Unification of Fundamental Interactions

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Abstract: In this work, we study the possibility of unification of electromagnetic, weak and strong interactions. In particular, we discuss the value of the unification scale for the Standard Model and for the newer Supersymmetry theory. Eventually, our results prove that no unification is possible in the Standard Model, but an unified theory can be built in a supersymmetric theory.

I. INTRODUCTION

The present understanding of fundamental particle interactions is described by the Standard Model (SM) (for a review see e.g. [1], for an introduction see e.g. [2, 3]), which describes the electromagnetic, weak and strong interactions. Gravity is described by General Relativity and will not be discussed in this work.

All three interactions are mediated by different gauge bosons, which interact with quarks and leptons in many ways. Moreover, each interaction can be undestood mathematically as the theory of a specific gauge group, as it is discussed below, and SM as the theory with symmetry group $\mathrm{U}(1)_Y \times \mathrm{SU}(2) \times \mathrm{SU}(3)$. In order to make the discussion easier, the concept of *color* charge is used, which are actually independent numerical quantities associated to each *color*.

The theory for the strong interaction is quantum chromodynamics, which can be studied as the coupling of eight gauge bosons, named gluons, to three *color* charges. Then, a coupling constant, g_3, arises so as to describe this theory. Mathematically, the theory of gauge group for strong interaction is SU(3).

The theory for the weak interaction includes two *color* charges, which are different from the strong interaction ones, and three bosons $\mathrm{W}^{\pm}$, W^0. As before, a coupling constant, g_2, emerges. Then, weak interaction is the theory of gauge group SU(2).

The electromagnetic interaction theory, known as quantum electrodynamics theory, is the theory of a single gauge boson, the photon, coupled to a single *color* charge with coupling constant g_{em} and it can be understood as the theory of gauge group U(1). Though, in SM one must introduce the theory of gauge group $\mathrm{U}_Y(1)$, where Y is the hypercharge, with gauge boson B and coupling constant g_Y. Here, the photon is a linear combination of bosons W^0 and B. For this reason, the coupling constants g_{em}, g_2, g_Y are related to each other by the weak mixing angle θ_W, known as Weinberg angle (see Eq. (9) and Eq. (10)).

Hence, all three theories can be studied as the coupling of gauge bosons to *color* charges with a coupling constant for each specific theory. Table I shows the SM particle content and their respective interactions. Since the three interactions are described with analogous theories, the idea of a possible unification of the three of them emerges.

		SU(3)	**SU(2)**	**U$_Y$(1)**
Spin 1/2 (fermions)	$\begin{pmatrix} u \\ d \end{pmatrix}_L$	$\surd$	$\surd$	$+\frac{1}{3}$
	u_R	$\surd$	$\times$	$+\frac{4}{3}$
	d_R	$\surd$	$\times$	$+\frac{2}{3}$
	$\begin{pmatrix} \nu \\ e \end{pmatrix}$	$\times$	$\surd$	-1
	e_R	$\times$	$\times$	-2
Higgs boson	$\begin{pmatrix} H^0 \\ H^- \end{pmatrix}$	$\times$	$\surd$	$+1$
Gluons	g	$\surd$	$\times$	0
EW gauge bosons	W	$\times$	$\surd$	0
	B	$\times$	$\times$	0

TABLE I: Particles in the SM, their interactions and their respective hypercharge values.

However, if such an unified theory exists, the $\mathrm{U}(1)_Y \times \mathrm{SU}(2) \times \mathrm{SU}(3)$ gauge theories must be included in it so as to include the three gauge groups. In addition, the electromagnetic charge is dependent on the other five *color* charges, so five charges are enough to build such a theory (see [4]). In order to have it, one should have just one gauge coupling instead of three.

At this point, quantum field theory plays a crucial role discovering that the value of the coupling constants is not constant, but changes with energy ([5, 6], for an introduction see e.g. [4]). Then, it may exist an energy scale Λ_{GUT} for which the values of the three different coupling constants are equal. For greater energy scales than Λ_{GUT} the constants remain as one in a unified the-

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ory while for smaller energy scales we have the separate $U_Y(1)\times SU(2)\times SU(3)$ theories, each with its own constant. Then, being able to create an unified theory means to run the coupling constants with the aim to obtain Λ_{GUT} and the unified coupling. Besides, a symmetry relation describing how the three constants unify and a way in which it breaks into the three initial ones should be introduced.

Finally, an equation describing the variation of the constants with the energy is needed for the purpose of running them. As already mentioned above, quantum field theory and its study of renormalization group introduced the main result used to do so (see [5, 6]). This result is the renormalization group equation (RGE, Eq. (1)), which gives us how gauge couplings change with energy, known as β-functions:

$$\frac{dg_i}{dt} = \beta_i(g_i) \ , \tag{1}$$

where $t = \ln \mu$, and μ is the renormalization group scale, that can be written such as $t = \ln \dfrac{\mu}{\Lambda_{GUT}}$.

II. THEORETICAL CALCULATION OF UNIFICATION SCALE

The Eq. (1) has not a trivial solution, but the β-functions can be expanded using perturbation theory. Restricting our work to 1-loop expansion, we are left with a simpler equation to start with [5, 6]:

$$\frac{dg_i}{dt} = -\frac{b_i}{(4\pi)^2} g_i^3 \Rightarrow \frac{d\alpha_i}{dt} = -\frac{b_i}{2\pi}\alpha_i^2 \ , \tag{2}$$

where we introduced $\alpha_i = \dfrac{g_i^2}{4\pi}$. As one can see, the value of the constants depend only on the b_i coefficients, which are independent of α_i and are obtained with the generators of SU(N) groups. Solving Eq. (2), we obtain:

$$\frac{1}{\alpha_i(Q)} = \frac{1}{\alpha_i(Q_0)} + \frac{b_i}{2\pi}\ln\frac{Q}{Q_0} \ . \tag{3}$$

Now, knowing that our goal is to obtain the scale Λ_{GUT}, we have to compute the scale for which we get $\alpha_i = \alpha_j$ for $i \neq j$ and $i, j = 1, 2, 3$. That is, indeed, computing Λ_{GUT} using Eq. (4) (notice here that we use Λ^{ij}_{GUT} when referring to the energy scale obtained when $\alpha_i = \alpha_j$).

$$\begin{aligned} 0 &= \frac{1}{\alpha_i(\Lambda^{ij}_{GUT})} - \frac{1}{\alpha_j(\Lambda^{ij}_{GUT})} = \\ &= \frac{1}{\alpha_i(Q_0)} - \frac{1}{\alpha_j(Q_0)} + \frac{1}{2\pi}(b_j - b_i)\ln\frac{\Lambda^{ij}_{GUT}}{Q_0} \ . \end{aligned} \tag{4}$$

Finally, in order to get Λ_{GUT} one still has to calculate the b_i coefficients.

A. Calculation of the b_i coefficients

As mentioned above, the b_i coefficients depend on the group structure. Thus, gauge bosons contribute to the coefficints with $\frac{11}{3}N$ for SU(N) and 0 for U(1) [7]. Each fermion contributes with $-\frac{2}{3}T_R$ and each complex scalar with $-\frac{1}{3}T_R$, where for the fundamental representation one has $T_R = \frac{1}{2}$ for SU(N) and $T_R = Y^2$ for $U(1)_Y$ with hypercharge Y, and for the adjoint representation one has $T_R = N$ for SU(N).

Since the normalization of $U_Y(1)$ is arbitrary and we are seeking for a gauge group G_{GUT} such that $U_Y(1)\times SU(2)\times SU(3) \subset G_{GUT}$ all generators from $U_Y(1)$ must be normalized as SU(2) and SU(3) generators are normalized. This results in $\alpha_1 = \frac{5}{3}\alpha_Y$ and $b_1 = \frac{3}{5}b_Y$.

Working in the SM, where particles are in the fundamental representation, we have:

$$b_1 = \frac{3}{5}b_Y = \frac{3}{5}\left(-\frac{20}{9}N_G - \frac{1}{6}N_H\right) = \tag{5}$$

$$= -\frac{4}{3}N_G - \frac{1}{10}N_H \ , \tag{6}$$

$$b_2 = \frac{11}{3}2 - \frac{4}{3}N_G - \frac{1}{6}N_H \ , \tag{7}$$

$$b_3 = \frac{11}{3}3 - \frac{4}{3}N_G \ , \tag{8}$$

where N_G is the number of generations of fermions, each of them formed by four fermions, and N_H is the number of complex scalars.

Then, in the SM there are 3 generations of fermions and one complex scalar (Higgs boson), so the value of the coefficients are $b_1 = -\dfrac{41}{10}, b_2 = \dfrac{19}{6}$ and $b_3 = 7$.

One can notice here that a full generation of fermions has the same contribution to all b_i's. Then, in Eq. (4), there will not be any change in the term $b_j - b_i$ if we have different numbers of generations since it will always cancel. This result implies that if a new theory introduces extra generations of fermions, it will not make any difference when it comes to unification since the only variation will be the value of $\alpha_i(\Lambda_{GUT})$.

III. NUMERICAL ANALYSIS

The main part of this work is to compute Λ_{GUT} from experimental values. Then, one needs to fix an energy scale from which to start running. Here, the initial conditions used are at the energy scale of M_Z, resulting in the values of Table II.

The existence of electroweak theory, which unifies electromagnetic and weak theories, must be taken into account because leaves us with just two independent coupling constants that can be related using the weak mixing angle θ_W as in Eq. (9) and Eq. (10) (see e.g. [2, 3]).

M_Z [GeV]	91.1876	$\pm$ 0.0021
$\sin^2\theta_W(M_Z)$	0.23122	$\pm$ 0.00004
$\alpha_3(M_Z)$	0.1185	$\pm$ 0.0016
$\alpha_{em}^{-1}(M_Z)$	127.951	$\pm$ 0.009

TABLE II: Experimental magnitudes used to fix the initial conditions, from [1].

$$\alpha_1 = \frac{5}{3}\alpha_Y = \frac{5}{3}\frac{\alpha_{em}}{1-\sin^2\theta_W} \tag{9}$$

$$\alpha_2 = \frac{\alpha_{em}}{\sin^2\theta_W} \tag{10}$$

Finally, using the initial conditions from Table II, Eq. (9), Eq. (10) and Eq. (4) we get three different values of the unification scale for each couple of gauge couplings.

As it can be easily noticed in Table III, the unification scale for each couple of constants is completely different, with a considerable difference between their orders of magnitude. This result is graphically presented in Fig. 1, where this contrast becomes evident. Thus, we can conclude that unification in the SM is not possible.

IV. SUPERSYMMETRY

The lack of possibility to have unification of the three interactions in the SM made physicists search for new theories that do unify. In this work, we'll study the chance of unification using one of those theories, named Supersymmetry (SUSY) (for an introduction see e.g. [8]).

SUSY is a symmetry that transforms particles with a given spin to particles with the same properties but different spin, named superpartners. To be more specific, it just changes the value of the spin of a particle by $\hbar/2$. This results in a transformation from bosons to fermions and viceversa that keeps the rest of quantum numbers invariant. Then, each particle has its own superpartner.

As it was already mentioned, the b_i coefficients depend on the group structure, which will change with the adding of these new particles. Thus, when calculating b_i for SUSY one has the particles of SM and the following new particles:

- Complete generations of s-quarks and s-leptons, SUSY partners of SM fermions, which are complex scalars in the fundamental representation.

- Gluinos, SUSY partners of SM gluons, which are fermions in the adjoint representation.

- A Higgs doublet, which introduces an extra Higgs boson in the fundamental representation.

- Higginos, SUSY partners of Higgs boson, which are fermions in the fundamental representation.

Λ_{GUT}^{12}	$= 1.030 \cdot 10^{13}$	$\pm\ 6 \cdot 10^{10}$
Λ_{GUT}^{13}	$= 2.48 \cdot 10^{14}$	$\pm\ 1 \cdot 10^{12}$
Λ_{GUT}^{23}	$= 1.030 \cdot 10^{17}$	$\pm\ 8 \cdot 10^{14}$

TABLE III: Values of the unification scales in the SM for each pair of coupling constants. All values are in GeV.

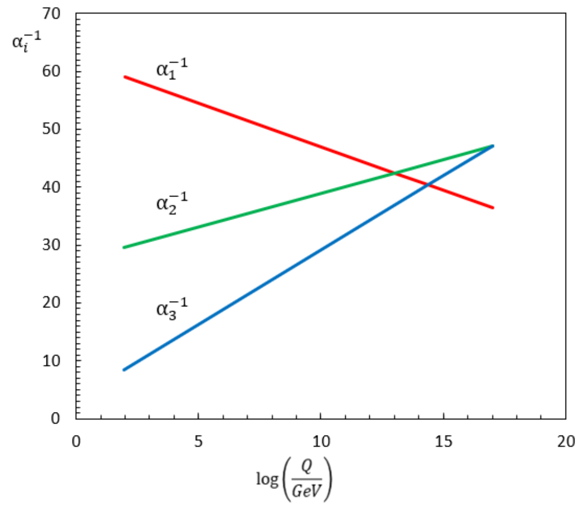

FIG. 1: Evolution of the inverse of the three coupling constants with respect of the logarithm of energy scale in the SM. The points were the lines cross are the unification scales for each couple of couplings.

- Electroweak gauginos, SUSY partners of electroweak gauge bosons, which are fermions in the adjoint representation.

Consequently, using the discussion above, in SUSY the value of the b_i coefficients are $b_1 = -\frac{20}{3}, b_2 = -1$ and $b_3 = 3$.

Now, using again the initial conditions from Table II, Eq. (9), Eq. (10) and Eq. (4) with the values of the b_i coefficients for SUSY we get three different values of the unification scale, just as before.

In Table IV, we observe how close the three values are to each other. As we can see, all three values have the same order of magnitude. However, they do not have the exact same value even if we consider the errors. Even with this small discrepancy in the values, we could conclude that they unify at $\Lambda_{GUT} \sim 2\cdot10^{16}$ GeV. In order to visualize the evolution of the couplings with energy and to graphically confirm the unification one can see Fig. 2.

One last remark before one is ready to afirm that the value of Λ_{GUT} is correct has to do with a lower limit for it (for a review see e.g. [9]). Grand Unification implies the existence of new particles and interactions which mediate proton decay, whose lifetime grows with the new particle masses. Since proton decay has never been detected, the proton lifetime must be at least the age of the universe. This implies that unification must occur at an energy

Λ_{GUT}^{12}	$= 2.009 \cdot 10^{16}$	$\pm\ 1.2 \cdot 10^{14}$
Λ_{GUT}^{13}	$= 2.174 \cdot 10^{16}$	$\pm\ 1.3 \cdot 10^{14}$
Λ_{GUT}^{23}	$= 2.429 \cdot 10^{16}$	$\pm\ 1.9 \cdot 10^{14}$

TABLE IV: Values of the unification scales in SUSY for each pair of coupling constants. All values are in GeV.

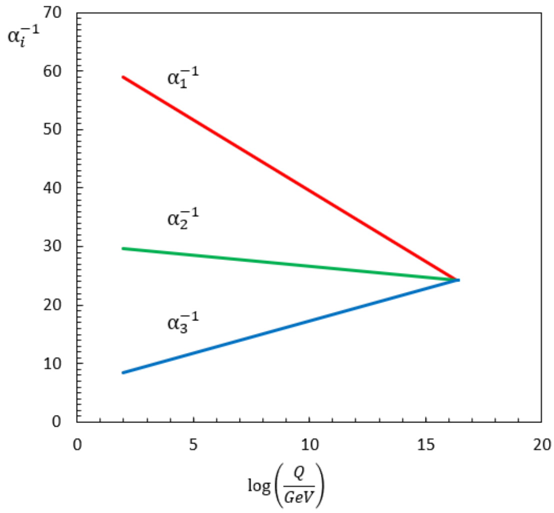

FIG. 2: Evolution of the inverse of the three coupling constants with respect of the logarithm of the energy scale in SUSY. The point were the lines cross is the unification scale.

scale $\geq 10^{16}$ GeV. Hence, if there is a scale Λ_{GUT} for which the three theories unify, then $\Lambda_{GUT} \geq 10^{16}$ GeV.

Thus, since we have $\Lambda_{GUT} \sim 2 \cdot 10^{16} > 10^{16}$ GeV, we can conclude that in SUSY the three interactions do unify.

V. CONCLUSIONS

In this work, we conclude that an unified theory is not possible in SM, but it is in SUSY. In order to do so, we started from the renormalization group equation at a 1-loop expansion and the discussion of the value of the b_i coefficients depending on the group structure.

Using this known results, we solved the differential equation so we could compare the evolution of the three coupling constants with respect to the energy scale. We also computed the value of the b_i coefficients both in SM and SUSY.

Then, using experimental results and how the weak mixing angle relates α_{em} with α_1 and α_2, we were able to compute the unification scales for each couple of coupling constants.

In one hand, by comparing the different unification scale values obtained when working in the SM, we noticed a big gap in their orders of magnitudes. Thus, we concluded that an unified theory is not possible in the SM.

On the other hand, when comparing those values when working in SUSY, the gap disappears and all three gauge couplings meet at $\sim 2 \cdot 10^{16}$ GeV. An important remark to be made is that this result is $\geq 10^{16}$ GeV, which means that is coherent with the proton lifetime. Hence, we found that an unified theory in SUSY is possible.

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