle)$ = Min$\{a_i\}$. More generally, if $w_k = 1$ and $w_j = 0$, for all $j \neq k$, we get for any λ, $IGOWA(\langle u_1,a_1\rangle, \ldots, \langle u_n,a_n\rangle) = b_k$, where b_k is the the a_i value of the IGOWA pair $\langle u_i,a_i\rangle$ having the kth largest u_i. The generalized mean is found when $w_j = 1/n$, for all a_i. The weighted generalized mean is obtained if $u_i > u_{i+1}$, for all i, and the GOWA operator is obtained if the ordered position of u_i is the same than the ordered position of b_j such that b_j is the jth largest of a_i.

Other families of IGOWA operators could be obtained by using a different weighting vector. For example, when $w_j = 1/m$ for $k \leq j \leq k + m - 1$ and $w_j = 0$ for $j > k + m$ and $j < k$, we are using the window-IGOWA operator that it is based on the window-OWA operator (Yager, 1993). Note that k and m must be positive integers such that $k + m - 1 \leq n$. Also note that if $m = k = 1$, and the initial position of the highest u_i is also the initial position of the highest a_i, then, the window-IGOWA is transformed in the maximum. If $m = 1$, $k = n$, and the initial position of the lowest u_i is also the initial position of the lowest a_i, then, the window-IGOWA becomes the minimum. And if $m = n$ and $k = 1$, the window-IGOWA becomes the generalized mean.

If $w_1 = w_n = 0$, and for all others $w_j = 1/(n - 2)$, we are using the olympic induced generalized average that it is based on the olympic average (Yager, 1996). Note that if $n = 3$ or $n = 4$, the olympic induced generalized average is transformed in the IGOWA median and if $m = n - 2$ and $k = 2$, the window-IGOWA is transformed in the olympic induced generalized average. Also note that the olympic induced generalized average is transformed in the olympic generalized average if $w_p = w_q = 0$, such that $u_p = \mathrm{Max}_\mathrm{i}\{a_i\}$ and $u_q = \mathrm{Min}_\mathrm{i}\{a_i\}$, and for all others $w_j = 1/(n - 2)$.

Another type of aggregation that could be used is the E-Z IGOWA weights that it is based on the E-Z OWA weights (Yager, 2003). In this case, we should distinguish between two classes. In the first class, we assign $w_j = (1/k)$ for $j = 1$ to k and $w_j = 0$ for $j > k$, and in the second class, we assign $w_j = 0$ for $j = 1$ to $n - k$ and $w_j = (1/k)$ for $j = n - k + 1$ to n. Note that the E-Z IGOWA weights becomes the E-Z GOWA weights for the first class if the ordered position of u_i is the same than the ordered position of b_j such that b_j is the jth largest of a_i, from $j = 1$ to k. And for the second class, the E-Z IGOWA weights becomes the E-Z GOWA weights if the ordered position of u_i is the same than the ordered position of b_j such that b_j is the jth largest of a_i, from $j = n - k + 1$ to n.

We note that the generalized median and the weighted generalized median (Yager, 1994) can also be used as induced aggregation operators. For the IGOWA median, if n is odd we assign $w_{(n+1)/2} = 1$ and $w_j = 0$ for all others, and this affects the argument a_i with the $[(n + 1)/2]$th largest u_i. If n is even we assign for example, $w_{n/2} = w_{(n/2)+1} = 0.5$, and this affects the arguments with the $(n/2)$th and $[(n/2) + 1]$th largest u_i. For the weighted IGOWA median, we select the argument a_i that has the kth largest inducing variable u_i, such that the sum of the weights from 1 to k is equal or higher than 0.5 and the sum of the weights from 1 to $k - 1$ is less than 0.5. Note that if the ordered position of u_i is the same than the ordered position of b_j such that b_j is the jth largest of a_i, then, the IGOWA median and the weighted IGOWA median become the GOWA median and the weighted GOWA median respectively.

Another interesting family is the S-IGOWA operator based on the S-OWA operator (Yager, 1993; Yager and Filev, 1994). It can be divided in three classes, the "orlike", the "andlike" and the generalized S-IGOWA operator. The "orlike" S-IGOWA operator is found when $w_p = (1/n)(1 - \alpha) + \alpha$, $u_p = \text{Max}\{a_i\}$, and $w_j = (1/n)(1 - \alpha)$ for all $j \neq p$ with $\alpha \in [0, 1]$. Note that if $\alpha = 0$, we get the arithmetic mean and if $\alpha = 1$, we get the maximum. The "andlike" S-IGOWA operator is found when $w_q = (1/n)(1 - \beta) + \beta$, $u_q = \text{Min}\{a_i\}$, and $w_j = (1/n)(1 - \beta)$ for all $j \neq q$ with $\beta \in [0, 1]$. Note that in this class, if $\beta = 0$ we get the average and if $\beta = 1$, we get the minimum. Finally, the generalized S-IGOWA operator is obtained when $w_p = (1/n)(1 - (\alpha + \beta) + \alpha$, with $u_p = \text{Max}\{a_i\}$; $w_q = (1/n)(1 - (\alpha + \beta) + \beta$, with $u_q = \text{Min}\{a_i\}$; and $w_j = (1/n)(1 - (\alpha + \beta)$ for all $j \neq p,q$ where $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$. Note that if $\alpha = 0$, the generalized S-IGOWA operator becomes the "andlike" S-IGOWA operator and if $\beta = 0$, it becomes the "orlike" S-IGOWA operator.

Other families of IGOWA operators could be developed such as the weights that depend on the aggregated objects (Yager, 1993). For example, we could develop the BADD-IGOWA operator that it is based on the OWA version developed in (Yager, 1993).

$$w_j = \frac{b_j^{\alpha}}{\sum_{j=1}^{n} b_j^{\alpha}} \tag{22}$$

where $\alpha \in (-\infty, \infty)$, b_j is the jth largest element of the arguments a_i. Note that the sum of the weights is 1 and $w_j \in [0,1]$. Also note that if $\alpha = 0$, we get the generalized mean and if $\alpha = \infty$, we get the maximum. Another family of IGOWA operator that depends on the aggregated objects is

$$w_j = \frac{(1-b_j)^{\alpha}}{\sum_{j=1}^{n} (1-b_j)^{\alpha}} \tag{23}$$

where $\alpha \in (-\infty, \infty)$, b_j is the jth largest element of the arguments a_i. Note that in this case if $\alpha = 0$, we also get the generalized mean and if $\alpha = \infty$, we get the minimum. A third family of IGOWA operator that depends on the aggregated objects is

$$w_j = \frac{(1/b_j)^{\alpha}}{\sum_{j=1}^{n} (1/b_j)^{\alpha}} \tag{24}$$

where $\alpha \in (-\infty, \infty)$, b_j is the jth largest element of the arguments a_i. In this case, we also get the generalized mean if $\alpha = 0$. If $\alpha = 1$, we obtain the harmonic mean and if $\alpha = \infty$, we get the minimum.

A very useful approach for obtaining the weights that it is also applicable for the IGOWA operator is the functional method introduced by Yager (1996) for the OWA operator. It can be summarized as follows. Let f be a function f: $[0, 1] \to [0, 1]$ such that $f(0) = f(1)$ and $f(x) \geq f(y)$ for $x > y$. We call this function a basic unit interval monotonic function (BUM). Using this BUM function we obtain the IGOWA weights w_j for $j = 1$ to n as

$$w_j = f\left(\frac{j}{n}\right) - f\left(\frac{j-1}{n}\right) \tag{25}$$

It can easily be shown that using this method, the w satisfy that the sum of the weights is 1 and $w_j \in [0,1]$.

Another family of aggregation operators that could be used in the IGOWA operator is the centered IGOWA weights. This type of operator has been suggested by Yager (2007a) for the OWA operator. Following the same methodology, we could say that an IGOWA operator is a centered aggregation operator if it is symmetric, strongly decaying and inclusive. It is symmetric if $w_j = w_{j+n-1}$. It is strongly decaying when $i < j \leq (n + 1)/2$ then $w_i < w_j$ and when $i > j \geq (n + 1)/2$ then $w_i < w_j$. It is inclusive if $w_j > 0$. Note that it is possible to consider a softening of the second condition by using $w_i \leq w_j$ instead of $w_i < w_j$. We shall refer to this as softly decaying centered IGOWA operator. Note that the generalized mean is an example of this particular case of centered IGOWA operator. Another particular situation of the centered IGOWA operator appears if we remove the third condition. We shall refer to it as a non-inclusive centered IGOWA operator. For this situation, we find the IGOWA median as a particular case.

A special type of centered IGOWA operator is the Gaussian IGOWA weights which follows the same methodology than the Gaussian OWA weights suggested by Xu (2005). In order to define it, we have to consider a Gaussian distribution η(μ, σ) where

$$\mu_n = \frac{1}{n} \sum_{j=1}^{n} j = \frac{n+1}{2} \tag{26}$$

$$\sigma_n = \sqrt{\frac{1}{n} \sum_{j=1}^{n} (j - \mu_n)^2} \tag{27}$$

Assuming that

$$\eta(j) = \frac{1}{\sqrt{2\Pi \sigma_n}} e^{-(j-\mu_n)^2 / 2\sigma_n^2} \tag{28}$$

we define the IGOWA weights as

$$w_j = \frac{\eta_j}{\sum_{j=}^{n} \eta(j)} = \frac{e^{-(j-\mu_n)^2 / 2\sigma_n^2}}{\sum_{j=1}^{n} e^{-(j-\mu_n)^2 / 2\sigma_n^2}} \tag{29}$$

Note that the sum of the weights is 1 and $w_j \in [0,1]$.

Other families of IGOWA operators could be obtained in the weighting vector following a similar methodology as developed for the OWA operator such as those developed in (Amin, 2007; Ahn, 2008; Liu, 2007; Liu and Han, 2008; Wang et al., 2007; Wang and Parkan, 2007; Wu et al., 2007; Xu, 2005; Yager, 2007b).

4.2. Analysing the parameter λ

Analyzing the If we analyze different values of the parameter λ in the IGOWA operator, we obtain another group of particular cases such as the usual IOWA operator, the induced OWG (IOWG) operator (Chiclana et al., 2004; Xu and Da, 2003), the induced OWHA (IOWHA) operator and the induced OWQA (IOWQA) operator.

When $\lambda = 1$, the IGOWA operator becomes the IOWA operator.

$$IGOWA(\langle u_1,a_1\rangle, \langle u_2,a_2\rangle, \ldots, \langle u_n,a_n\rangle) = \sum_{j=1}^{n} w_j b_j \tag{30}$$

From a generalized perspective of the reordering step we have to distinguish between the DIOWA operator and the AIOWA operator. In both cases, the formulation is the same with the difference that the DIOWA operator has a descending order and the AIOWA operators an ascending order.

When $\lambda = 0$, the IGOWA operator becomes the IOWG operator.

$$IGOWA(\langle u_1,a_1\rangle, \langle u_2,a_2\rangle, \ldots, \langle u_n,a_n\rangle) = \prod_{j=1}^{n} b_j^{\,w_j} \tag{31}$$

With the DIGOWA operator we obtain the descending IOWG (DIOWG) operator and with the AIGOWA operator, the ascending IOWG (AIOWG) operator.

When $\lambda = -1$, we get the IOWHA operator.

$$IGOWA(\langle u_1,a_1\rangle, \langle u_2,a_2\rangle, \ldots, \langle u_n,a_n\rangle) = \frac{1}{\sum_{j=1}^{n} \frac{w_j}{b_j}} \tag{32}$$

Note that from a generalized perspective of the reordering step we get the descending IOWHA (DIOWHA) operator and the ascending IOWHA (AIOWHA) operator.

When $\lambda = 2$, we get the IOWQA operator.

$$IGOWA(\langle u_1,a_1\rangle, \langle u_2,a_2\rangle, \ldots, \langle u_n,a_n\rangle) = \left(\sum_{j=1}^{n} w_j b_j^2 \right)^{1/2} \tag{33}$$

In this case, we obtain the descending IOWQA (DIOWQA) operator and the ascending IOWQA (AIOWQA) operator.

Note that other families could be obtained by using different values in the parameter λ. Note also that it is possible to study these families individually. Then, we could develop for each case, a similar analysis as it has been developed in Section 3 and 4.1, where we study different properties and families of the induced aggregation operators.

5. Induced Quasi-OWA operator

As it is explained in (Beliakov, 2005), a further generalization of the GOWA operator is possible by using quasi-arithmetic means. Following a similar methodology, we can suggest a similar generalization of the IGOWA operator by using quasi-arithmetic means. Then, we will get the Quasi-IOWA operator. It can be defined as follows.

Definition 6. A Quasi-IOWA operator of dimension n is a mapping $QIOWA$: $R^n \rightarrow R$ that has an associated weighting vector W of dimension n such that the sum of the weights is 1 and $w_j \in [0,1]$, then:

$$QIOWA(\langle u_1,a_1\rangle, \langle u_2,a_2\rangle, \ldots, \langle u_n,a_n\rangle) = g^{-1}\left(\sum_{j=1}^{n} w_j \, g\left(b_{(j)}\right)\right) \quad (34)$$

where b_j is the a_i value of the Quasi-IOWA pair $\langle u_i,a_i\rangle$ having the jth largest u_i, u_i is the order inducing variable, a_i is the argument variable and $g(b)$ is a general strictly monotone function. As we can see, we replace b^{λ} with a general continuous strictly monotone function $g(b)$.

Note that in this case we can also distinguish between descending (Quasi-DIOWA) and ascending (Quasi-AIOWA) orders. The weights of these operators are related by $w_j = w^*_{n+1-j}$, where w_j is the jth weight of the Quasi-DIOWA (or Quasi-IOWA) operator and w^*_{n+1-j} the jth weight of the Quasi-AIOWA operator.

Note also that all the properties and particular cases commented in the IGOWA operator are also applicable in this generalization. Then, the Quasi-IOWA operator is monotonic, bounded, idempotent and commutative. The problem of ties is solved by replacing the tied arguments by the quasi-arithmetic mean. And different families of Quasi-IOWA operator can be studied such as the olympic-Quasi-IOWA, the S-Quasi-IOWA, , the IOWQA, etc.

A further interesting aspect is that the Quasi-IOWA operator includes a lot of other particular cases that are not included in the IGOWA operator. For example, we could mention the trigonometric IOWA operator, the exponential IOWA operator and the radical IOWA operator.

The trigonometric IOWA is found when $g_1(t) = \sin((\pi/2)\ t)$, $g_2(t) = \cos((\pi/2)\ t)$ and $g_3(t) = \tan((\pi/2)\ t)$ are the generating functions. Then, the trigonometric IOWA functions are:

$$QIOWA(\langle u_1,a_1\rangle, \langle u_2,a_2\rangle, \ldots, \langle u_n,a_n\rangle) = \frac{2}{\pi}\arcsin\left(\sum_{j=1}^{n} w_j \sin\left(\frac{\pi}{2} b_j\right)\right) \tag{35}$$

$$QIOWA(\langle u_1,a_1\rangle, \langle u_2,a_2\rangle, \ldots, \langle u_n,a_n\rangle) = \frac{2}{\pi}\arccos\left(\sum_{j=1}^{n} w_j \cos\left(\frac{\pi}{2} b_j\right)\right) \tag{36}$$

$$QIOWA(\langle u_1,a_1\rangle, \langle u_2,a_2\rangle, \ldots, \langle u_n,a_n\rangle) = \frac{2}{\pi}\arctan\left(\sum_{j=1}^{n} w_j \tan\left(\frac{\pi}{2} b_j\right)\right) \tag{37}$$

The exponential IOWA is found when $g(t) = \gamma^t$, if $\gamma \neq 1$, and $g(t) = t$, if $\gamma = 1$. Then, the exponential IOWA operator is: $\log_\gamma\left(\sum_{j=1}^{n} w_j \gamma^{b_j}\right)$, if $\gamma \neq 1$, and the IOWA if $\gamma = 1$.

The radical IOWA is found if $\gamma > 0$, $\gamma \neq 1$, and the generating function is $g(t) = \gamma^{1/t}$. Then, the radical IOWA operator is:

$$QIOWA(\langle u_1,a_1\rangle, \langle u_2,a_2\rangle, \ldots, \langle u_n,a_n\rangle) = \left(\log_\gamma\left(\sum_{j=1}^{n} w_j \gamma^{1/b_j}\right)\right)^{-1} \tag{38}$$

Finally, note that in these cases it is also possible to study their properties and different particular cases as it has been explained in Section 3 and 4.1.

6. Illustrative example

In the following, we are going to develop an illustrative example of the new approach in a decision making problem. We will study an investment selection problem where an investor is looking for an optimal investment. Note that other decision making applications could be developed such as the selection of financial products (Merigó and Gil-Lafuente, 2007), etc.

We will analyze different particular cases of the IGOWA operator such as the AM, the WA, the OWA, the AOWA, the IOWA, the AIOWA, the QA, the IOWG, the IOWQA, the step-IOWA (k = 2), the median-IOWA and the olympic-IOWA. Note that with this analysis, we can analyze the optimal choice depending on the aggregation operator used. Then, we will see that each aggregation operator may lead to different results and decisions. Obviously, the question, as in other decision making problems, is the selection of the aggregation operator. By now, the answer we can give is that each decision maker will select one or more aggregation operators according to its interests. And depending on the aggregation operator used, his decisions will be different. The main advantage of the IGOWA is that it includes a wide range of particular cases that can be considered in the decision making problem. Then, the decision maker can consider a lot of possibilities and select the aggregation operator that is in accordance with its interests.

Assume an investor wants to invest some money in an enterprise in order to get high profits. Initially, he considers five possible alternatives.

- A_1 is a computer company.
- A_2 is a chemical company.
- A_3 is a food company.
- A_4 is a car company.
- A_5 is a TV company.

In order to evaluate these investments, the investor uses a group of experts. This group considers that the key factor is the economic environment of the economy. After careful analysis, they consider five possible situations for the economic environment: S_1 = Negative growth rate, S_2 = Growth rate near 0, S_3 = Low growth rate, S_4 = Medium growth rate, S_5 = High growth rate. The expected results depending on the situation S_i and the alternative A_k are shown in Table 1.

Table 1. Payoff matrix

	S_1	S_2	S_3	S_4	S_5
A_1	80	50	70	40	60
A_2	60	30	80	80	40
A_3	70	50	20	70	90
A_4	50	40	60	60	70
A_5	20	50	50	80	80

In this problem, the experts assume the following weighting vector: W = (0.1, 0.2, 0.2, 0.2, 0.3). Due to the fact that the attitudinal character is very complex because it involves the opinion of different members of the board of directors, the experts use order inducing variables to express it. The results are represented in Table 2.

Table 2. Inducing variables

	S_1	S_2	S_3	S_4	S_5
A_1	17	10	15	22	12
A_2	15	20	22	25	13
A_3	24	18	20	22	15
A_4	16	19	21	25	28
A_5	18	12	26	23	21

With this information, we can aggregate the expected results for each state of nature in order to take a decision. In Table 3 and 4, we present different results obtained by using different types of IGOWA operators.

Table 3. Aggregated results 1

	AM	WA	OWA	AOWA	IOWA	AIOWA
A_1	60	58	56	64	61	59
A_2	58	56	53	63	54	62
A_3	60	62	53	67	62	58
A_4	56	58	53	59	54	58
A_5	56	62	50	62	56	56

Table 4. Aggregated results 2

	QA	IOWQA	IOWG	Step	Median	Olympic
A_1	56.92	62.36	59.58	80	70	70
A_2	61.48	57.44	50.41	80	30	56.6
A_3	64.49	66.93	54.92	70	20	46.6
A_4	56.92	54.77	53.19	60	60	53.3
A_5	60.33	60.33	50.23	80	80	60

If we establish an ordering of the alternatives, a typical situation if we want to consider more than one alternative, then, we get the following results shown in Table 5. Note that the first alternative in each ordering is the optimal choice.

Table 5. Ordering of the investments

	Ordering		Ordering
AM	$A_1=A_3 \succ A_2 \succ A_4=A_5$	QA	$A_3 \succ A_2 \succ A_5 \succ A_1=A_4$
WA	$A_3=A_5 \succ A_1=A_4 \succ A_2$	IOWQA	$A_3 \succ A_1 \succ A_5 \succ A_2 \succ A_4$
OWA	$A_1 \succ A_2=A_3=A_4 \succ A_5$	IOWG	$A_1 \succ A_3 \succ A_4 \succ A_2 \succ A_5$
AOWA	$A_3 \succ A_1 \succ A_2 \succ A_5 \succ A_4$	Step-IOWA	$A_1=A_2=A_5 \succ A_3 \succ A_4$
IOWA	$A_3 \succ A_1 \succ A_5 \succ A_2=A_4$	Median-IOWA	$A_5 \succ A_1 \succ A_4 \succ A_2 \succ A_3$
AIOWA	$A_2 \succ A_1 \succ A_3=A_4 \succ A_5$	Olympic-IOWA	$A_1 \succ A_5 \succ A_2 \succ A_4 \succ A_3$

As we can see, depending on the aggregation operator used, the ordering of the investments may be different. Then, the decision about which investment or investments select may be also different.

7. Conclusions

In this paper, we have presented the IGOWA operator. It uses the main characteristics of two well known aggregation operators: the GOWA and the IOWA operator. Therefore, this operator uses generalized means and order inducing variables in the reordering of the arguments. Then, it can be seen from two different points of view: as a generalization of the IOWA operator by using generalized means or as an extension of the GOWA operator that uses order inducing variables in the reordering process. With the IGOWA operator, we have been able to generalize a wide range of OWA operators that include all the cases of the IOWA and the GOWA operator, and a lot of other cases such as the IOWG and the IOWQA operator. Moreover, we have further generalized the IGOWA operator by using quasi-arithmetic means. As a result, we have obtained the Quasi-IOWA operator which is a wider generalization that includes the IGOWA operator as a particular case and a lot of other cases.

We have also developed a numerical example of the new approach in order to see the applicability of the IGOWA operator in a financial decision making problem. The main advantage of this aggregation operator is that it includes a wide range of special cases. Then, depending on the special case used, the results and decisions may be different.

In future research, we expect to develop further extensions by adding new characteristics in the problem such as the use of uncertain information represented in the form of interval numbers, fuzzy numbers, linguistic variables, etc. We will also consider other decision making problems such as strategic decision making, product management, etc.

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