

Magnetic geodesics on Brinkmann spacetimes from parallel spinors

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ABSTRACT: The presence of spinorial structures in spacetime constrains its geometry and, consequently, influences the motion of particles. To describe this relation, we review the necessary elements of spin geometry, from Clifford algebras to $\text{spin}^{\mathbb{C}}$ structures and parallel spinors. We then consider a four-dimensional supersymmetric Einstein–Maxwell background arising from Freedman’s gauged supergravity. In this setting, the existence of a parallel spinor determines a null parallel Dirac current, which forces the spacetime to be of Brinkmann type. To probe this spacetime, we study the dynamics of a charged test particle. Starting from the minimally coupled Lagrangian and exploiting the cyclic coordinates, we perform a Routh reduction, obtaining an effective Lagrangian on the two-sphere. The resulting dynamics contains not only the electromagnetic interaction but also an effective contribution induced by the spacetime geometry itself. Moreover, the symmetries of the system allow us to reduce it further to a one-dimensional problem. We integrate the resulting equations to obtain the magnetic geodesics for charged particles in this background. We exhaustively classify the admissible solutions across the parameter space. The resulting branches describe constant, oscillatory, and asymptotic trajectories.

RESUM: La presència d’estructures espinorials en l’espai-temps restringeix la seva geometria i, en conseqüència, influeix en el moviment de les partícules. Per descriure aquesta relació, revisem els elements necessaris de la geometria espinorial, des de les àlgebres de Clifford fins a les estructures $\text{spin}^{\mathbb{C}}$ i els espinors paral·lels. A continuació, considerem un fons supersimètric d’Einstein–Maxwell en quatre dimensions que sorgeix de la supergravetat gaujada de Freedman. En aquest context, l’existència d’un espinor paral·lel determina un corrent de Dirac nul i paral·lel, que força l’espai-temps a ser de tipus Brinkmann. Tot seguit, explorem aquest espai-temps mitjançant la dinàmica d’una partícula de prova carregada. Partint del lagrangià amb acoblament mínim i aprofitant les coordenades cícliques, efectuем una reducció de Routh i obtenim un lagrangià efectiu sobre la 2-esfera. La dinàmica resultant conté no només la interacció electromagnètica, sinó també una contribució efectiva induïda per la mateixa geometria de l’espai-temps. A més, les simetries del sistema ens permeten reduir-lo encara més a un problema unidimensional. Integrem les equacions resultants per obtenir les geodèsiques magnètiques de partícules carregades en aquest fons. Classifiquem exhaustivament les solucions admissibles en tot l’espai de paràmetres. Les branques resultants descriuen trajectòries constants, oscil·latòries i asimptòtiques.

KEYWORDS: Clifford algebras, differential geometry, $\text{spin}^{\mathbb{C}}$ structures, parallel spinors, Einstein–Maxwell Brinkmann spacetimes, magnetic geodesics

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1 Introduction

Modern theoretical physics is deeply geometrical. It is well known that, in general relativity, gravity is not described as a force on a fixed background, but as the curvature of spacetime itself [1]. Remarkably, the same geometric point of view also underlies the gauge theories of particle physics. Since the fundamental work of Yang and Mills [2], gauge fields have become central objects in theoretical physics, especially in the description of electromagnetic, weak, and strong interactions. Later developments made explicit the geometric nature of gauge theory: gauge potentials can be understood as connections on fibre bundles, while the corresponding physical fields arise as the curvatures of these connections [3, 4]. This is one of the central bridges between mathematics and physics.

Spinors provide one of the most useful realizations of this bridge between geometry and physics. They first appeared in relativistic quantum mechanics, where the Dirac equation describes spin-1/2 particles [5]. However, their relevance goes beyond their original use: they can be used to encode geometric information about the underlying spacetime, and the existence of special classes of spinor fields can impose additional restrictions on the metric [6]. In particular, parallel spinors play a crucial role in the study of supergravity [7, 8]. The proper formulation of these ideas requires several mathematical tools, ranging from Clifford algebras and Spin groups to spinor bundles and spinorial connections [6, 9].

The aim of this thesis is to study a specific case of the geometric role of spinors in mathematical physics. More precisely, we consider an Einstein–Maxwell background which admits a parallel spinor, a condition which restricts the geometry to a Brinkmann-type spacetime [8, 10]. Since this background also carries a $U(1)$ gauge field, the dynamics of charged test particles provides a way to probe its structure. Thus, the work should be understood as a thesis in mathematical physics: an important part of it is devoted to mathematical structures such as Clifford algebras, Spin groups, spinor bundles, and $\text{spin}^{\mathbb{C}}$ connections, but they are introduced with a physical purpose. They provide the language to construct and understand supersymmetric Einstein–Maxwell backgrounds [8].

The thesis is organized as follows. We first review the basics of Clifford algebras, Spin groups, and spinors. We then globalize these structures on manifolds through $\text{spin}^{\mathbb{C}}$ structures, spinor bundles, and spinorial connections. This allows us to define the notion of a parallel spinor. After introducing this language, we proceed to present the spacetime of interest obtained in [8]. Finally, we study the magnetic geodesics for charged particles in this background, using them as probes of the gravitational and electromagnetic geometry. In this way, the thesis follows the path from Clifford algebras and spin geometry to the study of a concrete Einstein–Maxwell spacetime and charged-particle dynamics.

2 Clifford algebras and spinors

2.1 Real Clifford algebras, the Spin group, and spinors

Let V be a vector space over a field $\mathbb{K}$ and q a quadratic form on V . Let us denote by

$$T(V) := \bigoplus_{r=0}^{\infty} V^{\otimes r} = \mathbb{K} \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \dots$$

the tensor algebra of V and by $I(V, q) \subset T(V)$ the two-sided ideal generated by elements of the form $v \otimes v + q(v)1_{T(V)}$ for all $v \in V$.

Definition 2.1. The **Clifford algebra** associated with (V, q) is defined to be the quotient

$$Cl(V, q) := T(V)/I(V, q).$$

Consider the natural inclusion $i : V \hookrightarrow T(V)$ which identifies each $v \in V$ with the homogeneous element of degree 1 represented by v . Since i is injective, we write v instead of $i(v)$ in $T(V)$. Consider also the canonical projection to the quotient, $\pi_q : T(V) \rightarrow Cl(V, q)$. The composition

$$j_q := \pi_q \circ i : V \hookrightarrow Cl(V, q)$$

is the natural embedding of V into $Cl(V, q)$. It can be shown that j_q is injective. Hence, by abuse of notation, we identify V with its image under j_q . Accordingly, for $v \in V$ we still write v for its class $j_q(v) \in Cl(V, q)$ and we simply write $V \subset Cl(V, q)$.

We denote the product on $Cl(V, q)$ by $\cdot$ and refer to it as the Clifford product. Thus, if $v, w \in V$, $v \cdot w := [v \otimes w]$ is the class of $v \otimes w$ in the quotient. In particular,

$$v \cdot v = -q(v)1_{Cl(V, q)}.$$

With this convention, $Cl(V, q)$ is the associative algebra generated by V and the identity $1_{Cl(V, q)}$, subject to the above relation. Equivalently, if h denotes the symmetric bilinear form associated with q , so that $q(v) = h(v, v)$, and if the characteristic of the field satisfies $\text{char}(\mathbb{K}) \neq 2$, then for all $v, w \in V$,

$$v \cdot w + w \cdot v = -2h(v, w)1_{Cl(V, q)}.$$

We now consider finite-dimensional real vector spaces ($\mathbb{K} = \mathbb{R}$) endowed with a non-degenerate symmetric bilinear form. This is precisely the algebraic setting needed to study spinors on pseudo-Riemannian manifolds, particularly Lorentzian manifolds modelling spacetime. For such a manifold M and $p \in M$, the tangent space $T_p M$ is a finite-dimensional real vector space because M is finite-dimensional. Moreover, the metric g induces a non-degenerate symmetric bilinear form g_p on $T_p M$, equivalently a non-degenerate quadratic form. Hence, the relevant Clifford algebras are those associated with finite-dimensional non-degenerate real quadratic spaces. In a suitable basis, this quadratic form is represented by a diagonal matrix with r entries equal to 1 and s entries equal to -1 , where $r + s = \dim_{\mathbb{R}} T_p M$. By Sylvester's law of inertia [11], the numbers of positive and negative diagonal entries are basis-independent. The ordered pair (r, s) is the signature of the quadratic form. In the Lorentzian case relevant to spacetime geometry, the signature is $(n - 1, 1)$ in our convention. Thus, up to isometry, the relevant quadratic spaces are the standard quadratic spaces

$$\mathbb{R}^{r, s} := (\mathbb{R}^{r+s}, q_{r, s}),$$

with $q_{r, s}(x) = x_1^2 + \cdots + x_r^2 - x_{r+1}^2 - \cdots - x_{r+s}^2$. We will abuse notation and use interchangeably $\mathbb{R}^{r+s}$ and V , and occasionally write $\mathbb{R}^{r, s}$ for its underlying vector space when no confusion arises. The corresponding Clifford algebras are denoted by

$$Cl_{r, s} := Cl(\mathbb{R}^{r+s}, q_{r, s}) = Cl(\mathbb{R}^{r, s}).$$

These real Clifford algebras provide the geometric starting point. Their complexifications will be introduced later in order to define complex spinor representations.

Let $n := \dim_{\mathbb{R}} \mathbb{R}^{r,s} = r + s$ and let $\{e_1, \dots, e_{r+s}\}$ be the canonical orthogonal basis of $\mathbb{R}^{r,s}$. Then $Cl_{r,s}$ is generated by $e_1, \dots, e_{r+s}$ subject to the relations

$$e_i \cdot e_j = \begin{cases} -e_j \cdot e_i, & i \neq j \\ -1_{Cl_{r,s}}, & i = j \in \{1, \dots, r\} \\ 1_{Cl_{r,s}}, & i = j \in \{r+1, \dots, r+s\}. \end{cases}$$

Moreover, the monomials

$$e_{i_1} \cdot e_{i_2} \cdot \dots \cdot e_{i_k}, \quad 1 \leq i_1 < i_2 < \dots < i_k \leq r+s,$$

together with $1_{Cl_{r,s}}$, form a basis of $Cl_{r,s}$. In particular, $\dim_{\mathbb{R}} Cl_{r,s} = 2^{r+s}$.

The real Clifford algebras $Cl_{r,s}$ are classified, up to isomorphism, by the value of $r-s$ modulo eight. Since the full classification is not needed in the main argument, the corresponding table, isomorphisms, and low-dimensional examples are collected in Appendix A.

Tensor-degree parity gives $Cl_{r,s}$ a natural $\mathbb{Z}_2$ -grading, making it a superalgebra:¹

$$Cl_{r,s} = Cl_{r,s}^0 \oplus Cl_{r,s}^1,$$

where $Cl_{r,s}^0$ and $Cl_{r,s}^1$ denote, respectively, the even and odd parts of the algebra, and if $a \in Cl_{r,s}^i$ and $b \in Cl_{r,s}^j$, then $a \cdot b \in Cl_{r,s}^{i+j \pmod{2}}$. In terms of a basis $\{e_1, \dots, e_{r+s}\}$ of $\mathbb{R}^{r,s}$, the even and odd parts are given by

$$Cl_{r,s}^0 = \text{Span}_{\mathbb{R}} \left(\{1_{Cl_{r,s}}\} \cup \{e_{i_1} \cdots e_{i_k} \mid k \text{ even}, 1 \leq i_1 < \dots < i_k \leq r+s\} \right),$$

and

$$Cl_{r,s}^1 = \text{Span}_{\mathbb{R}} \{e_{i_1} \cdots e_{i_k} \mid k \text{ odd}, 1 \leq i_1 < \dots < i_k \leq r+s\}.$$

Notice that $Cl_{r,s}^0 \subset Cl_{r,s}$ is a subalgebra.

We define the multiplicative group of units of the Clifford algebra, which is the largest subset of $Cl_{r,s}$ forming a group under the Clifford product, by

$$Cl_{r,s}^{\times} := \{\varphi \in Cl_{r,s} \mid \exists \varphi^{-1} \in Cl_{r,s} \text{ such that } \varphi \varphi^{-1} = \varphi^{-1} \varphi = 1_{Cl_{r,s}}\}.$$

Now we turn to the Pin and Spin groups. The appeal of Clifford algebras lies not only in the fact that they encode the geometric content of the quadratic form through the Clifford relation, but also in that they naturally encode orthogonal transformations of the associated quadratic space. In particular, they provide an algebraic framework for describing orientation-preserving orthogonal transformations, namely the elements of $SO(r, s)$ (see Appendix B.1), which play a central role in geometry and physics.

As explained in Appendix B.2, the Clifford-Lipschitz group $\Gamma_{r,s} \subset Cl_{r,s}^{\times}$ consists of those invertible elements whose adjoint action preserves $V \subset Cl_{r,s}$. Restricting this action to V gives a group homomorphism

$$\text{Ad}^V : \Gamma_{r,s} \longrightarrow O(r, s).$$

¹Such $\mathbb{Z}_2$ -graded structures underlie the algebraic formulation of supersymmetric theories [12].

However, $\Gamma_{r,s}$ is too large for constructing the desired covering groups, since the kernel of Ad^V is larger than $\{\pm 1\}$. Multiplying an element by any non-zero scalar leaves its adjoint action unchanged. Restricting to products of unit vectors removes this continuous rescaling freedom, while the sign ambiguity remains because the negative of a unit vector is also a unit vector and changes the product only by an overall sign. This leads naturally to the Pin and Spin groups.

The details of the adjoint construction, including the reason why unit vectors suffice, are given in Appendix B.2.

Definition 2.2. The Pin **group** is defined by

$$\text{Pin}(r, s) := \{v_1 \dots v_k \mid k \geq 0, v_i \in V, q_{r,s}(v_i) = \pm 1 \forall i\} \leq Cl_{r,s}^\times.$$

Definition 2.3. The Spin **group** is defined by

$$\text{Spin}(r, s) := \text{Pin}(r, s) \cap Cl_{r,s}^0.$$

Thus,

$$\text{Spin}(r, s) = \{v_1 \dots v_{2k} \mid k \geq 0, v_i \in V, q_{r,s}(v_i) = \pm 1 \forall i\} \leq \text{Pin}(r, s),$$

so $\text{Spin}(r, s)$ consists of products of an even number of unit vectors in $\mathbb{R}^{r,s}$. As explained in Appendix B.3, each such vector determines a reflection of the quadratic space and therefore products of an even number of them are naturally associated with orientation-preserving orthogonal transformations, i.e. with $\text{SO}(r, s)$. Since $\text{Spin}(r, s) \subset \Gamma_{r,s}$, the adjoint representation introduced in Appendix B.2 restricts to a homomorphism

$$\text{Ad}^V|_{\text{Spin}(r,s)} : \text{Spin}(r, s) \longrightarrow \text{SO}(r, s), \quad (2.1)$$

which links the algebraic construction of $\text{Spin}(r, s)$ to $\text{SO}(r, s)$ transformations.

The importance of the Spin group lies in the fact that, although φ and $-\varphi$ induce the same transformation of $\mathbb{R}^{r,s}$, they may act differently on other representation spaces of $\text{Spin}(r, s)$. Such representations do not descend to $\text{SO}(r, s)$; their representation spaces are called spinor spaces and provide the algebraic origin of spinors. Let us formalize this.

Let Σ be a real or complex vector space and let (Σ, γ) be an irreducible representation of $Cl_{r,s}$, i.e. a unital algebra homomorphism

$$\begin{aligned} \gamma : Cl_{r,s} &\longrightarrow \text{End}(\Sigma) \\ x &\longmapsto \gamma(x) \end{aligned}$$

such that $\gamma(1) = \text{Id}_\Sigma$ and Σ has no non-trivial invariant subspaces under the $Cl_{r,s}$ -action.

Definition 2.4. The vector space Σ is a **spinor space** and an element $\psi \in \Sigma$ is a **spinor**.

Remark 2.5. The action $x \cdot \psi := \gamma(x)\psi$ makes every spinor space Σ a left $Cl_{r,s}$ -module, hence a Clifford module.

Since $\text{Spin}(r, s) \subset Cl_{r,s}^\times \subset Cl_{r,s}$, the restriction of γ to $\text{Spin}(r, s)$ takes values in $\text{GL}(\Sigma)$.

Definition 2.6. The **spin representation** is the group homomorphism

$$\gamma|_{\text{Spin}(r,s)} : \text{Spin}(r, s) \longrightarrow \text{GL}(\Sigma).$$

2.2 Complex Clifford algebras, the $\text{Spin}^{\mathbb{C}}$ group, and complex spinors

We now introduce the complex Clifford algebra, which will be the algebraic setting used for complex spinor representations.

Definition 2.7. The **complex Clifford algebra** associated with $\mathbb{R}^{r,s}$ is obtained by extending scalars from $\mathbb{R}$ to $\mathbb{C}$:

$$\mathbb{C}l_{r,s} := Cl_{r,s} \otimes_{\mathbb{R}} \mathbb{C}.$$

Every complex representation $\gamma : Cl_{r,s} \rightarrow \text{End}_{\mathbb{C}}(\Sigma)$ extends uniquely to a complex-linear representation

$$\gamma^{\mathbb{C}} : \mathbb{C}l_{r,s} \longrightarrow \text{End}_{\mathbb{C}}(\Sigma), \quad \gamma^{\mathbb{C}}(x \otimes z) := z\gamma(x),$$

and conversely every representation of $\mathbb{C}l_{r,s}$ restricts to a complex representation of $Cl_{r,s}$ [9, Ch. I, Remark 5.2].

By the universal property of Clifford algebras (see [9, Prop. 1.1]), the complexification of $Cl_{r,s}$ is isomorphic to the Clifford algebra of the complexified quadratic space,

$$\mathbb{C}l_{r,s} \cong Cl(\mathbb{C}^{r+s}, q_{r,s}^{\mathbb{C}}),$$

where $q_{r,s}^{\mathbb{C}}$ is obtained from the complexification of the bilinear form $h_{r,s}$ associated with $q_{r,s}$ [6]. Namely, for $v, w \in \mathbb{R}^{r,s}$, $z_1, z_2 \in \mathbb{C}$, and $z \in \mathbb{C}^{r+s}$,

$$h_{r,s}^{\mathbb{C}}(v \otimes z_1, w \otimes z_2) := z_1 z_2 h_{r,s}(v, w), \quad q_{r,s}^{\mathbb{C}}(z) := h_{r,s}^{\mathbb{C}}(z, z).$$

Remark 2.8. Over $\mathbb{C}$, all non-degenerate quadratic forms of the same dimension are equivalent. Hence, up to isomorphism, $\mathbb{C}l_{r,s}$ depends only on n and not on the signature (r, s) .

Therefore, the classification of complex Clifford algebras is much simpler than in the real case. If $n = 2k$ or $n = 2k + 1$, with $k \in \mathbb{Z}_{\geq 0}$, then [6, 13]

$$\mathbb{C}l_{2k} \cong \mathcal{M}(2^k, \mathbb{C}), \quad \mathbb{C}l_{2k+1} \cong \mathcal{M}(2^k, \mathbb{C}) \oplus \mathcal{M}(2^k, \mathbb{C}).$$

Thus, $\mathbb{C}l_{2k}$ has an irreducible representation on $\mathbb{C}^{2^k}$, whereas each matrix factor of $\mathbb{C}l_{2k+1}$ defines an irreducible representation on $\mathbb{C}^{2^k}$. This leads to the following definition.

Definition 2.9. The **complex n -spinor space**, also called the **Dirac spinor space**, is the complex vector space

$$\Sigma_n := \mathbb{C}^{2^k} \cong \underbrace{\mathbb{C}^2 \otimes \dots \otimes \mathbb{C}^2}_{k \text{ times}}.$$

Its elements are called **complex n -spinors** or **Dirac spinors**.

We fix a complex Clifford representation $\gamma : \mathbb{C}l_{r,s} \longrightarrow \text{End}(\Sigma_n)$. Its restriction to $\text{Spin}(r, s) \subset Cl_{r,s} \subset \mathbb{C}l_{r,s}$ gives the corresponding spin representation

$$\gamma|_{\text{Spin}(r,s)} : \text{Spin}(r, s) \longrightarrow \text{GL}(\Sigma_n).$$

Assume now that $n = 2k$ is even and let $\{e_1, \dots, e_{2k}\}$ be an orthonormal basis of $\mathbb{R}^{r,s}$. The product $e_1 \cdots e_{2k}$ is the volume element of the Clifford algebra. In our sign convention

$$(e_1 \cdots e_{2k})^2 = (-1)^{\frac{2k(2k-1)}{2}} e_1^2 \cdots e_{2k}^2 = (-1)^{k(2k-1)} (-1)^r = (-1)^{k+r}.$$

We therefore define the complex volume element by $\omega_{\mathbb{C}} := i^{k+r} e_1 \cdots e_{2k} \in \mathbb{C}l_{r,s}$, so that $\omega_{\mathbb{C}}^2 = i^{2(k+r)} (e_1 \cdots e_{2k})^2 = (-1)^{k+r} (-1)^{k+r} = 1$. The element $\omega_{\mathbb{C}}$ commutes with the even Clifford algebra $Cl_{r,s}^0$ and in particular with $\text{Spin}(r, s) \subset Cl_{r,s}^0$. Hence, the endomorphism $\Gamma := \gamma(\omega_{\mathbb{C}})$ satisfies

$$\Gamma^2 = \gamma(\omega_{\mathbb{C}}) \gamma(\omega_{\mathbb{C}}) = \gamma(\omega_{\mathbb{C}}^2) = \gamma(1) = \text{Id}_{\Sigma_{2k}}.$$

Therefore, Γ is diagonalizable with eigenvalues ± 1 and the Dirac spinor space decomposes into the corresponding eigenspaces

$$\Sigma_{2k} = \Sigma_{2k}^+ \oplus \Sigma_{2k}^-, \quad \Sigma_{2k}^{\pm} := \{\psi \in \Sigma_{2k} \mid \Gamma(\psi) = \pm \psi\}.$$

Definition 2.10. The elements of Σ_{2k}^+ and Σ_{2k}^- are called **positive** and **negative Weyl spinors**, respectively. They are also called **chiral spinors**.

We now introduce the $\text{Spin}^{\mathbb{C}}$ group. Inside $\mathbb{C}l_{r,s}$ we have $\text{Spin}(r, s) \subset Cl_{r,s} \subset \mathbb{C}l_{r,s}$ and $S^1 := \{z \in \mathbb{C} \mid |z| = 1\} \subset \mathbb{C}l_{r,s}$. Moreover, $\text{Spin}(r, s) \cap S^1 = \{-1, 1\}$. Thus, if $a \in \text{Spin}(r, s)$ and $z \in S^1$, then both pairs (a, z) and $(-a, -z)$ determine the same product in $\mathbb{C}l_{r,s}$, since $az = (-a)(-z)$. Thus, in order to combine $\text{Spin}(r, s)$ with the unit complex scalars without counting the same product twice, one identifies (a, z) with $(-a, -z)$, leading to the following definition.

Definition 2.11. The $\text{Spin}^{\mathbb{C}}$ **group** is defined as

$$\text{Spin}^{\mathbb{C}}(r, s) := (\text{Spin}(r, s) \times S^1) / \{(1, 1), (-1, -1)\}.$$

3 Spinors on manifolds

Section 2 was purely algebraic. We now explain how its constructions are globalized over a pseudo-Riemannian manifold. The required differential-geometric background is reviewed in Appendix C, mainly following Lee [14].

Let (M, g) be an oriented pseudo-Riemannian manifold of signature (r, s) . For every $p \in M$, the tangent space $T_p M$, endowed with the metric g_p , is a real quadratic vector space $(T_p M, g_p)$, so the constructions of the previous section can be applied pointwise. In particular, the Clifford algebra $Cl(T_p M, g_p)$ can be constructed at each point. The non-trivial problem is to assemble these pointwise constructions smoothly over M .

The algebraic construction provides a fixed spinor space Σ_n carrying a spin representation. Since a global oriented orthonormal frame may fail to exist, we choose one on each open set U_{α} of a suitable cover of M . Each such frame identifies $(TM, g)|_{U_{\alpha}}$ with $U_{\alpha} \times \mathbb{R}^{r,s}$ and allows Σ_n to be used as the local model for spinors.

On an overlap $U_\alpha \cap U_\beta$, two oriented orthonormal frames are related by a transition function $\Omega_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \text{SO}(r, s)$. Accordingly, a spinor field would be described in the two frames by generally different maps $\psi_\alpha : U_\alpha \rightarrow \Sigma_n$ and $\psi_\beta : U_\beta \rightarrow \Sigma_n$. Changing the frame also changes the identification of tangent vectors with $\mathbb{R}^{r,s}$ and hence the local expression of Clifford multiplication. Therefore, the local descriptions must be related on the overlap through the spin representation of a lift of $\Omega_{\alpha\beta}$ to $\text{Spin}(r, s)$, or to $\text{Spin}^\mathbb{C}(r, s)$ in the complex case, so that Clifford multiplication is independent of the chosen frame. A spin or $\text{spin}^\mathbb{C}$ structure encodes precisely a coherent choice of such lifts.

Once such a structure is chosen, the trivial bundles $U_\alpha \times \Sigma_n$ can be glued through the lifted transition functions to form a spinor bundle $S \rightarrow M$, whose fibre over p is a spinor space associated with $(T_p M, g_p)$. A spinor field is then a smooth section of S , i.e. an element of $\Gamma(S)$.

This issue is also reflected in the local expression of the Dirac operator [15, Ch. 2]. In flat space $\mathbb{R}^{r,s}$, if $\{e_1, \dots, e_n\}$ is the canonical orthonormal basis, the Dirac operator is

$$D = \sum_{i,j=1}^n g^{ij} \gamma_i \frac{\partial}{\partial x^j}, \quad \gamma_i := \gamma(e_i) \in \text{End}(\Sigma_n),$$

where $g_{ij} := g(e_i, e_j)$, (g^{ij}) is the inverse matrix of (g_{ij}) , and γ_i denotes Clifford multiplication by the basis vector e_i . In physics notation, with $\gamma^\mu := g^{\mu\nu} \gamma_\nu$, this is the operator

$$\not{D} = \gamma^\mu \partial_\mu.$$

On a pseudo-Riemannian manifold, the corresponding expression is therefore written locally using a frame $\{e_1, \dots, e_n\}$ and a covariant derivative on the spinor bundle:

$$D\psi = \sum_{i,j=1}^n g^{ij} e_i \cdot \nabla_{e_j}^S \psi.$$

Here $e_i \cdot \psi$ denotes Clifford multiplication induced by γ . The compatibility of Clifford multiplication and ∇^S with changes of frame makes this expression independent of the chosen local frame and hence globally defined.

We now formalize this discussion. We first define a $\text{spin}^\mathbb{C}$ structure and its associated spinor bundle. The Levi-Civita connection, viewed as a principal connection on the bundle of positively oriented orthonormal frames, lifts in the spin case through the covering $\text{Spin}(r, s) \rightarrow \text{SO}(r, s)$. In the $\text{spin}^\mathbb{C}$ case, it must first be combined with a connection on a $U(1)$ -principal bundle and the resulting product connection is then lifted to the $\text{Spin}^\mathbb{C}(r, s)$ -principal bundle. The lifted connection then induces the spinorial connection on the associated spinor bundle and hence the notion of a parallel spinor. The required material on principal bundles is reviewed in Appendix C.2. The construction follows Friedrich [6, Secs. 2.4–2.5 and 3.1], adapted here to the pseudo-Riemannian setting.

Let $P_{\text{SO}}(M, g) \rightarrow M$ be the $\text{SO}(r, s)$ -principal bundle of positively oriented orthonormal frames and consider the natural projection

$$\begin{aligned} \lambda : \text{Spin}^\mathbb{C}(r, s) &\longrightarrow \text{SO}(r, s) \\ [a, z] &\longmapsto \lambda([a, z]) := \text{Ad}^V(a). \end{aligned}$$

Definition 3.1. A $\text{spin}^\mathbb{C}$ structure on $P_{\text{SO}}(M, g)$ is a pair $(P_{\text{Spin}^\mathbb{C}}(M, g), \Lambda)$ consisting of a $\text{Spin}^\mathbb{C}(r, s)$ -principal bundle $P_{\text{Spin}^\mathbb{C}}(M, g)$ over M and a map $\Lambda : P_{\text{Spin}^\mathbb{C}}(M, g) \rightarrow P_{\text{SO}}(M, g)$ such that the following diagram commutes:

$$\begin{array}{ccc} P_{\text{Spin}^\mathbb{C}}(M, g) \times \text{Spin}^\mathbb{C}(r, s) & \longrightarrow & P_{\text{Spin}^\mathbb{C}}(M, g) \\ \Lambda \times \lambda \downarrow & & \downarrow \Lambda \\ P_{\text{SO}}(M, g) \times \text{SO}(r, s) & \longrightarrow & P_{\text{SO}}(M, g) \end{array} \quad \begin{array}{c} \nearrow \pi_{\text{Spin}^\mathbb{C}} \\ \searrow \pi_{\text{SO}} \end{array} \quad M$$

Definition 3.2. Given a $\text{spin}^\mathbb{C}$ structure $(P_{\text{Spin}^\mathbb{C}}(M, g), \Lambda)$ and complex spin representation $\gamma : \text{Spin}^\mathbb{C}(r, s) \rightarrow \text{GL}(\Sigma_n)$, its **associated spinor bundle** is the complex vector bundle

$$S := P_{\text{Spin}^\mathbb{C}}(M, g) \times_\gamma \Sigma_n.$$

Concretely, an element of S is an equivalence class $[u, \psi]$, where $u \in P_{\text{Spin}^\mathbb{C}}(M, g)$ is a $\text{spin}^\mathbb{C}$ frame and $\psi \in \Sigma_n$ is a spinor in the fixed spinor space. For every $a \in \text{Spin}^\mathbb{C}(r, s)$, the associated-bundle construction identifies

$$(u, \psi) \sim (u, \psi) \cdot a = (u \cdot a, \gamma(a)^{-1}\psi),$$

and therefore $[u, \psi] = [u \cdot a, \gamma(a)^{-1}\psi]$. Thus, relative to the chosen $\text{spin}^\mathbb{C}$ frame u , the geometric spinor is represented by $\psi \in \Sigma_n$, whereas relative to the frame $u \cdot a$, the same geometric spinor is represented by $\gamma(a)^{-1}\psi \in \Sigma_n$. Hence, both pairs describe the same element of the fibre $S_{\pi_{\text{Spin}^\mathbb{C}}(u)}$. See Appendix C.2.

We now introduce the spinorial connection. The Levi-Civita connection is the unique torsion-free metric connection on TM . As a covariant derivative on vector fields, it is denoted by ∇ . As a principal connection on $P_{\text{SO}}(M, g)$, it is an $\mathfrak{so}(r, s)$ -valued 1-form

$$Z : TP_{\text{SO}}(M, g) \longrightarrow \mathfrak{so}(r, s).$$

Let $(P_{\text{Spin}^\mathbb{C}}(M, g), \Lambda)$ be a $\text{spin}^\mathbb{C}$ structure, let $P_1 \rightarrow M$ be its characteristic $U(1)$ -principal bundle, and consider the 2-fold covering onto the fibre product $P_{\text{SO}}(M, g) \tilde{\times} P_1$,

$$\pi : P_{\text{Spin}^\mathbb{C}}(M, g) \longrightarrow P_{\text{SO}}(M, g) \tilde{\times} P_1.$$

Finally, we also fix a connection 1-form on the principal bundle P_1 ,

$$A : TP_1 \longrightarrow i\mathbb{R},$$

where we identify the Lie algebra of the group $S^1 = U(1)$ with the purely imaginary numbers $i\mathbb{R}$. The connections Z and A define the product connection on $P_{\text{SO}}(M, g) \tilde{\times} P_1$,

$$Z \times A : T(P_{\text{SO}}(M, g) \tilde{\times} P_1) \longrightarrow \mathfrak{so}(r, s) \oplus i\mathbb{R}.$$

The product connection $Z \times A$ lifts uniquely through the 2-fold covering π to a connection $\widetilde{Z \times A}$ on the $\text{Spin}^\mathbb{C}(r, s)$ -principal bundle $P_{\text{Spin}^\mathbb{C}}(M, g)$. The lifted connection is characterized by the commutative diagram

$$\begin{array}{ccc} TP_{\text{Spin}^\mathbb{C}}(M, g) & \xrightarrow{\widetilde{Z \times A}} & \mathfrak{spin}^\mathbb{C}(r, s) \cong \mathfrak{spin}(r, s) \oplus i\mathbb{R} \\ d\pi \downarrow & & \downarrow p_* \\ T(P_{\text{SO}}(M, g) \tilde{\times} P_1) & \xrightarrow{Z \times A} & \mathfrak{so}(r, s) \oplus i\mathbb{R} \end{array}$$

where $p_* := (dp)_e$ is the differential of the 2-fold covering $p : \text{Spin}^{\mathbb{C}}(r, s) \rightarrow \text{SO}(r, s) \times S^1$, defined by $[a, z] \mapsto (\text{Ad}^V(a), z^2)$.

Once the connection $\widetilde{Z \times A}$ has been defined on $P_{\text{Spin}^{\mathbb{C}}}(M, g)$, it induces a connection on the associated spinor bundle S . By the equivalence relation defining S , each section $\psi \in \Gamma(S)$ is equivalently represented by a map

$$\widehat{\psi} : P_{\text{Spin}^{\mathbb{C}}}(M, g) \longrightarrow \Sigma_n$$

satisfying $\widehat{\psi}(u \cdot a) = \gamma(a)^{-1} \widehat{\psi}(u)$ for $a \in \text{Spin}^{\mathbb{C}}(r, s)$. Indeed, $\psi(\pi_{\text{Spin}^{\mathbb{C}}}(u)) = [u, \widehat{\psi}(u)]$ is independent of the chosen $\text{spin}^{\mathbb{C}}$ frame u , since $[u \cdot a, \widehat{\psi}(u \cdot a)] = [u \cdot a, \gamma(a)^{-1} \widehat{\psi}(u)] = [u, \widehat{\psi}(u)]$.

The covariant exterior derivative of $\widehat{\psi}$ with respect to the lifted connection $\widetilde{Z \times A}$ is

$$D^A \widehat{\psi} = d\widehat{\psi} + \gamma_* \left(\widetilde{Z \times A} \right) \widehat{\psi},$$

where $\gamma_* := (d\gamma)_e : \mathfrak{spin}^{\mathbb{C}}(r, s) \rightarrow \text{End}(\Sigma_n)$. Since $D^A \widehat{\psi}$ is horizontal and equivariant, it descends to a section $\nabla^A \psi \in \Gamma(T^*M \otimes S)$, thereby defining the covariant derivative (see Definition C.74) [4, Def. 1.2.11, Props. 1.2.12 and 1.4.3, and Defs. 1.5.1–1.5.2]

$$\nabla^A : \Gamma(S) \longrightarrow \Gamma(T^*M \otimes S).$$

Definition 3.3. The covariant derivative ∇^A on the spinor bundle S induced by the lifted connection $\widetilde{Z \times A}$ is called the **spinorial connection**.

Definition 3.4. A spinor field $\psi \in \Gamma(S)$ is called ∇^A -**parallel**, or simply **parallel**, if

$$\nabla^A \psi = 0.$$

4 From parallel spinors to magnetic geodesics on a Brinkmann spacetime

We now apply the spinorial framework introduced above to a four-dimensional Lorentzian manifold. The starting point is the spinorial system studied in [8], where the existence of a parallel spinor together with a condition involving a $U(1)$ gauge field leads to a family of Brinkmann spacetimes. Our goal in this section is to take the resulting metric and gauge field as a fixed background, and study the motion of charged particles in it.

4.1 The spinorial system and the Brinkmann spacetime

Let M be a connected oriented four-dimensional manifold and let P be a $U(1)$ -principal bundle over M . Let g be a mostly-plus Lorentzian metric on M and let Q be a $\text{Spin}_0^{\mathbb{C}}(3, 1)$ -structure whose characteristic $U(1)$ -bundle is P , where $\text{Spin}_0^{\mathbb{C}}(3, 1)$ denotes the identity component of $\text{Spin}^{\mathbb{C}}(3, 1)$ [8, p. 7]. Hence, (M, g) is strongly $\text{spin}^{\mathbb{C}}$ [8, Def. 3.7]. Let $S = S^+ \oplus S^-$ be the complex spinor bundle associated with Q , decomposed into its two irreducible chiral components. We write S^μ , with $\mu \in \mathbb{Z}_2$, for either chiral spinor bundles.

Given a connection A on P , we denote by $\nabla^{g, A}$ the spinorial connection on S induced by the Levi-Civita connection of g and by the connection A . We denote by $F_A \in \Omega^2(M)$ the curvature of A , regarded as a real two-form on M , and we define the complex combination

$$F_A^\mu := \frac{1}{2}(F_A + i\mu *_g F_A) \in \Omega_{\mathbb{C}}^2(M),$$

where $*_g$ is the Hodge star operator induced by the metric g .

Definition 4.1. A **supersymmetric configuration** of Freedman's gauged supergravity on (M, P) is a triple (g, A, η) , with $\eta \in \Gamma(S^\mu)$, such that

$$\nabla^{g,A}\eta = 0, \quad (F_A^\mu - \lambda) \cdot \eta = 0, \quad (4.1)$$

for some non-zero complex constant $\lambda \in \mathbb{C}^*$.

The first equation in (4.1) is usually called the gravitino equation and it states that η is parallel with respect to the $\text{spin}^\mathbb{C}$ connection $\nabla^{g,A}$. The second equation in (4.1) is usually called the gaugino equation and imposes an algebraic condition on the curvature of the $U(1)$ -connection.

Let $\text{Lor}_0(M)$ be the set of time-orientable, strongly $\text{spin}^\mathbb{C}$, Lorentzian metrics on M , and let $C(P)$ be the affine space of connections on P . We define the symmetric tensor $F_A \circ_g F_A$ by $(F_A \circ_g F_A)(w_1, w_2) := \langle F_A(w_1), F_A(w_2) \rangle_g$, with $w_1, w_2 \in \Gamma(TM)$ [8, p. 39].

Definition 4.2. A pair $(g, A) \in \text{Lor}_0(M) \times C(P)$ is called a **supersymmetric solution** of Freedman's gauged supergravity on (M, P) if (g, A, η) is a supersymmetric configuration for some $\eta \in \Gamma(S^\mu)$ and the pair (g, A) satisfies the following differential system:

$$\text{Ric}^g - \frac{1}{2}s_g g = \Lambda g + \mathfrak{e}^2 F_A \circ_g F_A - \frac{\mathfrak{e}^2}{2}|F_A|_g^2 g, \quad d *_g F_A = 0, \quad (4.2)$$

for some non-zero real constants $\Lambda, \mathfrak{e} \in \mathbb{R}^*$.

Remark 4.3. Equations (4.2) are precisely the Einstein–Maxwell equations with cosmological constant Λ and coupling constant $\mathfrak{e}$.

We will not reproduce the full procedure of spinorial forms developed in [8]. However, since this construction is formulated in terms of differential forms, we will follow its convention, letting, at each point $p \in M$, the pointwise Clifford algebra be defined on the cotangent space T_p^*M endowed with the induced metric g_p^{-1} , or equivalently, working with the Clifford bundle $\mathbb{C}l(T^*M, g^{-1})$. We use the musical isomorphisms determined by the metric g ,

$$(\cdot)^\flat : TM \longrightarrow T^*M, \quad (\cdot)^\sharp : T^*M \longrightarrow TM.$$

Thus, if $X \in \Gamma(TM)$, we first take its flat $X^\flat \in \Gamma(T^*M) = \Omega^1(M)$ and then let this one-form act on spinors. We denote by $\langle \cdot, \cdot \rangle$ the natural non-degenerate Hermitian pairing on the spinor bundle S , taken to be linear in the first argument. With this convention, $\langle X^\flat \cdot \eta, \eta \rangle$ is purely imaginary [16].

The key point is that a spinor determines a vector field called its Dirac current [16] and in the present setting this vector field is null and parallel, imposing a strong restriction on the underlying Lorentzian manifold, which will be a Brinkmann spacetime [10, 17].

Definition 4.4. Let $\eta \in \Gamma(S^\mu)$ be a spinor. The **Dirac current** of η is the vector field $V_\eta \in \Gamma(TM)$ defined by

$$g(V_\eta, X) = i\langle X^\flat \cdot \eta, \eta \rangle, \quad X \in \Gamma(TM). \quad (4.3)$$

Definition 4.5. A Lorentzian manifold (M, g) is called a **Brinkmann** or **pp-wave spacetime** if it admits a non-zero vector field which is null and parallel. In other words, there exists $V \in \Gamma(TM)$, $V \neq 0$, such that

$$g(V, V) = 0, \quad \nabla^g V = 0. \quad (4.4)$$

Proposition 4.6. Let (g, A, η) be a supersymmetric configuration of Freedman's gauged supergravity on (M, P) , with $\eta \neq 0$. Then the Dirac current V_η is null and parallel with respect to the Levi-Civita connection:

$$g(V_\eta, V_\eta) = 0, \quad \nabla^g V_\eta = 0. \quad (4.5)$$

In particular, (M, g) is a Brinkmann spacetime.

Proof. We first prove that the Dirac current is parallel. Let $X, Y \in \Gamma(TM)$. Since the Levi-Civita connection is compatible with the metric, $\nabla^g g = 0$,

$$(\nabla_Y^g g)(V_\eta, X) = 0.$$

Using the property of the covariant derivative applied to a (0,2)-tensor [18, p. 96],

$$(\nabla_Y^g g)(V_\eta, X) = Y(g(V_\eta, X)) - g(\nabla_Y^g V_\eta, X) - g(V_\eta, \nabla_Y^g X).$$

Therefore,

$$g(\nabla_Y^g V_\eta, X) = Y(g(V_\eta, X)) - g(V_\eta, \nabla_Y^g X).$$

Let us see that the RHS is zero. Using the definition of the Dirac current in the first term,

$$Y(g(V_\eta, X)) = Y(i\langle X^\flat \cdot \eta, \eta \rangle) = iY(\langle X^\flat \cdot \eta, \eta \rangle).$$

By the compatibility of the spin^C connection $\nabla^{g,A}$ with the Hermitian product, we have

$$(\nabla_Y^{g,A} \langle \cdot, \cdot \rangle)(X^\flat \cdot \eta, \eta) = Y(\langle X^\flat \cdot \eta, \eta \rangle) - \langle \nabla_Y^{g,A} (X^\flat \cdot \eta), \eta \rangle - \langle X^\flat \cdot \eta, \nabla_Y^{g,A} \eta \rangle = 0.$$

Hence,

$$\begin{aligned} Y(g(V_\eta, X)) &= i\langle \nabla_Y^{g,A} (X^\flat \cdot \eta), \eta \rangle + i\langle X^\flat \cdot \eta, \nabla_Y^{g,A} \eta \rangle \\ &= i\langle \nabla_Y^g X^\flat \cdot \eta, \eta \rangle + i\langle X^\flat \cdot \nabla_Y^{g,A} \eta, \eta \rangle + i\langle X^\flat \cdot \eta, \nabla_Y^{g,A} \eta \rangle, \end{aligned}$$

where in the last step we have used that $\nabla^{g,A}$ acts by module derivations with respect to Clifford multiplication. Now, we can use the gravitino equation in (4.1), $\nabla^{g,A} \eta = 0$,

$$Y(g(V_\eta, X)) = i\langle \nabla_Y^g X^\flat \cdot \eta, \eta \rangle.$$

The RHS would be a Dirac current if $\nabla_Y^g X^\flat = (\nabla_Y^g X)^\flat$, but this is in fact true. Indeed,

$$\begin{aligned} (\nabla_Y^g X^\flat)(Z) &= Y(X^\flat(Z)) - X^\flat(\nabla_Y^g Z) = Y(g(X, Z)) - g(X, \nabla_Y^g Z) \\ &= g(\nabla_Y^g X, Z) + g(X, \nabla_Y^g Z) - g(X, \nabla_Y^g Z) = (\nabla_Y^g X)^\flat(Z). \end{aligned}$$

Therefore, using the definition of Dirac current (4.3),

$$Y(g(V_\eta, X)) = i\langle (\nabla_Y^g X)^\flat \cdot \eta, \eta \rangle = g(V_\eta, \nabla_Y^g X).$$

Finally, we see that, indeed,

$$g(\nabla_Y^g V_\eta, X) = g(V_\eta, \nabla_Y^g X) - g(V_\eta, \nabla_Y^g X) = 0.$$

Since this holds for every $X, Y \in \Gamma(TM)$ and g is non-degenerate, we conclude that

$$\nabla^g V_\eta = 0.$$

Let us now prove that the Dirac current V_η is a null vector. Let us consider the one-form associated with V_η via the musical isomorphism, $u_\eta := V_\eta^\flat$. By the description of the Dirac current of a spinor in [8, 19], the one-form u_η is isotropic, hence $g^{-1}(u_\eta, u_\eta) = 0$. Since $g^{-1}(u_\eta, u_\eta) = g^{-1}(V_\eta^\flat, V_\eta^\flat) = g((V_\eta^\flat)^\sharp, (V_\eta^\flat)^\sharp) = g(V_\eta, V_\eta)$, we have

$$g(V_\eta, V_\eta) = 0.$$

Finally, since η is parallel and not identically zero, it is nowhere vanishing. Indeed, if η vanished at some point, parallel transport would imply that it vanishes identically on the connected manifold M . Then, by [19, Prop. 3.26], the associated Dirac current one-form u_η is non-zero. Since $u_\eta = V_\eta^\flat$, this implies that V_η is a non-zero vector field.

Hence, V_η is a non-zero, null, and parallel vector field. By Definition 4.5, (M, g) is a Brinkmann spacetime. $\square$

Having established that the supersymmetry conditions force the underlying Lorentzian manifold to be of Brinkmann type, we now focus on a particular explicit family of such geometries. More precisely, we restrict to the explicit stationary quasi-supersymmetric spacetime classified in [8, Cor. 7.20]. In this case, we have

$$M \cong \mathbb{R}^2 \times S^2.$$

We use (u, v) as Cartesian coordinates on $\mathbb{R}^2$ and (θ, ϕ) as spherical coordinates on S^2 . In these coordinates, the metric reads

$$g = -(R^{-2}\psi^2 + c)du \otimes du + du \odot (dv + *_h d\psi) + h, \quad (4.6)$$

where $\odot$ denotes the symmetrized tensor product defined in Appendix D, $c \in \mathbb{R}$ is a constant, h is the round metric on S^2 of radius R , and ψ is a first eigenfunction of the Laplacian on (S^2, h) , i.e. $\Delta_h \psi = 2R^{-2}\psi$, and it can be written as

$$\psi(\theta, \phi) = c_1 \sin(\theta) \cos(\phi) + c_2 \sin(\theta) \sin(\phi) + c_3 \cos(\theta),$$

for constants $c_1, c_2, c_3 \in \mathbb{R}$. In these standard Brinkmann coordinates, the parallel null one-form is du , and its dual vector field is ∂_v , since (4.6) gives $g(\partial_v, \cdot) = du$. A local representative of the $U(1)$ -connection, in these coordinates, reads

$$A = \psi du - \mu \lambda_I R^2 \cos(\theta) d\phi, \quad (4.7)$$

where λ_I is the imaginary part of the complex constant λ appearing in (4.1).

Hence, we take (g, A) as the fixed background for charged-particle dynamics.

4.2 The charged-particle Lagrangian

We now study the motion of a charged probe particle on the fixed background (M, g, A) . This provides a dynamical probe of the geometry. Let

$$\gamma : I \subseteq \mathbb{R} \longrightarrow M$$

be a smooth worldline in M . In the local chart (U, x) , with $U \subseteq M$ and $x = (\mathbf{u}, \mathbf{v}, \theta, \phi)$,

$$(x \circ \gamma)(\tau) = (\mathbf{u}(\tau), \mathbf{v}(\tau), \theta(\tau), \phi(\tau)),$$

which we simply denote by $\gamma(\tau) = (\mathbf{u}(\tau), \mathbf{v}(\tau), \theta(\tau), \phi(\tau))$.

The free Lagrangian determined by the pseudo-Riemannian metric g is [20, p. 224]

$$L_{\text{free}} = \frac{1}{2}g(\dot{\gamma}, \dot{\gamma}) = \frac{1}{2}g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu,$$

per unit mass, where the second equality is its expression in local coordinates x^μ .

The interaction with the $U(1)$ gauge field, namely the electromagnetic field, is introduced by minimal coupling. Viewing gauge fields as connections on principal bundles [2–4], let $\omega \in \Omega^1(P, \mathfrak{u}(1))$ be a connection form on the $U(1)$ -principal bundle $P \rightarrow M$. In the terminology of [4, Sec. 6.1], M is spacetime, $U(1)$ is the gauge group, ω represents the gauge potential geometrically, and its curvature Ω is the field strength.

A local gauge potential A is obtained by choosing a local section $s : U \subseteq M \longrightarrow \pi^{-1}(U) \subseteq P$ and pulling back the connection form to the spacetime manifold, while the local field strength is obtained by pulling back its curvature:

$$A = s^*\omega \in \Omega^1(U, \mathfrak{u}(1)), \quad F = s^*\Omega \in \Omega^2(U, \mathfrak{u}(1)).$$

In this sense, the gauge potential A is not the connection form on P , but its local spacetime representative. For physical applications, after identifying $\mathfrak{u}(1) \cong \mathbb{R}$, A can be regarded as a real-valued one-form on $U \subseteq M$.

In classical relativistic mechanics, the interaction of a point particle of charge q with an electromagnetic field is governed by the Lorentz force [21, Secs. 16–17]. The dynamics of the particle can be recovered by adding a minimal coupling term to the free action, namely the integral of the local gauge potential A over the worldline, assuming $\gamma(I) \subseteq U$. Formally, this is defined as the integral of the pullback of A to the interval I [22, pp. 438–439, Thm. 16.34 and Cor. 16.36]

$$S_{\text{int}}[\gamma] = q \int_\gamma A = q \int_I \gamma^* A.$$

Using $A = A_\mu dx^\mu$, the pullback to the worldline is

$$\gamma^* A = \gamma^*(A_\mu dx^\mu) = \gamma^*(A_\mu) \gamma^*(dx^\mu) = (A_\mu \circ \gamma) d(x^\mu \circ \gamma),$$

where we have used that the pullback of a function is its composition, $\gamma^*(A_\mu) = A_\mu \circ \gamma$ [22, p. 77], and that it commutes with the exterior derivative, $\gamma^*(dx^\mu) = d(\gamma^*x^\mu) = d(x^\mu \circ \gamma)$ [23, Prop. 4.1.7]. Therefore, the interaction action takes the explicit form

$$S_{\text{int}}[\gamma] = q \int_I A_\mu(\gamma(\tau)) \dot{x}^\mu(\tau) d\tau,$$

and we identify the interaction Lagrangian as

$$L_{\text{int}} = qA_\mu(x)\dot{x}^\mu = qA(\dot{\gamma}).$$

Hence, the total charged-particle Lagrangian is

$$L = L_{\text{free}} + L_{\text{int}} = \frac{1}{2}g(\dot{\gamma}, \dot{\gamma}) + qA(\dot{\gamma}) = \frac{1}{2}g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu + qA_\mu(x)\dot{x}^\mu. \quad (4.8)$$

An important simplification is obtained by rotating the S^2 so that the z -axis coincides with $\mathbf{c} = (c_1, c_2, c_3)$. In these adapted spherical coordinates, which we still denote by (θ, ϕ) , the eigenfunction ψ becomes

$$\psi = \rho \cos(\theta), \quad \rho := \sqrt{c_1^2 + c_2^2 + c_3^2},$$

whence $\psi_\theta := \partial_\theta \psi = -\rho \sin(\theta)$ and $\psi_\phi := \partial_\phi \psi = 0$.

The derivation of the general metric components is detailed in Appendix D.

After the rotation, substitution of the above expressions into the general expression (D.3) yields the metric matrix with respect to the coordinates $(\mathbf{u}, \mathbf{v}, \theta, \phi)$:

$$(g_{\mu\nu}) = \begin{pmatrix} -(R^{-2}\rho^2 \cos^2(\theta) + \mathbf{c}) & 1 & 0 & -\rho \sin^2(\theta) \\ 1 & 0 & 0 & 0 \\ 0 & 0 & R^2 & 0 \\ -\rho \sin^2(\theta) & 0 & 0 & R^2 \sin^2(\theta) \end{pmatrix}. \quad (4.9)$$

Similarly, the gauge potential (4.7) becomes

$$A = \rho \cos(\theta) d\mathbf{u} - \mu \lambda_I R^2 \cos(\theta) d\phi. \quad (4.10)$$

Substituting (4.9) and (4.10) into (4.8), the charged-particle Lagrangian becomes

$$L = L_{S^2} - \frac{1}{2} \left(\frac{\rho^2}{R^2} \cos^2(\theta) + \mathbf{c} \right) \dot{\mathbf{u}}^2 + \dot{\mathbf{u}}\dot{\mathbf{v}} - \rho \sin^2(\theta) \dot{\mathbf{u}}\dot{\phi} + q\rho \cos(\theta) \dot{\mathbf{u}} - q\mu \lambda_I R^2 \cos(\theta) \dot{\phi}, \quad (4.11)$$

where $L_{S^2} := \frac{1}{2}R^2(\dot{\theta}^2 + \sin^2(\theta)\dot{\phi}^2)$ is the free-particle Lagrangian on the round sphere S^2 .

4.3 Effective reduction to S^2 and magnetic geodesics

Since the Lagrangian (4.11) does not depend explicitly on $\mathbf{u}$, $\mathbf{v}$, and ϕ , these variables are cyclic coordinates. Therefore, their conjugate momenta are constants of motion:

$$\pi_{\mathbf{v}} := \frac{\partial L}{\partial \dot{\mathbf{v}}} = \dot{\mathbf{u}} =: \alpha, \quad (4.12)$$

$$\pi_{\mathbf{u}} := \frac{\partial L}{\partial \dot{\mathbf{u}}} = - \left(\frac{\rho^2}{R^2} \cos^2(\theta) + \mathbf{c} \right) \dot{\mathbf{u}} + \dot{\mathbf{v}} - \rho \sin^2(\theta) \dot{\phi} + q\rho \cos(\theta) =: \beta, \quad (4.13)$$

$$\pi_\phi := \frac{\partial L}{\partial \dot{\phi}} = R^2 \sin^2(\theta) \dot{\phi} - \rho \sin^2(\theta) \dot{\mathbf{u}} - q\mu \lambda_I R^2 \cos(\theta) =: \ell, \quad (4.14)$$

where $\alpha, \beta, \ell \in \mathbb{R}$. From (4.12), we immediately obtain

$$\mathbf{u}(\tau) = \alpha\tau + \mathbf{u}_0. \quad (4.15)$$

Solving (4.13) and (4.14) for the velocities associated with the remaining cyclic coordinates gives

$$\dot{\phi} = \frac{\ell}{R^2 \sin^2(\theta)} + \frac{\alpha \rho}{R^2} + \frac{q\mu\lambda_I \cos(\theta)}{\sin^2(\theta)}, \quad (4.16)$$

$$\begin{aligned} \dot{\mathbf{v}} &= \left(\frac{\rho^2}{R^2} \cos^2(\theta) + \mathbf{c} \right) \alpha + \rho \sin^2(\theta) \dot{\phi} - q\rho \cos(\theta) + \beta \\ &= q\rho(\mu\lambda_I - 1) \cos(\theta) + \frac{\alpha \rho^2}{R^2} + \frac{\rho \ell}{R^2} + \alpha \mathbf{c} + \beta. \end{aligned} \quad (4.17)$$

We now apply Routh's procedure to eliminate the cyclic coordinates $\mathbf{u}$, $\mathbf{v}$, and ϕ . We define the Routhian, with the sign convention opposite to that of [24, Sec. 8.3, pp. 347–349], by

$$\mathcal{R} := L - \pi_{\mathbf{u}} \dot{\mathbf{u}} - \pi_{\mathbf{v}} \dot{\mathbf{v}} - \pi_{\phi} \dot{\phi}.$$

Substituting the conserved momenta and velocities, a direct computation gives

$$\mathcal{R} = \frac{1}{2} R^2 \dot{\theta}^2 - \frac{(\ell + q\mu\lambda_I R^2 \cos(\theta))^2}{2R^2 \sin^2(\theta)} - q\alpha\rho(\mu\lambda_I - 1) \cos(\theta) - \frac{\alpha \rho \ell}{R^2} - \frac{\alpha^2 \rho^2}{2R^2} - \frac{1}{2} \alpha^2 \mathbf{c} - \alpha\beta. \quad (4.18)$$

The last four terms are constant along the dynamics and therefore do not affect the Euler–Lagrange equation for θ . Thus, for the dynamics of this remaining non-cyclic coordinate, we may use the one-dimensional effective Lagrangian

$$L_{\text{eff}} := \frac{1}{2} R^2 \dot{\theta}^2 - \frac{(\ell + q\mu\lambda_I R^2 \cos(\theta))^2}{2R^2 \sin^2(\theta)} - q\alpha\rho(\mu\lambda_I - 1) \cos(\theta). \quad (4.19)$$

Remark 4.7. It is useful to perform an intermediate step in Routh's procedure. Eliminating $\mathbf{u}$ and $\mathbf{v}$ while keeping both coordinates (θ, ϕ) on S^2 , we obtain an effective Lagrangian describing the projected motion on the sphere. Namely, $\mathcal{R}_{S^2} := L - \pi_{\mathbf{u}} \dot{\mathbf{u}} - \pi_{\mathbf{v}} \dot{\mathbf{v}}$. Using $\pi_{\mathbf{v}} = \alpha$, $\pi_{\mathbf{u}} = \beta$, and $\dot{\mathbf{u}} = \alpha$, we obtain

$$\mathcal{R}_{S^2} = L_{S^2} - \frac{1}{2} \left(\frac{\rho^2}{R^2} \cos^2(\theta) + \mathbf{c} \right) \alpha^2 - \alpha \rho \sin^2(\theta) \dot{\phi} + q\alpha\rho \cos(\theta) - q\mu\lambda_I R^2 \cos(\theta) \dot{\phi} - \alpha\beta.$$

Dropping the constants, we can write

$$L_{\text{eff}}^{S^2} = L_{S^2} - \alpha \rho \sin^2(\theta) \dot{\phi} - q\mu\lambda_I R^2 \cos(\theta) \dot{\phi} - \frac{\alpha^2 \rho^2}{2R^2} \cos^2(\theta) + q\alpha\rho \cos(\theta). \quad (4.20)$$

This equation has a natural geometric interpretation. Let us define the projection of the worldline onto the S^2 by $\tilde{\gamma} := \text{pr}_{S^2} \circ \gamma$, where $\text{pr}_{S^2} : \mathbb{R}^2 \times S^2 \rightarrow S^2$ is the projection onto the S^2 factor. Hence, $\tilde{\gamma}(\tau) = (\theta(\tau), \phi(\tau))$ and $\dot{\tilde{\gamma}} = \dot{\theta} \partial_{\theta} + \dot{\phi} \partial_{\phi}$. The first term in (4.20) is then

$$L_{S^2} = \frac{1}{2} h(\dot{\tilde{\gamma}}, \dot{\tilde{\gamma}}).$$

The terms that are linear in $\dot{\phi}$ can be re-expressed, using (D.2), as

$$-\alpha \rho \sin^2(\theta) \dot{\phi} - q\mu\lambda_I R^2 \cos(\theta) \dot{\phi} = \alpha *_h d\psi(\dot{\tilde{\gamma}}) + qa(\dot{\tilde{\gamma}}),$$

where we have defined $a := -\mu\lambda_I R^2 \cos(\theta) d\phi$ as the S^2 component of A . Finally, we define the remaining terms as an effective potential,

$$V_{\text{eff}}^{S^2}(\theta) := \frac{\alpha^2 \rho^2}{2R^2} \cos^2(\theta) - q\alpha\rho \cos(\theta).$$

Therefore, the effective Lagrangian on S^2 can be written more compactly as

$$L_{\text{eff}}^{S^2} = \frac{1}{2} h(\dot{\tilde{\gamma}}, \dot{\tilde{\gamma}}) + \alpha *_h d\psi(\dot{\tilde{\gamma}}) + qa(\dot{\tilde{\gamma}}) - V_{\text{eff}}^{S^2}. \quad (4.21)$$

Equivalently, at the level of the action, the linear terms in the velocities come from the pullback of the one-forms $\alpha *_h d\psi$ and qa along $\tilde{\gamma}$. The term $\alpha *_h d\psi$ is present even if $q = 0$ and it comes from the off-diagonal part of the spacetime metric, whereas the term qa is the electromagnetic contribution coming from the gauge potential. Consequently, the projection onto S^2 of a geodesic with $q = 0$ and $\alpha \neq 0$ is not, in general, a free geodesic of the round sphere. Instead, its dynamics contains a magnetic term and an effective potential.

Returning to the effective one-dimensional Lagrangian (4.19), since it has no explicit dependence on τ , the Hamiltonian associated with L_{eff} is conserved:

$$H_{\text{eff}} := \dot{\theta} \frac{\partial L_{\text{eff}}}{\partial \dot{\theta}} - L_{\text{eff}} =: E.$$

Using $\pi_{\theta} := \partial L_{\text{eff}} / \partial \dot{\theta} = R^2 \dot{\theta}$ and substituting (4.19), we obtain

$$E = \frac{1}{2} R^2 \dot{\theta}^2 + V_{\text{eff}}(\theta), \quad V_{\text{eff}}(\theta) := \frac{(\ell + q\mu\lambda_I R^2 \cos(\theta))^2}{2R^2 \sin^2(\theta)} + q\alpha\rho(\mu\lambda_I - 1) \cos(\theta). \quad (4.22)$$

The effective potential V_{eff} arises from the Routh reduction and is not present in the original spacetime Lagrangian. Thus, this dynamics is equivalent to the motion of a one-dimensional particle of effective mass R^2 in the effective potential V_{eff} , with allowed regions determined by $E \geq V_{\text{eff}}(\theta)$.

The velocity associated with θ is

$$\dot{\theta}^2 = \frac{2}{R^2} \left[E - \frac{(\ell + q\mu\lambda_I R^2 \cos(\theta))^2}{2R^2 \sin^2(\theta)} - q\alpha\rho(\mu\lambda_I - 1) \cos(\theta) \right]. \quad (4.23)$$

Before integrating, let us define the concept of magnetic geodesic. We follow [25], adapting it to our conventions. Let (M, g) be a pseudo-Riemannian manifold and let $F \in \Omega^2(M)$ be a closed two-form (which in our case is the electromagnetic field strength).

Definition 4.8. For each $q \in \mathbb{R}$, we define the **Lorentz force** associated with qF as the bundle map $Y_{q,F} : TM \rightarrow TM$ determined by

$$qF_x(V, U) = g_x((Y_{q,F})_x(U), V), \quad (4.24)$$

for every $x \in M$ and every $U, V \in T_x M$. A curve $\gamma : I \rightarrow M$ is called a **magnetic geodesic** of charge q with magnetic field F if $\nabla_{\dot{\gamma}}^g \dot{\gamma} = Y_{q,F}(\dot{\gamma})$, or, equivalently, by (4.24),

$$g(\nabla_{\dot{\gamma}}^g \dot{\gamma}, X) = qF(X, \dot{\gamma}) \quad (4.25)$$

for every $X \in \Gamma(\gamma^* TM)$.

We now check that our charged-particle Lagrangian (4.8) gives precisely (4.25). Writing $A = A_\mu dx^\mu$, we have

$$\begin{aligned} F &= dA = d(A_\mu dx^\mu) = d(A_\mu) \wedge dx^\mu + A_\mu d(dx^\mu) = \partial_\nu A_\mu dx^\nu \wedge dx^\mu = \partial_\mu A_\nu dx^\mu \wedge dx^\nu \\ &= \frac{1}{2}(\partial_\mu A_\nu dx^\mu \wedge dx^\nu + \partial_\nu A_\mu dx^\nu \wedge dx^\mu) = \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu) dx^\mu \wedge dx^\nu \\ &= \frac{1}{2}(\partial_\mu A_\nu - \partial_\nu A_\mu)(dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu) = (\partial_\mu A_\nu - \partial_\nu A_\mu) dx^\mu \otimes dx^\nu. \end{aligned}$$

Then, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Now consider our original Lagrangian (4.8). The Euler–Lagrange equations give

$$\begin{aligned} 0 &= \frac{d}{d\tau} \left(\frac{\partial L}{\partial \dot{x}^\mu} \right) - \frac{\partial L}{\partial x^\mu} = g_{\mu\nu} \ddot{x}^\nu + \partial_\alpha g_{\mu\nu} \dot{x}^\alpha \dot{x}^\nu - \frac{1}{2} \partial_\mu g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta + q (\partial_\alpha A_\mu - \partial_\mu A_\alpha) \dot{x}^\alpha \\ &= g_{\mu\nu} \left(\ddot{x}^\nu + \Gamma_{\alpha\beta}^\nu \dot{x}^\alpha \dot{x}^\beta \right) + q (\partial_\alpha A_\mu - \partial_\mu A_\alpha) \dot{x}^\alpha = g_{\mu\nu} \left(\nabla_\gamma^g \dot{x}^\nu \right) - q F_{\mu\alpha} \dot{x}^\alpha. \end{aligned}$$

Therefore, $g_{\mu\nu} \left(\nabla_\gamma^g \dot{x}^\nu \right) = q F_{\mu\alpha} \dot{x}^\alpha$, or equivalently,

$$g(\nabla_\gamma^g \dot{x}, X) = q F(X, \dot{x})$$

for every $X \in \Gamma(\gamma^* TM)$, which is exactly (4.25). Hence, the Euler–Lagrange solutions of L are magnetic geodesics. Therefore, the solutions $\gamma(\tau) = (\mathbf{u}(\tau), \mathbf{v}(\tau), \theta(\tau), \phi(\tau))$ are magnetic geodesics on (M, g) with magnetic field F and charge q .

4.4 Integration of the equations of motion

We now integrate the first-order system obtained above. The coordinate $\mathbf{u}$ has already been integrated in (4.15). Moreover, the equations for $\mathbf{v}$ and ϕ , given in (4.17) and (4.16), depend on θ . Therefore, we first solve the equation for θ and then the remaining coordinates.

Let us introduce $x := \cos(\theta)$. Then, the differential equation (4.23) is equivalent to

$$\dot{x}^2 = a_3 x^3 + a_2 x^2 + a_1 x + a_0 =: P(x), \quad (4.26)$$

where

$$\begin{aligned} a_0 &= \frac{2E}{R^2} - \frac{\ell^2}{R^4}, & a_1 &= -\frac{2q}{R^2} [\alpha \rho (\mu \lambda_I - 1) + \mu \lambda_I \ell], \\ a_2 &= -\left(\frac{2E}{R^2} + q^2 \mu^2 \lambda_I^2 \right), & a_3 &= \frac{2}{R^2} q \alpha \rho (\mu \lambda_I - 1). \end{aligned} \quad (4.27)$$

The admissible solutions are subject to the constraints $x \in [-1, 1]$, $P(x) \geq 0$.

For the specific coefficients (4.27), the values of P at the boundary points $x = \pm 1$ of the admissible interval are given by

$$P(1) = -\frac{(\ell + q\mu\lambda_I R^2)^2}{R^4} \leq 0, \quad P(-1) = -\frac{(\ell - q\mu\lambda_I R^2)^2}{R^4} \leq 0.$$

Since a physical solution must satisfy $P(x) \geq 0$, it can reach the north or south pole only if the corresponding boundary point is a root of P . Therefore,

$$x = 1 \implies \ell = -q\mu\lambda_I R^2, \quad x = -1 \implies \ell = q\mu\lambda_I R^2.$$

Thus, the poles can only be reached for these values of the conserved angular momentum.

The complete integration of (4.26) is given in Appendix E. There, the equation is first classified algebraically according to the degree and real root structure of P , and the constraints coming from the specific coefficients (4.27) are imposed later. The physically realizable branches are summarized in Table 2.

The non-constant physically realizable branches are the following. In each formula, C denotes an integration constant fixed by the initial conditions. The roots appearing below are real roots of P , ordered increasingly. In the cubic cases, we write $r_1 \leq r_2 \leq r_3$.

- For the quadratic oscillatory branch, with $a_2 < 0$ and $-1 \leq r_1 < r_2 \leq 1$,

$$x(\tau) = r_1 + (r_2 - r_1) \sin^2 \left(\frac{\sqrt{-a_2}}{2} \tau + C \right). \quad (4.28)$$

- For the cubic Jacobi oscillatory branches, with three distinct real roots $r_1 < r_2 < r_3$, the first formula corresponds to $a_3 > 0$ and $-1 \leq r_1 < r_2 \leq 1$, while the second one corresponds to $a_3 < 0$ and $-1 \leq r_2 < r_3 \leq 1$:

$$x(\tau) = r_1 + (r_2 - r_1) \operatorname{sn}^2 \left(\frac{\sqrt{a_3(r_3 - r_1)}}{2} \tau + C, \sqrt{\frac{r_2 - r_1}{r_3 - r_1}} \right), \quad (4.29)$$

$$x(\tau) = r_3 - (r_3 - r_2) \operatorname{sn}^2 \left(\frac{\sqrt{-a_3(r_3 - r_1)}}{2} \tau + C, \sqrt{\frac{r_3 - r_2}{r_3 - r_1}} \right). \quad (4.30)$$

- For the cubic hyperbolic asymptotic branches, the first formula corresponds to $a_3 < 0$ and $-1 \leq r_1 = r_2 < r_3 \leq 1$, while the second one corresponds to $a_3 > 0$ and $-1 \leq r_1 < r_2 = r_3 \leq 1$:

$$x(\tau) = r_3 - (r_3 - r_1) \tanh^2 \left(\frac{\sqrt{-a_3(r_3 - r_1)}}{2} \tau + C \right), \quad (4.31)$$

$$x(\tau) = r_1 + (r_3 - r_1) \tanh^2 \left(\frac{\sqrt{a_3(r_3 - r_1)}}{2} \tau + C \right). \quad (4.32)$$

The constant branches are

$$x(\tau) = x_*, \quad x_* \in [-1, 1]. \quad (4.33)$$

If $P(x) = 0$ for all x , every such x_* is admissible. If P is not identically zero, the associated second-order equation $\ddot{x} = \frac{1}{2}P'(x)$ requires $P(x_*) = P'(x_*) = 0$, and hence constant branches correspond to multiple roots of P , while simple roots are turning points of non-constant branches, where the velocity vanishes but acceleration remains non-zero.

Once one of the above branches has been chosen, the original angular coordinate is recovered from $\theta(\tau) = \arccos(x(\tau))$. Finally, the remaining coordinates are obtained by substituting the chosen branch $x(\tau)$ into (4.17) and (4.16):

$$\phi(\tau) = \phi_0 + \frac{\alpha\rho}{R^2}(\tau - \tau_0) + \int_{\tau_0}^{\tau} \frac{\ell/R^2 + q\mu\lambda_I x(\tau')}{1 - x(\tau')^2} d\tau', \quad (4.34)$$

$$\mathbf{v}(\tau) = \mathbf{v}_0 + \left(\frac{\alpha\rho^2}{R^2} + \frac{\rho\ell}{R^2} + \alpha\mathbf{c} + \beta \right) (\tau - \tau_0) + q\rho(\mu\lambda_I - 1) \int_{\tau_0}^{\tau} x(\tau') d\tau'. \quad (4.35)$$

For a numerical validation of these expressions and for the visualization of projected magnetic geodesics on S^2 , we refer to the supplementary repository [26]. It contains the Mathematica notebook used to compare the analytic branches for $x(\tau)$, equivalently for $\theta(\tau)$, with direct `NDSolve` integrations, together with the plots of $x(\tau)$, $\theta(\tau)$, $\phi(\tau)$, the numerical error, and the three-dimensional animation on S^2 .

5 Conclusions

In this thesis, we have developed the mathematical framework required to relate spinorial structures to the geometry of spacetime. First, we have introduced Clifford algebras, Spin groups, and spinors. We have then globalized these notions through $\text{spin}^{\mathbb{C}}$ structures, spinor bundles, and spinorial connections. This has allowed us to define the notion of a parallel spinor on a Lorentzian manifold and to explain how, in the Einstein–Maxwell background considered here, the existence of such a spinor determines a null parallel Dirac current and therefore forces the spacetime to be of Brinkmann type.

We have then studied this background through the dynamics of a charged test particle. Starting from the minimally coupled Lagrangian, we have eliminated the cyclic coordinates by means of a Routh reduction, obtaining an effective dynamical system on the two-sphere and, subsequently, a one-dimensional equation for the polar motion. We have integrated the resulting equation and exhaustively classified its physically realizable branches. These branches describe constant, oscillatory, and asymptotic trajectories. We have also compared the analytic solutions with numerical integrations and implemented them in an interactive program for visualizing the corresponding trajectories.

A notable feature of the reduced system is that its effective dynamics contains contributions of two distinct origins: the electromagnetic coupling of the charged particle and terms induced by the geometry of the spacetime itself. In particular, the geometrically induced terms can remain present even in the neutral case $q = 0$. Moreover, the root structure of the polynomial directly determines the qualitative behaviour of the projected motion on S^2 : bounded intervals between simple roots produce oscillatory trajectories, non-constant branches associated with double roots produce asymptotic behaviours, and multiple roots which satisfy the appropriate initial conditions give rise to constant solutions. In this sense, the geometric restrictions imposed by the parallel spinor are indirectly reflected in the dynamics of the test particles.

Several directions remain open. First, it would be natural to study the completeness of the magnetic geodesics. Second, the system could be formulated globally in Hamiltonian and symplectic terms on T^*S^2 , avoiding the coordinate singularities associated with spherical coordinates. This would also provide the appropriate framework for determining when the projected trajectories are closed. Finally, the interpretation of this background as a gyratonic Brinkmann gravitational wave with spherical wave fronts in the presence of a positive physical cosmological constant deserves further investigation, particularly regarding its relation to de Sitter geometry.

A Classification and low-dimensional examples of real Clifford algebras

In this section, we recall the classification of real Clifford algebras and explicitly describe some low-dimensional examples as matrix algebras over the fields $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. We follow the standard literature [6, 9, 11, 13].

Theorem A.1. *Let $Cl_{r,s}$ be the real Clifford algebra of signature (r, s) . Then, $Cl_{r,s}$ is isomorphic to a matrix algebra over $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ according to the following table:*

$r - s \pmod{8}$	$Cl_{r,s}$
0	$\mathcal{M}(2^{[n/2]}, \mathbb{R})$
1	$\mathcal{M}(2^{[n/2]}, \mathbb{C})$
2	$\mathcal{M}(2^{[n/2]-1}, \mathbb{H})$
3	$\mathcal{M}(2^{[n/2]-1}, \mathbb{H}) \oplus \mathcal{M}(2^{[n/2]-1}, \mathbb{H})$
4	$\mathcal{M}(2^{[n/2]-1}, \mathbb{H})$
5	$\mathcal{M}(2^{[n/2]}, \mathbb{C})$
6	$\mathcal{M}(2^{[n/2]}, \mathbb{R})$
7	$\mathcal{M}(2^{[n/2]}, \mathbb{R}) \oplus \mathcal{M}(2^{[n/2]}, \mathbb{R})$

Table 1. Classification of real Clifford algebras.

where $n = r + s$, $[n/2]$ denotes the integer part of $n/2$, and $\mathcal{M}(d, \mathbb{K})$ denotes the algebra of $d \times d$ matrices over $\mathbb{K}$.

The classification can be obtained from the lower-dimensional examples given below together with the standard algebra isomorphisms

$$\begin{aligned}
 Cl_{r+1,s+1} &\cong Cl_{1,1} \otimes Cl_{r,s} \\
 Cl_{s+2,r} &\cong Cl_{2,0} \otimes Cl_{r,s} \\
 Cl_{s,r+2} &\cong Cl_{0,2} \otimes Cl_{r,s}.
 \end{aligned}$$

These isomorphisms can be found, for example, in [13].

A.1 Clifford algebra $Cl_{0,0}$

If $V = \mathbb{R}^{0,0}$, then $V = \{0\}$ and there are no generators. Therefore, the Clifford algebra is generated only by the identity element. Hence every element of $Cl_{0,0}$ is of the form

$$a,$$

with $a \in \mathbb{R}$. It follows immediately that

$$Cl_{0,0} \cong \mathbb{R}.$$

A.2 Clifford algebra $Cl_{1,0}$

If $V = \mathbb{R}^{1,0}$, then $Cl_{1,0}$ is generated by one element e_1 such that

$$e_1^2 = -1.$$

Thus every element of $Cl_{1,0}$ is of the form

$$a + be_1,$$

with $a, b \in \mathbb{R}$, and the map

$$a + be_1 \mapsto a + bi$$

defines an algebra isomorphism

$$Cl_{1,0} \cong \mathbb{C}.$$

A.3 Clifford algebra $Cl_{0,1}$

If $V = \mathbb{R}^{0,1}$, then $Cl_{0,1}$ is generated by one element e_1 such that

$$e_1^2 = 1.$$

Hence every element of $Cl_{0,1}$ is of the form

$$a + be_1,$$

with $a, b \in \mathbb{R}$. In this case,

$$Cl_{0,1} \cong \mathbb{R} \oplus \mathbb{R},$$

via the algebra isomorphism

$$a + be_1 \mapsto (a + b, a - b).$$

To verify that this is an algebra isomorphism, it suffices to check it on the basis $\{1, e_1\}$. First,

$$\varphi(1) = (1, 1),$$

which is the multiplicative identity in $\mathbb{R} \oplus \mathbb{R}$. Moreover,

$$\varphi(e_1) = (1, -1),$$

and therefore

$$\varphi(e_1)^2 = (1, -1)^2 = (1, 1) = \varphi(1).$$

Thus the defining relation $e_1^2 = 1$ is preserved. Since the map is linear and preserves the multiplication on the generators, it defines an algebra isomorphism

$$Cl_{0,1} \cong \mathbb{R} \oplus \mathbb{R}.$$

A.4 Clifford algebra $Cl_{1,1}$

We next consider the case $Cl_{1,1}$. Since $\dim_{\mathbb{R}} Cl_{1,1} = 4$, a basis is given by

$$1, \quad e_1, \quad e_2, \quad e_1e_2,$$

where

$$e_1^2 = -1, \quad e_2^2 = 1, \quad e_1e_2 = -e_2e_1.$$

Define

$$\varphi : Cl_{1,1} \rightarrow \mathcal{M}(2, \mathbb{R})$$

by

$$\varphi(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \varphi(e_1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \varphi(e_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$\varphi(e_1)^2 = -\text{Id}, \quad \varphi(e_2)^2 = \text{Id}, \quad \varphi(e_1)\varphi(e_2) = -\varphi(e_2)\varphi(e_1),$$

so φ respects the defining relations and extends to an algebra homomorphism. Moreover,

$$\varphi(e_1e_2) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore, the images of

$$1, \quad e_1, \quad e_2, \quad e_1e_2$$

are linearly independent in $\mathcal{M}(2, \mathbb{R})$. Since both algebras have dimension 4, it follows that φ is an isomorphism. Hence,

$$Cl_{1,1} \cong \mathcal{M}(2, \mathbb{R}).$$

A.5 Clifford algebra $Cl_{2,0}$

We now consider $Cl_{2,0}$. Since $\dim_{\mathbb{R}} Cl_{2,0} = 4$, a basis is given by

$$1, \quad e_1, \quad e_2, \quad e_1e_2,$$

where

$$e_1^2 = -1, \quad e_2^2 = -1, \quad e_1e_2 = -e_2e_1.$$

It follows that

$$(e_1e_2)^2 = e_1e_2e_1e_2 = -e_1^2e_2^2 = -(-1)(-1) = -1.$$

Therefore, the elements

$$e_1, \quad e_2, \quad e_1e_2$$

satisfy the same multiplication rules as the quaternionic units

$$i, \quad j, \quad k,$$

namely

$$i^2 = j^2 = k^2 = -1, \quad ij = k, \quad ji = -k.$$

Define

$$\varphi : Cl_{2,0} \rightarrow \mathbb{H}$$

by

$$\varphi(1) = 1, \quad \varphi(e_1) = i, \quad \varphi(e_2) = j, \quad \varphi(e_1 e_2) = k.$$

Since the defining relations are preserved, this defines an algebra homomorphism. As both $Cl_{2,0}$ and $\mathbb{H}$ are 4-dimensional over $\mathbb{R}$ and the images of

$$1, \quad e_1, \quad e_2, \quad e_1 e_2$$

are linearly independent in $\mathbb{H}$, it follows that φ is an isomorphism. Hence,

$$Cl_{2,0} \cong \mathbb{H}.$$

B Algebraic constructions related to Clifford algebras

In this section, we present some algebraic constructions related to Clifford algebras for completeness. Throughout this section, the following references are used extensively: [13, 27].

B.1 Linear and orthogonal groups of a quadratic vector space

Let (V, h) be a finite-dimensional real vector space endowed with a non-degenerate symmetric bilinear form of signature (r, s) , with $\dim_{\mathbb{R}} V = r + s = n$.

We define the general linear group of V as

$$\mathrm{GL}(V) := \{T : V \rightarrow V \mid T \text{ is linear and invertible}\}.$$

This is a group under composition, also denoted by $\mathrm{Aut}(V)$.

One can be naturally interested in linear transformations preserving the bilinear form. In that case, we define the orthogonal group of (V, h) as

$$\mathrm{O}(V, h) := \{T \in \mathrm{GL}(V) \mid h(T(v), T(w)) = h(v, w) \forall v, w \in V\}.$$

This is also called the isometry group. The subgroup of orientation-preserving transformations is called the special orthogonal group, denoted by

$$\mathrm{SO}(V, h) := \{T \in \mathrm{O}(V, h) \mid \det(T) = 1\},$$

where $\det(T) := \det([T]_{\mathcal{B}})$ can be computed in any basis since it is invariant under a change of basis.

Let $\mathcal{B}$ be a basis of V . Then each $T \in \mathrm{GL}(V)$ is represented by a matrix $A = [T]_{\mathcal{B}} \in \mathcal{M}(n, \mathbb{R})$, and

$$\mathrm{GL}(V) \cong \mathrm{GL}(n, \mathbb{R}) = \{A \in \mathcal{M}(n, \mathbb{R}) \mid A \text{ is invertible}\}.$$

Let $H := [h]_{\mathcal{B}}$ be the associated matrix of the bilinear form in this basis. Then, we have

$$h(v, w) = x^T H y,$$

where x and y are the coordinate column vectors of v and w in the basis $\mathcal{B}$. Since the coordinate vectors of $T(v)$ and $T(w)$ are Ax and Ay , the condition

$$h(T(v), T(w)) = h(v, w) \quad \forall v, w \in V$$

is equivalent to

$$(Ax)^T H(Ay) = x^T H y \quad \text{or} \quad x^T A^T H A y = x^T H y$$

for all $x, y \in \mathbb{R}^n$. Hence,

$$A^T H A = H.$$

Then,

$$\mathrm{O}(V, h) \cong \mathrm{O}(r, s) := \{A \in \mathrm{GL}(n, \mathbb{R}) \mid A^T H A = H\}.$$

Let us remark that in the case of signature $(3, 1)$, this group is the Lorentz group. For this reason, the Lorentz group is sometimes denoted by $\mathrm{O}(3, 1)$. Finally,

$$\mathrm{SO}(V, h) \cong \mathrm{SO}(r, s) := \{A \in \mathrm{O}(r, s) \mid \det(A) = 1\}.$$

B.2 Clifford group and adjoint representation

The guiding idea is to identify elements of the Clifford algebra whose action on the subspace $V \subset Cl_{r,s}$ reproduces orthogonal transformations of the quadratic space (V, h) . Since elements of a group can act by conjugation and conjugation preserves the group operation, we consider this natural action of the invertible elements of $Cl_{r,s}$ on $Cl_{r,s}$. This leads to the adjoint action, which we then restrict to those invertible elements whose adjoint action preserves the subspace $V \subset Cl_{r,s}$.

Recall that $Cl_{r,s}^\times$ denotes the multiplicative group of units of the Clifford algebra, which is the largest subset of $Cl_{r,s}$ forming a group under the Clifford product.

For every $\varphi \in Cl_{r,s}^\times$ we can define

$$\begin{aligned} \mathrm{Ad}_\varphi : Cl_{r,s} &\longrightarrow Cl_{r,s} \\ x &\longmapsto \mathrm{Ad}_\varphi(x) := \varphi x \varphi^{-1}. \end{aligned}$$

This map is an automorphism. Indeed, it is bijective, with inverse $\mathrm{Ad}_{\varphi^{-1}}$, and it preserves the Clifford product, since

$$\mathrm{Ad}_\varphi(x \cdot y) = \varphi x y \varphi^{-1} = \varphi x \varphi^{-1} \varphi y \varphi^{-1} = \mathrm{Ad}_\varphi(x) \cdot \mathrm{Ad}_\varphi(y).$$

Therefore, one obtains a group homomorphism

$$\begin{aligned} \mathrm{Ad} : Cl_{r,s}^\times &\longrightarrow \mathrm{Aut}(Cl_{r,s}) \\ \varphi &\longmapsto \mathrm{Ad}(\varphi) := \mathrm{Ad}_\varphi, \end{aligned}$$

called the adjoint representation or the adjoint action of $Cl_{r,s}^\times$ on $Cl_{r,s}$.

But we are especially interested in those invertible elements $\varphi \in Cl_{r,s}^\times$ whose adjoint action preserves the subspace $V \subset Cl_{r,s}$, i.e. such that $\mathrm{Ad}_\varphi(V) \subseteq V$. This leads to the definition of the Clifford-Lipschitz group

$$\Gamma_{r,s} := \{\varphi \in Cl_{r,s}^\times \mid \varphi V \varphi^{-1} \subseteq V\} \leq Cl_{r,s}^\times.$$

For every $\varphi \in \Gamma_{r,s}$, the map Ad_φ preserves the subspace $V \subset Cl_{r,s}$. Hence, its restriction to V defines a linear map

$$\begin{aligned} \text{Ad}_\varphi|_V : V &\longrightarrow V \\ x &\longmapsto \varphi x \varphi^{-1}. \end{aligned}$$

Moreover, this map is invertible, with inverse $\text{Ad}_{\varphi^{-1}}|_V$, and therefore

$$\text{Ad}_\varphi|_V \in \text{Aut}(V) = \text{GL}(V).$$

Thus, one obtains a group homomorphism

$$\begin{aligned} \text{Ad}^V : \Gamma_{r,s} &\longrightarrow \text{GL}(V) \\ \varphi &\longmapsto \text{Ad}_\varphi|_V \end{aligned}$$

from $\Gamma_{r,s}$ to $\text{GL}(V)$, which describes how elements of the Clifford-Lipschitz group induce linear transformations of the quadratic space V .

The image of this homomorphism actually lies in the orthogonal group $\text{O}(V, h)$. Indeed, let $\varphi \in \Gamma_{r,s}$ and let $v, w \in V$. Since Ad_φ is an algebra automorphism,

$$\text{Ad}_\varphi(v \cdot w + w \cdot v) = \text{Ad}_\varphi(v) \cdot \text{Ad}_\varphi(w) + \text{Ad}_\varphi(w) \cdot \text{Ad}_\varphi(v).$$

Using the Clifford relation,

$$v \cdot w + w \cdot v = -2h(v, w)1_{Cl_{r,s}},$$

we obtain

$$-2h(v, w)1_{Cl_{r,s}} = \text{Ad}_\varphi(v) \cdot \text{Ad}_\varphi(w) + \text{Ad}_\varphi(w) \cdot \text{Ad}_\varphi(v) = -2h(\text{Ad}_\varphi(v), \text{Ad}_\varphi(w))1_{Cl_{r,s}}.$$

Therefore,

$$h(\text{Ad}_\varphi(v), \text{Ad}_\varphi(w)) = h(v, w),$$

and finally $\text{Ad}_\varphi|_V \in \text{O}(V, h)$.

Hence, we obtain a group homomorphism

$$\begin{aligned} \text{Ad}^V : \Gamma_{r,s} &\longrightarrow \text{O}(r, s) \\ \varphi &\longmapsto \text{Ad}^V(\varphi) := \text{Ad}_\varphi|_V. \end{aligned}$$

In this way, the Clifford-Lipschitz group naturally induces orthogonal transformations of the quadratic space (V, h) by conjugation within the Clifford algebra.

The ambiguity between φ and $-\varphi$ is unavoidable, as both elements induce the same orthogonal transformation. Indeed, for every $v \in V$,

$$\text{Ad}_{-\varphi}(v) = (-\varphi)v(-\varphi)^{-1} = \varphi v \varphi^{-1} = \text{Ad}_\varphi(v).$$

In particular, we have

$$\{\pm 1\} \subseteq \ker(\text{Ad}^V).$$

However, $\Gamma_{r,s}$ is too large for the purposes of constructing the double covering of the orthogonal group, since the kernel of Ad^V is larger than $\{\pm 1\}$. Indeed,

$$\ker(\text{Ad}^V) = \{x \in \Gamma_{r,s} \mid xvx^{-1} = v \ \forall v \in V\} = \{x \in Cl_{r,s}^\times \mid xv = vx \ \forall v \in V\} = Z(Cl_{r,s})^\times.$$

To justify the last equality, recall that V generates $Cl_{r,s}$ as a unital algebra. Therefore, any element $x \in Cl_{r,s}^\times$ which commutes with every $v \in V$ also commutes with every Clifford monomial and, by linearity, with every element of $Cl_{r,s}$. Hence,

$$x \in Z(Cl_{r,s}) \cap Cl_{r,s}^\times = Z(Cl_{r,s})^\times.$$

Conversely, every invertible central element clearly commutes with all $v \in V$ and therefore lies in $\ker(\text{Ad}^V)$. Finally, the kernel is strictly larger than $\{\pm 1\}$. For example,

$$2 \, 1_{Cl_{r,s}} \in \ker(\text{Ad}^V),$$

since for every $v \in V$,

$$(2 \, 1_{Cl_{r,s}})v(2 \, 1_{Cl_{r,s}})^{-1} = v,$$

but $2 \, 1_{Cl_{r,s}} \notin \{\pm 1_{Cl_{r,s}}\}$.

This motivates the introduction of more refined subgroups of $Cl_{r,s}^\times$, such as the Pin and Spin groups, which play a central role. Before giving an alternative characterization of these groups, let us explain why unit vectors naturally enter their definition. Let $v \in V$ be non-null. Since

$$v^2 = -q_{r,s}(v)1_{Cl_{r,s}},$$

the vector v is invertible, with inverse

$$v^{-1} = -\frac{v}{q_{r,s}(v)}.$$

Moreover, for every non-zero $\lambda \in \mathbb{R}$ and every $w \in V$,

$$(\lambda v)w(\lambda v)^{-1} = vwv^{-1}.$$

Hence, the orthogonal transformation induced by a non-null vector depends only on its direction and not on its normalization. Every non-null vector can therefore be rescaled so that

$$q_{r,s}(v) = \pm 1.$$

This leads naturally to the Pin and Spin groups, defined using products of unit vectors.

B.3 Pin and Spin groups

In this subsection, we formulate an alternative and more abstract description of the Pin and Spin groups. To do so, it is convenient to introduce an anti-automorphism of the Clifford algebra.

Definition B.1. The **principal anti-involution** or **reversion** is the linear map

$$\tau : Cl_{r,s} \longrightarrow Cl_{r,s}$$

defined on monomials by

$$\tau(v_1 \cdots v_k) := v_k \cdots v_1, \quad v_1, \dots, v_k \in V,$$

and extended linearly to all elements of $Cl_{r,s}$.

This map is indeed an anti-automorphism since it satisfies

$$\tau(xy) = \tau(y)\tau(x)$$

for all $x, y \in Cl_{r,s}$.

Therefore, we can characterize the Pin and Spin groups as

$$\text{Pin}(r, s) = \{x \in Cl_{r,s}^\times \mid x\tau(x) = \pm 1, \ xV\tau(x) \subseteq V\},$$

and since $\text{Spin}(r, s) := \text{Pin}(r, s) \cap Cl_{r,s}^0$,

$$\text{Spin}(r, s) = \{x \in (Cl_{r,s}^0)^\times \mid x\tau(x) = \pm 1, \ xV\tau(x) \subseteq V\}.$$

We now make explicit the relation between these groups and reflections and clarify the geometric meaning of the adjoint action.

Proposition B.2. *Let $v \in V$ satisfy $h(v, v) \neq 0$. Then for every $w \in V$,*

$$-\text{Ad}_v(w) = -v w v^{-1} = w - 2 \frac{h(v, w)}{h(v, v)} v.$$

In particular, the map

$$\begin{aligned} R_v : V &\longrightarrow V \\ w &\longmapsto R_v(w) := -\text{Ad}_v(w) \end{aligned}$$

is the reflection with respect to the hyperplane $v^\perp := \{u \in V \mid h(u, v) = 0\}$.

Proof. Since $v^{-1} = -v/h(v, v)$, we have

$$-\text{Ad}_v(w) = -v w v^{-1} = \frac{v w v}{h(v, v)}.$$

Now, using the Clifford relation

$$v w + w v = -2h(v, w),$$

we obtain

$$v w v = v(-v w - 2h(v, w)) = -v^2 w - 2h(v, w)v.$$

Since $v^2 = -h(v, v)$, it follows that

$$v w v = h(v, v)w - 2h(v, w)v.$$

Dividing by $h(v, v)$, we conclude that

$$-\text{Ad}_v(w) = w - 2 \frac{h(v, w)}{h(v, v)} v.$$

This is precisely the formula for the reflection of w with respect to the hyperplane orthogonal to v . Indeed, writing $w = w^\perp + \lambda v$ with $w^\perp \in v^\perp$ and $\lambda = h(v, w)/h(v, v)$, one obtains

$$-\text{Ad}_v(w) = w - 2\lambda v = (w^\perp + \lambda v) - 2\lambda v = w^\perp - \lambda v = R_v(w),$$

which is precisely the reflection with respect to $v^\perp$. $\square$

Thus, the adjoint action of a non-null vector $v \in V$ reproduces, up to a sign, the reflection with respect to the hyperplane $v^\perp$. In this way, the adjoint representation introduced in Appendix B.2 provides an algebraic realization of reflections within the Clifford algebra.

Since every orthogonal transformation can be written as a composition of reflections (by the Cartan–Dieudonné theorem), it follows that products of such vectors naturally encode elements of the orthogonal group $O(r, s)$. In particular, elements of $\text{Spin}(r, s)$, being products of an even number of unit vectors, correspond to orientation-preserving orthogonal transformations and are therefore naturally associated with $\text{SO}(r, s)$.

This establishes a conceptual bridge between the algebraic structure of $Cl_{r,s}$ and the geometry of the quadratic space (V, h) .

Since $\text{Spin}(r, s) \subset \Gamma_{r,s}$, the adjoint representation constructed in Appendix B.2 restricts to a group homomorphism

$$\text{Ad}^V|_{\text{Spin}(r,s)} : \text{Spin}(r, s) \longrightarrow O(r, s).$$

Moreover, since elements of $\text{Spin}(r, s)$ are products of an even number of unit vectors, their image consists of orientation-preserving orthogonal transformations. Hence,

$$\text{Ad}^V|_{\text{Spin}(r,s)} : \text{Spin}(r, s) \longrightarrow \text{SO}(r, s).$$

Furthermore, this homomorphism is surjective by the Cartan–Dieudonné theorem. Indeed, every orthogonal transformation can be written as a composition of reflections, and by Proposition B.2 each reflection is induced by a non-null vector in the Clifford algebra. If the transformation is orientation-preserving, then the number of reflections must be even, and therefore it is induced by an element of $\text{Spin}(r, s)$.

Moreover, its kernel is

$$\ker(\text{Ad}^V|_{\text{Spin}(r,s)}) = \{\pm 1\}.$$

Indeed, let $x \in \text{Spin}(r, s)$ be in the kernel. Then

$$xvx^{-1} = v \quad \forall v \in V,$$

and hence

$$xv = vx \quad \forall v \in V.$$

In particular, x commutes with each basis vector e_j . Since $x \in \text{Spin}(r, s) \subset Cl_{r,s}^0$, we may write x as a linear combination of even basis monomials. If an even monomial $e_{i_1} \cdots e_{i_{2k}}$ with $k \geq 1$ appears in x , then it contains some basis vector e_j . Conjugating by this e_j changes the sign of that monomial, whereas scalar terms are unchanged. Hence such a monomial cannot occur if x commutes with every e_j . Therefore x must be a scalar,

$$x = \lambda 1_{Cl_{r,s}}.$$

Since $x \in \text{Spin}(r, s)$ satisfies the normalization condition $x\tau(x) = \pm 1_{Cl_{r,s}}$, we get

$$x\tau(x) = \lambda 1_{Cl_{r,s}} \lambda 1_{Cl_{r,s}} = \lambda^2 1_{Cl_{r,s}} = \pm 1_{Cl_{r,s}}.$$

Since $\lambda \in \mathbb{R}$, then $\lambda = \pm 1$. Therefore

$$\ker(\text{Ad}^V|_{\text{Spin}(r,s)}) = \{\pm 1_{Cl_{r,s}}\}.$$

Therefore, $\text{Spin}(r, s)$ is a double cover of $\text{SO}(r, s)$.

This covering is minimal, in the sense that the kernel reduces precisely to $\{\pm 1\}$.

C Review of differential geometry, fibre bundles, and connections

This appendix reviews the basic notions of differential geometry, fibre bundle theory, and connections, mainly following Lee [14], Rudolph and Schmidt [4], and Friedrich [6].

C.1 Manifolds and pseudo-Riemannian geometry

In this subsection, we begin with topological manifolds, the most basic type of manifold, and successively endow them with additional structure, leading to smooth, pseudo-Riemannian, and Lorentzian manifolds.

Definition C.1. Let M be a topological space. Then M is a **topological manifold of dimension n** or a **topological n -manifold** if it satisfies the following properties:

- (i) M is Hausdorff,
- (ii) M is second-countable, and
- (iii) M is locally Euclidean of dimension n , i.e. for every $p \in M$, there exist an open neighbourhood $U \subseteq M$ of p , an open subset $\widehat{U} \subseteq \mathbb{R}^n$, and a homeomorphism

$$\varphi : U \longrightarrow \widehat{U}.$$

Let M be a topological n -manifold.

Definition C.2. A **coordinate chart**, or simply a **chart**, on M is a pair (U, φ) , where $U \subseteq M$ is an open subset and $\varphi : U \rightarrow \widehat{U}$ is a homeomorphism from U to an open subset $\widehat{U} = \varphi(U) \subseteq \mathbb{R}^n$. The set U is called a **coordinate domain** or a **coordinate neighbourhood** of each of its points. The map φ is called a **local coordinate map**, and its component functions $(x^1, \dots, x^n)$, defined by

$$\varphi(p) = (x^1(p), \dots, x^n(p)),$$

are called **local coordinates** on U .

Definition C.3. Let (U, φ) and (V, ψ) be two charts of M such that $U \cap V \neq \emptyset$. The composite map

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \longrightarrow \psi(U \cap V)$$

is called the **transition map**, or **transition function**, from φ to ψ .

Remark C.4. Since a transition map is a composition of homeomorphisms, it is itself a homeomorphism.

Definition C.5. An **atlas** of M is a set of charts $\mathcal{A} := \{(U_i, \varphi_i)\}_{i \in I}$ covering M , i.e.

$$\bigcup_{i \in I} U_i = M.$$

The topological structure introduced above is sufficient for studying properties invariant under homeomorphisms, but it does not provide a notion of differentiability. Let $\widehat{U} \subseteq \mathbb{R}^n$ and $\widehat{V} \subseteq \mathbb{R}^m$ be open subsets. A map $F : \widehat{U} \rightarrow \widehat{V}$ is said to be **smooth**, or of class C^∞ , if each of its component functions has continuous partial derivatives of all orders. If, in addition, F is bijective and its inverse is also smooth, then F is called a **diffeomorphism**. In particular, every diffeomorphism is a homeomorphism. Since the local descriptions of a manifold are related by transition maps, requiring these maps to be diffeomorphisms allows differential calculus to be consistently transferred to the manifold.

Definition C.6. Two charts (U, φ) and (V, ψ) of M are said to be **smoothly compatible** if either $U \cap V = \emptyset$ or the transition map

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \longrightarrow \psi(U \cap V)$$

is a diffeomorphism.

Definition C.7. An atlas $\mathcal{A}$ of M is called a **smooth atlas** if any two charts in $\mathcal{A}$ are smoothly compatible.

Definition C.8. A smooth atlas $\mathcal{A}$ on M is called a **maximal smooth atlas** if it is not properly contained in any larger smooth atlas.

Definition C.9. A **smooth structure** on M is a maximal smooth atlas.

Definition C.10. A **smooth manifold of dimension n** , or a **smooth n -manifold**, is a pair $(M, \mathcal{A})$, where M is a topological n -manifold and $\mathcal{A}$ is a smooth structure on M .

From now on, we simply write M for the smooth manifold $(M, \mathcal{A})$. The purpose of introducing a smooth structure is to extend ordinary differential calculus from Euclidean spaces to manifolds. This is achieved by expressing functions and maps in smooth local coordinates and requiring their coordinate representations to be smooth in the usual Euclidean sense.

Definition C.11. Let M and N be smooth manifolds. A map $F : M \rightarrow N$ is called a **smooth map** if, for every $p \in M$, there exist smooth charts (U, φ) of M and (V, ψ) of N such that $p \in U$, $F(p) \in V$, $F(U) \subseteq V$, and the map

$$\psi \circ F \circ \varphi^{-1} : \varphi(U) \longrightarrow \psi(V)$$

is smooth. The map $\psi \circ F \circ \varphi^{-1}$ is called the **coordinate representation** of F with respect to the charts (U, φ) and (V, ψ) . Since the transition maps between smooth charts are diffeomorphisms, the smoothness of F is independent of the particular charts chosen.

Definition C.12. Let $k \in \mathbb{Z}_{>0}$. A smooth map

$$f : M \longrightarrow \mathbb{R}^k,$$

where $\mathbb{R}^k$ is endowed with its standard smooth structure, is called a **smooth $\mathbb{R}^k$ -valued function**. In the particular case $k = 1$, it is called a **smooth real-valued function**, or a **smooth scalar field**, and the set of all such functions is denoted by $C^\infty(M)$ or by $\mathcal{F}(M)$.

Definition C.13. Let M and N be smooth manifolds. A **diffeomorphism** from M to N is a bijective smooth map $F : M \rightarrow N$ whose inverse $F^{-1} : N \rightarrow M$ is also smooth. Two smooth manifolds are said to be **diffeomorphic** if there exists a diffeomorphism between them, and we write $M \cong N$.

Definition C.14. A **derivation** on M at a point $p \in M$ is a linear map $v : C^\infty(M) \rightarrow \mathbb{R}$ satisfying the Leibniz rule

$$v(fg) = vf \cdot g(p) + f(p) \cdot vg,$$

for all $f, g \in C^\infty(M)$.

Remark C.15. The set of all derivations at p is a vector space under pointwise addition and scalar multiplication.

Definition C.16. This vector space is denoted by $T_p M$ and is called the **tangent space** to M at p . An element of $T_p M$ is called a **tangent vector** at p .

Definition C.17. Let M and N be smooth manifolds, let $F : M \rightarrow N$ be a smooth map, and let $p \in M$. The **differential of F at p** is the map

$$(dF)_p : T_p M \longrightarrow T_{F(p)} N$$

defined by

$$((dF)_p v)f := v(f \circ F),$$

for every $v \in T_p M$ and $f \in C^\infty(N)$.

Remark C.18. The differential $(dF)_p : T_p M \rightarrow T_{F(p)} N$ is linear. Moreover, if F is a diffeomorphism, then $(dF)_p$ is an isomorphism of vector spaces, and

$$(dF)_p^{-1} = (dF^{-1})_{F(p)}.$$

Let (U, φ) be a smooth chart of M containing p . Since

$$\varphi : U \longrightarrow \widehat{U} \subseteq \mathbb{R}^n$$

is a diffeomorphism, its differential is a linear isomorphism. Using the natural identifications $T_p U \cong T_p M$ and $T_{\varphi(p)} \widehat{U} \cong T_{\varphi(p)} \mathbb{R}^n$, its inverse allows us to transport the standard coordinate vectors of $\mathbb{R}^n$ to $T_p M$. This motivates the following definition.

Definition C.19. Let (U, φ) be a smooth chart of M containing p , with local coordinates $(x^1, \dots, x^n)$. For each $i \in \{1, \dots, n\}$, the **i -th coordinate vector at p** is defined by

$$\left. \frac{\partial}{\partial x^i} \right|_p := (d\varphi)_p^{-1} \left(\left. \frac{\partial}{\partial x^i} \right|_{\varphi(p)} \right).$$

Proposition C.20. Let M be a smooth n -manifold, let $p \in M$, and let (U, φ) be a smooth chart containing p , with local coordinates $(x^1, \dots, x^n)$. Then the coordinate vectors

$$\left. \frac{\partial}{\partial x^1} \right|_p, \dots, \left. \frac{\partial}{\partial x^n} \right|_p$$

form a basis of $T_p M$, called the **coordinate basis** associated with the chart (U, φ) . In particular, $\dim(T_p M) = n$, and every tangent vector $v \in T_p M$ can be written uniquely as

$$v = \sum_{i=1}^n v^i \left. \frac{\partial}{\partial x^i} \right|_p,$$

where $v^1, \dots, v^n \in \mathbb{R}$ are called the **components** of v with respect to the coordinate basis.

Proof. Since

$$(d\varphi)_p : T_p M \longrightarrow T_{\varphi(p)} \mathbb{R}^n$$

is a linear isomorphism and $T_{\varphi(p)} \mathbb{R}^n$ has dimension n , it follows that

$$\dim(T_p M) = n.$$

Moreover, the standard coordinate vectors

$$\left. \frac{\partial}{\partial x^1} \right|_{\varphi(p)}, \dots, \left. \frac{\partial}{\partial x^n} \right|_{\varphi(p)}$$

form a basis of $T_{\varphi(p)} \mathbb{R}^n$. Therefore, their preimages under $(d\varphi)_p$,

$$\left. \frac{\partial}{\partial x^1} \right|_p, \dots, \left. \frac{\partial}{\partial x^n} \right|_p,$$

form a basis of $T_p M$. Consequently, every $v \in T_p M$ can be written uniquely as

$$v = \sum_{i=1}^n v^i \left. \frac{\partial}{\partial x^i} \right|_p,$$

for some unique $v^1, \dots, v^n \in \mathbb{R}$. □

Definition C.21. The **tangent bundle** of M , denoted by TM , is the disjoint union of the tangent spaces at all points of M :

$$TM := \bigsqcup_{p \in M} T_p M = \{(p, v) \mid p \in M, v \in T_p M\}.$$

We also define the natural **projection map**,

$$\begin{aligned} \pi : TM &\longrightarrow M \\ (p, v) &\longmapsto \pi(p, v) := p. \end{aligned}$$

Proposition C.22. *Let M be a smooth n -manifold. Then the tangent bundle TM admits a natural topology and smooth structure that make it into a smooth manifold of dimension $2n$. With respect to this structure, the natural projection*

$$\pi : TM \longrightarrow M$$

is smooth.

Proof. See [14, pp. 66–67, Prop. 3.18]. □

Definition C.23. Let $I \subseteq \mathbb{R}$ be an interval. A **curve** in M is a continuous map

$$\gamma : I \longrightarrow M.$$

The curve is called **smooth** if γ is a smooth map.

Definition C.24. Let $\gamma : I \rightarrow M$ be a smooth curve and let $t_0 \in I$. The **velocity vector of γ at t_0** is the tangent vector

$$\dot{\gamma}(t_0) := (d\gamma)_{t_0} \left(\left. \frac{d}{dt} \right|_{t_0} \right) \in T_{\gamma(t_0)} M,$$

where $d/dt|_{t_0}$ is the standard coordinate basis vector in $T_{t_0}\mathbb{R}$.

Remark C.25. Let (U, φ) be a smooth chart of M with local coordinates $(x^1, \dots, x^n)$. If $\gamma(t_0) \in U$, then, for t sufficiently close to t_0 , the coordinate representation of γ is

$$(\varphi \circ \gamma)(t) = (x^1(t), \dots, x^n(t)),$$

where $x^i(t) := (x^i \circ \gamma)(t)$. Writing the velocity vector in the coordinate basis as

$$\dot{\gamma}(t_0) = \sum_{i=1}^n v^i \left. \frac{\partial}{\partial x^i} \right|_{\gamma(t_0)},$$

we can determine its j -th component by applying it to the coordinate function x^j :

$$\dot{\gamma}(t_0)(x^j) = \sum_{i=1}^n v^i \left. \frac{\partial}{\partial x^i} \right|_{\gamma(t_0)} x^j = \sum_{i=1}^n v^i \delta_i^j = v^j.$$

On the other hand, by the definitions of the velocity vector and the differential,

$$\dot{\gamma}(t_0)(x^j) = \left[(d\gamma)_{t_0} \left(\frac{d}{dt} \Big|_{t_0} \right) \right] (x^j) = \frac{d}{dt} \Big|_{t_0} (x^j \circ \gamma) = \dot{x}^j(t_0).$$

Therefore, $v^j = \dot{x}^j(t_0)$, and hence

$$\dot{\gamma}(t_0) = \sum_{i=1}^n \dot{x}^i(t_0) \frac{\partial}{\partial x^i} \Big|_{\gamma(t_0)}.$$

Introducing the abbreviated notation

$$\partial_i := \frac{\partial}{\partial x^i},$$

this expression becomes

$$\dot{\gamma}(t_0) = \sum_{i=1}^n \dot{x}^i(t_0) \partial_i|_{\gamma(t_0)},$$

or, along the curve,

$$\dot{\gamma} = \sum_{i=1}^n \dot{x}^i \partial_i.$$

Definition C.26. A **smooth vector field** on M is a smooth map

$$X : M \longrightarrow TM$$

such that $X_p := X(p) \in T_p M$ for every $p \in M$. The space of smooth vector fields on M is denoted by $\mathfrak{X}(M)$.

Remark C.27. Smooth $\mathbb{R}^k$ -valued functions should not be confused with vector fields on M . Smooth $\mathbb{R}^k$ -valued functions take values in the fixed vector space $\mathbb{R}^k$, whereas a vector field X assigns to each point $p \in M$ a vector $X_p \in T_p M$.

Remark C.28. The condition $X(p) \in T_p M$ for every $p \in M$ is equivalent to

$$\pi \circ X = \text{Id}_M,$$

where $\pi : TM \rightarrow M$ is the natural projection. A smooth map satisfying this condition is called a **smooth section** of π . Thus, in the language of fibre bundles, smooth vector fields are precisely the smooth sections of the tangent bundle, and hence

$$\mathfrak{X}(M) = \Gamma(TM).$$

The notion of a section will be introduced more generally below in the context of fibre bundles, in Subsection C.2.

Remark C.29. Let (U, φ) be a smooth chart of M with local coordinates $(x^1, \dots, x^n)$. Since the coordinate vectors form a basis of each tangent space, there exist unique smooth functions

$$X^i : U \longrightarrow \mathbb{R}, \quad i = 1, \dots, n,$$

such that

$$X_p = \sum_{i=1}^n X^i(p) \left. \frac{\partial}{\partial x^i} \right|_p$$

for every $p \in U$. Thus, on U , the vector field can be written as

$$X = \sum_{i=1}^n X^i \frac{\partial}{\partial x^i}.$$

The functions X^i are called the **component functions** of X with respect to the given chart. Using the abbreviated notation introduced above, the local expression of X becomes

$$X = \sum_{i=1}^n X^i \partial_i.$$

Remark C.30. Let (U, φ) and (V, ψ) be two smooth charts of M such that $U \cap V \neq \emptyset$, and let $p \in U \cap V$. The vectors

$$(d\varphi)_p(X_p) \in T_{\varphi(p)}\mathbb{R}^n$$

and

$$(d\psi)_p(X_p) \in T_{\psi(p)}\mathbb{R}^n$$

are the coordinate representations of the same tangent vector X_p . Since $\psi = (\psi \circ \varphi^{-1}) \circ \varphi$ on $U \cap V$, the chain rule gives

$$(d\psi)_p(X_p) = (d(\psi \circ \varphi^{-1}))_{\varphi(p)}((d\varphi)_p(X_p)).$$

This is the **compatibility condition** between the two coordinate representations of X_p .

More explicitly, let $(x^1, \dots, x^n)$ and $(\tilde{x}^1, \dots, \tilde{x}^n)$ be the local coordinates associated with φ and ψ , respectively. If

$$X_p = \sum_{i=1}^n X^i(p) \left. \frac{\partial}{\partial x^i} \right|_p = \sum_{j=1}^n \tilde{X}^j(p) \left. \frac{\partial}{\partial \tilde{x}^j} \right|_p,$$

then

$$\tilde{X}^j(p) = \sum_{i=1}^n \frac{\partial \tilde{x}^j}{\partial x^i}(p) X^i(p), \quad j = 1, \dots, n.$$

Thus, the transformation law for the components of X_p is the coordinate expression of the preceding compatibility condition.

Definition C.31. Let $p \in M$. The **cotangent space** to M at p is the dual vector space of $T_p M$,

$$T_p^* M := (T_p M)^*.$$

Its elements are called **covectors** at p .

Remark C.32. Let (U, φ) be a smooth chart of M containing p , with local coordinates $(x^1, \dots, x^n)$. For each $i \in \{1, \dots, n\}$, the differential of the coordinate function x^i at p is the covector

$$dx^i|_p := (dx^i)_p \in T_p^*M.$$

These covectors satisfy

$$dx^i|_p \left(\frac{\partial}{\partial x^j} \Big|_p \right) = \delta_j^i, \quad i, j = 1, \dots, n.$$

Therefore,

$$dx^1|_p, \dots, dx^n|_p$$

form the basis of T_p^*M dual to the coordinate basis of T_pM . It is called the **coordinate covector basis** at p .

Definition C.33. The **cotangent bundle** of M , denoted by T^*M , is the disjoint union of the cotangent spaces at all points of M :

$$T^*M := \bigsqcup_{p \in M} T_p^*M = \{(p, \omega) \mid p \in M, \omega \in T_p^*M\}.$$

We also define the natural projection map

$$\begin{aligned} \pi : T^*M &\longrightarrow M \\ (p, \omega) &\longmapsto \pi(p, \omega) := p. \end{aligned}$$

Proposition C.34. *Let M be a smooth n -manifold. Then the cotangent bundle T^*M admits a natural topology and smooth structure that make it into a smooth manifold of dimension $2n$. With respect to this structure, the natural projection*

$$\pi : T^*M \longrightarrow M$$

is smooth.

Proof. See [14, pp. 276–278]. □

Definition C.35. A **smooth covector field**, or a **differential 1-form**, on M is a smooth map

$$\omega : M \longrightarrow T^*M$$

such that $\omega_p := \omega(p) \in T_p^*M$ for every $p \in M$. The space of differential 1-forms on M is denoted by $\Omega^1(M)$.

Remark C.36. The condition $\omega(p) \in T_p^*M$ for every $p \in M$ is equivalent to

$$\pi \circ \omega = \text{Id}_M.$$

Thus, differential 1-forms are precisely the smooth sections of the cotangent bundle, and hence

$$\Omega^1(M) = \Gamma(T^*M).$$

Remark C.37. Let (U, φ) be a smooth chart of M with local coordinates $(x^1, \dots, x^n)$. Since the coordinate covectors form a basis of each cotangent space, there exist unique smooth functions

$$\omega_i : U \longrightarrow \mathbb{R}, \quad i = 1, \dots, n,$$

such that

$$\omega_p = \sum_{i=1}^n \omega_i(p) dx^i|_p$$

for every $p \in U$. Thus, on U , the 1-form can be written as

$$\omega = \sum_{i=1}^n \omega_i dx^i.$$

The functions ω_i are called the **component functions** of ω with respect to the given chart.

Remark C.38. Let $f \in C^\infty(M)$. For every $p \in M$, the differential of f at p is a linear map

$$(df)_p : T_p M \longrightarrow T_{f(p)} \mathbb{R} \cong \mathbb{R},$$

and hence $(df)_p \in T_p^* M$. Consider the assignment

$$\begin{aligned} df : M &\longrightarrow T^* M \\ p &\longmapsto (df)_p. \end{aligned}$$

Let (U, φ) be a smooth chart with local coordinates $(x^1, \dots, x^n)$. Since the coordinate covectors form a basis of each cotangent space, there exist unique functions $a_i : U \rightarrow \mathbb{R}$ such that

$$(df)_p = \sum_{i=1}^n a_i(p) dx^i|_p$$

for every $p \in U$. Evaluating this equality on the coordinate vector $\partial_j|_p$ gives

$$\frac{\partial f}{\partial x^j}(p) = (df)_p \left(\partial_j|_p \right) = \sum_{i=1}^n a_i(p) dx^i|_p \left(\partial_j|_p \right) = \sum_{i=1}^n a_i(p) \delta_j^i = a_j(p).$$

Therefore,

$$(df)_p = \sum_{i=1}^n \frac{\partial f}{\partial x^i}(p) dx^i|_p,$$

and hence, on U ,

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i.$$

Since the component functions $\partial f / \partial x^i$ are smooth, the assignment $df : M \rightarrow T^* M$ is smooth. Thus, it defines a differential 1-form on M , called the **differential of f** , and $df \in \Omega^1(M)$.

Remark C.39. Let (U, φ) and (V, ψ) be two smooth charts of M such that $U \cap V \neq \emptyset$, with local coordinates $(x^1, \dots, x^n)$ and $(\tilde{x}^1, \dots, \tilde{x}^n)$, respectively, and let $p \in U \cap V$. If

$$\omega_p = \sum_{i=1}^n \omega_i(p) dx^i|_p = \sum_{j=1}^n \tilde{\omega}_j(p) d\tilde{x}^j|_p,$$

then, evaluating both expressions on the coordinate vector $\partial/\partial\tilde{x}^j|_p$, we obtain

$$\tilde{\omega}_j(p) = \sum_{i=1}^n \frac{\partial x^i}{\partial \tilde{x}^j}(p) \omega_i(p), \quad j = 1, \dots, n.$$

Thus, the components of a 1-form transform with the inverse Jacobian of the coordinate transformation.

Given a smooth map $F : M \rightarrow N$, a pullback is, intuitively, a way of using an object defined on N to construct a corresponding object on M . For a real-valued function f on N , i.e. $f : N \rightarrow \mathbb{R}$, this is achieved simply by composition, namely $F^*f = f \circ F$. For objects that act on tangent vectors, however, the differential of F is also required to compare the tangent spaces at $p \in M$ and $F(p) \in N$. In particular, if $\omega \in T_{F(p)}^*N$ and $v \in T_pM$, the vector v is first mapped to $(dF)_p(v) \in T_{F(p)}N$, on which ω can act. This defines a covector in T_p^*M . Thus, covectors, and more generally covariant tensors, admit a natural pullback by any smooth map. We begin with the case of covectors, which is the simplest non-trivial example of this construction.

Definition C.40. Let $F : M \rightarrow N$ be a smooth map and let $p \in M$. The differential

$$(dF)_p : T_pM \longrightarrow T_{F(p)}N$$

induces the dual linear map

$$(dF)_p^* : T_{F(p)}^*N \longrightarrow T_p^*M,$$

defined by

$$((dF)_p^*\omega)(v) = \omega((dF)_p(v))$$

for every $\omega \in T_{F(p)}^*N$ and $v \in T_pM$. This map is called the **pointwise pullback** by F at p , or the **cotangent map** of F at p .

Remark C.41. Although the differential of F maps each tangent vector $v \in T_pM$ to a tangent vector $(dF)_p(v) \in T_{F(p)}N$, it does not, in general, define the pushforward of a vector field on M to a vector field on N . Indeed, to define a vector field at a point $q \in N$, one would have to choose a point $p \in M$ such that $F(p) = q$, and such a point may fail to exist or may not be unique. This ambiguity disappears when pulling back covector fields: for every $p \in M$, the covector field on N has a uniquely determined value at $F(p)$, which can be pulled back by $(dF)_p^*$. Thus, every covector field on N admits a natural pullback to M .

Definition C.42. Let $F : M \rightarrow N$ be a smooth map and let $\omega \in \Omega^1(N)$. The **pullback** of ω by F is the assignment

$$F^*\omega : M \longrightarrow T^*M$$

defined by

$$(F^*\omega)_p := (dF)_p^* (\omega_{F(p)})$$

for every $p \in M$. Equivalently,

$$(F^*\omega)_p(v) = \omega_{F(p)}((dF)_p(v))$$

for every $p \in M$ and $v \in T_pM$.

Proposition C.43. Let $F : M \rightarrow N$ be a smooth map, let $u \in C^\infty(N)$ and let $\omega \in \Omega^1(N)$. Then

$$F^*(u\omega) = (u \circ F)F^*\omega.$$

Moreover,

$$F^*(du) = d(u \circ F).$$

Proof. See [14, Prop. 11.25, p. 285]. □

Proposition C.44. Let $F : M \rightarrow N$ be a smooth map and let $\omega \in \Omega^1(N)$. Then $F^*\omega$ is a differential 1-form on M . Therefore, pullback defines a map

$$F^* : \Omega^1(N) \longrightarrow \Omega^1(M).$$

Proof. See [14, Prop. 11.26, pp. 285–286]. □

Definition C.45. Let $p \in M$ and let $r, s \in \mathbb{Z}_{\geq 0}$. A **tensor of type (r, s) at p** is a multilinear map

$$A_p : \underbrace{T_p^*M \times \cdots \times T_p^*M}_r \times \underbrace{T_pM \times \cdots \times T_pM}_s \longrightarrow \mathbb{R}.$$

The vector space of tensors of type (r, s) at p is denoted by $T^{(r,s)}(T_pM)$. By convention,

$$T^{(0,0)}(T_pM) := \mathbb{R}.$$

Definition C.46. A **tensor field of type (r, s)** on M is an assignment

$$p \longmapsto A_p \in T^{(r,s)}(T_pM).$$

It is called **smooth** if, for every $\omega^1, \dots, \omega^r \in \Omega^1(M)$ and every $X_1, \dots, X_s \in \mathfrak{X}(M)$, the function

$$p \longmapsto A_p(\omega_p^1, \dots, \omega_p^r, (X_1)_p, \dots, (X_s)_p)$$

is smooth.

Remark C.47. A tensor of type $(0, 0)$ at a point is simply a real number. Consequently, smooth tensor fields of type $(0, 0)$ are precisely smooth real-valued functions on M . Smooth tensor fields of type $(1, 0)$ are vector fields, those of type $(0, 1)$ are differential 1-forms, and those of type $(0, 2)$ are smooth fields of bilinear forms on the tangent spaces. A tensor field A of type $(0, 2)$ is called **symmetric** if

$$A_p(v, w) = A_p(w, v)$$

for every $p \in M$ and every $v, w \in T_p M$.

Definition C.48. A **pseudo-Riemannian metric** on a smooth manifold M is a smooth symmetric tensor field g of type $(0, 2)$ such that its value at each point $p \in M$ is a non-degenerate bilinear form

$$\begin{aligned} g_p : T_p M \times T_p M &\longrightarrow \mathbb{R} \\ (v, w) &\longmapsto g_p(v, w). \end{aligned}$$

The signature of g_p is required to be the same for all $p \in M$.

Remark C.49. Equivalently, if $\mathfrak{X}(M)$ denotes the space of smooth vector fields on M and $C^\infty(M)$ denotes the algebra of smooth real-valued functions on M , then g defines a $C^\infty(M)$ -bilinear map

$$\begin{aligned} g : \mathfrak{X}(M) \times \mathfrak{X}(M) &\longrightarrow C^\infty(M) \\ (X, Y) &\longmapsto g(X, Y), \end{aligned}$$

where $g(X, Y)(p) := g_p(X_p, Y_p)$.

Definition C.50. A **pseudo-Riemannian manifold** of signature (r, s) is a pair (M, g) , where M is a smooth manifold and g is a pseudo-Riemannian metric on M of signature (r, s) .

Definition C.51. A **Lorentzian manifold** is a pseudo-Riemannian manifold (M, g) of signature $(n - 1, 1)$, where $n = \dim M$. In this case, g is called a **Lorentzian metric**.

C.2 Fibre bundles, principal and associated fibre bundles, and connections

We restrict our study to smooth fibre bundles, i.e. fibre bundles in the category of smooth manifolds. From now on, we simply refer to smooth fibre bundles as fibre bundles.

Definition C.52. Let E , M , and F be smooth manifolds, and let $\pi : E \rightarrow M$ be a smooth surjection. Let $\bigcup_{i \in I} U_i = M$ be an open cover of M . The 4-tuple (E, π, M, F) is called a **(smooth) fibre bundle** with **standard fibre** F if for each $i \in I$, there exists a diffeomorphism $\phi_i : \pi^{-1}(U_i) \rightarrow U_i \times F$ such that the following diagram commutes:

$$\begin{array}{ccc} E \supset \pi^{-1}(U_i) & \xrightarrow{\phi_i} & U_i \times F \\ & \searrow \pi & \downarrow \text{proj}_1 \\ & & U_i \end{array}$$

Then E is called **total space**, M its **base space**, and $E_p := \pi^{-1}(p) \cong F$ the fibre over $p \in M$. The pair (U_i, ϕ_i) is called a **local trivialization**.

Remark C.53. Local triviality does not imply global triviality. For example, consider real line bundles over S^1 . The trivial line bundle is

$$S^1 \times \mathbb{R}.$$

On the other hand, the Möbius line bundle can be obtained from $[0, 1] \times \mathbb{R}$ by identifying

$$(0, t) \sim (1, -t).$$

Both bundles are locally of the form $U \times \mathbb{R}$ for sufficiently small open subsets $U \subseteq S^1$, but they are globally different: the first bundle is trivial, whereas the Möbius line bundle is not. Thus, local trivializations describe the bundle locally, while the way in which these local products are glued together determines its global structure.

Definition C.54. Let G be a Lie group. A fibre bundle (P, π, M, G) with standard fibre G is called a **G -principal bundle** if

- (i) P is endowed with a smooth free right action of G ,

$$\begin{aligned} P \times G &\longrightarrow P \\ (u, g) &\longmapsto u \cdot g; \end{aligned}$$

- (ii) for every $u_1, u_2 \in P$,

$$\pi(u_1) = \pi(u_2) \iff \text{there exists } g \in G \text{ such that } u_1 \cdot g = u_2;$$

- (iii) its local trivializations

$$\phi_i : \pi^{-1}(U_i) \longrightarrow U_i \times G$$

can be chosen to be G -equivariant, where the right action of G on $U_i \times G$ is defined by $(p, h) \cdot g := (p, hg)$. Thus,

$$\phi_i(u \cdot g) = \phi_i(u) \cdot g$$

for every $u \in \pi^{-1}(U_i)$ and every $g \in G$.

Remark C.55. Condition (ii) states that the fibres of π are precisely the orbits of the right action of G . Indeed, the implication $u_1 \cdot g = u_2 \Rightarrow \pi(u_1) = \pi(u_2)$ means that the action preserves each fibre, whereas the converse implication means that the action is transitive on each fibre. Thus, for every $p \in M$ and every $u \in P_p$,

$$u \cdot G := \{u \cdot g \mid g \in G\} = P_p.$$

Together with the freeness of the action in condition (i), this shows that, for every $u_1, u_2 \in P_p$, there exists a unique $g \in G$ such that $u_1 \cdot g = u_2$. Equivalently, for every fixed $u \in P_p$, the map $G \rightarrow P_p, g \mapsto u \cdot g$, is a bijection, and condition (iii) ensures that it is a diffeomorphism. Hence, each fibre is diffeomorphic to G , although this identification is not canonical.

Definition C.56. Let (P_1, π_1, M, G_1) and (P_2, π_2, M, G_2) be principal bundles over the same base space M . Their **fibre product** is defined by

$$P_1 \times_M P_2 := \{(u_1, u_2) \in P_1 \times P_2 \mid \pi_1(u_1) = \pi_2(u_2)\}.$$

Remark C.57. The fibre product $P_1 \times_M P_2$ is a $(G_1 \times G_2)$ -principal bundle over M , with projection

$$\begin{aligned} \pi_{12} : P_1 \times_M P_2 &\longrightarrow M \\ (u_1, u_2) &\longmapsto \pi_1(u_1) = \pi_2(u_2), \end{aligned}$$

and right action

$$(u_1, u_2) \cdot (g_1, g_2) := (u_1 \cdot g_1, u_2 \cdot g_2).$$

Thus, unlike the Cartesian product $P_1 \times P_2$, the fibre product $P_1 \times_M P_2$ contains only pairs of elements lying over the same point of M .

Definition C.58. Let (E, π, M, F) be a fibre bundle. A **section** of E is a smooth map

$$s : M \longrightarrow E$$

such that $\pi \circ s = \text{id}_M$. More generally, if $U \subseteq M$ is open, a **local section over U** is a smooth map

$$s : U \longrightarrow \pi^{-1}(U)$$

such that $\pi \circ s = \text{id}_U$. The space of smooth sections of E is denoted by $\Gamma(E)$.

One of the main reasons why principal bundles are useful is that they provide a common geometric structure from which many other fibre bundles can be constructed. If a Lie group G acts smoothly on a manifold F , a G -principal bundle P over M determines an associated fibre bundle with standard fibre F , where the action of G is used to glue the local copies of F consistently over M . Different actions of G may therefore give rise to different associated bundles. In particular, when F is a vector space and the action is given by a representation, the resulting bundle is a vector bundle. For example, spinor bundles are obtained in this way from spin or $\text{spin}^{\mathbb{C}}$ principal bundles and their spin representations.

Let (P, π, M, G) be a G -principal bundle, and let F be a smooth manifold endowed with a smooth left action of G , $G \times F \rightarrow F$. Consider the smooth right action of G on $P \times F$ defined by

$$(u, f) \cdot g := (u \cdot g, g^{-1} \cdot f).$$

We denote the corresponding quotient by

$$P \times_G F := (P \times F)/G.$$

Its elements are equivalence classes satisfying

$$[u, f] = [u \cdot g, g^{-1} \cdot f]$$

for every $g \in G$, and the map

$$\begin{aligned} \pi_F : P \times_G F &\longrightarrow M \\ [u, f] &\longmapsto \pi(u) \end{aligned}$$

is well-defined. The 4-tuple $(P \times_G F, \pi_F, M, F)$ is a fibre bundle with standard fibre F .

Definition C.59. The fibre bundle $(P \times_G F, \pi_F, M, F)$ is called the **associated fibre bundle** of the G -principal bundle P via the given action of G on F .

Definition C.60. Let V be a finite-dimensional real or complex vector space. A fibre bundle (E, π, M, V) with standard fibre V is called a **vector bundle** if each fibre E_p is a vector space over the same field as V and its local trivializations

$$\phi_i : \pi^{-1}(U_i) \longrightarrow U_i \times V$$

can be chosen such that, for every $p \in U_i$, the induced map

$$\phi_{i,p} := \text{proj}_2 \circ \phi_i|_{E_p} : E_p \longrightarrow V$$

is linear.

Let (P, π, M, G) be a G -principal bundle, let V be a finite-dimensional real or complex vector space, and let $\rho : G \rightarrow \text{GL}(V)$ be a smooth representation. The corresponding left action of G on V is given by $g \cdot v := \rho(g)v$. Therefore, the induced right action of G on $P \times V$ is

$$(u, v) \cdot g := (u \cdot g, \rho(g^{-1})v) = (u \cdot g, \rho(g)^{-1}v),$$

where in the last equality we have used that ρ is a group homomorphism. The associated fibre bundle

$$P \times_\rho V := (P \times V)/G = P \times_G V$$

is a vector bundle with standard fibre V .

Definition C.61. The vector bundle $P \times_\rho V$ is called the **associated vector bundle** of the G -principal bundle P via the representation ρ .

Example C.62. Let M be an n -dimensional smooth manifold. Its **frame bundle** is

$$\text{Fr}(M) := \{u : \mathbb{R}^n \longrightarrow T_p M \mid p \in M \text{ and } u \text{ is a linear isomorphism}\},$$

with projection

$$\begin{aligned} \pi_{\text{Fr}} : \quad \text{Fr}(M) &\longrightarrow M \\ u : \mathbb{R}^n \rightarrow T_p M &\longmapsto p. \end{aligned}$$

Equivalently, a frame u over $p \in M$ can be identified with the ordered basis $(u_1, \dots, u_n)$ of $T_p M$ defined by $u_i := u(e_i)$, where $(e_1, \dots, e_n)$ is the standard basis of $\mathbb{R}^n$. Thus,

$$u(x) = x^i u_i \in T_p M$$

for every $x = x^i e_i \in \mathbb{R}^n$.

The frame bundle is a $\text{GL}(n, \mathbb{R})$ -principal bundle with right action

$$u \cdot A := u \circ A.$$

Let $\text{GL}(n, \mathbb{R})$ act on $\mathbb{R}^n$ through its standard representation, $A \cdot x := Ax$. Then the map

$$\begin{aligned} \Phi : \text{Fr}(M) \times_{\text{GL}(n, \mathbb{R})} \mathbb{R}^n &\longrightarrow TM \\ [u, x] &\longmapsto \Phi([u, x]) := (\pi_{\text{Fr}}(u), u(x)) = (\pi_{\text{Fr}}(u), x^i u_i) \end{aligned}$$

is well-defined, since

$$\Phi([u \cdot A, A^{-1}x]) = (\pi_{\text{Fr}}(u \cdot A), (u \circ A)(A^{-1}x)) = (\pi_{\text{Fr}}(u), u(x)) = \Phi([u, x]),$$

and defines an isomorphism of vector bundles. Hence,

$$TM \cong \text{Fr}(M) \times_{\text{GL}(n, \mathbb{R})} \mathbb{R}^n.$$

Thus, the tangent bundle can be regarded as the vector bundle associated with the frame bundle via the standard representation of $\text{GL}(n, \mathbb{R})$ on $\mathbb{R}^n$.

Example C.63. Let (M, g) be an oriented Lorentzian four-manifold endowed with a $\text{spin}^{\mathbb{C}}$ structure, and let Q be the corresponding $\text{Spin}_0^{\mathbb{C}}(3, 1)$ -principal bundle. Let

$$\rho : \text{Spin}_0^{\mathbb{C}}(3, 1) \longrightarrow \text{GL}(\Sigma)$$

be the complex spin representation on the spinor space Σ . The associated vector bundle

$$S := Q \times_{\rho} \Sigma$$

is called the **spinor bundle** of (M, g) . Its smooth sections,

$$\eta \in \Gamma(S),$$

are called **spinor fields**.

If $\Sigma = \Sigma^+ \oplus \Sigma^-$ is the chiral decomposition of the spinor space, then

$$S = S^+ \oplus S^-, \quad S^{\pm} := Q \times_{\rho^{\pm}} \Sigma^{\pm},$$

where $\rho^{\pm}$ denotes the restriction of the spin representation to $\Sigma^{\pm}$. Hence, a spinor field of chirality $\mu \in \mathbb{Z}_2 = \{\pm 1\}$ is a section

$$\eta \in \Gamma(S^{\mu}).$$

We now introduce connections on fibre bundles, focusing on principal bundles and the connections that they induce on associated bundles.

The fibre-bundle structure alone does not determine how a motion on the base manifold should be lifted to the total space. To see this, consider the trivial fibre bundle

$$\pi : S^1 \times \mathbb{R} \longrightarrow S^1.$$

Given a point $(p, y) \in S^1 \times \mathbb{R}$ and a tangent vector $X \in T_p S^1$, there are infinitely many tangent vectors in $T_{(p, y)}(S^1 \times \mathbb{R})$ that project onto X . Indeed, using the canonical identification

$$T_{(p, y)}(S^1 \times \mathbb{R}) \cong T_p S^1 \oplus T_y \mathbb{R} \cong T_p S^1 \oplus \mathbb{R},$$

any vector of the form (X, a) , with $a \in \mathbb{R}$, satisfies

$$(d\pi)_{(p, y)}(X, a) = X.$$

More explicitly,

$$\begin{aligned} (d\pi)_{(p,y)} : T_{(p,y)}(S^1 \times \mathbb{R}) &\cong T_p S^1 \oplus \mathbb{R} \longrightarrow T_p S^1 \\ (X, a) &\longmapsto (d\pi)_{(p,y)}(X, a) = X. \end{aligned}$$

Consequently, all the vectors (X, a) , with $a \in \mathbb{R}$, are lifts of X . Thus, the bundle structure alone does not determine which lift of X should be preferred.

A connection removes this ambiguity infinitesimally by distinguishing, at every point of the total space, the directions tangent to the fibre from a complementary family of directions, called horizontal directions. In the case of a principal bundle, this choice must also be compatible with the action of the structure group.

Let (P, π, M, G) be a G -principal bundle, and let $u \in P$. The differential of the bundle projection is the linear map

$$\begin{aligned} (d\pi)_u : T_u P &\longrightarrow T_{\pi(u)} M \\ Y &\longmapsto (d\pi)_u(Y). \end{aligned}$$

A tangent vector $Y \in T_u P$ produces no motion on the base precisely when $(d\pi)_u(Y) = 0$. Such vectors are therefore tangent to the fibre through u .

Definition C.64. The **vertical subspace** of P at $u \in P$ is

$$V_u P := \ker(d\pi)_u.$$

It is precisely the tangent space to the fibre through u , that is,

$$V_u P = T_u P_{\pi(u)}.$$

In a principal bundle, the vertical directions can be described in terms of the infinitesimal action of the structure group. Indeed, since the fibre through u is precisely the orbit

$$P_{\pi(u)} = u \cdot G,$$

every $A \in \mathfrak{g} := T_e G$ determines a curve in this fibre given by

$$t \longmapsto u \cdot \exp(tA).$$

Definition C.65. Let $A \in \mathfrak{g}$. The **Killing vector field** generated by A is defined by

$$(A_*)_u := \left. \frac{d}{dt} \right|_{t=0} u \cdot \exp(tA), \quad u \in P.$$

Since the fibres of a principal bundle are precisely the orbits of the G -action,

$$\pi(u \cdot \exp(tA)) = \pi(u)$$

for every t . Therefore,

$$(d\pi)_u((A_*)_u) = \left. \frac{d}{dt} \right|_{t=0} \pi(u \cdot \exp(tA)) = \left. \frac{d}{dt} \right|_{t=0} \pi(u) = 0,$$

and hence $(A_*)_u \in V_u P$.

Moreover, every vertical vector at u is generated by a unique element of $\mathfrak{g}$. More precisely, the map

$$\begin{aligned} \mathfrak{g} &\longrightarrow V_u P \\ A &\longmapsto (A_*)_u \end{aligned}$$

is a linear isomorphism. Indeed, the orbit map

$$\begin{aligned} \Phi_u : G &\longrightarrow P_{\pi(u)} \\ g &\longmapsto u \cdot g \end{aligned}$$

is a diffeomorphism, and the above linear map is precisely its differential at the identity (see Remark C.18). Thus, once u is fixed, the infinitesimal action identifies the Lie algebra $\mathfrak{g}$ with the vertical directions at u . In particular, every $Y^V \in V_u P$ can be written uniquely as

$$Y^V = (A_*)_u$$

for some $A \in \mathfrak{g}$. This identification will later allow the connection form associated with a principal connection to encode the vertical component of a tangent vector by an element of the Lie algebra, as formalized in Definition C.71.

We now return to the ambiguity in lifting tangent vectors from the base manifold. Given $X \in T_{\pi(u)} M$, any two vectors in $T_u P$ that project onto X differ by a vertical vector. A connection removes this ambiguity by choosing a subspace $H_u P \subseteq T_u P$ complementary to $V_u P$, whose elements are called horizontal vectors.

Definition C.66. A **connection** on the G -principal bundle (P, π, M, G) , or a **principal connection**, is a smooth assignment

$$u \in P \longmapsto H_u P \subseteq T_u P$$

such that

$$T_u P = H_u P \oplus V_u P$$

for every $u \in P$, and

$$(dR_g)_u (H_u P) = H_{u \cdot g} P$$

for every $u \in P$ and every $g \in G$, where $R_g : P \rightarrow P$ is the right translation $R_g(u) := u \cdot g$. The subspace $H_u P$ is called the **horizontal subspace** at u .

Remark C.67. The vertical subspace

$$V_u P = \ker(d\pi)_u$$

is completely determined by the bundle projection. In contrast, the condition

$$T_u P = H_u P \oplus V_u P$$

does not determine $H_u P$ uniquely, since a vector subspace may admit many different complementary subspaces. For example, if

$$V = \text{span}\{(0, 1)\} \subseteq \mathbb{R}^2,$$

then both

$$H_1 = \text{span} \{(1, 0)\} \quad \text{and} \quad H_2 = \text{span} \{(1, 1)\}$$

satisfy

$$\mathbb{R}^2 = H_i \oplus V.$$

Thus, a principal connection consists precisely in choosing, smoothly and compatibly with the right action of G , one such complementary subspace $H_u P$ at every $u \in P$.

Remark C.68. Since

$$T_u P = H_u P \oplus V_u P,$$

every tangent vector $Y \in T_u P$ admits a unique decomposition

$$Y = Y^H + Y^V, \quad Y^H \in H_u P, \quad Y^V \in V_u P.$$

The vectors Y^H and Y^V are called the **horizontal component** and the **vertical component** of Y , respectively.

Proposition C.69. *Let (P, π, M, G) be a G -principal bundle endowed with a principal connection. Then, for every $u \in P$, the restriction*

$$\begin{aligned} (d\pi)_u|_{H_u P} : H_u P &\longrightarrow T_{\pi(u)} M \\ Y^H &\longmapsto (d\pi)_u(Y^H) \end{aligned}$$

is a linear isomorphism. Consequently, for every $X \in T_{\pi(u)} M$, there exists a unique horizontal vector $X_u^H \in H_u P$ such that

$$(d\pi)_u(X_u^H) = X.$$

Proof. Since $V_u P = \ker(d\pi)_u$ and $T_u P = H_u P \oplus V_u P$, one has

$$\ker((d\pi)_u|_{H_u P}) = H_u P \cap \ker(d\pi)_u = H_u P \cap V_u P = \{0\},$$

so the restriction is injective. It is also surjective because π is a submersion. Indeed, given $X \in T_{\pi(u)} M$, there exists $Y \in T_u P$ such that $(d\pi)_u(Y) = X$. Writing $Y = Y^H + Y^V$, we obtain

$$X = (d\pi)_u(Y) = (d\pi)_u(Y^H + Y^V) = (d\pi)_u(Y^H) + (d\pi)_u(Y^V) = (d\pi)_u(Y^H),$$

since $(d\pi)_u(Y^V) = 0$. □

Definition C.70. The unique vector $X_u^H \in H_u P$ satisfying

$$(d\pi)_u(X_u^H) = X$$

is called the **horizontal lift** of $X \in T_{\pi(u)} M$ at u .

The decomposition into horizontal and vertical components can be encoded by a differential form with values in the Lie algebra $\mathfrak{g}$. Indeed, every vector $Y \in T_u P$ admits a unique decomposition

$$Y = Y^H + Y^V,$$

where $Y^H \in H_u P$ and $Y^V \in V_u P$. As shown above, the map

$$\begin{aligned} \mathfrak{g} &\longrightarrow V_u P \\ A &\longmapsto (A_*)_u \end{aligned}$$

is a linear isomorphism. Therefore, every vertical component $Y^V \in V_u P$ can be written uniquely as

$$Y^V = (A_*)_u$$

for some $A \in \mathfrak{g}$. The connection form assigns to Y precisely the unique element $A \in \mathfrak{g}$ that generates its vertical component.

Definition C.71. Let (P, π, M, G) be a G -principal bundle endowed with a principal connection. The **connection form** associated with this principal connection is the $\mathfrak{g}$ -valued one-form

$$\omega \in \Omega^1(P, \mathfrak{g})$$

defined pointwise by

$$\begin{aligned} \omega_u : T_u P &\longrightarrow \mathfrak{g} \\ Y &\longmapsto \omega_u(Y) := ((d\Phi_u)_e)^{-1}(Y^V) = A, \end{aligned}$$

where $A \in \mathfrak{g}$ is the unique element satisfying

$$Y^V = (A_*)_u.$$

Remark C.72. Although the connection form ω is globally defined on the total space P , it can be described locally by a differential form on the base manifold. Let $U \subseteq M$ be an open subset and let

$$s : U \longrightarrow \pi^{-1}(U)$$

be a local section. The $\mathfrak{g}$ -valued 1-form

$$\mathcal{A} := s^* \omega \in \Omega^1(U, \mathfrak{g})$$

is called the **local representative** of the connection form ω with respect to the section s . Explicitly, for every $p \in U$ and every $X \in T_p M$,

$$\mathcal{A}_p(X) = \omega_{s(p)}((ds)_p(X)).$$

Thus, the choice of a local section allows the globally defined connection form on P to be expressed as a $\mathfrak{g}$ -valued 1-form on an open subset of the base manifold. In gauge theory, $\mathcal{A}$ is interpreted as the local gauge potential. It turns out that a connection on P is uniquely characterized in terms of its local representatives on the base space M [4, Cor. 1.3.12].

The connection form ω is defined on the total space P , whereas the fields appearing in applications are usually sections of vector bundles associated with P . We now explain how ω induces a covariant derivative on every associated vector bundle. In particular, this construction gives rise to the spinorial covariant derivative on a spinor bundle [6, Sec. 3.1].

Let (P, π, M, G) be a G -principal bundle endowed with a connection form

$$\omega \in \Omega^1(P, \mathfrak{g}),$$

let

$$\rho : G \longrightarrow \mathrm{GL}(V)$$

be a finite-dimensional representation, and let

$$E := P \times_{\rho} V$$

be the corresponding associated vector bundle.

Every section $\psi \in \Gamma(E)$ corresponds uniquely to a smooth equivariant map

$$\widehat{\psi} : P \longrightarrow V$$

satisfying

$$\widehat{\psi}(u \cdot g) = \rho(g)^{-1} \widehat{\psi}(u)$$

for every $u \in P$ and every $g \in G$. The correspondence is determined by

$$\psi(\pi(u)) = [u, \widehat{\psi}(u)].$$

The equivariance condition ensures that this expression is independent of the chosen point u in the fibre [4, Prop. 1.2.6].

Let

$$\rho_* := (d\rho)_e : \mathfrak{g} \longrightarrow \mathrm{End}(V)$$

be the representation of the Lie algebra $\mathfrak{g}$ induced by ρ . Recall that the connection form determines the decomposition

$$Y = Y^H + Y^V$$

of every vector $Y \in T_u P$. More precisely, if $A := \omega_u(Y)$, then

$$Y^V = (A_*)_u, \quad Y^H = Y - (A_*)_u.$$

Definition C.73. The **covariant exterior derivative** of $\widehat{\psi}$ with respect to ω is the V -valued 1-form

$$D^\omega \widehat{\psi} \in \Omega^1(P, V)$$

defined by

$$\left(D^\omega \widehat{\psi} \right)_u (Y) := (d\widehat{\psi})_u (Y^H)$$

for every $u \in P$ and every $Y \in T_u P$ [4, Def. 1.4.1].

Using the equivariance of $\widehat{\psi}$ and the definition of the connection form, Proposition 1.4.3 gives the equivalent expression

$$D^\omega \widehat{\psi} = d\widehat{\psi} + \rho_*(\omega)\widehat{\psi}.$$

By Definition 1.4.1, $D^\omega \widehat{\psi}$ is horizontal, and Proposition 1.4.3 shows that it is of type ρ . By Definition 1.2.11, this means that it vanishes on vertical vectors and satisfies

$$R_g^* (D^\omega \widehat{\psi}) = \rho(g^{-1}) \circ D^\omega \widehat{\psi}$$

for every $g \in G$ [4, Def. 1.2.11, Def. 1.4.1, and Prop. 1.4.3]. Equivalently,

$$\left(D^\omega \widehat{\psi} \right)_{u \cdot g} ((dR_g)_u(Y)) = \rho(g^{-1}) \left(D^\omega \widehat{\psi} \right)_u (Y)$$

for every $u \in P$, every $g \in G$, and every $Y \in T_u P$.

At this point, $D^\omega \widehat{\psi}$ is a V -valued 1-form defined on the total space P . In order to obtain a covariant derivative of the section ψ , we must associate with it an E -valued 1-form on the base manifold M . Let $x \in M$ and let $X \in T_x M$. Choose a point $u \in P_x$ and a vector $Y \in T_u P$ satisfying

$$(d\pi)_u(Y) = X.$$

We define

$$(\nabla^\omega \psi)_x(X) := \left[u, \left(D^\omega \widehat{\psi} \right)_u (Y) \right] \in E_x.$$

We first verify that this expression is independent of the chosen lift Y . If $Y' \in T_u P$ is another vector satisfying $(d\pi)_u(Y') = X$, then

$$(d\pi)_u(Y' - Y) = 0,$$

and hence $Y' - Y \in V_u P$ is vertical. Since $D^\omega \widehat{\psi}$ is horizontal,

$$\left(D^\omega \widehat{\psi} \right)_u (Y') - \left(D^\omega \widehat{\psi} \right)_u (Y) = \left(D^\omega \widehat{\psi} \right)_u (Y' - Y) = 0.$$

Therefore, the result does not depend on the choice of Y .

We now verify that it is also independent of the chosen point $u \in P_x$. Any other point in the same fibre is of the form

$$u' = u \cdot g$$

for some $g \in G$. Since $\pi \circ R_g = \pi$, the vector

$$Y' := (dR_g)_u(Y) \in T_{u \cdot g} P$$

also projects onto X . Using the equivariance of $D^\omega \widehat{\psi}$ and the equivalence relation defining the associated bundle, we obtain

$$\left[u', \left(D^\omega \widehat{\psi} \right)_{u'} (Y') \right] = \left[u \cdot g, \rho(g^{-1}) \left(D^\omega \widehat{\psi} \right)_u (Y) \right] = \left[u, \left(D^\omega \widehat{\psi} \right)_u (Y) \right].$$

Thus, the expression depends only on x and X , and therefore defines an E -valued 1-form on M . This is precisely the correspondence of Proposition 1.2.12, specialized to the case of 1-forms [4, Prop. 1.2.12]. We denote this form by

$$\nabla^\omega \psi \in \Omega^1(M, E) \cong \Gamma(T^*M \otimes E).$$

Definition C.74. The resulting map

$$\begin{aligned}\nabla^\omega : \Gamma(E) &\longrightarrow \Gamma(T^*M \otimes E) \\ \psi &\longmapsto \nabla^\omega \psi\end{aligned}$$

is called the **covariant derivative on E induced by the principal connection ω** [4, Defs. 1.5.1–1.5.2].

The induced covariant derivative is linear and satisfies the Leibniz rule

$$\nabla^\omega(f\psi) = df \otimes \psi + f\nabla^\omega \psi$$

for every $f \in C^\infty(M)$ and every $\psi \in \Gamma(E)$ [4, Def. 1.5.2 and Prop. 1.5.4].

Remark C.75. A covariant derivative is often equivalently described in terms of differentiation along vector fields. For every $X \in \mathfrak{X}(M) = \Gamma(TM)$ and every $\psi \in \Gamma(E)$, one defines

$$\nabla_X^\omega \psi := (\nabla^\omega \psi)(X) \in \Gamma(E),$$

or, pointwise,

$$(\nabla_X^\omega \psi)(x) := (\nabla^\omega \psi)_x(X_x).$$

Hence, the covariant derivative may equivalently be viewed as a map

$$\begin{aligned}\nabla^\omega : \mathfrak{X}(M) \times \Gamma(E) &\longrightarrow \Gamma(E) \\ (X, \psi) &\longmapsto \nabla_X^\omega \psi.\end{aligned}$$

This formulation satisfies

$$\nabla_{fX}^\omega \psi = f\nabla_X^\omega \psi$$

and

$$\nabla_X^\omega(f\psi) = X(f)\psi + f\nabla_X^\omega \psi$$

for every $f \in C^\infty(M)$, every $X \in \mathfrak{X}(M)$, and every $\psi \in \Gamma(E)$ [4, Eq. (1.5.10) and Prop. 1.5.8].

Definition C.76. A section $\psi \in \Gamma(E)$ is called **parallel** with respect to ∇^ω , or simply **parallel** when the connection is understood, if

$$\nabla^\omega \psi = 0.$$

D Metric components

In this appendix, we compute the explicit components of the metric

$$g = -(R^{-2}\psi^2 + \mathfrak{c})du \otimes du + du \odot (d\mathfrak{b} + *_h d\psi) + h. \quad (\text{D.1})$$

For one-forms $\alpha, \beta \in \Omega^1(M)$, we use the conventions

$$\alpha \wedge \beta := \alpha \otimes \beta - \beta \otimes \alpha, \quad \alpha \odot \beta := \alpha \otimes \beta + \beta \otimes \alpha.$$

First, let us compute $*_h d\psi$. The round metric on S^2 of radius R in spherical coordinates is

$$h = R^2 d\theta \otimes d\theta + R^2 \sin^2(\theta) d\phi \otimes d\phi.$$

We choose the orientation for which an oriented orthonormal coframe $\{e^1, e^2\}$ satisfies

$$*_h e^1 = e^2, \quad *_h e^2 = -e^1.$$

The one-forms $d\theta$ and $d\phi$ are orthogonal but not normalized. Indeed, using the inverse matrix of h ,

$$|d\theta|_h = \sqrt{\langle d\theta, d\theta \rangle_h} = \sqrt{\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{R^2} & 0 \\ 0 & \frac{1}{R^2 \sin^2(\theta)} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}} = \frac{1}{R},$$

and

$$|d\phi|_h = \sqrt{\langle d\phi, d\phi \rangle_h} = \sqrt{\begin{pmatrix} 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{R^2} & 0 \\ 0 & \frac{1}{R^2 \sin^2(\theta)} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix}} = \frac{1}{R \sin(\theta)},$$

while

$$\langle d\theta, d\phi \rangle_h = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{R^2} & 0 \\ 0 & \frac{1}{R^2 \sin^2(\theta)} \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0.$$

Hence, an orthonormal coframe is given by

$$e^1 := R d\theta, \quad e^2 := R \sin(\theta) d\phi.$$

Therefore,

$$*_h d\theta = *_h \left(\frac{1}{R} e^1 \right) = \frac{1}{R} e^2 = \sin(\theta) d\phi,$$

and

$$*_h d\phi = *_h \left(\frac{1}{R \sin(\theta)} e^2 \right) = -\frac{1}{R \sin(\theta)} e^1 = -\csc(\theta) d\theta.$$

Since

$$d\psi = \psi_\theta d\theta + \psi_\phi d\phi,$$

where $\psi_\theta := \partial_\theta \psi$ and $\psi_\phi := \partial_\phi \psi$, we obtain

$$*_h d\psi = \psi_\theta *_h d\theta + \psi_\phi *_h d\phi = -\psi_\phi \csc(\theta) d\theta + \psi_\theta \sin(\theta) d\phi. \quad (\text{D.2})$$

Substituting this expression into (D.1), we get

$$\begin{aligned} g &= -(R^{-2}\psi^2 + \mathbf{c}) du \otimes du + du \odot d\mathbf{v} + du \odot *_h d\psi + R^2 d\theta \otimes d\theta + R^2 \sin^2(\theta) d\phi \otimes d\phi \\ &= -(R^{-2}\psi^2 + \mathbf{c}) du \otimes du + du \otimes d\mathbf{v} + d\mathbf{v} \otimes du - \psi_\phi \csc(\theta) (du \otimes d\theta + d\theta \otimes du) \\ &\quad + \psi_\theta \sin(\theta) (du \otimes d\phi + d\phi \otimes du) + R^2 d\theta \otimes d\theta + R^2 \sin^2(\theta) d\phi \otimes d\phi. \end{aligned}$$

Therefore, with respect to the coordinates $(u, \mathbf{v}, \theta, \phi)$, the matrix of metric components is

$$(g_{\mu\nu}) = \begin{pmatrix} -(R^{-2}\psi^2 + \mathbf{c}) & 1 & -\psi_\phi \csc(\theta) & \psi_\theta \sin(\theta) \\ 1 & 0 & 0 & 0 \\ -\psi_\phi \csc(\theta) & 0 & R^2 & 0 \\ \psi_\theta \sin(\theta) & 0 & 0 & R^2 \sin^2(\theta) \end{pmatrix}. \quad (\text{D.3})$$

E Integration of the equation $\dot{x}^2 = P(x)$

In this appendix, we give the explicit integration of the equation for θ . Let us introduce $x := \cos(\theta)$. Then, the differential equation (4.23) is equivalent to

$$\dot{x}^2 = a_3 x^3 + a_2 x^2 + a_1 x + a_0 =: P(x), \quad (\text{E.1})$$

where

$$\begin{aligned} a_0 &= \frac{2E}{R^2} - \frac{\ell^2}{R^4}, & a_1 &= -\frac{2q}{R^2} [\alpha \rho (\mu \lambda_I - 1) + \mu \lambda_I \ell], \\ a_2 &= -\left(\frac{2E}{R^2} + q^2 \mu^2 \lambda_I^2 \right), & a_3 &= \frac{2}{R^2} q \alpha \rho (\mu \lambda_I - 1). \end{aligned} \quad (\text{E.2})$$

The admissible solutions are subject to the constraints

$$x(\tau) \in [-1, 1], \quad P(x(\tau)) \geq 0. \quad (\text{E.3})$$

Once $x(\tau)$ is known, the angular variable is recovered as

$$\theta(\tau) = \arccos(x(\tau)). \quad (\text{E.4})$$

We first classify (E.1) as a real differential equation with arbitrary real coefficients a_i . Only then will we impose the specific form (E.2) of the coefficients coming from the present problem.

Before starting the classification, let us clarify the role of (E.1). The actual reduced dynamics comes from the one-dimensional effective Lagrangian (4.19), which can be written as

$$L_{\text{eff}}(\theta, \dot{\theta}) = \frac{1}{2} R^2 \dot{\theta}^2 - V_{\text{eff}}(\theta).$$

Therefore, the Euler–Lagrange equation is a second-order equation,

$$R^2 \ddot{\theta} = -\frac{dV_{\text{eff}}}{d\theta}.$$

On the other hand, since L_{eff} does not depend explicitly on τ , the effective energy is conserved:

$$E = \frac{1}{2} R^2 \dot{\theta}^2 + V_{\text{eff}}(\theta).$$

After introducing $x := \cos(\theta)$, this conservation law becomes precisely (E.1), $\dot{x}^2 = P(x)$. Thus, (E.1) is the energy equation associated with the second-order reduced dynamics. This is a key point because the equation $\dot{x}^2 = P(x)$ only fixes the square of the velocity. It does not determine, by itself, whether the motion goes towards increasing or decreasing values of x , nor whether a point with $\dot{x} = 0$ is an equilibrium or a turning point. In particular, a root of P formally gives $\dot{x} = 0$, but this only means that the velocity vanishes at that value. For this value to define a constant branch, the acceleration must vanish as well. The extra condition needed is obtained from the associated second-order equation.

Let us therefore write the second-order equation in terms of the variable x . In what follows, we write $V_{\text{eff}}(x)$ instead of $V_{\text{eff}}(\arccos(x))$. Since

$$\dot{x} = -\sin(\theta)\dot{\theta},$$

we have

$$\ddot{x} = -\cos(\theta)\dot{\theta}^2 - \sin(\theta)\ddot{\theta}.$$

After using $x = \cos(\theta)$, the energy equation, the Euler–Lagrange equation, and the chain rule

$$\frac{dV_{\text{eff}}}{d\theta} = \frac{dV_{\text{eff}}}{dx} \frac{dx}{d\theta} = -\sin(\theta) \frac{dV_{\text{eff}}}{dx},$$

the equation above becomes

$$\ddot{x} = -\frac{2x}{R^2} (E - V_{\text{eff}}(x)) - \frac{1-x^2}{R^2} \frac{dV_{\text{eff}}}{dx}.$$

On the other hand, from the energy equation,

$$\dot{x}^2 = \frac{2}{R^2} (1-x^2) (E - V_{\text{eff}}(x)),$$

and comparing with (E.1), we identify the RHS with $P(x)$, and hence

$$\frac{1}{2}P'(x) = -\frac{2x}{R^2} (E - V_{\text{eff}}(x)) - \frac{1-x^2}{R^2} \frac{dV_{\text{eff}}}{dx}.$$

Therefore, the second-order equation in the variable x is

$$\ddot{x} = \frac{1}{2}P'(x). \tag{E.5}$$

The above discussion is the key point for the treatment of constant branches. If $x(\tau) = x_*$ is constant, then

$$\dot{x} = 0, \quad \ddot{x} = 0.$$

Therefore, on the one hand, the energy equation gives

$$P(x_*) = 0,$$

and, on the other hand, the second-order equation (E.5) gives

$$P'(x_*) = 0.$$

Hence, unless $P(x) = 0$ for all x , a constant branch must correspond to a multiple root of P . Simple roots are not equilibrium points of the reduced dynamics, but rather turning points of non-constant branches.

There is also a related uniqueness subtlety. If one writes (E.1) as

$$\dot{x} = \pm\sqrt{P(x)},$$

then the RHS is not Lipschitz at a simple root of P . Indeed,

$$\frac{\partial}{\partial x} \left(\pm\sqrt{P(x)} \right) = \pm \frac{P'(x)}{2\sqrt{P(x)}},$$

which diverges at a simple root. Hence, the Picard–Lindelöf uniqueness theorem for first-order systems cannot be applied directly at a turning point. The changes of variables used below regularize the motion at the turning points and leave the Cauchy problem well defined [28, Sec. 2.2].

E.1 Algebraic classification for arbitrary coefficients

E.1.1 Constant polynomial

Assume first that $\deg P = 0$. We need to solve the following system:

$$\dot{x}^2 = a_0 = P(x) \geq 0, \quad x \in [-1, 1].$$

Case $a_0 > 0$. In this case,

$$\dot{x} = \pm\sqrt{a_0},$$

and hence

$$x(\tau) = \pm\sqrt{a_0}\tau + C.$$

This branch is unbounded and therefore leaves the admissible interval $x \in [-1, 1]$. Hence, it is discarded as an admissible branch.

Case $a_0 = 0$. In this case $\dot{x} = 0$, and therefore

$$x(\tau) = C, \quad -1 \leq C \leq 1. \quad (\text{E.6})$$

This is the static solution and it is admissible.

E.1.2 Linear polynomial

Assume now that $\deg P = 1$. We need to solve the following system:

$$\dot{x}^2 = a_1x + a_0 = P(x) \geq 0, \quad x \in [-1, 1], \quad a_1 \neq 0.$$

Let

$$r_1 := -\frac{a_0}{a_1}$$

be the unique real root. Then,

$$\dot{x}^2 = a_1(x - r_1) = P(x) \geq 0.$$

Case $a_1 > 0$. The allowed region is

$$x \geq r_1.$$

Taking the square root,

$$\dot{x} = \pm\sqrt{a_1}\sqrt{x - r_1}.$$

Integrating,

$$\int \frac{dx}{\sqrt{x - r_1}} = \pm\sqrt{a_1} \int d\tau,$$

and hence,

$$2\sqrt{x - r_1} = \pm\sqrt{a_1}\tau + C'.$$

After redefining the integration constant, we obtain

$$x(\tau) = r_1 + \frac{a_1}{4}(\tau + C)^2.$$

This parabola opens upwards and diverges to $+\infty$, so it is not an admissible branch.

Case $a_1 < 0$. The allowed region is

$$x \leq r_1.$$

Taking the square root,

$$\dot{x} = \pm \sqrt{-a_1} \sqrt{r_1 - x},$$

and integrating,

$$x(\tau) = r_1 + \frac{a_1}{4}(\tau + C)^2.$$

Since $a_1 < 0$, the parabola opens downwards and diverges to $-\infty$. This branch is also discarded as an admissible branch.

Thus, in the linear case there are no admissible branches.

E.1.3 Quadratic polynomial

Assume now that $\deg P = 2$. We need to solve the following system:

$$\dot{x}^2 = a_2 x^2 + a_1 x + a_0 = P(x) \geq 0, \quad x \in [-1, 1], \quad a_2 \neq 0.$$

We distinguish the cases according to the real root structure of P .

Two distinct real roots. Assume that P has two distinct real roots $r_1 < r_2$. Then,

$$\dot{x}^2 = a_2(x - r_1)(x - r_2) = P(x) \geq 0.$$

Case $a_2 > 0$: In this case,

$$P(x) \geq 0 \iff x \in (-\infty, r_1] \cup [r_2, \infty).$$

On the right branch, $x \in [r_2, \infty)$, we set

$$x(\tau) = r_1 + (r_2 - r_1) \cosh^2(\varphi(\tau)),$$

because

$$\varphi = 0 \implies x = r_2, \quad \varphi \rightarrow \infty \implies x \rightarrow \infty.$$

Substituting in the differential equation yields

$$\dot{\varphi} = \pm \frac{\sqrt{a_2}}{2}.$$

Remark E.1. We now see that the apparent violation of the Picard–Lindelöf theorem at the turning points of the original equation is a coordinate artifact. With this change of variable, the problem disappears because the derivative of the RHS with respect to φ does not diverge, guaranteeing the existence and uniqueness of the solution.

Integrating,

$$\varphi(\tau) = \pm \frac{\sqrt{a_2}}{2} \tau + C,$$

and therefore,

$$x(\tau) = r_1 + (r_2 - r_1) \cosh^2 \left(\frac{\sqrt{a_2}}{2} \tau + C \right).$$

This branch satisfies $x \geq r_2$ and diverges to $+\infty$.

On the left branch, $x \in (-\infty, r_1]$, we set

$$x(\tau) = r_2 - (r_2 - r_1) \cosh^2(\varphi(\tau)),$$

because

$$\varphi = 0 \implies x = r_1, \quad \varphi \rightarrow \infty \implies x \rightarrow -\infty.$$

This gives, by analogous calculation,

$$x(\tau) = r_2 - (r_2 - r_1) \cosh^2 \left(\frac{\sqrt{a_2}}{2} \tau + C \right).$$

This branch satisfies $x \leq r_1$ and diverges to $-\infty$. Both branches are therefore discarded as admissible branches on the sphere, again because they leave $x \in [-1, 1]$.

Case $a_2 < 0$: In this case,

$$P(x) \geq 0 \iff x \in [r_1, r_2].$$

We set

$$x(\tau) = r_1 + (r_2 - r_1) \sin^2(\varphi(\tau)),$$

since

$$\varphi = 0 \implies x = r_1, \quad \varphi = \frac{\pi}{2} \implies x = r_2.$$

Then,

$$\dot{x}^2 = 4(r_2 - r_1)^2 \sin^2(\varphi) \cos^2(\varphi) \dot{\varphi}^2,$$

whereas

$$x - r_1 = (r_2 - r_1) \sin^2(\varphi), \quad x - r_2 = -(r_2 - r_1) \cos^2(\varphi).$$

Hence,

$$\dot{\varphi}^2 = -\frac{a_2}{4},$$

and therefore

$$\varphi(\tau) = \pm \frac{\sqrt{-a_2}}{2} \tau + C.$$

Since $\sin^2$ is even, we can absorb the sign into the integration constant and obtain

$$x(\tau) = r_1 + (r_2 - r_1) \sin^2 \left(\frac{\sqrt{-a_2}}{2} \tau + C \right), \quad -1 \leq r_1 < r_2 \leq 1. \quad (\text{E.7})$$

This solution oscillates between the two turning points r_1 and r_2 . It is admissible provided

$$-1 \leq r_1 < r_2 \leq 1.$$

Double real root. Assume now that P has a double real root r . Then,

$$\dot{x}^2 = a_2(x - r)^2 = P(x) \geq 0.$$

Case $a_2 > 0$: In this case,

$$P(x) \geq 0 \iff x \in \mathbb{R}.$$

Hence,

$$\dot{x} = \pm\sqrt{a_2}(x - r),$$

and therefore,

$$x(\tau) = r + Ce^{\pm\sqrt{a_2}\tau}.$$

The exponentially growing branch leaves the admissible interval. The decaying branch can be written, for an initial condition $x(0) = x_0$, as

$$x(\tau) = r + (x_0 - r)e^{-\sqrt{a_2}\tau}, \quad -1 \leq r \leq 1, \quad -1 \leq x_0 \leq 1. \quad (\text{E.8})$$

This solution approaches the double root r asymptotically as $\tau \rightarrow \infty$. It is admissible for $\tau \geq 0$ provided

$$-1 \leq r \leq 1, \quad -1 \leq x_0 \leq 1.$$

Case $a_2 < 0$: In this case,

$$P(x) \geq 0 \iff x = r.$$

Therefore,

$$x(\tau) = r, \quad -1 \leq r \leq 1. \quad (\text{E.9})$$

No real roots. Assume finally that the quadratic polynomial has no real roots. Then, we can write

$$P(x) = a_2Q(x), \quad Q(x) := x^2 + px + q, \quad p^2 - 4q < 0.$$

Since Q has no real roots and has positive leading coefficient, we have

$$Q(x) > 0 \quad \text{for every } x \in \mathbb{R}.$$

Case $a_2 > 0$: In this case,

$$\dot{x}^2 = a_2Q(x) > 0 \quad \text{for every } x \in \mathbb{R}.$$

Thus, the velocity never vanishes and the motion has no turning points. Consequently, the corresponding branch cannot remain confined in the admissible interval $x \in [-1, 1]$ and is discarded.

Case $a_2 < 0$: In this case,

$$\dot{x}^2 = a_2Q(x) < 0 \quad \text{for every } x \in \mathbb{R}.$$

Therefore there are no real solutions, and this case is discarded.

E.1.4 Cubic polynomial

Assume now that $\deg P = 3$. We need to solve the following system:

$$\dot{x}^2 = a_3x^3 + a_2x^2 + a_1x + a_0 = P(x) \geq 0, \quad x \in [-1, 1], \quad a_3 \neq 0.$$

We distinguish the cases according to the real root structure of P .

Three distinct real roots. Assume first that P has three distinct real roots $r_1 < r_2 < r_3$. Then,

$$\dot{x}^2 = a_3(x - r_1)(x - r_2)(x - r_3) = P(x) \geq 0.$$

Case $a_3 > 0$: In this case,

$$P(x) \geq 0 \iff x \in [r_1, r_2] \cup [r_3, \infty).$$

On the unbounded branch, $x \in [r_3, \infty)$, we set

$$x(\tau) = r_3 + (r_3 - r_2) \tan^2(\varphi(\tau)),$$

because

$$\varphi = 0 \implies x = r_3, \quad \varphi \rightarrow \frac{\pi}{2} \implies x \rightarrow \infty.$$

Substituting in the differential equation yields

$$\dot{\varphi} = \pm \sqrt{\frac{a_3(r_3 - r_1)}{4}} \sqrt{1 - \frac{r_2 - r_1}{r_3 - r_1} \sin^2(\varphi)}.$$

Hence,

$$F\left(\sqrt{\frac{r_2 - r_1}{r_3 - r_1}}, \varphi\right) = \pm \frac{\sqrt{a_3(r_3 - r_1)}}{2} \tau + C,$$

where $F(k, \varphi)$ denotes the incomplete elliptic integral of the first kind in Schaum's convention [29, Eq. 35.1]. Therefore,

$$\varphi(\tau) = \text{am}\left(\pm \frac{\sqrt{a_3(r_3 - r_1)}}{2} \tau + C, \sqrt{\frac{r_2 - r_1}{r_3 - r_1}}\right).$$

Using

$$\tan^2(\varphi(\tau)) = \frac{\sin^2(\varphi(\tau))}{\cos^2(\varphi(\tau))},$$

together with the definitions of the Jacobi elliptic functions [29, Eqs. 35.11–35.12],

$$\sin(\text{am}(u, k)) = \text{sn}(u, k), \quad \cos(\text{am}(u, k)) = \text{cn}(u, k),$$

we get

$$\tan^2(\varphi(\tau)) = \frac{\text{sn}^2\left(\pm \frac{\sqrt{a_3(r_3 - r_1)}}{2} \tau + C, \sqrt{\frac{r_2 - r_1}{r_3 - r_1}}\right)}{\text{cn}^2\left(\pm \frac{\sqrt{a_3(r_3 - r_1)}}{2} \tau + C, \sqrt{\frac{r_2 - r_1}{r_3 - r_1}}\right)}.$$

Finally, since the square removes the sign and the integration constant can be redefined if necessary, we obtain

$$x(\tau) = r_3 + (r_3 - r_2) \frac{\operatorname{sn}^2 \left(\frac{\sqrt{a_3(r_3 - r_1)}}{2} \tau + C, \sqrt{\frac{r_2 - r_1}{r_3 - r_1}} \right)}{\operatorname{cn}^2 \left(\frac{\sqrt{a_3(r_3 - r_1)}}{2} \tau + C, \sqrt{\frac{r_2 - r_1}{r_3 - r_1}} \right)}.$$

This parametrization describes the unbounded component $[r_3, \infty)$. Since the quotient between sn^2 and cn^2 diverges when the denominator vanishes, the branch eventually leaves the admissible interval $x \in [-1, 1]$ and is discarded.

On the bounded branch, $x \in [r_1, r_2]$, we set

$$x(\tau) = r_1 + (r_2 - r_1) \sin^2(\varphi(\tau)),$$

since

$$\varphi = 0 \implies x = r_1, \quad \varphi = \frac{\pi}{2} \implies x = r_2.$$

Substituting in the differential equation yields

$$\dot{\varphi} = \pm \sqrt{\frac{a_3(r_3 - r_1)}{4}} \sqrt{1 - \frac{r_2 - r_1}{r_3 - r_1} \sin^2(\varphi)}.$$

Hence,

$$F \left(\sqrt{\frac{r_2 - r_1}{r_3 - r_1}}, \varphi \right) = \pm \frac{\sqrt{a_3(r_3 - r_1)}}{2} \tau + C.$$

Therefore,

$$\varphi(\tau) = \operatorname{am} \left(\pm \frac{\sqrt{a_3(r_3 - r_1)}}{2} \tau + C, \sqrt{\frac{r_2 - r_1}{r_3 - r_1}} \right).$$

Using the definition of the Jacobi elliptic sine and absorbing the sign into the integration constant, we obtain

$$x(\tau) = r_1 + (r_2 - r_1) \operatorname{sn}^2 \left(\frac{\sqrt{a_3(r_3 - r_1)}}{2} \tau + C, \sqrt{\frac{r_2 - r_1}{r_3 - r_1}} \right), \quad -1 \leq r_1 < r_2 \leq 1. \quad (\text{E.10})$$

This solution oscillates between the two turning points r_1 and r_2 . It is admissible provided

$$-1 \leq r_1 < r_2 \leq 1.$$

Case $a_3 < 0$: In this case,

$$P(x) \geq 0 \iff x \in (-\infty, r_1] \cup [r_2, r_3].$$

On the unbounded branch, $x \in (-\infty, r_1]$, we set

$$x(\tau) = r_1 - (r_2 - r_1) \tan^2(\varphi(\tau)),$$

because

$$\varphi = 0 \implies x = r_1, \quad \varphi \rightarrow \frac{\pi}{2} \implies x \rightarrow -\infty.$$

Substituting in the differential equation yields

$$\dot{\varphi} = \pm \sqrt{\frac{-a_3(r_3 - r_1)}{4}} \sqrt{1 - \frac{r_3 - r_2}{r_3 - r_1} \sin^2(\varphi)}.$$

Hence,

$$F\left(\sqrt{\frac{r_3 - r_2}{r_3 - r_1}}, \varphi\right) = \pm \frac{\sqrt{-a_3(r_3 - r_1)}}{2} \tau + C.$$

Therefore,

$$\varphi(\tau) = \text{am}\left(\pm \frac{\sqrt{-a_3(r_3 - r_1)}}{2} \tau + C, \sqrt{\frac{r_3 - r_2}{r_3 - r_1}}\right).$$

Using the same argument as before, we obtain

$$x(\tau) = r_1 - (r_2 - r_1) \frac{\text{sn}^2\left(\frac{\sqrt{-a_3(r_3 - r_1)}}{2} \tau + C, \sqrt{\frac{r_3 - r_2}{r_3 - r_1}}\right)}{\text{cn}^2\left(\frac{\sqrt{-a_3(r_3 - r_1)}}{2} \tau + C, \sqrt{\frac{r_3 - r_2}{r_3 - r_1}}\right)}.$$

This parametrization describes the unbounded component $(-\infty, r_1]$. Since the quotient between sn^2 and cn^2 diverges when the denominator vanishes, the branch eventually leaves the admissible interval $x \in [-1, 1]$ and is discarded.

On the bounded branch, $x \in [r_2, r_3]$, we set

$$x(\tau) = r_3 - (r_3 - r_2) \sin^2(\varphi(\tau)),$$

since

$$\varphi = 0 \implies x = r_3, \quad \varphi = \frac{\pi}{2} \implies x = r_2.$$

Substituting in the differential equation yields

$$\dot{\varphi} = \pm \sqrt{\frac{-a_3(r_3 - r_1)}{4}} \sqrt{1 - \frac{r_3 - r_2}{r_3 - r_1} \sin^2(\varphi)}.$$

Hence,

$$F\left(\sqrt{\frac{r_3 - r_2}{r_3 - r_1}}, \varphi\right) = \pm \frac{\sqrt{-a_3(r_3 - r_1)}}{2} \tau + C.$$

Therefore,

$$\varphi(\tau) = \text{am}\left(\pm \frac{\sqrt{-a_3(r_3 - r_1)}}{2} \tau + C, \sqrt{\frac{r_3 - r_2}{r_3 - r_1}}\right).$$

Using the definition of the Jacobi elliptic sine and absorbing the sign into the integration constant, we obtain

$$x(\tau) = r_3 - (r_3 - r_2) \text{sn}^2\left(\frac{\sqrt{-a_3(r_3 - r_1)}}{2} \tau + C, \sqrt{\frac{r_3 - r_2}{r_3 - r_1}}\right), \quad -1 \leq r_2 < r_3 \leq 1. \quad (\text{E.11})$$

This solution oscillates between the two turning points r_2 and r_3 . It is admissible provided

$$-1 \leq r_2 < r_3 \leq 1.$$

One double root and one simple root. Assume now that the cubic polynomial has one double real root and one simple real root.

If r_d denotes the double root, then

$$x(\tau) = r_d, \quad -1 \leq r_d \leq 1. \quad (\text{E.12})$$

Case $r_1 = r_2 < r_3$: In this case,

$$\dot{x}^2 = a_3(x - r_1)^2(x - r_3) = P(x) \geq 0.$$

If $a_3 > 0$, then

$$P(x) \geq 0 \iff x = r_1 \quad \text{or} \quad x \geq r_3.$$

The value $x = r_1$ gives the static solution. On the non-constant branch, $x \in [r_3, \infty)$, we set

$$x(\tau) = r_3 + (r_3 - r_1) \tan^2(\varphi(\tau)).$$

Substituting in the differential equation yields

$$\dot{\varphi} = \pm \frac{\sqrt{a_3(r_3 - r_1)}}{2}.$$

Integrating,

$$\varphi(\tau) = \pm \frac{\sqrt{a_3(r_3 - r_1)}}{2} \tau + C.$$

Therefore,

$$x(\tau) = r_3 + (r_3 - r_1) \tan^2 \left(\frac{\sqrt{a_3(r_3 - r_1)}}{2} \tau + C \right).$$

This parametrization describes the unbounded component $[r_3, \infty)$. Since $\tan^2$ diverges, the branch eventually leaves the admissible interval $x \in [-1, 1]$ and is discarded. The static solution $x(\tau) = r_1$ is admissible if $r_1 \in [-1, 1]$.

If $a_3 < 0$, then

$$P(x) \geq 0 \iff x \leq r_3.$$

Since $r_1 = r_2$ is a double root, we distinguish the three regions

$$x < r_1, \quad x = r_1, \quad r_1 < x \leq r_3.$$

First, consider the region $x < r_1$. In this region the motion belongs to the unbounded component $(-\infty, r_1)$. Equivalently, one may parametrize it by

$$x(\tau) = r_3 - (r_3 - r_1) \coth^2(\varphi(\tau)),$$

because

$$\varphi \rightarrow \infty \implies x \rightarrow r_1, \quad \varphi \rightarrow 0 \implies x \rightarrow -\infty.$$

Substituting in the differential equation yields

$$\dot{\varphi} = \pm \frac{\sqrt{-a_3(r_3 - r_1)}}{2}.$$

Therefore,

$$x(\tau) = r_3 - (r_3 - r_1) \coth^2 \left(\frac{\sqrt{-a_3(r_3 - r_1)}}{2} \tau + C \right).$$

This parametrization describes the unbounded component $(-\infty, r_1)$. Since $\coth^2$ has a pole, the branch eventually leaves the admissible interval $x \in [-1, 1]$ and is discarded.

Second, if $x = r_1$, then

$$\dot{x} = 0,$$

and we obtain the static solution

$$x(\tau) = r_1.$$

This solution is admissible if $r_1 \in [-1, 1]$.

Finally, consider the bounded region $r_1 < x \leq r_3$. Since this is an interval between two roots, we may first try the change of variable

$$x(\tau) = r_1 + (r_3 - r_1) \sin^2(\varphi(\tau)).$$

Then,

$$x - r_1 = (r_3 - r_1) \sin^2(\varphi), \quad x - r_3 = -(r_3 - r_1) \cos^2(\varphi),$$

and

$$\dot{x}^2 = 4(r_3 - r_1)^2 \sin^2(\varphi) \cos^2(\varphi) \dot{\varphi}^2.$$

Substituting in

$$\dot{x}^2 = a_3(x - r_1)^2(x - r_3),$$

we obtain

$$4(r_3 - r_1)^2 \sin^2(\varphi) \cos^2(\varphi) \dot{\varphi}^2 = -a_3(r_3 - r_1)^3 \sin^4(\varphi) \cos^2(\varphi).$$

Therefore, for $r_1 < x < r_3$,

$$\dot{\varphi} = \pm \frac{\sqrt{-a_3(r_3 - r_1)}}{2} \sin(\varphi).$$

Hence,

$$\int \frac{d\varphi}{\sin(\varphi)} = \pm \frac{\sqrt{-a_3(r_3 - r_1)}}{2} \int d\tau.$$

Using

$$\int \frac{d\varphi}{\sin(\varphi)} = \log \left| \tan \left(\frac{\varphi}{2} \right) \right|,$$

we get

$$\tan \left(\frac{\varphi}{2} \right) = C \exp \left(\pm \frac{\sqrt{-a_3(r_3 - r_1)}}{2} \tau \right).$$

After redefining the integration constant, this can be written as

$$\tan \left(\frac{\varphi}{2} \right) = \exp \left(\frac{\sqrt{-a_3(r_3 - r_1)}}{2} \tau + C \right).$$

Using the identity

$$\sin(\varphi) = \frac{2 \tan(\varphi/2)}{1 + \tan^2(\varphi/2)},$$

and writing

$$u := \frac{\sqrt{-a_3(r_3 - r_1)}}{2} \tau + C,$$

we obtain

$$\begin{aligned} \sin^2(\varphi(\tau)) &= \left(\frac{2e^u}{1 + e^{2u}} \right)^2 \\ &= \left(\frac{2}{e^{-u} + e^u} \right)^2 \\ &= \operatorname{sech}^2(u). \end{aligned}$$

Therefore,

$$x(\tau) = r_1 + (r_3 - r_1) \operatorname{sech}^2 \left(\frac{\sqrt{-a_3(r_3 - r_1)}}{2} \tau + C \right).$$

Equivalently, using $\operatorname{sech}^2(u) = 1 - \tanh^2(u)$, we obtain

$$x(\tau) = r_3 - (r_3 - r_1) \tanh^2 \left(\frac{\sqrt{-a_3(r_3 - r_1)}}{2} \tau + C \right), \quad -1 \leq r_1 = r_2 < r_3 \leq 1. \quad (\text{E.13})$$

This solution reaches the simple root r_3 as a turning point and approaches the double root r_1 asymptotically. Since $\tanh^2$ is bounded between 0 and 1, the whole branch remains in $[r_1, r_3]$. It is admissible provided

$$-1 \leq r_1 = r_2 < r_3 \leq 1.$$

Remark E.2. Equivalently, one can obtain the final expression more directly by using the reparametrization

$$x(\tau) = r_3 - (r_3 - r_1) \tanh^2(\varphi(\tau)).$$

However, starting from the trigonometric parametrization makes clear that, although the branch lies between two real roots, the double root prevents an oscillatory solution and produces an asymptotic approach instead.

Case $r_1 < r_2 = r_3$: In this case,

$$\dot{x}^2 = a_3(x - r_1)(x - r_3)^2 = P(x) \geq 0.$$

If $a_3 < 0$, then

$$P(x) \geq 0 \iff x \leq r_1 \quad \text{or} \quad x = r_3.$$

The value $x = r_3$ gives the static solution. On the non-constant branch, $x \in (-\infty, r_1]$, we set

$$x(\tau) = r_1 - (r_3 - r_1) \tan^2(\varphi(\tau)).$$

Substituting in the differential equation yields

$$\dot{\varphi} = \pm \frac{\sqrt{-a_3(r_3 - r_1)}}{2}.$$

Integrating,

$$\varphi(\tau) = \pm \frac{\sqrt{-a_3(r_3 - r_1)}}{2} \tau + C.$$

Therefore,

$$x(\tau) = r_1 - (r_3 - r_1) \tan^2 \left(\frac{\sqrt{-a_3(r_3 - r_1)}}{2} \tau + C \right).$$

This parametrization describes the unbounded component $(-\infty, r_1]$. Since $\tan^2$ diverges, the branch eventually leaves the admissible interval $x \in [-1, 1]$ and is discarded. The static solution $x(\tau) = r_3$ is admissible if $r_3 \in [-1, 1]$.

If $a_3 > 0$, then

$$P(x) \geq 0 \iff x \geq r_1.$$

The unbounded part $x > r_3$ is discarded. On the bounded branch, $x \in [r_1, r_3]$, we set

$$x(\tau) = r_1 + (r_3 - r_1) \tanh^2(\varphi(\tau)),$$

because

$$\varphi = 0 \implies x = r_1, \quad \varphi \rightarrow \infty \implies x \rightarrow r_3.$$

Substituting in the differential equation yields

$$\dot{\varphi} = \pm \frac{\sqrt{a_3(r_3 - r_1)}}{2}.$$

Integrating,

$$\varphi(\tau) = \pm \frac{\sqrt{a_3(r_3 - r_1)}}{2} \tau + C.$$

Hence,

$$x(\tau) = r_1 + (r_3 - r_1) \tanh^2 \left(\frac{\sqrt{a_3(r_3 - r_1)}}{2} \tau + C \right), \quad -1 \leq r_1 < r_2 = r_3 \leq 1. \quad (\text{E.14})$$

This solution reaches the simple root r_1 as a turning point and approaches the double root r_3 asymptotically. It is admissible provided

$$-1 \leq r_1 < r_2 = r_3 \leq 1.$$

Triple real root. Assume now that P has a triple real root r . Then,

$$\dot{x}^2 = a_3(x - r)^3 = P(x) \geq 0.$$

Case $a_3 > 0$: In this case,

$$P(x) \geq 0 \iff x \geq r.$$

We distinguish the two possibilities

$$x = r, \quad x > r.$$

First, if $x = r$, then

$$\dot{x} = 0,$$

and we obtain the static solution

$$x(\tau) = r, \quad -1 \leq r \leq 1. \quad (\text{E.15})$$

This solution is admissible if $r \in [-1, 1]$.

Now consider the non-constant branch $x > r$. Taking the square root,

$$\dot{x} = \pm \sqrt{a_3}(x - r)^{3/2}.$$

Hence,

$$\int \frac{dx}{(x - r)^{3/2}} = \pm \sqrt{a_3} \int d\tau,$$

and we get

$$-\frac{2}{\sqrt{x - r}} = \pm \sqrt{a_3} \tau + C'.$$

After redefining the integration constant, this gives

$$x(\tau) = r + \frac{4}{a_3(\tau + C)^2}.$$

This parametrization describes the unbounded component (r, ∞) . Since it has a pole at $\tau = -C$, the branch is not confined in the admissible interval $x \in [-1, 1]$ and is discarded.
Case $a_3 < 0$: In this case,

$$P(x) \geq 0 \quad \Longleftrightarrow \quad x \leq r.$$

We distinguish the two possibilities

$$x < r, \quad x = r.$$

Again, if $x = r$, then $x(\tau) = r$ gives the static solution (E.15). This solution is admissible if $r \in [-1, 1]$.

Now consider the non-constant branch $x < r$. Taking the square root,

$$\dot{x} = \pm \sqrt{-a_3}(r - x)^{3/2}.$$

Hence,

$$\int \frac{dx}{(r - x)^{3/2}} = \pm \sqrt{-a_3} \int d\tau,$$

and we get

$$\frac{2}{\sqrt{r - x}} = \pm \sqrt{-a_3} \tau + C'.$$

After redefining the integration constant, this gives

$$x(\tau) = r - \frac{4}{(-a_3)(\tau + C)^2}.$$

This parametrization describes the unbounded component $(-\infty, r)$. Since it has a pole at $\tau = -C$, the branch is not confined in the admissible interval $x \in [-1, 1]$ and is discarded.

One real root and two complex conjugate roots. Assume finally that the cubic polynomial has one real root and two complex conjugate roots. Then, we can write

$$P(x) = a_3(x - r_1)Q(x), \quad Q(x) := x^2 + px + q, \quad p^2 - 4q < 0.$$

Since Q has no real roots and has positive leading coefficient, we have

$$Q(x) > 0 \quad \text{for every } x \in \mathbb{R}.$$

Hence, the sign of P is controlled by the factor $a_3(x - r_1)$.

Case $a_3 > 0$: In this case,

$$P(x) \geq 0 \iff x \geq r_1.$$

Thus, the admissible real region of the differential equation is the unbounded component

$$[r_1, \infty).$$

The point $x = r_1$ is the only point in this component where the velocity can vanish. Hence, it is the only possible turning point. Since there is no second real turning point, a non-constant branch cannot remain confined in a bounded interval. Therefore, the branch eventually leaves the admissible interval $x \in [-1, 1]$ and is discarded.

Case $a_3 < 0$: In this case,

$$P(x) \geq 0 \iff x \leq r_1.$$

Thus, the admissible real region of the differential equation is the unbounded component

$$(-\infty, r_1].$$

Again, the point $x = r_1$ is the only point in this component where the velocity can vanish. Hence, it is the only possible turning point. Since there is no second real turning point, a non-constant branch cannot remain confined in a bounded interval. Therefore, the branch eventually leaves the admissible interval $x \in [-1, 1]$ and is discarded.

E.2 Physically realizable branches

Once the mathematically admissible solutions have been obtained, we must determine which of them are physically realizable. To do so, we impose the specific form of the coefficients (E.2), rather than treating them as arbitrary real numbers.

The cubic coefficient is

$$a_3 = \frac{2}{R^2} q \alpha \rho (\mu \lambda_I - 1).$$

Thus, the degree-three case occurs precisely when

$$q \alpha \rho (\mu \lambda_I - 1) \neq 0.$$

In this case, the bounded cubic branches found above can be realized, provided that the two roots bounding the motion lie in $[-1, 1]$. This gives the two Jacobi oscillatory branches (E.10) and (E.11), together with the two hyperbolic asymptotic branches (E.13)

and (E.14). Static solutions at multiple roots are also admissible if the corresponding root lies in $[-1, 1]$.

If

$$q\alpha\rho(\mu\lambda_I - 1) = 0,$$

then $a_3 = 0$ and the polynomial has degree at most two. In this case, the last term of the energy (4.22) vanishes and we have

$$E = \frac{1}{2}R^2\dot{\theta}^2 + \frac{(\ell + q\mu\lambda_IR^2\cos(\theta))^2}{2R^2\sin^2(\theta)} \geq 0.$$

Therefore,

$$a_2 = -\left(\frac{2E}{R^2} + q^2\mu^2\lambda_I^2\right) \leq 0.$$

Consequently, the quadratic asymptotic branch (E.8), which requires $a_2 > 0$, cannot be realized by the coefficients of the problem.

Moreover, the degree-one case is not realized. Indeed, if P had degree one, then we would have

$$a_3 = a_2 = 0, \quad a_1 \neq 0.$$

But we have just shown that, under the condition $a_3 = 0$, $a_2 \leq 0$. Therefore, $a_2 = 0$ forces

$$E = 0, \quad q\mu\lambda_I = 0.$$

Together with $q\alpha\rho(\mu\lambda_I - 1) = 0$, this gives

$$a_1 = -\frac{2q}{R^2}[\alpha\rho(\mu\lambda_I - 1) + \mu\lambda_I\ell] = 0.$$

This contradicts $a_1 \neq 0$. Therefore, P cannot have degree one.

In conclusion, after imposing the coefficients of the problem, the physically realizable branches are the following:

Degree	Root structure	Branch
0	$P = 0$	constant, (E.6)
2	$r_1 < r_2$, $a_2 < 0$	oscillatory $\sin^2$, (E.7)
2	double root, $a_2 < 0$	constant, (E.9)
3	$r_1 < r_2 < r_3$, $a_3 > 0$	Jacobi oscillatory sn^2 , (E.10)
3	$r_1 < r_2 < r_3$, $a_3 < 0$	Jacobi oscillatory sn^2 , (E.11)
3	one double root and one simple root	constant at the double root, (E.12)
3	$r_1 = r_2 < r_3$, $a_3 < 0$	hyperbolic asymptotic $\tanh^2$, (E.13)
3	$r_1 < r_2 = r_3$, $a_3 > 0$	hyperbolic asymptotic $\tanh^2$, (E.14)
3	triple root	constant, (E.15)

Table 2. Classification of the physically realizable branches according to the degree and root structure of P .

In each case, the root conditions appearing in the corresponding equation are understood, so that the branch remains inside the admissible interval $x \in [-1, 1]$.

The supplementary repository [26] provides a numerical validation of the branches obtained in this appendix, together with visualizations of the corresponding projected magnetic geodesics on S^2 .

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