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**REGULARITY OF LIPSCHITZ
DOMAINS IN TERMS OF THEIR
POISSON KERNEL**

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Abstract

We study the regularity of Lipschitz domains in terms of the regularity of their Poisson kernel. More precisely, we present a proof of the fact that if Ω is a bounded Lipschitz domain and its Poisson kernel is a positive $C^{k,\alpha}$ function, then $\partial\Omega$ is $C^{k+1,\alpha}$. The argument is based on the study of Lipschitz free boundaries for weak solutions of the Alt–Caffarelli problem. We prove optimal regularity and nondegeneracy estimates for weak solutions, establish an ϵ -regularity theorem, and study blow-ups at the free boundary using Weiss’ monotonicity formula. The Lipschitz geometry of the free boundary allows us to show that all blow-up limits are half-plane solutions, which implies the desired regularity of the free boundary.

Resum

Estudiem la regularitat dels dominis Lipschitz en termes de la regularitat del seu nucli de Poisson. Més precisament, presentem una demostració del fet que, si Ω és un domini Lipschitz fitat i el seu nucli de Poisson és una funció positiva de classe $C^{k,\alpha}$, aleshores $\partial\Omega$ és $C^{k+1,\alpha}$. L’argument es basa en l’estudi de les fronteres lliures Lipschitz de solucions febles del problema d’Alt–Caffarelli. Demostrem estimacions de regularitat òptima i de no-degeneració per a solucions febles, establim un teorema de ϵ -regularitat, i estudiem *blow-ups* a la frontera lliure mitjançant la fórmula de monotonia de Weiss. La geometria Lipschitz de la frontera lliure ens permet demostrar que tots els límits de *blow-up* són solucions de semiespai, fet que implica la regularitat de la frontera lliure.

Notation: Throughout this work, the ambient space is $\mathbb{R}^n$, $n \geq 2$.

- $x = (x', x_n)$: point in $\mathbb{R}^n$, with $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$.
- $|x|$: modulus of the point x , $|x| = \sqrt{x_1^2 + \dots + x_n^2}$.
- $e_1, \dots, e_n$: standard basis vectors of $\mathbb{R}^n$.
- $x \cdot y$: Euclidean scalar product in $\mathbb{R}^n$.
- $B_r(x)$: open ball of radius r centred at x in $\mathbb{R}^n$.
- $B_r := B_r(0)$.
- $B'_r(y)$: open ball of radius r centred at y in $\mathbb{R}^{n-1}$.
- $B'_r := B'_r(0)$.
- $\mathbb{R}_+^n := \{x \in \mathbb{R}^n : x_n > 0\}$, $\mathbb{R}_-^n := \{x \in \mathbb{R}^n : x_n < 0\}$.
- $B_r^+ := B_r \cap \mathbb{R}_+^n$, $B_r^- := B_r \cap \mathbb{R}_-^n$.
- $S^{n-1} := \partial B_1$: the unit sphere in $\mathbb{R}^n$.
- For a set $E \subset \mathbb{R}^n$, ∂E , $\bar{E}$, E° , and E^c denote respectively its boundary, closure, interior, and complement in the ambient space. For a pair of sets $E, F \subset \mathbb{R}^n$, $E \triangle F$ denotes the symmetric difference, $E \triangle F := (E \setminus F) \cup (F \setminus E)$.
- $D \subset\subset U$: $\bar{D}$ is compact and contained in U .
- $\mu|_E$: restriction of a measure μ to a set E .
- $|E|$: n -dimensional Lebesgue measure of a set $E \subset \mathbb{R}^n$.
- $\mathcal{H}^s$: s -dimensional Hausdorff measure.
- $\mathcal{L}^m$: m -dimensional Lebesgue measure.
- “a.e.” means almost everywhere, with respect to the measure indicated by context.
- $\text{dist}(x, E) := \inf\{|x - y| : y \in E\}$: distance from x to a set E . We also write $d_\Omega(x) := \text{dist}(x, \partial\Omega)$.
- $f|_E$: restriction of a function f to a set E .
- χ_E : characteristic function of a set E .
- $\{f > 0\} := \{x \in \mathbb{R}^n : f(x) > 0\}$: positivity set of f .
- $t_+ := \max\{t, 0\}$, $t_- := \max\{-t, 0\}$.
- $\text{supp } f$: support of a function f .
- $\text{osc}_E f := \sup_E f - \inf_E f$: oscillation of f on E .

- $\oint_E f d\mu$: average of f over the set E with respect to a measure μ ,

$$\oint_E f d\mu := \frac{1}{\mu(E)} \int_E f d\mu.$$

- $C(D)$: space of continuous functions on D .
- $C^k(D)$, $C^{k,\alpha}(D)$, and $C^{0,1}(D)$: respectively the spaces of k -times continuously differentiable, $C^{k,\alpha}$ -Hölder, and Lipschitz functions on D .
- $C_c(D)$, $C_c^\infty(D)$, and $C_c^{0,1}(D)$: continuous, smooth, and Lipschitz functions, respectively, with compact support in D .
- $L^p(D)$, $H^1(D)$: the usual Lebesgue and Sobolev spaces.
- loc : $f \in X_{\text{loc}}(D)$ means that $f \in X(K)$ for every $K \subset\subset D$, whenever this notation is meaningful.
- $\|\cdot\|_X$: norm in the space X .
- $\partial_{x_i} := \frac{\partial}{\partial x_i}$, and

$$\nabla f = (\partial_{x_1} f, \dots, \partial_{x_n} f), \quad \Delta f = \text{div}(\nabla f) = \sum_{i=1}^n \partial_{x_i}^2 f.$$

Subscripts on differential operators indicate the variable with respect to which the operator is taken; for example, ∇_y and Δ_y .

- Df : differential of f . If f is scalar-valued, Df is identified with ∇f ; if f is vector-valued, Df is its Jacobian matrix.
- $D^k f$: collection of all k -th order partial derivatives of f . In particular, $D^2 f$ denotes the Hessian matrix of f . $|D^k f(x)|$ denotes any equivalent norm of $D^k f(x)$.
- For a vector field ξ , $D\xi$ denotes its Jacobian matrix and $\text{div } \xi := \text{tr}(D\xi)$.
- If ν is a unit vector, then $\partial_\nu f := \nabla f \cdot \nu$ and $\partial_{-\nu} f := \nabla f \cdot (-\nu)$.
- P. V.: Cauchy principal value integral.
- Γ : gamma function,

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad x \in \mathbb{R}^+.$$

- Q : Continuous function in $\mathbb{R}^n$ such that $0 < Q_{\min} \leq Q \leq Q_{\max} < \infty$.
- C, c : positive constants whose value may change from line to line. Dependencies are indicated in parentheses or subscripts, for example $C(n)$, c_θ , or C_Q .

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Chapter 1

Introduction

The Laplacian

$$-\Delta := -(\partial_{x_1}^2 + \cdots + \partial_{x_n}^2)$$

is among the most important differential operators in analysis. For functions $u \in C^2$, the Taylor expansion of u around x can be used to prove that

$$u(x) - \oint_{B_r(x)} u(y) dy = \frac{r^2}{2(n+2)}(-\Delta u(x)) + o(r^2) \quad (r \rightarrow 0), \quad (1.1)$$

showing that $-\Delta u(x)$ quantifies, to second order, the deviation of u from its local average.

Because the Laplacian measures such a natural property of functions, differential equations involving it arise throughout mathematics and mathematical modelling. In particular, solutions to $-\Delta u = 0$, called harmonic functions, are especially important and appear, for instance, in electrostatics (as equilibrium potentials), in heat conduction (as steady-state temperature distributions), in the study of Brownian motion, and in the theory of minimal surfaces (see [28, Chapter 12] and [20, Section 1.1] for further examples).

Moreover, in the field of partial differential equations (PDE), the Laplacian is the simplest example of a second-order partial differential operator. As such, it serves as a prototype for a large class of equations: many important ideas and techniques in PDE were first developed for it before being extended to more complicated operators.

1.1 The Poisson kernel

A basic problem associated with the Laplacian is the Dirichlet problem for the Poisson equation. Given a domain $\Omega \subset \mathbb{R}^n$, a source function f in Ω , and a boundary datum g on $\partial\Omega$, one asks whether there is a function u such that

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \quad (1.2)$$

One way to obtain solutions to (1.2) is through a representation formula, first derived by the self-taught mathematician George Green in his 1828 essay [31].

If Ω is a sufficiently regular bounded domain, then for each $x \in \Omega$ there exists a function $G_\Omega(x, \cdot)$, called the Green function for Ω with pole at x , such that

$$\begin{cases} -\Delta_y G_\Omega(x, y) = \delta_x & \text{in } \Omega, \\ G_\Omega(x, y) = 0 & \text{on } \partial\Omega, \end{cases}$$

where δ_x denotes the Dirac delta function at x .

The solution u of (1.2) can then be represented as

$$u(x) = \int_{\Omega} G_\Omega(x, y) f(y) dy + \int_{\partial\Omega} \partial_\nu G_\Omega(x, y) g(y) d\mathcal{H}^{n-1}(y), \quad (1.3)$$

where $\partial_\nu G_\Omega(x, y) := \nabla_y G_\Omega(x, y) \cdot \nu(y)$ denotes the inward normal derivative of G_Ω with respect to the variable y .

When $f = 0$, the representation formula becomes

$$u(x) = \int_{\partial\Omega} \partial_\nu G_\Omega(x, y) g(y) d\mathcal{H}^{n-1}(y).$$

Therefore, harmonic functions in Ω can be recovered from their boundary values by integration against the Poisson kernel,

$$P_\Omega^x(y) := \partial_\nu G_\Omega(x, y). \quad (1.4)$$

By the representation formula, the study of solutions to (1.2) is reduced to the study of the Green function, provided we can construct it for the domain in question. For example, in the case of the unit ball, the Green function is given by

$$G_{B_1}(x, y) = \begin{cases} \frac{1}{(n-2)\mathcal{H}^{n-1}(\partial B_1)} \left(|x-y|^{2-n} - |y|^{2-n} \left| x - \frac{y}{|y|^2} \right|^{2-n} \right), & n \geq 3, \\ \frac{1}{2\pi} \log \left(\frac{|y| \left| x - \frac{y}{|y|^2} \right|}{|x-y|} \right), & n = 2, \end{cases} \quad (1.5)$$

with the expression understood by taking the limit when $y = 0$, and the resulting Poisson kernel is

$$P_{B_1}^x(y) = \frac{1}{\mathcal{H}^{n-1}(\partial B_1)} \frac{1 - |x|^2}{|x-y|^n}, \quad y \in \partial B_1. \quad (1.6)$$

Observe, in particular, that for the unit ball the Poisson kernel $P_{B_1}^x$ turns out to be a smooth function of y .

Although such explicit formulas are exceptional, one can still study the relationship between the geometry of a domain and the regularity properties of its Poisson kernel.

1.2 Regularity of Lipschitz domains in terms of their Poisson kernel

In a celebrated 1977 paper, Dahlberg proved that if Ω is a bounded Lipschitz domain,¹ then G_Ω exists and, for every $x \in \Omega$, the Poisson kernel $P_\Omega^x = \partial_\nu G_\Omega(x, \cdot)$ exists and is positive $\mathcal{H}^{n-1}$ -a.e. on $\partial\Omega$ [12] (see also [44, Theorem 14.14]).

¹The definitions and basic properties of Lipschitz domains, Hölder domains, and Hölder spaces used below are recalled in Appendix A.

Moreover, regularity of the domain implies regularity of the Poisson kernel. Indeed, Schauder regularity theory for the Laplacian (see, e.g., [26, Chapter 2]), developed in the 1930s, implies that if Ω is a bounded $C^{k,\alpha}$ domain with $k \geq 1$, then for every $x \in \Omega$, the Poisson kernel P_Ω^x is a positive function of class $C^{k-1,\alpha}(\partial\Omega)$ [44, Theorems 14.6 and 14.9].²

In this thesis, we address the converse question: *does the regularity of the Poisson kernel imply regularity of the domain?*

More precisely, we will explain the proof of the following theorem.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, and let $x \in \Omega$. If, for some $k \geq 0$ and $0 < \alpha < 1$, the Poisson kernel P_Ω^x is a positive function of class $C^{k,\alpha}(\partial\Omega)$, then $\partial\Omega$ is $C^{k+1,\alpha}$.*

1.2.1 Reduction to the Alt–Caffarelli problem

The proof is based on the fact that the Green function in a Lipschitz domain is the solution of an overdetermined boundary value problem.

Let Ω be a Lipschitz domain and $x_0 \in \Omega$. Fix a boundary point $y_0 \in \partial\Omega$ and let $r > 0$ be small enough so that $x_0 \notin B_r(y_0)$; by rescaling and translating the coordinate system if necessary, we may assume $B_r(y_0) = B_1$. Then $G(y) := G_\Omega(x_0, y)$ is positive and harmonic in $\Omega \cap B_1$ and vanishes on $\partial\Omega \cap B_1$. Assume that the Poisson kernel $P_\Omega^{x_0}$ is of class $C^{k,\alpha}(\partial\Omega)$, that is, that there is a function $Q \in C^{k,\alpha}(\mathbb{R}^n)$ such that $P_\Omega^{x_0} = Q|_{\partial\Omega}$. Since $\partial_\nu G = P_\Omega^{x_0}$ $\mathcal{H}^{n-1}$ -a.e. on $\partial\Omega$, G is, in some sense, a nonnegative solution to the following problem:

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega \cap B_1, \\ u = 0 & \text{on } \partial\Omega \cap B_1, \\ \partial_\nu u = Q & \text{on } \partial\Omega \cap B_1, \end{cases} \quad (1.7)$$

known as the Alt–Caffarelli or one-phase Bernoulli problem.

It turns out that, for proving Theorem 1.1, the only relevant property of G is that it solves (1.7). Because harmonic functions are determined by their boundary values, it is in general not possible to prescribe both the value of u and the value of $\partial_\nu u$ on $\partial\Omega$. Thus the domain Ω itself becomes an unknown of the problem, and we can derive regularity results for the free boundary $\partial\Omega$.

Since the domain is only assumed to be Lipschitz, the normal derivative in (1.7) cannot be interpreted classically, and we must define an appropriate notion of solution. To this end, recall the following integration by parts formula. If D is a sufficiently regular domain and $u, v: D \rightarrow \mathbb{R}$ are sufficiently regular functions,

$$-\int_D \nabla u \cdot \nabla v \, dx = \int_D \Delta u \, v \, dx + \int_{\partial D} \partial_\nu u \, v \, d\mathcal{H}^{n-1},$$

where ν denotes the inward unit normal to D . By [44, Lemma 8.6], this formula can be

²More precisely, $P_\Omega^x = h|_{\partial\Omega}$ for some $h \in C^{k-1,\alpha}$.

applied to the Green function of a Lipschitz domain to obtain

$$-\int_{B_1} \nabla G \cdot \nabla \phi \, dx = \int_{\partial\Omega \cap B_1} P_\Omega^{x_0} \phi \, d\mathcal{H}^{n-1} \quad \forall \phi \in C_c^\infty(B_1). \quad (1.8)$$

With this in mind, we define the following notion of solution to (1.7).

Definition 1.2 (Weak solution). Let $D \subset \mathbb{R}^n$ be a domain and let $Q: \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuous function. A nonnegative function $u \in C(D)$ is a weak solution of the Alt–Caffarelli problem in D for Q if u is harmonic in $\{u > 0\}$, the positivity set $\{u > 0\}$ is a Lipschitz domain, and

$$-\int_D \nabla u \cdot \nabla \phi \, dx = \int_{\partial\{u>0\} \cap D} Q \phi \, d\mathcal{H}^{n-1} \quad \text{for every } \phi \in C_c^\infty(D). \quad (1.9)$$

Remark 1.3. We will prove in Lemma 2.3 that any weak solution u is locally Lipschitz in D , and thus, by Rademacher’s theorem, is differentiable a.e. in D . Before this result is established, ∇u must be understood in a weak sense. That is, $\nabla u \in L_{\text{loc}}^2(D; \mathbb{R}^n)$ is the vector field characterized by

$$-\int_D \nabla u \cdot \Phi \, dx = \int_D u \operatorname{div} \Phi \, dx \quad \forall \Phi \in C_c^\infty(D; \mathbb{R}^n).$$

By [2, Remark 4.2], if $u \in C(D)$ is nonnegative and harmonic in $D \cap \{u > 0\}$, then $u \in H_{\text{loc}}^1(D)$, the Sobolev space of $L_{\text{loc}}^2(D)$ functions whose weak gradient exists and is in $L_{\text{loc}}^2(D; \mathbb{R}^n)$ (see also [5, Chapter 9]).

For Lipschitz functions, the weak gradient agrees a.e. with the classical gradient [24, Theorem 6.5]. In particular, $C_{\text{loc}}^{0,1}(D) \subset H_{\text{loc}}^1(D)$.

Remark 1.4. Definition 1.2 extends the classical notion of solution, in the sense that weak solutions with a smooth free boundary satisfy (1.7) in the classical sense. Indeed, if $\partial\{u > 0\}$ is $C^{1,\alpha}$, Schauder estimates ([30, Corollary 8.36]) yield $u \in C_{\text{loc}}^{1,\alpha}(\overline{\{u > 0\}})$. Thus, integration by parts gives

$$-\int_D \nabla u \cdot \nabla \phi \, dx = \int_{\partial\{u>0\} \cap D} \partial_\nu u \phi \, d\mathcal{H}^{n-1} \quad \forall \phi \in C_c^\infty(D),$$

where ν is the inward unit normal to $\{u > 0\}$. Then (1.9) implies that

$$\int_{\partial\{u>0\} \cap D} (Q - \partial_\nu u) \phi \, d\mathcal{H}^{n-1} = 0 \quad \forall \phi \in C_c^\infty(D),$$

and therefore $\partial_\nu u = Q$ on $\partial\{u > 0\}$.

Returning to Theorem 1.1, by the discussion preceding (1.7) and by (1.8), the Green function G , after a change of coordinates and extension by 0 in $B_1 \setminus \Omega$, is a weak solution to (1.7) with $D = B_1$ and boundary datum $Q = P_\Omega^{x_0}$. Thus, Theorem 1.1 will follow by applying the following theorem locally at each boundary point.

Theorem 1.5 (Lipschitz free boundaries are smooth). *Let k be a nonnegative integer and $0 < \alpha < 1$. Let $Q \in C^{k,\alpha}(\mathbb{R}^n)$ be such that $0 < Q_{\min} \leq Q \leq Q_{\max} < \infty$. If u is a weak solution in B_1 for Q in the sense of Definition 1.2, then*

$$\partial\{u > 0\} \cap B_{1/2} \text{ is } C^{k+1,\alpha}.$$

In particular, if Q is smooth, then the free boundary $\partial\{u > 0\}$ is smooth.

1.3 The theorem in context

The problem of characterising the regularity of Lipschitz domains in terms of their Poisson kernel has a long history. Caffarelli proved in 1987 that if Q is smooth and the free boundary of a certain type of solution to (1.7) is Lipschitz, then the free boundary is smooth [7]. Theorem 1.1 is often attributed to this paper (see, e.g., [36]); however, Caffarelli's results are for so-called viscosity solutions to (1.7), and it is not clear how they apply to the Poisson kernel. This discrepancy was only recently settled by Domingo-Pasarin and Ros-Oton in [21], where they proved Theorem 1.5.

We now turn to the hypotheses of Theorem 1.1. The assumption that Ω is Lipschitz is natural, since for less regular $C^{0,\alpha}$ domains the Poisson kernel might not be a well-defined function. Indeed, for general domains Ω there is still a representation formula for harmonic functions, but it is in terms of a probability measure ω_Ω^x , called the harmonic measure for Ω with pole at x : if u_g is the harmonic function in Ω with boundary values $g \in C(\partial\Omega)$, then

$$u_g(x) = \int_{\partial\Omega} g d\omega_\Omega^x.$$

The harmonic measure is an important object lying at the intersection of harmonic analysis, PDE, and geometric measure theory, and the relationship between the measure-theoretic regularity properties of ω_Ω^x and the geometry of Ω has been intensely studied in the past decades; see, e.g., [44] and the references therein.³ However, the techniques there are very different from those for the Poisson kernel, and we will not discuss this topic further in this thesis.

Returning to the Lipschitz assumption, it is in fact not merely technical: there are non-Lipschitz domains for which the Poisson kernel, understood as the density of harmonic measure with respect to surface measure, is a smooth positive function. In particular, Lewis and Vogel proved that for any dimension $n \geq 3$, there is a nontrivial domain $\Omega \subset \mathbb{R}^n$, with boundary homeomorphic to the unit sphere, such that, for some $x \in \Omega$ and a constant $a > 0$,

$$\omega_\Omega^x = a\mathcal{H}^{n-1}\Big|_{\partial\Omega}.$$

Moreover, the homeomorphism and its inverse can be taken to be $C^{0,1-\epsilon}$, for every small $\epsilon > 0$ [39, 40].

The assumption that Q is positive on the free boundary in Theorem 1.5 is also necessary. For instance, the function $u(x, y) = x_+ y_+$ is a weak solution in $\mathbb{R}^2$ for $Q = x + y$, yet the free boundary $\partial\{u > 0\} = \{x = 0, y \geq 0\} \cup \{x \geq 0, y = 0\}$ is not C^1 at the origin.

The result also does not hold for the limiting value $\alpha = 0$. If P_Ω^x is only assumed to be C^k , the domain Ω need not be C^{k+1} . In fact, even the converse implication does not hold in this case: if the domain is only assumed to be C^k , the Poisson kernel need not be C^{k-1} ; see [26, Section 2.2].

On the other hand, if one strengthens the assumptions so that Ω is $C^{1,\alpha}$ a priori,

³See also [13] for an application of related questions to a model of mammalian lungs.

then higher regularity of the free boundary can be obtained by more standard PDE techniques, in the spirit of Schauder estimates. In fact, a method for proving higher regularity in free boundary problems was developed by Kinderlehrer and Nirenberg in [37], before Caffarelli's breakthrough results for regularity of arbitrary free boundaries in [6].

We conclude this section by noting that Theorem 1.5 is of independent interest, since the Alt–Caffarelli problem arises in a variety of other contexts. Hans Wilhelm Alt and Luis Caffarelli, after whom the problem is named, first studied the problem in the context of the calculus of variations [2].⁴ Specifically, minimisers of the Alt–Caffarelli functional

$$F_Q(u; D) = \int_D (|\nabla u|^2 + Q^2 \chi_{\{u>0\}}) dx \quad (1.10)$$

in an appropriate function space are also solutions to (1.7).

The Alt–Caffarelli problem also appears in physics. For instance, it arises in flow problems of jets and cavities [29, Chapter 3]; the term *Bernoulli problem* comes from this context. It is partly these connections between analysis, geometry, and physics that have made free boundary problems one of the most active and fruitful areas in modern PDE theory.

1.4 Main ideas in the proof

We now outline the main ideas behind the proof of Theorem 1.5. The strategy described has its roots in the regularity theory for minimal surfaces; see, e.g., [11]. It has since been successfully applied in many other problems, especially free boundary problems and boundary regularity problems for PDE (see, e.g., [14, 46]).

The first observation is that, due to the homogeneity of the problem, if u is a weak solution in D and $x_0 \in \partial\{u > 0\}$, then the rescaling

$$u_r(x) := \frac{1}{r} u(x_0 + rx)$$

is a weak solution in $\frac{1}{r}(D - x_0)$ for $Q_r(x) := Q(x_0 + rx)$.

If the free boundary is sufficiently smooth near x_0 , then $\{u > 0\}$ has an inward unit normal ν at x_0 , and the boundary condition $\partial_\nu u(x_0) = Q(x_0)$ is satisfied in the classical sense. In that case the rescalings u_r converge locally uniformly, as $r \rightarrow 0$, to the *half-plane solution* $u_0(x) = Q(x_0)(x \cdot \nu)_+$.

The key idea is to study these rescalings even before any regularity of the free boundary is known. Using linear growth estimates away from the free boundary, one proves that a subsequence u_{r_j} , with $r_j \rightarrow 0$, converges to a nontrivial *blow-up limit* u_0 . This leads to a division of the free boundary into two subsets. We call a free boundary point *regular* if at least one blow-up limit at that point is a half-plane solution; the remaining free boundary points are called *singular*.

⁴The calculus of variations is the area of mathematical analysis dedicated to the study of minimisers of functionals.

Two basic questions arise. The first question is whether regular points are genuinely regular. This is answered through a “flat implies $C^{1,\alpha}$ ” theorem: if, at some small scale, a weak solution is sufficiently close to a half-plane solution, then its free boundary must be a regular deformation of a hyperplane. Such results are often called ϵ -regularity theorems, or improvement of flatness theorems.

The second question is whether the singular set, if it is non-empty, is small in some sense compared to the free boundary. Since these blow-up limits arise by zooming in at smaller and smaller scales, we expect them to be weak solutions in the full space $\mathbb{R}^n$, and we would like to have a Liouville-type theorem that classifies these objects.

Such Liouville-type theorems, however, are difficult to prove. In order to classify blow-up limits, we often need extra information arising from the blow-up procedure. In the Alt–Caffarelli problem, a monotonicity formula discovered by Weiss [49] allows us to prove that blow-ups are 1-homogeneous, and hence that $\partial\{u_0 > 0\}$ is a cone.

With this in hand, a family of dimension reduction arguments pioneered by Federer allows one to prove that if all blow-up limits in dimension k are regular, then the singular set is a relatively closed subset of the free boundary of dimension at most $n - k - 1$. In many free boundary problems, the objective then becomes proving regularity of blow-up limits in dimension k as large as possible.

In our case, the Lipschitz geometry of the free boundary and the fact that derivatives of harmonic functions satisfy the maximum principle allow us to push these arguments further and prove that if the free boundary is a Lipschitz cone, it must be a hyperplane. This implies that all free boundary points are regular and proves Theorem 1.5.

1.5 Organisation of the thesis

In Chapter 2, following [21], we prove optimal regularity and nondegeneracy estimates for weak solutions, and we use them to show precompactness for blow-up sequences of weak solutions. We also state more refined lemmas on the growth of u away from the free boundary that will be needed in Chapter 3.

In Chapter 3, following the original approach of [2], we prove an improvement of flatness theorem and deduce regularity of the free boundary near regular points.

In Chapter 4, we study blow-ups of weak solutions. Following [21], we use the Weiss monotonicity formula to prove that blow-ups are 1-homogeneous. The arguments in [21] are written assuming $Q \in C^\infty$; we show how a small modification of them yields the results for $Q \in C^{0,\alpha}$ as well. We conclude the chapter by proving the classification of blow-ups obtained in [21], thereby completing the proofs of Theorem 1.5 and Theorem 1.1.

Finally, in Chapter 5, we discuss some related open problems.

Chapter 2

Weak solutions of the Alt–Caffarelli problem

In this chapter, we prove some essential properties of weak solutions of the Alt–Caffarelli problem.

We remark that, in the literature, the term *weak solution* of the Alt–Caffarelli problem usually refers to [2, Definition 5.1], where, instead of assuming that the free boundary is Lipschitz, one asks for the regularity and nondegeneracy estimates in Proposition 2.2.

Throughout the thesis, Q denotes a continuous function such that $0 < Q_{\min} \leq Q \leq Q_{\max} < \infty$.

2.1 Optimal regularity and nondegeneracy

The first observation is that any weak solution is subharmonic, meaning that

$$-\int_D \nabla u \cdot \nabla \phi \, dx \geq 0$$

for every nonnegative $\phi \in C_c^\infty(D)$; see, for instance, [20, Section 3.2].

Lemma 2.1. *Any weak solution u is subharmonic.*

Proof. Let $\phi \in C_c^\infty(D)$ be nonnegative. By (1.9),

$$-\int_D \nabla u \cdot \nabla \phi \, dx = \int_{\partial\{u>0\} \cap D} Q \phi \, d\mathcal{H}^{n-1} \geq 0.$$

□

The rest of this section is dedicated to proving optimal regularity and nondegeneracy estimates for u .

In the following proposition, the upper bound says that, on average, u grows at most linearly away from $\partial\{u > 0\}$, while the lower bound says that this growth cannot degenerate to a smaller order.

Proposition 2.2 (Linear growth at the free boundary). *Let u be a weak solution in a domain D . For every open set $D_0 \subset\subset D$ there are constants $0 < c \leq C < \infty$ such that, for every $x \in \partial\{u > 0\} \cap D_0$ and every ball $B_r(x) \subset D_0$,*

$$c \leq \frac{1}{r} \int_{\partial B_r(x)} u d\mathcal{H}^{n-1} \leq C. \quad (2.1)$$

Consequently,

$$\sup_{B_r(x)} u \geq cr, \quad \int_{B_r(x)} u dy \leq Cr.$$

Proof. Fix $D_0 \subset\subset D$. Since $\{u > 0\}$ is a Lipschitz domain, for every $x \in \partial\{u > 0\} \cap D_0$ and every $B_r(x) \subset D_0$,

$$cr^{n-1} \leq \mathcal{H}^{n-1}(\partial\{u > 0\} \cap B_r(x)) \leq Cr^{n-1}, \quad (2.2)$$

for some constants $c, C > 0$ depending on D_0 . This follows by covering $\partial\{u > 0\} \cap D_0$ by finitely many Lipschitz graphs.

Let

$$\zeta(y) := \frac{1}{2n} (r^2 - |y - x|^2)_+ \in C_c^{0,1}(D).$$

By a standard density argument, ζ can be used as a test function in (1.9). Since $\Delta\zeta = -1$ in $B_r(x)$ and $\partial_\nu\zeta = r/n$ on $\partial B_r(x)$, integration by parts gives

$$- \int_{B_r(x)} \nabla u \cdot \nabla \zeta dy = \frac{r}{n} \int_{\partial B_r(x)} u d\mathcal{H}^{n-1} - \int_{B_r(x)} u dy.$$

On the other hand, (1.9) gives

$$- \int_{B_r(x)} \nabla u \cdot \nabla \zeta dy = \int_{\partial\{u>0\} \cap B_r(x)} Q\zeta d\mathcal{H}^{n-1}.$$

Combining the two identities and dividing by $|B_r|$, we obtain the useful formula

$$\int_{\partial B_r(x)} u d\mathcal{H}^{n-1} = \int_{B_r(x)} u dy + \frac{1}{|B_r|} \int_{\partial\{u>0\} \cap B_r(x)} Q\zeta d\mathcal{H}^{n-1}. \quad (2.3)$$

We first prove the lower bound. In $B_{r/2}(x)$ we have $\zeta \geq c_n r^2$. Therefore, by (2.2) and (2.3),

$$\int_{\partial B_r(x)} u d\mathcal{H}^{n-1} \geq \frac{Q_{\min}}{|B_r|} \int_{\partial\{u>0\} \cap B_{r/2}(x)} \zeta d\mathcal{H}^{n-1} \geq cr.$$

Dividing by r , we obtain the left-hand inequality in (2.1).

For the upper bound, we use boundary Hölder estimates for harmonic functions in Lipschitz domains. Since u is harmonic in $\{u > 0\}$, continuous in D , and vanishes on $D \setminus \{u > 0\}$, by [30, Corollary 9.29], $u \in C^{0,\alpha}(D_0)$, for some $0 < \alpha < 1$. In particular, for fixed $x \in \partial\{u > 0\} \cap D_0$, the quantity

$$\Phi(s) := s^{-\alpha} \int_{\partial B_s(x)} u d\mathcal{H}^{n-1}$$

is bounded for $0 < s < \text{dist}(x, \partial D_0)$. Rewriting (2.3) in terms of Φ , we get

$$\begin{aligned}\Phi(r) &= \frac{n}{r^{n+\alpha}} \int_0^r s^{n+\alpha-1} \Phi(s) ds + \frac{1}{r^\alpha |B_r|} \int_{\partial\{u>0\} \cap B_r(x)} Q \zeta d\mathcal{H}^{n-1} \\ &\leq \frac{n}{n+\alpha} \sup_{0 < s < r} \Phi(s) + Cr^{1-\alpha},\end{aligned}$$

where, in the last inequality, we used $Q \leq Q_{\max}$, $\zeta \leq Cr^2$, and the upper bound in (2.2). Taking the supremum over $0 < s < r$ gives

$$\left(1 - \frac{n}{n+\alpha}\right) \sup_{0 < s < r} \Phi(s) \leq Cr^{1-\alpha}.$$

Hence

$$\int_{\partial B_r(x)} u d\mathcal{H}^{n-1} \leq Cr,$$

which proves the right-hand side of (2.1).

The lower bound for $\sup_{B_r(x)} u$ follows immediately from the lower bound for the spherical average. Finally, integrating the upper bound in spherical coordinates gives

$$\int_{B_r(x)} u dy = \frac{n}{r^n} \int_0^r s^{n-1} \int_{\partial B_s(x)} u d\mathcal{H}^{n-1} ds \leq Cr.$$

□

Solutions to the Alt–Caffarelli problem cannot be expected to be smoother than Lipschitz continuous across the free boundary. Indeed, if u is a classical solution to (1.7), the inward normal derivative of u on $\{u > 0\}$ equals Q , while, on the zero side, the function is identically zero. Thus Lipschitz regularity is the optimal global regularity for u .

Lemma 2.3 (Optimal regularity). *Let $u \in C(D)$ be a nonnegative function. Assume that u is harmonic in $\{u > 0\}$ and satisfies the upper bound in (2.1). Then u is locally Lipschitz in D .*

Proof. Let $D_0 \subset\subset D$. We prove that $|\nabla u|$ is bounded in $\{u > 0\} \cap D_0$, with a bound depending only on D_0 , u , and the constants in Proposition 2.2. Since u vanishes continuously on $\partial\{u > 0\}$, this implies that u is Lipschitz in D_0 ; see [24, Theorem 4.5].

Let D_1 be an open set such that $D_0 \subset\subset D_1 \subset\subset D$, and set $\delta := \text{dist}(D_0, \partial D_1)/10$. Let $z \in \{u > 0\} \cap D_0$, and denote $d := \text{dist}(z, \partial\{u > 0\})$. There are two cases.

If $d \geq \delta$, then u is harmonic in $B_\delta(z)$. The classical interior estimate for harmonic functions ([23, Theorem 2.2.7]) gives

$$|\nabla u(z)| \leq \frac{C_n}{\delta^{n+1}} \int_{B_\delta(z)} u dx \leq \frac{C_n}{\delta} \|u\|_{L^\infty(D_1)}.$$

If $d < \delta$, let $y \in \partial\{u > 0\}$ be such that $|z - y| = d$. Then u is harmonic in $B_d(z)$, and the interior gradient estimate gives

$$|\nabla u(z)| \leq \frac{C_n}{d^{n+1}} \int_{B_d(z)} u dx.$$

Since $B_d(z) \subset B_{2d}(y) \subset D_1$, the integrated upper bound in Proposition 2.2 yields

$$|\nabla u(z)| \leq \frac{C_n}{d} \int_{B_{2d}(y)} u \, dx \leq C.$$

Thus $|\nabla u|$ is uniformly bounded in $\{u > 0\} \cap D_0$. $\square$

2.2 Blow-up sequences of weak solutions

The optimal regularity and nondegeneracy estimates allow us to study the behaviour of u near free boundary points by means of blow-up sequences. Given $x_0 \in \partial\{u > 0\}$, we consider

$$u_{r_k}(x) := \frac{1}{r_k} u(x_0 + r_k x), \quad r_k \rightarrow 0.$$

These rescalings zoom in on u near x_0 while preserving uniform bounds on the gradient. The optimal regularity estimate gives precompactness for such sequences, and the nondegeneracy estimate ensures that their limits are nontrivial.

For minimisers of the Alt–Caffarelli functional and for viscosity solutions, it is not hard to prove that the blow-up limit is again a minimiser or a viscosity solution. For weak solutions, this is more delicate, and we will not be able to show that blow-up limits are weak solutions until we have proved the ϵ -regularity theorem in Chapter 3.

Lemma 2.4. *Let u be a weak solution in a domain D . For $x_0 \in \partial\{u > 0\} \cap D$ and $r_k \rightarrow 0$, consider the rescalings*

$$u_{r_k}(x) := \frac{1}{r_k} u(x_0 + r_k x).$$

Then there is a blow-up limit $u_0 \in C_{\text{loc}}^{0,1}(\mathbb{R}^n)$ that satisfies $-\Delta u_0 = 0$ in $\{u_0 > 0\}$, $0 \in \partial\{u_0 > 0\}$, and, up to a rotation, $\{u_0 > 0\} = \{x_n > \phi_0(x')\}$ for some Lipschitz function $\phi_0: \mathbb{R}^{n-1} \rightarrow \mathbb{R}$, and such that, up to a subsequence,

1. $u_{r_k} \rightarrow u_0$ in $C_{\text{loc}}^{0,\alpha}(\mathbb{R}^n)$ for every $0 < \alpha < 1$,
2. $u_{r_k} \rightarrow u_0$ in $H_{\text{loc}}^1(\mathbb{R}^n)$,
3. $\chi_{\{u_{r_k} > 0\}} \rightarrow \chi_{\{u_0 > 0\}}$ in $L_{\text{loc}}^1(\mathbb{R}^n)$,
4. $\partial\{u_{r_k} > 0\} \rightarrow \partial\{u_0 > 0\}$ locally in the Hausdorff distance in $\mathbb{R}^n$.

Remark 2.5. We say that a sequence of functions $f_k \rightarrow f$ converges in $H_{\text{loc}}^1(\mathbb{R}^n)$ if $f_k \rightarrow f$ in $L_{\text{loc}}^2(\mathbb{R}^n)$ and $\nabla f_k \rightarrow \nabla f$ in $L_{\text{loc}}^2(\mathbb{R}^n; \mathbb{R}^n)$; see [5, Section 9.1].

For a sequence of closed sets $X_k \subset \mathbb{R}^n$, $X_k \rightarrow X$ locally in the Hausdorff distance in $\mathbb{R}^n$ if, for every compact set K , $\sup_{x \in X_k \cap K} \text{dist}(x, X) \rightarrow 0$ and $\sup_{x \in X \cap K} \text{dist}(x, X_k) \rightarrow 0$.

Proof of Lemma 2.4. Let $K \subset \subset \mathbb{R}^n$. For k large enough, $x_0 + r_k K \subset \subset D$. By Lemma 2.3, u is Lipschitz in a fixed neighbourhood of x_0 . Therefore the functions u_{r_k} are uniformly Lipschitz on K . Since $u_{r_k}(0) = 0$, they are also uniformly bounded on K . By Arzelà–Ascoli and a diagonal argument, there exist a subsequence, still denoted by u_{r_k} , and a locally Lipschitz function u_0 such that $u_{r_k} \rightarrow u_0$ locally uniformly in $\mathbb{R}^n$.

The uniform Lipschitz bound also implies convergence in $C_{\text{loc}}^{0,\alpha}$ for every $0 < \alpha < 1$; see [26, Property (H8)].

We now prove the Hausdorff convergence of the free boundaries. Suppose first that $y_k \in \partial\{u_{r_k} > 0\}$ and $y_k \rightarrow y$. Since $u_{r_k}(y_k) = 0$ and $u_{r_k} \rightarrow u_0$ locally uniformly, we have $u_0(y) = 0$. On the other hand, the nondegeneracy estimate applied at y_k gives, for every fixed $\rho > 0$,

$$\sup_{B_\rho(y)} u_0 \geq \limsup_{k \rightarrow \infty} \sup_{B_{\rho/2}(y_k)} u_{r_k} \geq c\rho.$$

Thus $y \in \partial\{u_0 > 0\}$.

Conversely, let $y \in \partial\{u_0 > 0\}$. We claim that every small ball $B_\rho(y)$ must contain points of $\partial\{u_{r_k} > 0\}$ for all large k . Indeed, suppose not. By the locally uniform convergence $u_{r_k} \rightarrow u_0$, $B_\rho(y)$ cannot be contained in $\{u_{r_k} = 0\}$ for all large k . We may pass to a subsequence, still denoted by r_k , such that $u_{r_k} > 0$ in $B_\rho(y)$. By Harnack's inequality, $u_{r_k}(z) \leq C u_{r_k}(y)$ for $z \in B_{\rho/2}(y)$, with $C > 0$ independent of k . Since $u_{r_k}(y) \rightarrow u_0(y) = 0$, this implies that $u_{r_k} \rightarrow 0$ uniformly in $B_{\rho/2}(y)$, contradicting $y \in \partial\{u_0 > 0\}$. Hence the free boundaries converge locally in the Hausdorff distance.

By the locally uniform convergence and the Hausdorff convergence of the free boundaries, if $B_\rho(y) \subset\subset \{u_0 > 0\}$, then $B_\rho(y) \subset\subset \{u_{r_k} > 0\}$ for all large k . Since each u_{r_k} is harmonic in $\{u_{r_k} > 0\}$ and by the locally uniform convergence $u_{r_k} \rightarrow u_0$, $-\Delta u_0 = 0$ in $\{u_0 > 0\}$.

For reasons of space, we omit the proof of the convergence in H_{loc}^1 and refer the reader to [21, Lemma 2.8].

It remains to show that $\{u_0 > 0\}$ is Lipschitz and that the characteristic functions converge in L_{loc}^1 . After translating and rotating, we may assume $x_0 = 0$ and, in a ball $B_\rho \subset\subset D$,

$$\{u > 0\} = \{(x', x_n) : x_n > \phi(x')\}, \quad \phi(0) = 0, \quad \text{Lip } \phi \leq L.$$

Hence, for every fixed $R > 0$ and all k sufficiently large, $\partial\{u_{r_k} > 0\} \cap B_R$ is given by the graph of $\phi_k(x') := \frac{1}{r_k} \phi(r_k x')$, with $\text{Lip } \phi_k \leq L$.

By Arzelà–Ascoli and a diagonal argument, there is a Lipschitz function ϕ_0 such that, up to a further subsequence, $\phi_k \rightarrow \phi_0$ locally uniformly. Thus, by the Hausdorff convergence of the free boundaries,

$$\partial\{u_0 > 0\} = \{x \in \mathbb{R}^n : x_n = \phi_0(x')\}. \quad (2.4)$$

Since $u_{r_k} = 0$ in $\{x_n < \phi_k(x')\}$, by the locally uniform convergence of $u_{r_k} \rightarrow u_0$, we have $\{x_n \leq \phi_0(x')\} \subset \{u_0 = 0\}$. Thus, by (2.4),

$$\{u_0 > 0\} = \{x_n > \phi_0(x')\}. \quad (2.5)$$

Since the graph of ϕ_0 has zero Lebesgue measure, the locally uniform convergence $\phi_k \rightarrow \phi_0$ and (2.5) imply that for every $\epsilon > 0$ and every compact set K , the symmetric difference

$$\{(u_{r_k} > 0) \triangle \{u_0 > 0\}\} \cap K$$

has Lebesgue measure at most ϵ for all large k . Thus $\chi_{\{u_{r_k} > 0\}} \rightarrow \chi_{\{u_0 > 0\}}$ in L_{loc}^1 . $\square$

Observe that if the free boundary were C^1 in a neighbourhood of x_0 , then all blow-up limits at x_0 would be the half-plane solution $u_0 = Q(x_0)(x \cdot \nu)_+$, where ν denotes the inward unit normal to $\{u > 0\}$ at x_0 . This motivates the following definition.

Definition 2.6. Let u be a weak solution. We call $x_0 \in \partial\{u > 0\}$ a *regular point* if there exists a blow-up limit of u at x_0 , in the sense of Lemma 2.4, that is a half-plane solution $u_0 = Q(x_0)(x \cdot \nu)_+$ for some unit vector $\nu \in \mathbb{R}^n$. A free boundary point that is not regular is called a *singular point*.

2.3 Growth away from the free boundary

The estimates in this section refine the linear growth bounds proved above. Proposition 2.2 says that the solution grows like a linear function at the free boundary, but it does not identify the slope. The next two results say that the slope is controlled from below and from above by Q . They will be used in the next chapter to ensure that weak solutions with sufficiently flat free boundaries are close to the corresponding half-plane solution.

We say that a ball B touches the boundary of a set $E \subset \mathbb{R}^n$ at $x \in \partial E$ if $B \subset E$ and $\partial B \cap \partial E = \{x\}$. The first result says that, at points $x_0 \in \partial\{u > 0\}$ where there is a ball $B \subset \{u = 0\}$ touching the free boundary at x_0 , u grows away from the free boundary with slope at least $Q(x_0)$.

Lemma 2.7. *If u is a weak solution and B is a ball in $\{u = 0\}$ touching the free boundary $\partial\{u > 0\}$ at x_0 , then*

$$\limsup_{\substack{x \rightarrow x_0 \\ u(x) > 0}} \frac{u(x)}{\text{dist}(x, B)} \geq Q(x_0).$$

As $|\nabla u|$ is subharmonic in $\{u > 0\}$, stronger estimates can be obtained from above than from below. Note that the following holds at every free boundary point.

Theorem 2.8. *If u is a weak solution in B_1 , then for every open set $D \subset\subset B_1$ there are constants $\alpha > 0$ and $C < \infty$ depending on n , D , $Q_{\min}$ and $Q_{\max}$ such that*

$$\sup_{B_r(x)} |\nabla u| \leq \sup_{B_R(x)} Q + C \left(\frac{r}{R} \right)^\alpha$$

for balls $B_r(x) \subset B_R(x) \subset D$ with $x \in \partial\{u > 0\}$.

If Q is Hölder continuous, this becomes:

Corollary 2.9. *If u is a weak solution in B_1 and if Q is Hölder continuous, then for $D \subset\subset B_1$ there are constants $\alpha > 0$ and $C < \infty$ depending on n , D , $Q_{\min}$, $Q_{\max}$, and the Hölder exponent and norm of Q on $\overline{D}$ such that*

$$|\nabla u(x)| \leq Q(x) + Cr^\alpha$$

for $B_r(x) \subset D \cap \{u > 0\}$ touching the free boundary.

For reasons of space, we will not prove these estimates here; we refer the reader to [2, Lemma 4.10, Theorem 6.3, Corollary 6.5].

Chapter 3

Flatness implies $C^{1,\alpha}$

In this chapter, following the original proof of Alt and Caffarelli [2, Theorem 8.1], we prove that if Q is Hölder continuous, then the free boundary is $C^{1,\alpha}$ near sufficiently flat free boundary points. From this, we deduce the smoothness of the free boundary near regular points.

The regularity of the free boundary is obtained through an improvement of flatness scheme: if the free boundary is sufficiently flat in a ball $B_\rho(x)$, then in a smaller ball $B_{\bar{\rho}}(x)$, it becomes even flatter by a fixed factor θ . An iteration of this result then yields the $C^{1,\alpha}$ regularity of the free boundary.

The fact that such an improvement of flatness implies $C^{1,\alpha}$ regularity is a general principle. In fact, improvement of flatness and improvement of oscillation schemes have been among the main methods for proving regularity results in PDE since the work of De Giorgi and Nash on Hilbert's XIX problem in the 1950s; see, e.g., [26, 35]. The main difficulty, therefore, lies in establishing the improvement of flatness lemma.

This is done by a compactness argument that goes as follows: assume that the flatness of the free boundary is not improved, no matter how flat the free boundary is, and consider a sequence of counterexamples whose free boundaries become arbitrarily flat yet for which the improvement of flatness fails. Rescaling each free boundary in the “flat” direction, we obtain in the limit the graph of a function that does satisfy an improvement of flatness property, yielding a contradiction.

Alt and Caffarelli's proof follows the main ideas of the proof of the analogous result for minimal surfaces presented in [25, Section 5.3]. In the minimal surface case, there is a later proof that also follows a similar strategy [47, Section 5.2], which readers unfamiliar with the theory of k -currents might find easier to follow; the interested reader can also find expositions of it in [11, 43].

Recall that, in [2], the assumption that the free boundary is a Lipschitz graph is replaced by weaker regularity and nondegeneracy assumptions on the solution u . Thus, the Lipschitz assumption is not essential to the results in this chapter.

As in the previous chapter, $Q : \mathbb{R}^n \rightarrow \mathbb{R}$ denotes a continuous function such that $0 < Q_{\min} \leq Q \leq Q_{\max} < \infty$.

3.1 Flat free boundary points

The flatness of the free boundary will be quantified by means of the following classes. Note that, in this chapter, we follow the conventions in [2]: $\{u > 0\}$ is thought of as being below $\partial\{u > 0\}$, and ν generally denotes the outward unit normal to $\{u > 0\}$.

Definition 3.1 (Flat free boundary). Let $0 < \sigma_+, \sigma_- \leq 1$, $\tau > 0$, and ν be a unit vector. We say that u is of class

$$F(\sigma_+, \sigma_-; \tau) \text{ in } B_\rho(x_0) \text{ in the direction } \nu$$

if

(i) u is a weak solution in $B_\rho(x_0)$ for some Q ;

(ii) $x_0 \in \partial\{u > 0\}$, and

$$u(x) = 0 \quad \text{for } (x - x_0) \cdot \nu \geq \sigma_+ \rho, \quad (3.1)$$

$$u(x) \geq -Q(x_0)((x - x_0) \cdot \nu + \sigma_- \rho) \quad \text{for } (x - x_0) \cdot \nu \leq -\sigma_- \rho; \quad (3.2)$$

(iii) the following bounds hold:

$$|\nabla u| \leq Q(x_0)(1 + \tau) \quad \text{in } B_\rho(x_0), \quad (3.3)$$

$$\text{osc}_{B_\rho(x_0)} Q \leq Q(x_0)\tau. \quad (3.4)$$

If we do not specify the direction ν , we take it to be e_n .

Observe that if the free boundary were completely flat, i.e. if it were exactly the hyperplane $\{(x - x_0) \cdot \nu = 0\}$, and Q were constant, then the weak solution u would simply be the half-plane solution $u(x) = Q(x_0)((x - x_0) \cdot \nu)_-$.

The condition $u \in F(\sigma_+, \sigma_-; \tau)$ means that the free boundary is trapped between the two parallel hyperplanes $(x - x_0) \cdot \nu = \sigma_+ \rho$ and $(x - x_0) \cdot \nu = -\sigma_- \rho$. Moreover, it tells us that u is at least as large as the half-plane solution corresponding to $\partial\{u > 0\} = \{(x - x_0) \cdot \nu = -\sigma_- \rho\}$ and $Q \equiv Q(x_0)$, and that its growth, both away from the free boundary (3.3) and along it (3.4), deviates by a multiplicative factor of at most $1 + \tau$ from the growth of this half-plane solution.

The following lemma will be used throughout the proof. It tells us that if the free boundary of a weak solution u is flat from above in a ball of radius ρ (and we have some control on the growth of u away from the free boundary), it must also be flat from below in the concentric ball of radius $\frac{\rho}{2}$. It is the key result that allows us to pass to the limit in the nonhomogeneous blow-up in Lemma 3.3.

Lemma 3.2. *There exist a large constant $C = C(n) < \infty$ and a small constant $\sigma_0 = \sigma_0(n) > 0$ such that if*

$$u \in F(\sigma, 1; \tau) \text{ in } B_\rho$$

with $\sigma \leq \sigma_0$ and $\tau \leq \sigma$, then

$$u \in F(2\sigma, C\sigma; \tau) \text{ in } B_{\rho/2}.$$

Proof. By rescaling we may assume that $\rho = 1$ and $Q(0) = 1$. Note that, since u is of class $F(\sigma, 1; \tau)$ in B_1 , u is of class $F(2\sigma, 1; \tau)$ in $B_{1/2}$.

We must prove that u is flat from below, that is,

$$u(x) \geq -x_n - C(n)\sigma.$$

The idea is as follows. In the limiting case where $\sigma = 0$, the free boundary would be $\{x_n = 0\}$, and if $Q \equiv 1$, the weak solution would be the half-plane solution $u_0 := \max(-x_n, 0)$. By (3.3) and (3.4), the deviation of the growth of our weak solution u from the growth of u_0 is of order $\tau \leq \sigma$. Since u is flat from above, it can become positive at most at a distance of order σ from $\{u_0 > 0\}$. Since u is harmonic in $\{u > 0\}$, we therefore expect the difference between u and u_0 to also be of order σ .

Instead of working in $\{u > 0\}$, we will work in a smoother domain D whose boundary still contains a free boundary point near the origin. We define

$$\eta(y) := \begin{cases} \exp\left(-\frac{9|y|^2}{1-9|y|^2}\right), & |y| < \frac{1}{3}, \\ 0, & |y| \geq \frac{1}{3}, \end{cases}$$

and choose the largest $s \geq 0$ such that

$$B_1 \cap \{u > 0\} \subset D := \{x \in B_1 : x_n < \sigma - s\eta(x')\}.$$

By the choice of s , there is a free boundary point $z \in B_{1/2} \cap \partial D \cap \partial\{u > 0\}$. Note that, since $0 \in \partial\{u > 0\}$, we must have $s \leq \sigma$.

We start by bounding the growth of u away from the free boundary at z from below. Let

$$l := \limsup_{x \rightarrow z} \frac{u(x)}{\text{dist}(x, \partial D)}.$$

By the construction of D , there is a ball $B \subset \{u = 0\}$ touching the free boundary at z , so that, by Lemma 2.7, (3.4), and the assumption $\tau \leq \sigma$,

$$l \geq Q(z) \geq Q(0) - \text{osc}_{B_1} Q \geq 1 - \tau \geq 1 - \sigma. \quad (3.5)$$

We now compare u with the harmonic function v in D with boundary values

$$\begin{aligned} v(x) &= 0 && \text{on } \partial D \cap B_1, \\ v(x) &= (1 + 2\sigma)(\sigma - x_n) && \text{on } \partial D \setminus B_1. \end{aligned}$$

Our goal is to estimate u from below by v , up to a constant of order σ . We argue by contradiction: if $u - v$ could not be bounded from below by a constant of order σ , u would grow too slowly away from the free boundary, contradicting (3.5).

Near points ξ far away from the free boundary, since $-\xi_n$ is bounded below, a lower bound for $u - v$ by a function comparable to $C\sigma x_n$ near ξ suffices to obtain a bound of order $C\sigma$.

For this, for any point $\xi \in \partial B_{3/4}$ with $\xi_n \leq -\frac{1}{2}$, consider the function w_ξ that is harmonic in $D \setminus \bar{B}_{1/10}(\xi)$ with boundary values

$$\begin{cases} w_\xi(x) = 0 & \text{on } \partial D, \\ w_\xi(x) = -x_n & \text{on } \partial B_{1/10}(\xi). \end{cases}$$

Fix $d = d(n) > 0$, to be chosen sufficiently large. Suppose that

$$u(x) \leq v(x) + d\sigma x_n \quad \text{in } B_{1/10}(\xi). \quad (3.6)$$

Observe that $u \leq v$ in D : u is subharmonic (Lemma 2.1), and by the construction of D and (3.3), $u \leq v$ on ∂D . Therefore, by the maximum principle,

$$u \leq v - d\sigma w_\xi \quad \text{in } D \setminus B_{1/10}(\xi),$$

which implies

$$l \leq \partial_{-\nu} v(z) - d\sigma \partial_{-\nu} w_\xi(z), \quad (3.7)$$

where ν denotes the inward unit normal to B at z .

Since D has a C^2 boundary in a neighbourhood of z , there is a radius $r > 0$ and a ball $B_r(x_z) \subset D \setminus \bar{B}_{1/10}(\xi)$ touching ∂D at z . By the barrier proof of the Hopf lemma (see, e.g., [23, Section 6.4.2]),

$$\partial_{-\nu} w_\xi(z) \geq c(n, r) \inf_{\partial B_{r/2}(x_z)} w_\xi > 0.$$

Since the family $D(s, \sigma)$ has C^2 boundaries uniformly for $0 \leq s \leq \sigma \leq \sigma_0$ in $B_{1/2}$ and the points z remain a fixed positive distance from $B_{1/10}(\xi)$, we can choose $r = r(n)$. Since $w_\xi = -x_n \geq 2/5$ on $\partial B_{1/10}(\xi)$, a comparison argument on a fixed annulus around $B_{1/10}(\xi)$, followed by the Harnack inequality along a uniformly bounded chain of balls, gives

$$\inf_{\partial B_{r/2}(x_z)} w_\xi \geq c(n).$$

Hence

$$\partial_{-\nu} w_\xi(z) \geq c(n) > 0. \quad (3.8)$$

Thus, for (3.7) to contradict (3.5), we need to prove that

$$\partial_{-\nu} v(z) \leq 1 + C(n)\sigma.$$

To do so, we compare v with a half-plane solution and use boundary Schauder estimates. Let $h(x) := v(x) - (1 + 2\sigma)(\sigma - x_n)$. Observe that h is harmonic in D and has boundary values

$$\begin{aligned} h(x) &= -(1 + 2\sigma)s\eta(x') & \text{on } x \in \partial D \cap B_1, \\ h(x) &= 0 & \text{on } x \in \partial D \setminus B_1. \end{aligned}$$

Extending the boundary datum on $\partial D \cap B_1$ to the function $\phi := -(1 + 2\sigma)s\eta(x')$ on D , we can apply [30, Corollary 8.36] in a small ball $B_r(z)$ to obtain

$$\|h\|_{C^{1,\alpha}(B_r(z) \cap D)} \leq C(\|h\|_{L^\infty(D)} + \|\phi\|_{C^{1,\alpha}(D)}), \quad (3.9)$$

where, since the family $D(s, \sigma)$ has uniformly $C^{1,\alpha}$ boundaries, we can choose C depending only on n . Since

$$\|h\|_{L^\infty(D)} = \|h\|_{L^\infty(\partial D)} \leq (1 + 2\sigma)\sigma \leq 3\sigma, \quad (3.10)$$

$$\|\phi\|_{C^{1,\alpha}(D)} = (1 + 2\sigma)s\|\eta\|_{C^{1,\alpha}(D)} \leq c_0\sigma, \quad (3.11)$$

substituting (3.10) and (3.11) into (3.9), we get

$$\partial_{-v}v(z) = \partial_{-v}((1 + 2\sigma)(\sigma - x_n) + h)(z) \leq 1 + C\sigma. \quad (3.12)$$

Now, from (3.6), using (3.12), (3.8), and (3.7), we obtain

$$l \leq \partial_{-v}v(z) - d\sigma\partial_{-v}w_\xi(z) \leq 1 + C\sigma - dc\sigma,$$

which contradicts (3.5), if d is chosen sufficiently large. Denoting this large constant by C , there are points $x_\xi \in B_{1/10}(\xi)$ with

$$u(x_\xi) \geq v(x_\xi) + C\sigma x_{\xi n}, \quad (3.13)$$

as desired.

To finish the proof, we transfer this estimate to all $x \in B_{1/2}$. Note that u is positive, and therefore harmonic, in $B_{1/5}(\xi)$. Indeed, by (3.3), the fundamental theorem of calculus, and the inequalities

$$v(x) \geq -(1 + \sigma)x_n \quad \text{and} \quad \xi_n \leq -\frac{1}{2},$$

we get for $x \in B_{1/5}(\xi)$

$$u(x) \geq u(x_\xi) - \frac{3}{10}(1 + \sigma) \geq -x_{\xi n} - \frac{3}{10} - C\sigma \geq \frac{1}{10} - C\sigma > 0,$$

if σ is small. Therefore, $v - u$ is nonnegative and harmonic in $B_{1/5}(\xi)$ and, by Harnack's inequality and (3.13), $(v - u)(\xi) \leq C(n)(v - u)(x_\xi) \leq C\sigma$. Hence

$$u(\xi) \geq -\xi_n - C\sigma. \quad (3.14)$$

Finally, using (3.3) and the fact that $\tau \leq \sigma$, integrating along vertical lines, and then using (3.14), we get that for any $\alpha > 0$ such that $\xi + \alpha e_n \in B_1$,

$$u(\xi + \alpha e_n) \geq u(\xi) - (1 + \sigma)\alpha \geq -(\xi_n + \alpha) - C\sigma.$$

Since any point in $B_{1/2}$ can be expressed as $\xi + \alpha e_n$, with α and ξ as above, we have proved that u is of class $F(2\sigma, C\sigma; \tau)$ in $B_{1/2}$. $\square$

3.2 A nonhomogeneous blow-up

In this section, we consider sequences of free boundaries that become arbitrarily flat and study their limit. Since, in the limit, the free boundary would simply be $\partial\{u > 0\} =$

$\{x_n = 0\}$, we rescale in the flatness direction at each step so that the free boundaries remain nontrivial.

We start by proving compactness for such sequences. We consider two separate limit operations, one in which we take the limit at the same time as an upper semicontinuous envelope and one in which we do so simultaneously with a lower semicontinuous envelope, and then prove, using Lemma 3.2, that they coincide along a subsequence. From this, the locally uniform convergence of the subsequence is easily obtained.

Lemma 3.3 (Nonhomogeneous blow-up). *Let $(u_k)_k$ be a sequence of weak solutions of class $F(\sigma_k, \sigma_k; \tau_k)$ in B_{ρ_k} with $\sigma_k \rightarrow 0$ and $\tau_k \sigma_k^{-2} \rightarrow 0$. Define for $y \in B'_1$*

$$\begin{aligned} f_k^+(y) &:= \sup\{h : (\rho_k y, \sigma_k \rho_k h) \in \partial\{u_k > 0\}\}, \\ f_k^-(y) &:= \inf\{h : (\rho_k y, \sigma_k \rho_k h) \in \partial\{u_k > 0\}\}, \end{aligned}$$

and the semi-relaxed limits

$$\begin{aligned} f^+(y) &:= \limsup_{k \rightarrow \infty} \left\{ f_m^+(z) : m \geq k, |y - z| \leq \frac{1}{k} \right\}, \\ f^-(y) &:= \liminf_{k \rightarrow \infty} \left\{ f_m^-(z) : m \geq k, |y - z| \leq \frac{1}{k} \right\}. \end{aligned}$$

Then, for a subsequence,

$$f(y) := f^+(y) = f^-(y).$$

Proof. By rescaling we may assume that $\rho_k = 1$. Note that, since f_k^+ and f_k^- are uniformly bounded by 1, so are f^+ and f^- . We can also verify that f^+ is upper semicontinuous. Indeed, otherwise there would be $y_k \rightarrow y$, with $y_k, y \in B'_1$, such that for some $\epsilon > 0$ and all k large enough, $f^+(y_k) \geq f^+(y) + \epsilon$. By the definition of f^+ , for each k there is a point $z_k \in B'_1$ such that $|z_k - y_k| \leq 1/k$, and an index $m_k \geq k$ such that

$$f_{m_k}^+(z_k) \geq f^+(y_k) - \frac{\epsilon}{2} \geq f^+(y) + \epsilon/2,$$

for sufficiently large k . Therefore

$$\limsup_{k \rightarrow \infty} f_{m_k}^+(z_k) > f^+(y),$$

which contradicts the definition of f^+ , since $z_k \rightarrow y$. Similarly, f^- is lower semicontinuous.

Clearly, $f^+ \geq f^-$. It remains to show that $f^- \geq f^+$. Consider the blown-up positivity sets

$$D_k := \{(y, h) \in B'_1 \times [-1, 1] : (y, \sigma_k h) \in B_1 \cap \{u_k > 0\}\}.$$

The space of nonempty compact subsets of a compact space, endowed with the Hausdorff distance, is a compact metric space. Therefore, we can choose a subsequence, which we still index by k , such that $(\overline{D}_k)_k$ converges in the Hausdorff distance to $\overline{D}$, the set of limit points of sequences $(d_k)_k$ with $d_k \in \overline{D}_k$, see, e.g., [34]. Note that the graph of f_k^+ over B'_1 is contained in $\overline{D}_k$ and that the graph of f^+ over B'_1 is contained in $\overline{D}$.

For any $y_0 \in B'_1$, and every k in the subsequence, there are $y_k \in B'_1$ such that

$$y_k \rightarrow y_0 \text{ and } f_k^+(y_k) \rightarrow f^+(y_0). \quad (3.15)$$

Indeed, by the Hausdorff convergence of $(\overline{D}_k)_k$, for any $y_0 \in B'_1$ there is a sequence $((y_k, h_k))_k$ with $(y_k, h_k) \in \overline{D}_k$ such that

$$(y_k, h_k) \rightarrow (y_0, f^+(y_0)).$$

Since $f_k^+(y_k) \geq h_k$, we have

$$\liminf_{k \rightarrow \infty} f_k^+(y_k) \geq f^+(y_0).$$

On the other hand, by the definition of f^+ ,

$$\limsup_{k \rightarrow \infty} f_k^+(y_k) \leq f^+(y_0).$$

Hence

$$f_k^+(y_k) \rightarrow f^+(y_0).$$

For k large enough, we will prove that $\partial\{u_k > 0\}$ is flat from above near $(y_k, \sigma_k f_k^+(y_k))$, and then use Lemma 3.2 to conclude the proof.

We claim that for any small $\epsilon > 0$, there exists $\delta > 0$ such that, for all sufficiently large k ,

$$\left(\overline{B}'_\delta(y_k) \times [f_k^+(y_k) + \epsilon, \infty)\right) \cap \overline{D}_k = \emptyset. \quad (3.16)$$

Indeed, otherwise there would be an $\epsilon > 0$ such that for arbitrarily large k and arbitrarily small $\delta > 0$ there is a point $(z_k, h_k) \in \overline{D}_k$ with $z_k \in B'_\delta(y_k)$ and $h_k \geq f_k^+(y_k) + \epsilon$. Since $(\overline{D}_k)_k$ converges to $\overline{D}$ in the Hausdorff distance, for large k there is $(z, h) \in \overline{D}$ with $|z - z_k| < \delta$ and $|h - h_k| < \delta$.

On the one hand,

$$h > f_k^+(y_k) + \epsilon - \delta \rightarrow f^+(y_0) + \epsilon - \delta. \quad (3.17)$$

On the other hand, for large enough k , $|y_0 - y_k| < \delta$, and therefore $z \in B'_{3\delta}(y_0)$. Since f^+ is upper semicontinuous, for small enough δ we have

$$f^+(z) < f^+(y_0) + \frac{\epsilon}{2} \text{ in } B'_{3\delta}(y_0),$$

so

$$h \leq f^+(z) < f^+(y_0) + \epsilon/2. \quad (3.18)$$

Combining (3.17) and (3.18), and choosing $\delta < \frac{\epsilon}{2}$ in (3.17), we obtain

$$f^+(y_0) + \frac{\epsilon}{2} < h < f^+(y_0) + \frac{\epsilon}{2},$$

a contradiction. Therefore (3.16) holds, which implies

$$B'_\delta(y_k) \times [\sigma_k f_k^+(y_k) + \sigma_k \epsilon, \infty) \cap \{u_k > 0\} = \emptyset,$$

or, in other words, u_k is of class $F(\sigma_k \epsilon / \delta, 1; \tau_k)$ in $B_\delta(y_k, \sigma_k f_k^+(y_k))$.

If k is large enough, we can now apply Lemma 3.2 to obtain

$$u_k \in F(\sigma_k \epsilon / \delta, C \sigma_k \epsilon / \delta; \tau_k) \quad \text{in } B_{\delta/2}(y_k, \sigma_k f_k^+(y_k)),$$

which, by (3.2), means that

$$\{x \in B_{\delta/2}(y_k, \sigma_k f_k^+(y_k)) : x_n < \sigma_k(f_k^+(y_k) - C\epsilon)\} \subset \{u_k > 0\}.$$

Thus, for large k ,

$$\{(y, h) \in B_1 : |y - y_k| < \frac{\delta}{4} \text{ and } h < f_k^+(y_k) - C\epsilon\} \subset D_k,$$

which means that

$$f_k^-(y) \geq f_k^+(y_k) - C\epsilon \quad \text{for } y \in B'_{\delta/4}(y_k).$$

Letting $k \rightarrow \infty$ and $\epsilon \rightarrow 0$, we obtain $f^-(y_0) \geq f^+(y_0)$, as desired. □

Corollary 3.4. *f is continuous with $f(0) = 0$, and $f_k^+ \rightarrow f$ and $f_k^- \rightarrow f$ locally uniformly.*

Proof. Observe that f is continuous since it is both upper and lower semicontinuous. Suppose that $(f_k^+)_k$ does not converge locally uniformly to f . That is, there exist a compact set $K \subset B'_1$, an $\epsilon > 0$, indices $k_j \rightarrow \infty$ and points $y_{k_j} \in K$ such that

$$|f(y_{k_j}) - f_{k_j}^+(y_{k_j})| > \epsilon. \quad (3.19)$$

Since K is compact, by passing to a subsequence, we may assume that $y_{k_j} \rightarrow y \in K$ with $|y - y_{k_j}| \leq \frac{1}{k_j}$. Since

$$f^-(y) \leq \liminf_{j \rightarrow \infty} f_{k_j}^+(y_{k_j}) \leq \limsup_{j \rightarrow \infty} f_{k_j}^+(y_{k_j}) \leq f^+(y),$$

by Lemma 3.3,

$$\lim_{j \rightarrow \infty} f_{k_j}^+(y_{k_j}) = f(y).$$

Since f is continuous,

$$\lim_{j \rightarrow \infty} f(y_{k_j}) = f(y).$$

Thus, passing to the limit in (3.19),

$$|f(y) - f(y)| > \epsilon,$$

which is impossible. The same argument yields the locally uniform convergence of $(f_k^-)_k$.

Since 0 is a free boundary point for every k ,

$$f_k^-(0) \leq 0 \leq f_k^+(0),$$

and hence $f(0) = 0$. □

In the rest of this section, we study the properties of the function f , culminating in Lemma 3.8 with an improvement of flatness lemma for f .

By analogy with the theory of minimal surfaces, where the nonhomogeneous blow-up limit is harmonic, we start by proving that f is subharmonic.

Lemma 3.5. *f is subharmonic.*

Proof. Assume for the sake of contradiction that f is not subharmonic. Then there is a point $y_0 \in B'_1$ and a radius $0 < \rho < 1$ such that

$$f(y_0) > \int_{\partial B'_\rho(y_0)} f d\mathcal{H}^{n-2}.$$

Let g be the harmonic extension of f in $B'_\rho(y_0)$ plus a sufficiently small positive constant. Then g is harmonic in $B'_\rho(y_0)$ and satisfies

$$g > f \quad \text{on } \partial B'_\rho(y_0), \quad f(y_0) > g(y_0).$$

In what follows, Z will denote the cylinder $B'_\rho(y_0) \times \mathbb{R}$ (from the German *Zylinder*), and

$$Z^+(\phi) := \{(y, h) \in Z : h > \phi(y)\},$$

with $Z^0(\phi)$ and $Z^-(\phi)$ defined analogously.

In order to isolate the deviation of $\{u_k > 0\}$ from a positivity set whose boundary is given by a harmonic function, we consider the Lipschitz domains

$$E_k := \{u_k > 0\} \cup Z^-(\sigma_k g).$$

On the one hand, since harmonic graphs are first-order approximations of minimal surfaces, for large k , we prove the following excess area estimate

$$\mathcal{H}^{n-1}(Z \cap \partial E_k) \geq \mathcal{H}^{n-1}(Z^0(\sigma_k g)) + c\sigma_k^2. \quad (3.20)$$

On the other hand, we use (1.9) to relate the area of ∂E_k in $Z^+(\sigma_k g)$ and the flux of ∇u_k through $Z^0(\sigma_k g)$, obtaining the upper bound

$$\mathcal{H}^{n-1}(Z \cap \partial E_k) \leq \frac{1 + \tau_k}{1 - \tau_k} \mathcal{H}^{n-1}(Z^0(\sigma_k g)), \quad (3.21)$$

which contradicts (3.20) and the assumption $\tau_k = o(\sigma_k^2)$ in Lemma 3.3.

We start by proving (3.21). Since $\partial\{u_k > 0\}$ has finite $\mathcal{H}^{n-1}$ measure, the set $Z^0(\sigma_k(g + c)) \cap \partial\{u_k > 0\}$ has positive $\mathcal{H}^{n-1}$ measure for at most countably many $c \in \mathbb{R}$. Thus, replacing g with $g + c$ (c as small as we like) if necessary, we may assume that

$$\mathcal{H}^{n-1}(Z^0(\sigma_k g) \cap \partial\{u_k > 0\}) = 0. \quad (3.22)$$

Therefore,

$$\mathcal{H}^{n-1}(Z \cap \partial E_k) = \mathcal{H}^{n-1}(Z^+(\sigma_k g) \cap \partial\{u_k > 0\}) + \mathcal{H}^{n-1}(Z^0(\sigma_k g) \cap \{u_k = 0\}). \quad (3.23)$$

In order to apply (1.9) in $Z^+(\sigma_k g)$, we use the following Lipschitz approximations of characteristic functions of subsets $A \subset \mathbb{R}^n$:

$$d_\delta(A)(x) := \min \left(\frac{1}{\delta} \text{dist}(x, \mathbb{R}^n \setminus A), 1 \right).$$

If A has a Lipschitz boundary and $\phi \in C_c(\mathbb{R}^n)$, then, as $\delta \rightarrow 0$,

$$\int_{\mathbb{R}^n} \phi d_\delta(A) dx \rightarrow \int_A \phi dx \quad \text{and} \quad \int_{\mathbb{R}^n} \phi \nabla d_\delta(A) dx \rightarrow \int_{\partial A} \phi \nu d\mathcal{H}^{n-1}, \quad (3.24)$$

where ν is the inward unit normal to A .

Indeed, the first limit is a direct consequence of the dominated convergence theorem, and the second limit follows from the local L^1 convergence $d_\delta(A) \rightarrow \chi_A$, together with [25, Lemma 3.2.38] and [3, Proposition 3.13].

Since $d_\delta(Z^+(\sigma_k g))$ restricted to $\overline{\{u_k > 0\}}$ has bounded support, there exist compactly supported functions $\tilde{d}_\delta(Z^+(\sigma_k g)) \in C_c^{0,1}(B_1)$ coinciding with $d_\delta(Z^+(\sigma_k g))$ on $\overline{\{u_k > 0\}}$. By a standard density argument, (1.9) extends to test functions in $C_c^{0,1}(B_1)$. Using $\tilde{d}_\delta$ as test functions in (1.9) and letting $\delta \rightarrow 0$, we obtain, by (3.24),

$$\int_{Z^+(\sigma_k g) \cap \partial\{u_k > 0\}} Q_{u_k} d\mathcal{H}^{n-1} = \int_{\partial Z^+(\sigma_k g) \cap \{u_k > 0\}} \nabla u_k \cdot \nu d\mathcal{H}^{n-1},$$

and therefore, by (3.4) and (3.3),

$$(1 - \tau_k) \mathcal{H}^{n-1}(Z^+(\sigma_k g) \cap \partial\{u_k > 0\}) \leq (1 + \tau_k) \mathcal{H}^{n-1}(\partial Z^+(\sigma_k g) \cap \{u_k > 0\}).$$

For large k , Corollary 3.4 gives

$$\partial Z^+(\sigma_k g) \cap \{u_k > 0\} = Z^0(\sigma_k g) \cap \{u_k > 0\},$$

and therefore

$$\mathcal{H}^{n-1}(Z^+(\sigma_k g) \cap \partial\{u_k > 0\}) \leq \frac{1 + \tau_k}{1 - \tau_k} \mathcal{H}^{n-1}(Z^0(\sigma_k g) \cap \{u_k > 0\}). \quad (3.25)$$

Combining (3.23) and (3.25), we obtain (3.21).

We now prove the excess area estimate (3.20).

Since f is continuous and $f(y_0) > g(y_0)$, we can find a smooth superharmonic function η with $f > \eta \geq g$ near y_0 . For example, choose $\epsilon > 0$, let ϕ be a smooth function such that $\chi_{B'_\epsilon(y_0)} \leq \phi \leq \chi_{B'_{2\epsilon}(y_0)}$, and let η be such that

$$\begin{cases} -\Delta \eta = \phi & \text{in } B'_\rho(y_0), \\ \eta = g & \text{on } \partial B'_\rho(y_0). \end{cases}$$

By Green's representation formula (see e.g., [23, Theorem 2.12]),

$$\sup_{B'_\rho(y_0)} (\eta - g) \rightarrow 0 \text{ as } \epsilon \rightarrow 0,$$

and choosing ϵ small enough, we have $f > \eta$ in $B'_{2\epsilon}(y_0)$.

We first prove a lower bound for

$$\mathcal{H}^{n-1}(Z \cap \partial E_k) - \mathcal{H}^{n-1}(Z^0(\sigma_k \eta)).$$

For this, we use the following vector field in Z , equal to the inward unit normal to $Z^+(\sigma_k \eta)$ on $Z^0(\sigma_k \eta)$

$$\hat{n}_k := \frac{(-\sigma_k \nabla \eta(y), 1)}{\sqrt{1 + |\sigma_k \nabla \eta(y)|^2}}.$$

Observe that, since $D^2 \eta$ is bounded and $-\Delta \eta \geq 0$, we have

$$\operatorname{div} \hat{n}_k = \frac{-\sigma_k \Delta \eta}{\sqrt{1 + \sigma_k^2 |\nabla \eta|^2}} + \frac{\sigma_k^3 \nabla \eta^T D^2 \eta \nabla \eta}{(1 + \sigma_k^2 |\nabla \eta|^2)^{3/2}} \geq -C \sigma_k^3. \quad (3.26)$$

By a similar argument to the one used for (3.22), we may assume that

$$\mathcal{H}^{n-1}(Z^0(\sigma_k \eta) \cap \partial\{u_k > 0\}) = 0. \quad (3.27)$$

To estimate the upper part of $\partial E_k \cap Z$, we use the Gauss-Green theorem (see, e.g., [42, Remark 9.5]) and (3.27) to compute

$$\begin{aligned} \mathcal{H}^{n-1}(Z^+(\sigma_k \eta) \cap \partial E_k) &\geq \int_{\partial E_k \cap Z^+(\sigma_k \eta)} \hat{n}_k \cdot \nu \, d\mathcal{H}^{n-1} \\ &= \int_{E_k \cap Z^+(\sigma_k \eta)} \operatorname{div} \hat{n}_k \, dx + \int_{E_k \cap \partial Z^+(\sigma_k \eta)} \hat{n}_k \cdot \hat{n}_k \, d\mathcal{H}^{n-1}, \end{aligned}$$

where ν is the inward unit normal to E_k . Since $u_k \in F(\sigma_k, \sigma_k; \tau_k)$, (3.1) and (3.2) imply that

$$|E_k \cap Z^+(\sigma_k \eta)| \leq \mathcal{H}^{n-1}(B'_1) \cdot 2\sigma_k,$$

and, using (3.26), we conclude

$$\mathcal{H}^{n-1}(Z^+(\sigma_k \eta) \cap \partial E_k) \geq \mathcal{H}^{n-1}(Z^0(\sigma_k \eta) \cap E_k) - C \sigma_k^4. \quad (3.28)$$

By construction of η and Corollary 3.4, for large k , we have

$$\{x : |x' - y_0| < 2\epsilon, x_n \leq \sigma_k \eta(x')\} \subset E_k.$$

Therefore, in $Z^-(\sigma_k \eta) \setminus E_k$, we have $-\Delta \eta = 0$, and thus, computing as in (3.26), $|\operatorname{div} \hat{n}_k| \leq C \sigma_k^3$. Arguing as above, we can estimate the lower part of $\partial E_k \cap Z$ by

$$\mathcal{H}^{n-1}(Z^-(\sigma_k \eta) \cap \partial E_k) \geq \mathcal{H}^{n-1}(Z^0(\sigma_k \eta) \setminus E_k) - C \sigma_k^4. \quad (3.29)$$

Adding (3.28) and (3.29), we obtain

$$\mathcal{H}^{n-1}(Z \cap \partial E_k) \geq \mathcal{H}^{n-1}(Z^0(\sigma_k \eta)) - C \sigma_k^4. \quad (3.30)$$

To finish the proof, we compare the areas of the graphs of g and η . Since g is harmonic and $g = \eta$ on $\partial B'_\rho(y_0)$, integration by parts gives

$$\int_{B'_\rho(y_0)} |\nabla \eta|^2 - |\nabla g|^2 \, d\mathcal{L}^{n-1} = \int_{B'_\rho(y_0)} |\nabla(\eta - g)|^2 \, d\mathcal{L}^{n-1}.$$

A Taylor expansion gives

$$\begin{aligned}
\mathcal{H}^{n-1}(Z^0(\sigma_k \eta)) - \mathcal{H}^{n-1}(Z^0(\sigma_k g)) &= \int_{B'_\rho(y_0)} \left(\sqrt{1 + |\sigma_k \nabla \eta|^2} - \sqrt{1 + |\sigma_k \nabla g|^2} \right) d\mathcal{L}^{n-1} \\
&\geq \int_{B'_\rho(y_0)} \left(1 + \frac{\sigma_k^2}{2} |\nabla \eta|^2 - \left(1 + \frac{\sigma_k^2}{2} |\nabla g|^2 \right) \right) d\mathcal{L}^{n-1} \\
&\quad - C\sigma_k^4 \\
&\geq c\sigma_k^2 \int_{B'_\rho(y_0)} |\nabla(\eta - g)|^2 d\mathcal{L}^{n-1} - C\sigma_k^4 \\
&\geq c\sigma_k^2 - C\sigma_k^4,
\end{aligned}$$

where in the last step we used that since $\eta - g \not\equiv 0$, the last integral is bounded below by a positive constant depending only on the fixed choice of ϵ . Together with (3.30), this yields (3.20). $\square$

In the case of minimal surfaces, the harmonicity of the blow-up limit is enough to prove the improvement of flatness for f . Since, in our case, the blow-up limit is only subharmonic, we need to prove further regularity properties of f .

Using an appropriate Green function as a test function in (1.9), Alt and Caffarelli found that the average of f over $\partial B'_r(y)$ converges to $f(y)$ faster than linearly as $r \rightarrow 0$. This means that f is better than Lipschitz continuous, in the sense that the average over $\partial B'_r(y)$ of a general Lipschitz function converges only linearly as $r \rightarrow 0$.

Lemma 3.6. *There exists a constant $C = C(n) < \infty$ such that for every $y \in B'_{1/2}$,*

$$\int_0^{1/4} \frac{1}{r^2} \left(\oint_{\partial B'_r(y)} f d\mathcal{H}^{n-2} - f(y) \right) dr \leq C.$$

Proof. Observe that, for large k , the functions u_k in Lemma 3.3 are of class $F(8\sigma_k, 8\sigma_k; \tau_k)$ in $B_{1/4}((y, \sigma_k f_k^+(y)))$. Thus, after translating, we may assume $y = 0$.

In order to obtain local information on $\partial\{u_k > 0\}$, we use a Green function with pole approaching the free boundary as a test function in (1.9).

For small $h > 0$, let G_h denote the odd extension to $B_{1/2}$ of the Green function of $B_{1/2} \cap \{x_n < 0\}$ with pole at $-he_n$. Set

$$G_h^k(x) := G_h(x + \sigma_k e_n), \quad B_{1/2}^k := B_{1/2}(-\sigma_k e_n).$$

By a standard approximation argument (see, e.g., the proof of [44, Lemma 3.7]), we can use G_h^k in $B_{1/2}^k$ as a test function in (1.9) and obtain

$$\begin{aligned}
\int_{B_{1/2}^k \cap \partial\{u_k > 0\}} G_h^k Q_{u_k} d\mathcal{H}^{n-1} &= - \int_{B_{1/2}^k} \nabla u_k \cdot \nabla G_h^k dx \\
&= \int_{\partial B_{1/2}^k} u_k \partial_{-\nu} G_h^k d\mathcal{H}^{n-1} - u_k(-(\sigma_k + h)e_n).
\end{aligned} \tag{3.31}$$

We compare (3.31) with the case of the half-plane solution $u_0 = (-x_n)_+$. As in (3.22), we may assume that $\mathcal{H}^{n-1}(\partial B_{1/2}^k \cap \partial\{u_k > 0\}) = 0$ (otherwise we replace the radius $1/2$ with a suitable radius nearby) so that, using Green's third identity on the domain $B_{1/2}^k \cap \{u_k > 0\}$ and the function $-x_n$ (or by the Gauss-Green theorem [42, Remark 9.5]), we obtain

$$\int_{B_{1/2}^k \cap \partial\{u_k > 0\}} (G_h^k e_n - x_n \nabla G_h^k) \cdot \nu_{u_k} d\mathcal{H}^{n-1} = - \int_{\partial B_{1/2}^k \cap \{u_k > 0\}} x_n \partial_{-\nu} G_h^k d\mathcal{H}^{n-1} - (\sigma_k + h), \quad (3.32)$$

where ν_{u_k} is the inward unit normal to $\{u_k > 0\}$, and ν is the inward unit normal to $B_{1/2}^k$. We divide (3.31) by $1 - \tau_k$, subtract it from (3.32), and divide by σ_k to obtain

$$\begin{aligned} \frac{1}{\sigma_k} \int_{B_{1/2}^k \cap \partial\{u_k > 0\}} x_n \partial_{-\nu_{u_k}} G_h^k d\mathcal{H}^{n-1} &= \frac{1}{\sigma_k} \left(\frac{u_k}{1 - \tau_k} + x_n \right) \Big|_{x = -(\sigma_k + h)e_n} \\ &\quad - \frac{1}{\sigma_k} \int_{\partial B_{1/2}^k \cap \{u_k > 0\}} \left(\frac{u_k}{1 - \tau_k} + x_n \right) \partial_{-\nu} G_h^k d\mathcal{H}^{n-1} \\ &\quad + \frac{1}{\sigma_k} \int_{B_{1/2}^k \cap \partial\{u_k > 0\}} G_h^k \left(\frac{Q_{u_k}}{1 - \tau_k} - \nu_{u_k} \cdot e_n \right) d\mathcal{H}^{n-1}. \end{aligned} \quad (3.33)$$

Observe that on the left-hand side we have a weighted integral of the height of $\partial\{u_k > 0\}$, with the weight becoming singular at the origin when $h \rightarrow 0$, whereas the right-hand side is a function of $\frac{u_k}{1 - \tau_k} - (-x_n)$.

We now pass to the limit in k . We begin by estimating each term in the right-hand side of (3.33). For the first term, observe that if $x_n \leq 0$ and $x' = 0$, we can use (3.3) and the fundamental theorem of calculus to obtain

$$u_k(x) = \int_{x_n}^0 -\partial_n u_k(0, t) dt \leq -x_n(1 + \tau_k),$$

and therefore, since we assumed $\tau_k = o(\sigma_k)$,

$$\frac{1}{\sigma_k} \left(\frac{u_k}{1 - \tau_k} + x_n \right) \Big|_{x = -(\sigma_k + h)e_n} \leq \frac{2\tau_k \sigma_k + h}{\sigma_k (1 - \tau_k)} \rightarrow 0,$$

and we can neglect the first term for $k \rightarrow \infty$. For the second term, first observe that for $x_n \leq 0$, (3.2) implies

$$\frac{u_k(x)}{1 - \tau_k} + x_n \geq \frac{u_k(x) + x_n}{1 - \tau_k} \geq -\frac{\sigma_k}{1 - \tau_k}.$$

Thus,

$$-\frac{1}{\sigma_k} \left(\frac{u_k}{1 - \tau_k} + x_n \right) \leq \frac{1}{1 - \tau_k}. \quad (3.34)$$

Furthermore, since G_h is the difference of the Green function for $B_{1/2}^-$ with pole at $-he_n$ and the Green function for $B_{1/2}^+$ with pole at he_n , we can compute it explicitly to obtain the estimate

$$\begin{aligned} \partial_{-\nu} G_h(x) &= C(n) \left(\frac{1}{4} - h^2 \right) \left(\frac{1}{|x + he_n|^n} - \frac{1}{|x - he_n|^n} \right) \\ &\leq C \left(\frac{1}{|-\frac{1}{2} + h|^n} - \frac{1}{|-\frac{1}{2} - h|^n} \right) \leq Ch, \end{aligned} \quad (3.35)$$

for small h and uniformly in $\partial B_{1/2}$ (cf. [23, Section 2.2.4 (a)] and (1.4)). Therefore, we can use (3.34) and (3.35) to obtain, letting $k \rightarrow \infty$,

$$\begin{aligned} -\frac{1}{\sigma_k} \int_{\partial B_{1/2}^k \cap \{u_k > 0\}} \left(\frac{u_k}{1 - \tau_k} + x_n \right) \partial_{-v} G_h^k d\mathcal{H}^{n-1} &\leq \frac{1}{1 - \tau_k} \int_{\partial B_{1/2}^k \cap \{u_k > 0\}} \partial_{-v} G_h^k d\mathcal{H}^{n-1} \\ &\rightarrow \int_{\partial B_{1/2} \cap \{x_n < 0\}} \partial_{-v} G_h d\mathcal{H}^{n-1} \leq C(n)h, \end{aligned}$$

where we used that $\chi_{\{u_k(x - \sigma_k e_n) > 0\}} \rightarrow \chi_{\{x_n < 0\}}$ pointwise a.e. and the dominated convergence theorem. Finally, the third term is nonpositive since $G_h^k \leq 0$ in $\{x_n \geq -\sigma_k\}$ and we have $1 - Q_{u_k} \leq \tau_k$ by (3.4). Thus, for large k , we have shown that

$$\int_{B_{1/2}^k \cap \partial \{u_k > 0\}} \frac{x_n}{\sigma_k} \partial_{-v_{u_k}} G_h^k d\mathcal{H}^{n-1} \leq Ch. \quad (3.36)$$

For the left-hand side, we use that, since $\chi_{\{u_k > 0\}} \rightarrow \chi_{\{x_n < 0\}}$ in $L^1(B_1)$, [3, Proposition 3.13] implies that

$$\int_{B_1 \cap \partial \{u_k > 0\}} \eta_k \cdot \nu_{u_k} d\mathcal{H}^{n-1} \rightarrow \int_{B'_1} \eta(y, 0) \cdot e_n d\mathcal{L}^{n-1}(y) \quad (3.37)$$

for Borel fields η_k with uniformly compact support in B_1 converging uniformly to $\eta \in C_c(B_1, \mathbb{R}^n)$. We can apply this to the functions

$$\eta_k := \chi_{B_{1/2}^k} \frac{x_n}{\sigma_k} \nabla G_h^k.$$

Indeed, by Lemma 3.3,

$$\sup_{x \in B_{1/2} \cap \partial \{u_k > 0\}} \left| \frac{x_n}{\sigma_k} \nabla G_h^k(x) - f(x') \nabla G_h(x', 0) \right| \rightarrow 0,$$

as $k \rightarrow \infty$, so η_k converges uniformly to $f(x') \nabla G_h((x', 0))$, which is continuous by Corollary 3.4. Since, on $B'_{1/2} \times \{0\}$, $\nabla G_h = \partial_n G_h e_n$ and, on $\partial B'_{1/2} \times \{0\}$, the explicit formula gives $\nabla G_h = 0$, we can use (3.37) to pass to the limit in (3.36) and obtain

$$\frac{1}{h} \int_{B'_{1/2}} f \partial_{-n} G_h d\mathcal{L}^{n-1} \leq C. \quad (3.38)$$

We now pass to the limit in h . Since G_h is symmetric around the x_n -axis, $\partial_{-n} G_h$ is radially symmetric in $B'_{1/2}$. Write

$$\partial_{-n} G_h(y, 0) =: g_h(|y|) \quad \text{for } y \in B'_{1/2}.$$

Integrating in spherical coordinates in (3.38), we obtain

$$\frac{1}{h} \int_0^{1/2} r^{n-2} g_h(r) \int_{\partial B'_r} f d\mathcal{H}^{n-2} dr \leq C. \quad (3.39)$$

Using the explicit formula for G_h as we did for (3.35), we can compute g_h explicitly and obtain, for $r \leq 1/4$ and small h ,

$$g_h(r) = C(n) \left(1 - \left(\frac{r^2 + h^2}{4h^2 r^2 + \frac{1}{4}} \right)^{\frac{n}{2}} \right) \frac{h}{(r^2 + h^2)^{\frac{n}{2}}} \geq \frac{c(n)h}{(r^2 + h^2)^{\frac{n}{2}}}. \quad (3.40)$$

Since, by Lemma 3.5 and Corollary 3.4, f is subharmonic and $f(0) = 0$, the average $\oint_{\partial B'_r} f d\mathcal{H}^{n-2}$ is nonnegative. Since $g_h(r)$ is also nonnegative, by (3.39), we have

$$\frac{1}{h} \int_0^{1/4} r^{n-2} g_h(r) \oint_{\partial B'_r} f d\mathcal{H}^{n-2} dr \leq C, \quad (3.41)$$

and we can substitute (3.40) into (3.41) to obtain

$$\int_0^{1/4} \frac{r^{n-2}}{(r^2 + h^2)^{n/2}} \oint_{\partial B'_r} f d\mathcal{H}^{n-2} dr \leq C.$$

Letting $h \rightarrow 0$ and using Fatou's lemma,

$$\int_0^{1/4} \frac{1}{r^2} \left(\oint_{\partial B'_r} f d\mathcal{H}^{n-2} \right) dr \leq C.$$

□

Lemma 3.7. f is Lipschitz continuous in $\overline{B'}_{1/4}$ with Lipschitz constant at most $C(n)$.

Proof. Let G_y be the Green function of $B'_{1/2}$ with pole at y . Since f is subharmonic (Lemma 3.5), by the representation theorem (see, e.g., [33, Theorem 3.14]),

$$f(y) = \int_{\partial B'_{1/2}} f \partial_{-\nu} G_y d\mathcal{H}^{n-2} - \int_{B'_{1/2}} G_y d\lambda,$$

where $\lambda := \Delta f$ is a (positive) Radon measure; see, e.g., [42, Chapter 2] for the definition of Radon measure.

The first integral is the harmonic extension of f in $B'_{1/2}$, which, by interior estimates (see, e.g., [23, Theorem 2.2.7]) and the fact that $|f| \leq 1$, is a Lipschitz function of y with Lipschitz constant bounded by $C(n)$.

We prove the bound

$$\int_{B'_{1/2}} \frac{d\lambda(z)}{|z - y|^{n-2}} \leq C(n), \quad \forall y \in B'_{1/4}. \quad (3.42)$$

This implies the Lipschitz continuity of the second integral, since, for $y_0, y_1 \in B'_{1/4}$, we can estimate (cf. [44, Sections 2.4 and 3.4])

$$|G_{y_0}(z) - G_{y_1}(z)| \leq C(n) |y_0 - y_1| \left(\frac{1}{|z - y_0|^{n-2}} + \frac{1}{|z - y_1|^{n-2}} \right).$$

By the representation theorem, for $y \in B'_{1/2}$ and $0 < r \leq 1/4$,

$$f(y) = \int_{\partial B'_r(y)} \partial_{-\nu} G^r f d\mathcal{H}^{n-2} - \int_{B'_r(y)} G^r d\lambda,$$

where G^r denotes the Green function of $B'_r(y)$ with pole at y . Since

$$\partial_{-\nu} G^r \equiv \frac{1}{\mathcal{H}^{n-2}(\partial B'_r)} \quad \text{on} \quad \partial B'_r(y),$$

we obtain

$$\oint_{\partial B'_r(y)} (f - f(y)) d\mathcal{H}^{n-2} = \int_{B'_r(y)} G^r d\lambda,$$

and, by Lemma 3.6 and Tonelli's theorem, we obtain

$$C \geq \int_0^{1/4} \frac{1}{r^2} \int_{B'_r(y)} G^r d\lambda dr \geq \int_{B'_{1/8}(y)} \int_{|z-y|}^{1/4} \frac{1}{r^2} G^r(z) dr d\lambda(z). \quad (3.43)$$

For $z \in B'_{1/8}(y)$, $z \neq y$, we claim that

$$\int_{|z-y|}^{1/4} \frac{1}{r^2} G^r(z) dr \geq \frac{c}{|z-y|^{n-2}}. \quad (3.44)$$

Indeed, for $n \geq 4$ (the cases $n = 3$ and $n = 2$ are analogous), we know that (see e.g., [23, Section 2.2.4 (c)])

$$G^r(z) = C \left(\frac{1}{|z-y|^{n-3}} - \frac{1}{r^{n-3}} \right),$$

and using the change of variables $r = |z-y|t$, we obtain

$$\int_{|z-y|}^{1/4} \frac{1}{r^2} G^r(z) dr \geq \frac{C}{|z-y|^{n-2}} \int_1^2 \frac{1-t^{3-n}}{t^2} dt \geq \frac{c}{|z-y|^{n-2}}.$$

Therefore, from (3.43) and (3.44), we obtain, for every $y \in B'_{1/2}$,

$$\int_{B'_{1/8}(y)} \frac{d\lambda(z)}{|z-y|^{n-2}} \leq C(n). \quad (3.45)$$

In particular, since $|z-y| \leq 1/8$ in $B'_{1/8}(y)$, this implies

$$\lambda(B'_{1/8}(y)) \leq C(n).$$

Covering $B'_{1/2}$ by finitely many balls $B'_{1/8}(y_j)$, with $y_j \in B'_{1/2}$, we obtain

$$\lambda(B'_{1/2}) \leq C(n). \quad (3.46)$$

Now fix $y \in B'_{1/4}$. Splitting the integral in (3.42), we get

$$\int_{B'_{1/2}} \frac{d\lambda(z)}{|z-y|^{n-2}} = \int_{B'_{1/8}(y)} \frac{d\lambda(z)}{|z-y|^{n-2}} + \int_{B'_{1/2} \setminus B'_{1/8}(y)} \frac{d\lambda(z)}{|z-y|^{n-2}}.$$

The first term is bounded by (3.45). For the second term, we use $|z-y| \geq 1/8$ and (3.46), obtaining

$$\int_{B'_{1/2} \setminus B'_{1/8}(y)} \frac{d\lambda(z)}{|z-y|^{n-2}} \leq C(n) \lambda(B'_{1/2}) \leq C(n).$$

This proves (3.42). $\square$

We can now prove an improvement of flatness property for f . For $\theta > 0$ as small as we like, we can find a scale $r > 0$ such that, at this scale, f can be approximated from above by a linear function with error at most θ . Note the necessity of Lemma 3.6: for a general Lipschitz function f , we cannot expect an improvement of the linear approximation at smaller scales.

Lemma 3.8 (Improvement of flatness from above for f). *There is a large constant $C = C(n) < \infty$ such that for every $\theta > 0$ there is a small constant $c_\theta = c(\theta, n) > 0$, a ball B'_r , and a vector $l \in \mathbb{R}^{n-1}$ with*

$$c_\theta \leq r \leq \theta, |l| \leq C, \text{ and } f(y) \leq l \cdot y + \frac{\theta}{2}r \text{ for } |y| \leq r.$$

Proof. Let $s > 0$ and let h be the harmonic extension of f in B'_s . We bound f from above by h and then apply Taylor's theorem, bounding each term using the previous lemmas.

We first bound $h(0)$. Let $\delta > 0$, and $c = c(\delta) \leq \frac{1}{4}$ to be chosen later. Recall that, by Corollary 3.4 and Lemma 3.5, $f(0) = 0$ and f is subharmonic. Thus

$$\int_{\partial B'_r} f d\mathcal{H}^{n-2} \geq 0.$$

Lemma 3.6 then implies

$$\begin{aligned} \log \frac{1}{4c} \inf_{c \leq s \leq \frac{1}{4}} \frac{1}{s} \int_{\partial B'_s} f d\mathcal{H}^{n-2} &= \int_c^{1/4} \frac{1}{r} dr \inf_{c \leq s \leq \frac{1}{4}} \frac{1}{s} \int_{\partial B'_s} f d\mathcal{H}^{n-2} \\ &\leq \int_0^{1/4} \frac{1}{r^2} \int_{\partial B'_r} f d\mathcal{H}^{n-2} dr \leq C. \end{aligned}$$

Choosing $c(\delta) := \frac{1}{4} \exp(-\frac{C}{\delta})$, we obtain a radius $c \leq s \leq \frac{1}{4}$ with

$$h(0) = \int_{\partial B'_s} f d\mathcal{H}^{n-2} \leq \delta s. \quad (3.47)$$

We now bound $|\nabla h(0)|$ and $\sup_{B'_{s/2}} |D^2 h|$. Since f is Lipschitz continuous by Lemma 3.7, and $f(0) = 0$, we have

$$\sup_{B'_s} |h| = \sup_{\partial B'_s} |f| \leq Cs,$$

and by interior estimates (see, e.g., [23, Theorem 2.2.7]),

$$|\nabla h(0)| \leq \frac{C}{s} \sup_{B'_s} |h| \leq C, \quad (3.48)$$

$$|D^2 h| \leq \frac{C}{s^2} \sup_{B'_s} |h| \leq \frac{C}{s} \quad \text{in } B'_{s/2}. \quad (3.49)$$

Since f is subharmonic, $f \leq h$ in B'_s . Therefore, Taylor's theorem together with (3.47) and (3.49) implies that for $y \in B'_{s/2}$,

$$f(y) \leq h(y) \leq \nabla h(0) \cdot y + \left(\delta s + \frac{C}{s} |y|^2 \right).$$

Taking $y \in B'_{\sqrt{\delta}s}$,

$$f(y) \leq \nabla h(0) \cdot y + C\sqrt{\delta}\sqrt{\delta}s. \quad (3.50)$$

Finally, we choose δ such that $\theta = 2C\sqrt{\delta}$, and set $r := \sqrt{\delta}s$, $l := \nabla h(0)$ and $c_\theta := \sqrt{\delta}s$. By the definition of s and (3.48),

$$c_\theta \leq r \leq \theta \quad \text{and} \quad |l| \leq C.$$

Then (3.50) gives

$$f(y) \leq l \cdot y + \frac{\theta}{2}r \quad \text{for } |y| \leq r.$$

□

3.3 Improvement of flatness

In this section, we show how the improvement of flatness from above for f yields an improvement of flatness for sufficiently flat free boundaries.

We start by proving that the flatness is improved from above. Note that, in general, the improvement can only be obtained if we allow the flatness direction to change; however, the discrepancy between these directions cannot be too large.

Lemma 3.9. *There is a large constant $C = C(n) < \infty$ such that for $\theta > 0$ there are small constants $c_\theta = c(\theta, n) > 0$ and $\sigma_\theta = \sigma(\theta, n) > 0$ with the following property: if*

$$u \in F(\sigma, \sigma; \tau) \text{ in } B_\rho \text{ in the direction } \nu$$

with $\sigma \leq \sigma_\theta$ and $\tau \leq \sigma_\theta \sigma^2$, then

$$u \in F(\theta\sigma, 1; \tau) \text{ in } B_{\bar{\rho}} \text{ in the direction } \bar{\nu},$$

for some $\bar{\rho}$ and $\bar{\nu}$ with $c_\theta \rho \leq \bar{\rho} \leq \theta\rho$ and $|\bar{\nu} - \nu| \leq C\sigma$.

Proof. By rescaling and rotating we may assume that $\nu = e_n$, $\rho = 1$, and $Q(0) = 1$.

Assume for the sake of contradiction that no such σ_θ exists. Then there is a sequence of counterexamples

$$u_k \in F(\sigma_k, \sigma_k; \tau_k) \text{ in } B_1$$

with $\sigma_k < \frac{1}{k}$ and $\tau_k < \frac{1}{k}\sigma_k^2$ for which the conclusion does not hold. Let f_k^+ and f be as in Lemma 3.3. By Lemma 3.8, there is a radius $c_\theta \leq r \leq \theta$ and a vector $l \in \mathbb{R}^{n-1}$ with $|l| \leq C$ such that

$$f(y) \leq l \cdot y + \frac{\theta}{2}r \quad \text{for } |y| \leq r.$$

Since, by Corollary 3.4, $(f_k^+)_k$ converges locally uniformly along a subsequence to f , for some large k we have that

$$f_k^+(y) \leq l \cdot y + \theta r \quad \text{for } |y| \leq r.$$

Therefore $u_k(x) = 0$ for $x \in B_r$ with $x_n \geq \sigma_k l \cdot x' + \theta\sigma_k r$, or equivalently,

$$x \cdot \bar{\nu} \geq \bar{\sigma}r,$$

with

$$\bar{\sigma} := \frac{\theta\sigma_k}{\sqrt{1 + |\sigma_k l|^2}}, \quad \bar{\nu} := \frac{(-\sigma_k l, 1)}{\sqrt{1 + |\sigma_k l|^2}}.$$

That is,

$$u_k \in F(\bar{\sigma}, 1; \tau_k) \text{ in } B_r \text{ in the direction } \bar{\nu},$$

with $\bar{\sigma} \leq \theta\sigma_k$ and $|\bar{\nu} - e_n| \leq C\sigma_k$. This contradicts the choice of u_k , proving the lemma.

□

Combining Lemma 3.2 and Lemma 3.9, we obtain the improvement of flatness. To use Lemma 3.2, we also need to improve the value of τ , for which we need some mild regularity assumptions on Q . In the next section, we will show that the assumption of a small deviation parameter $\tau \leq \sigma_\theta \sigma^2$ is automatically satisfied by all sufficiently flat free boundaries if Q is Hölder continuous.

Lemma 3.10. *Let ω be a continuous, increasing function with $\omega(0) = 0$ and $\theta > 0$. Then there are constants $\sigma_\theta = \sigma(n, \theta, \omega) > 0$, $c_\theta = c(n, \theta, \omega) > 0$, and $C = C(n) < \infty$ such that if*

$$u \in F(\sigma, 1; \tau) \text{ in } B_\rho \text{ in the direction } v$$

with $\sigma \leq \sigma_\theta$ and $\tau \leq \sigma_\theta \sigma^2$, and if for $0 \leq s \leq 1$, $\text{osc}_{B_{s\rho}} Q \leq \omega(s)\tau Q(0)$, then

$$u \in F(\theta\sigma, \theta\sigma; \theta^2\tau) \text{ in } B_{\bar{\rho}} \text{ in the direction } \bar{v}$$

for some $\bar{\rho}, \bar{v}$ with $c_\theta \rho \leq \bar{\rho} \leq \frac{1}{4}\rho$ and $|\bar{v} - v| \leq C\sigma$.

Proof. By rescaling we may assume that $\rho = 1$ and $Q(0) = 1$. If σ_θ is small enough, we can first apply Lemma 3.2 to get that

$$u \in F(C\sigma, C\sigma; \tau) \text{ in } B_{1/2} \text{ in the direction } v,$$

and then, for $0 < \theta_1 \leq \frac{1}{2}$, apply Lemma 3.9 to obtain

$$u \in F(C\theta_1\sigma, 1; \tau) \text{ in } B_{2r_1} \text{ in the direction } v_1, \quad (3.51)$$

for some r_1 and v_1 such that

$$c_{\theta_1} \leq 4r_1 \leq \theta_1, \text{ and } |v_1 - v| \leq C\sigma. \quad (3.52)$$

Applying this result repeatedly, we will be able to make the free boundary arbitrarily flat from above. In order to apply Lemma 3.2 and obtain flatness from below, we must also improve the value of τ .

We start by proving the bound $|\nabla u| \leq 1 + c\tau$ for some small c independent of u . To do so, consider the function

$$U := \begin{cases} \max(|\nabla u| - \sup_{B_{3r_1}} Q, 0) & \text{in } B_{2r_1} \cap \{u > 0\} \\ 0 & \text{in } B_{2r_1} \cap \{u = 0\} \end{cases}.$$

U is subharmonic in $\{u > 0\}$ since it is the maximum of two subharmonic functions. Moreover, since Q is continuous, by Theorem 2.8 we know that

$$\sup_{B_r(x)} |\nabla u| \leq \sup_{B_R(x)} Q + C \left(\frac{r}{R} \right)^\alpha \leq \sup_{B_{3r_1}} Q + C \left(\frac{r}{R} \right)^\alpha,$$

for some $\alpha > 0$ and all $B_r(x) \subset B_R(x) \subset B_{3r_1}$, and so U is continuous in $\bar{B}_{2r_1}$. By [44, Lemma 5.6], U is subharmonic in the entire ball B_{2r_1} . Since $u \in F(C\theta_1\sigma, 1; \tau)$ in B_{2r_1} in the direction v_1 , we get that $U \leq \tau$ in this ball, and, by taking σ_θ sufficiently small, we also have $U = 0$ in

$$B := B_{r_1/4} \left(\frac{r_1}{2} v_1 \right).$$

Therefore, we can estimate U from above by the harmonic function h in $B_{2r_1} \setminus \bar{B}$ with boundary values τ on ∂B_{2r_1} and 0 on ∂B to obtain that

$$U \leq h \leq (1 - c)\tau \quad \text{in } B_{r_1},$$

with the constant c depending only on n . Using the oscillation assumption on Q , (3.52), and the monotonicity of ω , we get that in B_{r_1} ,

$$|\nabla u| \leq \sup_{B_{3r_1}} Q + U \leq 1 + \omega(3r_1)\tau + (1 - c)\tau \leq 1 + (1 - c + \omega(\theta_1))\tau, \quad (3.53)$$

as desired.

For any $\theta_0 > 0$, we can write

$$\text{osc}_{B_{r_1}} Q \leq \omega(r_1)\tau \leq (\theta_0^{-2}\omega(\theta_1))\theta_0^2\tau. \quad (3.54)$$

Combining (3.51), (3.53) and (3.54), setting $\theta_0^2 := 1 - c/2$, and choosing θ_1 so small that $2C\theta_1 \leq \theta_0$, $\omega(\theta_1) \leq c/2$, we obtain

$$u \in F(\theta_0\sigma, 1; \theta_0^2\tau) \text{ in } B_{r_1} \text{ in the direction } \nu_1.$$

To finish the proof, we iterate this argument m times (choosing a small enough σ_θ at each step), and get that

$$u \in F(\theta_0^m\sigma, 1; \theta_0^{2m}\tau) \text{ in } B_{r_1 \dots r_m} \text{ in the direction } \nu_m$$

with r_j and ν_m such that

$$c_{\theta_j} \leq 4r_j \leq \theta_j, \quad \text{and} \quad |\nu_m - \nu| \leq \frac{C}{1 - \theta_0}\sigma.$$

Choosing m large enough so that $\theta_0^m \leq \theta$, and applying Lemma 3.2, we obtain that, for $\bar{\rho} := \frac{r_1 \dots r_m}{2}$, $\bar{\nu} := \nu_m$, and $c_\theta := \frac{1}{2}(c_{\theta_1} \dots c_{\theta_m})4^{-m}$,

$$u \in F(\theta\sigma, \theta\sigma; \theta^2\tau) \text{ in } B_{\bar{\rho}} \text{ in the direction } \bar{\nu},$$

with $c_\theta \leq \bar{\rho} \leq 1/4$ and $|\bar{\nu} - \nu| \leq C\sigma$. Undoing the initial scaling gives the conclusion for general ρ . $\square$

3.4 Smoothness of the free boundary

We now use the improvement of flatness result from the previous section to prove the main theorem of this chapter.

Theorem 3.11. *Suppose that Q is Hölder continuous and u is a weak solution for Q in a domain $\Omega \subset \mathbb{R}^n$. Then for every $D \subset\subset \Omega$ there are constants $\alpha > 0$, $\beta > 0$, $\sigma_0 > 0$, $\tau_0 > 0$ and $C < \infty$ such that if*

$$u \in F(\sigma, 1; \infty) \text{ in } B_\rho(x_0) \subset D \text{ in the direction } \nu$$

with $\sigma \leq \sigma_0$ and $\rho \leq \tau_0 \sigma^{2/\beta}$, then

$$\partial\{u > 0\} \cap B_{\rho/4}(x_0) \text{ is } C^{1,\alpha},$$

and for $x, y \in \partial\{u > 0\} \cap B_{\rho/4}(x_0)$,

$$|\nu(x) - \nu(y)| \leq C\sigma \left| \frac{x - y}{\rho} \right|^\alpha,$$

where $\nu(x)$ denotes the unit normal to the free boundary at x . The constants depend only on n , D , $Q_{\min}$, $Q_{\max}$, and the Hölder exponent and norm of Q .

Proof. We prove the result by iterating Lemma 3.10. In order to do so, we first need to establish that u is of class $F(\sigma, 1; \tau_\rho)$ in $B_\rho(x_0)$ in the direction ν for an appropriate τ_ρ . By Corollary 2.9, there is $\beta > 0$ that is a Hölder exponent for Q and such that

$$|\nabla u(x)| \leq Q(x) + Cr^\beta$$

for $B_r(x) \subset B_\rho(x_0)$ touching the free boundary. The Hölder continuity of Q yields, for $x \in B_\rho(x_0) \cap \{u > 0\}$,

$$\begin{aligned} |\nabla u(x)| &\leq Q(x) + C \operatorname{dist}(x, \partial\{u > 0\})^\beta \leq Q(x_0) + C|x - x_0|^\beta \\ &\leq Q(x_0)(1 + C\rho^\beta), \end{aligned}$$

and

$$\operatorname{osc}_{B_\rho(x_0)} Q \leq Q(x_0)C\rho^\beta.$$

(Recall that C may change from line to line and is allowed to depend on $\|Q\|_{C^\beta(D)}$.) Thus, for $\tau_\rho := C\rho^\beta$,

$$u \in F(\sigma, 1; \tau_\rho) \text{ in } B_\rho(x_0) \text{ in the direction } \nu. \quad (3.55)$$

Choosing $\tau_0 \leq (\frac{\sigma_0}{C})^{1/\beta}$, we get that $\tau_\rho \leq \sigma_0 \sigma^{2/\beta}$ if $\rho \leq \tau_0 \sigma^{2/\beta}$.

Let $x_1 \in \partial\{u > 0\} \cap B_{\rho/2}(x_0)$. We claim that, if σ_0 and τ_0 are sufficiently small, for any $x \in \partial\{u > 0\} \cap B_{\rho/4}(x_1)$ there is a unit vector $\nu(x_1)$ such that

$$|(x - x_1) \cdot \nu(x_1)| \leq \frac{C\sigma}{\rho^\alpha} |x - x_1|^{1+\alpha}. \quad (3.56)$$

We first show that

$$u \in F(C\sigma, C\sigma; \tau) \text{ in } B_{\rho/4}(x_1) \text{ in the direction } \nu, \quad (3.57)$$

for $\tau := C(\frac{\rho}{4})^\beta$. By (3.55), we can apply Lemma 3.2 to u in $B_\rho(x_0)$ to obtain, provided σ_0 is sufficiently small,

$$u \in F(2\sigma, C\sigma; \tau) \text{ in } B_{\rho/2}(x_0) \text{ in the direction } \nu, \quad (3.58)$$

and in particular,

$$-C\sigma\rho \leq (x_1 - x_0) \cdot \nu.$$

Then, for $x \in B_{\rho/2}(x_1) \cap \partial\{u > 0\}$,

$$(x - x_1) \cdot \nu = (x - x_0) \cdot \nu + (x_0 - x_1) \cdot \nu \leq C\sigma\rho,$$

and we can use Corollary 2.9 and the Hölder continuity of Q exactly as above to obtain

$$u \in F(C\sigma, 1; \tau) \text{ in } B_{\rho/2}(x_1) \text{ in the direction } \nu,$$

so that, after choosing τ_0 and σ_0 sufficiently small, we can apply Lemma 3.2 to obtain (3.57).

We now iteratively apply Lemma 3.10 with the function $\omega(s) = Cs^\beta$ and some $0 < \theta < 1$, so that we get ρ_m and ν_m with

$$c_\theta \rho_m \leq \rho_{m+1} \leq \frac{\rho_m}{4}, \quad \text{and} \quad |\nu_{m+1} - \nu_m| \leq \theta^m C\sigma,$$

where $\rho_0 := \frac{\rho}{4}$, and $\nu_0 := \nu$, such that

$$u \in F(\theta^m C\sigma, \theta^m C\sigma; \theta^{2m} \tau) \text{ in } B_{\rho_m}(x_1) \text{ in the direction } \nu_m. \quad (3.59)$$

To do so, we set

$$\sigma_0 \leq \frac{\sigma_\theta}{C}, \quad \tau_0 \leq 4 \left(\frac{\sigma_\theta}{C} \right)^{\frac{1}{\beta}}, \quad \text{and} \quad \theta := 2^{-\beta} < 1,$$

so that

$$\theta^m C\sigma \leq \sigma_\theta \quad \text{and} \quad \theta^{2m} \tau \leq \sigma_\theta (\theta^m C\sigma)^2.$$

Since $\rho_m \leq 4^{-m} \rho_0$ and Q is Hölder continuous, we also have

$$\text{osc}_{B_{\rho_m}(x_1)} Q \leq Q(x_1) \omega(s) \theta^{2m} \tau.$$

Observe that (3.59) implies

$$|(x - x_1) \cdot \nu_m| \leq \theta^m C\sigma \rho_m \quad \text{for } x \in B_{\rho_m}(x_1) \cap \partial\{u > 0\}.$$

Moreover, since

$$\sum_{m=0}^{\infty} |\nu_{m+1} - \nu_m| \leq \sum_{m=0}^{\infty} \theta^m C\sigma < \infty,$$

we also get convergence of the normal vectors: the limit $\nu(x_1) := \lim_{m \rightarrow \infty} \nu_m$ exists and satisfies

$$|\nu(x_1) - \nu_m| \leq \frac{C\theta^m}{1 - \theta} \sigma. \quad (3.60)$$

We can now prove (3.56). Let $x \in B_{\rho/4}(x_1) \cap \partial\{u > 0\}$. If $x = x_1$, the estimate is trivial. Otherwise, choose m such that $\rho_{m+1} \leq |x - x_1| \leq \rho_m$. Then $|x - x_1| \geq c_\theta \rho_m$ and $|x - x_1| \geq c_\theta^{m+1} \rho_0$. We can use (3.59) and (3.60) to estimate

$$\begin{aligned} |(x - x_1) \cdot \nu(x_1)| &\leq |(x - x_1) \cdot \nu_m| + |(x - x_1) \cdot (\nu(x_1) - \nu_m)| \\ &\leq \theta^m C\sigma \left(\rho_m + \frac{|x - x_1|}{1 - \theta} \right) \\ &\leq C\theta^m \sigma \left(\frac{1}{c_\theta} + \frac{1}{1 - \theta} \right) |x - x_1|, \end{aligned}$$

since

$$\theta^{m+1} = c_\theta^{(m+1)\alpha} \leq \left(\frac{|x - x_1|}{\rho_0} \right)^\alpha \quad \text{for } \alpha := \frac{\log \theta}{\log c_\theta} = \frac{\beta \log 2}{\log c_\theta^{-1}},$$

we have proved (3.56).

We now use these estimates to prove that the free boundary is a $C^{1,\alpha}$ graph. After a rotation, we may assume that $\nu = e_n$.

First, we prove that $\partial\{u > 0\} \cap B_{\rho/4}(x_0)$ is a graph in the direction e_n . By (3.58), each vertical line over $B'_{\rho/4}(x'_0)$ intersects the free boundary at least once. We claim that this intersection is unique. Indeed, suppose that $x, y \in \partial\{u > 0\} \cap B_{\rho/4}(x_0)$ and $x' = y'$. Then, by (3.58), $|x - y| \leq C\sigma\rho$. In particular, if σ_0 is sufficiently small, then $x \in B_{\rho/4}(y)$. Using (3.56) with base point y , together with (3.60) with $m = 0$, we obtain

$$\begin{aligned} |x_n - y_n| &= |(x - y) \cdot e_n| \leq |(x - y) \cdot \nu(y)| + |x - y| |\nu(y) - e_n| \\ &\leq C\sigma\rho^{-\alpha} |x - y|^{1+\alpha} + C\sigma |x - y|. \end{aligned}$$

Since $|x - y| \leq C\rho$, this gives

$$|x_n - y_n| \leq C\sigma |x - y|.$$

But $x' = y'$, so $|x - y| = |x_n - y_n|$. Choosing σ_0 sufficiently small, we obtain $x = y$. Thus the free boundary is the graph of a function g in $B'_{\rho/4}(x'_0)$ in the direction e_n .

We next show that g is Lipschitz. Let

$$x = (x', g(x')), \quad y = (y', g(y'))$$

be two points on the graph. If $|x' - y'| \leq \rho/8$, then we may again apply (3.56) to obtain

$$|g(x') - g(y')| \leq C\sigma |x - y| \leq C\sigma (|x' - y'| + |g(x') - g(y')|).$$

Choosing σ_0 sufficiently small gives

$$|g(x') - g(y')| \leq |x' - y'|.$$

If instead $|x' - y'| > \rho/8$, then (3.58) gives

$$|g(x') - g(y')| \leq C\sigma\rho \leq |x' - y'|,$$

after decreasing σ_0 if necessary. Hence g is 1-Lipschitz in $B'_{\rho/4}(x'_0)$.

It remains to prove that $g \in C^{1,\alpha}$. Let $x'_1 \in B'_{\rho/4}(x'_0)$, and write

$$x_1 = (x'_1, g(x'_1)).$$

Set

$$\nu(x_1) = (\nu'(x_1), \nu_n(x_1)).$$

Since $|\nu(x_1) - e_n| \leq C\sigma$, we have $\nu_n(x_1) \geq 1/2$, provided σ_0 is sufficiently small. Define the affine function

$$\ell_{x'_1}(x') := g(x'_1) - \frac{\nu'(x_1)}{\nu_n(x_1)} \cdot (x' - x'_1).$$

For $x' \in B'_{\rho/4}(x'_0)$, set $x = (x', g(x'))$. If $|x' - x'_1|$ is sufficiently small, then $x \in B_{\rho/4}(x_1)$, and (3.56) gives

$$|g(x') - \ell_{x'_1}(x')| = \frac{|(x - x_1) \cdot \nu(x_1)|}{\nu_n(x_1)} \leq C \frac{\sigma}{\rho^\alpha} |x - x_1|^{1+\alpha}.$$

Since g is Lipschitz, $|x - x_1| \leq C|x' - x'_1|$. Therefore

$$|g(x') - \ell_{x'_1}(x')| \leq C \frac{\sigma}{\rho^\alpha} |x' - x'_1|^{1+\alpha}.$$

For larger radii the same estimate follows from (3.58) and the bound $|\nabla \ell_{x'_1}| \leq C\sigma$. Hence, for every ball $B'_r(x'_1) \subset B'_{\rho/4}(x'_0)$,

$$\sup_{B'_r(x'_1)} |g - \ell_{x'_1}| \leq C \frac{\sigma}{\rho^\alpha} r^{1+\alpha}.$$

By Lemma A.3, this implies

$$g \in C^{1,\alpha}(B'_{\rho/4}(x'_0)) \quad \text{and} \quad [\nabla g]_{C^{0,\alpha}(B'_{\rho/4}(x'_0))} \leq C \frac{\sigma}{\rho^\alpha}.$$

Finally, the unit normal to the free boundary is given by

$$\frac{(-\nabla g(x'), 1)}{\sqrt{1 + |\nabla g(x')|^2}}.$$

Since the map

$$p \mapsto \frac{(-p, 1)}{\sqrt{1 + |p|^2}}$$

is Lipschitz on bounded sets, we obtain, for $x, y \in \partial\{u > 0\} \cap B_{\rho/4}(x_0)$,

$$|\nu(x) - \nu(y)| \leq C |\nabla g(x') - \nabla g(y')| \leq C\sigma \left| \frac{x - y}{\rho} \right|^\alpha.$$

This completes the proof. $\square$

Corollary 3.12. *Let u be a weak solution in B_1 for a Hölder continuous Q , and let $x_0 \in \partial\{u > 0\} \cap B_1$ be a regular point. Then $\partial\{u > 0\}$ is $C^{1,\alpha}$ in a neighbourhood of x_0 .*

Proof. Let $D \subset\subset B_1$ be a domain with $x_0 \in D$. Fix $0 < \sigma \leq \sigma_0$, where σ_0 is the constant in Theorem 3.11. Since x_0 is regular, there is a unit vector ν and a blow-up sequence $r_k \rightarrow 0$ such that $u_{r_k}(x) \rightarrow Q(x_0)(x \cdot \nu)_-$ in the sense of Lemma 2.4. Hence, for k large enough,

$$u_{r_k} = 0 \quad \text{in } B_1 \cap \{x \cdot \nu \geq \sigma\}.$$

Indeed, the cap $\overline{B_1} \cap \{x \cdot \nu \geq \sigma\}$ lies a positive distance from the limiting free boundary. Thus the local convergence of $\partial\{u_{r_k} > 0\} \rightarrow \{x \cdot \nu = 0\}$ in the Hausdorff distance implies that for large k , $\{x \cdot \nu \geq \sigma\} \cap \partial\{u_{r_k} > 0\} = \emptyset$. Since the cap is connected and contains no free boundary points, it is either contained in $\{u_{r_k} > 0\}$ or in $\{u_{r_k} = 0\}$. That $\chi_{\{u_{r_k} > 0\}} \rightarrow \chi_{\{x \cdot \nu < 0\}}$ in L^1_{loc} rules out $B_1 \cap \{x \cdot \nu \geq \sigma\} \subset \{u_{r_k} > 0\}$. Thus

$$u_{r_k} \in F(\sigma, 1; \infty) \quad \text{in } B_1 \quad \text{in the direction } \nu$$

for all large k . Returning to the original scale, this says that

$$u \in F(\sigma, 1; \infty) \quad \text{in } B_{r_k}(x_0) \quad \text{in the direction } \nu.$$

For k large enough, $B_{r_k}(x_0) \subset \subset D$ and

$$r_k \leq \tau_0 \sigma^{2/\beta},$$

where β and τ_0 are the constants in Theorem 3.11. Applying Theorem 3.11, we conclude that

$$\partial\{u > 0\} \cap B_{r_k/4}(x_0) \text{ is } C^{1,\alpha}.$$

□

We can also prove that the set of regular points is relatively open in the free boundary and has full $\mathcal{H}^{n-1}$ measure.

Corollary 3.13. *Let u be a weak solution in B_1 for a Hölder continuous Q and let $\partial_{\text{diff}}\{u > 0\}$ be the set of points where the free boundary $\partial\{u > 0\}$ has a well-defined normal. Then $\partial_{\text{diff}}\{u > 0\}$ is relatively open in $\partial\{u > 0\}$, $\mathcal{H}^{n-1}(\partial\{u > 0\} \setminus \partial_{\text{diff}}\{u > 0\}) = 0$, and $\partial\{u > 0\}$ is a $C^{1,\alpha}$ surface in a neighbourhood of every point in $\partial_{\text{diff}}\{u > 0\}$.*

Proof. The local regularity of $\partial_{\text{diff}}\{u > 0\}$ follows from the same argument as in Corollary 3.12 using the existence of a well-defined normal at x_0 in place of the convergence of blow-ups to the half-plane solution. The relative openness follows from the local regularity. The measure statement is true for any Lipschitz domain as a consequence of Rademacher's theorem. □

To conclude this chapter, we state the higher regularity theorem proved in [37]. Doubts that subsequently arose concerning how the results in [37] apply to the Alt–Caffarelli problem were addressed in [38, Appendix].

Theorem 3.14 (Higher regularity). *Let u be a classical solution of the Alt–Caffarelli problem in B_1 . If $Q \in C^{k,\alpha}$ and $\partial\{u > 0\} \cap B_1$ is $C^{k,\alpha}$ for some $0 < \alpha < 1$ and $k \geq 1$, then $\partial\{u > 0\} \cap B_{1/2}$ is $C^{k+1,\alpha}$.*

Remark 3.15. From the methods in [38], it also follows that if u is a classical solution of the Alt–Caffarelli problem for $Q \in C^{0,\alpha}(B_1)$ and $\partial\{u > 0\} \cap B_1$ is $C^{1,\beta}$ for some $0 < \beta < \alpha < 1$, then $\partial\{u > 0\} \cap B_{1/2}$ is $C^{1,\alpha}$.

Chapter 4

Blow-up limits of weak solutions

In this chapter, we study blow-up limits of weak solutions. Following [21], we use the improvement of flatness results to prove that blow-up limits are weak solutions. We then introduce the Weiss monotonicity formula and show how it implies that blow-up limits are 1-homogeneous. This, together with the assumption that the free boundaries are Lipschitz, allows us to prove that all free boundary points are regular and conclude the proof of Theorem 1.5.

Throughout this chapter, Q is always assumed to be Hölder continuous, as well as satisfying $0 < Q_{\min} \leq Q \leq Q_{\max} < \infty$.

4.1 Convergence of blow-ups

Proposition 4.1. *Let u be a weak solution in B_1 for Q . For any $x_0 \in \partial\{u > 0\} \cap B_1$ and any blow-up sequence at x_0*

$$u_{r_k}(x) = \frac{1}{r_k} u(x_0 + r_k x), \quad r_k \rightarrow 0,$$

there exists a blow-up limit $u_0 \in C_{\text{loc}}^{0,1}(\mathbb{R}^n)$ such that

1. u_0 is a weak solution in $\mathbb{R}^n$ for $Q \equiv Q(x_0)$;
2. up to a subsequence,
 - (a) $u_{r_k} \rightarrow u_0$ in $C_{\text{loc}}^{0,\alpha}(\mathbb{R}^n)$ for every $0 < \alpha < 1$,
 - (b) $u_{r_k} \rightarrow u_0$ in $H_{\text{loc}}^1(\mathbb{R}^n)$,
 - (c) $\chi_{\{u_{r_k} > 0\}} \rightarrow \chi_{\{u_0 > 0\}}$ in $L_{\text{loc}}^1(\mathbb{R}^n)$,
 - (d) $\partial\{u_{r_k} > 0\} \rightarrow \partial\{u_0 > 0\}$ locally in the Hausdorff distance in $\mathbb{R}^n$.

Proof. By Lemma 2.4, it remains only to prove that u_0 satisfies (1.9).

We denote by $\partial_{\text{diff}}\{u_0 > 0\}$ the set of points where $\partial\{u_0 > 0\}$ has a well-defined unit normal. Let $x_1 \in \partial_{\text{diff}}\{u_0 > 0\}$. We claim that the boundary $\partial\{u_0 > 0\}$ is $C^{1,\alpha}$ in a neighbourhood of x_1 and

$$\partial_\nu u_0 = Q(x_0) \quad \text{on } \partial_{\text{diff}}\{u_0 > 0\}, \tag{4.1}$$

where the equality holds in the classical sense.

By Lemma 2.4, we can rotate the coordinate system so that

$$\{u_0 > 0\} = \{x_n > \phi_0(x')\},$$

for some Lipschitz function $\phi_0: \mathbb{R}^{n-1} \rightarrow \mathbb{R}$. Since $x_1 \in \partial_{\text{diff}}\{u_0 > 0\}$, ϕ_0 is differentiable at x'_1 . Therefore, for every $\epsilon > 0$ there exists $s > 0$ such that

$$\partial\{u_0 > 0\} \cap B_s(x_1) \subset \{|(x - x_1) \cdot \nu| \leq \epsilon s\},$$

where ν denotes the inward unit normal to $\{u_0 > 0\}$ at x_1 .

Arguing as in the proof of Lemma 2.4, there is a sequence of Lipschitz functions ϕ_k such that, for large k ,

$$\{u_{r_k} > 0\} \cap B_{2s}(x_1) = \{x \in B_{2s}(x_1) : x_n > \phi_k(x')\},$$

and $\phi_k \rightarrow \phi_0$ uniformly on $B'_{2s}(x'_1)$. Therefore

$$u_{r_k} = 0 \quad \text{in } B_s(x_{1,k}) \cap \{(x - x_{1,k}) \cdot \nu \leq -2\epsilon s\},$$

where $x_{1,k} = (x'_1, \phi_k(x'_1))$. Thus, $u_{r_k} \in F(2\epsilon, 1; \infty)$ in $B_s(x_{1,k})$ in the direction $-\nu$, with constants independent of k . Choosing ϵ and s sufficiently small, Theorem 3.11 gives that $\partial\{u_{r_k} > 0\} \cap B_\rho(x_{1,k})$ is $C^{1,\alpha}$ for some small radius ρ , with estimates uniform in k . Passing to the limit, we obtain that $\partial\{u_0 > 0\}$ is $C^{1,\alpha}$ in a neighbourhood of x_1 .

Now, by Corollary 3.13 and Remark 1.4,

$$\partial_{\nu_k} u_{r_k} = Q_k := Q(x_0 + r_k x) \quad \text{on } \partial_{\text{diff}}\{u_{r_k} > 0\},$$

in the classical sense, where ν_k is the inward unit normal to $\{u_{r_k} > 0\}$. Since Q is continuous, $Q_k \rightarrow Q(x_0)$ locally uniformly. By Schauder estimates ([30, Corollary 8.36]), u_0 is locally $C^{1,\alpha}$ up to $\partial_{\text{diff}}\{u_0 > 0\}$, and therefore the C^1 convergence of $\partial_{\text{diff}}\{u_{r_k} > 0\}$ to $\partial_{\text{diff}}\{u_0 > 0\}$ yields (4.1).

We now verify that u_0 satisfies (1.9) in $\mathbb{R}^n$ with $Q \equiv Q(x_0)$. Let $\eta \in C_c^\infty(\mathbb{R}^n)$. As in Corollary 3.13, $\mathcal{H}^{n-1}(\partial\{u_0 > 0\} \setminus \partial_{\text{diff}}\{u_0 > 0\}) = 0$. For every $\delta > 0$, we can find a finite covering

$$\text{supp } \eta \cap (\partial\{u_0 > 0\} \setminus \partial_{\text{diff}}\{u_0 > 0\}) \subset \bigcup_{i=1}^{N_\delta} B_{t_i}(y_i),$$

such that $\sum_{i=1}^{N_\delta} t_i^{n-1} \leq \delta$. Denote $U_\delta := \bigcup_{i=1}^{N_\delta} B_{t_i}(y_i)$. We can split the left-hand side of (1.9) into

$$\int_{\mathbb{R}^n} \nabla u_0 \cdot \nabla \eta \, dx = \int_{\{u_0 > 0\} \cap U_\delta} \nabla u_0 \cdot \nabla \eta \, dx + \int_{\{u_0 > 0\} \setminus U_\delta} \nabla u_0 \cdot \nabla \eta \, dx =: I_1 + I_2.$$

The first integral satisfies

$$|I_1| \leq C \sum_{i=1}^{N_\delta} t_i^n \leq C \delta^{\frac{n}{n-1}}.$$

For the second integral, since $\partial\{u_0 > 0\} \setminus U_\delta$ is locally a finite union of $C^{1,\alpha}$ surfaces in $\text{supp } \eta$ and $\Delta u_0 = 0$ in $\{u_0 > 0\}$, we can integrate by parts and obtain, using (4.1),

$$I_2 = - \int_{\partial\{u_0 > 0\} \setminus U_\delta} Q(x_0) \eta \, d\mathcal{H}^{n-1} - \int_{\{u_0 > 0\} \cap \partial U_\delta} \eta \, \partial_\nu u_0 \, d\mathcal{H}^{n-1}.$$

The last term is bounded in absolute value by $C \sum_i t_i^{n-1} \leq C\delta$. Letting $\delta \rightarrow 0$ gives (1.9), as desired. $\square$

4.2 Homogeneity of blow-up limits

In this section, we take advantage of the variational structure of the Alt–Caffarelli problem to prove that blow-up limits are homogeneous.

We follow the arguments in [21], which are written assuming $Q \in C^\infty$. When Q is only assumed to be Hölder continuous, the term $\operatorname{div}(Q^2 \xi)$ appearing in the definition of stationary solution in [21, Definition 5.1] is not well-defined, and some small modifications are needed to account for this. For reasons of space, we mainly highlight these modifications and refer the reader to [21, Section 5] for the remaining details.

4.2.1 Stationary solutions

By [2, Theorem 5.5], minimisers of the Alt–Caffarelli functional (1.10) are weak solutions of the Alt–Caffarelli problem in the sense of [2, Definition 5.1]. Conversely, we show that weak solutions are critical points of (1.10), in a sense made precise below.

The idea is to compare the energy (1.10) of u with the energy of a competitor constructed by perturbing the domain of u : we consider $\xi \in C_c^\infty(B_1; \mathbb{R}^n)$, and set $\Phi_t(x) = x + t\xi(x)$ and $u_t := u \circ \Phi_t^{-1}$. If $Q \in C^1$, differentiating $t \mapsto F_Q(u_t; B_1)$ at 0 and using the divergence theorem [42, Remark 9.5], we obtain the following expression (see [48, Lemma 9.5]), which we take as a definition also for $Q \in C^{0,\alpha}$.

Definition 4.2. We define the *inner variation of F_Q at u in the direction ξ* as

$$\delta F_Q(u, B_1)[\xi] := \int_{B_1} \left(-2\nabla u \cdot D\xi \nabla u + |\nabla u|^2 \operatorname{div} \xi \right) dx + \int_{\partial\{u>0\}} Q^2 \xi \cdot \nu d\mathcal{H}^{n-1},$$

where ν denotes the outward unit normal to $\{u > 0\}$.

If $F_Q(u; B_1) \leq F_Q(u_t; B_1)$ for all small t , the function $t \mapsto F_Q(u_t; B_1)$ has a minimum at 0. Thus, if u is a minimiser, F_Q has zero inner variation at u in any direction ξ (see [48, Lemma 9.6]).

Definition 4.3. Let $u \in H^1(B_1)$ be a nonnegative function. We say that u is a stationary solution for F_Q if the inner variation of F_Q vanishes at u , that is, if

$$\delta F_Q(u, B_1)[\xi] = \int_{B_1} \left(-2\nabla u \cdot D\xi \nabla u + |\nabla u|^2 \operatorname{div} \xi \right) dx + \int_{\partial\{u>0\}} Q^2 \xi \cdot \nu d\mathcal{H}^{n-1} = 0, \quad (4.2)$$

for every vector field $\xi \in C_c^\infty(B_1; \mathbb{R}^n)$.

Proposition 4.4. *Every weak solution in B_1 for Q is a stationary solution.*

Proof. Let $\xi \in C_c^\infty(B_1; \mathbb{R}^n)$ be fixed. Let $\partial_{\text{diff}}\{u > 0\}$ denote the set of points where $\partial\{u > 0\}$ has a well-defined normal. By Corollary 3.13,

$$\mathcal{H}^{n-1}(\partial\{u > 0\} \setminus \partial_{\text{diff}}\{u > 0\}) = 0,$$

and $\partial_{\text{diff}}\{u > 0\}$ is locally $C^{1,\alpha}$ and relatively open in $\partial\{u > 0\}$. Set

$$\Sigma := \partial\{u > 0\} \setminus \partial_{\text{diff}}\{u > 0\}.$$

Since $\text{supp } \xi \cap \Sigma$ is compact and has zero $\mathcal{H}^{n-1}$ -measure, for every $\delta > 0$ we may cover it by finitely many balls

$$\text{supp } \xi \cap \Sigma \subset \bigcup_{i=1}^{N_\delta} B_{r_i}(x_i),$$

such that $x_i \in \Sigma$ and $\sum_{i=1}^{N_\delta} r_i^{n-1} \leq \delta$. Write $B_i := B_{r_i}(x_i)$ and split

$$\begin{aligned} \frac{1}{2} \delta F_Q(u, B_1)[\xi] &= \int_{\{u>0\} \cap \bigcup_i B_i} \left(\frac{1}{2} |\nabla u|^2 \operatorname{div} \xi - \nabla u \cdot D\xi \nabla u \right) dx \\ &\quad + \left[\int_{\{u>0\} \setminus \bigcup_i B_i} \left(\frac{1}{2} |\nabla u|^2 \operatorname{div} \xi - \nabla u \cdot D\xi \nabla u \right) dx \right. \\ &\quad \left. + \frac{1}{2} \int_{\partial\{u>0\} \setminus \bigcup_i B_i} Q^2 \xi \cdot \nu d\mathcal{H}^{n-1} \right] \\ &\quad + \frac{1}{2} \int_{\partial\{u>0\} \cap \bigcup_i B_i} Q^2 \xi \cdot \nu d\mathcal{H}^{n-1} \\ &=: I_1 + I_2 + I_3. \end{aligned}$$

We first estimate I_1 and I_3 . Since u is locally Lipschitz, Q is bounded, and ξ is smooth and compactly supported, the integrands are bounded on $\text{supp } \xi$. Hence, using that $r_i^{n-1} \leq \sum_{i=1}^{N_\delta} r_i^{n-1} \leq \delta$,

$$|I_1| \leq C \sum_{i=1}^{N_\delta} r_i^n \leq C \delta^{\frac{n}{n-1}}. \quad (4.3)$$

Since $\partial\{u > 0\}$ is Lipschitz, $\mathcal{H}^{n-1}(\partial\{u > 0\} \cap B_i) \leq C r_i^{n-1}$ and we obtain similarly that

$$|I_3| \leq C \delta. \quad (4.4)$$

It remains to estimate I_2 . Since u is harmonic in $\{u > 0\}$,

$$\frac{1}{2} |\nabla u|^2 \operatorname{div} \xi - \nabla u \cdot D\xi \nabla u = \operatorname{div} \left(\frac{1}{2} |\nabla u|^2 \xi - (\xi \cdot \nabla u) \nabla u \right) \quad \text{in } \{u > 0\}.$$

Also, by Remark 1.4,

$$\nabla u = -Q\nu \quad \text{on } \partial\{u > 0\} \cap \left(\text{supp } \xi \setminus \bigcup_{i=1}^{N_\delta} B_i \right).$$

Therefore, since $\text{supp } \xi \cap \left(\{u > 0\} \setminus \bigcup_{i=1}^{N_\delta} B_i \right)$ is a piecewise C^1 domain, we may apply the divergence theorem to obtain

$$I_2 = \int_{\{u>0\} \cap \partial(\bigcup_i B_i)} \left(\frac{1}{2} |\nabla u|^2 \xi - (\xi \cdot \nabla u) \nabla u \right) \cdot \nu d\mathcal{H}^{n-1},$$

where ν denotes the outward unit normal to $\bigcup_{i=1}^{N_\delta} B_i$.

Since u is locally Lipschitz and ξ is bounded,

$$|I_2| \leq C \sum_{i=1}^{N_\delta} \mathcal{H}^{n-1}(\partial B_i) \leq C \sum_{i=1}^{N_\delta} r_i^{n-1} \leq C\delta. \quad (4.5)$$

Combining (4.3), (4.4), and (4.5), and then letting $\delta \rightarrow 0$, we obtain (4.2), as desired. $\square$

4.2.2 Weiss monotonicity formula

In the theory of minimal surfaces, the normalised area of a minimal surface is monotone with respect to scale; see [11, Theorem 3.6.6]. Weiss showed in [49] that, although the Alt–Caffarelli energy (1.10) does not by itself yield a monotone quantity for weak solutions, one can consider instead the boundary-adjusted energy below.

Definition 4.5 (Weiss boundary-adjusted energy). Given a boundary datum Q and a nonnegative $u \in H^1(B_1)$, we define

$$W_Q(u) := \int_{B_1} |\nabla u|^2 dx + Q(0)^2 |\{u > 0\} \cap B_1| - \int_{\partial B_1} u^2 d\mathcal{H}^{n-1}.$$

When Q is constant, the Weiss energy is monotone. For variable Q , it is only almost monotone, but this is sufficient for our purpose.

Proposition 4.6 (Weiss monotonicity formula). *Let u be a weak solution in B_1 , let $x_0 \in B_1$ and $r_{x_0} = \frac{1}{2} \text{dist}(x_0, \partial B_1)$. Consider the scaling*

$$u_r(x) := \frac{1}{r} u(x_0 + rx), \quad Q_r(x) = Q(x_0 + rx).$$

Then the function $r \mapsto W_{Q_r}(u_r)$ is continuous and a.e. differentiable in $(0, r_{x_0})$. Moreover, we have

$$\frac{\partial}{\partial r} W_{Q_r}(u_r) \geq \frac{2}{r} \int_{\partial B_1} |x \cdot \nabla u_r - u_r|^2 d\mathcal{H}^{n-1} - C_Q r^{\alpha-1},$$

where C_Q is a positive constant that depends on $\|u\|_{L^\infty(B_1)}$ and $\|Q\|_{C^{0,\alpha}(B_1)}$.

Proof. The continuity and a.e. differentiability follow by arguing as in [48, Lemma 9.1, Lemma 9.2]. Moreover, by [48, Lemma 9.2], we have the following explicit expression for the derivative of $W_{Q_r}(u_r)$:

$$\frac{\partial}{\partial r} W_{Q_r}(u_r) = \frac{n}{r} (W_{Q_r}(z_r) - W_{Q_r}(u_r)) + \frac{1}{r} \int_{\partial B_1} |\nabla u_r \cdot x - u_r|^2 d\mathcal{H}^{n-1}, \quad (4.6)$$

where $z_r := |x| u_r(\frac{x}{|x|})$ is the 1-homogeneous extension of u_r from ∂B_1 to B_1 .

By translating the coordinates, we may assume $x_0 = 0$ and write $W_{Q_r}(u_r) = W_Q(u_r)$. Arguing as in (3.22), we may also assume that

$$\mathcal{H}^{n-1}(\partial\{u > 0\} \cap \partial B_r) = 0.$$

For every $\epsilon > 0$, consider the radial function $\phi_\epsilon \in C_c^{0,1}(B_1)$ such that $\chi_{B_{(1-\epsilon)r}} \leq \phi_\epsilon \leq \chi_{B_r}$ and

$$\phi_\epsilon = \frac{r - |x|}{\epsilon r} \quad \text{for } (1 - \epsilon)r < |x| < r.$$

By Proposition 4.4, u is a stationary solution in B_1 . Since (4.2) extends by density to test functions in $C_c^{0,1}(B_1; \mathbb{R}^n)$, we may use $\xi_\epsilon = \phi_\epsilon(x)x$ as a test vector field in (4.2). Letting $\epsilon \rightarrow 0$, we obtain, for a.e. $r \in (0, r_{x_0})$,

$$\begin{aligned} 0 &= (n-2) \int_{\{u>0\} \cap B_r} |\nabla u|^2 dx \\ &\quad + r \int_{\{u>0\} \cap \partial B_r} \left(2 \left(\nabla u \cdot \frac{x}{|x|} \right)^2 - |\nabla u|^2 \right) d\mathcal{H}^{n-1} \\ &\quad + \int_{\partial\{u>0\} \cap B_r} Q^2 x \cdot \nu d\mathcal{H}^{n-1}. \end{aligned} \quad (4.7)$$

We estimate the last term by comparing it with the case $Q \equiv Q(0)$:

$$\int_{\partial\{u>0\} \cap B_r} Q^2 x \cdot \nu d\mathcal{H}^{n-1} = \int_{\partial\{u>0\} \cap B_r} Q(0)^2 x \cdot \nu d\mathcal{H}^{n-1} + E_Q(r), \quad (4.8)$$

where

$$E_Q(r) := \int_{\partial\{u>0\} \cap B_r} (Q^2 - Q(0)^2) x \cdot \nu d\mathcal{H}^{n-1}.$$

Since $Q \in C^{0,\alpha}$, $|x| \leq r$ on B_r , and $\partial\{u > 0\}$ is Lipschitz, we can estimate

$$|E_Q(r)| \leq 2\|Q\|_{L^\infty(B_r)} [Q]_{C^{0,\alpha}(B_r)} r^{1+\alpha} \mathcal{H}^{n-1}(\partial\{u > 0\} \cap B_r) \leq C_Q r^{n+\alpha}. \quad (4.9)$$

Moreover, by the divergence theorem [42, Remark 9.5], we can compute

$$\int_{\partial\{u>0\} \cap B_r} Q(0)^2 x \cdot \nu d\mathcal{H}^{n-1} = -rQ(0)^2 \mathcal{H}^{n-1}(\{u > 0\} \cap \partial B_r) + nQ(0)^2 |\{u > 0\} \cap B_r|. \quad (4.10)$$

Since u is harmonic in $\{u > 0\}$, integration by parts yields

$$\int_{\{u>0\} \cap B_r} |\nabla u|^2 dx = \int_{\{u>0\} \cap \partial B_r} u \nabla u \cdot \frac{x}{|x|} d\mathcal{H}^{n-1}. \quad (4.11)$$

Inserting (4.8), (4.10), and (4.11) into (4.7), and then rescaling to B_1 , we obtain

$$\begin{aligned} 0 &= n \int_{B_1} \left(|\nabla u_r|^2 + Q(0)^2 \chi_{\{u_r>0\}} \right) dx \\ &\quad + \int_{\partial B_1} \left(-2u_r x \cdot \nabla u_r + 2(x \cdot \nabla u_r)^2 - |\nabla u_r|^2 - Q(0)^2 \chi_{\{u_r>0\}} \right) d\mathcal{H}^{n-1} + r^{-n} E_Q(r). \end{aligned} \quad (4.12)$$

On the other hand, using that $z_r = u_r$ on ∂B_1 and computing the integrals involving z_r in polar coordinates, we can compute

$$\begin{aligned} W_Q(z_r) - W_Q(u_r) &= \frac{1}{n} \int_{\partial B_1} \left(u_r^2 + |\nabla u_r|^2 - (x \cdot \nabla u_r)^2 + Q(0)^2 \chi_{\{u_r>0\}} \right) d\mathcal{H}^{n-1} \\ &\quad - \int_{B_1} (|\nabla u_r|^2 + Q(0)^2 \chi_{\{u_r>0\}}) dx. \end{aligned} \quad (4.13)$$

Solving (4.12) for the second term on the right-hand side of (4.13), substituting the result into (4.13), and using (4.9), we obtain

$$W_Q(z_r) - W_Q(u_r) \geq \frac{1}{n} \int_{\partial B_1} |x \cdot \nabla u_r - u_r|^2 d\mathcal{H}^{n-1} - C_Q r^\alpha.$$

Substituting into (4.6) yields the result. $\square$

When available, Weiss-type monotonicity formulas are among the central tools for classifying blow-ups in free boundary problems. Indeed, Proposition 4.6 yields the following result, which greatly restricts the possible blow-up limits.

Proposition 4.7. *Let u be a weak solution in B_1 , and let $x_0 \in \partial\{u > 0\} \cap B_1$ be a free boundary point. If u_0 is a blow-up limit for a blow-up sequence u_{r_k} of u at x_0 , then u_0 is a 1-homogeneous function.*

Proof. By translating the coordinates if necessary, we may assume $x_0 = 0$ and write $W_Q(u_r) = W_{Q_r}(u_r)$. By Proposition 4.6, for a.e. $r \in (0, r_{x_0})$,

$$\frac{\partial}{\partial r} W_Q(u_r) \geq \frac{2}{r} \int_{\partial B_1} |x \cdot \nabla u_r - u_r|^2 d\mathcal{H}^{n-1} - Cr^{\alpha-1}. \quad (4.14)$$

In particular, $r \mapsto W_Q(u_r) + \frac{C}{\alpha} r^\alpha$ is nondecreasing.

Since $x_0 \in \partial\{u > 0\}$ and u is Lipschitz in $B_{r_{x_0}}(x_0)$, we have $u(x_0) = 0$ and hence $|u_r(x)| \leq L|x|$ for all $0 < r < r_{x_0}$. Therefore

$$W_Q(u_r) \geq - \int_{\partial B_1} u_r^2 d\mathcal{H}^{n-1} \geq -C_L,$$

and it follows that the following limit exists and is finite:

$$\ell := \lim_{r \rightarrow 0} W_Q(u_r).$$

Fix $0 < S < R < \infty$. For k large enough, $Rr_k < r_{x_0}$. Integrating (4.14) from Sr_k to Rr_k , using polar coordinates and the rescaling $\rho = r_k|x|$, we obtain

$$\begin{aligned} W_Q(u_{Rr_k}) - W_Q(u_{Sr_k}) &\geq \int_{Sr_k}^{Rr_k} \frac{2}{\rho} \int_{\partial B_1} |x \cdot \nabla u_\rho - u_\rho|^2 d\mathcal{H}^{n-1} d\rho - \frac{C}{\alpha} ((Rr_k)^\alpha - (Sr_k)^\alpha) \\ &= \int_{B_R \setminus B_S} 2|x|^{-n-2} |x \cdot \nabla u_{r_k} - u_{r_k}|^2 dx - \frac{C}{\alpha} ((Rr_k)^\alpha - (Sr_k)^\alpha). \end{aligned}$$

Letting $k \rightarrow \infty$, using $W_Q(u_{Rr_k}) \rightarrow \ell$, $W_Q(u_{Sr_k}) \rightarrow \ell$, and $u_{r_k} \rightarrow u_0$ in $H^1(B_R)$, we get

$$\int_{B_R \setminus B_S} 2|x|^{-n-2} |x \cdot \nabla u_0 - u_0|^2 dx = 0.$$

Since $0 < S < R < \infty$ were arbitrary, it follows that $x \cdot \nabla u_0 - u_0 = 0$ a.e. in $\mathbb{R}^n$. Therefore, u_0 is 1-homogeneous; see, e.g., [48, Lemma 9.3]. $\square$

4.3 Classification of blow-ups

In this section, we prove that all points of the free boundary are regular, concluding the proof of Theorem 1.5.

Definition 4.8. A Lipschitz cone u is a 1-homogeneous weak solution for $Q \equiv Q_0$ for some constant $Q_0 > 0$. We say that a Lipschitz cone is trivial if there is a unit vector $v \in \mathbb{R}^n$ such that

$$u = Q_0(x \cdot v)_+.$$

Note that, if u is a Lipschitz cone, then $\partial\{u > 0\}$ is a cone; if u is a trivial Lipschitz cone, then $\partial\{u > 0\}$ is a hyperplane.

By Proposition 4.1 and Proposition 4.7, if u is a weak solution, at any point on the free boundary there is a blow-up limit that is a Lipschitz cone. Therefore, to prove that all free boundary points are regular, it is enough to prove that all Lipschitz cones are trivial. We start by proving two dimension reduction lemmas.

Lemma 4.9. *Let $Q_0 > 0$ be constant. Let $u: \mathbb{R}^n \rightarrow \mathbb{R}$ be constant in the e_n -direction, and write $u(x) = v(x')$, for some $v: \mathbb{R}^{n-1} \rightarrow \mathbb{R}$. Then v is a weak solution in $\mathbb{R}^{n-1}$ for $Q \equiv Q_0$ if and only if u is a weak solution in $\mathbb{R}^n$ for $Q \equiv Q_0$.*

Proof. Since $\Delta u = \Delta_{x'}v + \partial_{x_n}^2 u = \Delta_{x'}v$, u is harmonic in $\{u > 0\}$ if and only if v is harmonic in $\{v > 0\}$. Also, since the positivity set $\{u > 0\}$ is the cylinder $\{v > 0\} \times \mathbb{R}$, $\{u > 0\}$ is a Lipschitz domain if and only if $\{v > 0\}$ is a Lipschitz domain. We also have that on $\partial\{u > 0\}$, $d\mathcal{H}^{n-1}(x) = d\mathcal{H}^{n-2}(x') dx_n$.

Assume u is a weak solution in $\mathbb{R}^n$ for $Q \equiv Q_0$. Let $\psi \in C_c^\infty(\mathbb{R}^{n-1})$. Let $\eta \in C_c^\infty(\mathbb{R})$ with $\int_{\mathbb{R}} \eta = 1$, and set $\phi := \eta(x_n)\psi(x')$. Then, by (1.9) and the observations above,

$$\begin{aligned} - \int_{\mathbb{R}^n} \nabla u \cdot \nabla \phi dx &= Q_0 \int_{\partial\{u>0\}} \phi d\mathcal{H}^{n-1} \\ - \int_{\mathbb{R}^n} (\nabla_{x'}v, 0) \cdot (\eta \nabla_{x'}\psi, \psi \eta') dx &= Q_0 \int_{\mathbb{R}} \int_{\partial\{u>0\} \cap \{x_n=t\}} \eta \psi d\mathcal{H}^{n-2} dt \\ - \int_{\mathbb{R}^{n-1}} \nabla_{x'}v \cdot \nabla_{x'}\psi dx' &= Q_0 \int_{\partial\{v>0\}} \psi d\mathcal{H}^{n-2}. \end{aligned}$$

Suppose that v is a weak solution in $\mathbb{R}^{n-1}$ for $Q \equiv Q_0$. For any $\phi \in C_c^\infty(\mathbb{R}^n)$, let $\psi := \int_{\mathbb{R}} \phi dx_n \in C_c^\infty(\mathbb{R}^{n-1})$. Then, by (1.9) and the observations above,

$$\begin{aligned} - \int_{\mathbb{R}^{n-1}} \nabla_{x'}v \cdot \nabla_{x'}\psi dx' &= Q_0 \int_{\partial\{v>0\}} \psi d\mathcal{H}^{n-2} \\ - \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \nabla_{x'}v \cdot \nabla_{x'}\phi dx_n dx' &= Q_0 \int_{\mathbb{R}} \int_{\partial\{u>0\} \cap \{x_n=t\}} \phi d\mathcal{H}^{n-2} dt \\ - \int_{\mathbb{R}^n} \nabla_x u \cdot \nabla_x \phi dx &= Q_0 \int_{\partial\{u>0\}} \phi d\mathcal{H}^{n-1}. \end{aligned}$$

□

Lemma 4.10. *Let u be a Lipschitz cone in $\mathbb{R}^n$ such that $x_0 \in \partial\{u > 0\} \setminus \{0\}$. Then any blow-up sequence*

$$u_r(x) = \frac{1}{r}u(x_0 + rx),$$

has a subsequence u_{r_k} , $r_k \rightarrow 0$ which converges locally uniformly to

$$u_0(x_1, \dots, x_n) = v(x_1, \dots, x_{n-1}),$$

with v a Lipschitz cone in $\mathbb{R}^{n-1}$. Moreover, if x_0 is a singular point for $\partial\{u > 0\}$, then v is a nontrivial cone.

Proof. By rescaling and rotating we may assume $x_0 = e_n$. Let $Q_0 > 0$ be the boundary datum for u . By Proposition 4.1 and Proposition 4.7, we can extract a subsequence $r_k \rightarrow 0$ such that $u_{r_k} \rightarrow u_0$ locally uniformly, and u_0 is a 1-homogeneous weak solution for Q_0 in $\mathbb{R}^n$. Moreover, u_0 is constant in the e_n -direction. Indeed, for any $t \in \mathbb{R}$, we can take r_k small enough so that $1 + tr_k > 0$ and, using that u is 1-homogeneous,

$$u_{r_k}(x) = \frac{1}{r_k(1 + tr_k)} u((1 + tr_k)(e_n + r_k x)) = \frac{1}{1 + tr_k} u_{r_k}(te_n + (1 + tr_k)x),$$

and letting $r_k \rightarrow 0$, $u_0(x) = u_0(te_n + x)$. Therefore, by Lemma 4.9, $v(x') := u_0(x', 0)$ is a weak solution in $\mathbb{R}^{n-1}$ with boundary datum Q_0 . Since, for any $\lambda > 0$,

$$v(\lambda x') = u_0(\lambda x', 0) = \lambda u_0(x', 0) = \lambda v(x'),$$

v is a Lipschitz cone in $\mathbb{R}^{n-1}$.

Finally, if v is a trivial cone, there is a unit vector $v' \in \mathbb{R}^{n-1}$ such that $v = Q_0(x' \cdot v')_+$. Since $u_0(x) = v(x') = Q_0(x' \cdot v')_+ = Q_0(x \cdot v)_+$, where $v := (v', 0)$ is a unit vector in $\mathbb{R}^n$, x_0 is a regular point. $\square$

The following lemma is proved in [21] as a consequence of a general property of harmonic functions on Lipschitz domains [9, Lemma 11.12]. We present an alternative proof tailored to this setting. Note that this is the only point in the proof where the Lipschitz assumption on the free boundary is essential.

Lemma 4.11. *Let u be a Lipschitz cone in $\mathbb{R}^n$. Assume that $\partial\{u > 0\}$ is smooth outside the origin, and that $\{u > 0\} \cap B_1$ is, in B'_1 , the epigraph in the e_n -direction of a Lipschitz function with optimal Lipschitz constant L . Then there is a direction τ that is tangent to $\partial\{u > 0\}$ at some point $x_0 \in \partial\{u > 0\} \setminus \{0\}$ such that $\partial_\tau u \geq 0$ in $\{u > 0\}$.*

Proof. We first consider directions entering $\{u > 0\}$ at the origin, that is,

$$\sigma \in \{\xi = (\xi', \xi_n) \in \mathbb{S}^{n-1} : \xi_n > L|\xi'|\} =: \mathcal{C}.$$

Since u is 1-homogeneous, u_σ is 0-homogeneous in $\{u > 0\}$, and it is enough to prove that $\partial_\sigma u \geq 0$ in the annular section

$$D := (B_1 \setminus \overline{B}_{1/2}) \cap \{u > 0\}.$$

Since $\partial\{u > 0\}$ is smooth outside the origin, Remark 1.4 implies that $\partial_\sigma u$ is a harmonic function in D smooth up to $\overline{D}$. By the Hopf lemma (see, e.g., [23, Section 6.4.2])

$$u_\nu > 0 \quad \text{on } \overline{D} \cap \partial\{u > 0\},$$

where ν denotes the inward unit normal to $\{u > 0\}$. Since $\nu \cdot \sigma > 0$ and, on the free boundary, $\nabla u = u_\nu \nu$,

$$u_\sigma = u_\nu \nu \cdot \sigma > 0 \quad \text{on } \overline{D} \cap \partial\{u > 0\}. \quad (4.15)$$

By the maximum principle, u_σ attains its minimum on ∂D . If the minimum is attained on $\partial\{u > 0\}$, by (4.15) $u_\sigma \geq 0$ in D and we are done. Otherwise, the minimum is

attained on $(\partial B_1 \cup \partial B_{1/2}) \cap \{u > 0\}$, and since u_σ is 0-homogeneous, this implies that u_σ attains the minimum at an interior point. By the strong maximum principle and (4.15), $u_\sigma \geq 0$.

By the continuity of the derivatives of u in $\{u > 0\}$, $u_\sigma \geq 0$ also for $\sigma \in \bar{\mathcal{C}}$. By the definition of L , the conclusion is satisfied for some $\tau \in \partial \mathcal{C}$. □

Proposition 4.12. *All Lipschitz cones are trivial.*

Proof. We prove the proposition by induction on n .

The case $n = 1$ is obvious. Assume the statement holds for $n - 1$ and let u be a Lipschitz cone in $\mathbb{R}^n$. We may assume, without loss of generality, that $\partial\{u > 0\}$ is the graph in B'_1 in the direction e_n of a Lipschitz function with Lipschitz constant L . Moreover, by the induction hypothesis and Lemma 4.10, all points outside the origin in $\partial\{u > 0\}$ are regular, and therefore, by Corollary 3.12 and Theorem 3.14, $\partial\{u > 0\}$ is smooth outside the origin.

By Lemma 4.11, there is a direction τ that is tangent to $\partial\{u > 0\}$ at some point $x_0 \in \partial\{u > 0\} \setminus \{0\}$ and such that $\partial_\tau u \geq 0$ in $\{u > 0\}$.

Suppose that $\partial_\tau u > 0$ in $\{u > 0\}$. Since $\partial\{u > 0\}$ is smooth outside the origin, by Remark 1.4, there is a neighbourhood of x_0 in $\{u > 0\}$ in which u is smooth up to $\partial\{u > 0\}$ and $\partial_\nu u$ is constant on $\partial\{u > 0\}$, where ν denotes the inward unit normal to $\{u > 0\}$. Since τ is tangent to $\partial\{u > 0\}$ at x_0 ,

$$\partial_\nu \partial_\tau u(x_0) = 0.$$

On the other hand, since τ is tangent to $\partial\{u > 0\}$ at x_0 , $\partial_\tau u(x_0) = 0$ and the Hopf lemma (see, e.g., [23, Section 6.4.2]) implies that $\partial_\nu \partial_\tau u(x_0) > 0$, a contradiction.

Therefore, $\partial_\tau u = 0$ at some point in $\{u > 0\}$, and, by the strong maximum principle, $\partial_\tau u \equiv 0$. Thus u is constant in the τ -direction. Lemma 4.9 reduces the problem to $n - 1$ dimensions, and the induction hypothesis implies that u is trivial. □

Proof of Theorem 1.5. Let k be a nonnegative integer and $0 < \alpha < 1$. Suppose that $Q \in C^{k,\alpha}(\mathbb{R}^n)$ is such that $0 < Q_{\min} \leq Q \leq Q_{\max} < \infty$, and let u be a weak solution in B_1 for Q . Let $x_0 \in \partial\{u > 0\} \cap B_1$. By Proposition 4.1 and Proposition 4.7, there is a blow-up limit u_0 that is a Lipschitz cone. By Proposition 4.12, there is a unit vector $\nu \in \mathbb{R}^n$ such that $u_0 = Q(x_0)(x \cdot \nu)_+$. Therefore, x_0 is a regular point. By Corollary 3.12 and Theorem 3.14 (or Remark 3.15 if $k = 0$), $\partial\{u > 0\} \cap B_{1/2}$ is $C^{k+1,\alpha}$. □

Proof of Theorem 1.1. Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $x \in \Omega$, and $y \in \partial\Omega$. Suppose that $P_\Omega^x \in C^{k,\alpha}(\partial\Omega)$ is positive. By the discussion in Section 1.2.1, after a rescaling and a translation that brings y to 0, the Green function $G_\Omega(x, \cdot)$ is a weak solution in B_1 for P_Ω^x . Therefore, we can apply Theorem 1.5 to obtain that $\partial\Omega$ is $C^{k+1,\alpha}$ in a neighbourhood of y . Since $y \in \partial\Omega$ is arbitrary, we conclude that $\partial\Omega$ is $C^{k+1,\alpha}$. □

Chapter 5

Further results and open problems

5.1 Weak solutions and viscosity solutions

As mentioned in the introduction, Caffarelli proved in [7] that Lipschitz free boundaries of viscosity solutions to the Alt–Caffarelli problem are smooth (see also [9, Chapter 4]). His proof is quite different from the one for weak solutions. The key step is to prove a geometric improvement of the Lipschitz constant of the free boundary at smaller scales, which yields $C^{1,\alpha}$ regularity when iterated. The proof for viscosity solutions uses considerably less machinery than the blow-up classification used here: the main tools are the Harnack inequality and a pointwise comparison principle at the free boundary, which is built into the definition of a viscosity solution. It would be interesting to find an analogous direct proof for weak solutions.

5.2 The nonlocal case

A natural class of free boundary problems arises by replacing the Laplacian by a nonlocal operator L , meaning that the value of $Lu(x)$ may depend on the values of u far from x . Such operators arise in models involving anomalous diffusion, where diffusion is driven microscopically by stochastic processes other than Brownian motion, as well as in models involving long-range interactions or forces. In recent decades, and in part because of their many applications, nonlocal elliptic equations have become a major area of research in PDE theory; we refer the reader to [27] for a general introduction to their theory.

A particularly important class consists of symmetric $2s$ -stable integro-differential elliptic operators. These operators have the form

$$Lu(x) = 2 \text{P. V.} \int_{\mathbb{R}^n} (u(x) - u(y))K(x - y) dy, \quad (5.1)$$

where $K: \mathbb{R}^n \setminus \{0\} \rightarrow [0, \infty)$ satisfies

$$\lambda|h|^{-n-2s} \leq K(h) \leq \Lambda|h|^{-n-2s}, \quad K(h) = K(-h), \quad K(h) = \frac{K\left(\frac{h}{|h|}\right)}{|h|^{n+2s}}, \quad (5.2)$$

for some $0 < \lambda \leq \Lambda < \infty$, some $s \in (0, 1)$, and every $h \neq 0$. In the isotropic case

$$K(h) = \frac{c_{n,s}}{2}|h|^{-n-2s}, \quad c_{n,s} = 2^{2s}s \frac{\Gamma\left(\frac{n+2s}{2}\right)}{\Gamma(1-s)}\pi^{-\frac{n}{2}},$$

the operator is the fractional Laplacian $(-\Delta)^s$.

The natural one-phase problem associated with such an operator is

$$\begin{cases} Lu = 0 & \text{in } \Omega \cap B_1, \\ u = 0 & \text{on } B_1 \setminus \Omega, \\ \frac{u}{d_\Omega^s} = A(v) & \text{on } \partial\Omega \cap B_1, \end{cases} \quad (5.3)$$

where d_Ω denotes the distance to $\partial\Omega$ and

$$A(v) := \kappa_{n,s} \left(\int_{S^{n-1}} K(\theta) |\theta \cdot v|^{2s} d\theta \right)^{-1/2}, \quad \text{where } \kappa_{n,s} > 0.$$

This nonlocal one-phase problem also has a variational formulation, in which one studies minimisers of the energy

$$F(u) := \iint_{(B_1^c \times B_1^c)^c} |u(x) - u(y)|^2 K(x - y) dy dx + |\{u > 0\} \cap B_1|. \quad (5.4)$$

For the fractional Laplacian, the Caffarelli–Silvestre extension [10] allows one to translate the nonlocal one-phase problem to a local *thin* problem, in which the free boundary is a codimension-two object in the extended space $\mathbb{R}^{n+1}$.

This thin one-phase problem was first introduced in [8], where the optimal regularity and nondegeneracy of weak solutions were proved. The regularity theory for the free boundary was developed in [15, 16, 17, 18] for $s = \frac{1}{2}$, and the case of general $0 < s < 1$ has been studied in [19, 22, 41]. In particular, it is known that for $L = (-\Delta)^{\frac{1}{2}}$, Lipschitz free boundaries of viscosity solutions to (5.3) are smooth.

For the more general class of symmetric $2s$ -stable integro-differential elliptic operators, purely nonlocal techniques must be used, and progress has only come very recently. A flatness-implies- $C^{1,\alpha}$ result was obtained in [45], and a higher regularity result starting from $C^{2,\alpha}$ was proved in [4]. The step from $C^{1,\alpha}$ to $C^{2,\alpha}$ remains open.

One can also ask for an analogue of Theorem 1.1 in the nonlocal setting:

Question 5.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain and let $x \in \Omega$. Let L be a $2s$ -stable integro-differential elliptic operator. Consider a well-posed Dirichlet boundary-value problem for L in Ω , and suppose that the corresponding Poisson kernel is smooth. Does it follow that $\partial\Omega$ is C^∞ ?*

For the fractional Laplacian in $C^{1,1}$ domains, the natural boundary-value Dirichlet problem is given by $Lu = 0$ in Ω , $u = 0$ in $\bar{\Omega}^c$, and $u/d_\Omega^{s-1} = g$ on $\partial\Omega$. For $x \in \Omega$, the Poisson kernel corresponding to this problem is given by

$$P^x := \frac{G^x}{d_\Omega^s} \Big|_{\partial\Omega},$$

where G^x is the Green function satisfying $(-\Delta)^s G^x = \delta_x$ in Ω and $G^x = 0$ in Ω^c ; see [1, 32]. Near boundary points and away from the pole x , after translating and rescaling if necessary, G^x solves (5.3) with P^x in place of the right-hand side in the third equation. Higher regularity results were also developed for this problem in [4], and thus, if Ω is $C^{2,\alpha}$ and P^x is smooth, then $\partial\Omega$ is smooth.

In more general situations, the answer to Question 5.1 is still unknown.

Appendix A

Regularity classes for functions and domains

Definition A.1. Let k be a nonnegative integer and $0 < \alpha \leq 1$. Given a domain $D \subset \mathbb{R}^n$, the space $C^{k,\alpha}(\overline{D})$ is the set of functions $f \in C^k(\overline{D})$ such that the following norm is finite:

$$\|f\|_{C^{k,\alpha}(\overline{D})} := \|f\|_{C^k(\overline{D})} + [D^k f]_{C^{0,\alpha}(\overline{D})},$$

where $[\cdot]_{C^{0,\alpha}(\overline{D})}$ is the Hölder seminorm

$$[f]_{C^{0,\alpha}(\overline{D})} := \sup_{\substack{x,y \in \overline{D} \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.$$

We will often write $C^{k,\alpha}(D)$ instead of $C^{k,\alpha}(\overline{D})$, $[\cdot]_{C^{k,\alpha}(D)}$ instead of $[\cdot]_{C^{k,\alpha}(\overline{D})}$, and $\|\cdot\|_{C^{k,\alpha}(D)}$ instead of $\|\cdot\|_{C^{k,\alpha}(\overline{D})}$.

If $f \in C^{0,\alpha}$, we say that f is Hölder continuous with Hölder exponent α .

Definition A.2. We say that $f \in C^{k,\alpha}(\partial D)$ if there is $g \in C^{k,\alpha}(\mathbb{R}^n)$ such that $f = g|_{\partial D}$.

The following is a useful sufficient condition for $C^{1,\alpha}$ regularity.

Lemma A.3. Assume that for every $x \in B_1$ there is a linear function $\ell_x(y) = a_x + b_x \cdot (y - x)$ such that

$$\|u - \ell_x\|_{L^\infty(B_r(x))} \leq C_0 r^{1+\alpha} \quad \text{for all } B_r(x) \subset \overline{B}_1.$$

Then $u \in C^{1,\alpha}(\overline{B}_1)$ and $[Du]_{C^{0,\alpha}(\overline{B}_1)} \leq CC_0$, with C depending only on n and α .

Proof. See [26, Property (H4)]. □

Definition A.4 (Lipschitz function). Functions in $C^{0,1}(\overline{D})$ are called Lipschitz functions. We will often write $\text{Lip } f$ instead of $[f]_{C^{0,1}(\overline{D})}$. If $\text{Lip } f = L$, we will sometimes say that f is an L -Lipschitz function.

Remark A.5. The restriction of a Lipschitz function to any line segment is absolutely continuous.

Theorem A.6 (Rademacher's theorem). If $f \in C^{0,1}(\overline{D})$, then f is $\mathcal{L}^n$ -a.e. differentiable in D .

Proof. See [24, Theorem 3.2]. □

Definition A.7. Let $k \geq 0$ be a nonnegative integer and $0 < \alpha \leq 1$. We say that a domain Ω is $C^{k,\alpha}$ if, locally around every boundary point and after a rigid change of coordinates, Ω is the epigraph of a $C^{k,\alpha}$ function.

We will often say that $\partial\Omega$ is $C^{k,\alpha}$ instead of saying that Ω is $C^{k,\alpha}$. We call $C^{0,1}$ domains Lipschitz domains.

Remark A.8. As a consequence of Rademacher's theorem, Lipschitz domains have an $\mathcal{H}^{n-1}$ -a.e. defined inward unit normal.

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