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On the Bergman Kernel and the Suita Conjecture

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A mi Gelinucas.

Abstract

The primary object of study in this thesis is the Bergman kernel of a domain in the complex plane. This object can be examined from multiple perspectives, naturally fostering a deep interplay between complex analysis, functional analysis, and potential theory. The work begins with a foundational overview of reproducing kernel Hilbert spaces (RKHS), before specializing to unweighted Bergman spaces. Here, we detail the extremal properties of the Bergman kernel, the Bergman projection, and their critical role in establishing the boundary regularity of conformal mappings. The thesis then develops essential concepts from potential theory, exploring the Bergman kernel's strong connections with objects such as the Green's function and the logarithmic capacity. These potential-theoretic tools are subsequently used to characterize domains with a trivial Bergman space. This mathematical synergy culminates in the final chapter, which is dedicated to the Suita conjecture. We present Zbigniew Błocki's purely one-dimensional proof of this geometric inequality which establishes a sharp bound of the logarithmic capacity of a domain by its Bergman kernel.

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Introduction

Given a domain Ω in $\mathbb{C}$ and $0 < p < \infty$, the Bergman space $A^p(\Omega)$ is defined as the intersection of holomorphic functions in Ω and the Lebesgue space L^p with respect to the area measure A; $A^p(\Omega) = \mathscr{H}(\Omega) \cap L^p(\Omega, dA)$. When $p = 2$, the space $A^2(\Omega)$ possesses a powerful structure: it is a reproducing kernel Hilbert space. For this reason, we dedicate Chapter §1 to a general study of these functional spaces. Each of these spaces has a unique reproducing kernel, which allows us to define the well-known Bergman kernel $K : \Omega \times \Omega \to \mathbb{C}$. In Chapter §2 we study its intrinsic connection to the Riemann Mapping Theorem and its invariance under conformal mappings. We deduce an explicit expression for the derivative of the Riemann mapping $\varphi_\zeta : \Omega \to \mathbb{D}$ (for a domain that is simply connected) via the kernel:

$$\varphi'_\zeta(z) = \sqrt{\pi(K\zeta, \zeta)^{-1}} K(z, \zeta).$$

Furthermore, we analyze the boundary regularity of such mappings through the Bergman projection, going further than Carathéodory's Theorem. Virtually, this theorem says that the biholomorphism φ_ζ extends continuously and one-to-one to the boundary $\partial\Omega$ if it is bounded by a single Jordan curve. Using the Bergman projection, defined for $f \in L^2(\Omega, dA)$ as

$$Pf(z) = \int_\Omega f(\xi) K(z, \xi)\, dA(\xi),$$

to prove Painlevé's Theorem, which says that if our domain Ω has smooth boundary, $\varphi_\zeta \in \mathscr{C}^\infty(\overline{\Omega})$ and $\varphi_\zeta^{-1} \in \mathscr{C}^\infty(\overline{\mathbb{D}})$.

We dedicate Chapter §3 to characterize the domains Ω admitting a nontrivial Bergman space $A^2(\Omega)$. Namely, $A^2(\Omega) = \{0\}$ if and only if $\mathbb{C} \setminus \Omega$ is a polar set. In order to prove this, we introduce some notions of Potential Theory as the Green's function of a domain $G : \Omega \times \Omega \to [-\infty, \infty)$. One special property of this function is that $G(z, w) - \log|z - w|$ is harmonic throughout Ω, which it allows us to define other concepts such as the Robin constant or the logarithmic capacity of a set.

Chapter §4 is devoted to the Suita conjecture, which states, given K_Ω the Bergman kernel of $A^2(\Omega)$ and $c_\Omega(w)$ the logarithmic capacity of $\mathbb{C} \setminus \Omega$ with respect to a point $w \in \Omega$

$$c_\Omega(w)^2 \leq \pi K_\Omega(w, w).$$

We follow the proof of Z. Błocki, which uses an EDO to find the optimal constant of the Ohsawa–Takegoshi L^2 extension Theorem, but the conjecture can be stated in a more general setting for Riemann surfaces as an inequality of metrics. We also deduce, using similar tecniques, the following expression for the Bergman kernel:

$$K_\Omega(w, w) = \frac{1}{\pi^2} \inf \left\{ \int_\Omega \frac{|\partial\alpha(z)|^2}{|z - w|^2}\, dA(z) : \alpha \in \mathscr{C}_c^\infty(\Omega),\ \alpha(w) = 1 \right\}.$$

Finally, Appendix §5 serves as a technical supplement, collecting foundational notions of complex analysis. We recall key results concerning harmonic, subharmonic, and superharmonic functions, and briefly discuss the Dirichlet problem and regular points. We finish giving some properties of hyperbolic domains (domains which admit Green's function).

1. Introduction to reproducing kernel Hilbert spaces

In this section, we introduce in general terms the reproducing kernel Hilbert spaces (RKHS). These are special types of Hilbert spaces, where every function behaves "nicely" when we evaluate it at a specific point. We work always in the same framework:

Let $\mathcal{H}$ be an inner product space of functions $f : X \to \mathbb{C}$, X non-empty. For a fixed $x \in X$, the map $\delta_x : \mathcal{H} \to \mathbb{C}$, $\delta_x(f) = f(x)$ is called the *evaluation functional at* x. It is clear that evaluation functionals are always linear: given $f, g \in \mathcal{H}$ and $\alpha, \beta \in \mathbb{C}$,

$$\delta_x(\alpha f + \beta g) = (\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x) = \alpha \delta_x(f) + \beta \delta_x(g).$$

Hence, the natural question is whether they are also continuous (recall that this is the same as being bounded, since we are working in normed spaces). This is exactly how reproducing kernel Hilbert space is defined.

Definition 1.1. A *reproducing kernel Hilbert space (on X)* (RKHS) is a Hilbert space $\mathcal{H}$ of functions $f : X \to \mathbb{C}$, X non-empty such that the evaluation functional δ_x is continuous for all $x \in X$.

As a consequence, RKHS are well behaved, relative to other Hilbert spaces. In particular, the following fundamental property follows directly from the definition.

Proposition 1.2. *The norm convergence in a RKHS $\mathcal{H}$ implies the pointwise convergence.*

Proof. If $\{f_n\}_n \subset \mathcal{H}, f \in \mathcal{H}$ are such that $\lim_{n\to\infty} \|f_n - f\|_{\mathcal{H}} = 0$, then for any $x \in X$, $|f_n(x) - f(x)| = |\delta_x(f_n) - \delta_x(f)| \leq \|\delta_x\| \|f_n - f\|_{\mathcal{H}}$, and therefore $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in X$. □

In a general Hilbert space this does not happen. Take, for instance, $\mathcal{H} = L^2([0,1])$ with the norm

$$\|f - g\| = \left(\int_0^1 |f(x) - g(x)|^2 dx \right)^{1/2},$$

and consider the sequence given by $g_n = x^n$, $n \geq 1$. Then

$$\lim_{n\to\infty} \|g_n - 0\| = \lim_{n\to\infty} \frac{1}{\sqrt{2n+1}} = 0,$$

but $g_n(1) = 1$ for all n. Therefore, $g_n \xrightarrow{\mathcal{H}} 0$, but $g_n(1) \not\xrightarrow{\mathcal{H}} 0$.

1.1. On reproducing kernels.

At this point, one might notice that although we call these reproducing kernel Hilbert spaces, the definition has not mentioned a "kernel" yet. We are going to define what a kernel is and show how it naturally connects with the continuity of the evaluation function.

Definition 1.3. Let $\mathcal{H}$ be a Hilbert space of functions $f : X \to \mathbb{C}$, X non-empty. A function $K : X \times X \to \mathbb{C}$ is called a *reproducing kernel of* $\mathcal{H}$ if it satisfies

(1) $K(\cdot, x) \in \mathcal{H}$, $\forall x \in X$;
(2) $f(x) = \langle f, K(\cdot, x) \rangle_{\mathcal{H}}$, $\forall x \in X$, $f \in \mathcal{H}$. (reproducing property)

In particular, for any $x, y \in X$,

$$K(x,y) = \langle K(\cdot,y), K(\cdot,x)\rangle_{\mathcal{H}}. \tag{1}$$

We see now that every RKHS has a unique reproducing kernel. While the uniqueness follows directly from the definition, the existence is ensured by the Riesz representation theorem.

Proposition 1.4. *If the reproducing kernel of a space $\mathcal{H}$ exists, it is unique.*

Proof. Assume that $\mathcal{H}$ has two reproduciong kernels K_1 and K_2. Then for all $f \in \mathcal{H}$ and $x \in X$,

$$\langle f, K_1(\cdot,x) - K_2(\cdot,x)\rangle_{\mathcal{H}} = \langle f, K_1(\cdot,x)\rangle_{\mathcal{H}} - \langle f, K_2(\cdot,x)\rangle_{\mathcal{H}} = f(x) - f(x) = 0.$$

In particular, taking $f = K_1(\cdot,x) - K_2(\cdot,x)$, we obtain that $\|K_1(\cdot,x) - K_2(\cdot,x)\|^2 = 0$ for all $x \in X$, i.e., $K_1 = K_2$. □

Theorem 1.5. *$\mathcal{H}$ is a RKHS if and only if $\mathcal{H}$ has a reproducing kernel.*

Proof. Given a Hilbert space $\mathcal{H}$ that has a reproducing kernel K, and $f \in \mathcal{H}$, $x \in X$,

$$\begin{aligned} |\delta_x f| = |f(x)| &= |\langle f, K(\cdot,x)\rangle_{\mathcal{H}}| \le \|f\|_{\mathcal{H}}\|K(\cdot,x)\|_{\mathcal{H}} \\ &= \langle K(\cdot,x), K(\cdot,x)\rangle_{\mathcal{H}}^{1/2}\|f\|_{\mathcal{H}} = K(x,x)^{1/2}\|f\|_{\mathcal{H}}, \end{aligned}$$

where we have used the Cauchy-Schwarz inequality and (1).

Conversely, assume $\delta_x : \mathcal{H} \to \mathbb{C}$ is a bounded functional. The Riesz representation theorem states that there exists an element $f_x \in \mathcal{H}$ such that for all $f \in \mathcal{H}$, $\delta_x(f) = \langle f, f_x\rangle_{\mathcal{H}}$. Define $K(y,x) := f_x(y)$, for all $x, y \in X$. Clearly $K(\cdot,x) = f_x \in \mathcal{H}$ and $\langle f, K(\cdot,x)\rangle_{\mathcal{H}} = \delta_x(f) = f(x)$. Thus, K is the reproducing kernel. □

Proposition 1.6. *Let K be the reproducing kernel of a RKHS $\mathcal{H}$. Then*

i) For all $x, y \in \Omega$ we have that $K(x,y) = \overline{K(y,x)}$;
ii) For all $x \in X$, one has $K(x,x) = \|K(\cdot,x)\|_{\mathcal{H}}^2$;
iii) For all $x, y \in X$, $|K(x,y)|^2 \le K(x,x)K(y,y)$.

Proof. *i*): From (1), $K(x,y) = \langle K(\cdot,y), K(\cdot,x)\rangle_{\mathcal{H}} = \overline{\langle K(\cdot,x), K(\cdot,y)\rangle}_{\mathcal{H}} = \overline{K(y,x)}$. *ii*): Let $x \in X$. Again by (1) we have $K(x,x) = \langle K(\cdot,x), K(\cdot,x)\rangle_{\mathcal{H}} = \|K(\cdot,x)\|_{\mathcal{H}}^2$. *iii*): For any $x, y \in X$

$$|K(x,y)|^2 = |\langle K(\cdot,y), K(\cdot,x)\rangle|^2 \overset{C.S.}{\le} \|K(\cdot,x)\|_{\mathcal{H}}^2\|K(\cdot,y)\|_{\mathcal{H}}^2 \overset{ii)}{=} K(x,x)K(y,y).$$

□

Let X be any subset. Then $\mathcal{H}$ is a Hilbert space and therefore admits a countable orthonormal basis (recall that it is a maximal orthonormal subset). That is, there exists $\{\phi_j\}_{j\in J}$ satisfying

$$\langle \phi_i, \phi_j\rangle = \delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \ne j \end{cases}$$

and the usual identities in a Hilbert space hold:

Let $f \in \mathcal{H}$:

1. The function f can be expressed as the well-known "Fourier" series

$$f = \sum_{j \in J} \langle f, \phi_j \rangle \phi_j, \tag{2}$$

and the sum converges with respect to the $\mathcal{H}$ norm.

Following the Remark 2.4, we deduce the uniform convergence (of the partial sums to f) on compact subsets of Ω.

2. We have the Parseval's identity:

$$\|f\|_{\mathcal{H}}^2 = \sum_{j \in J} |\langle f, \phi_j \rangle|^2.$$

3. For another $g \in \mathcal{H}$ we obtain, as a consequence of (2),

$$\langle f, g \rangle = \sum_{j \in J} \langle f, \phi_j \rangle \langle \phi_j, g \rangle.$$

Theorem 1.7. *If K is the reproducing kernel of the RKHS $\mathcal{H}$, with orthonormal basis $\{\phi_j\}_{j \in J}$, then for all $x \in \Omega$:*

$$K(x, x) = \sum_{j \in J} |\phi_j(x)|^2,$$

where the sum converges absolutely for each $x \in X$.

Proof. To show the absolute convergence, note that for each $x \in X$, the values $\phi_j(x) = \langle \phi_j, K(\cdot, x) \rangle$ are the Fourier coefficients of the element $K(\cdot, x) \in \mathcal{H}$ with respect to the orthonormal basis $\{\phi_j\}_{j \in J}$. Since $K(x, x) \geq 0$, the terms in the sum are $|\phi_j(x)|^2 \geq 0$. By Parseval's identity applied to the element $K(\cdot, x)$:

$$\sum_{j \in J} |\langle K(\cdot, x), \phi_j \rangle|^2 = \|K(\cdot, x)\|_{\mathcal{H}}^2.$$

Substituting the reproducing property $\langle K(\cdot, x), \phi_j \rangle = \overline{\phi_j(x)}$, we obtain:

$$\sum_{j \in J} |\phi_j(x)|^2 = \|K(\cdot, x)\|_{\mathcal{H}}^2 = K(x, x).$$

Since $K(x, x) < \infty$ for all $x \in X$, the series $\sum_{j \in J} |\phi_j(x)|^2$ consists of non-negative terms and has a finite sum. Thus, it converges absolutely (and unconditionally) for each $x \in X$. □

Proposition 1.8. *Let K be the reproducing kernel of the RKHS $\mathcal{H}$ and let $\{\phi_j\}_{j \in J}$ be an orthonormal basis of $\mathcal{H}$. Then*

$$K(x, y) = \sum_{j \in J} \phi_j(x) \overline{\phi_j(y)}, \tag{3}$$

where the sum converges absolutely for each $x, y \in X$.

Proof. Since K is the reproducing kernel of $\mathcal{H}$ we can write, as before,

$$K(x, y) = \langle K(\cdot, y), K(\cdot, x) \rangle \overset{3.}{=} \sum_{j \in J} \langle K(\cdot, y), \phi_j \rangle \langle \phi_j, K(\cdot, x) \rangle = \sum_{j \in J} \overline{\phi_j(y)} \phi_j(x).$$

Now fix $x, y \in X$. In general, we have $\phi_j(z) = \langle \phi_j, K(\cdot, z)\rangle_{\mathcal{H}}$, so $\phi_j(x)$ is the Fourier coefficient of $K(\cdot, x)$ with respect to the base $\{\phi_j\}$. $\overline{\phi_j(y)}$ is the conjugate of the Fourier coefficient of $K(\cdot, y)$ with respect to the same base. Therefore, working on $\ell^2(J)$ and applying C.S.

$$\sum_{j\in J} |\phi_j(x)||\phi_j(y)| \leq \left(\sum_{j\in J} |\phi_j(x)|^2\right)^{1/2} \left(\sum_{j\in J} |\phi_j(y)|^2\right)^{1/2}.$$

Using Theorem 1.7, $\sum_{j\in J} |\phi_j(x)\overline{\phi_j(y)}| \leq \sqrt{K(x,x)}\sqrt{K(y,y)} < \infty$. □

1.2. **pre-RKHS.** Although we have already defined the reproducing kernel Hilbert spaces, it is useful for the proof of Theorem 1.18 to recall the notion of a pre-RKHS. Doing so allows us to isolate the fundamental algebraic properties of the kernel before invoking the completeness requirements of a full Hilbert space.

Definition 1.9. A *pre-reproducing kernel Hilbert space (on X)* (pre-RKHS) is an inner product space $\mathcal{H}_0$ of functions $f : X \to \mathbb{C}$, X non-empty such that:

(1) The evaluation functional δ_x is continuous for all $x \in X$;
(2) Every Cauchy sequence $\{f_n\}_n \subset \mathcal{H}_0$ that converges pointwise to 0, also converges in $\mathcal{H}_0$-norm to 0.

We also define $\mathcal{C}(\mathcal{H}_0) \supset \mathcal{H}_0$ to be the set of functions $f : X \to \mathbb{C}$ for which there exists an $\mathcal{H}_0$-Cauchy sequence $\{f_n\}_n \subset \mathcal{H}_0$ converging pointwise to f, called the *completion of* $\mathcal{H}_0$.

As a consequence , the pointwise convergence of a Cauchy sequence to a function $f \in \mathcal{H}_0$ in a pre-RKHS $\mathcal{H}_0$ implies the $\mathcal{H}_0$-norm convergence to the same function f.[1] Our goal is to prove that $\mathcal{C}(\mathcal{H}_0)$ is a RKHS. Specifically, we want to prove Proposition 1.14. While one implication is almost trivial, the other one requires some previous results.

Lemma 1.10. *Let $f, g \in \mathcal{C}(\mathcal{H}_0)$, with $\{f_n\}_n, \{g_n\}_n \subset \mathcal{H}_0$ Cauchy sequences (w.r.t. the $\mathcal{H}_0$-norm) that converge pointwise to f and g, respectively. Then the following map defines an inner product on $\mathcal{C}(\mathcal{H}_0)$:*

$$\langle f, g\rangle := \lim_{n\to\infty} \langle f_n, g_n\rangle_{\mathcal{H}_0}. \tag{4}$$

Proof. Take $\varepsilon > 0$ and define the sequence given by $\alpha_n = \langle f_n, g_n\rangle_{\mathcal{H}_0}$. Since every Cauchy sequence is bounded, there exist $A, B \in \mathbb{R}$ such that $\|f_n\|_{\mathcal{H}_0} \leq A$ and $\|g_n\|_{\mathcal{H}_0} \leq B$, for all $n \in \mathbb{N}$. Moreover, there exist $N_1 \in \mathbb{N}$ such that $\|f_n - f_m\|_{\mathcal{H}_0} < \varepsilon/(2B)$ for $n, m \geq N_1$ and $N_2 \in \mathbb{N}$ such that $\|g_n - g_m\|_{\mathcal{H}_0} < \varepsilon/(2A)$ for $n, m \geq N_2$. Taking $n, m \geq \max\{N_1, N_2\}$,

$$\begin{aligned}
|\alpha_n - \alpha_m| &= |\langle f_n, g_n\rangle_{\mathcal{H}_0} - \langle f_m, g_m\rangle_{\mathcal{H}_0}| = |\langle f_n - f_m, g_n\rangle_{\mathcal{H}_0} - \langle f_m, g_n - g_m\rangle_{\mathcal{H}_0}| \\
&\leq |\langle f_n - f_m, g_n\rangle_{\mathcal{H}_0}| + |\langle f_m, g_n - g_m\rangle_{\mathcal{H}_0}| \\
&\leq \|f_n - f_m\|_{\mathcal{H}_0}\|g_n\|_{\mathcal{H}_0} + \|f_m\|_{\mathcal{H}_0}\|g_n - g_m\|_{\mathcal{H}_0} \\
&\leq B\frac{\varepsilon}{2B} + A\frac{\varepsilon}{2A} = \varepsilon.
\end{aligned}$$

[1] Consider the Cauchy sequence $\{f_n - f\}_n \subset \mathcal{H}_0$.

Therefore, $\{\alpha_n\}_n$ is a Cauchy sequence in $\mathbb{C}$, which is complete, so it is convergent and consequently (4) exists and is finite. It is well defined because the limit is independent of the Cauchy sequence chosen. Indeed, if $\{F_n\}, \{G_n\}$ are $\mathcal{H}_0$-Cauchy sequences that also converge pointwise to f and g, respectively, then denoting by $A_n = \langle F_n, G_n\rangle_{\mathcal{H}_0}$,

$$\begin{aligned}|\alpha_n - A_n| &= |\langle f_n, g_n\rangle_{\mathcal{H}_0} - \langle F_n, G_n\rangle_{\mathcal{H}_0}| = |\langle f_n - F_n, g_n\rangle_{\mathcal{H}_0} - \langle F_n, g_n - G_n\rangle_{\mathcal{H}_0}| \\ &\leq |\langle f_n - F_n, g_n\rangle_{\mathcal{H}_0}| + |\langle F_n, g_n - G_n\rangle_{\mathcal{H}_0}| \\ &\leq \|f_n - F_n\|_{\mathcal{H}_0}\|g_n\|_{\mathcal{H}_0} + \|F_n\|_{\mathcal{H}_0}\|g_n - G_n\|_{\mathcal{H}_0} \to 0.\end{aligned}$$

The convergence in the $\mathcal{H}_0$-norm is ensured since both $\{f_n\}_n, \{F_n\}_n$ converge pointwise to f and therefore $\{f_n - F_n\}$ converges pointwise to 0 (recall $\mathcal{H}_0$ is pre-RKHS), and so does $\{g_n - G_n\}$. Hence, $\{\alpha_n\}_n$ and $\{A_n\}_n$ have the same limit.

We prove now that it is an inner product: Let $f, g, h \in \mathcal{C}(\mathcal{H}_0)$ and $a, b \in \mathbb{C}$,

(1) $\langle f, g\rangle = \lim_{n\to\infty}\langle f_n, g_n\rangle_{\mathcal{H}_0} = \lim_{n\to\infty}\overline{\langle g_n, f_n\rangle_{\mathcal{H}_0}} = \overline{\lim_{n\to\infty}\langle f_n, g_n\rangle_{\mathcal{H}_0}} = \overline{\langle g, f\rangle}$.

(2) $\langle af + bg, h\rangle = \lim_{n\to\infty}\langle af_n + bg_n, h_n\rangle_{\mathcal{H}_0}$
$= a\lim_{n\to\infty}\langle f_n, h_n\rangle_{\mathcal{H}_0} + b\lim_{n\to\infty}\langle g_n, h_n\rangle_{\mathcal{H}_0} = a\langle f, h\rangle + b\langle g, h\rangle$.

(3) If $\langle f, f\rangle = 0$, then for all $x \in X$,

$$|f(x)| = \left|\lim_{n\to\infty} f_n(x)\right| = \lim_{n\to\infty}|\delta_x(f_n)| \leq \lim_{n\to\infty}\|\delta_x\|\|f_n\|_{\mathcal{H}_0} = \|\delta_x\|\lim_{n\to\infty}\|f_n\|_{\mathcal{H}_0} = 0.$$

The evaluation functional is bounded and $f_n \overset{\mathcal{H}_0}{\to} 0$ since we are working in a pre-RKHS. □

Lemma 1.11. *A pre-RKHS $\mathcal{H}_0$ is dense in $\mathcal{C}(\mathcal{H}_0)$.*

Proof. Since we have defined in Lemma 1.10 an inner product for $\mathcal{C}(\mathcal{H}_0)$, we can now endow a norm there given by $\|f\|^2 := \langle f, f\rangle = \lim_{n\to\infty}\langle f_n, f_n\rangle_{\mathcal{H}_0} = \lim_{n\to\infty}\|f_n\|^2_{\mathcal{H}_0}$. Let $\{f_n\}_n$ a $\mathcal{H}_0$-Cauchy sequence that converge pointwise to $f \in \mathcal{C}(\mathcal{H}_0)$. Then for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|f_n - f_k\|_{\mathcal{H}_0} < \varepsilon$ for all $n, k \geq N$. If we fix $k \geq N$, the sequence $\{f_n - f_k\}_k$ converges pointwise to $f - f_k$, and then $\|f - f_k\|^2 = \lim_{n\to\infty}\|f_n - f_k\|^2_{\mathcal{H}_0} \leq \varepsilon^2$, which means that for all $k \geq N$, $\|f - f_k\| \leq \varepsilon$, so $f_n \overset{\mathcal{C}(\mathcal{H}_0)}{\to} f$. □

At this point we may think that $\mathcal{C}(\mathcal{H}_0)$ is already a complete metric space, but at this stage, we have only defined $\mathcal{C}(\mathcal{H}_0)$ as a set of functions obtained via pointwise limits of Cauchy sequences in $\mathcal{H}_0$. We must still prove that this set is closed under the convergence of any Cauchy sequence (not just those in $\mathcal{H}_0$). For that, we now show that the evaluation functionals are continuous on the completion of $\mathcal{H}_0$.

Lemma 1.12. *Given a pre-RKHS $\mathcal{H}_0$, the evaluation functionals are continuous on $\mathcal{C}(\mathcal{H}_0)$.*

Proof. Let $x \in X$ fix. It is enough to show δ_x is continuous at 0. We know δ_x continuous on $\mathcal{H}_0$, so for all $\varepsilon > 0$, there exists $\iota > 0$ such that if $\|h\|_{\mathcal{H}_0} < \iota$, then $|\delta_x(h)| = |h(x)| < \varepsilon/2$.

Take $f \in \mathcal{C}(\mathcal{H}_0)$ with $\|f\| \leq \iota/2$. By Lemma 1.11, there exists $\{f_n\}_n$ $\mathcal{H}_0$-Cauchy sequence that converge both pointwise and in $\|\cdot\|$ to f, so we can find N large enough so that

$$|f(x) - f_N(x)| < \frac{\varepsilon}{2}, \qquad \|f - f_N\| < \frac{\iota}{2}$$

And therefore $\|f_N\|_{\mathcal{H}_0} = \|f_N\| \leq \|f\| + \|f - f_N\| < \iota$, which implies

$$|f(x)| \leq |\delta_x(f_N)| + |f(x) - f_N(x)| < \varepsilon.$$

We have shown that $\|f\| \leq \iota/2 \implies |\delta_x(f)| < \varepsilon$, which means δ_x is $\|\cdot\|$-continuous at the origin. □

Lemma 1.13. *Given a pre-RKHS $\mathcal{H}_0$, its completion $\mathcal{C}(\mathcal{H}_0)$ is complete.*

Proof. Let $\{f_n\}_n$ be a $\mathcal{C}(\mathcal{H}_0)$-Cauchy sequence. Applying Lemma 1.12, we have that for all $x \in X$ and $n, m \in \mathbb{N}$,

$$|f_n(x) - f_m(x)| = |\delta_x(f_n - f_m)| \leq \|\delta_x\|\|f_n - f_m\|$$

This implies that $\{f_n(x)\}_n \subset \mathbb{C}$ is Cauchy and then converges to a point $\xi_x \in \mathbb{C}$. Define $f(x) := \xi_x = \lim_{n\to\infty} f_n(x)$ and since $\mathcal{H}_0$ is dense in $\mathcal{C}(\mathcal{H}_0)$, for all n choose $h_n \in \mathcal{H}_0$ such that $\|h_n - f_n\| < 1/n$. Hence,

$$\begin{aligned} |h_n(x) - f(x)| &\leq |h_n(x) - f_n(x)| + |f_n(x) - f(x)| \\ &= |\delta_x(h_n - f_n)| + |f_n(x) - f(x)| \\ &\leq \frac{\|\delta_x\|}{n} + |f_n(x) - f(x)| \overset{n\to\infty}{\longrightarrow} 0 \end{aligned}$$

and

$$\begin{aligned} \|h_n - h_m\|_{\mathcal{H}_0} = \|h_n - h_m\| &\leq \|h_n - f_n\| + \|f_n - f_m\| + \|f_m - h_m\| \\ &\leq \frac{1}{n} + \frac{1}{m} + \|f_n - f_m\|. \end{aligned}$$

This means that $\{h_n\}_n$ converges pointwise to f and $\{h_n\}_n$ is $\mathcal{H}_0$-Cauchy. Thus, $f \in \mathcal{C}(\mathcal{H}_0)$ by definition. It rests to prove that $\|f_n - f\| \to 0$:

$$\|f_n - f\| \leq \|f_n - h_n\| + \|h_n - f\| \leq \frac{1}{n} + \|h_n - f\| \overset{n\to\infty}{\longrightarrow} 0,$$

since $h_n \overset{\mathcal{C}(\mathcal{H}_0)}{\to} f$, and the aim follows. □

Proposition 1.14. *Let $\mathcal{H}_0$ be an inner product space. Then it is a pre-RKHS if and only if the following conditions hold:*

- *i) In $\mathcal{H}_0 \subset \mathcal{C}(\mathcal{H}_0)$, the topology induced by $\langle\cdot,\cdot\rangle_{\mathcal{H}_0}$ coincides with the topology induced by $\mathcal{C}(\mathcal{H}_0)$;*
- *ii) $\mathcal{C}(\mathcal{H}_0)$ is a RKHS.*

Proof. If both conditions are true, by *ii*), $\delta_x : \mathcal{C}(\mathcal{H}_0) \to \mathbb{C}$ is continuous for all $x \in X$. In particular, δ_x is continuous for all functions on $\mathcal{H}_0 \subset \mathcal{C}(\mathcal{H}_0)$. Moreover, if $\{f_n\}_n \subset \mathcal{H}_0$ is a Cauchy sequence such that converges pointwise to 0, by *i*),

$$\|f_n\|_{\mathcal{H}_0} \to 0 \iff \|f_n\|_{\mathcal{C}(\mathcal{H}_0)} \to 0.$$

Conversely, suppose that $\mathcal{H}_0$ is pre-RKHS. We define in $\mathscr{C}$ the inner product given in (4). By Lemma 1.13 we have that $\mathcal{C}(\mathcal{H}_0)$ is indeed a Hilbert space and by Lemma 1.12, we deduce that it is a RKHS, proving *ii*). The point *i*) follows by construction. □

1.3. **Moore–Aronszajn Theorem.** Theorem 1.5 allows for an alternative characterization of an RKHS: a Hilbert space $\mathcal{H}$ of complex-valued functions is an RKHS if and only if it admits a reproducing kernel. Our objective now is to shift our focus toward the kernels themselves. We shall first define kernels as functions that admit a representation via inner products in some feature space. Subsequently, we will establish the Moore–Aronszajn Theorem, which provides the fundamental link between these two notions.

Definition 1.15. Let X be a non-empty set. A function $K : X \times X \to \mathbb{C}$ is called a *kernel* (*of* $\mathcal{H}$) if there exists a complex Hilbert space $\mathcal{H}$ (called *feature space*) and a map $\zeta : X \to \mathcal{H}$ (called *feature map*[2]) such that for all $x, y \in X$, $K(x, y) = \langle \zeta(x), \zeta(y) \rangle_{\mathcal{H}}$.

Definition 1.16. Let X be a non-empty set, and let $g : X \times X \to \mathbb{C}$ be a hermitian function, i.e., $g(x, y) = \overline{g(y, x)}$ for all $x, y \in X$. We say that g is *positive-definite* if for every finite subset $\{x_i\}_{i=1}^n \subset X$ and complex numbers $\{c_j\}_{j=1}^n \subset \mathbb{C}$, we have

$$\sum_{i=1}^{n} \sum_{j=1}^{n} c_i \overline{c_j} g(x_i, x_j) \geq 0.$$

Lemma 1.17. *Every kernel of a Hilbert space $\mathcal{H}$ is a positive-definite function.*

Proof. Let $\zeta : X \to \mathcal{H}$ the feature map of the kernel K, and let $\{x_i\}_{i=1}^n \subset X$, $\{c_j\}_{j=1}^n \subset \mathbb{C}$. Then

$$\begin{aligned} \sum_{i,j} c_i \overline{c_j} K(x_i, x_j) &= \sum_{i,j} c_i \overline{c_j} \langle \zeta(x_i), \zeta(x_j) \rangle_{\mathcal{H}} = \sum_{i,j} \langle c_i \zeta(x_i), c_j \zeta(x_j) \rangle_{\mathcal{H}} \\ &= \left\langle \sum_{i=1}^{n} c_i \zeta(x_i), \sum_{j=1}^{n} c_j \zeta(x_j) \right\rangle_{\mathcal{H}} = \left\| \sum_{i=1}^{n} c_i \zeta(x_i) \right\|_{\mathcal{H}}^2 \geq 0. \end{aligned}$$

□

As a consequence, reproducing kernels are intrinsically positive-definite functions. By defining the feature map $\zeta : x \mapsto K(\cdot, x)$ as in Lemma 1.17, and since (1) holds, we can see that the reproducing kernel associated with any RKHS is always a (Hermitian) positive-definite function. The Moore–Aronszajn Theorem proves the converse: the mere existence of a (Hermitian) positive-definite function $K : X \times X \to \mathbb{C}$ is sufficient to guarantee the existence of a unique RKHS for which K acts as the reproducing kernel. This foundational result, although famously detailed and expanded upon by Aronszajn [1], acknowledges the original theorem to E.H. Moore.

Theorem 1.18 (**Moore–Aronszajn**). *Every positive-definite function $K : X \times X \to \mathbb{C}$ is associated with a unique RKHS $\mathcal{H}$ on X, for which K is its reproducing kernel.*

Proof. We define the set $\mathcal{H}_0 := \operatorname{span}\{K(\cdot, x) : x \in X\}$, and we endow it with the following inner product:

$$\langle f, g \rangle := \sum_{i=1}^{n} \sum_{j=1}^{m} a_i \overline{b_j} K(y_j, x_i), \tag{5}$$

[2]For a given kernel, there may be more than one feature map.

where $f = \sum_{i=1}^n a_i K(\cdot, x_i)$ and $g = \sum_{j=1}^m b_j K(\cdot, y_j)$. Firstly, it is independent of the choice of the coefficients a_i, b_j as

$$\langle f, g\rangle = \sum_{i=1}^n a_i \overline{g(x_i)} = \sum_{j=1}^m \overline{b_j} f(y_j).$$

Moreover, setting $g = K(\cdot, x)$, we get the reproducing property:

$$\langle f, K(\cdot, x)\rangle = \sum_{i=1}^n a_i \overline{K(x_i, x)} = \sum_{i=1}^n a_i K(x, x_i) = f(x). \tag{6}$$

And indeed (5) defines an inner product: the first conditions are satisfied because K is hermitian, positive-definite and the functions are complex linear combinations of those kernels. By (6), if $\langle f, f\rangle = 0$, then for all $x \in X$,

$$|f(x)| = |\langle f, K(\cdot, x)\rangle| \overset{\text{C.S.}}{\leq} \|f\| |\langle K(\cdot, x), K(\cdot, x)\rangle|^{1/2} = \|f\| |\langle K(x, x)\rangle|^{1/2} = 0. \tag{7}$$

In fact, $\mathcal{H}_0$ is a pre-RKHS. We have already seen in (7) that the evaluation functional is bounded in $\mathcal{H}_0$. Take a $\mathcal{H}_0$-Cauchy sequence $\{f_n\}_n$ that converges pointwise to 0. Let $\varepsilon > 0$. There exists $A > 0$ such that for all $n \in \mathbb{N}$, $\|f_n\| < A$. There exists $N_1 \in \mathbb{N}$ such that for all $n, m \geq N_1$, $\|f_n - f_m\| < \varepsilon/(2A)$. Let us say that

$$f_{N_1} = \sum_{i=1}^k c_i K(\cdot, z_i).$$

Take $N_2 \in \mathbb{N}$ so that $|f_n(z_i)| < \varepsilon/(2k|c_i|)$ for $1 \leq i \leq k$. Hence, for all $n \geq \max\{N_1, N_2\}$,

$$\begin{aligned}\|f_n\|^2 &\leq |\langle f_n, f_n\rangle + \langle f_{N_1}, f_n\rangle - \langle f_{N_1}, f_n\rangle| \leq |\langle f_n - f_{N_1}, f_n\rangle| + |\langle f_{N_1}, f_n\rangle| \\ &\leq \|f_n\| \|f_n - f_{N_1}\| + \sum_{i=1}^k |c_i f_n(z_i)| \leq A\frac{\varepsilon}{2A} + \sum_{i=1}^k |c_i| \frac{\varepsilon}{2k|c_i|} = \varepsilon.\end{aligned}$$

Thus, $\mathcal{H}_0$ is a pre-RKHS, and defining $\mathcal{H} := \mathcal{C}(\mathcal{H}_0)$, by Proposition 1.14, it is a RKHS. Furthermore, the reproducing kernel of $\mathcal{H}$ is K: let $f \in \mathcal{H}$ and let $\{f_n\}_n$ be the $\mathcal{H}_0$-Cauchy sequence that converges pointwise to f. Then

$$\langle f, K(\cdot, x)\rangle_{\mathcal{H}} \overset{(4)}{=} \lim_{n\to\infty} \langle f_n, K(\cdot, x)\rangle \overset{(6)}{=} \lim_{n\to\infty} f_n(x) = f(x).$$

Uniqueness: let $\mathcal{G}$ other RKHS with reproducing kernel K. For all $x, y \in X$, we have that

$$\langle K(\cdot, y), K(\cdot, x)\rangle_{\mathcal{H}} = K(x, y) = \langle K(\cdot, y), K(\cdot, x)\rangle_{\mathcal{G}}.$$

By linearity, $\langle \cdot, \cdot\rangle_{\mathcal{H}} = \langle \cdot, \cdot\rangle_{\mathcal{G}}$ in $\mathcal{H}_0$. Since G is complete containing $\mathcal{H}_0$, $\mathcal{H} \subset \mathcal{G}$. Take now $f \in \mathcal{G}$. Since $\mathcal{H}$ is closed in $\mathcal{G}$ (complete), we can write $f = f_1 + f_2$, where $f_1 \in H$ and $f_2 \in H^\perp$. For all $x \in X$,

$$f(x) = \langle f, K(\cdot, x)\rangle_{\mathcal{G}} = \langle f_1, K(\cdot, x)\rangle_{\mathcal{G}} + \langle f_2, K(\cdot, x)\rangle_{\mathcal{G}} \overset{(**)}{=} \langle f_1, K(\cdot, x)\rangle_{\mathcal{H}} = f_1(x),$$

where in $(**)$ we use that $K(\cdot, x) \in \mathcal{H}$ and $\langle \cdot, \cdot\rangle_{\mathcal{H}} = \langle \cdot, \cdot\rangle_{\mathcal{G}}$ in $\mathcal{H}_0$. Thus, $f = f_1 \in \mathcal{H}$, which means $\mathcal{H} \supset \mathcal{G}$. □

2. Bergman spaces and the Bergman kernel

Given a domain $\Omega \subset \mathbb{C}$, let $\mathscr{H}(\Omega)$ be the space of holomorphic functions defined in Ω. Given $0 < p < \infty$, the (unweighted) *Bergman spaces* on $\Omega \subset \mathbb{C}$ are defined as

$$A^p(\Omega) := \left\{ f \in \mathscr{H}(\Omega) : \|f\|^p_{A^p(\Omega)} = \int_\Omega |f(z)|^p dA(z) < \infty \right\} = \mathscr{H}(\Omega) \cap L^p(\Omega, dA)$$

where A is the two-dimensional Lebesgue measure in Ω[3] (given $z = x+iy = re^{i\theta}$, $dA(z) = dxdy = r\,drd\theta$ in rectangular and polar coordinates, respectively). The story of these spaces begins with Stephan Bergman [2], when he studied these spaces for the case $p = 2$. The study of Hardy spaces begun in 1915, when G.H. Hardy started studying properties of analytic functions in the unit disk through its integral values. Interest in this topic grew around 1960. During this time, L. Carleson, followed by Shapiro and Shields, solved the so called "universal interpolation problems". In 1922, Bergman introduced what we now call the Bergman kernel, focusing on A^2. The systematic study of A^p for $0 < p < \infty$ was largely pushed forward in the 1970s.

Remark 2.1. Firstly, we establish a straightforward characterization for the membership of these functions in the special case of $A^2(\{z : |z| > R\})$, based on their behavior at infinity. Let us formalize the notion of analyticity at the infinity. A function f is said to be analytic (or holomorphic) at $\infty \in \mathbb{C}_\infty$ if the auxiliary function $g(w) = f(1/w)$ is analytic in a neighborhood of the origin $w = 0$. In this case, we naturally define $f(\infty) := g(0)$, $f^{(k)}(\infty) := g^{(k)}(0)$.

The Lemma will be useful later (see Theorem 3.24).

Lemma 2.2. *If $H = \{z : |z| > R\} \subset \mathbb{C}_\infty$ and $f(z) = \sum_{n=0}^\infty a_n z^{-n}$ is analytic on H, then*

i) $f \in A^2(H)$ if and only if $a_0 = f(\infty) = 0$, $a_1 = f'(\infty) = 0$, and $\sum_{n=2}^\infty |a_n|^2 R^{-(2n-2)} < \infty$.

ii) If f is also bounded on H, $f \in A^2(H)$ if and only if $f(\infty) = f'(\infty) = 0$.

Proof. Since

$$|f(z)|^2 = \left(\sum_{n=0}^\infty \frac{a_n}{z^n} \right) \left(\sum_{m=0}^\infty \frac{\overline{a_m}}{\overline{z}^m} \right) = \sum_{n=0}^\infty \sum_{m=0}^\infty \frac{a_n \overline{a_m}}{z^n \overline{z}^m}$$

we have in polar coordinates that

$$\begin{aligned} \int_{|z|>R} |f|^2 dA &= \int_R^\infty \int_0^{2\pi} \left(\sum_{n=0}^\infty \sum_{m=0}^\infty \frac{a_n \overline{a_m}}{r^n r^m} \frac{e^{im\theta}}{e^{in\theta}} \right) r\, d\theta\, dr \\ &= \int_R^\infty \sum_{n=0}^\infty \sum_{m=0}^\infty \frac{a_n \overline{a_m}}{r^{n+m-1}} \left(\int_0^{2\pi} e^{i(m-n)\theta} d\theta \right) dr, \end{aligned}$$

[3] Some authors prefer to use the two-dimensional Lebesgue measure normalized to avoid constants in the formulas (e.g. In the case of $\mathbb{D}$, $dm(z) = \frac{1}{\pi} dA(z)$).

where the sum and the integral can be exchanged using that f is analytic in H, so the sum converges uniformly. We finally get

$$\int_{|z|>R} |f|^2 dA = 2\pi \sum_{n=0}^{\infty} |a_n|^2 \int_R^{\infty} \frac{dr}{r^{2n-1}}. \tag{8}$$

$i)$ follows from (8). The integral on the right-hand side converges iff $-2n+2<0$, so the first two terms of the series must be zero in order to have convergence.

For $ii)$ note that by increasing R we may assume that f is analytic in a neighborhood of the closure of H. If, in addition, $f(\infty)=f'(\infty)=0$, then $f(z)=z^{-2}g(z)$, where g is a bounded analytic function on H. □

2.1. **Growth of A^p functions.** Consider a domain $\Omega \subset \mathbb{C}$. In this section, we study the growth of the functions in Bergman spaces and some of its direct consequences.

Proposition 2.3. *Let $f \in A^p(\Omega)$, $0<p<\infty$. Then for every $z \in \Omega$,*

$$|f(z)| \leq \frac{1}{\pi^{1/p}} \operatorname{dist}(z, \partial\Omega)^{-2/p} \|f\|_{A^p(\Omega)}. \tag{9}$$

Proof. Fix $z \in \Omega$ and let $\delta < \operatorname{dist}(z, \partial\Omega)$. We consider the disk $D(z,\delta) = \{w \in \Omega : |w-z|<\delta\}$. Since f is holomorphic in Ω (and therefore $|f|^p$ is subharmonic in Ω) we have that for $0 \leq r < \delta$ (recall Definition 5.7),

$$|f(z)|^p \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z+re^{i\theta})|^p d\theta,$$

so

$$\begin{aligned} \frac{\delta^2}{2}|f(z)|^p &= \int_0^{\delta} |f(z)|^p r dr \leq \int_0^{\delta} \frac{1}{2\pi} \int_0^{2\pi} |f(z+re^{i\theta})|^p d\theta \, r dr \\ &= \frac{1}{2\pi} \int_{D(z,\delta)} |f(\xi)|^p dA(\xi) \leq \frac{1}{2\pi} \int_{\Omega} |f(\xi)|^p dA(\xi). \end{aligned}$$

It implies that $|f(z)| \leq \pi^{-1/p}\delta^{-2/p}\|f\|_{A^p(\Omega)}$. Taking the limit as δ approaches $\operatorname{dist}(z,\Omega)$, the claim follows. □

Remark 2.4. The growth of the functions in $A^p(\Omega)$ described in Proposition 2.3 allows us to deduce some consequences. Firstly, the convergence in $A^p(\Omega)$ implies the uniform convergence on the compact sets of the domain $\Omega \subset \mathbb{C}$.

As another consequence, we show that $A^p(\Omega)$ is a Banach space when $1 \leq p < \infty$ and a complete metric space when $0<p<1$. Of course $\|\cdot\|_{A^p(\Omega)}$ defines a norm when $1 \leq p < \infty$ and when $0<p<1$, $A^p(\Omega)$ is a metric space by defining $d(f,g) = \int_\Omega |f(z)-g(z)|^p dA(z) = \|f-g\|^p_{A^p(\Omega)}$ as usual.

Proposition 2.5. *For every $0<p<\infty$, the Bergman space $A^p(\Omega)$ is closed in $L^p(\Omega, dA)$.*

Proof. Let $\{f_n\}_n \subset A^p(\Omega)$ and assume $f_n \to f$ in $L^p(\Omega, dA)$. In particular, $\{f_n\}_n$ is Cauchy in $L^p(\Omega, dA)$. By Proposition 2.3, $\{f_n\}_n$ is uniformly Cauchy on compact sets in Ω, so it converges uniformly on compact sets in Ω (so pointwise) to some function

g. Since each $f_n \in \mathscr{H}(\Omega)$, this function g is also holomorphic. Since f_n converges to f in $L^p(\Omega, dA)$, there exists a subsequence $\{f_{n_k}\}_k$ of $\{f_n\}_n$ that converges (pointwise) to f almost everywhere in Ω. Therefore, $f = g$ almost everywhere in Ω, which means $f \in A^p(\Omega)$. □

Proposition 2.6. *Let $\{f_n\}_n \subset A^p(\Omega)$, $0 < p < \infty$, which is uniformly bounded in $A^p(\Omega)$. Then there exists a subsequence $\{f_{n_k}\}_k$ of $\{f_n\}_n$ that converges uniformly on compact sets in Ω to some function $f \in A^p(\Omega)$.*

Proof. From Proposition 2.3 and the uniform boundedness of the sequence, we have

$$|f_n(z)| \leq \frac{1}{\pi^{1/p}} \operatorname{dist}(z, \partial\Omega)^{-2/p} \|f_n\|_{A^p(\Omega)} \leq \frac{1}{\pi^{1/p}} \operatorname{dist}(z, \partial\Omega)^{-2/p} M,$$

for some $M > 0$ and for all $z \in \Omega$. Let $K \subset \Omega$ be a compact set and define $\delta_K = \inf_{z\in K} \operatorname{dist}(z, \partial\Omega) > 0$. Thus, for all $z \in K$,

$$|f_n(z)| \leq \frac{M}{\pi^{1/p} \cdot \delta_K^{2/p}} =: c = c(K).$$

This means that $\{f_n\}_n$ is uniformly bounded in compact subsets of Ω. Applying Montel's Theorem (Theorem 5.1), we have that $\{f_n\}_n$ is normal on Ω. Therefore, there exists a subsequence $\{f_{n_k}\}_k$ of $\{f_n\}_n$ such that it converges uniformly to f on the compact subsets of Ω to some holomorphic function defined in Ω. But $f \in A^p(\Omega)$ since it is the uniform limit of analytic functions and since $f_{n_k} \to f$ pointwise, by Fatou's lemma

$$\begin{aligned} \|f\|_{A^p(\Omega)} &= \int_\Omega |f(z)|^p \, dA(z) = \int_\Omega \lim_{k\to\infty} |f_{n_k}(z)|^p \, dA(z) \leq \liminf_{k\to\infty} \int_\Omega |f_{n_k}(z)|^p \, dA(z) \\ &= \liminf_{k\to\infty} \|f_{n_k}\|^p_{A^p(\Omega)} \leq c^p. \end{aligned}$$

□

Remark 2.7. Baring in mind the growth condition (9), $A^p(\mathbb{C}) = \{0\}$. To avoid trivial cases like this one, we implicitly assume that our domains (open and connected) are regions where the space is separating points from zero. That is, for every point z in the domain, there must exist at least one function in the space that does not vanish at that point. This ensures that the space is large enough to capture the local behavior of analytic functions at every point in the domain. Domains behaving like $\mathbb{C}$, that is, having a trivial Bergman space, are classified (for $p = 2$) in Theorem 3.24, in Chapter §3.

2.2. **The Bergman kernel.** In this section, we restrict our attention to the case $p = 2$. Let Ω be a domain such that $A^2(\Omega)$ is non-trivial. Then we can define an inner product as usual via

$$\langle f, g\rangle = \langle f, g\rangle_{A^2(\Omega)} := \int_\Omega f(z)\overline{g(z)} \, dA(z).$$

As a consequence of Proposition 2.5, the Bergman space $A^2(\Omega)$ is in fact a Hilbert space. Furthermore, it is a RKHS on Ω since the evaluation functionals δ_z are bounded for all $z \in \Omega$ by (9). From Theorem 1.5, there exists a unique reproducing kernel $K : \Omega \times \Omega \to \mathbb{C}$ of $A^2(\Omega)$. It is called the *Bergman kernel* of the space $A^2(\Omega)$.

As we know, it reproduces the value of any function as

$$f(z) = \int_\Omega f(\xi) K(z,\xi)\, dA(\xi) \tag{10}$$

and also satisfies the useful properties given in Proposition 1.6. In this section, we give special properties of this kernel. The first one is immediate: $z \mapsto K(z,\xi)$ is holomorphic and $\xi \mapsto K(z,\xi)$ is anti-holomorphic.

Our aim is to prove that for a fixed $z \in \Omega$, the value $K(z,z)$ can be characterized by $K(z,z) = \sup\{|f(z)|^2 : f \in A^2(\Omega), \|f\| \leq 1\}$. Intuitively speaking, if the kernel is too large at z, it means that the space $A^2(\Omega)$ allows functions with very high peaks at that point without their total norm exploding.

Let $\Omega \subset \mathbb{C}$ be any domain. Then $A^2(\Omega)$ admits a countable orthonormal basis. Indeed, we have seen that $A^2(\Omega)$ is a Hilbert space, so we can take an orthonormal basis β. The countability of the basis is ensured by the separability of $L^2(\Omega, dA)$, since in Proposition 2.5 we have seen that $A^2(\Omega)$ is closed in $L^2(\Omega, dA)$, which is separable, so it implies that $A^2(\Omega)$ is also separable. By a classical result in Hilbert spaces, we know that if the space is separable, then all orthonormal bases are countable, so we can assume $\beta = \{\phi_n\}_n$. Here, following Remark 2.4, the Fourier series given in (2) converges uniformly on compact subsets of Ω.

Theorem 2.8 (**Extremal property**). *If K is the Bergman kernel of $A^2(\Omega)$, with orthonormal basis $\{\phi_n\}_n$, then for all $z \in \Omega$:*

$$K(z,z) = \sum_{n=1}^{\infty} |\phi_n(z)|^2 = \sup_{\substack{f \in A^2(\Omega) \\ \|f\|_{A^2(\Omega)} \leq 1}} |f(z)|^2 < \infty.$$

The convergence is uniform on compact subsets of Ω.

Proof. Let $f \in A^2(\Omega)$ with $\|f\|_{A^2(\Omega)} \leq 1$ and fix $z \in \Omega$. Applying Cauchy-Schwarz,

$$|f(z)| = |\langle f, K(\cdot, z)\rangle| \leq \|f\|_{A^2(\Omega)} \|K(\cdot,z)\|_{A^2(\Omega)} \leq \|K(\cdot,z)\|_{A^2(\Omega)} = \sqrt{K(z,z)}.$$

Therefore $K(z,z)$ is an upper bound for the set $\{|f(z)|^2 : \|f\|_{A^2(\Omega)} \leq 1\}$. Consider now the function $f_z \in A^2(\Omega)$ defined by

$$f_z(w) = \frac{K(w,z)}{\|K(\cdot,z)\|_{A^2(\Omega)}} = \frac{K(w,z)}{\sqrt{K(z,z)}}.$$

Indeed it is holomorphic and

$$\int_\Omega |K(w,z)|^2 dA(w) = \langle K(\cdot,z), K(\cdot,z)\rangle_{A^2(\Omega)} = K(z,z).$$

Then $\|f_z\|_{A^2(\Omega)} = 1$ and $|f_z(z)|^2 = K(z,z)$, which confirms that the supremum[4] is indeed $K(z,z)$. From Theorem 1.7, we have the first equality. It remains to see the uniform convergence. Fix $L \subset \Omega$ be compact and $z \in L$. Using Proposition 2.3, there

[4] In fact, it is attained, so it is a maximum.

exists $C_L > 0$ such that for all $f \in A^2(\Omega)$ with $\|f\|_{A^2(\Omega)} < 1$, $|f(z)| \leq C_L$ for each $z \in \Omega$. Therefore,

$$K(z,z) = \sup_{\|f\|\leq 1} |f(z)|^2 \leq C_L^2 < \infty$$

and thus $\sum_{n=1}^{\infty} |\phi_n(z)|^2$ is uniformly bounded in L. Since $w \mapsto |\phi(w)|^2$ is continuous and $S_N(z) := \sum_{n=1}^{N} |\phi_n(z)|^2$ is increasing monotonically and converges pointwise to $K(z,z)$, the uniform convergence follows from Theorem 5.3. □

Proposition 2.9. *Let K be the Bergman kernel of $A^2(\Omega)$ and let $\{\phi_n\}_n$ be an orthonormal basis of $A^2(\Omega)$. Then*

$$K(z,\xi) = \sum_{n=0}^{\infty} \phi_n(z)\overline{\phi_n(\xi)}, \tag{11}$$

where the sum converges with respect to the $A^2(\Omega)$ norm for each variable and therefore uniformly in compact subsets of $\Omega \times \Omega$.

Proof. Proposition 1.8 gives the equality. Now fix $\xi \in \Omega$. The function $z \mapsto K(z,\xi)$ is in $A^2(\Omega)$. Since $\{\phi_n\}_n$ is an orthonormal basis of a Hilbert space, the Fourier series of any element converges in the norm of it. Hence, $S_N(z,\xi) = \sum_{n=0}^{N} \phi_n(z)\overline{\phi_n(\xi)}$ satisfies:

$$\lim_{N\to\infty} \|K(\cdot,\xi) - S_N(\cdot,\xi)\|_{A^2} = 0.$$

Fix now a compact $L \subset \Omega$. The uniform convergence follows from the fact that, for $(z,\zeta) \in L \times L$, we have

$$\sum_{n=0}^{\infty} |\phi_n(z)\overline{\phi_n(\zeta)}| \leq \left(\sum_{n\geq 1} |\phi_n(z)|^2\right)^{1/2} \left(\sum_{n\geq 1} |\phi_n(\zeta)|^2\right)^{1/2} \leq C_L$$

by Theorem 2.8. □

To motivate the following results, we first shift our focus to the case where $\Omega = \mathbb{D}$. By specializing to the unit disk, we can explicitly compute its Bergman kernel, providing a concrete foundation for the more general regularity results that will follow.

Proposition 2.10. *The set $\beta = \{\phi_n(z) = \sqrt{(n+1)/\pi}\, z^n\}_{n\geq 0}$ is an orthonormal basis of $A^2(\mathbb{D})$.*

Proof. It is orthonormal because for every $p, q \in \mathbb{N}$ we have that

$$\begin{aligned}
\langle \phi_p, \phi_q \rangle &= \frac{\sqrt{p+1}\sqrt{q+1}}{\pi} \int_{\mathbb{D}} z^p \overline{z^q}\, dA(z) = \frac{\sqrt{p+1}\sqrt{q+1}}{\pi} \int_0^1 \int_0^{2\pi} r(re^{i\theta})^p (re^{-i\theta})^q \, d\theta dr \\
&= \frac{\sqrt{p+1}\sqrt{q+1}}{\pi} \int_0^1 r^{p+q+1} dr \int_0^{2\pi} e^{i(p-q)\theta} d\theta = \frac{\sqrt{p+1}\sqrt{q+1}}{\pi} \frac{2\pi\delta_{pq}}{p+q+2} \\
&= \begin{cases} \frac{p+1}{2(p+1)} \frac{2\pi}{\pi} = 1, & \text{if } p = q \\ 0 & \text{if } p \neq q. \end{cases}
\end{aligned}$$

To see that it is complete (i.e., $\beta^{\perp} = \{0\}$) — and therefore a basis — we can equivalently show that Pareseval's identity holds for all $f \in A^2(\mathbb{D})$. However, we can directly prove

it by definition: let $f \in A^2(\mathbb{D})$. Since it is analytic, we can write $f(z) = \sum_k a_k z^k$, with $a_k \in \mathbb{C}$. Then

$$\langle f, z^n \rangle = \int_{\mathbb{D}} f(z)\overline{z^n}\, dA(z) = \int_0^1 \int_0^{2\pi} \left(\sum_{k=0}^{\infty} a_k r^k e^{ik\theta} \right) r^n e^{-in\theta} r\, d\theta\, dr$$

The sum converges uniformly in compact subsets of the unit disk, so considering $|z| \leq \rho$, with $0 < \rho < 1$, we can exchange the sum and the integral to get

$$\langle f, z^n \rangle = \sum_{k=0}^{\infty} a_k \int_0^1 r^{k+n+1} \left(\int_0^{2\pi} e^{i(k-n)\theta}\, d\theta \right) dr = \sum_{k=0}^{\infty} a_k \frac{2\pi\delta_{kn}}{k+n+2} = \pi \frac{a_n}{n+1}.$$

Therefore, if $\langle f, \phi_n \rangle = 0$ for all n, then $\langle f, z^n \rangle = 0$ for all n, so $a_n = 0$ for all $n \geq 0$, and letting $r \to 1^-$, we conclude that the orthonormal set β is complete. □

Proposition 2.11. *The Bergman kernel of $A^2(\mathbb{D})$ is given by*

$$K_{\mathbb{D}}(z, \xi) = \frac{1}{\pi} \frac{1}{(1 - z\bar{\xi})^2}. \tag{12}$$

Proof. Employing (11) and using the basis β given in Proposition 2.10,

$$K_{\mathbb{D}}(z, \xi) = \sum_{n=0}^{\infty} \phi_n(z)\overline{\phi_n(\xi)} = \frac{1}{\pi} \sum_{n=0}^{\infty} (n+1)(z\bar{\xi})^n = \frac{1}{\pi} \frac{1}{(1 - z\bar{\xi})^2}.$$

□

Proposition 2.12. *Let $U_1 \subset U_2 \subset \ldots$ be bounded domains in $\mathbb{C}$ with $U = \cup_j U_j$ bounded. If K_j the Bergman kernel of $A^2(U_j)$, then $K_j \to K$ uniformly in compact subsets of $U \times U$, K being the Bergman kernel of U.*

Proof. Since $U_j \subset U_{j+1} \subset U$, we have that if $f \in L^2(U_{j+1}, dA)$, then $f|_{U_j} \in L^2(U_j, dA)$. Even more important, we have for all j

$$A^2(U) \subset \cdots \subset A^2(U_{j+1}) \subset A^2(U_j),$$

since if $\mathscr{H}(U_j) \supset \mathscr{H}(U_{j+1})$. As a consequence of Theorem 2.8, we obtain the following:

$$K_1(z, z) \geq K_2(z, z) \geq \cdots \geq K(z, z) > 0.$$

Let $L \subset U$ be a compact subset. The compactness guarantees the existence of a natural N such that $L \subset U_N$. For all $j \geq N$, we know

$$0 < K_j(z, z) \leq K_N(z, z), \quad \forall z \in L.$$

By the Weierstrass Theorem, $z \mapsto K_N(z, z)$ reaches a maximum in the compact L. Thus, there exists $M_L > 0$ such that

$$\sup_{j \geq N} \sup_{z \in L} K_j(z, z) \leq M_L,$$

meaning that $\{z \mapsto K_j(z, z)\}$ is uniformly bounded in compact subsets of U. For each j and a fix point $\xi \in U$, let us define the holomorphic function $f_{j,\xi}(z) := K_j(z, \xi)$. We want to see that this family is normal. If $z \in L \subset U$ is compact,

$$|K_j(z, \xi)| \leq \sqrt{K_j(z, z)}\sqrt{K_j(\xi, \xi)} \leq \sqrt{M_L}\sqrt{K_j(\xi, \xi)} =: C_{\xi, L}.$$

Hence, for each ξ fixed, the family $\{z \mapsto K_j(z,\xi)\}_{j\geq N}$ is uniformly bounded in compact subsets of U. Applying Montel's Theorem (Theorem 5.1), we have that it is normal on U. Therefore, there exists a subsequence $\{K_{j_k}(z,\xi)\}_k$ such that it converges uniformly on the compact subsets of U to some holomorphic function defined in U, ι_ξ. Since $\{K_j(z,z)\}_j$ is a decreasing sequence of real numbers (for each z), the Monotone Convergence Theorem ensures the convergence pointwise $K_j(z,z) \downarrow \iota_\xi(z)$. It rests to check that it verifies the reproducing property in $A^2(U)$. For $f \in A^2(U)$, we know that for all j

$$f(z) = \int_{U_j} f(\xi) K_j(z,\xi)\, dA(\xi).$$

By Lebesgue's Dominated Convergence Theorem,

$$f(z) = \lim_{k\to\infty} \int_{U_{j_k}} f(\xi) K_{j_k}(z,\xi)\, dA(\xi) = \int_U f(\xi)\iota_\xi(z)\, dA(\xi).$$

The unicity of the Bergman kernel implies that $\iota_\xi(z) = K(z,\xi)$. □

Proposition 2.13 (**The Bergman projection**). *Let K be the Bergman kernel of $A^2(\Omega)$ and for $f \in L^2(\Omega, dA)$, define $Pf : \Omega \to \mathbb{C}$ by*

$$Pf : z \mapsto \int_\Omega f(\xi) K(z,\xi)\, dA(\xi). \tag{13}$$

The linear operator P is called the Bergman projection of $A^2(\Omega)$. It satisfies:

1) *The function $Pf \in \mathscr{H}(\Omega)$ is well-defined;*
2) *If in addition f is holomorphic (i.e., $f \in A^2(\Omega)$ already), $Pf = f$.*
3) *P is the orthogonal projection of $L^2(\Omega, dA)$ onto $A^2(\Omega)$.*

Proof. 1) Let us first note that the integral is well defined for every $f \in L^2(\Omega, dA)$. Indeed, fix $z \in \Omega$ and consider $\xi \mapsto \overline{K(z,\xi)}$, which is in $A^2(\Omega)$, so by Cauchy-Schwarz inequality

$$|Pf(z)| \leq \int_\Omega |K(z,\xi)f(\xi)|\, dA(\xi) \leq \|K(\cdot,z)\|_{A^2(\Omega)} \|f\|_2 < \infty.$$

To see that $Pf \in \mathscr{H}(\Omega)$, we can differentiate under the integral sign. Since $f \in L^2(\Omega, dA)$ and the mapping $\xi \mapsto K(z,\xi)$ satisfies the conditions for Lebesgue's Dominated Convergence Theorem on compact neighborhoods of z, we obtain

$$\frac{\partial}{\partial \bar{z}}(Pf)(z) = \int_\Omega f(\xi) \frac{\partial}{\partial \bar{z}} K(z,\xi)\, dA(\xi) = 0.$$

2) If $f \in A^2(\Omega)$, by (10) the claim follows.

3) Let $\Pi : L^2(\Omega, dA) \to A^2(\Omega)$ be the orthogonal projection onto the closed subspace $A^2(\Omega)$. We know that any $f \in L^2(\Omega, dA)$ can be decomposed uniquely as $f = \Pi f + f^\perp$, where $\Pi f \in A^2(\Omega)$ and $f^\perp \in A^2(\Omega)^\perp$. Using the definition of P and the inner product in L^2,

$$Pf(z) = \langle f, K(\cdot,z)\rangle = \langle \Pi f + f^\perp, K(\cdot,z)\rangle = \langle \Pi f, K(\cdot,z)\rangle + \langle f^\perp, K(\cdot,z)\rangle.$$

Since $K(\cdot,z) \in A^2(\Omega)$ and $f^\perp$ is orthogonal to $A^2(\Omega)$, we have $\langle f^\perp, K(\cdot,z)\rangle = 0$. Thus, $Pf(z) = \langle \Pi f, K(\cdot,z)\rangle$. But Πf is a function in $A^2(\Omega)$, so by 2), $\langle \Pi f, K(\cdot,z)\rangle = \Pi f(z)$.

Therefore, $Pf(z) = \Pi f(z)$ for all $z \in \Omega$, which implies $P = \Pi$. This confirms P is the orthogonal projection. □

Example 2.14. Using (12), for all $f \in L^2(\mathbb{D}, dA)$, the Bergman projection in the unit disk is

$$P_{\mathbb{D}} f(z) = \frac{1}{\pi} \int_{\mathbb{D}} \frac{f(\xi)}{(1 - z\bar{\xi})^2}\, dA(\xi),$$

so $Pf = f$ when $f \in A^2(\mathbb{D})$. ♣

2.3. **The connection with the Riemann Mapping Theorem and the boundary regularity of conformal maps.** In this section, we establish the foundational links between the Bergman kernel and the classical theory of conformal mappings. The Bergman kernel is not merely a reproducing tool; it is deeply intertwined with the Riemann Mapping Theorem. Furthermore, the Bergman projection serves as a robust framework for investigating the boundary regularity of these maps, offering elegant proofs for when and how a conformal map extends continuously or smoothly to the boundary of its domain.

A central question is how the Bergman kernel behaves under a change of variables. Specifically, if two domains are conformally equivalent, their kernels are not just related, but are tied together by a precise transformation law. After recalling the Riemann Mapping Theorem, we demonstrate that given a univalent (holomorphic and injective) mapping between two domains, we can explicitly relate their respective Bergman kernels using the derivative of the mapping.

Theorem 2.15 (**Riemann Mapping Theorem**). *If $\Omega \subset \mathbb{C}$ is a domain, properly contained in $\mathbb{C}$ and simply connected, then for all $\zeta \in \Omega$, there exists a unique biholomorphism $\varphi_\zeta : \Omega \to \mathbb{D}$ such that $\varphi_\zeta(\zeta) = 0$ and $\varphi'_\zeta(\zeta) > 0$.*

This mapping is called the Riemann map of Ω and ζ.

Theorem 2.16. *Let $\Omega_1, \Omega_2 \subset \mathbb{C}$ be two domains, and let $\varphi : \Omega_1 \to \Omega_2$ be a univalent mapping between them. If we denote by K_i the Bergman kernel of $A^2(\Omega_i)$, for $i = 1, 2$, then*

$$K_1(z, \xi) = K_2(\varphi(z), \varphi(\xi))\, \varphi'(z)\overline{\varphi'(\xi)}.$$

Proof. Define the linear operator $T : A^2(\Omega_2) \to A^2(\Omega_1)$ by $Tf(z) = (f \circ \varphi)(z)\varphi'(z)$. Then, since φ is univalent, we can write

$$\begin{aligned} \|f\|^2_{A^2(\Omega_2)} &= \int_{\Omega_2} |f(w)|^2 dA(w) = \int_{\Omega_1} |f(\varphi(z))|^2 |\det J_\varphi(z)|^2 dA(z) \\ &= \int_{\Omega_1} |T(f)(z)|^2 dA(z) = \|Tf\|^2_{A^2(\Omega_1)} \end{aligned}$$

and therefore T is an isometry. Consider now $f \in A^2(\Omega_2)$ and define $h(z) = (Tf)(z) = f(\varphi(z))\varphi'(z)$. By the reproducing property in Ω_1,

$$\begin{aligned} h(z) &= \int_{\Omega_1} h(\xi) K_1(z, \xi)\, dA(\xi) = \int_{\Omega_1} f(\varphi(\xi))\varphi'(\xi) K_1(z, \xi)\, dA(\xi) \\ &= \int_{\Omega_1} f(\varphi(\xi)) \frac{1}{\overline{\varphi'(\xi)}} K_1(z, \xi) |\varphi'(\xi)|^2 dA(\xi). \end{aligned}$$

Changing the variables $w = \varphi(\xi)$, $dA(w) = |\varphi'(\xi)|^2 dA(\xi)$, then

$$h(z) = \int_{\Omega_2} f(w) \frac{1}{\overline{\varphi'(\varphi^{-1}(w))}} K_1(z, \varphi^{-1}(w))\, dA(w). \tag{14}$$

But we know that, by the reproducing property in Ω_2 in the point $w = \varphi(z)$,

$$f(\varphi(z)) = \int_{\Omega_2} f(w) K_2(\varphi(z), w) dA(w).$$

Thus

$$h(z) = f(\varphi(z))\varphi'(z) = \int_{\Omega_2} f(w) K_2(\varphi(z), w)\varphi'(z) dA(w). \tag{15}$$

Finally, we obtain from (14) and (15) that, since the reproducing kernel is unique,

$$\overline{\varphi'(\varphi^{-1}(w))} K_1(z, \varphi^{-1}(w)) = K_2(\varphi(z), w)\varphi'(z).$$

Putting $\xi = \varphi^{-1}(w)$, we obtain the desired result. □

Example 2.17 (Bergman kernel of $\mathbb{H}$). Consider the upper half-plane $\mathbb{H} = \{\Im(z) > 0\}$. It is known that the Cayley transform, given by

$$\Phi : \mathbb{H} \ni z \mapsto \frac{z - i}{z + i} \in \mathbb{D} \tag{16}$$

is a biholomorphism between $\mathbb{H}$ and the unit disk. In this case we have

$$\Phi'(z) = \frac{(z+i) - (z-i)}{(z+i)^2} = \frac{2i}{(z+i)^2} \implies \Phi'(z)\overline{\Phi'(\xi)} = \frac{4}{(z+i)^2(\overline{\xi} - i)^2}.$$

Employing Theorem 2.16,

$$K_{\mathbb{H}}(z, \xi) = K_{\mathbb{D}}(\Phi(z), \Phi(\xi))\Phi'(z)\overline{\Phi'(\xi)},$$

but from (12) we know

$$K_{\mathbb{D}}(\varphi(z), \varphi(\xi)) = \frac{1}{\pi\left(1 - \frac{z-i}{z+i} \cdot \overline{\left(\frac{\xi - i}{\xi + i}\right)}\right)^2},$$

and doing some computations

$$1 - \frac{(z-i)(\overline{\xi} + i)}{(z+i)(\overline{\xi} - i)} = \frac{(z\overline{\xi} - zi + i\overline{\xi} + 1) - (z\overline{\xi} + zi - i\overline{\xi} + 1)}{(z+i)(\overline{\xi} - i)} = \frac{-2i(z - \overline{\xi})}{(z+i)(\overline{\xi} - i)}$$

we finaly arrive at

$$K_{\mathbb{H}}(z, \xi) = \frac{1}{\pi} \frac{(z+i)^2(\overline{\xi} - i)^2}{-4(z - \overline{\xi})^2} \frac{4}{(z+i)^2(\overline{\xi} - i)^2} = \frac{-1}{\pi(z - \overline{\xi})^2}. \tag{17}$$

♣

Remark 2.18. We will use this "invariance" property of the Bergman kernel to study either the boundary regularity of conformal mappings (see Theorem 2.36) or to explicitly compute the derivative of the Riemann mappings (see Corollary 2.19).

Corollary 2.19. *Let $\Omega \subsetneq \mathbb{C}$ be a simply connected domain and let K be the Bergman kernel of $A^2(\Omega)$. Let $\varphi_\zeta : \Omega \to \mathbb{D}$ be the Riemann map of Ω and $\zeta \in \Omega$. Then, for all $z \in \Omega$,*

$$\varphi'_\zeta(z) = \sqrt{\frac{\pi}{K(\zeta,\zeta)}} K(z,\zeta). \tag{18}$$

Proof. Fix $\zeta \in \Omega$. From Theorem 2.16 it follows that $K(z,\zeta) = J(\varphi_\zeta(z), \varphi_\zeta(\zeta))\varphi'_\zeta(z)\overline{\varphi'_\zeta(\zeta)}$, where $J(\omega,\xi)$ is the Bergman kernel of $A^2(\mathbb{D})$, given in (12). Thus, using that $\varphi_\zeta(\zeta) = 0$ and $\varphi'_\zeta(\zeta) > 0$,

$$K(z,\zeta) = \frac{1}{\pi} \frac{1}{(1 - \varphi_\zeta(z)\overline{\varphi_\zeta(\zeta)})^2} \varphi'_\zeta(z)\overline{\varphi'_\zeta(\zeta)} = \frac{1}{\pi}\varphi'_\zeta(z)\varphi'_\zeta(\zeta).$$

Therefore $K(\zeta,\zeta) = \frac{1}{\pi}(\varphi'_\zeta(\zeta))^2$ and the claim follows. □

This result shows that knowing the Bergman kernel of the domain Ω, we can explicitly give the biholomorphism between Ω and $\mathbb{D}$, since from (18) we deduce

$$\varphi_\zeta(z) = \sqrt{\frac{\pi}{K(\zeta,\zeta)}} \int_\zeta^z K(w,\zeta)\, dw. \tag{19}$$

Example 2.20. Fix $\zeta = i$. On Example 2.17 we have seen that the Bergman kernel of the upper half-plane is given by

$$K_{\mathbb{H}}(z,\xi) = \frac{-1}{\pi(z - \overline{\xi})^2}.$$

Using now (19),

$$\begin{aligned}\varphi_\zeta(z) &= \sqrt{\frac{\pi}{K_{\mathbb{H}}(\zeta,\zeta)}} \int_\zeta^z K_{\mathbb{H}}(w,\zeta)\, dw = \frac{-2\pi\Im(\zeta)}{\pi} \int_\zeta^z \frac{1}{(w - \overline{\zeta})^2}\, dw = 2\Im(\zeta)\frac{1}{w - \overline{\zeta}}\bigg|_{w=\zeta}^{z} \\ &= 2\Im(\zeta)\left(\frac{1}{z - \overline{\zeta}} - \frac{1}{\zeta - \overline{\zeta}}\right) = \left\{\frac{2\Im(\zeta) + i(z - \overline{\zeta})}{i(z - \overline{\zeta})}\right\}.\end{aligned}$$

Thus, the map

$$\varphi_i(z) = \left\{\frac{2 + i(z + i)}{i(z + i)}\right\} = \frac{z - i}{z + i}$$

is a biholomorphism between $\mathbb{H}$ and $\mathbb{D}$; the Cayley transform given in (16). ♣

Definition 2.21. Let $\Omega \subset \mathbb{C}$ be a domain. We say that Ω satisfies *Bell's condition R* if for each $k \in \{0,1,2,\ldots\}$, there exists $\ell \in \{0,1,2,\ldots\}$ such that whenever $f : \Omega \to \mathbb{C}$ has bounded derivatives up to order ℓ, then $Pf : \Omega \to \mathbb{C}$, where P the Bergman projection of $A^2(\Omega)$, has bounded derivatives up to order k.

Definition 2.22. Let $\Omega \subset \mathbb{C}$ be a domain. We say that a function $\Phi : \Omega \to \mathbb{C}$ is k *times boundedly continuously differentiable*, and we write $\phi \in \mathscr{C}_b^k(\Omega)$, if all partial derivatives

$$\left(\frac{\partial}{\partial x}\right)^s \left(\frac{\partial}{\partial y}\right)^t \Phi$$

of order $s+t \leq k$ on Ω exist, are continuous and bounded. In this case we set

$$\|\Phi\|_{\mathscr{C}_b^k(\Omega)} = \sum_{s+t\leq k} \sup_{\Omega} \left\| \left(\frac{\partial}{\partial x}\right)^s \left(\frac{\partial}{\partial y}\right)^t \Phi(x+iy) \right\|$$

and $\mathscr{C}_b^\infty(\Omega) = \cap_k \mathscr{C}_b^k(\Omega)$.

A function $\Phi : \overline{\Omega} \to \mathbb{C}$ is called $\mathscr{C}^k$ if all of its partial derivatives up to order k extend continuously to $\overline{\Omega}$. As before, $\mathscr{C}^\infty(\overline{\Omega}) = \cap_k \mathscr{C}^k(\overline{\Omega})$

Definition 2.23. If $f, g \in \mathscr{C}^k(\overline{\Omega})$, then we say that f and g *agree up to order k on $\partial\Omega$* if

$$\left(\frac{\partial}{\partial x}\right)^s \left(\frac{\partial}{\partial y}\right)^t (f(z) - g(z))\Big|_{\partial\Omega} = 0$$

for all s, t such that $s + t \leq k$.

We say that a bounded domain $\Omega \subset \mathbb{C}$ has $\mathscr{C}^\infty$ (or *smooth*) boundary if there is a function $\rho : \mathbb{C} \to \mathbb{R}$ such that

(1) ρ is $\mathscr{C}^\infty$;
(2) $\Omega = \{z \in \mathbb{C} : \rho(z) < 0\}$;
(3) $\nabla\rho$ is nowhere zero on $\partial\Omega$.

Example 2.24. In the case of the disk $\mathbb{D}$, $\rho(z) = |z|^2 - 1$. Notice ρ is $\mathscr{C}^\infty$ and $|\nabla\rho| = 2$ on $\partial\mathbb{D}$. Hence $\mathbb{D}$ has smooth boundary. ♣

Lemma 2.25. *Let $\Omega = \{z \in \mathbb{C} : \rho(z) < 0\} \subset \mathbb{C}$ be a bounded domain with $\mathscr{C}^\infty$ boundary, and let U be a neighborhood of $\partial\Omega$, on which $|\nabla\rho| \geq c > 0$. Then there exists a $\mathscr{C}^\infty$ function $\alpha : \overline{\Omega} \to \mathbb{C}$ such that:*

(1) $\alpha = 0$ on $\Omega \setminus U$;
(2) $\alpha = 1$ in a neighborhood of $\partial\Omega$.

Proof. Define the functions, for $t \in \mathbb{R}$,

$$h(t) = \begin{cases} e^{-1/t} & t > 0 \\ 0 & t \leq 0 \end{cases}, \qquad g(s) = \frac{h(s)}{h(s) + h(1-s)}.$$

Then $g(s) = 0$ if $s \leq 0$ and $g(s) = 1$ if $s \geq 1$. Since U is a neighborhood of $\partial\Omega$, there exists $\varepsilon > 0$ such that $\{z \in \overline{\Omega} : -\varepsilon < \rho(z) < 0\} \subset U$. Let $\varepsilon_1, \varepsilon_2$ such that $0 < \varepsilon_2 < \varepsilon_1 < \varepsilon$. Define α as $\alpha(z) = g\left(\frac{\rho(z)+\varepsilon_1}{\varepsilon_1 - \varepsilon_2}\right)$. Since both g and ρ are $\mathscr{C}^\infty$, α is also $\mathscr{C}^\infty$. If z is near the boundary, i.e., $\rho(z) > -\varepsilon_2$, then

$$\frac{\rho(z) + \varepsilon_1}{\varepsilon_1 - \varepsilon_2} > \frac{-\varepsilon_2 + \varepsilon_1}{\varepsilon_1 - \varepsilon_2} = 1,$$

so $\alpha(z) = 1$ in the neighborhood $\{z : \rho(z) > -\varepsilon_2\}$. If $z \in \Omega \setminus U$, $\rho(z) < -\varepsilon < -\varepsilon_1$. In this case

$$\frac{\rho(z) + \varepsilon_1}{\varepsilon_1 - \varepsilon_2} < \frac{-\varepsilon_1 + \varepsilon_1}{\varepsilon_1 - \varepsilon_2} = 0$$

and hence $\alpha(z) = 0$. □

Lemma 2.26. *Let $\Omega = \{z \in \mathbb{C} : \rho(z) < 0\}$ be a bounded domain with $\mathscr{C}^\infty$ boundary. Given $g \in \mathscr{C}^1(\overline{\Omega})$, choose $f = \rho g$. If $h \in A^2(\Omega)$, then*

$$\int_\Omega \overline{h(\zeta)} \frac{\partial}{\partial \zeta} f(\zeta)\, dA(\zeta) = 0.$$

Proof. The idea is to integrate by parts, but h is not defined in the boundary of Ω. Let $\varepsilon > 0$ small and let $\Omega_\varepsilon = \{z : \rho(z) < -\varepsilon\}$. Define $f_\varepsilon(z) = (\rho(z) + \varepsilon) g(z)$. It is clear that $f_\varepsilon \in \mathscr{C}^1(\overline{\Omega}_\varepsilon)$ and $f_\varepsilon\big|_{\partial\Omega_\varepsilon} = 0$. Since $h \in \mathscr{H}(\Omega) \subset \mathscr{H}(\overline{\Omega}_\varepsilon)$, we have that h is $\mathscr{C}^\infty(\overline{\Omega}_\varepsilon)$. By Green's Theorem (Theorem 5.2),

$$\int_{\Omega_\varepsilon} \overline{h(\zeta)} \frac{\partial}{\partial \zeta} f_\varepsilon(\zeta)\, dA(\zeta) = -\int_{\Omega_\varepsilon} \left(\frac{\partial}{\partial \zeta} \overline{h(\zeta)} \right) f_\varepsilon(\zeta)\, dA(\zeta).$$

The integral over the boundary is zero since $h \cdot f_\varepsilon\big|_{\partial\Omega_\varepsilon} = 0$. But $h \in A^2(\Omega)$, so $\partial \overline{h}/\partial \zeta = 0$, so this implies

$$\int_{\Omega_\varepsilon} \overline{h(\zeta)} \frac{\partial}{\partial \zeta} f_\varepsilon(\zeta)\, dA(\zeta) = 0. \tag{20}$$

Finally, if χ_ε denotes the characteristic function on Ω_ε,

$$\begin{aligned}
\left| \int_\Omega \overline{h(\zeta)} \frac{\partial}{\partial \zeta} f(\zeta)\, dA(\zeta) \right| &\overset{(20)}{=} \left| \int_\Omega \overline{h(\zeta)} \frac{\partial}{\partial \zeta} f(\zeta)\, dA(\zeta)) - \int_\Omega \overline{h(\zeta)} \chi_\varepsilon(\zeta) \frac{\partial}{\partial \zeta} f_\varepsilon(\zeta)\, dA(\zeta) \right| \\
&\leq \int_\Omega |\overline{h(\zeta)}| \left| \frac{\partial f}{\partial \zeta}(\zeta) \right| |1 - \chi_\varepsilon(\zeta)|\, dA(\zeta) \\
&\quad + \int_\Omega |\overline{h(\zeta)}| |\chi_\varepsilon(\zeta)| \cdot \left| \frac{\partial f}{\partial \zeta}(\zeta) - \frac{\partial f_\varepsilon}{\partial \zeta}(\zeta) \right| dA(\zeta).
\end{aligned}$$

Now, by the Cauchy-Schwarz inequality, this is

$$\begin{aligned}
&\leq \left(\int_\Omega |\overline{h(\zeta)}|^2 \left| \frac{\partial f}{\partial \zeta}(\zeta) \right|^2 dA(\zeta) \right)^{1/2} \cdot \left(\int_\Omega |1 - \chi_\epsilon(\zeta)|^2 dA(\zeta) \right)^{1/2} \\
&\quad + \left(\int_\Omega |\overline{h(\zeta)}|^2 dA(\zeta) \right)^{1/2} \cdot \left(\int_\Omega \left| \frac{\partial f}{\partial \zeta}(\zeta) - \frac{\partial f_\epsilon}{\partial \zeta}(\zeta) \right|^2 dA(\zeta) \right)^{1/2}.
\end{aligned}$$

Taking now the limit as $\varepsilon \to 0^+$, the term $\int_\Omega |1 - \chi_\varepsilon(\zeta)|^2 dA(\zeta)$ goes to zero. Also we see that $\partial f_\varepsilon/\partial \zeta \to \partial f/\partial \zeta$ uniformly. This is because Ω is a bounded domain with $\mathscr{C}^1$ boundary and g is $\mathscr{C}^1(\overline{\Omega})$, so $\partial g/\partial \zeta$ is a continuous function in a compact set. Therefore there exists $M = \sup_{z \in \overline{\Omega}} \left| \frac{\partial g}{\partial \zeta} \right| < \infty$ and in particular

$$\sup_{z \in \overline{\Omega}} \left| \frac{\partial f_\varepsilon}{\partial \zeta} - \frac{\partial f}{\partial \zeta} \right| \leq \varepsilon M.$$

Taking the limit as $\varepsilon \to 0^+$ the claim follows. ∎

Proposition 2.27 (Bell's lemma). *Let $\Omega = \{z \in \mathbb{C} : \rho(z) < 0\}$ be a bounded domain with $\mathscr{C}^\infty$ boundary, with $|\nabla \rho| \geq 0$ in a neighborhood of $\partial\Omega$. Let $u \in \mathscr{C}^\infty(\overline{\Omega})$ and $m \geq 1$. Then there is a $g \in \mathscr{C}^{m-1}(\overline{\Omega})$ which agrees with u up to order $m-1$ on $\partial\Omega$ and such that $Pg = 0$, where P the Bergman projection of $A^2(\Omega)$.*

Proof. Let α be as in Lemma 2.25. We will define the function g by induction. For the case $\mathscr{C}^1$, let

$$w_1(z) = \frac{\alpha(z)u(z)\rho(z)}{(\partial\rho/\partial z)(z)}$$

and

$$v_1(z) = \frac{\partial}{\partial z} w_1(z) = \alpha(z)u(z) + \rho(z) \underbrace{\frac{\partial}{\partial z}\left(\frac{\alpha(z)u(z)}{(\partial\rho/\partial z)(z)}\right)}_{=:\eta_1}.$$

The function η_1 is continuous on $\overline{\Omega}$, so $v_1 - u = \rho\,\eta_1$ near $\partial\Omega$. This means $(v_1-u)\big|_{\partial\Omega} = 0$, so v_1 and u agree to order zero on $\partial\Omega$. Also, by Lemma 2.26,

$$Pv_1(z) = \int_\Omega K(z,\zeta)\frac{\partial}{\partial\zeta} w_1(\zeta)\,dA(\zeta) = 0.$$

Suppose now that we have w_{k-1} and $v_{k-1} = \frac{\partial}{\partial z} w_{k-1}$ such that v_{k-1} agrees to order $(k-1)-1$ with u on $\partial\Omega$ and $P(v_{k-1}) = 0$. We shall construct a function w_k of the form

$$w_k = w_{k-1} + \theta_k \rho^k \tag{21}$$

such that $v_k = \frac{\partial}{\partial z} w_k$ agrees to order $k-1$ with u on $\partial\Omega$ and $Pv_k = 0$. We define the differential operator $\mathcal{D} : \mathscr{C}^k(\overline{\Omega}) \to \mathscr{C}^{k-1}(\overline{\Omega})$ by

$$\mathcal{D}(\phi) = \frac{\alpha}{|\partial\rho/\partial z|^2} \Re\left(\frac{\partial\rho}{\partial z}\frac{\partial\phi}{\partial\overline{z}}\right).$$

If $z \in \partial\Omega$,

$$\mathcal{D}(\rho)(z) = \frac{\alpha(z)}{|\partial\rho/\partial z|^2} \Re\left(\frac{\partial\rho}{\partial z}\overline{\frac{\partial\rho}{\partial z}}\right) = \frac{\alpha(z)}{|\partial\rho/\partial z|^2}\left|\frac{\partial\rho}{\partial z}\right|^2 = 1.$$

Defining now

$$\theta_k = \frac{\alpha\mathcal{D}^{k-1}(u - v_{k-1})}{k!\,\partial\rho/\partial z},$$

with w_k defined as in (21) and $v_k = \frac{\partial}{\partial z} w_k$, we have

$$\begin{aligned}\mathcal{D}^{k-1}(u - v_k) &= \mathcal{D}^{k-1}(u) - \mathcal{D}^{k-1}\left\{\frac{\partial}{\partial z}(w_{k-1} + \theta_k\rho^k)\right\} \\ &= \mathcal{D}^{k-1}(u - v_{k-1}) - \mathcal{D}^{k-1}\left\{\frac{\partial\theta_k}{\partial z}\rho^k\right\} - \mathcal{D}^{k-1}\left\{\theta_k k\frac{\partial\rho}{\partial z}\rho^{k-1}\right\}.\end{aligned}$$

In the boundary, using $\mathcal{D}^m(\rho^m) = m!(\mathcal{D}\rho)^m$, we obtain

$$\begin{aligned}\mathcal{D}^{k-1}(u-v_k) &= \mathcal{D}^{k-1}(u-v_{k-1}) - 0 - \left(k\theta_k \frac{\partial\rho}{\partial z}\right)(k-1)! \\ &= \mathcal{D}^{k-1}(u-v_{k-1}) - k!\frac{\mathcal{D}^{k-1}(u-v_{k-1})}{k!\partial\rho/\partial z}\frac{\partial\rho}{\partial z} \\ &= \mathcal{D}^{k-1}(u-v_{k-1}) - \mathcal{D}^{k-1}(u-v_{k-1}) = 0.\end{aligned} \tag{22}$$

Notice now that any directional derivative at a point $p \in \partial\Omega$ is a linear combination of

$$\mathcal{D} = a(p)\frac{\partial}{\partial x} + b(p)\frac{\partial}{\partial y} \quad \text{and} \quad \tau = -b(p)\frac{\partial}{\partial x} + a(p)\frac{\partial}{\partial y},$$

—so τ (tangent) is the orthogonal operator of $\mathcal{D}$ (normal)—. Therefore, we shall prove that

$$\tau^s\mathcal{D}^t(u-v_k)\Big|_{\partial\Omega} = 0$$

for all s, t such that $s+t = k-1$ (notice that the cases where $s+t < k-1$ follow from the inductive hypothesis and the explicit form of w_k in (21)). The case $s = 0, t = k-1$ is exactly (22). If $s \geq 1$, write

$$\tau^s\mathcal{D}^t(u-v_k) = \tau(\underbrace{\tau^{s-1}\mathcal{D}^t(u-v_k)}_{=:G}) \tag{23}$$

Since $s-1+t = k-2$, $G = 0$ in the boundary. Thus, the tangential derivative of G, $\tau(G)$, must also be zero on the boundary. This shows that (23) is zero in $\partial\Omega$.

We have constructed v_m, so it suffices to set $g := v_m \in \mathscr{C}^{m-1}(\overline{\Omega})$. Then

$$Pg(z) = \int_\Omega K(z,\xi)g(\xi)\,dA(\xi) = \int_\Omega K(z,\xi)\frac{\partial}{\partial\xi}w_m(\xi)\,dA(\xi) = 0$$

by Lemma 2.26. □

Using Bell's lemma (Proposition 2.27) on the disk, we can deduce the following facts:

(a) If $K_{\mathbb{D}}$ is the Bergman kernel of $A^2(\mathbb{D})$, then

$$\left|\left(\frac{\partial}{\partial z}\right)^k K_{\mathbb{D}}(z,w)\right| = \left|\frac{(k+1)!\overline{w}^k}{\pi(1-z\overline{w})^{k+2}}\right| \leq \frac{(k+1)!}{(1-|w|)^{k+2}}.$$

(b) If $u \in \mathscr{C}^k(\overline{\mathbb{D}})$ and vanishes to order k at $\partial\mathbb{D}$ (i.e., u agrees with the zero function to order k at $\partial\mathbb{D}$), then there exists $C > 0$ such that $|u(z)| \leq C(1-|z|)^k$.

Theorem 2.28 (**Bell's condition R for the disk**). *If $k \geq 1$ and $u \in \mathscr{C}^{k+2}(\overline{\mathbb{D}})$, then $\|P_{\mathbb{D}}(u)\|_{\mathscr{C}^k_b(\mathbb{D})} < \infty$.*

Proof. From Bell's lemma there exists a function $v \in \mathscr{C}^{k+2}(\overline{\mathbb{D}})$ which agrees with u to order $k+2$ on $\partial\mathbb{D}$ and $P_{\mathbb{D}}(v) = 0$. Then $P_{\mathbb{D}}(u-v) = P_{\mathbb{D}}(u)$ and $v-u$ vanishes to order

$k+2$ on $\partial\mathbb{D}$. In particular, by **(b)** above, $|u(z)-v(z)| \leq C(1-|z|)^{k+2}$, so for $j \leq k$ we have

$$\begin{aligned}\left|\left(\frac{\partial}{\partial z}\right)^j P_{\mathbb{D}}(u)(z)\right| &= \left|\left(\frac{\partial}{\partial z}\right)^j P_{\mathbb{D}}(u-v)(z)\right| \\ &= \left|\int_{\mathbb{D}} \left(\frac{\partial}{\partial z}\right)^j K_{\mathbb{D}}(z,\zeta)(u-v)(\zeta)\, dA(\zeta)\right| \\ &\overset{\text{(a)}}{\leq} \int_{\mathbb{D}} \frac{(j+1)!}{(1-|\zeta|)^{j+2}} C(1-|\zeta|)^{k+2}\, dA(\zeta) < \infty.\end{aligned}$$

Hence, it is bounded and the claim follows. □

Actually, the point **(b)** above holds on any bounded Ω with $\mathscr{C}^k$ boundary. The next result is given as a corollary of Theorem 2.16.

Corollary 2.29 (**Bell's projection formula**). *Let $\Omega_1, \Omega_2 \subset \mathbb{C}$ be two domains, and let $\Phi : \Omega_1 \to \Omega_2$ be a univalent mapping between them. If we denote by P_i the Bergman projection of $A^2(\Omega_i)$, for $i = 1, 2$, then*

$$P_2((\Phi^{-1})' \cdot (f \circ \Phi^{-1})) = (\Phi^{-1})' \cdot (P_1(f) \circ \Phi^{-1}).$$

From Theorem 2.15, we can infer that any two simply connected domains $\Omega_1, \Omega_2 \subsetneq \mathbb{C}$ are conformally equivalent; that is, there exists a biholomorphic mapping $\varphi : \Omega_1 \to \Omega_2$. However, for practical applications, the focus often shifts to whether φ (and its inverse φ^{-1}) can be extended continuously to the boundary $\partial\Omega_1$.One might intuitively expect that the smoother or "nicer" the boundaries of Ω_1 and Ω_2, the more likely such a continuous extension exists. In fact, this phenomenon is remarkably consistent: the extension exists in almost all standard cases, and its regularity increases in tandem with the regularity of the boundaries themselves. This relationship is a well-documented area of complex analysis, and the following theorem defines the specific conditions required for the domain's boundary:

Theorem 2.30 (**Carathéodory**). *Let $\Omega_1, \Omega_2 \subset \mathbb{C}$ be bounded domains in $\mathbb{C}$, each of which is bounded by a single Jordan (simple closed) curve*[5]*. If $\varphi : \Omega_1 \to \Omega_2$ is a biholomorphism, then φ extends continuously and one-to-one to $\partial\Omega_1$. That is, there is a continuous, injective function $\hat{\varphi} : \overline{\Omega}_1 \to \overline{\Omega}_2$ such that $\hat{\varphi}|_{\Omega_1} = \varphi$.*

The proof of Theorem 2.30 can be found in [12] or [6]. The first proof by Carathéodory use properties of Lebesgue measure on the circle. The remarkable fact is that, in fact, we can say a little bit more:

Let $\Omega = \{z \in \mathbb{C} : \rho(z) < 0\}$ be a bounded simply connected domain with $\mathscr{C}^\infty$ boundary, and let $F : \mathbb{D} \to \Omega$ be a conformal mapping. The continuous one-to-one extension of F to $\overline{\mathbb{D}}$, by Theorem 2.30, is actually in $\mathscr{C}^\infty(\overline{\mathbb{D}})$ (see Theorem 2.36). We introduce a few lemmas before to prove it.

[5]Notice that the domains are simply connected since each has complement with just one connected component.

Lemma 2.31 (**Hopf's lemma**). *Let $U \subset \mathbb{C}$ be a smoothly bounded (bounded and with $\mathscr{C}^\infty$ boundary) domain. Let $u : \overline{U} \to \mathbb{R}$ be a harmonic function on U, continuous on $\overline{U}$. Suppose that u assumes a local maximum value at $p \in \partial U$, and let ν be the unit outward normal vector to ∂U at p. Then the one-sided lower derivative $\partial u/\partial \nu$ is positive, i.e.,*

$$\frac{\partial u}{\partial \nu} := \liminf_{t \to 0^+} \frac{u(p) - u(p - t\nu)}{t} > 0.$$

Remark 2.32. Notice that the fact that p is a local maximum guaranties that the one-sided lower derivative is non-negative .The proof utilizes the Poisson kernel and Harnack's inequality; for a detailed treatment of these classical tools, we refer the reader to [7].

Lemma 2.33. *Let $\Omega \subset \mathbb{C}$ be a smoothly bounded simply connected domain and let $F : \mathbb{D} \to \Omega$ be a biholomorphism. Then there is a constant $C > 0$ such that for all $z \in \mathbb{D}$,*

$$\operatorname{dist}(F(z), \partial\Omega) \leq C(1 - |z|).$$

Proof. Define $u : w \mapsto \log |F^{-1}(w)|$. This function is well defined in $\Omega \setminus \{F(0)\}$. For all $w_0 \in \Omega \setminus \{F(0)\}$, we can choose an analytic branch of the logarithm, defined as

$$\operatorname{Log}(z) = \log(w_0) + \int_{w_0}^{z} \frac{d\xi}{\xi}.$$

Then we can write $u(w) = \Re\{\operatorname{Log}(F^{-1}(w))\}$. It is known that the real part of a holomorphic function is harmonic, and since F^{-1} and Log are holomorphic, in fact u it is harmonic there. Furthermore, it is continuous on $(\Omega \setminus \{F(0)\}) \cup \partial\Omega$ and $u \equiv 0$ on $\partial\Omega$. This is a consequence of Theorem 2.30: since Ω has a smooth boundary and is bounded, in particular $\partial\Omega$ is a Jordan curve. If we take $w \in \partial\Omega$, $F^{-1}(w) \in \partial\mathbb{D}$, so $u(w) = \log(1) = 0$. In particular, it attains a (global) maximum at every point of $\partial\Omega$, since $|F^{-1}(w)| < 1$ if $w \in \Omega \setminus \{F(0)\}$. We can apply therefore Hopf's lemma: for all point $p \in \partial\Omega$,

$$\frac{\partial u}{\partial \nu}(p) = \liminf_{t \to 0^t} \frac{u(p) - u(p - t\nu)}{t} = L > 0.$$

The points $w_t = p - t\nu$ verify that $\operatorname{dist}(w_t, \partial\Omega) = t$, so

$$\liminf_{t \to 0^+} \frac{0 - u(w_t)}{t} = L > 0.$$

By definition, there exists therefore $\delta > 0$ such that for all $0 < t = \operatorname{dist}(w_t, \partial\Omega) < \delta$, $-u(w_t) \geq c \operatorname{dist}(w_t, \partial\Omega)$. Hence,

$$\operatorname{dist}(F(z), \partial\Omega) \leq \frac{1}{c}|u(F(z))| = \frac{1}{c}|\log |z||$$

for $z \in \mathbb{D}$ near the boundary. This allows us to express the logarithm as $(1 - |z|)$, and the claim follows.[6] □

[6] Notice that no differentiability at boundary points is assumed in the lemma (and also in Lemma 2.31). The derivative estimates are on the "lower derivative" only, which, as a lim inf always exists.

Lemma 2.34. *Let $\Omega \subset \mathbb{C}$ be a smoothly bounded simply connected domain and let $F : \mathbb{D} \to \Omega$ be a biholomorphic mapping. Then*

(1) For $k \in \{0, 1, \ldots\}$, there exists a constant $C_k > 0$ such that for all $z \in \mathbb{D}$,

$$\left| \left(\frac{\partial}{\partial z} \right)^k F(z) \right| \leq \frac{C_k}{(1 - |z|)^k}.$$

(2) If $\psi \in \mathscr{C}^{2k+2}(\overline{\Omega})$ vanishes to order $2k + 2$ on $\partial\Omega$, then $F' \cdot (\psi \circ F) \in \mathscr{C}^k_b(\mathbb{D})$.

Proof. 1) F is bounded. It suffices to apply Cauchy estimates on the disk $D(z, (1-|z|))$.
2) For $j \leq k$ we have

$$\left| \left(\frac{\partial}{\partial z} \right)^j [F' \cdot (\psi \circ F)] \right| = \left| \sum_{\ell=0}^{j} \binom{j}{\ell} \left(\frac{\partial}{\partial z} \right)^\ell (F') \cdot \left(\frac{\partial}{\partial z} \right)^{j-\ell} (\psi \circ F) \right|. \tag{24}$$

But $\left(\frac{\partial}{\partial z}\right)^{j-\ell} (\psi \circ F)$ is a linear combination, with complex coefficients, of terms of the form

$$\left[\left(\left(\frac{\partial}{\partial z} \right)^m \psi \right) (F(z)) \right] \cdot \left(\frac{\partial}{\partial z} \right)^{n_1} F(z) \cdots \left(\frac{\partial}{\partial z} \right)^{n_k} F(z)$$

where $m \leq j - \ell$ and $n_1 + \cdots + n_k \leq j - \ell$. So (24) is dominated by

$$C \operatorname{dist}(F(z), \partial\Omega)^{2k+1-(j-\ell)} \prod_{i=1}^{k} \frac{1}{(1 - |z|)^{n_i}} \leq C \operatorname{dist}(F(z), \partial\Omega)^{k+1} \frac{1}{(1 - |z|)^{k+1}}$$
$$\leq C \frac{(1 - |z|)^{k+1}}{(1 - |z|)^{k+1}} = C.$$

where the last inequality follows from Lemma 2.33. □

Lemma 2.35. *Let $G : \mathbb{D} \to \mathbb{C}$ be holomorphic, such that*

$$\left| \left(\frac{\partial}{\partial z} \right)^j G(z) \right| \leq C_j < \infty$$

for $j = 0, \ldots, k + 1$. Then each $(\partial/\partial z)^j G$ extends continuously to $\overline{\mathbb{D}}$ for $j = 1, \ldots, k$.

Theorem 2.36 (Painlevé). *If $\Omega = \{z \in \mathbb{C} : \rho(z) < 0\}$ is a bounded, simply connected domain with $\mathscr{C}^\infty$ boundary[7], and $F : \mathbb{D} \to \Omega$ is a conformal mapping, then $F \in \mathscr{C}^\infty(\overline{\mathbb{D}})$ and $F^{-1} \in \mathscr{C}^\infty(\overline{\Omega})$.*

Proof. Let $k \geq 1$ and let P be the Bergman projection of $A^2(\Omega)$. By Proposition 2.27 applied to the function $u \equiv 1$ on Ω, there is a function $v \in \mathscr{C}^{2k+8}(\overline{\Omega})$ such that v agrees with u up to order $2k + 8$ on $\partial\Omega$ and $P(v) = 0$. If we define $\phi = 1 - v$, then

$$P(\phi) = P(1) - P(v) = \int_\Omega K(\cdot, \zeta) dA(\zeta) + 0 = 1.$$

[7] In particular, it is a Jordan curve. Therefore, by Theorem 2.30, we know $F \in \mathscr{C}^0(\overline{\mathbb{D}})$ and $F^{-1} \in \mathscr{C}^0(\overline{\Omega})$, while its existence is ensured by Theorem 2.15.

Moreover, ϕ vanishes to order $2k+8$ on $\partial\Omega$. By Lemma 2.34, $G' := F' \cdot (\phi \circ F) \in \mathscr{C}_b^{k+3}(\mathbb{D})$. Hence, each derivative is bounded:

$$\left| \left(\frac{\partial}{\partial z} \right)^j G'(z) \right| \leq C_j < \infty$$

for $j = 0, \ldots, k+3$. If $p \in \partial\mathbb{D}$, we define

$$G'(p) = \int_0^1 G''(tp) \cdot p \, dt + G'(0),$$

which clearly defines a continuous extension of G' to $\partial\mathbb{D}$. In fact, each $(\partial/\partial z)^j G'$ extends continuously to $\overline{\mathbb{D}}$ for $j = 1, \ldots, k+2$ (inductively), i.e. $G' \in \mathscr{C}^{k+2}(\overline{\mathbb{D}})$. Applying Theorem 2.28, $P_{\mathbb{D}}(F' \cdot (\phi \circ F)) \in \mathscr{C}^k(\mathbb{D})$ and has k bounded derivatives. Now, Bell's projection formula (Proposition 2.29) gives

$$P_{\mathbb{D}}(F' \cdot (\phi \circ F)) = F' \cdot (P(\phi) \circ F') = F',$$

so F' has bounded derivatives up to order k. Again, by the same reasoning as before, $F' \in \mathscr{C}^{k-1}(\overline{\mathbb{D}})$. Since k is arbitrary, we may conclude that $F \in \mathscr{C}^\infty(\overline{\mathbb{D}})$.

To show that $F^{-1} \in \mathscr{C}^\infty(\overline{\Omega})$, we shall prove that $|F'(z)| > 0$ for all $z \in \partial\mathbb{D}$. (we already know it is everywhere non-zero on $\mathbb{D}$). Since F is holomorphic, F satisfies the Cauchy-Riemann equations on $\overline{\mathbb{D}}$. Suppose then that there exists a point $p \in \partial\mathbb{D}$, with outward unit normal vector ν, such that $F'(p) = 0$. Consider $u = \rho \circ F$. Since $F \in \mathscr{C}^\infty(\overline{\mathbb{D}})$, $u \in \mathscr{C}^\infty(\overline{\mathbb{D}})$. Moreover, since we can assume ρ such that $\Delta\rho > 0$ near $\partial\Omega$, u is a subharmonic function on $\mathbb{D}$ that is continuous on $\overline{\mathbb{D}}$. By definition of the conformal map, $u < 0$ in $\mathbb{D}$ and $u|_{\partial\mathbb{D}} = 0$. Thus, u attains its global maximum value of 0 at every point of the boundary $\partial\mathbb{D}$, and in particular at p. Since $F'(p) = 0$, the chain rule yields:

$$\frac{\partial u}{\partial \nu}(p) = \nabla\rho(F(p)) \cdot DF(p)[\nu] = \nabla\rho(F(p)) \cdot F'(p)\nu = 0.$$

However, since u is a non-constant subharmonic function on $\mathbb{D}$ attaining its maximum at $p \in \partial\mathbb{D}$, Hopf's lemma (Lemma 2.31) says

$$\frac{\partial u}{\partial \nu}(p) > 0.$$

This yields a contradiction. Consequently, $F'(z) \neq 0$ for all $z \in \overline{\mathbb{D}}$. Since $F \in \mathscr{C}^\infty(\overline{\mathbb{D}})$ and its derivative never vanishes on the compact set $\overline{\mathbb{D}}$, it follows from the real Inverse Function Theorem up to the boundary that the global inverse mapping F^{-1} is uniquely determined and satisfies $F^{-1} \in \mathscr{C}^\infty(\overline{\Omega})$. This completes the proof. □

Remark 2.37. Clasically, Theorem 2.36 was proved with the study of Green's functions (see Definition 3.4). The ideas of this proof are due to S. Bell and E. Ligocka.

3. Characterization of domains with trivial Bergman space A^2

The main goal of this chapter is to prove Theorem 3.24, which characterizes the domains Ω admitting a nontrivial Bergman space $A^2(\Omega)$. In order to do so, we need to introduce several concepts from Potential Theory. The elements introduced here will also be fundamental in the treatment of the Suita conjecture in the final chapter of this thesis. We will outline only the properties strictly necessary for our study, presenting some of them without proof. The interested reader is referred to [7] and [13] for a more comprehensive treatment.

3.1. **The Green's function.** We introduce the Green's function, which serves as a fundamental tool for solving the Dirichlet problem (see Section §5.3). Historically, the Green's function $G(z, w)$ was designed to represent the potential at a point z generated by a unit impulse at a source point w, under the condition that the boundary $\partial\Omega$ is kept at zero potential.

Definition 3.1 (**Green's function I**)**.** Given a bounded domain $\Omega \subset \mathbb{C}$ and $w \in \Omega$, the *(negative) classical Green's function on* Ω *with pole at* w is a function $g_w \equiv g(\cdot, w) : \Omega \setminus \{w\} \to \mathbb{R}$ that satisfies

(1) g_w is harmonic on $\Omega \setminus \{w\}$;
(2) $g_w(\cdot) - \log|\cdot - w|$ is harmonic in a neighborhood of w;
(3) $\lim_{z\to p} g(z, w) = 0$ for all $p \in \partial\Omega$.

Remark 3.2. With this definition, we must discuss some properties of the function.

- Uniqueness: In fact, if there exists another function h_w verifying the properties, then by (2) $g_w - h_w$ is harmonic throughout Ω. But (3) implies that $g_w - h_w = 0$ on the boundary. From Theorem 5.8, it follows that $g_w = h_w$.
- Singularity: $g(\cdot, w)$ has a pole at w, with $g(z, w) \to -\infty$ as $z \to w$. This is a direct consequence of (2).
- Negativity: g_w is harmonic in $\Omega \setminus \{w\}$ and $\lim_{z\to w} g_w(z) = -\infty$ since $g_w(\cdot) - \log|\cdot - w|$ is harmonic at the point w. Again by Theorem 5.8, $g(\cdot, w) < 0$ for all $\Omega \setminus \{w\}$.
- Maximality: g_w is the maximal negative function such that $g_w(\cdot) - \log|\cdot - w|$ is harmonic in Ω; suppose there exists another function h_w such that $h_w < 0$ in $\Omega \setminus \{w\}$ and $h_w(\cdot) - \log|\cdot - w|$ is harmonic in Ω. Then consider $\Phi = h_w - g_w = (h_w(z) - \log|z - w|) - (g_w(z) - \log|z - w|)$, which is harmonic in Ω. Furthermore, for $p \in \partial\Omega$,

  $$\limsup_{z\to p} \Phi(z) = \limsup_{z\to p} (h_w(z) - g(z, w)) = \limsup_{z\to p} h_w(z) - 0 \leq 0.$$

  Since Φ is harmonic in Ω and $\Phi \leq 0$ in $\partial\Omega$, by Theorem 5.8, $\Phi \leq 0$ in Ω, so $h_w \leq g_w$, as we wanted to prove.

Proposition 3.3. *If $\Omega \subset \mathbb{C}$ is a bounded Dirichlet domain, then for each $w \in \Omega$ there is a (classical) Green's function on Ω with pole at w.*

Proof. Define the boundary function $f : \partial\Omega \to \mathbb{R}$ by $f(z) = -\log|z - w|$. By Corollary 5.19, we know there is a continuous function $u : \overline{\Omega} \to \mathbb{R}$ that solves this Dirichlet

problem. It is harmonic in Ω and $u = f$ in $\partial\Omega$, so the function $g_w(z) = u(z) + \log|z-w|$ is the Green's function on Ω with pole at w. Indeed, (1) is verified since $\log|\cdot - w| = \Re(\operatorname{Log}|\cdot - w|)$ is harmonic in $\mathbb{C}\setminus\{w\}$, where Log is an analytic branch of the logarithm. (2) is by construction: $g(z,w) - \log|z-w| = u(z)$, which is harmonic in the entire domain Ω as it solves the Dirichlet problem. Thus, it is harmonic in a neighborhood of w. (3) follows from the fact that for any $p \in \partial\Omega$, we have $G(p,w) = \log|p-w| - \log|p-w| = 0$. □

While Definition 3.1 provides a solid foundation for classical domains, its applicability is restricted to Dirichlet domains. In more general domains, a classical Green's function may not exist due to the presence of irregular boundary points. Therefore, we require a generalized definition to extend these concepts to any domain $\Omega \subset \mathbb{C}_\infty$ that possesses a non-constant positive superharmonic function (see Theorem 5.27).

Definition 3.4 (**Green's function II**)**.** Given a domain $\Omega \subset \mathbb{C}_\infty$, the *(negative) Green's function on* Ω is a function $G : \Omega \times \Omega \to [-\infty, \infty)$ that satisfies:

i) For each $w \in \Omega$, the function $G_w(\cdot) = G(\cdot, w)$ is negative and harmonic on $\Omega \setminus \{w\}$;
ii) For each $w \neq \infty$ in Ω, $z \mapsto G(z,w) - \log|z-w|$ is harmonic in a neighborhood of w. If $\infty \in \Omega$, $z \mapsto G(z,\infty) + \log|z|$ is harmonic in a neighborhood of ∞;
iii) G is the maximal function $G \times G \to [-\infty, \infty)$ that satisfies i) and ii).

At this point, some observations are necessary. If in ii) $\infty \notin \Omega$, then $G(\cdot, w) - \log|\cdot - w|$ is harmonic throughout Ω. Similarly, if $0 \notin \Omega$, then $G(\cdot,\infty) - \log|\cdot|$ is harmonic throughout Ω. Do not forget also that to say the function $z \mapsto G(z,\infty) + \log|z|$ is harmonic near ∞ is to say that $z \mapsto G(1/z,\infty) + \log|1/z| = G(1/z,\infty) - \log|z|$ is harmonic near 0.

Example 3.5. Consider the unit disk $\mathbb{D}$. Its Green's function is

$$G(z,w) = \log\left|\frac{z-w}{1-\overline{w}z}\right|.$$

Indeed, we know that the mapping

$$\phi_w(z) = \frac{z-w}{1-\bar{w}z}$$

is an automorphism of the disk. Therefore $|\phi_w(z)| < 1$ for all $z \in \mathbb{D}\setminus\{w\}$ and hence $G(z,w) = \log|\phi_w(z)| < 0$. Moreover, the map ϕ_w is in $\mathscr{H}(\mathbb{D})$, without zeros in $\mathbb{D}\setminus\{w\}$. Since the logarithm of the module of a holomorphic function that does not vanish is harmonic, $G(z,w)$ is harmonic on $\mathbb{D}\setminus\{w\}$.

If we write $G(z,w) = \log|z-w| - \log|1-\bar{w}z|$, then $G(z,w) - \log|z-w| = -\log|1-\bar{w}z|$. Now, the function $z \mapsto 1-\bar{w}z$ is holomorphic and non-vanishing at $\mathbb{D}$ (its unique zero is at $z = 1/\bar{w}$, and its outside $\mathbb{D}$ since $|w| < 1$), so $z \mapsto -\log|1-\bar{w}z|$ is harmonic in $\mathbb{D}$.

As $z \to \partial\mathbb{D}$, we obtain $|\phi_w(z)| \to 1$, so $G(z,w) \to \log(1) = 0$. If there is another function $f : \mathbb{D}\setminus\{w\} \to [-\infty,\infty)$ satisfying i) and ii), then $f - G_w = f - \log|\cdot - w| - (G_w - \log|\cdot - w|)$ is harmonic in $\mathbb{D}$. Finally, since $f < 0$ and $G(z,w) \to 0$ in the boundary, by Theorem 5.8 $f(z) \leq G(z,w)$ for all $z \in \mathbb{D}$. ♣

Proposition 3.6. *Consider a domain $\Omega \subset \mathbb{C}_\infty$.*

a) If there is a Green's function for Ω, then is unique.

b) If G is the Green's function for Ω, then

$$G_w(z) = \begin{cases} G(z,w), & \text{if } z \neq w \\ -\infty, & \text{if } z = w \end{cases}$$

defines a subharmonic function on Ω.

c) Let $\Omega_1, \Omega_2 \subset \mathbb{C}_\infty$ be two domains, such that there is a conformal mapping $\tau : \Omega_1 \to \Omega_2$. Let $w \in \Omega_1$ and $\alpha = \tau(w)$. If we denote G^1 and G^2 the Green's functions of Ω_1 and Ω_2 respectively, and $\tau(w) = \alpha$ for any $w \in \Omega_1$, then

$$G^1_w(z) = G^2_\alpha(\tau(z)).$$

d) If Ω is simply connected in the extended plane and for each $w \in \Omega$, $\tau_w : \Omega \to \mathbb{D}$ is the Riemann map for Ω and w, then $G(\cdot, w) := \log|\tau_w(\cdot)|$ is the Green's function for Ω.

e) If G is the Green's function for Ω and $w \in \Omega$, the function defined in a neighborhood of w by

$$H_w(z) = \begin{cases} G(z,w) - \log|z-w| & \text{if } w \neq \infty \\ G(z,\infty) + \log|z| & \text{if } w = \infty \end{cases}$$

is harmonic at $z = w$.

f) If $\infty \notin \Omega$ and Ω is a (bounded) Dirichlet domain with Green's function G, the function defined as $u : z \mapsto G(z,w) - \log|z-w|$ is the solution to the Dirichlet problem of Ω with boundary values $f : z \mapsto -\log|z-w|$.

g) If G is the Green's function for Ω, then it is symmetric.

Proof. *a*): Suppose we have G, G' two Green's functions on Ω. Then by iii), since G' satisfies i) and ii), $G'(\cdot, w) \leq G(\cdot, w)$ for $w \in \Omega$ fixed. By the same argument with G, it must be that $G(\cdot, w) \leq G'(\cdot, w)$, so $G = G'$.

b): By i), G_w is harmonic in $\Omega \setminus \{w\}$. In particular, it is subharmonic. Near w, $G_w(w) = -\infty$ and since G_w is negative, the integral

$$\frac{1}{2\pi} \int_0^{2\pi} G_w(w + re^{i\theta})\, d\theta = I$$

is either a negative real number or $-\infty$. In all those cases, $G_w(w) \leq I$, so G_w is subharmonic in Ω.

c): It suffices to assume that neither Ω_1 nor Ω_2 contain ∞. Let $\varphi : \Omega_1 \times \Omega_1 \to [-\infty, \infty)$ be defined by $\varphi = G^2 \circ (\tau \times \tau)$. Let us verify that φ has the properties of Definition 3.4. *i*) Clearly $\varphi(\cdot, w)$ is negative and harmonic in $\Omega_1 \setminus \{w\}$. *ii*) If $\tau(w) = \alpha$, consider the power series expansion of τ about w,

$$\tau(z) = \tau(w) + A_1(z-w) + A_2(z-w)^2 + \ldots = \alpha + (z-w)(A_1 + A_2(z-w) + \ldots),$$

so we deduce

$$\log|\tau(z) - \alpha| = \log|z-w| + h_1(z), \tag{25}$$

where $h_1(z) = \log|A_1 + A_2(z-w) + \dots|$ is harmonic near w, since $A_1 = \tau'(w) \neq 0$. We know $G^2_\alpha(\zeta) = \log|\zeta - \alpha| + h_2(\zeta)$, where h_2 is harmonic near $\alpha = \tau(w)$. Then by (25)

$$\varphi(z,w) = G^2_\alpha(\tau(z)) = \log|\tau(z) - \alpha| + H(\tau(z)) = \log|z-w| + h_1(z) + h_2 \circ \tau(z)$$

and therefore $\varphi(z,w) - \log|z-w| = H(z)$, where H is harmonic near $w \in \Omega_1$, since τ is conformal. *iii*) Let $f : \Omega_1 \times \Omega_1 \to [-\infty, \infty)$ be another function verifying i) and ii). Then $f \circ (\tau^{-1} \times \tau^{-1}) : \Omega_2 \times \Omega_2 \to [-\infty, \infty)$ also verify i) and ii) on $\Omega_2 \times \Omega_2$. Therefore $f \circ (\tau^{-1} \times \tau^{-1}) \leq G^2$, so $f \leq G^2 \circ (\tau \times \tau) = \varphi$.

d): Let $w \in \Omega$. Then for any $b \in \Omega$,

$$\tau_b(z) = c\frac{\tau_a(z) - \tau_a(b)}{1 - \overline{\tau_a(b)}\tau_a(z)}$$

for some constant c (depending on b) with $|c| = 1$. Also from Example 3.5 we have seen that the function

$$H(z,w) = \log\left|\frac{z-w}{1-\overline{w}z}\right|.$$

is the Green's function for the disk $\mathbb{D}$. Thus, using *c*)

$$G(z,w) = H(\tau_w(z), \tau_w(w)) = \log\left|\frac{\tau_w(z) - \tau_w(w)}{1 - \overline{\tau_w(w)}\tau_w(z)}\right| = \log|\tau_w(z)|.$$

e): It follows directly from ii) of the definition. *f*): Clearly u is harmonic in Ω. If $\zeta \in \partial\Omega$,

$$\lim_{z\to\zeta} u(z) = \lim_{z\to\zeta}[G(z,w) - \log|z-w|] = 0 - \log|\zeta - w|.$$

This shows that u is continuous on $\overline{\Omega}$ and in the boundary it is exactly the continuous function $f(\zeta) = -\log|\zeta - w|$.

f) : Let $J_w : \Omega \to \mathbb{R}$ the unique harmonic solution in Ω that solves the Dirichlet problem with boundary values $f(\zeta) = -\log|\zeta - w|$. Define $\tilde{G}(z,w) = J_w(z) + \log|z-w|$. Then let us check that the first two conditions are verified for this function $\tilde{G}$. ii) follows directly by construction. The negativity is given by the fact that $\lim_{z\to\zeta}\tilde{G}(z,w) = -\log|\zeta - w| + \log|\zeta - w| = 0$, for $\zeta \in \partial\Omega$. Also $\lim_{z\to w}\tilde{G}(z,w) = -\infty$, so by the Maximum Principle, $\tilde{G} < 0$. Therefore,

$$G(z,w) \geq \tilde{G}(z,w) \quad \text{for all } z \in \Omega \setminus \{w\}.$$

To prove the other inequality, consider the function $u = G(\cdot, w) - f$, which we know is harmonic in Ω. We have $u < -\log|\cdot - w|$ in Ω, so taking the limit superior as we approach any point $\zeta \in \partial\Omega$,

$$\limsup_{z\to\zeta} u(z) \leq -\log|\zeta - w| = f(z)$$

But we also know that on the boundary $J_w = f$, so $\limsup_{z\to\zeta}(u(z) - J_w(z)) \leq 0$ on the boundary. Again, by Proposition 5.8 we conclude $u \leq J_w$ in Ω, so $G_w = \tilde{G}_w$, and therefore $u = G(\cdot, w) + f = \tilde{G}(\cdot, w) + f = J_w$.

g) : The symmetry can be found in [7]. □

Proposition 3.7. *Let $U_1 \subset U_2 \subset \dots$ be domains in $\mathbb{C}$ with $U = \cup_j U_j$ and let $w \in U$ fixed. Suppose that U admits a Green's function G. If G_j is the Green's function of U_j, then $G_j(\cdot, w) \to G(\cdot, w)$ uniformly in compact subsets of $U \setminus \{w\}$.*

Proof. Fix $w \in U$ and consider the indices j for which $w \in U_j$. The function $G_{j+1}|_{U_j}$ satisfies conditions i) and ii) of Definition 3.4. Hence, $G_{j+1}|_{U_j}(\cdot, w) \leq G_j(\cdot, w)$, and by Theorem 5.4, $g(z, w) := \lim_n G_j(z, w)$ exists uniformly on compact subsets of $U \setminus \{w\}$, and either $g_w = g(\cdot, w)$ is harmonic or $g \equiv -\infty$. But again by the same argument $g|_{U_n}(\cdot, w) \leq G_n(\cdot, w)$ for all n. Thus $G(z, w) \leq g(z, w)$, so g_w is harmonic on $U \setminus \{w\}$.

Clearly $g \leq 0$ and $g(\cdot, w) - \log|z - w|$ is harmonic near w. Therefore, g satisfy i) and ii), so $g \leq G$ and the claim follows. □

3.2. **Other notions of Potential Theory.** Firstly, we introduce the notion of harmonic measure. From now on, we speak of *hyperbolic domains*, which are domains that admit a Green's function (see Section §5.4). If $\Omega \subset \mathbb{C}_\infty$ is an hyperbolic domain and f is a real-valued continuous function on $\partial_\infty \Omega$, then f is a solvable function for the Dirichlet problem.

Definition 3.8. For any hyperbolic domain $\Omega \subset \mathbb{C}_\infty$ and $a \in \Omega$, the unique probability measure ω_a^Ω supported on $\partial_\infty \Omega$ verifying that

$$\hat{f}(a) = \int_{\partial_\infty \Omega} f \, d\omega_a^\Omega \tag{26}$$

for each $f \in \mathscr{C}(\partial_\infty \Omega, \mathbb{R})$ is called the *harmonic measure* for Ω at a.

To ensure the existence and uniqueness of the harmonic measure, considering a point $a \in \Omega$ and the map $\mathscr{Q} : \mathscr{C}(\partial_\infty \Omega, \mathbb{R}) \ni f \mapsto \hat{f}(a) \in \mathbb{R}$ is a positive linear functional with $\|\mathscr{Q}\| = 1$. Indeed, the linearity of $f \mapsto \hat{f}$ is due to the fact that if u, v are finite-valued solvable functions on Ω, and $\alpha, \beta \in \mathbb{R}$, $\widehat{\alpha u + \beta v} = \alpha \hat{u} + \beta \hat{v}$. If f is a positive function in $\mathscr{C}(\partial_\infty \Omega, \mathbb{R})$, then $0 \in \hat{\mathcal{P}}(f, \Omega)$, so $\hat{f} \geq 0$. Thus, $\mathscr{Q}$ is a positive linear functional. Since $\hat{1} = 1$, $\|\mathscr{Q}\| = 1$.

In fact, (26) holds for all bounded Borel functions f. It can be proven that ω_a^Ω has no atoms and the following reult:

Lemma 3.9. *Let $\Omega \subset \mathbb{C}_\infty$ be hyperbolic such that $\infty \in \Omega$, $\alpha \in \Omega$ and $G(z, \alpha)$ the negative Green's function of Ω with singularity at $\alpha \neq \infty$. Then*

i) for all $z \neq \infty$

$$G(z, \alpha) = \log|z - \alpha| - \int_{\partial\Omega} \log|\xi - \alpha| \, d\omega_z^\Omega(\xi) + G(z, \infty).$$

ii) for any $\beta \notin \Omega$ and $z \neq \infty$,

$$G(z, \infty) = \int_{\partial\Omega} \log|\xi - \beta| \, d\omega_z^\Omega(\xi) - \log|z - \beta|.$$

We give now the definition of logarithmic potential. It provides a potential-theoretic framework to introduce the Robin constant and the logarithmic capacity of a set by

optimizing the functional energy over a class of unit measures. Firstly, consider a compact $K \subset \mathbb{C}$. Then for any $R > 1$,

$$\int_{|z|\leq R} |\log|z||dA(z) = \pi R^2 \left(\log R - \frac{1}{2}\right) + \pi,$$

so it means that $\int_K |\log|z||dA(z) < \infty$. It makes sense therefore to give the following definition.

Definition 3.10. For a compactly supported measure μ on $\mathbb{C}$, the *logarithmic potential* of μ is the function

$$L_\mu(z) = \int \log \frac{1}{|z-w|} d\mu(w).$$

So for any compactly supported measure μ, L_μ is defined at every point of $\mathbb{C}$. We have the following elementary properties, which follow from the properties of the Cauchy transform and some other observations.

Proposition 3.11. *If μ is a compactly supported measure, then*

(1) L_μ is a locally integrable function.
(2) L_μ is harmonic on $\mathbb{C}_\infty \setminus \operatorname{supp}(\mu)$.
(3) If $\hat{\mu}$ is the Cauchy transform of μ (see Definition 5.5*),*

$$\frac{\partial L_\mu}{\partial z} = -\frac{\hat{\mu}}{2}, \quad \frac{\partial L_\mu}{\partial \overline{z}} = -\frac{\hat{\overline{\mu}}}{2}.$$

(4) $\Delta L_\mu = -2\pi\mu$.
(5) If μ is also positive, with $\operatorname{supp}(\mu) \subset K$, K compact, and if $L_\mu < M$ in K, then $L_\mu < M$ in $\mathbb{C}$.

If E is any set, define $M(E)$ as the set of all real-valued, regular, Borel measures compactly supported on E and such that $L_\mu \in L^1(E, \mathbb{C}; |\mu|)$. Denote also $M_p(E)$ the set of all positive, regular, Borel, probability measures compactly supported on E. If $M(E) \neq \emptyset$, let $\mu \in M(E)$ and put

$$I(\mu) = \int L_\mu(z)\, d(z) = \int\int \log \frac{1}{|z-w|}\, d\mu(w) d\mu(z). \tag{27}$$

For such measures μ, $I(\mu)$ is well defined and is called the *energy integral* of μ. Now, since the logarithmic potential of a positive measure is bounded below on compact sets, we have that if μ is a positive, regular Borel measure compactly supported on E, then either $\mu \in M(E)$ or

$$\lim_{n\to\infty} \int \min\{L_\mu, n\}\, d\mu = +\infty.$$

It shows then that we can define $I(\mu) = \int L_\mu d\mu$ for all positive measures[8].

Definition 3.12. If E is any set such that $M(E) \neq \emptyset$, the *(classical) Robin constant* is defined as $v(E) = \inf\{I(\mu) : \mu \in M_p(E)\}$. The *(classical) logarithmic capacity* of such E is defined by

$$c(E) = e^{-v(E)}.$$

[8]We admit the possibility $I(\mu) = \infty$.

If $M(E) = \emptyset$, define $v(E) = \infty$ and $c(E) = 0$.

We obtain that if $E_1 \subset E_2$, then $c(E_1) < c(E_2)$ and $c(E) = \sup\{c(K) : K \subset E \text{ compact}\}$.

We briefly introduce now polar sets. Analytically, they represent sets that are "invisible" to bounded harmonic and subharmonic functions; they act as removable singularities. Geometrically, in $\mathbb{C}$, a polar set has zero Lebesgue measure and zero Hausdorff dimension in the spectrum of capacities, making it negligible as a single point or a countable collection of points, even though it might be uncountably infinite.

Definition 3.13. A set Z is *polar* if there exists a non-constant subharmonic function u on $\mathbb{C}$ such that $Z \subset \{z : u(z) = -\infty\}$.

Taking into account the function $z \mapsto \log|z - p|$, one can see that every singleton is polar. Moreover, from properties of subharmonic functions, every polar set must be Lebesgue measurable with $A(Z) = 0$. Indeed, $E = \{z : u(z) = -\infty\} = \cap_n\{z : u(z) = -n\}$, and since u is not identical to $-\infty$, $A(E) = 0$. In particular, polar sets have no interior.

Proposition 3.14. *The following statements hold.*

(1) If K is a non-trivial compact connected set, then K is not a polar set.
(2) If Z is polar, then every compact subset of Z is totally disconnected.

Proof. (2) follows from (1). Suppose u is a subharmonic function on $\mathbb{C}$ such that $u(z) = -\infty$ on K and let U be a bounded open subset of $\{z : u(z) < 0\}$ so that $U \supset K$. Using Theorem 2.15, we have the conformal mapping $\tau : \mathbb{D} \to \mathbb{C}_\infty \setminus K$ with $\tau(0) = \infty$. If $\{z_n\}_n \subset \mathbb{D}$ such that $|z_n| \to 1$, all the limit points of $\{\tau(z_n)\}_n$ will lie then in K, and hence $u \circ \tau$ is a subharmonic function on $\mathbb{D}$, $u \circ \tau \to -\infty$ as $z \to p$, p being any point in $\partial\mathbb{D}$. Using finally Theorem 5.8 we conclude $u \circ \tau \equiv -\infty$, and thus so is u. □

The proof of the following theorem is omitted, as it involves technical machinery and preliminary results that fall beyond the scope of this text. It can be found in Conway [7].

Theorem 3.15. *For a compact (totally disconnected) set $K \subset \mathbb{C}$, the following are equivalent.*

i) K is a polar set
ii) If Ω bounded with $K \subset \partial G$, then $\omega_a^\Omega(K) = 0$ for every $a \in \Omega$.
iii) $\mathbb{C} \setminus K$ is parabolic.
iv) There is no positive non-zero measure μ on K such that L_μ is bounded above.

Theorem 3.16. *If E is a Borel set, the following are equivalent:*

(a) $c(E) > 0$.
(b) There is a non-zero positive, regular, Borel measure compactly supported on E such that L_μ is bounded above.
(c) There is a compact subset of E that is not polar.
(d) There is a positive measure μ with $I(\mu) < \infty$ and $\mu(E) > 0$.

Proof. We can assume E to be a compact set K. We can also assume that $K \subset B(0,1/2)$. Indeed one has that if $T(z) = az + b$, $a \neq 0$, then $v(T(E)) = v(E) - \log|a|$ and $c(T(E)) = |a|c(E)$.

$(a) \implies (b)$ Let $\nu \in M_1(K)$ with $I(\nu) < \infty$. Since $K \subset B(0;1/2)$, $L_\nu \geq 0$ on K. Since $L_\nu \in L^1(K,\nu)$, there exists $M > 0$ such that if $F = \{z \in \operatorname{supp}(\nu) : L_\nu(z) \leq M\}$, then $\nu(F) > 0$. Since L_ν is a lower semicontinuous function, F is compact. If $\mu = \nu|_F$, then $L_\mu(z) \leq L_\nu(z) \leq M$ for all $z \in F$ and hence, by (5) of Proposition 3.11 throughout $\mathbb{C}$.

$(b) \implies (c)$. This is immediate from Theorem 3.15.

$(c) \implies (d)$. By Theorem 3.15, there exists a positive, regular, Borel measure compactly supported on K, μ, such that $L_\mu \leq M$, so $I(\mu) = \int L_\mu d\mu < \infty$.

$(d) \implies (a)$. If μ is a positive measure with $\mu(K) > 0$ and $I(\mu) < \infty$, then by definition, $v(K) < \infty$ and so $c(K) > 0$. □

Definition 3.12 provides a potential-theoretic framework to introduce the Robin constant and the logarithmic capacity of a set E by optimizing the functional energy over a class of unit measures. This approach captures the global electrostatic equilibrium of the system. Alternatively, when considering the exterior domain $\Omega = \mathbb{C}_\infty \setminus K$ of a compact set K, one can define these notions locally by evaluating the asymptotic behavior of the Green's function at a specific reference point.

Definition 3.17. If $K \subset \mathbb{C}$ is a compact subset such that $\Omega := \mathbb{C}_\infty \setminus K$ is hyperbolic with Green's function G, then for any fixed point $w \in \Omega$, the *Robin constant with respect to* w for K is defined by

$$\gamma_\Omega(w) = -\lim_{z \to w} H_w(z),$$

where $H_w(z)$ is the regular part of the Green's function as defined in Proposition 3.6. If $\Omega = \mathbb{C}_\infty \setminus K$ is parabolic, define $\gamma_\Omega(w) = \infty$.

Notice that in the special case when $w = \infty$, $\gamma_\Omega(\infty) = -\lim_{z\to\infty}(G(z,\infty) + \log|z|)$, for a number a, $\log|z| - \log|z-a| = \log|1 - a/z| \to 0$, as $z \to \infty$. Thus $\gamma_\Omega(\infty) = -\lim_{z\to\infty}(G(z,\infty) + \log|z-a|)$. This observation will come in handy later.

Proposition 3.18. *Let $K \subset \mathbb{C}$ be a compact connected subset, and let $\Omega = \mathbb{C}_\infty \setminus K$ and a point $w \in \Omega$. If $\tau_w : \Omega \to \mathbb{D}$ be the Riemann map for Ω, then*

$$\gamma_\Omega(w) = -\log|\tau_w'(w)|.$$

Proof. In Proposition 3.6 we have seen that $G(z,w) = \log|\tau_w(z)|$ for $z \in \Omega \setminus \{w\}$, where G is the Green's function for Ω. Thus,

$$\begin{aligned} -\gamma_\Omega(w) &= \lim_{z\to w}(G(z,w) - \log|z-w|) = \lim_{z\to w}(\log|\tau_w(z)| - \log|z-w|) \\ &= \lim_{z\to w} \log\frac{|\tau_w(z)|}{|z-w|} = \lim_{z\to w}\log\left|\frac{\tau_w(z)-\tau_w(w)}{z-w}\right| = \log|\tau_w'(w)|. \end{aligned}$$

□

Proposition 3.19. *Let $K \subset \mathbb{C}$ be a compact such that $U = \mathbb{C}_\infty \setminus K$ is hyperbolic, let $\gamma = \gamma_U(\infty)$, and let Ω be the component of U that contains ∞. If ω is harmonic measure for Ω at ∞ and L_w is its logarithmic potential, then*

$$L_w(\alpha) = \begin{cases} \gamma + G(\alpha, \infty), & \alpha \in \Omega \\ \gamma, & \alpha \notin \overline{\Omega} \end{cases}$$

Proof. We can assume WLOG that $K = \hat{K}$. Lemma 3.9 yields

$$G(z, \alpha) = \log|z - \alpha| - \int_{\partial\Omega} \log|\xi - \alpha| \, d\omega_z^\Omega(\xi) + G(z, \infty).$$

for all finite α and $z \neq \alpha$. But using Proposition 5.27 we get

$$L_{\omega_z^\Omega}(\alpha) = \int \log|\xi - \alpha|^{-1} \, d\omega_z^\Omega(\xi) = G(\alpha, z) - [G(z, \infty) + \log|z - \alpha|]$$

Letting now $z \to \infty$ and using that $L_{\omega_z^\Omega} \to L_\omega$, we obtain $L_\omega(\alpha) = G(\alpha, \infty) + \gamma$.

If $\alpha \notin \overline{\Omega}$, again by Lemma 3.9

$$-[G(z, \infty) + \log|z - \alpha|] = \int_{\partial\Omega} \log \frac{1}{|\xi - \alpha|} \, d\omega_z^\Omega(\xi).$$

Letting $z \to \infty$, $L_\omega(\alpha) = \gamma$. □

This constant $\gamma_\Omega(w)$ can be interpreted as the "self-energy" of a unit charge placed at w in the presence of the grounded boundary $\partial\Omega$. We can explicitly relate the Robin constant and the Bergman kernel in the diagonal:

Theorem 3.20. *Let $\Omega \subset \mathbb{C}$ be a hyperbolic domain. If K denotes the Bergman kernel of $A^2(\Omega)$, then*

$$K(z, z) = \frac{1}{\pi} \frac{\partial^2 \gamma_\Omega}{\partial z \partial \bar{z}}(z). \tag{28}$$

Proof. It follows from Theorem 5.28. Notice we have $\gamma_\Omega(\zeta) = \psi(\zeta, \zeta)$, where ψ is defined in (47), the regular part of the Green's function. Thus,

$$\frac{\partial \gamma_\Omega}{\partial \bar{\zeta}} = \left(\frac{\partial \psi}{\partial \bar{z}} \frac{\partial \bar{z}}{\partial \bar{\zeta}} + \frac{\partial \psi}{\partial \bar{w}} \frac{\partial \bar{w}}{\partial \bar{\zeta}} \right) \Bigg|_{z=w=\zeta} = \psi_{\bar{z}} + \psi_{\bar{w}}.$$

But we also have

$$\frac{\partial}{\partial \zeta}(\psi_{\bar{z}}) = \psi_{\bar{z}z} \frac{\partial z}{\partial \zeta} + \psi_{\bar{z}w} \frac{\partial w}{\partial \zeta} = \psi_{z\bar{z}} + \psi_{w\bar{z}}, \quad \frac{\partial}{\partial \zeta}(\psi_{\bar{w}}) = \psi_{\bar{w}z} \frac{\partial z}{\partial \zeta} + \psi_{\bar{w}w} \frac{\partial w}{\partial \zeta} = \psi_{z\bar{w}} + \psi_{w\bar{w}},$$

and therefore

$$\frac{\partial^2 \gamma_\Omega}{\partial \zeta \partial \bar{\zeta}} = \psi_{z\bar{z}} + 2\psi_{z\bar{w}} + \psi_{w\bar{w}}.$$

We know that for the singular part of the Green's function, $\log|z - w| = \frac{1}{2}\log(z - w) + \frac{1}{2}\log(\bar{z} - \bar{w})$. The mixed derivative $\frac{\partial^2}{\partial z \partial \bar{w}} \log|z - w|$ is strictly equal to 0 for $z \neq w$,

which implies that $\frac{\partial^2 G}{\partial z \partial \bar{w}} = \frac{\partial^2 \psi}{\partial z \partial \bar{w}} = \psi_{z\bar{w}}$. Finally, Theorem 5.28 and evaluating at the diagonal $z = w = \zeta$ yields:

$$\frac{1}{\pi}\frac{\partial^2 \gamma_\Omega}{\partial \zeta \partial \bar{\zeta}}(\zeta) = \frac{2}{\pi}\psi_{z\bar{w}}(\zeta,\zeta) = \frac{2}{\pi}\frac{\partial^2 G}{\partial z \partial \bar{w}}(\zeta,\zeta) = K(\zeta,\zeta).$$

□

Directly from Proposition 3.19, the classical Robin constant $v(K)$ is the local Robin constant at infinity, which means that $v(K) = \gamma_\Omega(\infty)$ whenever Ω contains the point at infinity. A similar identification can be made for the capacity. We can now introduce the local, geometric definition of capacity from the perspective of the Green's function. It can be shown that $c(K) = e^{-\gamma_\Omega(\infty)}$ whenever $\Omega = \mathbb{C}_\infty \setminus K$ contains the point at infinity (see Definition 3.21).

Definition 3.21. If $K \subset \mathbb{C}$ is a compact subset such that $\Omega := \mathbb{C}_\infty \setminus K$ is hyperbolic with Green's function G, then for any fixed point $w \in \Omega$, the *logarithmic capacity of K with respect to $w \in \Omega$* is defined as:

$$c_\Omega(w) = e^{-\gamma_\Omega(w)}.$$

If Ω is parabolic, we set $c_\Omega(w) = 0$.

In the same way, if Ω is the hyperbolic (connected) component of $\mathbb{C}_\infty \setminus K$ which contains ∞ the *logarithmic capacity of K with respect to ∞* is defined as $c_\Omega(\infty) = e^{-\gamma_\Omega(\infty)}$. If Ω is parabolic, we set $c_\Omega(\infty) = 0$.

Although Definition 3.21 is strictly formulated for compact sets, the notion of capacity extends naturally to cases where the complement $\mathbb{C}_\infty \setminus \Omega$ is not compact (for instance, when it is unbounded). For a point $w \in \Omega$, we can apply the Möbius transformation $z \mapsto (z-w)^{-1}$, which sends w to infinity and transforms the unbounded complement $\mathbb{C}_\infty \setminus \Omega$ into a strictly compact set $\tilde{K} \subset \mathbb{C}$. The capacity $c_\Omega(w)$ is then rigorously understood as the logarithmic capacity of this compact set $\tilde{K}$ with respect to infinity.

Lemma 3.22. *Let $K_1, K_2 \subset \mathbb{C}$ be compact subsets such that $\Omega_1 = \mathbb{C}_\infty \setminus K_1$ and $\Omega_2 = \mathbb{C}_\infty \setminus K_2$ are hyperbolic components and let $w \in \Omega_2$. If $K_1 \subset K_2$ (i.e. $\Omega_2 \subset \Omega_1$), then $c_{\Omega_1}(w) \leq c_{\Omega_2}(w)$. The inequality holds for $w = \infty$.*

Proof. Let G_1 and G_2 the Green's functions of Ω_1 and Ω_2, respectively. Since $\Omega_2 \subset \Omega_1$, we know in Ω_2 that $G_1(\cdot, w)|_{\Omega_2} \leq G_2(\cdot, w)$. Then adding in both sides $-\log|z-w|$ and taking the limit as $z \to w$, we finally get that $-\gamma_{\Omega_1}(w) \leq -\gamma_{\Omega_2}(w)$, so $c_{\Omega_1}(w) \leq c_{\Omega_2}(w)$. The case $w = \infty$ is similar. □

Proposition 3.23. *Let $\{K_n\}_n$ be a sequence of compact sets such that $K_n \supset K_{n+1}$ for all n and let $K = \cap_n K_n$. Define $\Omega_n = \mathbb{C}_\infty \setminus K_n$ and $\Omega = \mathbb{C}_\infty \setminus K$, and assume that Ω is hyperbolic. Then, for all $w \in \Omega$,*

$$c_{\Omega_n}(w) \to c_\Omega(w).$$

Proof. First, since $K_n \supset K_{n+1}$, taking complements in $\mathbb{C}_\infty$ yields an increasing sequence of domains $\Omega_n \subset \Omega_{n+1}$. Furthermore, $\cup_n \Omega_n = \cup_n (\mathbb{C}_\infty \setminus K_n) = \mathbb{C}_\infty \setminus \cap_n K_n = \mathbb{C}_\infty \setminus K = \Omega$. Since $w \in \Omega = \cup_n \Omega_n$, there exists an integer N such that $w \in \Omega_n$ for all $n \geq N$. We

can assume WLOG that this holds for all n. Let $G_n(\cdot, w)$ and $G(\cdot, w)$ be the Green's functions of Ω_n and Ω with pole at w, respectively. By Proposition 3.7, we know that $G_n(\cdot, w) \to G(\cdot, w)$ uniformly on compact subsets of $\Omega \setminus \{w\}$. Using the regular part of the Green's functions near w, these functions H_w^n and H_w are harmonic near w.

Choose a small closed disk $\overline{D}(w, r) \subset \Omega$. Since $\overline{D}(w, r) \setminus \{w\}$ is a compact subset of $\Omega \setminus \{w\}$, the sequence $G_n(\cdot, w)$ converges uniformly to $G(\cdot, w)$ on the boundary circle $\partial D(w, r)$. Therefore, $H_w^n \to H_w$ uniformly on $\partial D(w, r)$. By Theorem 5.8, this uniform convergence extends to the interior of the disk, meaning $H_w^n \to H_w$ uniformly on the whole closed disk $\overline{D}(w, r)$. In particular,

$$-\lim_{n\to\infty} \gamma_{\Omega_n}(w) = \lim_{n\to\infty} \lim_{z\to w} H_w^n(z) = \lim_{z\to w} H_w(z) = -\gamma_\Omega(w).$$

Finally, we conclude $c_{\Omega_n}(w) \to c_\Omega(w)$ as $n \to \infty$. □

In Remark 2.7 we saw that there exist domains in $\mathbb{C}$ such that its Bergman space $A^p(\Omega)$ is trivial. Here we characterize those domains in the case $p = 2$ following a proof due to Hedberg. While the theorem holds for any closed set, we will prove it under the assumption that K is compact. This restriction allows us to carry out the proof utilizing the potential-theoretic tools introduced thus far. A complete proof of the general statement requires heavy machinery from the theory of removable singularities, which falls outside the scope of this thesis. The interested reader can find a complete proof in [13].

Theorem 3.24. *Let $K \subset \mathbb{C}$ is a closed subset, then the following are equivalent:*

a) K is a polar set.
b) $A^2(\mathbb{C} \setminus K) = \{0\}$.

Proof. *b)* $\implies$ *a)*: WLOG we can assume $A(K) = 0$ and $\operatorname{diam}(K) < 1$. Indeed, we can assume that the area of K is zero: if not, define

$$f(z) = \int_K \frac{1}{\zeta - z} dA(\zeta).$$

This function is continuous in $\mathbb{C}$ and analytic in $\mathbb{C} \setminus K$. Its derivative f' is non-zero and belongs to $A^2(\mathbb{C} \setminus K)$. we can also assume $\operatorname{diam}(K) < 1$ because of the geometric invariance of polar sets and Bergman spaces, i.e., let $T(z) = cz + d$, with $c > 0$. If we choose c small enough, $\operatorname{diam}(T(K)) < 1$ (a set K is polar if and only if $cK + d$ is polar). If there is a function $g \in A^2(\mathbb{C} \setminus T(K))$, then $f(z) = g(cz + d)c \in A^2(\mathbb{C} \setminus K)$. Thus, $A^2(\mathbb{C} \setminus K) = \{0\}$ implies $A^2(\mathbb{C} \setminus K') = \{0\}$.

Assuming K not polar, by Theorem 3.16, $c(K) > 0$. We will find a non-zero function $f \in A^2(\mathbb{C} \setminus K)$. Let K_1, K_2 be disjoint compact subsets of K, each of which has positive capacity. For $j = 1, 2$ let μ_j be a probability measure on K_j with logarithmic potential that is bounded above and put $\mu = \mu_1 - \mu_2$. So μ is a non-zero measure carried by K and $\mu(K) = 0$. If $f = \hat{\mu}$, the Cauchy transform of μ, $f \in \mathscr{H}(\mathbb{C}_\infty \setminus K)$ and $0 = f(\infty) = f'(\infty)$ (Proposition 3.11). Choose $R > 1$ such that $K \subset B(0; R)$ and define $H = \{z : |z| > R\}$. By Lemma 2.2, $f \in A^2(H)$.

We have to prove that $\int_{B(0,R)} |f|^2 dA < \infty$. Let $z, w \in K$ with $z \neq w$ and for $0 < \varepsilon < |z - w|/2$ define $B_\varepsilon = B(w, \varepsilon) \cup B(z, \varepsilon)$. Now since $z \neq w$, the function

$\zeta \to [(\zeta - w)(\bar{\zeta} - \bar{z})]^{-1}$ is locally integrable (with respect to the two-dimensional Lebesgue measure A). Thus,

$$\int_{B(0,R)} \frac{dA(\zeta)}{(\zeta - w)(\bar{\zeta} - \bar{z})} = \lim_{\varepsilon \to 0} \int_{B(0,R)\setminus B_\varepsilon} \frac{dA(\zeta)}{(\zeta - w)(\bar{\zeta} - \bar{z})}.$$

Also note that $2(\partial/\partial\bar{\zeta}) \log|\zeta - z| = (\bar{\zeta} - \bar{z})^{-1}$. Therefore Theorem 5.2 yields

$$\begin{aligned}\int_{B(0,R)\setminus B_\varepsilon} \frac{dA(\zeta)}{(\zeta - w)(\bar{\zeta} - \bar{z})} &= 2\int_{B(0,R)\setminus B_\varepsilon} \frac{\partial}{\overline{\zeta - w}} \log|\zeta - z| dA(\zeta) = \frac{1}{i}\int_{|\zeta|=R} \frac{\log|\zeta - z|}{\zeta - w} d\zeta \\ &\quad - \frac{1}{i}\int_{|\zeta - w|=\varepsilon} \frac{\log|\zeta - z|}{\zeta - w} d\zeta - \frac{1}{i}\int_{|\zeta - z|=\varepsilon} \frac{\log|\zeta - z|}{\zeta - w} d\zeta \\ &= I - I_\varepsilon - J_\varepsilon.\end{aligned}$$

Now for these three integrals we have

$$I_\varepsilon = \frac{1}{i}\int_0^{2\pi} \log|w - z + \varepsilon e^{i\theta}|\, i\,\varepsilon\, e^{i\theta} d\theta = 2\pi \log|z - w|.$$

$$|J_\varepsilon| = \left|\frac{1}{i}\int_0^{2\pi} \frac{\log|\varepsilon e^{i\theta}|}{(z - w + \varepsilon e^{i\theta})}\, i\,\varepsilon\, e^{i\theta}\, d\theta\right| \le 2\pi \frac{\varepsilon \log(1/\varepsilon)}{|z - w| - \varepsilon}.$$

$$I = \int_0^{2\pi} \frac{\log|Re^{i\theta} - z|}{Re^{i\theta} - w}\, R\, e^{i\theta}\, d\theta.$$

Writing $d = \sup\{|z| : z \in K\}$, we obtain $d < R$ and $|Re^{i\theta} - w| \ge R - d$. Also $\log|Re^{i\theta} - z| \le \log 2R$. Thus, there is a constant C_1 that depends only on R and K such that $|I| \le C_1 \log 2R$. Therefore

$$\begin{aligned}\left|\int_{B(0,R)\setminus B_\varepsilon} \frac{dA(\zeta)}{(\zeta - w)(\bar{\zeta} - \bar{z})}\right| &\le |I| + |I_\varepsilon| - |J_\varepsilon| \\ &\le C_1 \log 2R + 2\pi \log\frac{1}{|z - w|} + 2\pi \frac{\varepsilon \log \varepsilon^{-1}}{|z - w| - \varepsilon}.\end{aligned}$$

Letting $\varepsilon \to 0$ we see that there is a constant C that depends only on R and K such that, whenever $z \ne w$,

$$\left|\int_{B(0,R)} \frac{dA(\zeta)}{(\zeta - w)(\bar{\zeta} - \bar{z})}\right| \le C \log\frac{2R}{|z - w|}. \tag{29}$$

Now the fact that the logarithmic potentials L_{μ_1} and L_{μ_2} are bounded above implies that the measure μ have no atoms. Indeed, if μ has an atom z_0, then

$$L_{\mu_1}(z) = \int_{\{z_0\}} \log\frac{1}{|z - w|}\, d\mu_1(w) + \int_{\mathbb{C}\setminus\{z_0\}} \log\frac{1}{|z - w|}\, d\mu_1(w),$$

so taking the limit as $z \to z_0$, the first integral becomes $c \log|z_0 - z_0|^{-1} = +\infty$, so it would imply that $L_{\mu_1}(z_0)$ is not bounded above. Hence $|\mu \times \mu|(\{(z, z) : z \in K\}) = 0$

by Fubini's Theorem. So (29) holds $|\mu\times\mu|$-a.e. on $K\times K$ and

$$\begin{aligned}\int_{K\times K}\left|\frac{dA(\zeta)}{(\zeta-w)(\bar\zeta-\bar z)}\right| d|\mu\times\mu|(z,w) &\le C\int_K\int_K \log\frac{2R}{|z-w|}\,d|\mu|(z)\,d|\mu|(w)\\ &= 4C\log 2R + I(|\mu|)<\infty,\end{aligned}$$

where $I(|\mu|)$ is the energy integral of $|\mu|$, defined in (27). From Fubini's Theorem we get that

$$\begin{aligned}\int_{B(0,R)}|f|^2\,dA &= \int_{B(0,R)}\left(\int_K\frac{d\mu(w)}{\zeta-w}\right)\left(\int_K\frac{d\mu(z)}{\bar\zeta-\bar z}\right)dA(\zeta)\\ &\le \int_{K\times K}\left|\int_{B(0,R)}\frac{dA(\zeta)}{(\zeta-w)(\bar\zeta-\bar z)}\right| d|\mu\times\mu|(z,w)<\infty,\end{aligned}$$

hence $f\in A^2(\mathbb{C}\setminus K)$.

a) $\Longrightarrow$ *b)*: Assume that K is polar, so K is totally disconnected (Proposition 3.14) and $\mathbb{C}\setminus K$ is connected. If $f\in A^2(\mathbb{C}\setminus K)$, then f has an expansion $f(z)=\sum_{n=2}^\infty a_n z^{-n}$. We shall prove that $a_n=0$ for all $n\ge 2$, or equivalently, show that for all $n\ge 1$, $\int_\Gamma z^n f(z)\,dz=0$, where Γ is any smooth Jordan system surrounding K.

WLOG, it can be assumed that $K\subset\mathbb{D}$, and let Γ be a finite collection of pairwise disjoint smooth positively oriented Jordan curves in $\mathbb{D}\setminus K$ such that $K\subset \operatorname{ins}\Gamma$; the area closed by Γ. Let $\gamma=\gamma_{\mathbb{C}_\infty\setminus\Gamma}(\infty)$ and put $W=\mathbb{C}_\infty\setminus[\Gamma\cup\operatorname{ins}\Gamma]=\operatorname{out}\Gamma$ and ω its harmonic measure evaluated at ∞, and let G be the Green function for W[9]. If $u=L_\omega/\gamma$, then u is harmonic on W and, by Proposition 3.19, $u(z)=1+G(z,\infty)/\gamma$ for all z in W, and $u(z)=1$ for all $z\notin W$ (because $\partial W=\Gamma$ is a system of curves and so each point of Γ is a regular point).

For $\varepsilon>0$, set $\Gamma_\varepsilon=\{z: u(z)=1-\varepsilon\}$ and $E_\varepsilon=\Gamma_\varepsilon\cup\operatorname{ins}\Gamma_\varepsilon$. The value of ε can be chosen as small as desired with Γ_ε a smooth positive Jordan system that contains Γ in its inside. If $U=\{z:u(z)>0\}\setminus E_\varepsilon$, U is a bounded open set that contains E_ε and has a smooth boundary. Note that if $|z|>2$, $\operatorname{dist}(z,\Gamma)>1$ and so $|z-w|^{-1}<1$ for all w in Γ; thus $u(z)<0$. Hence $U\subset 2\mathbb{D}$.

If g is any function analytic in a neighborhood of $\overline{\{z:u(z)>0\}\setminus\overline{\operatorname{ins}\Gamma}}$, then

$$\int_\Gamma g(z)\,dz=\int_{\Gamma_\varepsilon}g(z)\,dz=\frac{1}{1-\varepsilon}\int_{\partial U}g(z)\,u(z)\,dz.$$

Thus applying Theorem 5.2 we get

$$\int_\Gamma g(z)\,dz=\frac{2i}{1-\varepsilon}\int_U\frac{\partial}{\partial\bar z}[g(z)u(z)]\,dA(z)=\frac{2i}{1-\varepsilon}\int_U g\,\frac{\partial}{\partial\bar z}u(z)\,dA(z). \tag{30}$$

Taking $g=\partial u/\partial z$ in (30) and using the fact that u is real-valued, we get

$$\int_{\Gamma_\varepsilon}\frac{\partial}{\partial z}u(z)\,dz=\frac{2i}{1-\varepsilon}\int_U\left|\frac{\partial}{\partial z}u\right|^2 dA.$$

[9]The hyperbolicity is given by Proposition 3.15.

Now fix $n \geq 1$, let $f \in A^2(\mathbb{C} \setminus K)$, and let $g = z^n f$ in (30); this yields

$$\begin{aligned}\left|\int_\Gamma z^n f(z)\,dz\right|^2 &= \frac{4}{(1-\varepsilon)^2}\left|\int_U z^n f\,\frac{\partial}{\partial \bar{z}} u\,dA\right|^2 \\ &\leq \frac{4}{(1-\varepsilon)^2}\left(\int_U |z^n f|^2\,dA\right)\left(\int_U \left|\frac{\partial}{\partial z}u\right|^2\,dA\right) \\ &\leq \frac{2^{n+1}}{1-\varepsilon}\|f\|^2_{A^2(\mathbb{C}\setminus K)}\int_{\Gamma_\varepsilon}\frac{\partial}{\partial z}u(z)\,dz.\end{aligned}$$

According to Proposition 3.11, $\partial u/\partial z = -(2\gamma)^{-1}\hat{\omega}$, the Cauchy transform of ω. Thus

$$\int_{\Gamma_\varepsilon}\frac{\partial}{\partial z}u(z)\,dz = -\frac{1}{2\gamma}\int_{\Gamma_\varepsilon}\int_\Gamma\left(\frac{1}{\zeta - z}\,d\omega(\zeta)\right)dz = \frac{1}{2\gamma}\int_\Gamma\int_{\Gamma_\varepsilon}\left(\frac{1}{z-\zeta}\,dz\right)d\omega(\zeta) = \frac{1}{2\gamma},$$

since Γ is in the inside of Γ_ε. This combined with the fact that ε was arbitrary gives that

$$\left|\int_\Gamma z^n f(z)\,dz\right|^2 \leq \frac{2^n\|f\|^2_{A^2(\mathbb{C}\setminus K)}}{\gamma_{\mathbb{C}_\infty\setminus\Gamma}(\infty)}$$

for any such system Γ. Since $c(K) = 0$, we can get a sequence of such curve systems $\{\Gamma_k\}_k$ that squeeze down to K. Thus, by Proposition 3.23, $\gamma_{\mathbb{C}_\infty\setminus\Gamma_k}(\infty) \to \gamma_{\mathbb{C}_\infty\setminus K}(\infty) = \infty$; but $\int_{\Gamma_k} z^n f(z)\,dz$ remains constant. Thus, this integral must be zero, and so $f = 0$. □

4. The Suita conjecture

The Suita conjecture is a result in complex analysis that links the geometry of a domain (its logarithm capacity) to its Bergman kernel. Proposed by Nobuyuki Suita in 1972, it remained one of the most challenging open problems in the field for over 40 years. It reads as follows: given a domain $\Omega \subset \mathbb{C}$, let K_Ω be the Bergman kernel of $A^2(\Omega)$ and $c_\Omega(w)$ the logarithmic capacity of $\mathbb{C} \setminus \Omega$ with respect to w. Then, one has the inequality

$$c_\Omega(w)^2 \leq \pi K_\Omega(w, w). \tag{31}$$

This conjecture was proved to be true in 2013 by Z. Błocki [5] for bounded domains in the plane $\mathbb{C}$ by obtaining the optimal constant in the L^2-extension theorem (in $\mathbb{C}$):

Theorem 4.1 (**Ohsawa–Takegoshi L^2 extension Theorem**). *Let $\Omega \subset \mathbb{C}$ a bounded domain and let $w \in \Omega$. Consider φ a subharmonic function in Ω. For any value $\xi \in \mathbb{C}$, there exists a holomorphic function $f \in \mathscr{H}(\Omega)$ such that $f(w) = \xi$ and*

$$\int_\Omega |f(z)|^2 e^{-\varphi(z)}\, dA(z) \leq C\, |\xi|^2 e^{-\varphi(w)},$$

where C is a constant depending only on the geometry of Ω.

In a more general setting, discovered by T. Ohsawa and K. Takegoshi in 1987, this theorem guarantees that any L^2-holomorphic function defined on a bounded Stein submanifold —such as a lower-dimensional pseudoconvex domain in $\mathbb{C}^n$— can be holomorphically extended to a surrounding higher-dimensional domain while maintaining a strict bound on its L^2 growth.

The intuition behind it is the following; if we consider $\varphi \equiv 0$ and we choose $\xi = 1$, then there exists a function $f \in \mathscr{H}(\Omega)$ such that $f(w) = 1$ and $\int_\Omega |f|^2\, dA \leq C$. Therefore $f \in A^2(\Omega)$ and now by Theorem 2.8, we know

$$K_\Omega(w, w) = \sup_{g \in A^2(\Omega)} \frac{|g(w)|^2}{\|g\|^2_{A^2(\Omega)}} = \frac{1}{\inf\{\|g\|^2_{A^2(\Omega)} : g \in A^2(\Omega), |g(w)| = 1\}}.$$

and hence it gives $K_\Omega(w, w) \geq C^{-1}$. It is interesting, however, to use a weight function φ —which depends on the Green function of Ω— in order to prove the conjecture. Specifically, as we will see, $\varphi = 2G_w - \log|\cdot - w|^2 - \log(1 - e^{2G_w})$.

In 2015, Qi'an Guan and Xiangyu Zhou [11] provided the complete proof for general Riemann surfaces. However, we present here the purely one-dimensional approach following the proof of Błocki in [4].

Remark 4.2. Before continuing, we must make some remarks:

(1) Although we state the Suita conjecture for planar domains $\Omega \subset \mathbb{C}$ to work directly with the (classical) Bergman kernel for functions, the conjecture was originally formulated by Suita for arbitrary open Riemann surfaces. In a more general setting, such as domains $\Omega \subset \mathbb{C}_\infty$, both the capacity and the Bergman kernel must be treated as conformal metrics (differentials) rather than scalar functions.

(2) By virtue of the convergence properties established in Propositions 2.12, 3.7, and 3.23, together with the exhaustion guaranteed by Theorem 5.6, we may assume without loss of generality that the domain Ω is bounded and with smooth boundary.
(3) We are interested only in hyperbolic domains. Otherwise, the Bergman kernel would be zero according to Theorems 3.15, 3.24, and the logarithmic capacity would also be identically zero by definition, making (31) trivial.
(4) The equality of (31) holds for some point $w \in \Omega$ (and consequently for all points in Ω) if and only if Ω is conformally equivalent to $\mathbb{D} \setminus E$, where $\mathbb{D}$ is the unit disk and E is a closed polar set (i.e., a closed set of logarithmic capacity zero).
If $\tau_w : \Omega \to \mathbb{D}$ is the Riemann map for Ω, then directly from Proposition 3.18, $c_\Omega(w) = |\tau_w'(w)|$. Thus, by Corollary 2.19, $c_\Omega(w) = \sqrt{\pi K_\Omega(w,w)}$.

We expose here the Błocki's proof with detail, which is based on ideas from Berndtsson [3] and largely following the approach developed by Guan and Zhou [10]. The following description of the Bergman kernel as a solution to the Dirichlet problem will be useful. In some cases, rather than working directly with the Bergman kernel, we will work with this specific solution.

Proposition 4.3. *Let $\Omega \subset \mathbb{C}$ be a bounded domain with $\mathscr{C}^1$ boundary, $w \in \Omega$ and let K be the Bergman kernel of $A^2(\Omega)$. Considering the boundary function $z \in \partial\Omega \mapsto (\pi(\overline{z-w}))^{-1}$ the Dirichlet problem admits a solution, given by the harmonic (complex-valued) function $v : \overline{\Omega} \to \mathbb{C}$. Then*

$$K(z,w) = \frac{\partial}{\partial z} v(z).$$

Proof. Since v is harmonic on Ω,

$$0 = \Delta v = \frac{\partial}{\partial z \partial \overline{z}} v = \frac{\partial}{\partial \bar{z}} \left(\frac{\partial v}{\partial z} \right),$$

and hence $\xi \mapsto (\partial/\partial z)(\xi)$ is holomorphic. It suffices to show that it has the reproducing property. Let $h \in A^2(\Omega) \cap \mathscr{C}^1(\overline{\Omega})$. Then since f is holomorphic,

$$\frac{\partial}{\partial \overline{z}}(h\overline{v}) = h \frac{\partial \overline{v}}{\partial \overline{z}} \tag{32}$$

and therefore using the Cauchy integral formula, Theorem 5.2 and (32) we conclude that

$$\begin{aligned}
\langle h, \partial v/\partial z \rangle &= \int_\Omega h(z) \overline{\frac{\partial v}{\partial z}(z)}\, dA(z) = \int_\Omega h(z) \frac{\partial \bar{v}}{\partial \bar{z}}(z)\, dA(z) \\
&= \int_\Omega \frac{\partial}{\partial \bar{z}}(h\bar{v})\, dA(z) = \frac{1}{2i} \int_{\partial\Omega} h(z) \overline{v(z)}\, dz \\
&= \frac{1}{2i} \int_{\partial\Omega} \frac{h(z)}{\pi(z-w)}\, dz = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{h(z)}{z-w}\, dz = f(w).
\end{aligned}$$

Now, since $A^2(\Omega)\cap\mathscr{C}^1(\overline{\Omega})$ is dense in $A^2(\Omega)$[10] the claim follows. □

For simplicity, denote from now on $\partial=\partial/\partial z$ and $\overline{\partial}=\partial/\partial\overline{z}$. We want to work in weighted Bergman spaces, introducing a function $\varphi:\Omega\to\mathbb{R}$ to obtain an accurate bound to prove (31). Consider then, for a $\mathscr{C}^2(\overline{\Omega})$ function $\varphi:\Omega\to\mathbb{R}$, the following weighted scalar product

$$\langle\alpha,\beta\rangle_\varphi=\int_\Omega\alpha\overline{\beta}e^{-\varphi}dA. \tag{33}$$

We want to find the adjoint operator of $\overline{\partial}$ with respect to this inner product, which we denote by $\overline{\partial}^*_\varphi$. Assuming that $\alpha,\beta\in\mathscr{C}^2(\overline{\Omega})$ are such that on $\partial\Omega$ either $\alpha=0$ or $\beta=0$, the following must hold true: $\langle\overline{\partial}^*_\varphi\alpha,\beta\rangle_\varphi=\langle\alpha,\overline{\partial}\beta\rangle_\varphi$. But

$$\langle\alpha,\overline{\partial}\beta\rangle_\varphi=\int_\Omega\alpha\overline{\overline{\partial}\beta}e^{-\varphi}dA=\int_\Omega\alpha(\partial\overline{\beta})e^{-\varphi}dA.$$

Since $\partial(\alpha\overline{\beta}e^{-\varphi})=(\partial\alpha)\overline{\beta}e^{-\varphi}+\alpha(\partial\overline{\beta})e^{-\varphi}+\alpha\overline{\beta}(\partial e^{-\varphi})$, we obtain $\alpha\partial\overline{\beta}e^{-\varphi}=\partial(\alpha\overline{\beta}e^{-\varphi})-(\partial\alpha)\overline{\beta}e^{-\varphi}-\alpha\overline{\beta}(-\partial\varphi e^{-\varphi})$. Integrating over Ω,

$$\int_\Omega\alpha\partial\overline{\beta}e^{-\varphi}dA=\int_\Omega\partial(\alpha\overline{\beta}e^{-\varphi})dA+\int_\Omega-\partial\alpha\overline{\beta}e^{-\varphi}dA+\int_\Omega\alpha\overline{\beta}(\partial\varphi)e^{-\varphi}dA.$$

Applying Theorem 5.2, the first integral vanishes:

$$\int_\Omega\partial(\alpha\overline{\beta}e^{-\varphi})dA=\frac{i}{2}\oint_{\partial\Omega}\alpha\overline{\beta}e^{-\varphi}\,d\overline{z}=0.$$

Therefore

$$\langle\alpha,\overline{\partial}\beta\rangle_\varphi=\int_\Omega\left(-\partial\alpha+\alpha\partial\varphi\right)\overline{\beta}e^{-\varphi}dA=\langle-\partial\alpha+\alpha\partial\varphi,\beta\rangle_\varphi,$$

so $\overline{\partial}^*_\varphi$ is given by $\overline{\partial}^*_\varphi\alpha=-\partial\alpha+\alpha\partial\varphi$. In particular,

$$\overline{\partial}\,\overline{\partial}^*_\varphi\alpha=-\overline{\partial}\partial\alpha+\overline{\partial}(\alpha\partial\varphi)=-\overline{\partial}\partial\alpha+(\overline{\partial}\alpha)\partial\varphi+\alpha(\overline{\partial}\partial\varphi)=\overline{\partial}^*_\varphi\overline{\partial}\alpha+\alpha\partial\overline{\partial}\varphi. \tag{34}$$

Define now $\alpha(z):=e^{\varphi(z)}\left(1-\pi(\overline{z-w})v(z)\right)$, where v is as in Proposition 4.3 and φ to be determined. We are assuming that $\partial\Omega$ is smooth, so $v\in\mathscr{C}^\infty(\overline{\Omega})$. In $\partial\Omega$ we have that $\alpha=0$, and $\alpha(w)=e^{\varphi(w)}$. From Proposition 4.3, we get

$$K(w,w)=\langle K(\cdot,w),K(\cdot,w)\rangle_{A^2(\Omega)}=\int_\Omega|K(z,w)|^2dA(z)=\int_\Omega|\partial v|^2dA. \tag{35}$$

Since $\partial\alpha=\partial\left(e^{\varphi}\right)\left(1-\pi(\overline{\cdot-w})v\right)+e^{\varphi}\partial(1-\pi(\overline{\cdot-w})v)=(\partial\varphi)e^{\varphi}(1-\pi(\overline{\cdot-w})v)+e^{\varphi}(-\pi(\overline{\cdot-w})\partial v)$, we obtain $\overline{\partial}^*_\varphi\alpha=-\partial\alpha+\alpha\partial\varphi=\pi(\overline{z-w})\partial v\,e^{\varphi}$, so $|\overline{\partial}^*_\varphi\alpha|^2=\pi^2|z-w|^2|\partial v|^2e^{2\varphi}$. Finally, employing (35),

$$K_\Omega(w,w)=\frac{1}{\pi^2}\int_\Omega|\overline{\partial}^*_\varphi\alpha|^2\frac{e^{-2\varphi(z)}}{|z-w|^2}dA(z). \tag{36}$$

[10]By the Farrell-Markushevich Theorem (see, for instance, [9]), the analytic polynomials are dense in $A^2(\Omega)$ provided that the boundary of Ω is sufficiently regular. Since polynomials are smooth on $\overline{\Omega}$, it follows that $A^2(\Omega)\cap\mathscr{C}^1(\overline{\Omega})$ is dense in $A^2(\Omega)$.

Proposition 4.4. *Let $\mu, \varphi \in \mathscr{C}^2(\overline{\Omega})$ be real-valued functions and assume $\alpha \in \mathscr{C}^2(\overline{\Omega})$ vanishes on $\partial\Omega$. Then*

$$\int_\Omega \mu |\overline{\partial}^*_\varphi \alpha|^2 e^{-\varphi} dA = \int_\Omega \left(\mu |\overline{\partial}\alpha|^2 + |\alpha|^2 (\mu \partial\overline{\partial}\varphi - \partial\overline{\partial}\mu) + 2\Re(\overline{\alpha}\overline{\partial}\mu\overline{\partial}^*_\varphi \alpha) \right) e^{-\varphi} dA.$$

Proof. We have $\langle \mu\overline{\partial}^*_\varphi\alpha, \overline{\partial}^*_\varphi\alpha\rangle_\varphi = \langle \overline{\partial}\mu\overline{\partial}^*_\varphi\alpha, \alpha\rangle_\varphi + \langle \mu\overline{\partial\partial}^*_\varphi\alpha, \alpha\rangle_\varphi$ and by (34) and using the fact that μ is real-valued, $\langle \mu\overline{\partial\partial}^*_\varphi\alpha, \alpha\rangle_\varphi = \langle \overline{\partial}^*_\varphi\overline{\partial}\alpha, \mu\alpha\rangle_\varphi + \langle \alpha\partial\overline{\partial}\varphi, \mu\alpha\rangle_\varphi$. But this first term in the right-hand side can be written as

$$\langle \overline{\partial}^*_\varphi\overline{\partial}\alpha, \mu\alpha\rangle_\varphi = \langle \overline{\partial}\alpha, \mu\overline{\partial}\alpha\rangle_\varphi + \langle \overline{\partial}\alpha, \alpha\overline{\partial}\mu\rangle_\varphi = \langle \overline{\partial}\alpha, \mu\overline{\partial}\alpha\rangle_\varphi + \langle \alpha, \overline{\partial}\mu\overline{\partial}^*_\varphi\alpha\rangle_\varphi - \langle \alpha, \alpha\partial\overline{\partial}\mu\rangle_\varphi.$$

Finally, we get

$$\langle \mu\overline{\partial}^*_\varphi\alpha, \overline{\partial}^*_\varphi\alpha\rangle_\varphi = \langle \overline{\partial}\mu\overline{\partial}^*_\varphi\alpha, \alpha\rangle_\varphi + \langle \overline{\partial}\alpha, \mu\overline{\partial}\alpha\rangle_\varphi + \overline{\langle \overline{\partial}\mu\overline{\partial}^*_\varphi\alpha, \alpha\rangle_\varphi} - \langle \alpha, \alpha\partial\overline{\partial}\mu\rangle_\varphi + \langle \alpha\partial\overline{\partial}\varphi, \mu\alpha\rangle_\varphi,$$

which is the equality claimed. □

Corollary 4.5 (Nakano inequality). *Let $\varphi \in \mathscr{C}^2(\overline{\Omega})$ be a real-valued function and assume $\alpha \in \mathscr{C}^2(\overline{\Omega})$ vanishes on $\partial\Omega$. Let both $\mu_1 \in \mathscr{C}^2(\overline{\Omega})$ and an integrable function μ_2 be positive. Then*

$$\int_\Omega (\mu_1 + \mu_2) |\overline{\partial}^*_\varphi \alpha|^2 e^{-\varphi} dA \geq \int_\Omega |\alpha|^2 \left(\mu_1 \partial\overline{\partial}\varphi - \partial\overline{\partial}\mu_1 - \frac{|\partial\mu_1|^2}{\mu_2} \right) e^{-\varphi} dA.$$

Proof. By applying Proposition 4.4 to the function μ_1, we can write

$$\int_\Omega \mu_1 |\overline{\partial}^*_\varphi \alpha|^2 e^{-\varphi} dA = \int_\Omega \mu_1 |\overline{\partial}\alpha|^2 e^{-\varphi} dA + \int_\Omega \left(|\alpha|^2 (\mu_1 \partial\overline{\partial}\varphi - \partial\overline{\partial}\mu_1) + 2\Re(\overline{\alpha}\overline{\partial}\mu_1\overline{\partial}^*_\varphi\alpha) \right) e^{-\varphi} dA.$$

Since μ_1 is positive and $|\overline{\partial}\alpha|^2 \geq 0$, the first term on the right-hand side is non-negative, so we can discard it to establish the lower bound:

$$\int_\Omega \mu_1 |\overline{\partial}^*_\varphi \alpha|^2 e^{-\varphi} dA \geq \int_\Omega |\alpha|^2 (\mu_1 \partial\overline{\partial}\varphi - \partial\overline{\partial}\mu_1) e^{-\varphi} dA + 2\int_\Omega \Re(\overline{\alpha}\overline{\partial}\mu_1\overline{\partial}^*_\varphi\alpha) e^{-\varphi} dA. \tag{37}$$

To control the term involving the real part, we use the classical inequality $2ab < a^2 + b^2$. Specifically, for any positive function μ_2, we have the pointwise inequality:

$$2\left|\overline{\alpha}\overline{\partial}\mu_1\overline{\partial}^*_\varphi\alpha\right| = 2\left|\alpha\overline{\partial}^*_\varphi\alpha\frac{\partial\mu_1}{\sqrt{\mu_2}}\right| \cdot \sqrt{\mu_2} \leq \frac{|\alpha|^2|\partial\mu_1|^2}{\mu_2} + \mu_2|\overline{\partial}^*_\varphi\alpha|^2.$$

Since we also have $-2\Re(\overline{\alpha}\overline{\partial}\mu_1\overline{\partial}^*_\varphi\alpha) \leq 2\left|\overline{\alpha}\overline{\partial}\mu_1\overline{\partial}^*_\varphi\alpha\right|$, the inequality can be rewritten as:

$$2\Re(\overline{\alpha}\overline{\partial}\mu_1\overline{\partial}^*_\varphi\alpha) \geq -\frac{|\alpha|^2|\partial\mu_1|^2}{\mu_2} - \mu_2|\overline{\partial}^*_\varphi\alpha|^2. \tag{38}$$

Substituting (38) into the lower bound (37),

$$\int_\Omega \mu_1 |\overline{\partial}^*_\varphi \alpha|^2 e^{-\varphi} dA \geq \int_\Omega |\alpha|^2 (\mu_1 \partial\overline{\partial}\varphi - \partial\overline{\partial}\mu_1) e^{-\varphi} dA - \int_\Omega \frac{|\alpha|^2|\partial\mu_1|^2}{\mu_2} e^{-\varphi} dA - \int_\Omega \mu_2 |\overline{\partial}^*_\varphi \alpha|^2 e^{-\varphi} dA.$$

Finally, moving the last integral involving μ_2 to the left-hand side and grouping the terms, we conclude. □

From (36) we should use Corollary 4.5 with $\mu_1+\mu_2 : z \mapsto e^{-\varphi(z)}/|z-w|^2$. Denote as before $G_w = G(\cdot, w)$ and define the harmonic function in Ω, $\psi := 2G_w - \log|z-w|^2$. Then $c_\Omega(w)^2 = e^{\psi(w)}$. We will seek

$$\varphi = \psi + \chi(-2G_w), \qquad \mu_1 = e^{-\gamma(-2G_w)},$$

where $\chi(t), \gamma(t)$, defined for $t = -2G_w \geq 0$ will be determined later. Notice

$$\mu_2 = \frac{e^{-\varphi}}{|z-w|^2} - \mu_1 = \frac{e^{-\psi-\chi(t)}}{|z-w|^2} - e^{-\gamma(t)} = e^{t-\chi(t)} - e^{-\gamma(t)}.$$

However, since $\Delta(\log|\cdot - w|) = 2\partial\overline{\partial}(\log(\cdot - w)(\overline{\cdot - w})) = 2\partial(1/\overline{\cdot - w}) = 2\pi\delta_w$ and $G_w - \log|\cdot - w|$ is harmonic,

$$\partial\overline{\partial} G_w = \frac{2\pi}{4}\delta_w + 0 = \frac{\pi}{2}\delta_w.$$

Thus,

$$\begin{aligned}
&\mu_1\partial\overline{\partial}\varphi - \partial\overline{\partial}\mu_1 \\
&\quad = e^{-\gamma(t)}\left(\chi'(t)(-\pi\delta_w) + \chi''(t)4|\partial G_w|^2\right) - \left(\pi\delta_w\gamma'(t) + 4(\gamma'(t)^2 - \gamma''(t))|\partial G_w|^2\right)e^{-\gamma(t)} \\
&\quad = -e^{-\gamma(t)}\pi\delta_w\left(\chi'(t) + \gamma'(t)\right) + 4|\partial G_w|^2\left(\chi''(t) + \gamma''(t) - \gamma'(t)^2\right)e^{-\gamma(t)}.
\end{aligned}$$

and

$$-\frac{|\partial\mu_1|^2}{\mu_2} = -\frac{\left|-2(\partial G_w)\gamma'(t)e^{-\gamma(t)}\right|^2}{e^{t-\chi(t)} - e^{-\gamma(t)}} = -4\frac{|\partial G_w|^2(\gamma'(t))^2 e^{-2\gamma(t)}}{e^{t-\chi(t)} - e^{-\gamma(t)}},$$

so gathering everything and denoting $\eta = \chi + \gamma$,

$$\begin{aligned}
&\mu_1\partial\overline{\partial}\varphi - \partial\overline{\partial}\mu_1 - \frac{|\partial\mu_1|^2}{\mu_2} \\
&\quad = -e^{-\gamma(t)}\pi\delta_w\left(\chi'(t) + \gamma'(t)\right) + 4\left(\chi''(t) + \gamma''(t) - \frac{\gamma'(t)^2}{1 - e^{\chi(t)-\gamma(t)-t}}\right)e^{-\gamma(t)}|\partial G_w|^2 \\
&\quad = -e^{-\gamma(t)}\pi\delta_w\eta'(t) + 4\left(\eta''(t) - \frac{\gamma'(t)^2}{1 - e^{\eta(t)-2\gamma(t)-t}}\right)e^{-\gamma(t)}|\partial G_w|^2.
\end{aligned}$$

It is convenient to choose η and γ satisfying:

$$-\eta'(t)e^{-\gamma(t)} = 1, \qquad \eta''(t) - \frac{\gamma'(t)^2}{e^{t+2\gamma(t)-\eta(t)} - 1} = 0.$$

This ODE of second order has solution $\eta(t) = -\log(t + e^{-t} - 1)$, so $\gamma(t) = \eta(t) + \log(1 - e^{-t})$. Indeed,

$$\eta'(t) = -\frac{1}{t + e^{-t} - 1}\cdot\frac{d}{dt}(t + e^{-t} - 1) = -\frac{1 - e^{-t}}{t + e^{-t} - 1}$$

and we also have $e^{-\gamma(t)} = e^{-\eta(t)-\log(1-e^{-t})} = (t+e^{-t}-1)(1-e^{-t})^{-1}$, so

$$-\eta'(t)\cdot e^{-\gamma(t)} = -\left(-\frac{1-e^{-t}}{t+e^{-t}-1}\right)\left(\frac{t+e^{-t}-1}{1-e^{-t}}\right) = 1.$$

Morover,

$$\eta''(t) = \frac{(-e^{-t})(t+e^{-t}-1)-(e^{-t}-1)(1-e^{-t})}{(t+e^{-t}-1)^2} = \frac{1-(t+1)e^{-t}}{(t+e^{-t}-1)^2} \tag{39}$$

and

$$\gamma'(t) = \eta'(t) + \frac{e^{-t}}{1-e^{-t}} = \frac{e^{-t}-1}{t+e^{-t}-1} + \frac{e^{-t}}{1-e^{-t}}. \tag{40}$$

Since $t+2\gamma(t)-\eta(t) = t+2[\eta(t)+\log(1-e^{-t})]-\eta(t) = t+\eta(t)+2\log(1-e^{-t})$,

$$e^{t+2\gamma(t)-\eta(t)} = e^t e^{\eta(t)}(1-e^{-t})^2 = \frac{e^t(1-e^{-t})^2}{t+e^{-t}-1} = \frac{(e^t-1)^2}{te^t-(e^t-1)}. \tag{41}$$

From (39), (40) and (41) we deduce the ODE. Therefore, we have $\chi(t) = -\log(1-e^{-t})$ and $\varphi(t) = \psi - \log\left(1-e^{2G_w}\right) = \psi - \log\left(1-|z-w|^2e^{\psi}\right)$. By Corollary 4.5, since $\alpha(w) = e^{\varphi(w)} = e^{\psi(w)} = c_\Omega(w)^2$,

$$\int_\Omega |\overline{\partial}^*_\varphi \alpha|^2 \frac{e^{-2\varphi(z)}}{|z-w|^2} dA(z) \geq \int_\Omega |\alpha|^2 \left(-e^{-\gamma(t)}\pi\delta_w\eta'(t)\right) e^{-\varphi} dA = \pi c_\Omega(w)^2.$$

Finally, using (36), we obtain (31). This completes the proof.

Remark 4.6. Although we have used Corollary 4.5 for μ_1, which is not $\mathscr{C}^2$ at w —in fact it is of the form $\mu_1 = -2G_w + \rho$, where ρ is precisely

$$\rho = -1 + \frac{-2G_w}{e^{-2G_w}-1},$$

which is smooth ($\rho \to -1$ as we approach w)— by approximation it is clear that it holds also for such a function.

Using the ideas of the proof, we obtain the following representation of the Bergman kernel in the diagonal.

Theorem 4.7. *If $\Omega \subset \mathbb{C}$ is a domain and K is the Bergman kernel of $A^2(\Omega)$, then for all $w \in \Omega$,*

$$K(w,w) = \frac{1}{\pi^2}\inf\left\{\int_\Omega \frac{|\partial\alpha(z)|^2}{|z-w|^2}\, dA(z) : \alpha \in \mathscr{C}^\infty_c(\Omega),\ \alpha(w) = 1\right\}.$$

Proof. Fix $w \in \Omega$ and let $\alpha \in \mathscr{C}^\infty_c(\Omega)$, $f \in A^2(\Omega)$ with $\alpha(w) = f(w) = 1$. Then the function $u(z) := f(z)/(\pi(z-w))$ satisfies $\overline{\partial} u = \delta_w$. Indeed,

$$\overline{\partial}\left(\frac{f(z)}{\pi(z-w)}\right) = \frac{1}{\pi}(\overline{\partial} f)\cdot\frac{1}{z-w} + \frac{1}{\pi} f\cdot\left(\overline{\partial}\frac{1}{z-w}\right) = f\cdot\delta_w = f(w)\delta_w = \delta_w.$$

Therefore, since $\alpha(w) = \langle \delta_w, \alpha\rangle$,

$$1 = |\alpha(w)|^2 = \left|\int_\Omega \overline{\alpha}\overline{\partial} u\, dA\right|^2 = \left|-\int_\Omega u\overline{\partial\alpha}\, dA\right|^2 \leq \frac{1}{\pi^2}\|f\|^2_{A^2(\Omega)}\int_\Omega \frac{|\partial\alpha(z)|^2}{|z-w|^2}\, dA(z).$$

Taking now the function $f(z) = K(z,w)/K(w,w) \in A^2(\Omega)$, which satisfies $f(w) = 1$ and $\|f\|^2_{A^2(\Omega)} = 1/K(w,w)$, we obtain

$$K(w,w) \leq \frac{1}{\pi^2} \int_\Omega \frac{|\partial\alpha(z)|^2}{|z-w|^2}\, dA(z).$$

Taking the infimum over all $\alpha \in \mathscr{C}^\infty_c(\Omega)$ with $\alpha(w) = 1$, $\leq$ holds.

To prove $\geq$, assume that $\Omega = \{z \in \mathbb{C} : \rho(z) < 0\}$ is bounded and with smooth boundary. Let v be harmonic in Ω and such that $v(z) = 1/(\pi(\overline{z-w}))$ for all $z \in \partial\Omega$. Then $\alpha(z) := 1 - \pi(\overline{z-w})v(z)$ is smooth up to the boundary, vanishes there and $\alpha(w) = 1$. Proposition 4.3 yields

$$\begin{aligned} K(z,w) &= \partial\left(\frac{\alpha(z)-1}{-\pi(\overline{z-w})}\right) \\ &= \frac{1}{-\pi(\overline{z-w})}\partial(\alpha(z)-1) + (\alpha(z)-1)\partial\left(\frac{1}{-\pi(\overline{z-w})}\right) \\ &= -\frac{\partial\alpha(z)}{\pi(\overline{z-w})} - (\alpha(w)-1)\delta_w = -\frac{\partial\alpha(z)}{\pi(\overline{z-w})}. \end{aligned} \tag{42}$$

It is enough to show that α can be well approximated by test functions. We shall find a sequence $\{\alpha_n\}_n \subset \mathscr{C}^\infty_c(\Omega)$ such that $\alpha_n(w) = 1$ for all n and verifying

$$\int_\Omega \frac{|\partial\alpha_n(z)|^2}{|z-w|^2}\, dA(z) \longrightarrow \int_\Omega \frac{|\partial\alpha(z)|^2}{|z-w|^2}\, dA(z).$$

Consider for that a function $\chi \in \mathscr{C}^\infty(\mathbb{R})$ such that $\chi(t) = 0$ if $t \leq 1$ and $\chi(t) = 1$ if $t \geq 2$ and define for all n the test functions $\alpha_n := \chi(-n\rho)\alpha$. Then it has compact support because $\alpha_n(z) = 0$ when $0 > \rho(z) \geq -1/n$. Since $w \in \Omega$ is a fix point in $\operatorname{int}\Omega$, there exists $c > 0$ such that $-\rho(w) = c > 0$. For n large enough ($n > 2/c$), we have $-n\rho(w) > 2$. Thus, $\chi(-n\rho(w)) = 1$ and $\alpha_n(w) = \alpha(w) = 1$.

Now, applying the ∂ operator we get $\partial\alpha_n = \chi(-n\rho)\partial\alpha - n\chi'(-n\rho)\alpha\partial\rho$, so

$$\partial\alpha_n - \partial\alpha = (\chi(-n\rho)-1)\partial\alpha - n\chi'(n\rho)\alpha\partial\rho.$$

Notice that for $-\rho(z) \geq 2/n$, we have $\chi(-n\rho) = 1$ and $\chi'(-n\rho) = 0$. Therefore, the difference $\partial\alpha_n - \partial\alpha$ is only supported in the strip near the boundary given by

$$U_n = \left\{z \in \Omega : -\frac{2}{n} < \rho(z) < 0\right\}.$$

Since w is fixed away from the boundary, the term $z \mapsto |z-w|^{-2}$ is uniformly bounded in all U_n for n large. It rests to prove $\int_{U_n} |\partial\alpha_n - \partial\alpha|^2\, dA \to 0$. We have, since $A(U_n) \to 0$ as $n \to \infty$, that

$$\int_{U_n} |\chi(-n\rho)-1|^2 |\partial\alpha|^2\, dA \leq \int_{U_n} |\partial\alpha|^2\, dA \to 0,$$

where we have used Lebesgue's Dominated Convergence Theorem. For the other term, since $\alpha|_{\partial\Omega} = 0$ and α is smooth up to the boundary, there exists $C > 0$ such that

$$|\alpha(z)| \leq C\operatorname{dist}(z,\partial\Omega) \leq -C'\rho(z).$$

If we work in U_n, $n|\alpha| \leq 2C'$, and since both χ and $\partial\rho$ ar bounded in a neighborhood of $\overline{\Omega}$, we finally get

$$\int_{U_n} |n\chi'(n\rho)\alpha\partial\rho|^2 \leq M \cdot A(U_n) \to 0$$

for some $M > 0$. Using (42) we obtain

$$K(w,w) = \|K(\cdot,w)\|_{A^2(\Omega)} = \int_\Omega |K(z,w)|^2\, dA(z) = \frac{1}{\pi^2}\int_\Omega \frac{|\partial\alpha(z)|^2}{|z-w|^2}\, dA(z).$$

At the end, we get for all n

$$\begin{aligned}&\inf\left\{\frac{1}{\pi^2}\int_\Omega \frac{|\partial\beta(z)|^2}{|z-w|^2}\, dA(z) : \beta \in \mathscr{C}_c^\infty(\Omega),\, \beta(w) = 1\right\}\\ &\quad\leq \frac{1}{\pi^2}\int_\Omega \frac{|\partial\alpha_n(z)|^2}{|z-w|^2}\, dA(z) \longrightarrow \frac{1}{\pi^2}\int_\Omega \frac{|\partial\alpha(z)|^2}{|z-w|^2}\, dA(z) = K(w,w).\end{aligned}$$

□

5. APPENDIX

5.1. **Some basic results of Complex Analysis.** In this section, we recall some fundamental results from Complex Analysis that will be used throughout this work. We also use the following notation.

- $\mathbb{R}_\infty = \mathbb{R} \cup \{\pm\infty\}$ the two-point compactification of $\mathbb{R}$.
- The Riemann sphere $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ (the Alexandroff extension of $\mathbb{C}$). Given $\Omega \subset \mathbb{C}_\infty$, we denote by $\overline{\Omega}^\infty$ and $\partial_\infty \Omega$ its closure and boundary, respectively.

Theorem 5.1 (**Montel**). *Let Ω be an open subet of $\mathbb{C}$ and let $\mathcal{F} \subset \mathscr{H}(\Omega)$ be a family of holomorphic functions defined in Ω. The following statements are equivalent:*

(1) $\mathcal{F}$ is normal (i.e., any $\{f_n\}_n \subset \mathcal{F}$ has a subsequence which converges uniformly on compact subsets of Ω);
(2) $\mathcal{F}$ is uniformly bounded in compact subsets of Ω.

Theorem 5.2 (**Green**). *Let $\Omega \subseteq \mathbb{C}$ be a domain with $\mathscr{C}^1$ boundary. Let C denote the boundary of Ω, oriented positively. Suppose that $F(x,y) = u(x,y) + iv(x,y)$ is a function that is in $\mathscr{C}^1(\overline{\Omega})$. Then*

$$\oint_C (u\,dx + v\,dy) = \iint_\Omega (v_x - u_y)\,dxdy.$$

Alternatively, if $F = u + iv$,

$$\oint_C F(z)\,dz = 2i \int_\Omega \frac{\partial}{\partial \overline{z}} F(z)\,dA(z).$$

Theorem 5.3 (**Dini**). *Let X be a compact topological space, and let $\{f_n\}_n$ be a monotonically increasing sequence of continuous functions, $f_n : X \to \mathbb{R}$, such that converges pointwise to a continuous function $f : X \to \mathbb{R}$. Then, the convergence is uniform on X.*

Theorem 5.4 (**Harnack**). *Let $U \subset \mathbb{C}$ be a domain. Then*

i) The metric space of the harmonic functions defined on U (with the metric inherited from $\mathscr{C}(U, \mathbb{R})$) is complete.
ii) If $\{u_n\}_n$ is a sequence of harmonic functions defined on U such that $u_1 \leq u_2 \leq \ldots$, then either $u_n \to \infty$ uniformly on compact subsets of U or $u_n \to u$ (in the space of such harmonic functions), where u is harmonic on U.

Definition 5.5. For a compactly supported measure μ on $\mathbb{C}$, the *Cauchy transform* of μ is the function $\hat{\mu}$ defined a.e. by

$$\hat{\mu}(z) = \int \frac{1}{z - w} d\mu(w).$$

We have that $\hat{\mu}$ is locally integrable and analytic on $\mathbb{C}_\infty \setminus \operatorname{supp}(\mu)$.

Theorem 5.6. *Let $\Omega \subset \mathbb{C}$ be a domain. There exist subdomains $\{\Omega_n\}_n$ of Ω with a smooth boundary, $\cup_{n=1}^\infty \Omega_n = \Omega$, such that $\overline{\Omega}_n$ is compact and $\overline{\Omega}_n \subset \Omega_{n+1}$ for each n.*

5.2. **On harmonic, subharmonic and superharmonic functions.** We will introduce now some notions oriented to solve the Dirichlet problem and give some basic results —which will be quite helpful for our purposes— without proof. The reader can find them in any elementary book of Complex Analysis or Potential Theory.

Recall that a *harmonic function* defined in an open set $U \subset \mathbb{C}$ is a function $\mathscr{C}^2(U) \ni u : U \to \mathbb{R}$ such that

$$\Delta = 4\frac{\partial^2 u}{\partial z \partial \bar{z}} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Some elementary properties of harmonic functions are that if $f \in \mathscr{H}(U)$, then $\Re(f)$ and $\Im(f)$ are harmonic. Conversely, for every harmonic function $u : U \to \mathbb{R}$, there exists another harmonic function $v : U \to \mathbb{R}$, called the *harmonic conjugate* of u, such that $f(z) = u(x, y) + iv(x, y)$ is holomorphic in U for $z = x + iy$.

Moreover, if $u : U \to \mathbb{R}$ is harmonic, then it is infinitely differentiable. Also, from Cauchy's integral formula, we deduce that if u is harmonic in U, and $\overline{D}(a, r) \subset U$, then

$$u(a) = \frac{1}{2\pi} \int_0^{2\pi} u(a + re^{i\theta})\, d\theta. \tag{43}$$

We will say that a continuous function $u : U \to \mathbb{R}$ has the *Mean Value Property (MVP)* if whenever $\overline{D}(a, r) \subset U$, (43) is verified. Using the Poisson kernel, one can prove that every continuous function $u : U \to \mathbb{R}$ which has the MVP is harmonic.

Definition 5.7. Let $U \subset \mathbb{C}$ be an open set, and let $\varphi : U \to \mathbb{R}$ be a continuous function. We say that

(1) φ is a *subharmonic function on* U if whenever $\overline{D}(a, r) \subset U$,

$$\varphi(a) \leq \frac{1}{2\pi} \int_0^{2\pi} \varphi(a + re^{i\theta})\, d\theta.$$

(2) φ is a *superharmonic function on* U if whenever $\overline{D}(a, r) \subset U$,

$$\varphi(a) \geq \frac{1}{2\pi} \int_0^{2\pi} \varphi(a + re^{i\theta})\, d\theta.$$

Other properties are that φ is superharmonic if and only if $-\varphi$ is subharmonic. Because of this, we can only work with subharmonic functions. Another remark is that every harmonic function is both subharmonic and superharmonic. In fact, u is harmonic if and only if u is both subharmonic and superharmonic. We present the analogue of the Maximum Principle for harmonic functions, along with several related results without proof.

Theorem 5.8 (**Maximum Principle for subharmonic functions**). *Let $\varphi : U \to \mathbb{R}$ be subharmonic in a domain $U \subset \mathbb{C}_\infty$ and let $z_0 \in U$ be a point such that $\varphi(z_0) = \sup_{z \in U} \varphi(z)$. Then φ is constant in U.*

Corollary 5.9. *Let $U \subset \mathbb{C}_\infty$ be a domain and let $\varphi, \psi : U \to \mathbb{R}$ continuous and bounded, such that φ is subharmonic and ψ is superharmonic. If for each point $p \in \partial_\infty U$,*

$$\limsup_{z \to p} \varphi(z) \leq \liminf_{z \to p} \psi(z),$$

then either $\varphi < \psi$ or $\varphi = \psi$, being this function harmonic.

5.3. **The Dirichlet problem.** Given a domain[11] $\Omega \subset \mathbb{C}_\infty$ and a continuous function $f : \partial_\infty \Omega \to \mathbb{R}$, there is a continuous function $u : \overline{\Omega}^\infty \to \mathbb{R}$ such that u is harmonic on Ω and $u|_{\partial_\infty \Omega} = f$? This is known as the *Dirichlet problem of* Ω *with boundary values* f.

(1) The function f is called *boundary function/values*.
(2) If for a domain Ω and a boundary function f, the Dirichlet problem can be solved (there exists a function u as above), then Ω is called a *Dirichlet domain*.
(3) The function u is called a *(classical) solution of the Dirichlet problem*.

The Dirichlet problem is a classical subject in analysis with strong physical motivations (see [8]). Observe that we are allowing the domains to contain the infinity point. We must clarify the meaning of being harmonic (or subharmonic) at infinity.

Definition 5.10. Let $K \subset \mathbb{C}$ be a compact subset and $u : \mathbb{C} \setminus K \to \mathbb{R}$ harmonic. We say that u is *harmonic at infinity* if $z \mapsto u(1/z)$ has a harmonic extension to a neighborhood of 0.

Remark 5.11. Notice that for bounded domains, the Dirichlet problem has unique (classical) solution (if exists). Indeed, from Theorem 5.8, we deduce that if $U \subset \mathbb{C}$ is a bounded domain, and $u : \overline{U}^\infty \to \mathbb{R}$ continuous, harmonic on U, then $\max_{\overline{U}} u = \max_{\partial U} u$. Therefore, if u_1, u_2 are two solutions, then $u_1 - u_2$ is harmonic and zero on the boundary, so $u_1 - u_2 \equiv 0$.

Some conditions in $\partial\Omega$ are necessary in order to ensure a (classical) solution to the Dirichlet problem in Ω. The first to use subharmonic functions for the study of Dirichlet's problem was O. Perròn. Let $\Omega \subset \mathbb{C}_\infty$ be an open set, and let $f : \partial_\infty \Omega \to \mathbb{R}_\infty$ is any extended real-valued function. Then we define

$$\hat{\mathcal{P}}(f, \Omega) = \{\phi : \phi \text{ subharmonic on } \Omega, \text{ bounded above, and } \limsup_{z \to p} \phi(z) \leq f(p), \text{ for all } p \in \partial_\infty \Omega\},$$

$$\check{\mathcal{P}}(f, \Omega) = \{\phi : \phi \text{ superharmonic on } \Omega, \text{ bounded below, and } \liminf_{z \to p} \phi(z) \geq f(p), \text{ for all } p \in \partial_\infty \Omega\},$$

which are called the *lower* and *upper Perròn families associated with* f *and* Ω, respectively. Observe that $-\check{\mathcal{P}}(-f, \Omega) = \hat{\mathcal{P}}(f, \Omega)$. Also define the *lower* and *upper Perròn functions associated with* f *and* Ω, respectively:

$$\hat{f} = \sup_{\phi \in \hat{\mathcal{P}}(f,\Omega)} \phi, \qquad \check{f} = \sup_{\phi \in \check{\mathcal{P}}(f,\Omega)} \phi.$$

Remark 5.12. Notice here that $\hat{f}$ is well-defined, being possible that it can be the $-\infty$ function, since the $-\infty$ function is in $\hat{\mathcal{P}}(f, \Omega)$. Similarly $\check{f}$, and they are both defined in Ω. Furthermore, $\hat{f}$ and $\check{f}$ are harmonic functions on any component of Ω on which they

[11] While Conway [7] develops the theory for general open sets, he notes that it suffices to consider domains, as a function is solvable (see Definition 5.20) on an open set Ω if and only if it is solvable on each of its connected components.

are not identically $\pm\infty$. We now prove that if a classical solution exists (equivalently we are in a Dirichlet domain), it is given by $u = \hat{f} = \check{f}$.

Proposition 5.13. *If $f : \partial_\infty\Omega \to \mathbb{R}$ is continuous and $u : \overline{\Omega}^\infty \to \mathbb{R}$ is a classical solution of the Dirichlet problem of Ω with boundary values f, then $\hat{f} = \check{f} = u$.*

Proof. $u \leq \hat{f}$: We know that u is harmonic on Ω, so in particular it is subharmonic on Ω. We also have that, for $\zeta \in \partial_\infty\Omega$, $\lim_{z\to\zeta} u(z) = f(\zeta)$. Finally, since $\overline{\Omega}^\infty$ is compact in $\mathbb{C}_\infty$, u is bounded above. Therefore $u \in \hat{\mathcal{P}}(f, \Omega)$, so $u \leq \hat{f}$.

$\hat{f} \leq u$: Let $\phi \in \hat{\mathcal{P}}(f, \Omega)$ fix, and consider $h = \phi - u$. We have h subharmonic on Ω, and for all $\zeta \in \partial_\infty\Omega$

$$\limsup_{z\to\zeta}(\phi(z) - u(z)) = \limsup_{z\to\zeta} \phi(z) - f(\zeta) \leq f(\zeta) - f(\zeta) = 0.$$

By Theorem 5.8, $h(z) = \phi(z) - u(z) \leq 0$ for all $z \in \Omega$, so $\phi \leq u$, and the claim follows.

Similarly, using superharmonic functions, one can prove that $\hat{f} = u$. □

Definition 5.14. Let $U \subset \mathbb{C}$ be a domain and $p \in \partial_\infty U$. A *barrier for U at p* is a family of functions $\{\psi_r : r > 0\}$ such that

(1) ψ_r is defined and superharmonic on $U \cap D(p, r)$, with $0 \leq \psi_r \leq 1$;
(2) $\lim_{z\to p} \psi_r(z) = 0$;
(3) $\lim_{z\to w} \psi_r(z) = 1$ for $w \in U \cap \partial D(p, r)$.

In this case, we say that p is a *regular point*. Otherwise, it is called a *singular point*.

Example 5.15 (Bouligand barrier). Let's prove that every point $p \in \partial\mathbb{D}$ is regular:

Given a small $r > 0$, define $h(z) = \Re(1 - \bar{p}z)$. If $z = p$, then $\bar{p}p = |p|^2 = 1$, so $h(p) = 1 - 1 = 0$. For all $z \in \mathbb{D}$, $\Re(\bar{p}z) < 1$, which implies $h(z) > 0$. Let $M_r = \min\{h(z) : z \in \partial D(p, r) \cap \mathbb{D}\}$ and define

$$\psi_r(z) = \min\left(1, \frac{h(z)}{M_r}\right).$$

Since it is the minimum of harmonic functions, ψ_r is superharmonic. We also have $\lim_{z\to p} h(z) = 0$, so $\psi_r \to 0$ as $z \to p$. Finally, $\lim_{z\to w} \psi_r(z) = 1$ for $w \in U \cap \partial D(p, r)$ by construction. ♣

Lemma 5.16. *If Ω is a Dirichlet domain, then each point on $\partial_\infty\Omega$ is regular.*

Proof. Let $p \in \partial_\infty\Omega$, $(p \neq \infty)$ and consider

$$f(z) = \begin{cases} \frac{|z-p|}{1+|z-p|}, & \text{if } z \neq \infty \\ 1, & \text{if } z = \infty. \end{cases}$$

Then f is continuous on $\partial_\infty\Omega$, so there exists a continuous function $u : \overline{\Omega}^\infty \to \mathbb{R}$ such that u is harmonic on Ω and $u = f$ in the boundary. In particular, $u(p) = 0$, and, in fact, it is the unique zero of u in $\overline{\Omega}^\infty$: $f(z) \geq 0$ for all $z \in \partial_\infty\Omega$. Applying Theorem 5.8, $u \geq 0$ in $\overline{\Omega}^\infty$. Suppose that there exist $q \in \overline{\Omega}^\infty$ such that $u(q) = 0$ and $q \neq p$. If $q \in \partial_\infty\Omega$, this is a contradiction with the definition of f, because $f(w) > 0$ for all points $w \neq p$ in the boundary.

Let $c_r = \inf\{u(z) : |z-p| = r, z \in \Omega\} = \min\{u(z) : |z-p| = r, z \in \overline{\Omega}^{\infty}\} > 0$. Defining

$$\psi_r : D(p,r) \ni z \mapsto \frac{\min\{u(z), c_r\}}{c_r} \in \mathbb{R},$$

these are a barrier for p: Since $\min\{u(z), c_r\} \leq c_r$, clearly $0 \leq \psi_r \leq 1$. Also, the minimum of harmonic functions is superharmonic. We have

$$\lim_{z\to p} \frac{\min\{u(z), c_r\}}{c_r} = \frac{\min\{u(p), c_r\}}{c_r} = \frac{\min\{0, c_r\}}{c_r} = 0.$$

Let $w \in \Omega$ such that $|w-p| = r$. By the definition of c_r, $u(w) \geq c_r$. Hence, if $z \to w$, $\min\{u(z), c_r\} \to \min\{u(w), c_r\} = c_r$ and finally $\psi_r(w) = c_r/c_r = 1$. □

Theorem 5.17. *Let Ω be a domain, and let $p \in \partial_\infty\Omega$ be a regular point. If $f : \partial_\infty\Omega \to \mathbb{R}$ is continuous at p,*

$$\lim_{z\to p} \hat{f}(z) = f(p) = \lim_{z\to p} \check{f}(z).$$

Proof. Let $\{\psi_r : r > 0\}$ be a barrier for Ω at p and let us assume $p \neq \infty$ and $f(p) = 0$ (otherwise consider $f - f(p)$). Let $\varepsilon > 0$ and choose $\delta > 0$ such that $|f(w)| < \varepsilon$, for $w \in \partial_\infty\Omega$ with $|w-p| < 2\delta$. Define $\psi := \psi_\delta$ and

$$\hat{\psi} : \Omega \ni z \mapsto \begin{cases} \psi(z), & \text{if } z \in \Omega \cap D(p,\delta) \\ 1, & \text{if } z \in \Omega \setminus D(p,\delta). \end{cases}$$

Then $\hat{\psi}$ is superharmonic. If $|f(w)| \leq M$ for all $w \in \partial_\infty\Omega$ (which is compact), then $-M\hat{\psi} - \varepsilon$ is subharmonic. Our claim now is more: $-M\hat{\psi} - \varepsilon \in \hat{\mathcal{P}}(f,\Omega)$.

If $w \in \partial_\infty\Omega \setminus D(p,\delta)$, then $\limsup_{z\to w}[-M\hat{\psi}(z) - \varepsilon] = -M - \varepsilon < f(w)$. But $\hat{\psi}(z) > 0$, so $\limsup_{z\to w}[-M\hat{\psi}(z) - \varepsilon] \leq -\varepsilon$ for all $w \in \partial_\infty\Omega$. In particular, if $w \in \partial_\infty\Omega \cap D(p,\delta)$, then $\limsup_{z\to w}[-M\hat{\psi}(z) - \varepsilon] \leq -\varepsilon < f(w)$ by the choice of δ. This proves the claim and therefore

$$-M\hat{\psi} - \varepsilon \leq \hat{f} = u \tag{44}$$

in Ω. A similar analysis yields

$$\liminf_{z\to w}[M\hat{\psi}(z) + \varepsilon] \geq \limsup_{z\to w} \varphi(z),$$

for all $w \in \partial_\infty\Omega$ and $\varphi \in \hat{\mathcal{P}}(f,\Omega)$. By Corollary 5.9, $\varphi \leq M\hat{\psi} + \varepsilon$ for all $\varphi \in \hat{\mathcal{P}}(f,\Omega)$, so $u \leq M\hat{\psi} + \varepsilon$. Combining this with (44),

$$-M\hat{\psi} - \varepsilon \leq u \leq M\hat{\psi} + \varepsilon, \tag{45}$$

and since both $\hat{\psi} \to 0$ and $\psi \to 0$ as $z \to p$ and ε was arbitrary small, (45) gives $\lim_{z\to p} u(z) = 0 = f(p)$.

We conclude that $u = \hat{f}$ take the same values of f as we approach to the boundary. This function is known to be harmonic on Ω and continuous on $\overline{\Omega}^{\infty}$, so it is a (classical) solution. Applying symmetric arguments for the upper Perròn function associated, $\check{f}$, the claim follows. □

Remark 5.18. Theorem 5.17 agrees with Proposition 5.13, which says that in Dirichlet domains $u = \hat{f}$ is the classical solution, but further states that, even when the problem does not admit a classical solution, the function $u = \hat{f}$ has the correct boundary behavior at regular points.

Corollary 5.19 (**Perròn**). *A domain Ω is a Dirichlet domain if and only if every point $p \in \partial_\infty \Omega$ is regular.*

Although this is an important result, sometimes it is hard to verify whether a point is regular or not. It would be desired to find conditions under the boundary of the domain to characterize Dirichlet domains. However, we are interested in what is called "general" solutions of the Dirichlet problem. To this end, we shall focus our attention on the Perròn functions associated to the boundary values of the problem.

Proposition 5.13 shows that, in order to solve the Dirichlet problem, it must be that $\hat{f} = \check{f}$. If this happens, define $u : \overline{\Omega}^\infty \to \mathbb{R}$ as $u = \hat{f}$. However, we still have difficulties here. How do we know that u is continuous on $\overline{\Omega}^\infty$? Theorem 5.17 asserts that it will not always be true, so we can not always solve the Dirichlet problem. Secondly, how can we decide when $\hat{f} = \check{f}$? For this last question, requiring that all points are regular is too strong. Indeed, it can be shown that this happens in bounded domains when the collection of irregular points has harmonic measure zero (see Definition 3.8).

Definition 5.20. Let $\Omega \subset \mathbb{C}_\infty$ be a set. We say that a function $f : \partial_\infty \Omega \to \mathbb{R}_\infty$ is *solvable* if $\hat{f} = \check{f}$ and this function is finite-valued on Ω.

If f is a solvable function, then $\hat{f}$ will be called the *solution of the Dirichlet problem with boundary values f*[12]. If Ω is a domain such that every continuous function on $\partial_\infty \Omega$ is solvable, we say that Ω is a *solvable domain*.

Since $\hat{f}$ and $\check{f}$ are harmonic functions satisfying $\hat{f} \leq \check{f}$ on the domain Ω, the condition $\hat{f}(z_0) = \check{f}(z_0)$ for a single point $z_0 \in \Omega$ implies $\hat{f} \equiv \check{f}$ throughout Ω (consider the harmonic function $H = \check{f} - \hat{f} \geq 0$, which has a minimum at z_0. By Theorem 5.8, $H \equiv H(z_0) = 0$). Thus, for f to be solvable, it suffices that these functions coincide and are finite at one point of the domain.

Theorem 5.21. *Every Dirichlet domain is a solvable domain.*

Proof. Consider a Dirichlet domain $\Omega \subset \mathbb{C}_\infty$. Then for every continuous function $f : \partial_\infty \Omega \to \mathbb{R}$, there exists a classical solution u of the Dirichlet problem. According to Proposition 5.13, f is solvable. □

However, not every solvable domain is a Dirichlet domain. As an example,

Example 5.22. Let $U = \mathbb{D} \setminus \{0\}$. Then $\partial_\infty U = \partial U = \{z : |z| = 1\} \cup \{0\}$. Set

$$f(z) = \begin{cases} 1, & \text{if } |z| = 1 \\ 0, & \text{if } z = 0. \end{cases}$$

Then f is continuous on ∂U. If there is a continuous u on $\overline{U} = \overline{\mathbb{D}}$ which solves the Dirichlet problem, then u must be radial, i.e., it must verify $u(z_1) = u(z_2)$ whenever

[12]Sometimes known as the Perròn-Wiener-Brelot solution of the Dirichlet problem.

$|z_1| = |z_2|$. The reason for this is that if $z \mapsto u(z)$ is a solution and we fix $\theta \in \mathbb{R}$, $z \mapsto u(e^{i\theta}z)$ is still harmonic in U and equals f in ∂U. Then $z \mapsto u(e^{i\theta}z)$ is also a solution. Following Remark 5.11, since U is bounded, the Dirichlet problem has a unique solution, so $u(z) = u(e^{i\theta}z)$.

Using now the chain rule, the Laplace operator $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$, written in polar coordinates, takes the form

$$\Delta = \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

Since u is radial, it is independent of θ, so

$$0 = \Delta u = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right).$$

Hence $r(\partial u/\partial r) = C_1$, with $C_1 \in \mathbb{R}$. It implies $u = C_1 \log r + C_2$, where $C_2 \in \mathbb{R}$. Notice now that there is no possible choice of C_1 and C_2 so that $u|_{\partial U} \equiv f$: if we are in $\partial\mathbb{D} = \{r = 1\}$, it would imply $u(z) = C_2 = 1$, but therefore it does not matter if C_1 is zero or not, $u(0) \neq f(0)$. Therefore, U is not a Dirichlet domain.

However, take any continuous function $f : \partial U \to \mathbb{R}$ and define $g = f|_{\partial\mathbb{D}}$. Example 5.15 shows that all point in $\partial\mathbb{D}$ are regular, Corollary 5.19 ensures the existence of a harmonic function h on $\mathbb{D}$ such that

$$\lim_{z\to\zeta} h(z) = g(\zeta) = f(\zeta), \quad \forall \zeta \in \partial\mathbb{D}.$$

The problem arises with the point $z = 0$, which in fact is irregular. It turns out that this point does not affect us:

Since h is harmonic on $\mathbb{D}$ and coincides with f on $\{|z| = 1\}$, for each $\varepsilon > 0$, the function $v = h + \varepsilon \in \mathcal{P}(\check{f}, \mathbb{D})$. To cope with $f(0)$, we use the logarithmic function (harmonic in U), $w_\varepsilon(z) = h(z) + \varepsilon - \eta \log|z|$, for $\eta > 0$ small. We have $w_\varepsilon(z) \geq f$ in ∂U. Taking $\varepsilon, \eta \to 0$, $\check{f} \leq h$ (analogously $\hat{f} \geq h$. And we know $\hat{f} \leq \check{f}$, so U is solvable. ♣

Remark 5.23. Essentially, the difference between Dirichlet and solvable domains is that solvability only requires the lower and upper Perròn functions to coincide ($\hat{f} = \check{f}$) within the interior of the domain, ensuring the existence of a unique harmonic candidate. In contrast, a Dirichlet domain further demands that this function extends continuously to the entire boundary, satisfying the condition $\lim_{z\to p} u(z) = f(p)$ for every $p \in \partial_\infty\Omega$. A solvable domain may have irregular boundary points where the harmonic solution fails to recover the prescribed boundary value.

In order to characterize these regions, Conway [7] established that every bounded domain is solvable. Furthermore, he proved that any domain whose complement in $\mathbb{C}$ has a non-empty interior (i.e., it is not dense) is also solvable. We now give a result that will be useful to prove Theorem 5.28.

Theorem 5.24 (**Global Regularity for Harmonic Functions**). *Let $\Omega \subset \mathbb{C}$ be a bounded domain with boundary $\partial\Omega$ of class $\mathscr{C}^\infty$. If $u \in \mathscr{C}(\overline{\Omega}) \cap \mathscr{C}^2(\Omega)$ is the unique solution to the Dirichlet problem with boundary values $f \in \mathscr{C}^\infty(\partial\Omega)$, then $u \in \mathscr{C}^\infty(\overline{\Omega})$.*

5.4. **Hyperbolic domains.** Not every domain admits a Green's function. Notice that if Ω has a Green's function G, then there are non-constant positive superharmonic functions functions on Ω via $\Phi_w(z) = -G(z,w)$ for each $w \in \Omega$. This leads us to define the following sets:

Definition 5.25. A domain $\Omega \subset \mathbb{C}_\infty$ is *hyperbolic* if there is a superharmonic function on Ω that is bounded below and not constant on Ω. Otherwise Ω is called *parabolic*.

Remark 5.26. By the comments before the definition, every domain that admits a Green's function is hyperbolic. Notice also that any domain conformally equivalent to a hyperbolic domain is also hyperbolic (Proposition 3.6, c)). Moreover, if Ω is not dense in $\mathbb{C}_\infty$, there exists a disk D such that $\Omega \cap D = \emptyset$. We can send this disk outside $\mathbb{D}$ (using Möbius transform) and send Ω inside $\mathbb{D}$, which is hyperbolic (see Example 3.5 and Theorem 5.27 below), so Ω is hyperbolic.

Given a domain $\Omega \subset \mathbb{C}_\infty$, it can be shown that:

(1) If Ω is solvable and $\partial\Omega$ is not a singleton, then Ω is hyperbolic.
(2) If Ω is parabolic, then $K = \mathbb{C}_\infty \setminus \Omega$ is totally disconnected and $K = \partial K = \hat{K}$.
(3) $\mathbb{C}$ is parabolic.
(4) If $\partial_\infty\Omega$ has at least two points, then Ω is solvable if and only if it is hyperbolic.

The following equivalence is a fundamental tool that we will use. Consequently, from this point forward, we will almost exclusively work with our domains to be hyperbolic.

Theorem 5.27. *A domain $\Omega \subset \mathbb{C}_\infty$ admits a Green's function if and only if it is hyperbolic.*

Proof. Assume Ω hyperbolic and fix $w \in \Omega$, $w \neq \infty$. We will construct explicitly the Green's function with pole at w. Define u on $\partial_\infty\Omega \cup \{w\}$ by $u(w) = -\infty$ and $u(\xi) = 0$ for $\xi \in \partial_\infty\Omega$. Consider now the upper Perròn function of u on $\Omega \setminus \{w\}$; $G_w = \breve{u}$.

i) Clearly, it is a non-positive harmonic function on $\Omega \setminus \{w\}$.

ii) Choose $r > 0$ such that $D = \overline{D(w,r)} \subset \Omega$ and let $R > r$ such that $\overline{D(w,R)} \subset \Omega$. Observe that $\Omega \setminus D$ is solvable, let $\chi = \chi_{\partial D}$ be the characteristic function of the circle ∂D and put $h = \hat{\chi}$, the Perròn function of χ on $\Omega \setminus D$. It can be shown that since $0 \leq h \leq 1$, since we are in an hyperbolic domain, $h < 1$ in $\Omega \setminus D$. Thus $m = \max\{h(z) : |z-w| = R\} < 1$. Choose a positive constant M with $mM < M - \log(r/R)$ and define a function $\psi_0 : \Omega \to \mathbb{R}_\infty$ by

$$\psi_0(z) = \begin{cases} Mh(z), & z \notin \Omega \setminus D \\ M - \log \frac{r}{|z-w|}, & z \in D. \end{cases}$$

This function is superharmonic on Ω. Indeed, ψ_0 is lower semicontinuous. It is continuous and the only place where we have to check the integral condition is when $|z-w| = r$. To verify this, we first check that $Mh(\xi) < M - \log[r/|\xi - w|]$ for $r < |\xi - w| < R$. Indeed, if $|\xi - w| = r$, $h(\xi) = 1$ and so $Mh(\xi) = M = M - \log[r/|\xi - w|]$, so if $|\xi - w| = R$, $Mh(\xi) \leq Mm < M - \log[r/|\xi - w|]$. The desired inequality now holds by the Maximum Principle. Now, if $|z - w| = r$ and $\delta < R - r$, this implies that

$$\frac{1}{2\pi}\int_0^{2\pi} \psi_0(z + \delta e^{i\theta})\, d\theta \leq M - \log \frac{r}{|z-w|} = \psi_0(z).$$

This proves the superharmonicity. Therefore, the function $\psi_0 \in \check{\mathcal{P}}(u, \Omega \setminus \{w\})$ and so the harmonic function G_w satisfies $G_w(z) \leq \psi_0(z)$ for all $z \in \Omega \setminus \{w\}$. If $z \in D$, $z \neq w$, $G_w(z) - \log|z-w| \leq \psi_0(z) - \log|z-w| = M + \log r$, which is a constant. Thus, $G_w(z) - \log|z-w|$ admits a harmonic extension to $D(a,r)$.

iii) Assume that g_w is another function having properties i) and ii) of Definition 3.4, so g_w is a superharmonic function on Ω. Now $\lim_{z\to w} g_w(z) = \infty = u(w)$. Since g_w is negative, $\limsup_{z\to\zeta} g_w(z) \leq 0 = u(\zeta)$ for $\zeta \in \partial_\infty\Omega$. Therefore $g_w \in \check{\mathcal{P}}(u, G \setminus \{a\})$ and so $g_w \geq G$. Hence G_w is the Green function for Ω with singularity at w. □

The next result directly relates the Bergman kernel and Green's function for distinct points $z \neq w$.

Theorem 5.28 (**Schiffer**). *Let $\Omega \subset \mathbb{C}$ be a hyperbolic domain and let G be its Green's function. If K denotes the Bergman kernel of $A^2(\Omega)$, then for all $(z,w) \in \Omega \times \Omega$ with $z \neq w$, we have*

$$K(z,w) = \frac{2}{\pi}\frac{\partial^2 G}{\partial z \partial \bar{w}}(z,w). \tag{46}$$

Proof. By Propositions 2.12, 3.7 and Theorem 5.6, we may assume that Ω is bounded and $\partial\Omega$ is smooth. The function

$$\psi(z,w) = G(z,w) - \log|z-w| \tag{47}$$

is then smooth in $\overline{\Omega} \times \Omega$. Indeed, this is a consequence of Proposition 3.6 f), and Theorem 5.24. Fixing now $w_0 \in \Omega$, set

$$u = \frac{\partial \psi}{\partial \overline{w}}(\cdot, w_0).$$

Then u is harmonic in Ω, continuous on $\overline{\Omega}$ and in the boundary, $u(z) = (2(\overline{z-w}))^{-1}$. Proposition 4.3 yieds

$$K(\cdot, w_0) = \frac{2}{\pi}\frac{\partial u}{\partial z} = \frac{2}{\pi}\frac{\partial^2 G}{\partial z \partial \overline{w}}(\cdot, w_0).$$

□

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