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Constructive Methods in KAM Theory for Quasi-Periodic Time-Dependent Hamiltonian Systems and Applications

Pedro Porras Flores



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Constructive Methods in KAM Theory for Quasi-Periodic Time-Dependent
Hamiltonian Systems and Applications

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PEDRO PORRAS FLORES

DIRECTORES:

DR. RENATO CARLOS CALLEJA CASTILLO
IIMAS, UNAM

DR. ALEJANDRO HARO PROVINCIALE
M.I., UB

MIEMBROS DEL COMITÉ TUTOR:

DR. RICARDO MÉNDEZ FRAGOSO
FACULTAD DE CIENCIAS, UNAM
DR. JOSÉ M. MONDELO GONZÁLEZ
DEPARTAMENTO DE MATEMÁTICAS, UAB

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“The scientist does not study nature because it is useful; he studies it because he delights in it, and he delights in it because it is beautiful. If nature were not beautiful, it would not be worth knowing, and if nature were not worth knowing, life would not be worth living. I mean the intellectual beauty, which is a higher form of beauty, and which alone can satisfy the mind.”.

Henri Poincare

A papá y mamá

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Abstract

In this work, we prove a KAM theorem and present an algorithm formulated in an a-posteriori format, using the parameterization method to look invariant tori in non-autonomous Hamiltonian systems with n degrees of freedom that depend periodically or quasi-periodically (QP) on time, with ℓ external frequencies. Such a system is described by a Hamiltonian function in the $2n$ -dimensional phase space, $\mathcal{M}$, that depends also on ℓ angles, $\varphi \in \mathbb{T}^\ell$. We take advantage of the fibbered structure of the extended phase space $\mathcal{M} \times \mathbb{T}^\ell$. As a result of our approach, the parameterization of tori requires the last ℓ variables, to be precise φ , while the first $2n$ components are determined by an invariance equation. This reduction decreases the dimension of the problem where the unknown is a parameterization from $2(n + \ell)$ to $2n$.

We employ a quasi-Newton method, in order to prove the KAM theorem. This iterative method begins with an initial parameterization of an approximately invariant torus, meaning it approximately satisfies the invariance equation. The approximation is refined by applying corrections that reduce quadratically the invariance equation error. This process converges to a torus in a complex strip of size ρ_∞ , provided suitable Diophantine (γ, τ) conditions and a non-degeneracy condition on the torsion are met. Given the nature of the proof, this provides a numerical method that can be effectively implemented on a computer,

We exhibit the algorithm with two models. The first is a Tokamak model [[CVC⁺05](#), [VL21](#)], which proposes a control method to create barriers to the diffusion of magnetic field lines through a small modification in the magnetic perturbation. The second model [[dCN00](#)], known as the vorticity defect model, describes the nonlinear evolution of localized vorticity perturbations in a constant vorticity flow. This model was originally derived in the context of plasma physics and fluid dynamics.

Introduction

In 2015, the United Nations General Assembly adopted the 2030 Agenda, consisting of 17 interconnected global goals designed to be a *plan to achieve a better and more sustainable future for all*. The seventh of these goals focuses on Affordable and Clean Energy, a significant challenge where science has the responsibility to provide solutions, and governments have the obligation to act in collaboration with society.

Today, numerous efforts are underway to address this energy crisis, one of the most ambitious being the quest to harness energy through nuclear fusion. Hopes in this area are largely placed on the development of Tokamak reactors, which aim to confine plasma using magnetic field lines. The concept is promising, as just 250 kilograms of fuel — 125 kg of deuterium and 125 kg of tritium — could generate the same amount of energy as burning 250 million tons of fossil fuel, without producing radioactive waste, unlike nuclear fission reactors.

With this conviction, the ITER project (International Thermonuclear Experimental Reactor) is developing the first Tokamak. This project demands immense efforts across economic, political, social, and scientific fields. The scientific challenges, which in my opinion are the more straightforward ones, are nonetheless impressive, as they require contributions from a wide range of disciplines: engineering, material science, computer science, nuclear physics, massive computing, plasma physics and, most notably, KAM Theory.

One of the key challenges lies in creating magnetic field models to confine the plasma — these models are often time-dependent Hamiltonian systems. In this regard, it is important to demonstrate the system's stability. Here, is when KAM theory plays a critical role. Due to the physical, dynamical, and geometrical nature of Tokamaks, it is essential to demonstrate and calculate the existence of KAM tori in these non-autonomous Hamiltonian systems.

It is important to note that this modest thesis takes inspiration from the stability of plasma in Tokamak models. However, the techniques developed here are applicable to a wide range of fields, such as physics, biology, astrodynamics, and more.

The KAM Theory, developed by Kolmogorov, Arnold, and Moser [Kol54, Arn63, Mös62], plays a crucial role in understanding the dynamics of Hamiltonian systems, (see [CC06, Dum14] for a brief history). This theory states that, certain invariant tori in an integrable system survive after small perturbations, maintaining a quasiperiodic structure. Despite its importance, the perturbative nature of KAM theory limits its practical application, as it does not always allow for the explicit calculation of invariant manifolds within the physically relevant regime.

However, de la Llave, González, Jorba, and Villanueva introduced the parameterization method in KAM Theory [dLLGJV05], which offers a significant improvement. This method establishes connections between theoretical predictions and practical applications, making it practical and accessible for real-world scenarios. By solving an invariance equation to parameterize the invariant torus and employing iterative schemes with quasi-Newton methods, the parameterization approach addresses the limitations of the classical perturbative methods. This allows researchers to bridge the gap between theory and practice, effectively applying KAM theory in various contexts.

For instance, time-dependent Hamiltonians, which naturally arise in astrodynamics and celestial mechanics, provide a framework for studying the evolution of dynamical systems. A fundamental class of these systems is known as the restricted N -body problems, where a small-mass particle moves under the gravitational influence of a set of primary bodies without affecting their motion.

The Circular Restricted Three-Body Problem (CRTBP) is the simplest case, where two primary bodies orbit each other in circular paths around their common center of mass while the third body moves within their gravitational field [SG69]. A more general case is the Elliptic Restricted Three-Body Problem (ERTBP), where the primary bodies follow elliptical orbits, introducing explicit periodic time dependence.

The Quasi-Bicircular Problem (QBCP) further complicates the system by incorporating a third massive body in an approximately circular orbit, leading to additional resonances [And98, CC07]. These models are crucial for understanding dynamical stability, the existence of invariant tori in near-integrable systems, and the design of space missions [AS99, And03].

Building on this, Fernández-Mora, Haro and Mondelo [FMHM24a, FMHM24b] introduce a method to compute parameterizations of partially hyperbolic invariant tori and their invariant bundles in non-autonomous quasi-periodic Hamiltonian systems. The papers [FMHM24a, FMHM24b] generalize flow map parameterization techniques to the quasi-periodic setting. Introduces fiberwise isotropic tori and discusses symplectic deformations and moment maps to compute parameterizations of partially hyperbolic invariant tori and their invariant bundles in non-autonomous quasi-periodic Hamiltonian systems. These constructions are vital for guaranteeing the existence of solutions to cohomological equations, and their algorithms have been successfully applied to the elliptic restricted three-body problem to compute nonresonant 3-dimensional invariant tori and their invariant bundles around the L_1 point.

Other scenarios where periodic and quasi-periodic Hamiltonian systems naturally emerge include Tokamaks, developed in the 1950s by Tamm and Sakharov, toroidal devices designed for magnetic confinement fusion. Studies, such as those [CVC⁺05] by Chandre, Vittot, Ciraolo, Ghendrih, and Lima, or [dCN00] by del Castillo-Negrete, highlight the dynamics of magnetic field lines and transport

phenomena in these devices. Understanding the stability and structure of invariant tori in this context is essential for effective plasma confinement, as demonstrated in [MMP84], and reducing chaotic transport.

The stability of magnetic structures, such as quasi-periodic flux surfaces, is essential for maintaining confinement and preventing energy losses or plasma instabilities. KAM methods are employed to study the robustness of these surfaces, providing valuable insights into designing more efficient and stable confinement configurations [Boo04]. These developments demonstrate the broader impact of quasi-periodic Hamiltonians, as they not only enhance our understanding of intrinsic dynamics but also contribute to technological advancements in areas as diverse as space exploration and sustainable energy production. All these considerations have motivated the present work.

We present a KAM theorem for quasiperiodically time-dependent systems in a-posteriori format, using the techniques developed in [HCF⁺16, HL19, FH25]. Besides the algorithm suggested by the proof.

We present a proof in a-posteriori format using the parametrization method to obtain a Lagrangian torus in Hamiltonian systems that depend periodically or quasi-periodically on time, with external frequencies vector $\alpha \in \mathbb{R}^\ell$. One of the advantages of this constructive approach is that the proof suggests an algorithm that is implemented numerically on a computer in [CHP25b]. In this work, we also take advantage of the symplectic fibered structure of the extended phase space, using the angles that describe the periodic or quasi-periodic external action.

When adopting the symplectic formalism, the dimension of the extended phase space typically becomes $2(n + \ell)$. However, given our focus on systems with temporal dynamics that are either periodic or quasi-periodic, we redefine the setup. By introducing a variable $\varphi \in \mathbb{T}^\ell$, which parametrizes the external action, the dimension of the phase space is effectively reduced to $2n + \ell$. This adjustment not only simplifies the representation but also aligns naturally with the underlying dynamics, enabling a more efficient analysis of the system.

As a result of our approach, the parameterization of the torus requires the last ℓ variables to have prescribed periodic dynamics, while the n first components are determined by an invariance equation. This reduction allows us to bring down the dimension problem from $2(n + \ell)$ to $2n$. Moreover, we manage to obtain a symmetric expression for the torsion, that does not require computing derivatives of known functions but only their products, which represents a considerable numerical advantage. This feature makes the corresponding numerical method very efficient.

The novel expression for the torsion, given in equation (2.2.25), has a significant advantage over equation (4.60) in [HL19], as it avoids using derivatives and is based solely on the product of functions. This reformulation eliminates the need for Cauchy estimates, leading to improved estimates of the quasi-Newton step for convergence. From a numerical point of view, this modification makes the algorithm more efficient, reducing computational costs and favoring its implementation in high-precision computations, see [CHP25b].

The proof follows an a-posteriori approach, that employs a quasi-Newton method to establish the existence of invariant tori in Hamiltonian dynamical systems. The iterative method starts with an initial parameterization of an approximately invariant torus, in a space with complex strip of size

ρ , such that the invariance equation is approximately satisfied. This approximation is improved by adding a correction that reduces the invariance error to an error which is quadratic with respect to the original error. The process converges to a torus in a space with complex strip of size ρ_∞ , under suitable Diophantine (γ, τ) conditions and a non-degeneracy condition on the torsion, which allows us to prove the persistence of the invariant tori, if $\gamma^{-4}\rho^{-4\tau}$ times the norm is quadratic with respect to the original error. However, with some additional effort, the smallness factor can be relaxed to $\gamma^{-2}\rho^{-2\tau-1}$, like in [FH24]. The nature of the proof is constructive so it leads quite naturally to a numerical algorithm that can be implemented on a computer. The framework is also suitable for obtaining results in different scales of Banach spaces using hard implicit function theorems and smothing [CDIL10]. The theorem states that, under necessary and sufficient geometrical and dynamical hypothesis, there exists a constant $\mathfrak{C}_1$ such that the inequality (2.1.3) is satisfied. An additional advantage of this approach is that the constants $\mathfrak{C}_1$, (γ, τ) and ρ can be explicitly calculated for each problem. This makes it possible to determine the accuracy needed to calculate the invariance error, thus guaranteeing the existence of the torus. A great advantage of the a-posteriori approach is that the hypotheses of the proof depend on the existence of an approximately invariant solution that satisfies regularity and non-degeneracy conditions.

The approximate solution can be obtained in several ways, such as through numerical computations, Lindstedt series, the integrable case, etc. However, since physically relevant regimes are often far from integrability, the theorem provides a way to work in these settings, as long as a good approximation is available. In particular, one can take advantage of numerically computed results to estimate the constant $\mathfrak{C}_1$, the invariance error, and the Diophantine conditions. This framework allows for Computer-Assisted Proofs (CAPs), as demonstrated in the work of [FHL17], (see [CC87, VL21] for other approaches). This not only strengthens the validity of the theoretical results but also guarantees exceptional accuracy and robustness in verifying the existence of the torus. Furthermore, there are real-world problems where relevant solutions can be found. We note that our approach differs from other works that rely on powerful numerical integrations to validate flows and jets like [KS17, MB09]. Instead, we focus on validating the parameterization approximation and its respective estimations at each Newton step.

The work [CVC⁺05], focuses on diffusion control in Hamiltonian systems with one and a half degrees of freedom, introducing a method to create barriers to the diffusion of magnetic field lines in a Tokamak time-periodic model. These barriers, represented by invariant tori, are generated through localized chaos control using a correction term of a smaller order than the original perturbation. The technique finds critical applications in magnetic fusion scenarios, such as restoring confinement in stochastic regions.

On the other hand, in [dCN00] a transported field and a velocity field are dynamically coupled via a mean field term coming from the advection-diffusion equation. Specifically, the analysis focuses on the single-wave model, which reduces to a quasi-periodic pendulum system characterized by the interaction of two external frequencies and one internal frequency, forming a three-dimensional torus. This model involves self-consistent chaotic transport, offering a robust framework for studying the dynamics of coupled systems and their invariant structures.

The application of the KAM method for time-dependent periodic or quasi-periodic invariant tori is well-founded in the context of both scenarios, given their Hamiltonian structure and intrinsic time dependence. In the Tokamak case, KAM theory offers a rigorous framework to establish the persistence of invariant tori. In the second case, the quasi-periodic nature emerging from the single-wave model aligns naturally with the assumptions of KAM theory. This method predicts the persistence of invariant tori while preserving the structured transport properties of the system. By leveraging the KAM framework, both models benefit from its robust capacity to explain the stability and persistence of invariant structures in time-dependent dynamical systems, providing deep insights into their behavior and stability.

The structure of this thesis is as follows. In Chapter 1, we present the notation and tools needed to develop a KAM theorem for Hamiltonian systems with periodic or quasi-periodic time dependencies. This includes the use of the r -th order derivative of, analytic functions, norms, and the Diophantine condition. Additionally, we address nonautonomous Hamiltonians that depend periodically or quasi-periodically on time, focusing on invariant tori, the invariance equation, and the required dynamical and geometric properties.

In Chapter 2 we present the KAM theorem for Lagrangian tori and we prove foundational lemmas. These lemmas provide the necessary conditions and properties to demonstrate the persistence of invariant tori in the system. This lays the groundwork for proving the theorem under specific dynamical and geometric conditions. we provide estimates for various geometrical and dynamical properties of an approximately invariant torus. These approximations are crucial to demonstrate the quadratic convergence of the method.

In Chapter 3, we provide a proof of the KAM Theorem. For clarity, we begin by outlining the approach used to correct the parameterization of the torus. Specifically, Section 3.1 covers the approximate solution of the linearized equations within the symplectic framework constructed in Section 2.2.2, establishing a method for solving the invariance equation. Section 3.2 presents quantitative estimates for the objects derived from one iteration of this procedure. Section 3.3 discusses the convergence of the quasi-Newton method.

In Chapter 4, We apply the suggested algorithm from the proof to a couple of models, which are detailed below. In Section 4.2, we apply the method to a Tokamak model with one and a half degrees of freedom. Using the Poincare section and Birkhoff averages, we compute internal frequencies, using the techniques in [DSSY17], and construct a two-dimensional approximately invariant torus as the initial guess. We then refine this torus using the KAM method and analyze its continuation with respect to parameters. Furthermore, we examine the behavior of Sobolev norms as indicators of potential breakdowns of the invariant torus, see [CDIL10, CL10]. In Section 4.3, we study one more system: a quasiperiodic pendulum with two external perturbations, with three-dimensional invariant tori. We present three scenarios; quasi-periodically perturbed rotor (related to single wave model), quasi-periodically perturbed pendulum (rotational tori) and quasi-periodically perturbed pendulum (librational tori). We emphasize the challenges of implementing these three-dimensional objects numerically, paying special attention to optimizing computational resources, such as RAM memory, and ensuring code efficiency.

Finally, in the appendices of this work provide complementary developments that enhance the main analysis.

In Appendix A, introduces an extended symplectic formulation for quasi-periodic Hamiltonian systems, providing a mathematical framework that facilitates their study.

In Appendix B, is dedicated to the computation of a novel torsion expression, a key element in this work.

In Appendix C, compiles fundamental constants involved in the KAM theorem, These constants are explicit and are used to compute the constants $\mathfrak{C}_1$ and $\mathfrak{C}_2$, that appear in the KAM theorem.

In Appendix D, we present a table for the Tokamak model with the parameter value 0.0004. The table includes the time and memory usage percentages at each step of the quasi-Newton method iterations.

Through this thesis, we demonstrate not only the versatility of the KAM method in various scenarios but also the importance of a robust numerical approach for tackling problems in high-dimensional quasiperiodic Hamiltonian systems. This work lays the groundwork for future research on the optimization and application of KAM methods in physical and mathematical contexts. For instance, to compute invariant tori in the Restricted Three-Body Problem (RTBP), around the equilibrium points L_4 and L_5 .

Preliminaries on Hamiltonian Systems and KAM Theory

We follow [HL19], and include the definitions here for completeness.

1.1 Basic notation

We denote by $\mathbb{R}$ and $\mathbb{C}$ the fields of real and complex numbers, respectively. The spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ represent the real and complex vector spaces of dimension n , respectively, endowed with the norm

$$|v| := \max_{i=1,\dots,n} |v_i|.$$

Let $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ be open subsets of real vector spaces, and let $\rho > 0$ be a positive real number. The complex strip of size ρ , denoted by $U_\rho \times V_\rho$, is a subset of $\mathbb{C}^m \times \mathbb{C}^n$ defined as

$$U_\rho \times V_\rho := \{(x, y) \in \mathbb{C}^m \times \mathbb{C}^n \mid \operatorname{Re}(x, y) \in U \times V, |\operatorname{Im}(x, y)| < \rho\},$$

where $\operatorname{Re}(x, y)$ and $\operatorname{Im}(x, y)$ represent the real and imaginary parts of the complex vectors $x \in \mathbb{C}^m$ and $y \in \mathbb{C}^n$, respectively.

Furthermore, for two sets $X, Y \subset \mathbb{C}^m \times \mathbb{C}^n$, the distance between X and Y , denoted by $\operatorname{dist}(X, Y)$, is defined as

$$\operatorname{dist}(X, Y) := \inf \{|x - y| \mid x \in X, y \in Y\},$$

where $|x - y|$ denotes the Euclidean distance between points x and y in $\mathbb{C}^m \times \mathbb{C}^n$.

Moreover, we denote by $\mathbb{R}^{n_1 \times n_2}$ and $\mathbb{C}^{n_1 \times n_2}$ the spaces of $n_1 \times n_2$ matrices with entries in $\mathbb{R}$ and $\mathbb{C}$, respectively. We adopt the standard identifications $\mathbb{R}^m \simeq \mathbb{R}^{m \times 1}$ and $\mathbb{C}^m \simeq \mathbb{C}^{m \times 1}$, treating column vectors as $m \times 1$ matrices. The identity and zero matrices of size $n \times n$ are denoted by I_n and O_n , respectively, while the zero matrix of size $n_1 \times n_2$ is written as $O_{n_1 \times n_2}$. Additionally, the zero

column vector in $\mathbb{R}^n$ or $\mathbb{C}^n$ is denoted by 0_n . In both the real and complex matrix spaces, $\mathbb{R}^{n_1 \times n_2}$ and $\mathbb{C}^{n_1 \times n_2}$, we define the supremum norm or sup-norm, as a natural extension of the maximum norm for vectors. Given an $n_1 \times n_2$ matrix M with entries in $\mathbb{R}$ or $\mathbb{C}$, its norm is given by

$$|M| := \max_{i=1, \dots, n_1} \sum_{j=1, \dots, n_2} |M_{i,j}|,$$

where $M_{i,j}$ represents the (i, j) -th entry of the matrix. This norm measures the largest absolute value among all entries of the matrix, providing a straightforward way to quantify its size.

Beyond matrices, norms also play a fundamental role in the study of analytic functions. Consider a bounded analytic function

$$f : \mathcal{U} \times \mathcal{V} \rightarrow \mathbb{C},$$

where $\mathcal{U} \subset \mathbb{C}^n$ and $\mathcal{V} \subset \mathbb{C}^m$ are open sets. The supremum norm of f , denoted by $\|f\|_{\mathcal{U} \times \mathcal{V}}$, is defined as

$$\|f\|_{\mathcal{U} \times \mathcal{V}} := \sup_{(x,y) \in \mathcal{U} \times \mathcal{V}} |f(x,y)|.$$

It is particularly useful in the study of analytic extensions, convergence of function sequences, and the application of fixed-point theorems in functional analysis. The action of the r th-order derivative of f at a point $(x, y) \in \mathcal{U} \times \mathcal{V}$ on a collection of (column) vectors $v_1, \dots, v_r \in \mathbb{C}^m$, with $v_k = (v_{1k}, \dots, v_{mk})^\top$, is

$$D_x^r f(x, y)[v_1, \dots, v_r] := \sum_{\ell_1, \dots, \ell_r} \frac{\partial^r f}{\partial x_{\ell_1} \dots \partial x_{\ell_r}}(x, y) v_{\ell_1 1} \dots v_{\ell_r r},$$

where the indices $\ell_1, \dots, \ell_r$ run from 1 to m . And its norm is,

$$\|D_x^r f\|_{\mathcal{U} \times \mathcal{V}} := \sum_{\ell_1, \dots, \ell_r} \left\| \frac{\partial^r f}{\partial x_{\ell_1} \dots \partial x_{\ell_r}} \right\|_{\mathcal{U} \times \mathcal{V}}.$$

Given a matrix valued function $M : \mathcal{U} \times \mathcal{V} \subset \mathbb{C}^m \times \mathbb{C}^n \rightarrow \mathbb{C}^{n_1 \times n_2}$ (whose components, $M_{i,j}$, are analytic functions), a point $(x, y) \in \mathcal{U} \times \mathcal{V}$, and a collection of (column) vectors $v_1, \dots, v_r \in \mathbb{C}^m$, we obtain a $n_1 \times n_2$ matrix $D_x^r M(x, y)[v_1, \dots, v_r]$ such that

$$(D_x^r M(x, y)[v_1, \dots, v_r])_{i,j} := D_x^r M_{i,j}(x, y)[v_1, \dots, v_r],$$

since M is analytic in $\mathcal{U} \times \mathcal{V}$, we can define the norm.

$$\|D_x^r M\|_{\mathcal{U} \times \mathcal{V}} := \max_{i=1, \dots, n_1} \sum_{j=1, \dots, n_2} \|D_x^r M_{i,j}\|_{\mathcal{U} \times \mathcal{V}},$$

with,

$$\|D_x^r M_{i,j}\|_{\mathcal{U} \times \mathcal{V}} = \sum_{\ell_1, \dots, \ell_r} \left\| \frac{\partial^r M_{i,j}}{\partial x_{\ell_1} \dots \partial x_{\ell_r}} \right\|_{\mathcal{U} \times \mathcal{V}}.$$

Therefore, we have,

$$\begin{aligned}
\|D_x^r M[v_1, \dots, v_r]\|_{\mathcal{U} \times \mathcal{V}} &\leq \max_{i=1, \dots, n_1} \sum_{j=1}^{n_2} \|D_x^r M_{i,j}[v_1, \dots, v_r]\|_{\mathcal{U} \times \mathcal{V}} \\
&\leq \max_{i=1, \dots, n_1} \sum_{j=1}^{n_2} \left\| \sum_{\ell_1, \dots, \ell_r} \frac{\partial^r M_{i,j}}{\partial x_{\ell_1} \dots \partial x_{\ell_r}} v_{\ell_1 1} \dots v_{\ell_r r} \right\|_{\mathcal{U} \times \mathcal{V}} \\
&\leq \max_{i=1, \dots, n_1} \sum_{j=1}^{n_2} \sum_{\ell_1, \dots, \ell_r} \left\| \frac{\partial^r M_{i,j}}{\partial x_{\ell_1} \dots \partial x_{\ell_r}} \right\|_{\mathcal{U} \times \mathcal{V}} \max_{\ell=1, \dots, r} \|v_{\ell 1}\|_{\mathcal{U}} \dots \max_{\ell=1, \dots, r} \|v_{\ell r}\|_{\mathcal{U}}
\end{aligned}$$

There is also a similar bound for the action of the transpose,

$$\|(D_x^r M[v_1, \dots, v_r])^\top\|_{\mathcal{U} \times \mathcal{V}} \leq \|D_x^r M^\top\|_{\mathcal{U} \times \mathcal{V}} \|v_1\|_{\mathcal{U}} \dots \|v_r\|_{\mathcal{U}}.$$

We introduce the fundamental notation that will be used throughout this analysis. The d -dimensional standard torus is denoted by

$$\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d,$$

which represents the quotient space obtained by identifying points in $\mathbb{R}^d$ that differ by integer translations. Intuitively, the torus can be seen as a generalization of the unit circle $\mathbb{T}^1$ to higher dimensions, where each coordinate behaves as a periodic variable with period 1.

To extend this concept to the complex domain, we define the complex strip of the torus of width $\rho > 0$, denoted by

$$\mathbb{T}_\rho^d = \{\theta \in \mathbb{C}^d / \mathbb{Z}^d : |\operatorname{Im} \theta| < \rho\}.$$

This set consists of all points in the complexified torus whose imaginary components remain strictly bounded by ρ . The introduction of this complex extension plays an important role in the study of analytic functions, particularly in perturbation theory and the KAM (Kolmogorov–Arnold–Moser) theory, where analyticity in a strip around the real torus is a fundamental assumption.

Now, consider a real-valued function defined on the direct product of two tori:

$$u : \mathbb{T}^d \times \mathbb{T}^\ell \rightarrow \mathbb{R}.$$

Such a function is inherently periodic in all of its arguments. In many applications, it is convenient to analytically continue u to a complex domain. For this purpose, we define its complex extension over the product of complex strips:

$$u : \mathbb{T}_\rho^d \times \mathbb{T}_\rho^\ell \rightarrow \mathbb{R}.$$

This formulation allows us to study the analyticity properties of u in a controlled neighborhood of the real torus. The width ρ of the complex strip determines how far the function can be extended into the complex plane while maintaining analyticity. The larger ρ is, the more regular (or well-behaved)

the function u is expected to be.

Furthermore, because u is periodic in its toroidal variables, it admits a Fourier series expansion,

$$u(\theta, \varphi) = \sum_{k_1 \in \mathbb{Z}^d} \sum_{k_2 \in \mathbb{Z}^\ell} \hat{u}_{k_1, k_2} e^{2\pi i(k_1 \cdot \theta + k_2 \cdot \varphi)},$$

$$\hat{u}_{k_1, k_2} = \int_{\mathbb{T}^\ell} \int_{\mathbb{T}^d} u(\theta, \varphi) e^{-2\pi i(k_1 \cdot \theta + k_2 \cdot \varphi)} d\theta d\varphi,$$

and introduce the notation $\langle u \rangle := \hat{u}_{0,0}$ for the average. This representation is fundamental in the analysis of quasi-periodic dynamical systems, as it provides a natural way to decompose functions into their harmonic components. The notation above is extended to $n_1 \times n_2$ matrix-valued periodic functions $M : \mathbb{T}_\rho^d \times \mathbb{T}_\rho^\ell \rightarrow \mathbb{C}^{n_1 \times n_2}$, for which $\hat{M}_{k_1, k_2} \in \mathbb{C}^{n_1 \times n_2}$ denotes the Fourier coefficient with index $(k_1, k_2) \in \mathbb{Z}^d \times \mathbb{Z}^\ell$. Now, we denote the $\mathcal{A}(\mathbb{T}_\rho^d \times \mathbb{T}_\rho^\ell)$ to represent the Banach space of holomorphic functions $u : \mathbb{T}_\rho^d \times \mathbb{T}_\rho^\ell \rightarrow \mathbb{C}$ that can be continuously extended to $\bar{\mathbb{T}}_\rho^d \times \bar{\mathbb{T}}_\rho^\ell$ and whose values in $\mathbb{T}_\rho^d \times \mathbb{T}_\rho^\ell$, are real-analytic. The space $\mathcal{A}(\mathbb{T}_\rho^d \times \mathbb{T}_\rho^\ell)$ is endowed with the norm,

$$\|u\|_\rho = \|u\|_{\mathbb{T}_\rho^d \times \mathbb{T}_\rho^\ell} = \max_{|\operatorname{Im}(\theta, \varphi)| \leq \rho} |u(\theta, \varphi)|.$$

Using Cauchy estimates, we can determine the derivatives of these functions. For example, if $u \in \mathcal{A}(\mathbb{T}_\rho^d \times \mathbb{T}_\rho^\ell)$ and $\rho > 0$, then its partial derivative $\partial u / \partial x_\ell$ belongs to $\mathcal{A}(\mathbb{T}_{\rho-\delta}^d \times \mathbb{T}_{\rho-\delta}^\ell)$ for any $0 < \delta < \rho$ and we have the following bounds:

$$\left\| \frac{\partial u}{\partial x_\ell} \right\|_{\rho-\delta} \leq \frac{1}{\delta} \|u\|_\rho, \quad \|D_x u\|_{\rho-\delta} \leq \frac{d}{\delta} \|u\|_\rho, \quad \|(D_x u)^\top\|_{\rho-\delta} \leq \frac{1}{\delta} \|u\|_\rho.$$

The proof of these three inequalities relies on the Cauchy estimates for holomorphic functions in a complex strip. For the first inequality, applying the Cauchy integral formula to u in the variable x_ℓ implies that the norm of the derivative can be bounded by the supremum of u in a slightly larger domain, introducing a factor of $1/\delta$. The second inequality follows by applying this result to each component of the gradient $(D_x u)$ and summing the contributions of all d partial derivatives, which introduces an additional factor of d . Finally, the third inequality follows similarly as the case. For matrix valued functions $M : \mathbb{T}_\rho^d \times \mathbb{T}_\rho^\ell \rightarrow \mathbb{C}^{n_1 \times n_2}$ with components in $\mathcal{A}(\mathbb{T}_\rho^d \times \mathbb{T}_\rho^\ell)$, we have $\|D_x M\|_{\rho-\delta} \leq \frac{d}{\delta} \|M\|_\rho$. This result also applies to the transpose of M , $\|D_x M^\top\|_{\rho-\delta} \leq \frac{d}{\delta} \|M^\top\|_\rho$. In the particular case $w : \mathbb{T}_\rho^d \times \mathbb{T}_\rho^\ell \rightarrow \mathbb{C}^n \approx \mathbb{C}^{n \times 1}$, we obtain:

$$\|D_x w\|_{\rho-\delta} \leq \frac{d}{\delta} \|w\|_\rho, \quad \|D_x w^\top\|_{\rho-\delta} \leq \frac{d}{\delta} \|w^\top\|_\rho \leq \frac{nd}{\delta} \|w\|_\rho,$$

$$\|(D_x w)^\top\|_{\rho-\delta} \leq \frac{n}{\delta} \|w\|_\rho.$$

The set of Diophantine vectors is defined as

$$\mathcal{D}_{\gamma, \tau} = \left\{ (\omega, \alpha) \in \mathbb{R}^d \times \mathbb{R}^\ell : |k_1 \cdot \omega + k_2 \cdot \alpha| \geq \frac{\gamma}{|(k_1, k_2)|_1^\tau}, \forall (k_1, k_2) \in \mathbb{Z}^d \times \mathbb{Z}^\ell \setminus \{0\} \right\}. \quad (1.1.1)$$

An essential condition in this theorem is the assumption that the frequency vector (ω, α) for specific $\gamma > 0$ and $\tau \geq d + \ell - 1$, satisfies the Diophantine condition. The quantity $|(k_1, k_2)|_1$ represents the sum of the absolute values of each component of the vector (k_1, k_2) .

Remark 1.1.1. It is not possible to represent exact Diophantine vectors numerically, because of the limited precision of floating-point arithmetic. This limitation means that we cannot accurately define irrational numbers, and therefore true Diophantine vectors. Instead, we can approximate these vectors using the best possible numbers within the arithmetic precision we are working with. For each component of the frequency vector that is not an integer, we express it as a continued fraction. We replace the final part of the continued fraction with ones to improve the approximation and make the components more accurate. In this way, we could compute numerically Diophantine constants as in [Vil17, HL19]

Finally, we present notation regarding the cohomological equations which are central to KAM theory. This notation describes the relationship between a periodic or quasi-periodic function,

$$v(\theta, \varphi) = \sum_{k_1 \in \mathbb{Z}^d} \sum_{k_2 \in \mathbb{Z}^\ell} \hat{v}_{k_1, k_2} e^{2\pi i(k_1 \cdot \theta + k_2 \cdot \varphi)},$$

and the frequency vector, $(\omega, \alpha) \in \mathbb{R}^d \times \mathbb{R}^\ell$,

$$\mathfrak{L}_{\omega, \alpha} u = v - \langle v \rangle, \quad \mathfrak{L}_{\omega, \alpha} := - \left(\sum_{i=1}^d \omega_i \frac{\partial}{\partial \theta_i} + \sum_{j=1}^\ell \alpha_j \frac{\partial}{\partial \varphi_j} \right). \quad (1.1.2)$$

The formal solution of equation (1.1.2) can be expressed as

$$\mathfrak{R}_{\omega, \alpha} v(\theta, \varphi) = \sum_{k_1 \in \mathbb{Z}^d \setminus \{0\}} \sum_{k_2 \in \mathbb{Z}^\ell \setminus \{0\}} \hat{u}_{k_1, k_2} e^{2\pi i(k_1 \cdot \theta + k_2 \cdot \varphi)}, \quad \hat{u}_{k_1, k_2} = \frac{-\hat{v}_{k_1, k_2}}{2\pi i(k_1 \cdot \omega + k_2 \cdot \alpha)}. \quad (1.1.3)$$

with $(k_1, k_2) \neq (0, 0)$ and $\hat{u}_{0,0}$ is a degree of freedom. Notice that if $v = 0$, then $\mathfrak{R}_{\omega, \alpha} v = 0$.

For $r \in \mathbb{R}^+$, the Sobolev space H^r is a Banach space consisting of functions v . These functions are characterized by the norm

$$\|v\|_r := \sqrt{\sum_{k_1 \in \mathbb{Z}^d} \sum_{k_2 \in \mathbb{Z}^\ell} |2\pi i(k_1 \cdot \omega + k_2 \cdot \alpha)|^{2r} |\hat{v}_{k_1, k_2}|^2} < \infty. \quad (1.1.4)$$

Here, $|\cdot|$ denotes the maximum norm in the spaces $\mathbb{R}^n$ and $\mathbb{C}^n$. Similarly, this notation extends to real or complex matrices of arbitrary dimension, referring to the matrix norm induced by the corresponding vector norm.

The behavior of this norm is crucial in the study of the breakdown of quasi-periodic solutions. As discussed in [CDIL10, CL10], the Sobolev norm serves as a key diagnostic tool for identifying the loss of smoothness in invariant tori. A rapid growth in this norm indicates the onset of breakdown, linking it to the regularity properties of the system. These ideas provide a numerical and theoretical foundation for detecting critical thresholds in quasi-periodic structures.

The notation required for this thesis, including analytic function spaces, norms of these functions and their derivatives, Diophantine vectors, cohomological equations, and related concepts can be extended from the works [HCF⁺16, HL19, FH24] and is very similar to [FMHM24a]. For convenience, we refer the reader to these references.

1.2 QP Hamiltonian systems, symplectic bundles and invariant tori

We require an open set $\mathcal{M}$ of $\mathbb{R}^{2n}$ endowed with an exact symplectic form $\omega = d\alpha$, for a certain 1-form α called *action form*. This setting can be adapted to the case in which $\mathcal{M}$ is open subset of $\mathcal{M} \subset \mathbb{T}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^n$, with $n_1 + n_2 = n$. We will assume that ω is exact, so $\mathcal{M}$ is endowed with an exact symplectic structure. The matrix representations of α and ω are given by the matrix-valued functions

$$\begin{aligned} a : \mathcal{M} &\longrightarrow \mathbb{R}^{2n}, \\ z &\longmapsto a(z), \end{aligned}$$

and

$$\begin{aligned} \Omega : \mathcal{M} &\longrightarrow \mathbb{R}^{2n \times 2n}, \\ z &\longmapsto \Omega(z) = (Da(z))^\top - Da(z), \end{aligned}$$

the non-degeneracy of ω is equivalent to $\det \Omega(z) \neq 0$ for all $z \in \mathcal{M}$. If there exists a globally defined function $h : \mathcal{M} \rightarrow \mathbb{R}$ such that $i_{X_h} \omega = -dh$, then X_h is said to be a Hamiltonian vector field with respect to the symplectic form ω and h is a Hamiltonian function for X_h . In coordinates,

$$X_h(z) = \Omega(z)^{-1} (Dh(z))^\top.$$

In the non-autonomous case, the Hamiltonian can be expressed as follows,

$$\begin{aligned} h : \mathcal{M} \times \mathbb{R} &\longrightarrow \mathbb{R}, \\ (z, t) &\longmapsto h(z, t). \end{aligned}$$

The symplectic 2-form requires an even-dimensional phase space, while $\mathcal{M} \times \mathbb{R}$, is odd-dimensional. The typical way to overcome this issue is to introduce a on $\mathcal{M} \times \mathbb{R}$ using the form $\omega_h = \omega + dh \wedge dt$. However, this means that the phase space dimension is $2(n+1)$, and if we want to study time-dependent periodic or quasi-periodic dynamics with associated frequencies $\alpha \in \mathbb{R}^\ell$, the phase space dimension becomes $2(n+\ell)$, since the symplectic form on $\mathcal{M} \times \mathbb{T}^\ell$ is $\omega_h = \omega + dI \wedge d\varphi$, with I the fictitious variables conjugated to φ . This would be expensive from the numerical point of view so, an alternative approach is to consider that the time dependence is on an ℓ -dimensional torus. In other words,

$$\begin{aligned} h : \mathcal{M} \times \mathbb{T}^\ell &\longrightarrow \mathbb{R}, \\ (z, \varphi) &\longmapsto h(z, \varphi), \end{aligned}$$

where $\dot{\varphi} = \alpha$ and $\alpha \in \mathbb{R}^\ell$ are the external frequencies. We consider the evolution operator

$$\begin{aligned} \phi_t : \mathcal{I}(z, \varphi) \subset \mathbb{R} &\longrightarrow \mathcal{M} \\ t &\longmapsto \phi_t(z, \varphi), \end{aligned} \quad (1.2.1)$$

where $\mathcal{I}(\theta, \varphi)$ is the maximal interval of existence, and $\phi_t(z, \varphi)$ satisfies

$$\begin{aligned} \frac{\partial \phi_t}{\partial t}(z, \varphi) &= X_h(\phi_t(z, \varphi), \varphi + t\alpha), \\ \phi_0(z, \varphi) &= z, \end{aligned}$$

where X_H is a vector field. We define

$$Dom = \{(t, z, \varphi) | (z, \varphi) \in \mathcal{M} \times \mathbb{T}^\ell, t \in \mathcal{I}(z, \varphi)\}.$$

Now, we consider the following flow,

$$\begin{aligned} \tilde{\Phi}_t : Dom &\longrightarrow \mathcal{M} \times \mathbb{T}^\ell \\ (z, \varphi, t) &\longmapsto \tilde{\Phi}_t(z, \varphi) := \begin{pmatrix} \phi_t(z, \varphi) \\ \varphi + t\alpha \end{pmatrix}, \end{aligned} \quad (1.2.2)$$

Remark 1.2.1. The equation (1.2.2) satisfies the flow properties.

$$\tilde{\Phi}_0(z, \varphi) = \begin{pmatrix} \phi_0(z, \varphi) \\ \varphi \end{pmatrix} = \begin{pmatrix} z \\ \varphi \end{pmatrix},$$

and

$$\begin{aligned} \tilde{\Phi}_{t+s}(z, \varphi) &= \begin{pmatrix} \phi_{t+s}(z, \varphi) \\ \varphi + (t+s)\alpha \end{pmatrix} = \begin{pmatrix} (\phi_t \circ \phi_s)(z, \varphi) \\ \varphi + t\alpha + s\alpha \end{pmatrix} = \begin{pmatrix} \phi(\phi(z, \varphi), s), t \end{pmatrix} \\ &= \tilde{\Phi}_t \left(\begin{pmatrix} \phi_s(z, \varphi) \\ \varphi + s\alpha \end{pmatrix} \right) = \tilde{\Phi}_t \left(\tilde{\Phi}_s \left(\begin{pmatrix} z \\ \varphi \end{pmatrix} \right) \right) = \tilde{\Phi}_s \circ \tilde{\Phi}_t \left(\begin{pmatrix} z \\ \varphi \end{pmatrix} \right). \end{aligned}$$

Therefore, we can define the vector field of a non-autonomous system that depends periodically or quasi-periodically on time accordingly. As a result, we can determine the vector field of a non-autonomous system that depends periodically or quasi-periodically on time,

$$\begin{aligned} \tilde{Z} : \mathcal{M} \times \mathbb{T}^\ell &\longrightarrow \mathbb{R}^{2n} \times \mathbb{T}^\ell, \\ (z, \varphi) &\longmapsto \tilde{Z}(z, \varphi) := \begin{pmatrix} Z_h(z, \varphi) \\ \alpha \end{pmatrix}, \end{aligned}$$

with $Z_h(z, \varphi) = \Omega(z)^{-1}(\mathcal{D}_z h(z, \varphi))^\top$. We look for a parameterization $\tilde{K} : \mathbb{T}^d \times \mathbb{T}^\ell \rightarrow \mathcal{M} \times \mathbb{T}^\ell$, with $1 \leq d \leq n$, that satisfies the invariance equation

$$\tilde{\Phi}_t \left(\tilde{K}(\theta, \varphi) \right) = \tilde{K}(\theta + t\omega, \varphi + t\alpha), \quad (1.2.3)$$

where $\omega \in \mathbb{R}^d$ is a vector of internal frequencies, furthermore ω and α are non-resonant. In this work, we are interested in finding a parameterization for the Lagrangian Torus, as discussed in [Mos66, BHS96], i.e., we consider the case where $d = n$; hence, throughout this work, we focus exclusively on this scenario. From the above equation, $\tilde{K}$ is of the form,

$$\tilde{K}(\theta, \varphi) = \begin{pmatrix} K(\theta, \varphi) \\ \varphi \end{pmatrix}. \quad (1.2.4)$$

We denote by $\tilde{\mathcal{K}} = \{(K(\theta, \varphi), \varphi) | (\theta, \varphi) \in \mathbb{T}^n \times \mathbb{T}^\ell\}$, the embedded submanifold of $\mathcal{M} \times \mathbb{T}^\ell$. If π_φ is the projection on $\mathcal{M}$ with basepoint φ , then $\mathcal{K}_\varphi = \tilde{K}(\mathbb{T}^n, \varphi)$ is fiber of $\mathcal{K}$. Parameterization (1.2.4)

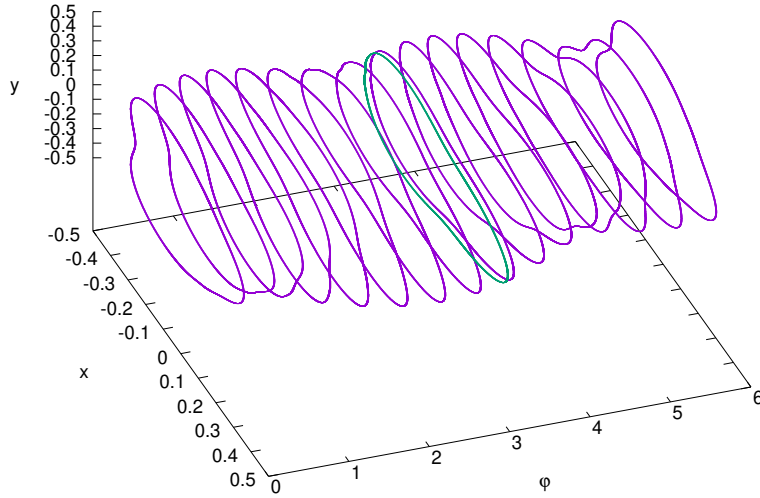


Figure 1.1: Two-dimensional torus expressed in cylindrical coordinates, each curve is a leaf of the torus fibration. The green leaf is the K parameterization for a given φ , $\mathcal{K}_\varphi$.

has the advantage that $K : \mathbb{T}^n \times \mathbb{T}^\ell \rightarrow \mathcal{M}$, now only has to solve equation,

$$Z_h(K(\theta, \varphi), \varphi) = D_\theta K(\theta, \varphi)\omega + D_\varphi K(\theta, \varphi)\alpha, \quad (1.2.5)$$

which is an infinitesimal version of (1.2.3) for the n -component of $\tilde{K}$. Notice that the equation for φ is trivial. Geometrically, we visualize that the manifold $\tilde{K}(\theta, \varphi)$ is foliated by $K(\theta, \varphi)$ for every φ , as shown in Figure 1.1. This means that the $(n + \ell)$ -dimensional manifold $\mathcal{K} = K(\mathbb{T}^n, \mathbb{T}^\ell)$ is invariant and the internal dynamics is given by the constant vector field (ω, α) , the invariance equation can be written in the form

$$Z_h(K(\theta, \varphi), \varphi) = -\mathcal{L}_{\omega, \alpha}(K(\theta, \varphi)). \quad (1.2.6)$$

We assume that the parametrization is regular, meaning that $D_\theta K(\theta, \varphi)$ has rank n . The way the

torus $\tilde{\mathcal{K}}$ is embeded in $\mathcal{M} \times \mathbb{T}^\ell$ is given by the *degree map* $\mathfrak{D} \in \mathbb{Z}^{n \times n_1}$ so that K is of the form

$$K(\theta, \varphi) = \begin{pmatrix} \mathfrak{D}\theta \\ 0_{n_2} \\ 0_n \end{pmatrix} + K_p(\theta, \varphi) = \begin{pmatrix} \mathfrak{D}\theta \\ 0_{n_2} \\ 0_n \end{pmatrix} + \begin{pmatrix} K_x^p(\theta, \varphi) \\ K_y^p(\theta, \varphi) \end{pmatrix},$$

where $K_p : \mathbb{T}^n \times \mathbb{T}^\ell \rightarrow \mathbb{R}^{2n}$ is 1-periodic in (θ, φ) ,

meaning $K_p(\theta + e_1, \varphi + e_2) = K_p(\theta, \varphi)$ for all $(e_1, e_2) \in \mathbb{Z}^n \times \mathbb{Z}^\ell$.

If the torus is approximately invariant, we define the *error of invariance* $E : \mathbb{T}^n \times \mathbb{T}^\ell \rightarrow \mathcal{M}$ given by

$$E(\theta, \varphi) = Z_h(K(\theta, \varphi), \varphi) + \mathfrak{L}_{\omega, \alpha}(K(\theta, \varphi)). \quad (1.2.7)$$

Remark 1.2.2. Another way to handle time-dependent quasiperiodic Hamiltonian systems,

$$\begin{aligned} h : \mathbb{R}^{2n} \times \mathbb{T}^\ell &\longrightarrow \mathbb{R}, \\ (z, \varphi) &\longmapsto h(z, \varphi), \end{aligned}$$

is to extend the symplectic formalism by introducing fictitious variables, $\mathbf{I}$, associated with momentum, which are conjugate to the quasi-periodic time variables, i.e.

$$\begin{aligned} \hat{h} : \mathbb{R}^{2n} \times \mathbb{T}^\ell \times \mathbb{R}^\ell &\longrightarrow \mathbb{R}, \\ (z, \varphi, \mathbf{I}) &\longmapsto \hat{h}(z, \varphi, \mathbf{I}) = h(z, \varphi) + \mathbf{I} \cdot \alpha, \end{aligned}$$

with $\alpha \in \mathbb{R}^\ell$ external frequencies. We obtain a parameterization of an invariant torus of dimension $2(n + \ell)$, as follows

$$\hat{K}(\theta, \varphi) = \begin{pmatrix} K(\theta, \varphi) \\ \varphi \\ \mathbf{I}(\theta, \varphi) \end{pmatrix}.$$

With this approach, the problem has dimension $2(n + \ell)$ instead of $2n$. Moreover, the proof becomes harder since the involved objects have more intricate expressions. For further details, see the Appendix A.

1.3 Linearized dynamics and reducibility

Let us introduce an operator for any m -dimensional vector subbundle, parameterized by $V : \mathbb{T}^n \times \mathbb{T}^\ell \rightarrow \mathbb{R}^{2n \times m}$, where $1 \leq m \leq 2n$. The operator is defined as

$$\mathcal{X}_V(\theta, \varphi) := D_z Z_h(K(\theta, \varphi), \varphi) V(\theta, \varphi) + \mathfrak{L}_{\omega, \alpha}(V(\theta, \varphi)). \quad (1.3.1)$$

We say that a subbundle is invariant under the linearized equations if $\mathcal{X}_V(\theta, \varphi) = O_{2n \times m}$ for all $(\theta, \varphi) \in \mathbb{T}^n \times \mathbb{T}^\ell$. We consider the map $L : \mathbb{T}^n \times \mathbb{T}^\ell \rightarrow \mathbb{R}^{2n \times n}$ given by

$$L(\theta, \varphi) = D_\theta K(\theta, \varphi),$$

we note that $\mathcal{X}_L(\theta, \varphi) = O_{2n \times n}$, for all $(\theta, \varphi) \in \mathbb{T}^n \times \mathbb{T}^\ell$. Moreover, the quasi-periodicity implies that the torus is fiberwise Lagrangian, i.e., the pull-back for all φ , $K_\varphi^* \omega = 0$ of the symplectic form on the torus vanishes. The matrix representation is

$$\Omega_L(\theta, \varphi) = (D_\theta K(\theta, \varphi))^\top \Omega(K(\theta, \varphi)) D_\theta K(\theta, \varphi) = (L(\theta, \varphi))^\top \Omega(K(\theta, \varphi)) L(\theta, \varphi). \quad (1.3.2)$$

Thus, $\mathcal{K}$ is isotropic if $\Omega_L(\theta, \varphi) = O_n$. For each $\varphi \in \mathbb{T}^\ell$, $\mathcal{K}_\varphi$ is a Lagrangian submanifold of $\mathcal{M}$, which is the case we study here. For this purpose, we introduce several geometric constructions that allow us to define adapted symplectic frames. Hence, we will obtain a map $N : \mathbb{T}^n \times \mathbb{T}^\ell \rightarrow \mathbb{R}^{2n \times n}$ such that the juxtaposed matrix

$$P(\theta, \varphi) = \begin{pmatrix} L(\theta, \varphi) & N(\theta, \varphi) \end{pmatrix},$$

satisfies $\text{rank } P(\theta, \varphi) = 2n$ for every $(\theta, \varphi) \in \mathbb{T}^n \times \mathbb{T}^\ell$ and

$$P(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) P(\theta, \varphi) = \begin{pmatrix} O_n & -I_n \\ I_n & O_n \end{pmatrix} := \Omega_0, \quad (1.3.3)$$

Ω_0 denote the matrix representation of the Darboux form. From equation (1.3.3), we derive the following geometric properties:

$$L(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) L(\theta, \varphi) = O_n, \quad (1.3.4)$$

$$L(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) N(\theta, \varphi) = -I_n, \quad (1.3.5)$$

$$N(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) L(\theta, \varphi) = I_n, \quad (1.3.6)$$

$$N(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) N(\theta, \varphi) = O_n. \quad (1.3.7)$$

In this case, we say that $P : \mathbb{T}^n \times \mathbb{T}^\ell \rightarrow \mathbb{R}^{2n \times 2n}$ is a symplectic frame. The use of these linear coordinates allows the linearized equation to be reduce to a triangular form:

$$\begin{aligned} \mathcal{X}_P(\theta, \varphi) &= D_z Z_h(K(\theta, \varphi), \varphi) P(\theta, \varphi) + \mathfrak{L}_{\omega, \alpha} P(\theta, \varphi) \\ &= \begin{pmatrix} D_z Z_h(K(\theta, \varphi), \varphi) L(\theta, \varphi) + \mathfrak{L}_{\omega, \alpha} L(\theta, \varphi) & D_z Z_h(K(\theta, \varphi), \varphi) N(\theta, \varphi) + \mathfrak{L}_{\omega, \alpha} N(\theta, \varphi) \end{pmatrix} \\ &= \begin{pmatrix} O_{2n \times n} & D_z Z_h(K(\theta, \varphi), \varphi) N(\theta, \varphi) + \mathfrak{L}_{\omega, \alpha} N(\theta, \varphi) \end{pmatrix} \\ &= P(\theta, \varphi) \Lambda(\theta, \varphi), \end{aligned}$$

with

$$\Lambda(\theta, \varphi) = \begin{pmatrix} O_n & T(\theta, \varphi) \\ O_n & O_n \end{pmatrix},$$

and $T : \mathbb{T}^n \times \mathbb{T}^\ell \rightarrow \mathbb{R}^{n \times n}$

$$\begin{aligned} T(\theta, \varphi) &= N(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) (D_z Z_h(K(\theta, \varphi), \varphi) N(\theta, \varphi) + \mathfrak{L}_{\omega, \alpha} N(\theta, \varphi)) \\ &= N(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) \mathcal{X}_N(\theta, \varphi). \end{aligned} \quad (1.3.8)$$

The matrix $T(\theta, \varphi)$ is usually called the *torsion matrix* and will play a role in the non-degeneracy condition.

Remark 1.3.1. In this remark, we consider the derivative of the expression $\Omega(z)Z_h(z, \varphi) = (D_z h(z, \varphi))^\top$,

$$\sum_{m=1}^{2n} \left(\frac{\partial \Omega_{j,m}(z)}{\partial z_i} Z_m(z, \varphi) + \Omega_{j,m}(z) \frac{\partial Z_m(z, \varphi)}{\partial z_i} \right) = \frac{\partial^2 h(z, \varphi)}{\partial z_i \partial z_j}.$$

Since the Hamiltonian is analytic, we have

$$\begin{aligned} 0 &= \frac{\partial^2 h(z, \varphi)}{\partial z_j \partial z_i} - \frac{\partial^2 h(z, \varphi)}{\partial z_i \partial z_j} \\ &= \sum_{m=1}^{2n} \left(\frac{\partial \Omega_{i,m}(z, \varphi)}{\partial z_j} Z_m(z, \varphi) + \Omega_{i,m}(z, \varphi) \frac{\partial Z_m(z, \varphi)}{\partial z_j} - \frac{\partial \Omega_{j,m}(z, \varphi)}{\partial z_i} Z_m(z, \varphi) - \Omega_{j,m}(z, \varphi) \frac{\partial Z_m(z, \varphi)}{\partial z_i} \right) \\ &= \sum_{m=1}^{2n} \left(\left(\frac{\partial \Omega_{i,m}(z)}{\partial z_j} - \frac{\partial \Omega_{j,m}(z)}{\partial z_i} \right) Z_m(z, \varphi) + \Omega_{i,m}(z) \frac{\partial Z_m(z, \varphi)}{\partial z_j} - \Omega_{j,m}(z) \frac{\partial Z_m(z, \varphi)}{\partial z_i} \right) \\ &= \sum_{m=1}^{2n} \left(\frac{\partial \Omega_{i,j}(z)}{\partial z_m} Z_m(z, \varphi) + \Omega_{i,m}(z) \frac{\partial Z_m(z, \varphi)}{\partial z_j} + \Omega_{m,j}(z) \frac{\partial Z_m(z, \varphi)}{\partial z_i} \right). \end{aligned}$$

In the last equality, we use the skew-symmetry of the symplectic form and the following geometric property,

$$\frac{\partial \Omega_{r,s}(z)}{\partial z_t} + \frac{\partial \Omega_{s,t}(z)}{\partial z_r} + \frac{\partial \Omega_{t,r}(z)}{\partial z_s} = 0,$$

for any triplet (r, s, t) . Then

$$D_z \Omega(z) [Z_h(z, \varphi)] + (D_z Z_h(z, \varphi))^\top \Omega(z) + \Omega(z) D_z Z_h(z, \varphi) = O_{2n}. \quad (1.3.9)$$

1.4 Construction of a geometrically adapted frame

In this section, we construct a symplectic frame for the Lagrangian torus on the bundle $\mathcal{T}_{\mathcal{K}}\mathcal{M}$ by complementing the column vectors of a map $L : \mathbb{T}^n \times \mathbb{T}^\ell \rightarrow \mathbb{R}^{2n \times n}$, that parameterizes a Lagrangian subbundle. To this end, we assume that we have a map $N^0 : \mathbb{T}^n \times \mathbb{T}^\ell \rightarrow \mathbb{R}^{2n \times n}$ such that.

$$\text{rank} \begin{pmatrix} L(\theta, \varphi) & N^0(\theta, \varphi) \end{pmatrix} = 2n \quad \Leftrightarrow \quad \det(L(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) N^0(\theta, \varphi)) \neq 0, \quad (1.4.1)$$

for every $(\theta, \varphi) \in \mathbb{T}^n \times \mathbb{T}^\ell$. Then, we complement the Lagrangian subspace generated by $L(\theta, \varphi)$ by means of a map $N : \mathbb{T}^n \times \mathbb{T}^\ell \rightarrow \mathbb{R}^{2n \times n}$ in the following manner,

$$N(\theta, \varphi) = L(\theta, \varphi)A(\theta, \varphi) + \tilde{N}(\theta, \varphi).$$

with,

$$\tilde{N}(\theta, \varphi) = N^0(\theta, \varphi)B(\theta, \varphi).$$

Using (1.3.3), we obtain,

$$B(\theta, \varphi) = -(L(\theta, \varphi)^\top \Omega(K(\theta, \varphi))N^0(\theta, \varphi))^{-1},$$

and

$$A(\theta, \varphi) = -\frac{1}{2}(\tilde{N}(\theta, \varphi)^\top \Omega(K(\theta, \varphi))\tilde{N}(\theta, \varphi)),$$

where A is a solution of

$$A(\theta, \varphi) - A(\theta, \varphi)^\top = -\tilde{N}(\theta, \varphi)^\top \Omega(K(\theta, \varphi))\tilde{N}(\theta, \varphi).$$

Following [HCF⁺16], assume that $\mathcal{M}$ is endowed with a Riemannian metric $\mathbf{g}$, represented in coordinates as the positive definite symmetric matrix-valued function $G : \mathcal{M} \rightarrow \mathbb{R}^{2n \times 2n}$. Then, we define the linear isomorphism $\mathbf{J} : T\mathcal{M} \rightarrow T\mathcal{M}$ such that $\omega_z(\mathbf{J}_z u, v) = \mathbf{g}_z(u, v)$, $\forall u, v \in T_z \mathcal{M}$. Observe also that $\mathbf{J}$ is antisymmetric with respect to $\mathbf{g}$, that is, $\mathbf{g}_z(u, \mathbf{J}_z v) = -\mathbf{g}_z(\mathbf{J}_z u, v)$. The matrix-valued function gives the matrix representation of $\mathbf{J}$, $J : \mathcal{M} \rightarrow \mathbb{R}^{2n \times 2n}$. Then, we have

$$\Omega^\top = -\Omega, \quad G^\top = G, \quad J = -\Omega^{-1}G, \quad (1.4.2)$$

to represent the symplectic form using the frame given by J . Then, we choose the map $N^0(\theta, \varphi)$ as follows

$$N^0(\theta, \varphi) := J(K(\theta, \varphi))L(\theta, \varphi) \quad (1.4.3)$$

and condition (1.4.1) is equivalent to

$$\det(L(\theta, \varphi)^\top G(K(\theta, \varphi))L(\theta, \varphi)) \neq 0 \quad \text{for all } (\theta, \varphi) \in \mathbb{T}^n \times \mathbb{T}^\ell.$$

Moreover, the matrix $B(\theta, \varphi)$ is expressed as follows

$$B(\theta, \varphi) = (L(\theta, \varphi)^\top G(K(\theta, \varphi))L(\theta, \varphi))^{-1}.$$

With this, we obtain the frame P . Next, we derive the following expression for the torsion, with details provided in the Appendix B.

$$\begin{aligned} T(\theta, \varphi) = & -\frac{1}{2}\tilde{N}(\theta, \varphi)^\top \left(T_h(K(\theta, \varphi), \varphi) + T_h(K(\theta, \varphi), \varphi)^\top \right) \tilde{N}(\theta, \varphi) \\ & + \tilde{N}(\theta, \varphi)^\top T_h(K(\theta, \varphi), \varphi)^\top N(\theta, \varphi) \\ & + N(\theta, \varphi)^\top T_h(K(\theta, \varphi), \varphi) \tilde{N}(\theta, \varphi), \end{aligned} \quad (1.4.4)$$

one advantage of (1.4.4), is that it does not depend on the Lie derivative term of $N(\theta, \varphi)$. Moreover, the object

$$T_h(z, \varphi) = \Omega(z) \left(D_z Z_h(z, \varphi) - D_z J(z) [Z_h(z, \varphi)] J(z)^{-1} - J(z) D_z Z_h(z, \varphi) J(z)^{-1} \right), \quad (1.4.5)$$

naturally emerges, linking the Hamiltonian vector field and the geometric structures Ω, G and J .

Remark 1.4.1. Another expression to new torsion. equation, (1.4.4)

$$\begin{aligned} T(\theta, \varphi) = & N(\theta, \varphi)^\top \left(\frac{1}{2} \left(T_h(K(\theta, \varphi), \varphi) + T_h(K(\theta, \varphi), \varphi)^\top \right) \right) N(\theta, \varphi) \\ & + A(\theta, \varphi)^\top L(\theta, \varphi)^\top T_h(K(\theta, \varphi), \varphi)^\top N(\theta, \varphi) \\ & + N(\theta, \varphi)^\top T_h(K(\theta, \varphi), \varphi)^\top L(\theta, \varphi) A(\theta, \varphi) \\ & - A(\theta, \varphi)^\top L(\theta, \varphi)^\top \left(\frac{1}{2} \left(T_h(K(\theta, \varphi), \varphi) + T_h(K(\theta, \varphi), \varphi)^\top \right) \right) L(\theta, \varphi) A(\theta, \varphi). \end{aligned} \quad (1.4.6)$$

This new torsion expression has several advantages that we discuss in further detail in the thesis. In Appendix B, you can find further derivation details. The objects computed so far, except for the new expression for the torsion, correspond to **Case II** in [HCF⁺16]. There is also the special case where the isomorphism J is anti-involutive. That is, $J^2 = -I$. Then, we say that the triplet (ω, g, J) is compatible and that J endows $\mathcal{M}$ with an almost complex structure. This is called **Case III** in chapter 4 of [HCF⁺16]. In coordinates, we have the following properties

$$J^2 = -I_{2n}, \quad \Omega = J^\top \Omega J, \quad G = J^\top G J.$$

In this particular case, we have

$$N^0(\theta, \varphi) = J(K(\theta, \varphi))L(\theta, \varphi), \quad A(\theta, \varphi) = O_n,$$

$$B(\theta, \varphi) = (L(\theta, \varphi)^\top G(K(\theta, \varphi))L(\theta, \varphi))^{-1},$$

and, the new torsion is,

$$T(\theta, \varphi) = N(\theta, \varphi)^\top T_h(K(\theta, \varphi), \varphi) N(\theta, \varphi). \quad (1.4.7)$$

and

$$T_h(z, \varphi) = \Omega(z) \left(D_z Z_h(z, \varphi) + D_z J(z) [Z_h(z, \varphi)] J(z) + J(z) D_z Z_h(z, \varphi) J(z) \right). \quad (1.4.8)$$

We observe that both cases $T(\theta, \varphi)$ and $T_h(\theta, \varphi)$ are symmetric.

Remark 1.4.2. To summarize, we have that, given a parameterization of a torus $\mathcal{K} = K(\mathbb{T}^n \times \mathbb{T}^\ell)$ (not necessarily invariant) and a tangent frame $L : \mathbb{T}^n \times \mathbb{T}^\ell \rightarrow \mathbb{R}^{2n \times n}$, the normal frame $N : \mathbb{T}^n \times \mathbb{T}^\ell \rightarrow \mathbb{R}^{2n \times n}$ is constructed as follows:

$$N(\theta, \varphi) := L(\theta, \varphi)A(\theta, \varphi) + \tilde{N}(\theta, \varphi), \quad (1.4.9)$$

where,

$$\tilde{N}(\theta, \varphi) = N^0(\theta, \varphi)B(\theta, \varphi), \quad (1.4.10)$$

$$N^0(\theta, \varphi) = J(K(\theta, \varphi))L(\theta, \varphi), \quad (1.4.11)$$

$$B(\theta, \varphi) = (L(\theta, \varphi)^\top G(K(\theta, \varphi))L(\theta, \varphi))^{-1}, \quad (1.4.12)$$

$$A(\theta, \varphi) = \begin{cases} -\frac{1}{2}(\tilde{N}(\theta, \varphi)^\top \Omega(K(\theta, \varphi))\tilde{N}(\theta, \varphi)), & \text{if \textbf{Case II};} \\ O_n. & \text{if \textbf{Case III}.} \end{cases} \quad (1.4.13)$$

The torsion matrix $T : \mathbb{T}^n \times \mathbb{T}^\ell \rightarrow \mathbb{R}^{n \times n}$, given by

$$T(\theta, \varphi) = \begin{cases} \begin{aligned} &-\frac{1}{2}\tilde{N}(\theta, \varphi)^\top (T_h(K(\theta, \varphi), \varphi) + T_h(K(\theta, \varphi), \varphi)^\top) \tilde{N}(\theta, \varphi) & \text{if \textbf{Case II};} \\ &+\tilde{N}(\theta, \varphi)^\top T_h(K(\theta, \varphi), \varphi)^\top N(\theta, \varphi) \\ &+N(\theta, \varphi)^\top T_h(K(\theta, \varphi), \varphi)\tilde{N}(\theta, \varphi), \end{aligned} \\ N(\theta, \varphi)^\top T_h(K(\theta, \varphi), \varphi)N(\theta, \varphi), & \text{if \textbf{Case III}.} \end{cases} \quad (1.4.14)$$

with, $T_h : \mathcal{M} \times \mathbb{T}^\ell \rightarrow \mathbb{R}^{2n \times 2n}$

$$T_h(z, \varphi) = \begin{cases} \begin{aligned} &\Omega(z) \left(D_z Z_h(z, \varphi) \right. & \text{if \textbf{Case II};} \\ &\quad -D_z J(z) [Z_h(z, \varphi)] J(z)^{-1} \\ &\quad \left. -J(z) D_z Z_h(z, \varphi) J(z)^{-1} \right), \end{aligned} \\ \begin{aligned} &\Omega(z) \left(D_z Z_h(z, \varphi) \right. & \text{if \textbf{Case III}.} \\ &\quad +D_z J(z) [Z_h(z, \varphi)] J(z) \\ &\quad \left. +J(z) D_z Z_h(z, \varphi) J(z) \right). \end{aligned} \end{cases} \quad (1.4.15)$$

We can now stated our main theorem.

KAM Theorem for Lagrangian Tori

In order to prove the KAM theorem for Lagrangian tori, it is necessary to establish a series of lemmas that will be the foundation for the proof. These lemmas will provide the conditions and properties required to demonstrate the persistence of invariant tori in a system. By stating these lemmas, we can lay the groundwork of the proof of the KAM theorem, which states that an invariant torus exists for certain dynamical and geometrical conditions.

2.1 KAM theorem for Lagrangian torus

Theorem 2.1.1 (KAM theorem for quasi-periodic Hamiltonian systems). *Let us consider an exact symplectic structure $\omega = d\alpha$ and a Riemannian metric g on the open set $\mathcal{M} \subset \mathbb{R}^{2n} \times \mathbb{T}^\ell$. Let $h \in \mathcal{A}(\mathcal{M}_\rho \times \mathbb{T}_\rho^\ell)$ be a Hamiltonian function that depends quasi-periodically on time, with $1 \leq \ell \leq n$ and $K : \mathbb{T}^n \times \mathbb{T}^\ell \rightarrow \mathcal{M}$ be a parameterization of an Lagrangian approximately invariant torus with frequency vector $(\omega, \alpha) \in \mathbb{R}^n \times \mathbb{R}^\ell$, furthermore consider the tangent frame $L : \mathbb{T}^n \times \mathbb{T}^\ell \rightarrow \mathbb{R}^{2n \times n}$ given by*

$$L(\theta, \varphi) = D_\theta K(\theta, \varphi). \quad (2.1.1)$$

Then, we assume that the following hypotheses hold.

H_1 *The global objects can be analytically extended to the complex domain $\mathcal{B} \subset \mathbb{C}^{2n}$, and there are constants that quantify the control of their analytic norms. For the geometric structures ω, g, J*

in $\mathcal{B}$, the matrix functions $\Omega, G, J : \mathcal{B} \rightarrow \mathbb{C}^{2n \times 2n}$ satisfy:

$$\begin{aligned} \|\Omega\|_{\mathcal{B} \times \mathbb{T}^\ell} &\leq c_{\Omega,0}, & \|D_z \Omega\|_{\mathcal{B} \times \mathbb{T}^\ell} &\leq c_{\Omega,1}, \\ \|G\|_{\mathcal{B} \times \mathbb{T}^\ell} &\leq c_{G,0}, & \|D_z G\|_{\mathcal{B} \times \mathbb{T}^\ell} &\leq c_{G,1}, & \|D_z^2 G\|_{\mathcal{B} \times \mathbb{T}^\ell} &\leq c_{G,2}, \\ \|J\|_{\mathcal{B} \times \mathbb{T}^\ell} &\leq c_{J,0}, & \|D_z J\|_{\mathcal{B} \times \mathbb{T}^\ell} &\leq c_{J,1}, & \|D_z^2 J\|_{\mathcal{B} \times \mathbb{T}^\ell} &\leq c_{J,2}, \\ \|J^\top\|_{\mathcal{B} \times \mathbb{T}^\ell} &\leq c_{J^\top,0}, & \|D_z J^\top\|_{\mathcal{B} \times \mathbb{T}^\ell} &\leq c_{J^\top,1}, \\ \|J^{-1}\|_{\mathcal{B} \times \mathbb{T}^\ell} &\leq c_{J^{-1},0}, \\ \|J^{-\top}\|_{\mathcal{B} \times \mathbb{T}^\ell} &\leq c_{J^{-\top},0}. \end{aligned}$$

For the Hamiltonian $h : \mathcal{B} \times \mathbb{T}^\ell \rightarrow \mathbb{C}$ and its corresponding vector field $Z_h : \mathcal{B} \times \mathbb{T}^\ell \rightarrow \mathbb{C}^{2n}$ and $T_h : \mathcal{M} \times \mathbb{T}^\ell \rightarrow \mathbb{C}^{2n}$ given by (1.4.5), we have:

$$\begin{aligned} \|D_z h\|_{\mathcal{B} \times \mathbb{T}^\ell} &\leq c_{h,1}, \\ \|Z_h\|_{\mathcal{B} \times \mathbb{T}^\ell} &\leq c_{Z_h,0}, & \|D_z Z_h\|_{\mathcal{B} \times \mathbb{T}^\ell} &\leq c_{Z_h,1}, \\ \|T_h\|_{\mathcal{B} \times \mathbb{T}^\ell} &\leq c_{T_h,0}, & \|D_z T_h\|_{\mathcal{B} \times \mathbb{T}^\ell} &\leq c_{T_h,1}, \\ \|T_h^\top\|_{\mathcal{B} \times \mathbb{T}^\ell} &\leq c_{T_h^\top,0}, & \|D_z T_h^\top\|_{\mathcal{B} \times \mathbb{T}^\ell} &\leq c_{T_h^\top,1}, \\ \|D_z^2 Z_h\|_{\mathcal{B} \times \mathbb{T}^\ell} &\leq c_{Z_h,2}, & \|D_z Z_h^\top\|_{\mathcal{B} \times \mathbb{T}^\ell} &\leq c_{Z_h^\top,1}. \end{aligned}$$

H_2 The parameterization K is analytic in a complex strip $\mathbb{T}_\rho^n \times \mathbb{T}_\rho^\ell$, with $\rho > 0$, which is contained in the global domain:

$$\text{dist}(K(\mathbb{T}_\rho^n \times \mathbb{T}_\rho^\ell), \partial(\mathcal{B} \times \mathbb{T}_\rho^\ell)) > 0.$$

Moreover, the components of K and $D_\theta K$ belong to $\mathcal{A}(\mathbb{T}_\rho^n \times \mathbb{T}_\rho^\ell)$, and there are constants σ_L and $\sigma_{L^\top}$ such that

$$\|D_\theta K\|_\rho < \sigma_L, \quad \|(D_\theta K)^\top\|_\rho < \sigma_{L^\top}.$$

H_3 We assume that $L(\theta, \varphi)$ given by (2.1.1) has maximum rank for every $(\theta, \varphi) \in \bar{\mathbb{T}}_\rho^n \times \bar{\mathbb{T}}_\rho^\ell$, given. Moreover, there exists a constant σ_B such that

$$\|B\|_\rho < \sigma_B,$$

where $B(\theta, \varphi)$ is given by (1.4.12).

H_4 There exists a constant σ_T such that

$$|\langle T \rangle^{-1}| < \sigma_T,$$

where $T(\theta, \varphi)$ is given by (1.4.14).

H_5 The frequency (ω, α) belongs to $\mathcal{D}_{\gamma, \tau}$, given by (1.1.1), for certain $\gamma > 0$ and $\tau \geq n + \ell - 1$.

Under the above hypotheses, for each $0 < \rho_\infty < \rho$ there exists a constant $\mathfrak{C}_1$ such that, if the error of invariance

$$E(\theta, \varphi) = Z_h(K(\theta, \varphi), \varphi) + \mathfrak{L}_{\omega, \alpha} K(\theta, \varphi), \quad (2.1.2)$$

satisfies

$$\frac{\mathfrak{C}_1 \|E\|_\rho}{\gamma^4 \rho^{4\tau}} < 1, \quad (2.1.3)$$

then there exists an invariant torus $\mathcal{K}_\infty = K_\infty(\mathbb{T}^n \times \mathbb{T}^\ell)$ with frequencies (ω, α) , satisfying $K_\infty \in \mathcal{A}(\mathbb{T}_{\rho_\infty}^n \times \mathbb{T}_{\rho_\infty}^\ell)$ and

$$\|D_\theta K_\infty\|_{\rho_\infty} < \sigma_L, \quad \|(D_\theta K_\infty)^\top\|_{\rho_\infty} < \sigma_{L^\top}, \quad \text{dist}(K_\infty(\mathbb{T}_{\rho_\infty}^n \times \mathbb{T}_{\rho_\infty}^\ell), \partial(\mathcal{B} \times \mathbb{T}_\rho^\ell)) > 0.$$

Furthermore, the objects are close to the original ones: there exist a constant $\mathfrak{C}_2$ such that

$$\|K_\infty - K\|_{\rho_\infty} < \frac{\mathfrak{C}_2 \|E\|_\rho}{\gamma^2 \rho^{2\tau}}, \quad (2.1.4)$$

The constants $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are given explicitly in Appendix C.

Remark 2.1.2. In the canonical case, we have $\Omega = \tilde{\Omega} = \Omega_0$, $G = I_{2n}$, and $J = \Omega_0$. Hence, we have $c_{\Omega,0} = 1$, $c_{\Omega,1} = 0$, $c_{\tilde{\Omega},0} = 1$, $c_{\tilde{\Omega},1} = 0$, $c_{\tilde{\Omega},2} = 0$, $c_{G,0} = 1$, $c_{G,1} = 0$, $c_{G,2} = 0$, $c_{J,0} = 1$, $c_{J,1} = 0$, $c_{J,2} = 0$, $c_{J^\top,0} = 1$, and $c_{J^\top,1} = 0$.

2.2 Some lemmas to control approximate geometric properties

In this section, we include the lemmas that are need to prove the main theorem.

The proof of the following lemma can be found in [Rüs75, Rüs76], as it is a recurring result in the context of KAM theorems. For refined estimates on the constant c_R in terms of δ , see [FHL18].

Lema 2.2.1 (Rüssmann estimates). *Let $(\omega, \alpha) \in \mathcal{D}_{\gamma, \tau}$ for some $\gamma > 0$ and $\tau \geq n + \ell - 1$. Then, for any $v \in \mathcal{A}(\mathbb{T}_\rho^n \times \mathbb{T}_\rho^\ell)$, with $\rho > 0$, there exists a unique zero-average solution of $\mathfrak{L}_{\omega, \alpha} u = v - \langle v \rangle$, denoted by $u = \mathfrak{R}_{\omega, \alpha} v$. Moreover, for any $0 < \delta < \rho$ we have that $u \in \mathcal{A}(\mathbb{T}_{\rho-\delta}^n \times \mathbb{T}_{\rho-\delta}^\ell)$ and the estimate*

$$\|u\|_{\rho-\delta} \leq \frac{c_R}{\gamma \delta^\tau} \|v\|_\rho,$$

where c_R is a constant that depends on $n + \ell$, τ and possibly on δ .

2.2.1 Approximate lagrangianity

Lema 2.2.2 (Approximate lagrangianity). *With the hypotheses of Theorem 2.1.1. Let us consider $\Omega_L : \mathcal{M} \times \mathbb{T}^\ell \rightarrow \mathbb{R}^{2n \times 2n}$, the matrix representation of the pull-back on $K_\varphi^* \omega$. We have*

$$\langle \Omega_L \rangle = O_n, \quad (2.2.1)$$

and the following estimate holds:

$$\|\Omega_L\|_{\rho-2\delta} \leq \frac{C_{\Omega_L}}{\gamma\delta^{\tau+1}} \|E\|_{\rho}, \quad (2.2.2)$$

where the constant C_{Ω_L} is provided in Table 4.1.

Proof. To prove that $\langle \Omega_L \rangle = O_n$, we recall that $\Omega_L(z, \varphi)$ is the matrix representation of $K_{\varphi}^* \omega = K^* \omega = d(K^* \alpha)$, so

$$\begin{aligned} \Omega_L(K(\theta, \varphi), \varphi) &= (D_{\theta}K(\theta, \varphi))^{\top} \Omega(K(\theta, \varphi)) D_{\theta}K(\theta, \varphi) \\ &= (D_{\theta}K(\theta, \varphi))^{\top} \left((D_z a(K(\theta, \varphi)))^{\top} - D_z a(K(\theta, \varphi)) \right) D_{\theta}K(\theta, \varphi) \\ &= (D_{\theta}(a(K(\theta, \varphi))))^{\top} D_{\theta}K(\theta, \varphi) - (D_{\theta}K(\theta, \varphi))^{\top} D_{\theta}(a(K(\theta, \varphi))), \end{aligned}$$

and so, the components of $\Omega_L(\theta, \varphi)$ are

$$\begin{aligned} (\Omega_L)_{i,j}(K(\theta, \varphi), \varphi) &= \sum_{m=1}^{2n} \left(\frac{\partial(a_m(K(\theta, \varphi)))}{\partial \theta_i} \frac{\partial K_m(\theta, \varphi)}{\partial \theta_j} - \frac{\partial(a_m(K(\theta, \varphi)))}{\partial \theta_j} \frac{\partial K_m(\theta, \varphi)}{\partial \theta_i} \right) \\ &= \sum_{m=1}^{2n} \left(\frac{\partial}{\partial \theta_i} \left(a_m(K(\theta, \varphi)) \frac{\partial K_m(\theta, \varphi)}{\partial \theta_j} \right) - \frac{\partial}{\partial \theta_j} \left(a_m(K(\theta, \varphi)) \frac{\partial K_m(\theta, \varphi)}{\partial \theta_i} \right) \right). \end{aligned}$$

We observe that the components of $\Omega_L(K(\theta, \varphi), \varphi)$ consist of sums of derivatives of periodic functions with respect to the variable θ , therefore $\langle \Omega_L \rangle = O_n$. We can now proceed to prove inequality (2.2.2). To do this, we consider the approximately invariant case, using the invariant error given by equation (2.1.2).

$$\begin{aligned} \mathfrak{L}_{\omega, \alpha}(\Omega(K(\theta, \varphi))) &= -D_{\theta}(\Omega(K(\theta, \varphi))) \omega - D_{\varphi}(\Omega(K(\theta, \varphi))) \alpha \\ &= -D_z \Omega(K(\theta, \varphi)) D_{\theta}K(\theta, \varphi) \omega - D_z \Omega(K(\theta, \varphi)) D_{\varphi}K(\theta, \varphi) \alpha \\ &= D_z \Omega(K(\theta, \varphi)) [\mathfrak{L}_{\omega, \alpha}(K(\theta, \varphi))] \\ &= D_z \Omega(K(\theta, \varphi)) [E(\theta, \varphi) - Z_h(K(\theta, \varphi), \varphi)], \end{aligned}$$

and

$$\mathfrak{L}_{\omega, \alpha}(D_{\theta}K(\theta, \varphi)) = D_{\theta}E(\theta, \varphi) - D_z Z_h(K(\theta, \varphi), \varphi) D_{\theta}K(\theta, \varphi),$$

Replacing the last expressions $\mathfrak{L}_{\omega, \alpha}(\Omega_L(\theta, \varphi))$, we have,

$$\begin{aligned} \mathfrak{L}_{\omega, \alpha}(\Omega_L(\theta, \varphi)) &= \mathfrak{L}_{\omega, \alpha} \left((D_{\theta}K(\theta, \varphi))^{\top} \Omega(K(\theta, \varphi)) D_{\theta}K(\theta, \varphi) \right) \\ &= \mathfrak{L}_{\omega, \alpha} \left((D_{\theta}K(\theta, \varphi))^{\top} \right) \Omega(K(\theta, \varphi)) D_{\theta}K(\theta, \varphi) \\ &\quad + (D_{\theta}K(\theta, \varphi))^{\top} \mathfrak{L}_{\omega, \alpha}(\Omega(K(\theta, \varphi))) D_{\theta}K(\theta, \varphi) \\ &\quad + (D_{\theta}K(\theta, \varphi))^{\top} \Omega(K(\theta, \varphi)) \mathfrak{L}_{\omega, \alpha}(D_{\theta}K(\theta, \varphi)) \\ &= D_{\theta}(E(\theta, \varphi))^{\top} \Omega(K(\theta, \varphi)) D_{\theta}K(\theta, \varphi) \\ &\quad + (D_{\theta}K(\theta, \varphi))^{\top} D_z \Omega(K(\theta, \varphi)) [E(\theta, \varphi)] D_{\theta}K(\theta, \varphi) \\ &\quad + (D_{\theta}K(\theta, \varphi))^{\top} \Omega(K(\theta, \varphi)) D_{\theta}E(\theta, \varphi). \end{aligned}$$

The expression $\mathfrak{L}_{\omega,\alpha}\Omega_L(\theta, \varphi)$, is controlled using H_1 , H_2 , the Banach algebra properties and Cauchy estimates as follows

$$\begin{aligned}
\|\mathfrak{L}_{\omega,\alpha}\Omega_L\|_{\rho-\delta} &\leq \|D_\theta E^\top (\Omega \circ K) D_\theta K\|_{\rho-\delta} + \|D_\theta K^\top D_z (\Omega \circ K) [E] D_\theta K\|_{\rho-\delta} \\
&\quad + \|D_\theta K^\top (\Omega \circ K) D_\theta E\|_{\rho-\delta} \\
&\leq \|(D_\theta E)^\top\|_{\rho-\delta} \|\Omega\|_{\mathcal{B} \times \mathbb{T}^\ell} \|D_\theta K\|_\rho + \|(D_\theta K)^\top\|_\rho \|D_z \Omega\|_{\mathcal{B} \times \mathbb{T}^\ell} \|E\|_\rho \|D_\theta K\|_\rho \\
&\quad + \|(D_\theta K)^\top\|_\rho \|\Omega\|_{\mathcal{B} \times \mathbb{T}^\ell} \|D_\theta E\|_{\rho-\delta} \\
&\leq \frac{2n}{\delta} \|E\|_\rho \|\Omega\|_{\mathcal{B} \times \mathbb{T}^\ell} \|D_\theta K\|_\rho + \|(D_\theta K)^\top\|_\rho \|D_z \Omega\|_{\mathcal{B} \times \mathbb{T}^\ell} \|E\|_\rho \|D_\theta K\|_\rho \\
&\quad + \frac{n}{\delta} \|(D_\theta K)^\top\|_\rho \|\Omega\|_{\mathcal{B} \times \mathbb{T}^\ell} \|E\|_\rho \\
&\leq \frac{2nc_{\Omega,0}\sigma_L + \sigma_{L^\top c_{\Omega,1}}\sigma_L\delta + n\sigma_{L^\top c_{\Omega,0}}}{\delta} \|E\|_\rho =: \frac{C_{\mathfrak{L}\Omega_L}}{\delta} \|E\|_\rho.
\end{aligned} \tag{2.2.3}$$

Using Rußmann estimates, we obtain

$$\|\Omega_L\|_{\rho-2\delta} \leq \frac{c_R}{\gamma\delta^\tau} \|\mathfrak{L}_{\omega,\alpha}\Omega_L\|_{\rho-\delta} \leq \frac{c_R C_{\mathfrak{L}\Omega_L}}{\gamma\delta^{\tau+1}} \|E\|_\rho =: \frac{C_{\Omega_L}}{\gamma\delta^{\tau+1}} \|E\|_\rho. \tag{2.2.4}$$

□

Remark 2.2.3. If the symplectic form depends explicitly on the φ coordinates, then the Lie's derivative of the reduced form is

$$\mathfrak{L}_{\omega,\alpha}(\Omega(K(\theta, \varphi), \varphi)) = D_z \Omega(K(\theta, \varphi), \varphi) \mathfrak{L}_{\omega,\alpha} K(\theta, \varphi) - D_\varphi \Omega(K(\theta, \varphi), \varphi) \alpha,$$

the additional term is an obstruction in proving in general that the Lie's derivative of the reduced form is zero. Since,

$$\mathfrak{L}_{\omega,\alpha}(\Omega_L(K(\theta, \varphi), \varphi)) = -(D_\theta K(\theta, \varphi))^\top (D_\varphi \Omega(K(\theta, \varphi), \varphi) \alpha) D_\theta K(\theta, \varphi),$$

this dependence of the fibered symplectic structure on φ suggests a more general case study that could be explored in the future.

2.2.2 Approximate symplectic frame

In this section, we demonstrate that if $\mathcal{K}$ is an approximately invariant torus, then we can construct an adapted frame that is approximately symplectic. The first step of this process is finding an approximate Lagrangian bundle that contains the tangent bundle $T_{\mathcal{K}_\varphi} \mathcal{M}$ for $\varphi \in \mathbb{T}^\ell$.

Lema 2.2.4. *With the hypotheses of Theorem 2.1.1, the map $L(\theta, \varphi)$ given by (2.1.1) satisfies*

$$\|L\|_\rho \leq C_L, \quad \|L^\top\|_\rho \leq C_{L^\top}, \quad \langle L^\top (\Omega \circ K) E \rangle = 0_n, \tag{2.2.5}$$

Furthermore, the object

$$G_L(\theta, \varphi) := L(\theta, \varphi)^\top G(K(\theta, \varphi)) L(\theta, \varphi), \quad (2.2.6)$$

is controlled as

$$\|G_L\|_\rho \leq C_{G_L}, \quad (2.2.7)$$

The above constants are given explicitly in Table 4.1.

Proof. Estimating $L(\theta, \varphi)$ and $L(\theta, \varphi)^\top$ is straightforward since the torus is Lagrangian, as indicated in (2.1.1).

$$\|L\|_\rho = \|D_\theta K\|_\rho < \sigma_L =: C_L \quad (2.2.8)$$

$$\|L^\top\|_\rho = \|(D_\theta K)^\top\|_\rho < \sigma_{L^\top} =: C_{L^\top} \quad (2.2.9)$$

To derive the final expression in equation (2.2.5), we perform the following computation,

$$\begin{aligned} L(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) E(\theta, \varphi) &= D_\theta K(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) (Z_h(K(\theta, \varphi), \varphi) + \mathfrak{L}_{\omega, \alpha}(K(\theta, \varphi))) \\ &= D_\theta K(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) Z_h(K(\theta, \varphi), \varphi) \\ &\quad + D_\theta K(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) \mathfrak{L}_{\omega, \alpha}(K(\theta, \varphi)) \\ &= D_\theta K(\theta, \varphi)^\top D_z h(K(\theta, \varphi), \varphi)^\top + D_\theta K(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) \mathfrak{L}_{\omega, \alpha}(K(\theta, \varphi)) \\ &= (D_z h(K(\theta, \varphi), \varphi) D_\theta K(\theta, \varphi))^\top + D_\theta K(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) \mathfrak{L}_{\omega, \alpha}(K(\theta, \varphi)) \\ &= (D_\theta (h(K(\theta, \varphi), \varphi)))^\top + D_\theta K(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) \mathfrak{L}_{\omega, \alpha}(K(\theta, \varphi)) \\ &= (D_\theta (h(K(\theta, \varphi), \varphi)))^\top - \Omega_L(\theta, \varphi) \omega - D_\theta K(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) D_\varphi K(\theta, \varphi) \alpha. \end{aligned}$$

Where we have used (2.1.2) and (1.3.2). We will now compute the average of the components of the three-term in the last expression. We analyze the components of the first term for an arbitrary φ .

$$\begin{aligned} (\langle D_\theta (h(K(\theta, \varphi), \varphi)) \rangle_\theta)_{i,j} &= \int_0^T \int_0^T \frac{\partial}{\partial \theta_i} (h(K(\theta, \varphi), \varphi)) d\theta_i d\varphi_j, \\ &= \int_0^T (h(K(T, \varphi), \varphi) - h(K(0, \varphi), \varphi)) d\varphi_j = 0, \end{aligned}$$

In the last equality, we use the periodicity of K . Therefore the average of the first term is zero. From lemma (2.2.2), we obtain that $\langle \Omega_L \rangle = 0$. We define $\check{\Omega}_{K_\varphi}(\theta, \varphi) := D_\theta K(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) D_\varphi K(\theta, \varphi)$ to prove that the last term has zero average. By proceeding in an analogous way to $\Omega_L(\theta, \varphi)$, we obtain that the components of $\check{\Omega}_K(\theta, \varphi)$ are given by

$$\begin{aligned} (\check{\Omega}_K)_{i,j} &= \sum_{m=1}^{2n} \left(\frac{\partial(a_m(K(\theta, \varphi)))}{\partial \theta_i} \frac{\partial K_m(\theta, \varphi)}{\partial \varphi_j} - \frac{\partial(a_m(K(\theta, \varphi)))}{\partial \varphi_j} \frac{\partial K_m(\theta, \varphi)}{\partial \theta_i} \right) \\ &= \sum_{m=1}^{2n} \left(\frac{\partial}{\partial \theta_i} \left(a_m(K(\theta, \varphi)) \frac{\partial K_m(\theta, \varphi)}{\partial \varphi_j} \right) - \frac{\partial}{\partial \varphi_j} \left(a_m(K(\theta, \varphi)) \frac{\partial K_m(\theta, \varphi)}{\partial \theta_i} \right) \right). \end{aligned}$$

therefore,

$$\begin{aligned}
(\langle \check{\Omega}_K \rangle)_{i,j} &= \int_0^T \int_0^T \sum_{m=1}^{2n} \left(\frac{\partial}{\partial \theta_i} \left(a_m(K(\theta, \varphi)) \frac{\partial K_m(\theta, \varphi)}{\partial \varphi_j} \right) - \frac{\partial}{\partial \varphi_j} \left(a_m(K(\theta, \varphi)) \frac{\partial K_m(\theta, \varphi)}{\partial \theta_i} \right) \right) d\theta_i d\varphi_j \\
&= \sum_{m=1}^{2n} \left(\int_0^T \int_0^T \frac{\partial}{\partial \theta_i} \left(a_m(K(\theta, \varphi)) \frac{\partial K_m(\theta, \varphi)}{\partial \varphi_j} \right) - \frac{\partial}{\partial \varphi_j} \left(a_m(K(\theta, \varphi)) \frac{\partial K_m(\theta, \varphi)}{\partial \theta_i} \right) \right) d\theta_i d\varphi_j \\
&= \sum_{m=1}^{2n} \int_0^T a_m(K(T, \varphi)) \frac{\partial K_m(T, \varphi)}{\partial \varphi_j} - a_m(K(0, \varphi)) \frac{\partial K_m(0, \varphi)}{\partial \varphi_j} d\varphi_j \\
&\quad - \sum_{m=1}^{2n} \int_0^T a_m(K(\theta, T)) \frac{\partial K_m(\theta, T)}{\partial \theta_i} - a_m(K(\theta, 0)) \frac{\partial K_m(\theta, 0)}{\partial \theta_i} d\theta_i \\
&= 0
\end{aligned}$$

The last equality holds because $K(\theta, \varphi)$ is periodic with respect to θ and φ . Using the above, we obtain the following expression

$$\langle D_\theta K(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) D_\varphi K(\theta, \varphi) \rangle = O_n.$$

Finally, by using the definition of G_L and the estimates for L and $L^\top$, we conclude

$$\|G_L\|_\rho \leq \|L^\top\|_\rho \|G\|_{\mathbb{B} \times \mathbb{T}^\ell} \|L\|_\rho \leq C_{L^\top} c_{G,0} C_L =: C_{G_L}. \quad (2.2.10)$$

□

The following lemma demonstrates that the geometric constructions detailed in subsection 1.4 enable us to obtain an approximately symplectic frame attached to an approximately invariant torus.

Lema 2.2.5. *With the hypotheses of Theorem 2.1.1, the map $N : \mathbb{T}^n \times \mathbb{T}^\ell \rightarrow \mathbb{R}^{2n \times n}$ given by (1.4.9) satisfies*

$$\|N\|_\rho \leq C_N, \quad \|N^\top\|_\rho \leq C_{N^\top}, \quad (2.2.11)$$

and the map $P : \mathbb{T}^n \times \mathbb{T}^\ell \rightarrow \mathbb{R}^{2n \times 2n}$ given by

$$P(\theta, \varphi) = \begin{pmatrix} L(\theta, \varphi) & N(\theta, \varphi) \end{pmatrix},$$

induces an approximately symplectic vector bundle isomorphism, i.e., the error map

$$E_{\text{sym}}(\theta, \varphi) := P(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) P(\theta, \varphi) - \Omega_0, \quad \Omega_0 = \begin{pmatrix} O_n & -I_n \\ I_n & O_n \end{pmatrix}, \quad (2.2.12)$$

is small in the following sense:

$$\|E_{\text{sym}}\|_{\rho-2\delta} \leq \frac{C_{\text{sym}}}{\gamma \delta^{\tau+1}} \|E\|_\rho. \quad (2.2.13)$$

The above constants are given explicitly in Table 4.1.

Proof. We shall consider equation (1.4.11) and the hypothesis H_1 to prove this result.

$$\|N^0\|_\rho \leq \|J \circ K\|_\rho \|L\|_\rho \leq \|J\|_{\mathcal{B} \times \mathbb{T}^\ell} \|L\|_\rho \leq c_{J,0} C_L =: C_{N^0}, \quad (2.2.14)$$

$$\|(N^0)^\top\|_\rho \leq \|L^\top\|_\rho \|J^\top\|_{\mathcal{B} \times \mathbb{T}^\ell} \leq C_{L^\top} c_{J^\top,0} =: C_{N^{0,\top}}, \quad (2.2.15)$$

$$\|\tilde{N}\|_\rho \leq \|N^0\|_\rho \|B\|_\rho \leq C_{N^0} \sigma_B =: C_{\tilde{N}}, \quad (2.2.16)$$

and

$$\|\tilde{N}^\top\|_\rho \leq \|B\|_\rho \|(N^0)^\top\|_\rho \leq \sigma_B C_{N^{0,\top}} =: C_{\tilde{N}^\top}. \quad (2.2.17)$$

We use (2.2.16) and (2.2.17), to estimate A .

$$\|A\|_\rho \leq \frac{1}{2} \|\tilde{N}^\top\|_\rho \|\Omega \circ K\|_\rho \|\tilde{N}\|_\rho \leq \frac{1}{2} C_{\tilde{N}^\top} c_{\Omega,0} C_{\tilde{N}} =: C_A. \quad (2.2.18)$$

Now, we compute the norm of $N(\theta, \varphi)$ using H_3 from equations (1.4.12) and (1.4.13). We used the fact that $B(\theta, \varphi)$ is symmetric, and $A(\theta, \varphi)$ is antisymmetric. These yields the following bounds

$$\|N\|_\rho \leq \|L\|_\rho \|A\|_\rho + \|\tilde{N}\|_\rho \leq C_L C_A + C_{\tilde{N}} =: C_N, \quad (2.2.19)$$

$$\|N^\top\|_\rho \leq \|A^\top\|_\rho \|L^\top\|_\rho + \|\tilde{N}^\top\|_\rho \leq C_A C_{L^\top} + C_{\tilde{N}^\top} =: C_{N^\top}. \quad (2.2.20)$$

Observe that,

$$\begin{aligned} L(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) N(\theta, \varphi) &= L(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) L(\theta, \varphi) A(\theta, \varphi) + L(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) N^0(\theta, \varphi) B(\theta, \varphi) \\ &= \Omega_L(K(\theta, \varphi), \varphi) A(\theta, \varphi) - L(\theta, \varphi)^\top G(K(\theta, \varphi)) L(\theta, \varphi) B(\theta, \varphi) \\ &= \Omega_L(K(\theta, \varphi), \varphi) A(\theta, \varphi) - I_n, \end{aligned} \quad (2.2.21)$$

where we use that $J^\top \Omega(K(\theta, \varphi), \varphi) = G(K(\theta, \varphi), \varphi)$ and (1.4.12). We also have from (1.4.13),

$$\begin{aligned} N(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) N(\theta, \varphi) &= B(\theta, \varphi)^\top N^0(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) L(\theta, \varphi) A(\theta, \varphi) \\ &\quad + B(\theta, \varphi)^\top N^0(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) N^0(\theta, \varphi) B(\theta, \varphi) \\ &\quad + A(\theta, \varphi)^\top L(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) L(\theta, \varphi) A(\theta, \varphi) \\ &\quad + A(\theta, \varphi)^\top L(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) N^0(\theta, \varphi) B(\theta, \varphi) \\ &= A(\theta, \varphi)^\top \Omega_L(K(\theta, \varphi), \varphi) A(\theta, \varphi) + A(\theta, \varphi) - A(\theta, \varphi)^\top \\ &\quad + B(\theta, \varphi)^\top L(\theta, \varphi)^\top \tilde{\Omega}(K(\theta, \varphi)) L(\theta, \varphi) B(\theta, \varphi) \\ &= A(\theta, \varphi)^\top \Omega_L(K(\theta, \varphi), \varphi) A(\theta, \varphi). \end{aligned} \quad (2.2.22)$$

We compute the error of the symplectic character of the frame,

$$\begin{aligned}
E_{sym}(\theta, \varphi) &= P(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) P(\theta, \varphi) - \Omega_0 \\
&= \begin{pmatrix} L(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) L(\theta, \varphi) & L(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) N(\theta, \varphi) + I_n \\ N(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) L(\theta, \varphi) - I_n & N(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) N(\theta, \varphi) \end{pmatrix} \\
&= \begin{pmatrix} \Omega_L(K(\theta, \varphi)), \varphi & \Omega_L(K(\theta, \varphi)), \varphi A(\theta, \varphi) \\ A(\theta, \varphi)^\top \Omega_L(K(\theta, \varphi)), \varphi & A(\theta, \varphi)^\top \Omega_L(K(\theta, \varphi)), \varphi A(\theta, \varphi) \end{pmatrix},
\end{aligned} \tag{2.2.23}$$

Therefore using (2.2.2), the control for the symplectic error is

$$\|E_{sym}\|_{\rho-2\delta} \leq \frac{(1 + C_A) \max\{1, C_A\} C_{\Omega_L}}{\gamma \delta^{\tau+1}} \|E\|_\rho =: \frac{C_{sym}}{\gamma \delta^{\tau+1}} \|E\|_\rho, \tag{2.2.24}$$

which yields (2.2.13). $\square$

Remark 2.2.6. The above estimates can be readily adapted to **Case III**, for which $A = O_n$. In this case we have,

$$N(\theta, \varphi) = N^0(\theta, \varphi) B(\theta, \varphi), \quad E_{sym}(\theta, \varphi) = \begin{pmatrix} \Omega_L(\theta, \varphi) & O_n \\ O_n & B(\theta, \varphi)^\top \Omega_L(\theta, \varphi) B(\theta, \varphi) \end{pmatrix}.$$

The corresponding estimates are given explicitly in Table 4.1.

2.2.3 Control of the torsion matrix

In this subsection, our objective is controlling the torsion matrix $T(\theta, \varphi)$ as defined in equation (1.4.4).

Although this expression may initially appear cumbersome, it does not rely on the Lie derivative. Instead, it depends solely on easily controllable geometric and dynamical objects.

Lema 2.2.7. *With the hypotheses of Theorem 2.1.1, the torsion matrix $T(\theta, \varphi)$, given by (1.4.4), has components in $\mathcal{A}(\mathbb{T}_\rho^n \times \mathbb{T}_\rho^\ell)$ and satisfies the estimate*

$$\|T\|_\rho \leq C_T,$$

where the constant C_T is provided in Table 4.1.

Proof. In this proof, we use hypothesis H_1 from Theorem 2.1.1 and estimates of the quantity $\tilde{N}$, given in equation (2.2.16) and (2.2.17).

Using all of the above and the torsion expression provided in equation (1.4.4), we obtain that,

$$\begin{aligned}
\|T\|_\rho &\leq \frac{1}{2} \left(\|\tilde{N}^\top\|_\rho \|T_h\|_\rho \|\tilde{N}\|_\rho + \|\tilde{N}^\top\|_\rho \|T_h^\top\|_\rho \|\tilde{N}\|_\rho \right) \\
&\quad + \|\tilde{N}^\top\|_\rho \|T_h^\top\|_\rho \|N\| + \|N^\top\|_\rho \|T_h\|_\rho \|\tilde{N}\|_\rho \\
&\leq \frac{1}{2} \left(C_{\tilde{N}^\top c_{T_h,0}} C_{\tilde{N}} + C_{\tilde{N}^\top c_{T_h^\top,0}} C_{\tilde{N}} \right) \\
&\quad + C_{\tilde{N}^\top c_{T_h^\top,0}} C_N + C_{N^\top c_{T_h,0}} C_{\tilde{N}} =: C_T,
\end{aligned} \tag{2.2.25}$$

which completes the proof. $\square$

Remark 2.2.8. For Case III, the constants in (2.2.25) are: $C_{\tilde{N}} = C_N$, $C_{\tilde{N}^\top} = C_{N^\top}$, $C_A = 0$ and $C_T = C_{N^\top c_{T_h,0}} C_N$.

Remark 2.2.9. Upon comparing Lemma 2.2.7 of this work with Lemma 4.8 of [HL19], we see the advantages of using the definition (1.4.4) for torsion instead of (1.3.8), which is similar to that in [HL19]. The torsion matrix in, (1.4.4), belong to $\mathcal{A}(\mathbb{T}_\rho^n \times \mathbb{T}_\rho^\ell)$, while the matrix in (1.3.8) belongs to $\mathcal{A}(\mathbb{T}_{\rho-\delta}^n \times \mathbb{T}_{\rho-\delta}^\ell)$. This indicates that first one is more regular than second one, which directly affects the numerical implementation resulting from the proof presented in this thesis.

Expressions (1.3.8) and (1.4.4) coincide if the torus is invariant. However, if the torus is approximately invariant, this does not hold. To show this fact, we can construct an expression for $T(\theta, \varphi)$, in the approximately invariant case.

Lema 2.2.10. *With the hypotheses of Theorem 2.1.1, the torsion matrix $T(\theta, \varphi)$, given by (1.4.4), has components in $\mathcal{A}(\mathbb{T}_\rho^n \times \mathbb{T}_\rho^\ell)$ and satisfies the estimate*

$$\|N^\top \Omega \mathcal{X}_N - T\|_{\rho-2\delta} \leq \frac{C_{E_T}}{\gamma \delta^{\tau+1}} \|E\|_\rho.$$

where the constant C_{E_T} is provided in Table 4.1.

Proof. We consider the invariance equation, (1.2.7), for an approximately invariant torus, i.e.

$$\mathfrak{L}_{\omega,\alpha} K(\theta, \varphi) = E(\theta, \varphi) - Z_h(K(\theta, \varphi), \varphi), \tag{2.2.26}$$

from which we obtain

$$\mathfrak{L}_{\omega,\alpha} L(\theta, \varphi) = D_\theta E(\theta, \varphi) - D_z Z_h(K(\theta, \varphi), \varphi) L(\theta, \varphi). \tag{2.2.27}$$

Using (2.2.27) the Lie derivative of J is,

$$\mathfrak{L}_{\omega,\alpha} J(K(\theta, \varphi)) = D_z J(K(\theta, \varphi)) [E(\theta, \varphi)] - D_z J(K(\theta, \varphi)) [Z_h(K(\theta, \varphi), \varphi)]. \tag{2.2.28}$$

Therefore,

$$\begin{aligned}\mathfrak{L}_{\omega,\alpha}\Omega(K(\theta,\varphi)) &= D_z\Omega(K(\theta,\varphi)) [E(\theta,\varphi)] \\ &\quad + (D_z Z_h(K(\theta,\varphi),\varphi))^\top \Omega(K(\theta,\varphi)) \\ &\quad + \Omega(K(\theta,\varphi)) D_z Z_h(K(\theta,\varphi),\varphi).\end{aligned}\tag{2.2.29}$$

Taking the Lie derivative of G , using (1.4.2), (2.2.28), (2.2.29), and (1.3.9) we have,

$$\begin{aligned}\mathfrak{L}_{\omega,\alpha}G(K(\theta,\varphi)) &= -D_z\Omega(K(\theta,\varphi)) [E(\theta,\varphi) - D_z Z_h(K(\theta,\varphi),\varphi)] J(K(\theta,\varphi)) \\ &\quad - \Omega(K(\theta,\varphi)) D_z J(\theta,\varphi) [E(\theta,\varphi) - D_z Z_h(K(\theta,\varphi),\varphi)] \\ &= D_z\Omega(K(\theta,\varphi)) [D_z Z_h(K(\theta,\varphi),\varphi)] J(K(\theta,\varphi)) \\ &\quad + \Omega(K(\theta,\varphi)) D_z J(\theta,\varphi) [D_z Z_h(K(\theta,\varphi),\varphi)] \\ &\quad - D_z\Omega(K(\theta,\varphi)) [E(\theta,\varphi)] J(K(\theta,\varphi)) \\ &\quad - \Omega(K(\theta,\varphi)) D_z J(\theta,\varphi) [E(\theta,\varphi)] \\ &= - (D_z Z_h(K(\theta,\varphi),\varphi))^\top \Omega(K(\theta,\varphi)) J(K(\theta,\varphi)) \\ &\quad - \Omega(K(\theta,\varphi)) D_z Z_h(K(\theta,\varphi),\varphi) J(K(\theta,\varphi)) \\ &\quad + \Omega(K(\theta,\varphi)) D_z J(K(\theta,\varphi)) [Z_h(K(\theta,\varphi),\varphi)] \\ &\quad - D_z\Omega(K(\theta,\varphi)) [E(\theta,\varphi)] J(K(\theta,\varphi)) \\ &\quad - \Omega(K(\theta,\varphi)) D_z J(K(\theta,\varphi)) [E(\theta,\varphi)].\end{aligned}\tag{2.2.30}$$

To calculate the Lie derivative for B , we use that $I_d = B^{-1}(\theta,\varphi)B(\theta,\varphi)$,

$$\begin{aligned}\mathfrak{L}_{\omega,\alpha}B(\theta,\varphi) &= B(\theta,\varphi)\mathfrak{L}_{\omega,\alpha}\left(L(\theta,\varphi)^\top\right)\Omega(K(\theta,\varphi))N(\theta,\varphi)^0 \\ &\quad - B(\theta,\varphi)L(\theta,\varphi)^\top\mathfrak{L}_{\omega,\alpha}(G(K(\theta,\varphi)))L(\theta,\varphi)B(\theta,\varphi) \\ &\quad + N(\theta,\varphi)^{0,\top}\Omega(K(\theta,\varphi))\mathfrak{L}_{\omega,\alpha}(L(\theta,\varphi))B(\theta,\varphi).\end{aligned}$$

Therefore, the Lie derivative of $B(\theta,\varphi)$ for an approximately invariant torus is,

$$\mathfrak{L}_{\omega,\alpha}B(\theta,\varphi) = \tilde{N}(\theta,\varphi)^\top J(K(\theta,\varphi))^{-\top} T_h(K(\theta,\varphi),\varphi) \tilde{N}(\theta,\varphi) + E_{\mathfrak{L}B}(\theta,\varphi),\tag{2.2.31}$$

with,

$$E_{\mathfrak{L}B}(\theta,\varphi) = \tilde{N}(\theta,\varphi)^\top J(K(\theta,\varphi))^{-\top} T_E(\theta,\varphi) \tilde{N}(\theta,\varphi),\tag{2.2.32}$$

and,

$$\begin{aligned}T_E(\theta,\varphi) &= D_z\Omega(K(\theta,\varphi)) [E(\theta,\varphi)] \\ &\quad + \Omega(K(\theta,\varphi)) D_z J(K(\theta,\varphi)) [E(\theta,\varphi)] J(K(\theta,\varphi))^{-1} \\ &\quad + J(K(\theta,\varphi))^\top \Omega(K(\theta,\varphi)) L(\theta,\varphi) B(\theta,\varphi) (D_\theta E(\theta,\varphi))^\top \Omega(K(\theta,\varphi)) \\ &\quad + J(K(\theta,\varphi))^\top \Omega(K(\theta,\varphi)) D_\theta E(\theta,\varphi) B(\theta,\varphi) L(\theta,\varphi)^\top \Omega(K(\theta,\varphi)),\end{aligned}\tag{2.2.33}$$

where we used equations (2.2.27), (2.2.30), and $I_n = \tilde{N}(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) L(\theta, \varphi)$.

Following a similar approach to that used for $B(\theta, \varphi)$, we determine the Lie derivative for $\tilde{N}(\theta, \varphi)$,

$$\begin{aligned} \mathfrak{L}_{\omega, \alpha} \tilde{N}(\theta, \varphi) = & \left(-D_z J(K(\theta, \varphi)) [Z_h(K(\theta, \varphi), \varphi)] J(K(\theta, \varphi))^{-1} \right. \\ & - J(K(\theta, \varphi)) D_z Z_h(K(\theta, \varphi), \varphi) J(K(\theta, \varphi))^{-1} \\ & + J(K(\theta, \varphi)) L(\theta, \varphi) \tilde{N}(\theta, \varphi)^\top J(K(\theta, \varphi))^{-\top} T_h(\theta, \varphi) \Big) \tilde{N}(\theta, \varphi) \\ & + E_{\mathfrak{L}\tilde{N}}(\theta, \varphi), \end{aligned} \quad (2.2.34)$$

where,

$$\begin{aligned} E_{\mathfrak{L}\tilde{N}}(\theta, \varphi) = & D_z J(K(\theta, \varphi)) [E(\theta, \varphi)] L(\theta, \varphi) B(\theta, \varphi) + J(K(\theta, \varphi)) D_\theta E(\theta, \varphi) B(\theta, \varphi) \\ & + J(K(\theta, \varphi)) L(\theta, \varphi) E_{\mathfrak{L}B}(\theta, \varphi). \end{aligned} \quad (2.2.35)$$

Now, we calculate $\mathfrak{L}_{\omega, \alpha} A(\theta, \varphi)$,

$$\mathfrak{L}_{\omega, \alpha} A(\theta, \varphi) = \mathfrak{L}_{\omega, \alpha} \bar{A}(\theta, \varphi) + E_{\mathfrak{L}A}(\theta, \varphi), \quad (2.2.36)$$

with,

$$\begin{aligned} \mathfrak{L}_{\omega, \alpha} \bar{A}(\theta, \varphi) &= \frac{1}{2} \tilde{N}(\theta, \varphi)^\top \left(T_h(\theta, \varphi)^\top - T_h(\theta, \varphi)^\top J(K(\theta, \varphi))^{-1} \tilde{N}(\theta, \varphi) L(\theta, \varphi)^\top J(K(\theta, \varphi))^\top \Omega(K(\theta, \varphi)) \right. \\ &\quad \left. - T_h(\theta, \varphi) - \Omega(K(\theta, \varphi)) J(K(\theta, \varphi)) L(\theta, \varphi) \tilde{N}(\theta, \varphi)^\top J(K(\theta, \varphi))^{-\top} T_h(\theta, \varphi) \right) \tilde{N}(\theta, \varphi), \end{aligned} \quad (2.2.37)$$

and

$$\begin{aligned} E_{\mathfrak{L}A}(\theta, \varphi) = & -\frac{1}{2} \left(E_{\mathfrak{L}\tilde{N}}(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) \tilde{N}(\theta, \varphi) + \tilde{N}(\theta, \varphi)^\top D_z \Omega(K(\theta, \varphi)) [E(\theta, \varphi)] \tilde{N}(\theta, \varphi) \right. \\ & \left. + \tilde{N}(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) E_{\mathfrak{L}\tilde{N}}(\theta, \varphi) \right). \end{aligned} \quad (2.2.38)$$

Using equations (2.2.27), (2.2.34) and (2.2.36), we obtain that the Lie derivative of $N(\theta, \varphi)$ is,

$$\begin{aligned}
\mathfrak{L}_{\omega,\alpha} N(\theta, \varphi) = & \frac{1}{2} D_z Z_h(K(\theta, \varphi), \varphi) L(\theta, \varphi) \tilde{N}(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) \tilde{N}(\theta, \varphi) \\
& + \frac{1}{2} L(\theta, \varphi) \tilde{N}(\theta, \varphi)^\top \left(T_h(\theta, \varphi)^\top - T_h(\theta, \varphi) \right. \\
& \quad - T_h(\theta, \varphi)^\top J(K(\theta, \varphi))^{-1} \tilde{N}(\theta, \varphi) L(\theta, \varphi)^\top J(K(\theta, \varphi))^\top \Omega(K(\theta, \varphi)) \\
& \quad \left. - \Omega(K(\theta, \varphi)) J(K(\theta, \varphi)) L(\theta, \varphi) \tilde{N}(\theta, \varphi)^\top J(K(\theta, \varphi))^{-\top} T_h(\theta, \varphi) \right) \tilde{N}(\theta, \varphi) \quad (2.2.39) \\
& - \left(D_z J(K(\theta, \varphi)) [Z_h(K(\theta, \varphi), \varphi)] J(K(\theta, \varphi))^{-1} \right. \\
& \quad + J(K(\theta, \varphi)) D_z Z_h(K(\theta, \varphi), \varphi) J(K(\theta, \varphi))^{-1} \\
& \quad \left. - J(K(\theta, \varphi)) L(\theta, \varphi) \tilde{N}(\theta, \varphi)^\top J(K(\theta, \varphi))^{-\top} T_h(\theta, \varphi) \right) \tilde{N}(\theta, \varphi) \\
& + E_{\mathfrak{L}N}(\theta, \varphi),
\end{aligned}$$

with,

$$E_{\mathfrak{L}N}(\theta, \varphi) = E_{\mathfrak{L}\tilde{N}}(\theta, \varphi) + L(\theta, \varphi) E_{\mathfrak{L}A}(\theta, \varphi) + D_\theta E(\theta, \varphi) A(\theta, \varphi). \quad (2.2.40)$$

Using the above construction in equation (1.4.14) and from the definition of torsion, equation (1.4.4), we determine the difference for an approximately invariant torus as follows

$$N(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) \mathfrak{X}_N(\theta, \varphi) = T(\theta, \varphi) + E_T(\theta, \varphi). \quad (2.2.41)$$

As we have done before, we define the error,

$$\begin{aligned}
E_T(\theta, \varphi) = & \frac{1}{2} A(\theta, \varphi)^\top \Omega_L(\theta, \varphi) \tilde{N}(\theta, \varphi)^\top \left(T_h(\theta, \varphi)^\top - T_h(\theta, \varphi) \right) \tilde{N}(\theta, \varphi) \\
& + A(\theta, \varphi)^\top \Omega_L(\theta, \varphi) \left(A(\theta, \varphi) T_h(\theta, \varphi) \tilde{N}(\theta, \varphi) - \tilde{N}(\theta, \varphi)^\top T_h(\theta, \varphi)^\top L(\theta, \varphi) A(\theta, \varphi) \right) \\
& + \left(A(\theta, \varphi)^\top L(\theta, \varphi)^\top + \tilde{N}(\theta, \varphi)^\top \right) \Omega(K(\theta, \varphi)) E_{\mathfrak{L}N}(\theta, \varphi). \quad (2.2.42)
\end{aligned}$$

We estimate the error in equation (2.2.41). Let us begin with $E_{\mathfrak{L}N}(\theta, \varphi)$ and its associated terms. From equation (2.2.33), we have

$$\begin{aligned}
\|T_E\|_{\rho-\delta} \leq & \|D_z \Omega\|_{\mathcal{B} \times \mathbb{T}^\ell} \|E\|_\rho + \|\Omega\|_{\mathcal{B} \times \mathbb{T}^\ell} \|D_z J\|_{\mathcal{B} \times \mathbb{T}^\ell} \|J^{-1}\|_{\mathcal{B} \times \mathbb{T}^\ell} \|E\|_\rho \\
& + \frac{2n}{\delta} \|J^\top\|_{\mathcal{B} \times \mathbb{T}^\ell} \|\Omega\|_{\mathcal{B} \times \mathbb{T}^\ell} \|L\|_\rho \|B\|_{\mathcal{B} \times \mathbb{T}^\ell} \|\Omega\|_{\mathcal{B} \times \mathbb{T}^\ell} \|E\|_\rho \\
& + \frac{n}{\delta} \|J^\top\|_{\mathcal{B} \times \mathbb{T}^\ell} \|\Omega\|_{\mathcal{B} \times \mathbb{T}^\ell} \|B\|_{\mathcal{B} \times \mathbb{T}^\ell} \|L^\top\|_\rho \|\Omega\|_{\mathcal{B} \times \mathbb{T}^\ell} \|E\|_\rho \quad (2.2.43) \\
\leq & \frac{c_{\Omega,1}\delta + c_{\Omega,0}c_{J,1}c_{J^{-1},0}\delta + 2nc_{J^\top,0}C_L\sigma_B c_{\Omega,0}^2 + nc_{J^\top,0}\sigma_B C_{L^\top}(c_{\Omega,0})^2}{\delta} \|E\|_\rho \\
= &: \frac{C_{T_E}}{\delta} \|E\|_\rho.
\end{aligned}$$

Analogously for $T_E^\top$, $E_{\mathfrak{L}B}$, $E_{\mathfrak{L}B}^\top$, $E_{\mathfrak{L}\tilde{N}}$, $E_{\mathfrak{L}\tilde{N}}^\top$, $E_{\mathfrak{L}A}$, $E_{\mathfrak{L}N}$ and E_T , i.e.

$$\begin{aligned} \|T_E^\top\|_{\rho-\delta} &\leq \frac{c_{\Omega,1}\delta + c_{J^{-\top},0}c_{J^\top,1}c_{\Omega,0}\delta + n\sigma_B C_{L^\top} c_{J,0} c_{\Omega,0}^2 + 2n C_L \sigma_B c_{J,0} (c_{\Omega,0})^2}{\delta} \|E\|_\rho \\ &=: \frac{C_{T_E^\top}}{\delta} \|E\|_\rho, \end{aligned} \quad (2.2.44)$$

$$\|E_{\mathfrak{L}B}\|_{\rho-\delta} \leq \frac{C_{\tilde{N}^\top} c_{J^{-\top},0} C_{T_E} C_{\tilde{N}}}{\delta} \|E\|_\rho =: \frac{C_{E_{\mathfrak{L}B}}}{\delta} \|E\|_\rho, \quad (2.2.45)$$

$$\|E_{\mathfrak{L}B}^\top\|_{\rho-\delta} \leq \frac{C_{\tilde{N}^\top} C_{T_E^\top} c_{J^{-1},0} C_{\tilde{N}}}{\delta} \|E\|_\rho =: \frac{C_{E_{\mathfrak{L}B}^\top}}{\delta} \|E\|_\rho, \quad (2.2.46)$$

$$\|E_{\mathfrak{L}\tilde{N}}\|_{\rho-\delta} \leq \frac{\delta c_{J,1} C_L \sigma_B + n c_{J,0} \sigma_B + c_{J,0} C_L C_{E_{\mathfrak{L}B}}}{\delta} \|E\|_\rho =: \frac{C_{E_{\mathfrak{L}\tilde{N}}}}{\delta} \|E\|_\rho, \quad (2.2.47)$$

$$\begin{aligned} \|E_{\mathfrak{L}\tilde{N}}^\top\|_{\rho-\delta} &\leq \frac{\delta \sigma_B C_{L^\top} c_{J^\top,1} + 2n \sigma_B c_{J^\top,0} + C_{E_{\mathfrak{L}B}^\top} C_{L^\top} c_{J^\top,0}}{\delta} \|E\|_\rho \\ &=: \frac{C_{E_{\mathfrak{L}\tilde{N}}^\top}}{\delta} \|E\|_\rho, \end{aligned} \quad (2.2.48)$$

$$\begin{aligned} \|E_{\mathfrak{L}A}\|_{\rho-\delta} &\leq \frac{1}{2} \left(\frac{C_{E_{\mathfrak{L}\tilde{N}}^\top} c_{\Omega,0} C_{\tilde{N}} + \delta C_{\tilde{N}^\top} c_{\Omega,1} C_{\tilde{N}} + C_{\tilde{N}^\top} c_{\Omega,0} C_{E_{\mathfrak{L}\tilde{N}}}}{\delta} \right) \|E\|_\rho \\ &=: \frac{C_{E_{\mathfrak{L}A}}}{\delta} \|E\|_\rho, \end{aligned} \quad (2.2.49)$$

$$\begin{aligned} \|E_{\mathfrak{L}N}\|_{\rho-\delta} &\leq \|E_{\mathfrak{L}\tilde{N}}\|_{\rho-\delta} + \|L\|_{\rho-\delta} \|E_{\mathfrak{L}A}\|_{\rho-\delta} + \|D_\theta E\|_{\rho-\delta} \|A\|_{\rho-\delta} \\ &\leq \frac{C_{E_{\mathfrak{L}\tilde{N}}} + C_L C_{E_{\mathfrak{L}A}} + n C_A}{\delta} \|E\|_\rho =: \frac{C_{E_{\mathfrak{L}N}}}{\delta} \|E\|_\rho, \end{aligned} \quad (2.2.50)$$

and

$$\begin{aligned} \|E_T\|_{\rho-2\delta} &\leq \frac{1}{2} \frac{C_A C_{\Omega_L} C_{\tilde{N}^\top} C_{T_h^\top} C_{\tilde{N}}}{\gamma \delta^{\tau+1}} \|E\|_\rho \\ &\quad + \frac{1}{2} \frac{C_A C_{\Omega_L} C_{\tilde{N}^\top} C_{T_h} C_{\tilde{N}}}{\gamma \delta^{\tau+1}} \|E\|_\rho \\ &\quad + \frac{C_A C_{\Omega_L} C_A C_{T_h} C_{\tilde{N}}}{\gamma \delta^{\tau+1}} \|E\|_\rho \\ &\quad + \frac{C_A C_{\Omega_L} C_{\tilde{N}^\top} C_{T_h^\top} C_{\tilde{N}} C_L C_A}{\gamma \delta^{\tau+1}} \|E\|_\rho \\ &\quad + \frac{\gamma \delta^\tau C_A C_{L^\top} c_{\Omega,0} C_{E_{\mathfrak{L}N}}}{\gamma \delta^{\tau+1}} \|E\|_\rho \\ &\quad + \frac{\gamma \delta^\tau C_{\tilde{N}^\top} c_{\Omega,0} C_{E_{\mathfrak{L}N}}}{\gamma \delta^{\tau+1}} \|E\|_\rho =: \frac{C_{E_T}}{\gamma \delta^{\tau+1}} \|E\|_\rho. \end{aligned} \quad (2.2.51)$$

To control equation (2.2.41), we use the previous estimates made previously, i.e.,

$$\|N^\top \Omega \mathcal{X}_N - T\|_{\rho-2\delta} \leq \frac{C_{E_T}}{\gamma \delta^{\tau+1}} \|E\|_\rho. \quad (2.2.52)$$

Refer to Table 4.1 for the latest set of constants. $\square$

2.2.4 Approximate reducibility

The following lemma estimates the reducibility error obtained after replacing the derivative of the vector field at the approximately invariant torus in the linearized equation for an upper triangular matrix in a new symplectic coordinate system.

Lema 2.2.11. *With the hypotheses of Theorem 2.1.1, the map $P : \mathbb{T}^d \times \mathbb{T}^\ell \rightarrow \mathbb{R}^{2n \times 2n}$, characterized in Lemma 2.2.5, approximately reduces the linearized equation associated with the vector field $DZ_h \circ K$ to a block-triangular matrix, i.e. the error map*

$$E_{\text{red}}(\theta, \varphi) := -\Omega_0 P(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) (D_z Z_h(K(\theta, \varphi)) P(\theta, \varphi) + \mathfrak{L}_{\omega, \alpha} P(\theta, \varphi)) - \Lambda(\theta, \varphi), \quad (2.2.53)$$

with

$$\Lambda(\theta, \varphi) = \begin{pmatrix} O_n & T(\theta, \varphi) \\ O_n & O_n \end{pmatrix} \quad (2.2.54)$$

and $T(\theta, \varphi)$ is given by (1.4.4), is small in the following sense:

$$\|E_{\text{red}}\|_{\rho-2\delta} \leq \frac{C_{\text{red}}}{\gamma \delta^{\tau+1}} \|E\|_\rho,$$

where the constant C_{red} is provided in Table 4.1.

Proof. By employing the notation introduced in equation (1.3.1), we represent the block components of equation (2.2.53), denoted as $E_{\text{red}}^{i,j}(\theta, \varphi)$, as follows:

$$E_{\text{red}}^{1,1}(\theta, \varphi) = N(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) \mathcal{X}_L(\theta, \varphi), \quad (2.2.55)$$

$$E_{\text{red}}^{1,2}(\theta, \varphi) = N(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) \mathcal{X}_N(\theta, \varphi) - T(\theta, \varphi), \quad (2.2.56)$$

$$E_{\text{red}}^{2,1}(\theta, \varphi) = -L(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) \mathcal{X}_L(\theta, \varphi), \quad (2.2.57)$$

$$E_{\text{red}}^{2,2}(\theta, \varphi) = -L(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) \mathcal{X}_N(\theta, \varphi). \quad (2.2.58)$$

In order to control the reducibility error, we estimate its components. We notice the presence of vector operators such as $\mathcal{X}_L$ or $\mathcal{X}_N$. Consequently, our next step involves estimating these operators. First, we look for an expression of the tangent vector bundle in terms of the invariance error. We find that,

$$\begin{aligned} \mathcal{X}_L(\theta, \varphi) &= D_z Z_h(K(\theta, \varphi), \varphi) L(\theta, \varphi) + \mathfrak{L}_{\omega, \alpha} L(\theta, \varphi) \\ &= D_\theta E(\theta, \varphi). \end{aligned} \quad (2.2.59)$$

Similarly, considering the transpose of the tangent vector bundle, we observe that

$$\begin{aligned}\mathcal{X}_L^\top(\theta, \varphi) &= L(\theta, \varphi)^\top D_z Z_h(K(\theta, \varphi), \varphi)^\top + \mathfrak{L}_{\omega, \alpha} L(\theta, \varphi)^\top \\ &= (D_\theta E(\theta, \varphi))^\top.\end{aligned}\tag{2.2.60}$$

Then, the objects, $\mathcal{X}_L(\theta, \varphi)$ and $\mathcal{X}_L(\theta, \varphi)^\top$ are controlled as follows

$$\|\mathcal{X}_L\|_{\rho-\delta} \leq \frac{n}{\delta} \|E\|_\rho =: \frac{C_{\mathcal{X}_L}}{\delta} \|E\|_\rho,\tag{2.2.61}$$

$$\|\mathcal{X}_L^\top\|_{\rho-\delta} \leq \frac{2n}{\delta} \|E\|_\rho =: \frac{C_{\mathcal{X}_L^\top}}{\delta} \|E\|_\rho.\tag{2.2.62}$$

Now, we take the Lie derivative of both sides on the expression (2.2.21).

$$\mathfrak{L}_{\omega, \alpha} \left(L(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) \right) N(\theta, \varphi) + L(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) (\mathfrak{L}_{\omega, \alpha} N(\theta, \varphi)) = \mathfrak{L}_{\omega, \alpha} (\Omega_L(\theta) A(\theta)).$$

Substituting the last expression into $E_{\text{red}}^{2,2}(\theta, \varphi)$, and incorporating (2.1.2) as well as the geometric property (1.3.9), we have the following,

$$\begin{aligned}E_{\text{red}}^{2,2}(\theta, \varphi) &= -L(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) (D_z Z_h(K(\theta, \varphi), \varphi) N(\theta, \varphi) + \mathfrak{L}_{\omega, \alpha} N(\theta, \varphi)) \\ &= L(\theta, \varphi)^\top D_z \Omega(K(\theta, \varphi)) [E(\theta, \varphi)] N(\theta, \varphi) + \mathcal{X}_L(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) N(\theta, \varphi) \\ &\quad - \mathfrak{L}_{\omega, \alpha} (\Omega_L(\theta, \varphi)) A(\theta, \varphi) - \Omega_L(\theta, \varphi) \mathfrak{L}_{\omega, \alpha} (A(\theta, \varphi)).\end{aligned}\tag{2.2.63}$$

We observe that, to control $E_{\text{red}}^{2,2}$, we only need to estimate $\Omega_L \mathfrak{L}_{\omega, \alpha} A$. That is,

$$\begin{aligned}\|\Omega_L \mathfrak{L}_{\omega, \alpha} A\|_{\rho-2\delta} &\leq \|\Omega_L \mathfrak{L}_{\omega, \alpha} \bar{A}\|_{\rho-2\delta} + \|\Omega_L E_{\mathfrak{L}A}\|_{\rho-2\delta} \\ &\leq \|\Omega_L\|_{\rho-2\delta} \|\mathfrak{L}_{\omega, \alpha} \bar{A}\|_{\rho-\delta} + \|\Omega_L\|_{\rho-2\delta} \|E_{\mathfrak{L}A}\|_{\rho-\delta} \\ &\leq \frac{C_{\Omega_L} C_{\mathfrak{L}A} + \gamma \delta^\tau \sigma_L^\top c_{\Omega, 0} \sigma_L C_{E_{\mathfrak{L}A}}}{\gamma \delta^{\tau+1}} \|E\|_\rho,\end{aligned}\tag{2.2.64}$$

where, we use expression (2.2.37),

$$\begin{aligned}\|\mathfrak{L}_{\omega, \alpha} \bar{A}\|_{\rho-\delta} &\leq \frac{1}{2} \left(C_{\tilde{N}^\top c_{T_h^\top, 0} C_{\tilde{N}}} + C_{\tilde{N}^\top c_{T_h^\top, 0} c_{J^{-1}, 0} C_{\tilde{N}}} C_{L^\top c_{J^\top, 0} c_{\Omega, 0} C_{\tilde{N}}} \right. \\ &\quad \left. + C_{\tilde{N}^\top c_{T_h, 0} C_{\tilde{N}}} + C_{\tilde{N}^\top c_{\Omega, 0} c_{J, 0} C_L C_{\tilde{N}^\top c_{J^{-\top}, 0} c_{T_h, 0} C_{\tilde{N}}} \right) =: C_{\mathfrak{L}A}.\end{aligned}\tag{2.2.65}$$

We note that the component $E_{\text{red}}^{1,2}(\theta, \varphi)$ coincides with equation (2.2.41), i.e.

$$E_{\text{red}}^{1,2}(\theta, \varphi) = E_T(\theta, \varphi).\tag{2.2.66}$$

Finally, we estimate the norms of the block components of E_{red} using the expressions (2.2.55),

(2.2.66), (2.2.57) and (2.2.63), and the previous estimates:

$$\|E_{\text{red}}^{1,1}\|_{\rho-2\delta} \leq \frac{C_{N^\top} c_{\Omega,0} C_{\mathcal{X}_L}}{\delta} \|E\|_\rho =: \frac{C_{\text{red}}^{1,1}}{\delta} \|E\|_\rho, \quad (2.2.67)$$

$$\|E_{\text{red}}^{1,2}\|_{\rho-2\delta} \leq \frac{C_{E_T}}{\gamma\delta^{\tau+1}} \|E\|_\rho =: \frac{C_{\text{red}}^{1,2}}{\gamma\delta^{\tau+1}} \|E\|_\rho, \quad (2.2.68)$$

$$\|E_{\text{red}}^{2,1}\|_{\rho-2\delta} \leq \frac{C_{L^\top} c_{\Omega,0} C_{\mathcal{X}_L}}{\delta} \|E\|_\rho =: \frac{C_{\text{red}}^{2,1}}{\delta} \|E\|_\rho, \quad (2.2.69)$$

$$\begin{aligned} \|E_{\text{red}}^{2,2}\|_{\rho-2\delta} &\leq \left(C_{L^\top} c_{\Omega,1} C_N + \frac{C_{\mathcal{X}_L^\top} c_{\Omega,0} C_N}{\delta} + \frac{C_{\mathcal{L}\Omega_L} C_A}{\delta} + \frac{C_{\Omega_L} C_{\mathcal{L}A} + \gamma\delta^\tau \sigma_{L^\top} c_{\Omega,0} \sigma_L C_{E_{\mathcal{L}A}}}{\gamma\delta^{\tau+1}} \right) \|E\|_\rho \\ &=: \frac{C_{\text{red}}^{2,2}}{\gamma\delta^{\tau+1}} \|E\|_\rho. \end{aligned} \quad (2.2.70)$$

Then, we end up with

$$\|E_{\text{red}}\|_{\rho-2\delta} \leq \frac{\max\{C_{\text{red}}^{1,1}\gamma\delta^\tau + C_{\text{red}}^{1,2}, C_{\text{red}}^{2,1}\gamma\delta^\tau + C_{\text{red}}^{2,2}\}}{\gamma\delta^{\tau+1}} \|E\|_\rho =: \frac{C_{\text{red}}}{\gamma\delta^{\tau+1}} \|E\|_\rho, \quad (2.2.71)$$

thus completing the proof. $\square$

Proof of the Lagrangian KAM Theorem

In this section we present a proof of Theorem 2.1.1. For convenience, we start by outlining the scheme used to correct the parameterization of the torus. That is, in Section 3.1 we discuss the approximate solution of the linearized equations in the symplectic frame constructed in Section 2.2.2. This establishes a method to obtain a solution of the invariance equation. In Section 3.2 we produce quantitative estimates for the objects obtained when performing one iteration of the previous procedure. Finally, in Section 3.3 we discuss the convergence of the quasi-Newton method.

3.1 The quasi-Newton method

As it is usual in the a-posteriori approach in KAM theory, the argument consists in refining an approximate solution $K(\theta, \varphi)$, by means of a quasi-Newton method. Let us consider the equations associated with the invariance error (2.1.2).

We obtain the new parameterization $\bar{K}(\theta, \varphi) = K(\theta, \varphi) + \Delta K(\theta, \varphi)$ by considering the linearized equation

$$DZ_h(K(\theta, \varphi), \varphi)\Delta K(\theta, \varphi) + \mathfrak{L}_{\omega, \alpha}\Delta K(\theta, \varphi) = -E(\theta, \varphi). \quad (3.1.1)$$

If we obtain a good enough approximation of the solution $\Delta K(\theta, \varphi)$ of (3.1.1), then $\bar{K}(\theta, \varphi)$ provides a parameterization of an approximately invariant torus with frequency (ω, α) , having a quadratic error in terms of $E(\theta, \varphi)$.

To solve the linearized equation (3.1.1), we make use of the approximately symplectic frame $P(\theta, \varphi)$, defined on the full tangent space, which has been characterized in Chapter 2 (see Lemma 2.2.5). In particular, we introduce the linear change of coordinates,

$$\Delta K(\theta, \varphi) = P(\theta, \varphi)\xi(\theta, \varphi), \quad (3.1.2)$$

where $\xi(\theta, \varphi)$ is the new unknown. Taking into account this expression, the linearized equation

becomes

$$(DZ_h(K(\theta, \varphi), \varphi)P(\theta, \varphi) + \mathfrak{L}_{\omega, \alpha}P(\theta, \varphi))\xi(\theta, \varphi) + P(\theta, \varphi)\mathfrak{L}_{\omega, \alpha}\xi(\theta, \varphi) = -E(\theta, \varphi), \quad (3.1.3)$$

We now multiply both sides of (3.1.3) by $-\Omega_0 P(\theta, \varphi)^\top \Omega(K(\theta, \varphi))$, and we use the geometric properties in Lemma 2.2.5 and Lemma 2.2.11, thus obtaining the equivalent equations,

$$\begin{aligned} (\Lambda(\theta, \varphi) + E_{\text{red}}(\theta, \varphi))\xi(\theta, \varphi) + (I_{2n} - \Omega_0 E_{\text{sym}}(\theta, \varphi))\mathfrak{L}_{\omega, \alpha}\xi(\theta, \varphi) \\ = \Omega_0 P(\theta, \varphi)^\top \Omega(K(\theta, \varphi))E(\theta, \varphi) := \begin{pmatrix} \eta^L(\theta, \varphi) \\ \eta^N(\theta, \varphi) \end{pmatrix}, \end{aligned} \quad (3.1.4)$$

$\Lambda(\theta, \varphi)$ is the triangular matrix-valued map given in (2.2.54). We notice that $\langle \eta^N \rangle = 0_n$, from the third equation in (2.2.5).

Then, it turns out that the solution of (3.1.4) are approximated by the solutions of a triangular system that involves solving two cohomological equations of the form (1.1.2) consecutively. Quantitative estimates for the solutions of such equations are obtained by applying Rüssmann estimates. This is summarized in the following standard statement.

Lema 3.1.1 (Upper triangular equations). *Let $(\omega, \alpha) \in \mathcal{D}_{\gamma, \tau}$ and let us consider a map $\eta = (\eta^L, \eta^N) : \mathbb{T}^n \times \mathbb{T}^\ell \rightarrow \mathbb{R}^{2n} \simeq \mathbb{R}^n \times \mathbb{R}^n$, with components in $\mathcal{A}(\mathbb{T}_\rho^n \times \mathbb{T}_\rho^\ell)$, and a map $T : \mathbb{T}^n \times \mathbb{T}^\ell \rightarrow \mathbb{R}^{n \times n}$, with components in $\mathcal{A}(\mathbb{T}_\rho^n \times \mathbb{T}_\rho^\ell)$. Assume that T satisfies the non-degeneracy condition $\det \langle T \rangle \neq 0$ and η satisfies the compatibility condition $\langle \eta^N \rangle = 0_n$. Then, for any $\xi_{0,0}^L \in \mathbb{R}^n$, the system of equations*

$$\begin{pmatrix} O_n & T(\theta, \varphi) \\ O_n & O_n \end{pmatrix} \begin{pmatrix} \xi^L(\theta, \varphi) \\ \xi^N(\theta, \varphi) \end{pmatrix} + \begin{pmatrix} \mathfrak{L}_{\omega, \alpha} \xi^L(\theta, \varphi) \\ \mathfrak{L}_{\omega, \alpha} \xi^N(\theta, \varphi) \end{pmatrix} = \begin{pmatrix} \eta^L(\theta, \varphi) \\ \eta^N(\theta, \varphi) \end{pmatrix} \quad (3.1.5)$$

has a solution of the form

$$\begin{aligned} \xi^N(\theta, \varphi) &= \xi_{0,0}^N + \mathfrak{R}_{\omega, \alpha}(\eta^N(\theta, \varphi)), \\ \xi^L(\theta, \varphi) &= \xi_{0,0}^L + \mathfrak{R}_{\omega, \alpha}(\eta^L(\theta, \varphi) - T(\theta, \varphi)\xi^N(\theta, \varphi)), \end{aligned}$$

where

$$\xi_{0,0}^N = \langle T \rangle^{-1} \langle \eta^L - T \mathfrak{R}_\omega(\eta^N) \rangle$$

and $\mathfrak{R}_{\omega, \alpha}$ is given by (1.1.3). Moreover, we have the estimates

$$\begin{aligned} |\xi_{0,0}^N| &\leq |\langle T \rangle^{-1}| \left(\|\eta^L\|_\rho + \frac{c_R}{\gamma \delta^\tau} \|T\|_\rho \|\eta^N\|_\rho \right), \\ \|\xi^N\|_{\rho-\delta} &\leq |\xi_{0,0}^N| + \frac{c_R}{\gamma \delta^\tau} \|\eta^N\|_\rho, \\ \|\xi^L\|_{\rho-2\delta} &\leq |\xi_{0,0}^L| + \frac{c_R}{\gamma \delta^\tau} \left(\|\eta^L\|_{\rho-\delta} + \|T\|_\rho \|\xi^N\|_{\rho-\delta} \right). \end{aligned}$$

Proof. From (3.1.5), we have,

$$\begin{aligned}\mathfrak{L}_{\omega,\alpha}\xi^N(\theta, \varphi) &= \eta^N(\theta, \varphi) \\ \mathfrak{L}_{\omega,\alpha}\xi^L(\theta, \varphi) &= \eta^L(\theta, \varphi) - T(\theta, \varphi)\xi^N(\theta, \varphi).\end{aligned}$$

By hypothesis we know that $\langle \eta^N \rangle = 0_n$, so

$$\xi^N(\theta, \varphi) = \xi_{0,0}^N + \mathfrak{R}_{\omega,\alpha}(\eta^N(\theta, \varphi)). \quad (3.1.6)$$

To solve the second equation, the average $\langle \eta^L - T\xi^N \rangle$ must be zero. i.e.

$$\begin{aligned}0_n &= \langle \eta^L - T\xi^N \rangle = \langle \eta^L - S\mathfrak{R}_{\omega,\alpha}(\eta^N) - S\xi_0^N \rangle \\ &= \langle \eta^L - S\mathfrak{R}_{\omega,\alpha}(\eta^N) \rangle - \langle S\xi_0^N \rangle \\ &= \langle \eta^L - T\mathfrak{R}_{\omega,\alpha}(\eta^N) \rangle - \langle T \rangle \xi_{0,0}^N.\end{aligned}$$

Then,

$$\xi_{0,0}^N = \langle T \rangle^{-1} \langle \eta^L - T\mathfrak{R}_{\omega,\alpha}(\eta^N) \rangle. \quad (3.1.7)$$

Back to the second equation, its solution is,

$$\xi^L(\theta, \varphi) = \xi_{0,0}^L + \mathfrak{R}_{\omega,\alpha}(\eta^L(\theta, \varphi) - T(\theta, \varphi)\xi^N(\theta, \varphi)), \quad (3.1.8)$$

in addition, we have estimates

$$\|\xi^N\|_{\rho-\delta} \leq |\xi_{0,0}^N| + \frac{c_R}{\gamma\delta^\tau} \|\eta^N\|_\rho,$$

now, for $\xi_{0,0}^N$

$$\begin{aligned}|\xi_{0,0}^N| &\leq |\langle T \rangle^{-1}| |\langle \eta^L - T\mathfrak{R}_{\omega,\alpha}(\eta^N) \rangle| \\ &\leq |\langle T \rangle^{-1}| \left(\|\eta^L\|_\rho + \frac{c_R}{\gamma\delta^\tau} \|T\|_\rho \|\eta^N\|_\rho \right).\end{aligned}$$

Finally,

$$\begin{aligned}\|\xi^L\|_{\rho-2\delta} &= \|\xi_{0,0}^L + \mathfrak{R}_{\omega,\alpha}(\eta^L - T\xi^N)\|_{\rho-2\delta} \\ &\leq |\xi_{0,0}^L| + \frac{c_R}{\gamma\delta^\tau} \|\eta^L - T\xi^N\|_{\rho-\delta} \\ &\leq |\xi_{0,0}^L| + \frac{c_R}{\gamma\delta^\tau} (\|\eta^L\|_{\rho-\delta} + \|T\|_\rho \|\xi^N\|_{\rho-\delta}),\end{aligned}$$

and this concludes the proof. $\square$

To approximate the solutions of (3.1.4) we invoke Lemma 3.1.1 taking

$$\begin{aligned}\eta^L(\theta, \varphi) &= -N(\theta, \varphi)^\top \Omega(K(\theta, \varphi))E(\theta, \varphi), \\ \eta^N(\theta, \varphi) &= L(\theta, \varphi)^\top \Omega(K(\theta, \varphi))E(\theta, \varphi),\end{aligned}\tag{3.1.9}$$

and $T(\theta, \varphi)$ given by (1.4.4). Note that $\langle \xi^N \rangle = \xi_{0,0}^N$ and we have the freedom of choosing any value for $\langle \xi^L \rangle = \xi_{0,0}^L \in \mathbb{R}^n$. For convenience in setting the phase, we choose $\xi_{0,0}^L = 0_n$, even though other choices are possible.

From Lemma 3.1.1 we read that $\|\xi\|_{\rho-2\delta} = \mathcal{O}(\|E\|_\rho)$ and, using the geometric properties characterized in Chapter 2, we have $\|E_{\text{red}}\|_{\rho-2\delta} = \mathcal{O}(\|E\|_\rho)$ and $\|E_{\text{sym}}\|_{\rho-2\delta} = \mathcal{O}(\|E\|_\rho)$. From these estimates, we conclude that the solution of equation (3.1.4) is approximated by the solution of the cohomological equation

$$\Lambda(\theta, \varphi)\xi(\theta, \varphi) + \mathfrak{L}_{\omega, \alpha}\xi(\theta, \varphi) = \eta(\theta, \varphi).\tag{3.1.10}$$

This, together with other estimates, will be suitably quantified in the next section.

3.2 One step of the iterative procedure

In this section we apply one correction of the quasi-Newton method described in Section 3.1 and we obtain sharp quantitative estimates for the new approximately invariant torus and related objects. We set sufficient conditions to preserve the control of the previous estimates.

Lema 3.2.1 (The Iterative Lemma). *With the hypotheses of Theorem 2.1.1, Then, there exist constants $C_{\Delta K}$, $C_{\Delta B}$, $C_{\Delta \langle T \rangle^{-1}}$ and C_E such that if the inequalities*

$$\frac{\hat{C}_\Delta \|E\|_\rho}{\gamma^2 \delta^{2\tau+1}} < 1, \quad \frac{C_E \|E\|_\rho}{\gamma^4 \delta^{4\tau}} < 1,\tag{3.2.1}$$

hold for some $0 < \delta < \rho$, where

$$\begin{aligned}\hat{C}_\Delta := \max \left\{ C_{\text{sym}} \gamma \delta^\tau, \frac{n C_{\Delta K}}{\sigma_L - \|DK\|_\rho}, \frac{2n C_{\Delta K}}{\sigma_{L^\top} - \|(DK)^\top\|_\rho}, \right. \\ \left. \frac{C_{\Delta B}}{\sigma_B - \|B\|_\rho}, \frac{C_{\Delta \langle T \rangle^{-1}}}{\sigma_T - |\langle T \rangle^{-1}|}, \frac{C_{\Delta K} \delta}{\text{dist}(K(\mathbb{T}_\rho^n \times \mathbb{T}_\rho^\ell), \partial(\mathcal{B} \times \mathbb{T}_\rho^\ell))} \right\}.\end{aligned}\tag{3.2.2}$$

then we have an approximate torus of the same frequency ω given by $\bar{K} = K + \Delta K$, with components in $\mathcal{A}(\mathbb{T}_{\rho-2\delta}^n \times \mathbb{T}_{\rho-2\delta}^\ell)$, that defines new objects $\bar{B}$ and $\bar{T}$ (obtained by replacing K by $\bar{K}$) satisfying

$$\|D\bar{K}\|_{\rho-3\delta} < \sigma_L,\tag{3.2.3}$$

$$\|(D\bar{K})^\top\|_{\rho-3\delta} < \sigma_{L^\top},\tag{3.2.4}$$

$$\|\bar{B}\|_{\rho-3\delta} < \sigma_B,\tag{3.2.5}$$

$$|\langle \bar{T} \rangle^{-1}| < \sigma_T,\tag{3.2.6}$$

$$\text{dist}(\bar{K}(\mathbb{T}_{\rho-2\delta}^n \times \mathbb{T}_{\rho-2\delta}^\ell), \partial(\mathcal{B} \times \mathbb{T}_\rho^\ell)) > 0,\tag{3.2.7}$$

and

$$\|\bar{K} - K\|_{\rho-2\delta} < \frac{C_{\Delta K}}{\gamma^2 \delta^{2\tau}} \|E\|_{\rho}, \quad (3.2.8)$$

$$\|\bar{B} - B\|_{\rho-3\delta} < \frac{C_{\Delta B}}{\gamma^2 \delta^{2\tau+1}} \|E\|_{\rho}, \quad (3.2.9)$$

$$|\langle \bar{T} \rangle^{-1} - \langle T \rangle^{-1}| < \frac{C_{\Delta \langle T \rangle^{-1}}}{\gamma^2 \delta^{2\tau+1}} \|E\|_{\rho}, \quad (3.2.10)$$

The new error of invariance is given by

$$\bar{E}(\theta, \varphi) = Z_h(\bar{K}(\theta, \varphi), \varphi) + \mathfrak{L}_{\omega, \alpha} \bar{K}(\theta, \varphi),$$

and satisfies

$$\|\bar{E}\|_{\rho-2\delta} < \frac{C_E}{\gamma^4 \delta^{4\tau}} \|E\|_{\rho}^2. \quad (3.2.11)$$

The above constants are collected in Table 4.2.

Proof. To obtain the estimates of the theorem requires many computations, so we break down the demonstration into seven steps.

Step 1: Control of the new parameterization. We start by considering a new parameterization $\bar{K}(\theta, \varphi) = K(\theta, \varphi) + \Delta K(\theta, \varphi)$, which is obtained from the system (3.1.5), with $\eta(\theta, \varphi)$ given by (3.1.9). We choose the solution that satisfies $\xi_{0,0}^L = 0$. Using the estimates obtained in Chapter 2, we have the following inequalities:

$$\|\eta^L\|_{\rho} \leq C_{N^\top c_{\Omega,0}} \|E\|_{\rho}, \text{ and } \|\eta^N\|_{\rho} \leq C_{L^\top c_{\Omega,0}} \|E\|_{\rho}.$$

Combining Lemma 2.2.3 and Lemma 3.1.1, we obtain estimates for the solution of the cohomological equations

$$\begin{aligned} |\xi_{0,0}^N| &\leq |\langle T \rangle^{-1}| \left(\|\eta^L\|_{\rho} + \frac{c_R}{\gamma \delta^\tau} \|T\|_{\rho} \|\eta^N\|_{\rho} \right) \\ &\leq \sigma_T \left(C_{N^\top c_{\Omega,0}} + \frac{c_R}{\gamma \delta^\tau} C_T C_{L^\top c_{\Omega,0}} \right) \|E\|_{\rho} =: \frac{C_{\xi_0^N}}{\gamma \delta^\tau} \|E\|_{\rho}, \end{aligned} \quad (3.2.12)$$

$$\begin{aligned} \|\xi^N\|_{\rho-\delta} &\leq |\xi_0^N| + \frac{c_R}{\gamma \delta^\tau} \|\eta^N\|_{\rho} \\ &\leq \frac{C_{\xi_0^N}}{\gamma \delta^\tau} \|E\|_{\rho} + \frac{c_R}{\gamma \delta^\tau} C_{L^\top c_{\Omega,0}} \|E\|_{\rho} =: \frac{C_{\xi^N}}{\gamma \delta^\tau} \|E\|_{\rho}, \end{aligned} \quad (3.2.13)$$

$$\begin{aligned} \|\xi^L\|_{\rho-2\delta} &\leq |\xi_0^L| + \frac{c_R}{\gamma \delta^\tau} \left(\|\eta^L\|_{\rho} + \|T\|_{\rho} \|\xi^N\|_{\rho-\delta} \right) \\ &\leq \frac{c_R}{\gamma \delta^\tau} \left(C_{N^\top c_{\Omega,0}} + C_T \frac{C_{\xi^N}}{\gamma \delta^\tau} \right) \|E\|_{\rho} =: \frac{C_{\xi^L}}{\gamma^2 \delta^{2\tau}} \|E\|_{\rho}. \end{aligned} \quad (3.2.14)$$

The magnitude of the vector $\xi(\theta, \varphi)$, which satisfies (3.1.10), is estimated as

$$\begin{aligned} \|\xi\|_{\rho-2\delta} &\leq \max\{\|\xi^L\|_{\rho-2\delta}, \|\xi^N\|_{\rho-\delta}\} \\ &\leq \frac{\max\{C_{\xi^L}, C_{\xi^N}\gamma\delta^\tau\}}{\gamma^2\delta^{2\tau}}\|E\|_\rho =: \frac{C_\xi}{\gamma^2\delta^{2\tau}}\|E\|_\rho. \end{aligned} \quad (3.2.15)$$

The new parameterization $\bar{K}(\theta, \varphi)$ and associated objects are estimated through conventional calculations. The control (3.2.8) stems directly from

$$\bar{K}(\theta, \varphi) - K(\theta, \varphi) = \Delta K(\theta, \varphi) = P(\theta, \varphi)\xi(\theta, \varphi) = L(\theta, \varphi)\xi^L(\theta, \varphi) + N(\theta, \varphi)\xi^N(\theta, \varphi).$$

In other words, by employing approximations provided in (3.2.13) and (3.2.14), we obtain

$$\|\bar{K} - K\|_{\rho-2\delta} \leq \frac{C_L C_{\xi^L} + C_N C_{\xi^N} \gamma \delta^\tau}{\gamma^2 \delta^{2\tau}} \|E\|_\rho =: \frac{C_{\Delta K}}{\gamma^2 \delta^{2\tau}} \|E\|_\rho. \quad (3.2.16)$$

To complete this step, we need to make sure that the value of $\bar{K}(\theta, \varphi)$ falls within the defined domain $\mathcal{B} \times \mathbb{T}_\rho^\ell$ of all the entities involved. This verification is essential as it enables us to calculate the new error $\bar{E}(\theta, \varphi)$ before proceeding with the remaining geometrical elements. To achieve this, we take note of the fact that:

$$\begin{aligned} \text{dist}(\bar{K}(\mathbb{T}_{\rho-2\delta}^n \times \mathbb{T}_{\rho-2\delta}^\ell), \partial(\mathcal{B} \times \mathbb{T}_\rho^\ell)) &\geq \text{dist}(K(\mathbb{T}_\rho^n \times \mathbb{T}_\rho^\ell), \partial(\mathcal{B} \times \mathbb{T}_\rho^\ell)) - \|\Delta K\|_{\rho-2\delta} \\ &\geq \text{dist}(K(\mathbb{T}_\rho^n \times \mathbb{T}_\rho^\ell), \partial(\mathcal{B} \times \mathbb{T}_\rho^\ell)) - \frac{C_{\Delta K}}{\gamma^2 \delta^{2\tau}} \|E\|_\rho > 0. \end{aligned} \quad (3.2.17)$$

The last inequality follows from Hypothesis (3.2.1), using the sixth term in (3.2.2). Thus, we have established the control in (3.2.7).

Step 2: Control of the new error of invariance. To guarantee that the corrected parameterization $\bar{K}$ is invariant, we control the quadratic terms neglected in equation (3.1.10), by first considering the error in solving the linearized equation (3.1.4),

$$E_{\text{lin}}(\theta, \varphi) = E_{\text{red}}(\theta, \varphi)\xi(\theta, \varphi) - \Omega_0 E_{\text{sym}}(\theta, \varphi) \mathfrak{L}_{\omega, \alpha} \xi(\theta, \varphi). \quad (3.2.18)$$

We control the term $\mathfrak{L}_{\omega, \alpha} \xi(\theta, \varphi)$ by leveraging the fact that $\xi(\theta, \varphi)$ satisfies the solution of the cohomological equation (3.1.10):

$$\|\mathfrak{L}_\omega \xi^N\|_\rho = \|\eta^N\|_\rho \leq C_{L^\top c_{\Omega, 0}} \|E\|_\rho =: C_{\mathfrak{L}_\omega \xi^N} \|E\|_\rho, \quad (3.2.19)$$

$$\begin{aligned} \|\mathfrak{L}_\omega \xi^L\|_{\rho-\delta} &= \|\eta^L - T\xi^N\|_{\rho-\delta} \leq \left(C_{N^\top c_{\Omega, 0}} + C_T \frac{C_{\xi^N}}{\gamma \delta^\tau} \right) \|E\|_\rho \\ &=: \frac{C_{\mathfrak{L}_\omega \xi^L}}{\gamma \delta^\tau} \|E\|_\rho, \end{aligned} \quad (3.2.20)$$

$$\|\mathfrak{L}_\omega \xi\|_{\rho-\delta} \leq \max \left\{ \frac{C_{\mathfrak{L}_\omega \xi^L}}{\gamma \delta^\tau}, C_{\mathfrak{L}_\omega \xi^N} \right\} \|E\|_\rho =: \frac{C_{\mathfrak{L}_\omega \xi}}{\gamma \delta^\tau} \|E\|_\rho. \quad (3.2.21)$$

Therefore, the control for $E_{\text{lin}}(\theta, \varphi)$ is given by

$$\begin{aligned} \|E_{\text{lin}}\|_{\rho-2\delta} &\leq \|E_{\text{red}}\|_{\rho-2\delta} \|\xi\|_{\rho-2\delta} + c_{\Omega,0} \|E_{\text{sym}}\|_{\rho-2\delta} \|\mathfrak{L}_{\omega}\xi\|_{\rho-2\delta} \\ &\leq \frac{C_{\text{red}}}{\gamma\delta^{\tau+1}} \frac{C_{\xi}}{\gamma^2\delta^{2\tau}} \|E\|_{\rho}^2 + c_{\Omega,0} \frac{C_{\text{sym}}}{\gamma\delta^{\tau+1}} \frac{C_{\xi}}{\gamma\delta^{\tau}} \|E\|_{\rho}^2 =: \frac{C_{\text{lin}}}{\gamma^3\delta^{3\tau+1}} \|E\|_{\rho}^2. \end{aligned} \quad (3.2.22)$$

After making the correction, the invariance error associated with the new parameterization can be determined.

$$\begin{aligned} \bar{E}(\theta, \varphi) &= Z_h(K(\theta, \varphi) + \Delta K(\theta, \varphi), \varphi) + \mathfrak{L}_{\omega,\alpha} K(\theta, \varphi) + \mathfrak{L}_{\omega,\alpha} \Delta K(\theta, \varphi) \\ &= Z_h(K(\theta, \varphi), \varphi) + DZ_h(K(\theta, \varphi), \varphi) \Delta K(\theta, \varphi) + \mathfrak{L}_{\omega,\alpha} K(\theta, \varphi) + \mathfrak{L}_{\omega,\alpha} \Delta K(\theta, \varphi) + \Delta^2 Z(\theta, \varphi) \\ &= DZ_h(K(\theta, \varphi), \varphi) \Delta K(\theta, \varphi) + \mathfrak{L}_{\omega,\alpha} \Delta K(\theta, \varphi) + E(\theta, \varphi) + \Delta^2 Z(\theta, \varphi) \\ &= \left(DZ_h(K(\theta, \varphi), \varphi) P(\theta, \varphi) + \mathfrak{L}_{\omega,\alpha} P(\theta, \varphi) \right) \xi(\theta, \varphi) + P(\theta, \varphi) \mathfrak{L}_{\omega,\alpha} \xi(\theta, \varphi) \\ &\quad + E(\theta, \varphi) + \Delta^2 Z_h(K(\theta, \varphi), \varphi) \\ &= (-\Omega_0 P(\theta, \varphi)^\top \Omega(K(\theta, \varphi)))^{-1} E_{\text{lin}}(\theta, \varphi) + \Delta^2 Z(\theta, \varphi) \\ &= P(\theta, \varphi) (I_{2n} - \Omega_0 E_{\text{sym}}(\theta, \varphi))^{-1} E_{\text{lin}}(\theta, \varphi) + \Delta_{\theta}^2 Z_h(K(\theta, \varphi), \varphi), \end{aligned} \quad (3.2.23)$$

where

$$\begin{aligned} \Delta_{\theta}^2 Z_h(K(\theta, \varphi), \varphi) &= Z_h(K(\theta, \varphi) + \Delta K(\theta, \varphi), \varphi) - Z_h(K(\theta, \varphi), \varphi) - D_z Z_h(K(\theta, \varphi), \varphi) \Delta K(\theta, \varphi) \\ &= \int_0^1 (1-s) D_z^2 Z_h(K(\theta, \varphi) + t \Delta K(\theta, \varphi), \varphi) [\Delta K(\theta, \varphi), \Delta K(\theta, \varphi)] ds, \end{aligned} \quad (3.2.24)$$

and we used equation (2.2.12) and (3.2.18). We use the mean value theorem to control $\Delta_{\theta}^2 Z(\theta, \varphi)$, i.e.,

$$\begin{aligned} \|\Delta_{\theta}^2 Z_h(\theta, \varphi)\|_{\rho-2\delta} &= \left\| \int_0^1 (1-s) D_z^2 Z_h(K(\theta, \varphi) + t \Delta K(\theta, \varphi), \varphi) [\Delta K(\theta, \varphi), \Delta K(\theta, \varphi)] ds \right\|_{\rho-2\delta} \\ &\leq \left| \int_0^1 (1-s) ds \right| \|D_z^2 Z_h\|_{\mathcal{B} \times \mathbb{T}^{\ell}} \|\Delta K\|_{\rho-2\delta} \|\Delta K\|_{\rho-2\delta} \\ &\leq \frac{1}{2} c_{Z_h,2} \frac{(C_{\Delta K})^2}{\gamma^4 \delta^{4\tau}} \|E\|_{\rho}^2. \end{aligned} \quad (3.2.25)$$

It is noteworthy that the error function equation (3.2.23), is well-defined as a result of the calculations in equation (3.2.17). We estimate the norm of the error function as follows

$$\|\bar{E}\|_{\rho-2\delta} \leq \|P\|_{\rho-2\delta} \|(I - \Omega_0 E_{\text{sym}})^{-1}\|_{\rho-2\delta} \|E_{\text{lin}}\|_{\rho-2\delta} + \|\Delta_{\theta}^2 Z_h\|_{\rho-2\delta}.$$

Then, using a Neumann series argument, we obtain

$$\|(I - \Omega_0 E_{\text{sym}})^{-1}\|_{\rho-2\delta} \leq \frac{1}{1 - \|\Omega_0 E_{\text{sym}}\|_{\rho-2\delta}} < 1,$$

where we used the inequality

$$\frac{C_{\text{sym}}}{\gamma\delta^{\tau+1}} \|E\|_{\rho} < 1,$$

that corresponds to the second term in (3.2.2) (Hypothesis (3.2.1)). Putting together the above estimates, we obtain

$$\|\bar{E}\|_{\rho-2\delta} \leq \left(\frac{2(C_L + C_N)C_{\text{lin}}}{\gamma^3\delta^{3\tau+1}} + \frac{1}{2}c_{Z_h,2} \frac{(C_{\Delta K})^2}{\gamma^4\delta^{4\tau}} \right) \|E\|_{\rho}^2 =: \frac{C_E}{\gamma^4\delta^{4\tau}} \|E\|_{\rho}^2. \quad (3.2.26)$$

We have obtained estimate (3.2.11).

Step 3: Control of the new frame $L(\theta, \varphi)$. Equation (3.2.16) can be combined with Cauchy estimates to derive the constraint presented in equation (3.2.3),

$$\|D_{\theta}\bar{K}\|_{\rho-3\delta} \leq \|D_{\theta}K\|_{\rho} + \|D_{\theta}\Delta K\|_{\rho-3\delta} \leq \|D_{\theta}K\|_{\rho} + \frac{nC_{\Delta K}}{\gamma^2\delta^{2\tau+1}} \|E\|_{\rho} < \sigma_L, \quad (3.2.27)$$

where the last inequality follows from Hypothesis (3.2.1) (this corresponds to the third term in (3.2.2)). Estimate (3.2.4) is analogous

$$\|(D_{\theta}\bar{K})^{\top}\|_{\rho-3\delta} \leq \|(D_{\theta}K)^{\top}\|_{\rho} + \frac{2nC_{\Delta K}}{\gamma^2\delta^{2\tau+1}} \|E\|_{\rho} < \sigma_{L^{\top}}. \quad (3.2.28)$$

After obtaining the estimates (3.2.27) and (3.2.28), it is clear that

$$\|\bar{L}\|_{\rho-3\delta} \leq C_L, \quad \|\bar{L}^{\top}\|_{\rho-3\delta} \leq C_{L^{\top}}.$$

We can control the norm of the corresponding corrections using Cauchy estimates, the mean value theorem, and the estimate (3.2.16):

$$\|\bar{L} - L\|_{\rho-3\delta} \leq \frac{nC_{\Delta K}}{\gamma^2\delta^{2\tau+1}} \|E\|_{\rho} =: \frac{C_{\Delta L}}{\gamma^2\delta^{2\tau+1}} \|E\|_{\rho}, \quad (3.2.29)$$

$$\|\bar{L}^{\top} - L^{\top}\|_{\rho-3\delta} \leq \frac{2nC_{\Delta K}}{\gamma^2\delta^{2\tau+1}} \|E\|_{\rho} =: \frac{C_{\Delta L^{\top}}}{\gamma^2\delta^{2\tau+1}} \|E\|_{\rho}. \quad (3.2.30)$$

Step 4: Control of the new transversality condition. To control $\bar{B}$, we take

$$\begin{aligned} G_L(\theta, \varphi) &= L(\theta, \varphi)^\top G(K(\theta, \varphi)) L(\theta, \varphi), \\ G_{\bar{L}}(\theta, \varphi) &= \bar{L}(\theta, \varphi)^\top G(\bar{K}(\theta, \varphi)) \bar{L}(\theta, \varphi), \end{aligned}$$

where we have used the notation introduced in (2.2.6). First, we compute

$$\begin{aligned} \|G \circ \bar{K} - G \circ K\|_{\rho-2\delta} &\leq \|DG\|_{\mathcal{B}} \|\bar{K} - K\|_{\rho-2\delta} \leq \frac{c_{G,1} C_{\Delta K}}{\gamma^2 \delta^{2\tau}} \|E\|_{\rho} \\ &=: \frac{C_{\Delta G}}{\gamma^2 \delta^{2\tau}} \|E\|_{\rho}, \end{aligned} \tag{3.2.31}$$

and

$$\begin{aligned} \|G_{\bar{L}} - G_L\|_{\rho-3\delta} &\leq \|\bar{L}^\top (G \circ \bar{K}) \bar{L} - \bar{L}^\top (G \circ K) L\|_{\rho-3\delta} \\ &\quad + \|\bar{L}^\top (G \circ \bar{K}) L - \bar{L}^\top (G \circ K) L\|_{\rho-3\delta} \\ &\quad + \|\bar{L}^\top (G \circ K) L - L^\top (G \circ K) L\|_{\rho-3\delta} \\ &\leq \|\bar{L}^\top\|_{\rho} \|G\|_{\mathcal{B}} \|\bar{L} - L\|_{\rho-3\delta} \\ &\quad + \|\bar{L}^\top\|_{\rho} \|G \circ \bar{K} - G \circ K\|_{\rho-3\delta} \|L\|_{\rho} \\ &\quad + \|\bar{L}^\top - L^\top\|_{\rho-3\delta} \|G\|_{\mathcal{B}} \|L\|_{\rho} \\ &\leq \frac{C_{L^\top} c_{G,0} C_{\Delta L} + C_{L^\top} C_{\Delta G} C_L \delta + C_{\Delta L^\top} c_{G,0} C_L}{\gamma^2 \delta^{2\tau+1}} \|E\|_{\rho} \\ &=: \frac{C_{\Delta G_L}}{\gamma^2 \delta^{2\tau+1}} \|E\|_{\rho}. \end{aligned} \tag{3.2.32}$$

Afterward, we introduce the constant

$$C_{\Delta B} := (\sigma_B)^2 C_{\Delta G_L},$$

and review and verify the condition (A.1) mentioned in Lemma A, as stated in [FH24]

$$\frac{(\sigma_B)^2 \|G_{\bar{L}} - G_L\|_{\rho-3\delta}}{\sigma_B - \|B\|_{\rho}} \leq \frac{C_{\Delta B}}{\sigma_B - \|B\|_{\rho}} \frac{\|E\|_{\rho}}{\gamma^2 \delta^{2\tau+1}} < 1. \tag{3.2.33}$$

The final inequality stems from the Hypothesis (3.2.1), corresponding to the fifth term in (3.2.2). Thus, with the Lemma A, as given in [FH24], we conclude that

$$\|\bar{B}\|_{\rho-3\delta} < \sigma_B, \quad \|\bar{B} - B\|_{\rho-3\delta} \leq \frac{(\sigma_B)^2 C_{\Delta G_L}}{\gamma^2 \delta^{2\tau+1}} \|E\|_{\rho} = \frac{C_{\Delta B}}{\gamma^2 \delta^{2\tau+1}} \|E\|_{\rho}, \tag{3.2.34}$$

which allows us to arrive at the estimates (3.2.5) and (3.2.9) for the new object.

Step 5: Control of the new frame $N(\theta, \varphi)$. For controlling the newly adapted normal frame $\bar{N}(\theta, \varphi)$, it is advantageous to adapt the notation:

$$\begin{aligned} N(\theta, \varphi) &= L(\theta, \varphi)A(\theta, \varphi) + \tilde{N}(\theta, \varphi), \\ \tilde{N}(\theta, \varphi) &= N^0(\theta, \varphi)B(\theta, \varphi), \\ N^0(\theta, \varphi) &= J(K(\theta, \varphi))L(\theta, \varphi), \\ A(\theta, \varphi) &= -\frac{1}{2}(\tilde{N}(\theta, \varphi)^\top \Omega(K(\theta, \varphi))\tilde{N}(\theta, \varphi)), \\ B(\theta, \varphi) &= (L(\theta, \varphi)^\top G(K(\theta, \varphi))L(\theta, \varphi))^{-1}. \end{aligned}$$

As usual, we obtain the new objects $\bar{N}(\theta, \varphi)$, $\bar{A}(\theta, \varphi)$, $\bar{N}^0(\theta, \varphi)$, $\bar{\tilde{N}}(\theta, \varphi)$ and $\bar{B}(\theta, \varphi)$ by replacing $K(\theta, \varphi)$ with $\bar{K}(\theta, \varphi)$. Note that we controlled $\bar{B}(\theta, \varphi)$ in Step 4. Now, we reproduce the computations in (3.2.31) and (3.2.32) for the matrix functions.

$$\|\Omega \circ \bar{K} - \Omega \circ K\|_{\rho-2\delta} \leq \frac{c_{\Omega,1}C_{\Delta K}}{\gamma^2\delta^{2\tau}}\|E\|_\rho =: \frac{C_{\Delta\Omega}}{\gamma^2\delta^{2\tau}}\|E\|_\rho, \quad (3.2.35)$$

$$\|J \circ \bar{K} - J \circ K\|_{\rho-2\delta} \leq \frac{c_{J,1}C_{\Delta K}}{\gamma^2\delta^{2\tau}}\|E\|_\rho =: \frac{C_{\Delta J}}{\gamma^2\delta^{2\tau}}\|E\|_\rho, \quad (3.2.36)$$

$$\|(J \circ \bar{K} - J \circ K)^\top\|_{\rho-2\delta} \leq \frac{c_{J^\top,1}C_{\Delta K}}{\gamma^2\delta^{2\tau}}\|E\|_\rho =: \frac{C_{\Delta J^\top}}{\gamma^2\delta^{2\tau}}\|E\|_\rho, \quad (3.2.37)$$

$$\begin{aligned} \|\bar{N}^0 - N^0\|_{\rho-3\delta} &\leq \|J\|_{\mathcal{B}}\|\bar{L} - L\|_{\rho-3\delta} + \|J \circ \bar{K} - J \circ K\|_{\rho-2\delta}\|L\|_\rho \\ &\leq \frac{c_{J,0}C_{\Delta L} + C_{\Delta J}C_L\delta}{\gamma^2\delta^{2\tau+1}}\|E\|_\rho \\ &=: \frac{C_{\Delta N^0}}{\gamma^2\delta^{2\tau+1}}\|E\|_\rho, \end{aligned} \quad (3.2.38)$$

$$\begin{aligned} \|(\bar{N}^0)^\top - (N^0)^\top\|_{\rho-3\delta} &\leq \frac{C_{\Delta L^\top}c_{J^\top,0} + C_{L^\top}C_{\Delta J^\top}\delta}{\gamma^2\delta^{2\tau+1}}\|E\|_\rho \\ &=: \frac{C_{\Delta N^{0\top}}}{\gamma^2\delta^{2\tau+1}}\|E\|_\rho, \end{aligned} \quad (3.2.39)$$

$$\|\bar{\tilde{N}} - \tilde{N}\|_{\rho-3\delta} \leq \frac{C_{\Delta N^0}\sigma_B + C_{N^0}C_{\Delta B}}{\gamma^2\delta^{2\tau+1}}\|E\|_\rho =: \frac{C_{\Delta \tilde{N}}}{\gamma^2\delta^{2\tau+1}}\|E\|_\rho, \quad (3.2.40)$$

$$\|\bar{\tilde{N}}^\top - \tilde{N}^\top\|_{\rho-3\delta} \leq \frac{\sigma_B C_{\Delta N^{0\top}} + C_{\Delta B}C_{N^{0,\top}}}{\gamma^2\delta^{2\tau+1}}\|E\|_\rho =: \frac{C_{\Delta \tilde{N}^\top}}{\gamma^2\delta^{2\tau+1}}\|E\|_\rho. \quad (3.2.41)$$

Now, we control the matrix $\bar{A}(\theta, \varphi)$ as follows

$$\|\bar{A} - A\|_{\rho-3\delta} \leq \frac{\frac{1}{2}(C_{\Delta \tilde{N}^\top}c_{\Omega,0}C_{\tilde{N}} + C_{\tilde{N}^\top}C_{\Delta\Omega}C_{\tilde{N}}\delta + C_{\tilde{N}^\top}c_{\Omega,0}C_{\Delta \tilde{N}})}{\gamma^2\delta^{2\tau+1}}\|E\|_\rho =: \frac{C_{\Delta A}}{\gamma^2\delta^{2\tau+1}}\|E\|_\rho. \quad (3.2.42)$$

In this instance, we have employed the fact that $B(\theta, \varphi)^\top = B(\theta, \varphi)$, along with the constants specified in equations (3.2.35), (3.2.40), and (3.2.41). Similarly, the corresponding control is maintained for $\|\bar{A}^\top - A^\top\|_{\rho-3\delta}$, as we observe that $A(\theta, \varphi)^\top = -A(\theta, \varphi)$. We notice that $C_{\Delta A} = 0$ in **Case III**.

Then, we control the correction of the adapted normal frame:

$$\begin{aligned}
\|\bar{N} - N\|_{\rho-3\delta} &\leq \frac{C_L C_{\Delta A} + C_{\Delta L} C_A + C_{\Delta \bar{N}}}{\gamma^2 \delta^{2\tau+1}} \|E\|_{\rho} \\
&=: \frac{C_{\Delta N}}{\gamma^2 \delta^{2\tau+1}} \|E\|_{\rho}, \tag{3.2.43}
\end{aligned}$$

$$\begin{aligned}
\|\bar{N}^{\top} - N^{\top}\|_{\rho-3\delta} &\leq \frac{C_A C_{\Delta L^{\top}} + C_{\Delta A} C_{L^{\top}} + C_{\Delta \bar{N}^{\top}}}{\gamma^2 \delta^{2\tau+1}} \|E\|_{\rho} \\
&=: \frac{C_{\Delta N^{\top}}}{\gamma^2 \delta^{2\tau+1}} \|E\|_{\rho}. \tag{3.2.44}
\end{aligned}$$

Finally, we have

$$\begin{aligned}
\|\bar{N}\|_{\rho-3\delta} &\leq \|J \circ \bar{K}\|_{\mathcal{B} \times \mathbb{T}_{\rho}^{\ell}} \|\bar{L}\|_{\rho-3\delta} \|\bar{B}\|_{\rho-3\delta} \leq c_{J,0} C_L \sigma_B \leq C_{\bar{N}}, \\
\|\bar{N}^{\top}\|_{\rho-3\delta} &\leq \|\bar{B}\|_{\rho-3\delta} \|\bar{L}^{\top}\|_{\rho-3\delta} \|(J \circ \bar{K})^{\top}\|_{\mathcal{B} \times \mathbb{T}_{\rho}^{\ell}} \leq C_{\bar{N}^{\top}}, \\
\|\bar{A}\|_{\rho-3\delta} &\leq \frac{1}{2} \|\bar{N}^{\top}\|_{\rho-3\delta} \|\Omega \circ \bar{K}\|_{\mathcal{B} \times \mathbb{T}_{\rho}^{\ell}} \|\bar{N}\|_{\rho-3\delta} \leq C_A, \\
\|\bar{N}\|_{\rho-3\delta} &\leq C_N, \\
\|\bar{N}^{\top}\|_{\rho-3\delta} &\leq C_{N^{\top}}.
\end{aligned}$$

Step 6: Control of the $T_h(K(\theta, \varphi), \varphi)$ operator. We consider the correction $T_h(\bar{K}(\theta, \varphi), \varphi)$ and $T_h(\bar{K}(\theta, \varphi), \varphi)^{\top}$, i.e.,

$$\|\bar{T}_h\|_{\rho} \leq c_{T_h,0}, \tag{3.2.45}$$

$$\|\bar{T}_h^{\top}\|_{\rho} \leq c_{T_h^{\top},0}. \tag{3.2.46}$$

The last two inequalities are satisfied since T_h is a global object defined in $\mathcal{B} \times \mathbb{T}_{\rho}^{\ell}$, hypothesis H_1 and Step 2. On the other hand, the control of the correction to the new $T_h(\theta, \varphi)$,

$$\begin{aligned}
\|T_h \circ \bar{K} - T_h \circ K\|_{\rho-2\delta} &\leq \|D_z T_h\|_{\rho-2\delta} \|\bar{K} - K\|_{\rho-2\delta} \\
&\leq \frac{c_{T_h,1} C_{\Delta K}}{\gamma^2 \delta^{2\tau}} \|E\|_{\rho} =: \frac{C_{\Delta T_h}}{\gamma^2 \delta^{2\tau}} \|E\|. \tag{3.2.47}
\end{aligned}$$

Finally,

$$\|(T_h \circ \bar{K})^{\top} - (T_h \circ K)^{\top}\|_{\rho-2\delta} \leq \frac{c_{T_h^{\top},1} C_{\Delta K}}{\gamma^2 \delta^{2\tau}} \|E\|_{\rho} =: \frac{C_{\Delta T_h^{\top}}}{\gamma^2 \delta^{2\tau}} \|E\|. \tag{3.2.48}$$

Step 7: Control of the new non-degeneracy condition. We now establish that the non-degeneracy (twist) condition with the new torsion matrix is controlled,

$$\begin{aligned}
\|\bar{T} - T\|_{\rho-3\delta} &\leq \frac{1}{2} \frac{C_{\Delta\tilde{N}^\top c_{T_h,0}} C_{\tilde{N}} + C_{\tilde{N}^\top C_{\Delta T_h}} C_{\tilde{N}} \delta + C_{\tilde{N}^\top c_{T_h,0}} C_{\Delta\tilde{N}}}{\gamma^2 \delta^{2\tau+1}} \|E\|_\rho \\
&\quad + \frac{1}{2} \frac{C_{\Delta\tilde{N}^\top c_{T_h^\top,0}} C_{\tilde{N}} + C_{\tilde{N}^\top C_{\Delta T_h^\top}} C_{\tilde{N}} \delta + C_{\tilde{N}^\top c_{T_h^\top,0}} C_{\Delta\tilde{N}}}{\gamma^2 \delta^{2\tau+1}} \|E\|_\rho \\
&\quad + \frac{C_{\Delta\tilde{N}^\top c_{T_h^\top,0}} C_N + C_{\tilde{N}^\top C_{\Delta T_h^\top}} C_N \delta + C_{\tilde{N}^\top c_{T_h^\top,0}} C_{\Delta N}}{\gamma^2 \delta^{2\tau+1}} \|E\|_\rho \\
&\quad + \frac{C_{\Delta N^\top c_{T_h,0}} C_{\tilde{N}} + C_{N^\top C_{\Delta T_h}} C_{\tilde{N}} \delta + C_{N^\top c_{T_h,0}} C_{\Delta\tilde{N}}}{\gamma^2 \delta^{2\tau+1}} \|E\|_\rho \\
&=: \frac{C_{\Delta T}}{\gamma^2 \delta^{2\tau+1}} \|E\|_\rho.
\end{aligned} \tag{3.2.49}$$

Now, we introduce the constant

$$C_{\Delta\langle T \rangle^{-1}} := (\sigma_T)^2 C_{\Delta T},$$

and verify condition (A.1) in Lemma A from [FH24]

$$\frac{(\sigma_T)^2 |\langle \bar{T} \rangle - \langle T \rangle|}{\sigma_T - |\langle T \rangle^{-1}|} \leq \frac{(\sigma_T)^2 \|\bar{T} - T\|_{\rho-3\delta}}{\sigma_T - |\langle T \rangle^{-1}|} \leq \frac{C_{\Delta\langle T \rangle^{-1}}}{\sigma_T - |\langle T \rangle^{-1}|} \frac{\|E\|_\rho}{\gamma^2 \delta^{2\tau+1}} < 1, \tag{3.2.50}$$

where the last inequality follows from Hypothesis (3.2.1) (this corresponds to the sixth term in (3.2.2)). Hence, by invoking Lemma A from [FH24], we conclude that

$$|\langle \bar{T} \rangle^{-1}| < \sigma_T, \quad |\langle \bar{T} \rangle^{-1} - \langle T \rangle^{-1}| \leq \frac{(\sigma_T)^2 C_{\Delta T}}{\gamma^2 \delta^{2\tau+1}} \|E\|_\rho =: \frac{C_{\Delta\langle T \rangle^{-1}}}{\gamma^2 \delta^{2\tau+1}} \|E\|_\rho, \tag{3.2.51}$$

and so, we obtain the estimates (3.2.5) and (3.2.10) on the new object. $\square$

3.3 Convergence of the iterative process

We are ready to prove our KAM theorem for Lagrangian tori. Once we established the quadratic procedure in Section 3.2, proving the convergence of the scheme follows by standard arguments. However, we will carefully explain the required computations as our goal is to present explicit conditions for the KAM theorem.

Proof of Theorem 2.1.1. Let us consider the approximate torus $K_0 := K$ with initial error $E_0 := E$. We also introduce $B_0 := B$ and $T_0 := T$ associated with the initial approximation. By applying Lemma 3.2.1 recursively, we obtain new objects $K_s := \bar{K}_{s-1}$, $E_s := \bar{E}_{s-1}$, $B_s := \bar{B}_{s-1}$ and $T_s := \bar{T}_{s-1}$. At each step, the domain of analyticity of these objects gets reduced. To characterize this, we introduce parameters $a_1 > 1$, $a_2 > 1$, $a_3 := 3 \frac{a_1}{a_1-1} \frac{a_2}{a_2-1}$ and define

$$\rho_0 := \rho, \quad \delta_0 := \frac{\rho_0}{a_3}, \quad \rho_s := \rho_{s-1} - 3\delta_{s-1}, \quad \delta_s := \frac{\delta_0}{a_1^s}, \quad \rho_\infty := \lim_{s \rightarrow \infty} \rho_s = \frac{\rho_0}{a_2}.$$

Assuming that we have applied the Iterative Lemma 3.2.1 s times successfully, we can denote the objects at the s -step of the quasi-Newton method as K_s , E_s , B_s , and T_s . It is important to note that condition (3.2.1) is required at every step, but the construction has been performed in a way that we can control $\|DK_s\|_{\rho_s}$, $\|(DK_s)^\top\|_{\rho_s}$, $\|B_s\|_{\rho_s}$, $\text{dist}(K_s(\mathbb{T}_{\rho_s}^n \times \mathbb{T}_{\rho_s}^\ell), \partial(\mathcal{B} \times \mathbb{T}_\rho^\ell))$, and $|\langle T_s \rangle^{-1}|$ uniformly with respect to s . This means that the constants obtained in Table 4.1 and Table 4.2 are the same for all steps, as we consider the worst value of δ_s , that is, $\delta_0 = \rho_0/a_3$.

The first computation is monitoring the sequence E_s of invariance errors

$$\begin{aligned} \|E_s\|_{\rho_s} &< \frac{C_E}{\gamma^4 \delta_{s-1}^{4\tau}} \|E_{s-1}\|_{\rho_{s-1}}^2 = \frac{C_E a_1^{4\tau(s-1)}}{\gamma^4 \delta_0^{4\tau}} \|E_{s-1}\|_{\rho_{s-1}}^2 \\ &< \left(\frac{a_1^{4\tau} C_E \|E_0\|_{\rho_0}}{\gamma^4 \delta_0^{4\tau}} \right)^{2^s - 1} a_1^{-4\tau s} \|E_0\|_{\rho_0} < a_1^{-4\tau s} \|E_0\|_{\rho_0}, \end{aligned} \quad (3.3.1)$$

where we used the sums $1+2+\dots+2^{s-1} = 2^s - 1$, and $1(s-1)+2(s-2)+2^2(s-3)\dots+2^{s-2}1 = 2^s - s - 1$. Notice that we also used the inequality

$$\frac{a_1^{4\tau} C_E \|E_0\|_{\rho_0}}{\gamma^4 \delta_0^{4\tau}} < 1, \quad (3.3.2)$$

which is included in (2.1.3). From equation (3.3.1), we verify Hypothesis (3.2.1) of the iterative Lemma, enabling us to proceed with step $s+1$. The necessary and sufficient condition will be incorporated into Hypothesis (2.1.3) of the KAM theorem. The second inequality in (3.2.1) reads

$$\frac{C_E \|E_s\|_{\rho_s}}{\gamma^4 \delta_s^{4\tau}} \leq \frac{C_E a_1^{-4\tau s} \|E_0\|_{\rho_0}}{\gamma^4 \delta_s^{4\tau}} = \frac{C_E \|E_0\|_{\rho_0}}{\gamma^4 \delta_0^{4\tau}} \leq a_1^{-4\tau} < 1,$$

where we used (3.3.1) and (3.3.2). The first term in (3.2.2) is guaranteed by performing the computation

$$\frac{2C_{\text{sym}} \|E_s\|_{\rho_s}}{\gamma \delta_s^{\tau+1}} \leq \frac{2C_{\text{sym}} a_1^{s(\tau+1)} a_1^{-4\tau s} \|E_0\|_{\rho_0}}{\gamma \delta_0^{\tau+1}} \leq \frac{2C_{\text{sym}} \|E_0\|_{\rho_0}}{\gamma \delta_0^{\tau+1}} < 1, \quad (3.3.3)$$

where we used (3.3.1), the fact that $1 - 3\tau < 0$, and we have included the last inequality in (2.1.3). The remaining conditions are similar. It is important to note that at the s -th step, the objects involved are DK_s , $(DK_s)^\top$, B_s and $\langle T_s \rangle^{-1}$. Therefore, it is necessary to establish a relationship between these conditions and the initial objects DK_0 , $(DK_0)^\top$, B_0 and $\langle T_0 \rangle^{-1}$. For example, the second term in (3.2.2) reads

$$\left(\frac{nC_{\Delta K}}{\sigma_L - \|DK_s\|_{\rho_s}} \right) \frac{\|E_s\|_{\rho_s}}{\gamma^2 \delta_s^{2\tau+1}} < 1,$$

and it is verified by conducting the following calculations

$$\begin{aligned}
\|DK_s\|_{\rho_s} + \frac{nC_{\Delta K}\|E_s\|_{\rho_s}}{\gamma^2\delta_s^{2\tau+1}} &< \|DK_0\|_{\rho_0} + \sum_{j=0}^s \frac{nC_{\Delta K}\|E_j\|_{\rho_j}}{\gamma^2\delta_j^{2\tau+1}} \\
&< \|DK_0\|_{\rho_0} + \sum_{j=0}^{\infty} \frac{nC_{\Delta K}a_1^{(1-2\tau)j}}{\gamma^2\delta_0^{2\tau+1}} \|E_0\|_{\rho_0} \\
&= \|DK_0\|_{\rho_0} + \frac{nC_{\Delta K}\|E_0\|_{\rho_0}}{\gamma^2\delta_0^{2\tau+1}} \left(\frac{1}{1-a_1^{1-2\tau}} \right) < \sigma_L. \tag{3.3.4}
\end{aligned}$$

The last inequality is incorporated in equation (2.1.3). After reproducing the same computations, we obtain the third, fourth, and fifth terms in (3.2.2), related to $(DK_s)^\top$, B_s and $\langle T_s \rangle^{-1}$. Finally, the sixth term in (3.2.2) is verified as follows

$$\begin{aligned}
\text{dist}(K_s(\mathbb{T}_{\rho_s}^n \times \mathbb{T}_{\rho_s}^\ell), \partial(\mathcal{B} \times \mathbb{T}_\rho^\ell)) - \frac{C_{\Delta K}}{\gamma^2\delta_s^{2\tau}} \|E_s\|_{\rho_s} &> \text{dist}(K_0(\mathbb{T}_{\rho_0}^n \times \mathbb{T}_{\rho_0}^\ell), \partial(\mathcal{B} \times \mathbb{T}_\rho^\ell)) \\
&\quad - \sum_{j=0}^{\infty} \frac{C_{\Delta K}a_1^{-2\tau j} \|E_0\|_{\rho_0}}{\gamma^2\delta_0^{2\tau}} \\
&= \text{dist}(K_0(\mathbb{T}_{\rho_0}^n \times \mathbb{T}_{\rho_0}^\ell), \partial(\mathcal{B} \times \mathbb{T}_\rho^\ell)) \\
&\quad - \frac{C_{\Delta K}\|E_0\|_{\rho_0}}{\gamma^2\delta_0^{2\tau}} \frac{1}{1-a_1^{-2\tau}} > 0, \tag{3.3.5}
\end{aligned}$$

the final inequality is included into (2.1.3).

Having guaranteed all hypotheses of Lemma 3.2.1, we collect the inequalities (3.3.2), (3.3.3), (3.3.4) and (3.3.5) that are included into hypothesis (2.1.3). This follows by introducing the constant $\mathfrak{C}_1$ as

$$\mathfrak{C}_1 := \max \{ (a_1 a_3)^{4\tau} C_E, (a_3)^{2\tau+1} \gamma^2 \rho^{2\tau-1} C_\Delta \} \tag{3.3.6}$$

where

$$\begin{aligned}
C_{\Delta,1} := \max \left\{ \frac{nC_{\Delta K}}{\sigma_L - \|DK\|_\rho}, \frac{2nC_{\Delta K}}{\sigma_{L^\top} - \|(DK)^\top\|_\rho}, \right. \\
\left. \frac{C_{\Delta B}}{\sigma_B - \|B\|_\rho}, \frac{C_{\Delta \langle T \rangle^{-1}}}{\sigma_T - |\langle T \rangle^{-1}|} \right\}, \tag{3.3.7}
\end{aligned}$$

$$C_{\Delta,2} := \frac{C_{\Delta K}\delta}{\text{dist}(K(\mathbb{T}_\rho^d), \partial\mathcal{B})}, \tag{3.3.8}$$

$$C_\Delta := \max \left\{ C_{\text{sym}} \gamma \delta^\tau, \frac{C_{\Delta,1}}{1-a_1^{1-2\tau}}, \frac{C_{\Delta,2}}{1-a_1^{-2\tau}} \right\}. \tag{3.3.9}$$

It is worth highlighting that we have reverted to the original notation, where $K = K_0$, $B = B_0$, $T = T_0$, $\rho = \rho_0$ and $\delta = \delta_0$ for the new initial objects.

Through induction, we can repeat the iterative process an infinite number of times. To clarify,

we have:

$$\|E_s\|_{\rho_s} < a_1^{-4\tau s} \|E\|_{\rho} \longrightarrow 0 \quad \text{when } s \rightarrow \infty.$$

The iterative scheme leads to the convergence of a true quasi-periodic torus, denoted by K_{∞} . As a result of the output of Lemma 3.2.1, this object satisfies $K_{\infty} \in \mathcal{A}(\mathbb{T}_{\rho_{\infty}}^n \times \mathbb{T}_{\rho_{\infty}}^{\ell})$ and

$$\|DK_{\infty}\|_{\rho_{\infty}} < \sigma_L, \quad \|(DK_{\infty})^{\top}\|_{\rho_{\infty}} < \sigma_{L^{\top}}, \quad \text{dist}(K_{\infty}(\mathbb{T}^n \times \mathbb{T}^{\ell}), \partial(\mathcal{B} \times \mathbb{T}_{\rho}^{\ell})) > 0.$$

To control the limiting objects, we repeat the computations in (3.3.5) as follows

$$\|K_{\infty} - K\|_{\rho_{\infty}} \leq \sum_{j=0}^{\infty} \|K_{j+1} - K_j\|_{\rho_{j+1}} < \frac{C_{\Delta K} \|E\|_{\rho}}{\gamma^2 \delta^{2\tau}} \frac{1}{1 - a_1^{-2\tau}} =: \frac{\mathfrak{C}_2 \|E\|_{\rho}}{\gamma^2 \rho^{2\tau}}, \quad (3.3.10)$$

thus obtaining the estimates in (2.1.4). This completes the proof of the KAM theorem to Lagrangian Tori. $\square$

Applications and Computational Algorithms of the KAM Theorem

In this Chapter, we present an algorithm to compute a fiberwise Lagrangian torus in quasi-periodic Hamiltonian systems, whose convergence is proved in the [CHP25a]. We exhibit the algorithm with two models. The first is a Tokamak model [CVC⁺05, VL21], which proposes a control method to create barriers to the diffusion of magnetic field lines through a small modification in the magnetic perturbation. The second model [dCN00], known as the vorticity defect model, describes the non-linear evolution of localized vorticity perturbations in a constant vorticity flow. This model was originally derived in the context of plasma physics and fluid dynamics.

4.1 KAM algorithms for Langrangian invariant tori

The constructuon of quasi-Newton method in [CHP25a] suggests to a numerical method. The approach involves starting with an approximately invariant torus, $K(\theta, \varphi)$, meaning one with a non-zero invariance error E . The method is as follows. We obtain a new parameterization of the torus as

$$\bar{K}(\theta, \varphi) = K(\theta, \varphi) + \Delta K(\theta, \varphi).$$

The new invariance error is given by

$$\bar{E}(\theta, \varphi) = Z_H(\bar{K}(\theta, \varphi), \varphi) + \mathfrak{L}_{\omega, \alpha} \bar{K}(\theta, \varphi).$$

Expanding last equation, we have,

$$\bar{E}(\theta, \varphi) = D_z Z_H(K(\theta, \varphi), \varphi) \Delta K(\theta, \varphi) + \mathfrak{L}_{\omega, \alpha} \bar{K}(\theta, \varphi) + E(\theta, \varphi) + \Delta_\theta^2 Z(\theta, \varphi).$$

To determine ΔK , we retain only terms up to first order, and solve the following equation,

$$D_z Z_H(K(\theta, \varphi), \varphi) \Delta K(\theta, \varphi) + \mathfrak{L}_{\omega, \alpha} \Delta K(\theta, \varphi) = -E(\theta, \varphi).$$

To tackle this equation, we use the approximately symplectic framework represented by

$$P(\theta, \varphi) = (L(\theta, \varphi) N(\theta, \varphi)) ,$$

such that, for some $\xi : \mathbb{T}^n \times \mathbb{T}^\ell \rightarrow \mathbb{R}^{2n}$

$$\Delta K(\theta, \varphi) = P(\theta, \varphi) \xi(\theta, \varphi) ,$$

i.e., $\xi(\theta, \varphi)$ is the new unknown.

Leveraging certain geometric properties, we obtain,

$$\begin{aligned} (\Lambda(\theta, \varphi) + E_{red}(\theta, \varphi)) \xi(\theta, \varphi) + (I_{2n} - \Omega_0 E_{sym}(\theta, \varphi)) \mathfrak{L}_{\omega, \alpha} \xi(\theta, \varphi) \\ = \Omega_0 P(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) E(\theta, \varphi) , \end{aligned}$$

$$\Lambda(\theta, \varphi) = \begin{pmatrix} O_n & T(\theta, \varphi) \\ O_n & O_n \end{pmatrix} . \quad (4.1.1)$$

Neglecting quadratic terms, the solution to the above equation is approximated by solving a triangular system that requires handling cohomological equations,

$$\begin{aligned} \Lambda(\theta, \varphi) \xi(\theta, \varphi) + \mathfrak{L}_{\omega, \alpha} \xi(\theta, \varphi) \\ = \Omega_0 P(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) E(\theta, \varphi) \\ = \begin{pmatrix} -N(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) E(\theta, \varphi) \\ L(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) E(\theta, \varphi) \end{pmatrix} =: \begin{pmatrix} \eta^L(\theta, \varphi) \\ \eta^N(\theta, \varphi) \end{pmatrix} . \end{aligned} \quad (4.1.2)$$

To solve these equations, $\det \langle T \rangle^{-1}$ must be different from zero, where,

$$\xi^N(\theta, \varphi) = \xi_{0,0}^N + \mathfrak{R}_{\omega, \alpha}(\eta^N(\theta, \varphi)) , \quad (4.1.3)$$

$$\xi^L(\theta, \varphi) = \xi_{0,0}^L + \mathfrak{R}_{\omega, \alpha}(\eta^L(\theta, \varphi) - T(\theta, \varphi) \xi^N(\theta, \varphi)) , \quad (4.1.4)$$

with

$$\xi_{0,0}^N = \langle T \rangle^{-1} \langle \eta^L - T \mathfrak{R}_{\omega, \alpha}(\eta^N) \rangle .$$

Remark 4.1.1. In the examples presented in this work, the symplectic form is canonical, therefore, $\Omega = \Omega_0$, $G = I_{2n}$, and $J = \Omega_0$. Hence, we have $c_{\Omega,0} = 1$, $c_{\Omega,1} = 0$, $c_{G,0} = 1$, $c_{G,1} = 0$, $c_{G,2} = 0$, $c_{J,0} = 1$, $c_{J,1} = 0$, $c_{J,2} = 0$, $c_{J^\top,0} = 1$, and $c_{J^\top,1} = 0$. So the torsion is

$$T(\theta, \varphi) = N(\theta, \varphi)^\top T_h(K(\theta, \varphi), \varphi) N(\theta, \varphi) . \quad (4.1.5)$$

with

$$T_h(K(\theta, \varphi), \varphi) = \Omega_0 D_z Z_h(K(\theta, \varphi), \varphi) - D_z Z_h(K(\theta, \varphi), \varphi) \Omega_0 . \quad (4.1.6)$$

In terms of the Hamiltonian, the torsion is expressed as follows

$$T(\theta, \varphi) = B(\theta, \varphi)^{-1} L(\theta, \varphi)^\top S_h(K(\theta, \varphi), \varphi) L(\theta, \varphi) B(\theta, \varphi)^{-1}. \quad (4.1.7)$$

with

$$S_h(K(\theta, \varphi), \varphi) = \begin{pmatrix} (D_{yy} - D_{xx}) h(K(\theta, \varphi), \varphi) & -2D_{xy} h(K(\theta, \varphi), \varphi) \\ -2D_{xy} h(K(\theta, \varphi), \varphi) & (D_{xx} - D_{yy}) h(K(\theta, \varphi), \varphi) \end{pmatrix}. \quad (4.1.8)$$

If the torus is approximately invariant. Therefore, the algorithm is following:

Algorithm 4.1.2 (Computation of the Adapted Frame and Torsion). *Let K satisfy the equation approximately. Compute the adapted frame P and the reduced dynamics by following these steps:*

1. Derive $L(\theta, \varphi)$ using (2.1.1).
2. Obtain $B(\theta, \varphi)$ from (1.4.12).
3. Calculate $A(\theta, \varphi)$ based on (1.4.13).
4. Evaluate $\tilde{N}(\theta, \varphi)$ according to (1.4.10).
5. Determine $N(\theta, \varphi)$ from (1.4.9).
6. Compute $T_h(\theta, \varphi)$ from (1.4.15).
7. Derive $T(\theta, \varphi)$ based on (1.4.14).
8. Construct the frame $P(\theta, \varphi)$ using equation (1.3.3).

Algorithm 4.1.3 (Correction of the Generating Torus). *Let K approximately satisfy the equation. Obtain the corrected generating torus by following these steps:*

1. Compute $P(\theta, \varphi)$ using Algorithm 4.1.2.
2. Derive $E(\theta, \varphi)$ using (1.2.7).
3. Evaluate $\eta(\theta, \varphi)$, the right-hand side of (3.1.5).
4. Solve the cohomological equation for ξ^N and ξ^L , from (4.1.3) and (4.1.4) respectively, verifying $\langle \eta^N \rangle = 0$.
5. Improve $K(\theta, \varphi) \leftarrow K(\theta, \varphi) + P(\theta, \varphi) \xi(\theta, \varphi)$.

4.1.1 Numerical method for torus continuation

Due to the nature of the method, an initial torus is required. Therefore, if the system depends on a family of parameters, we can use a previously computed torus for the next value of the parameter, meaning that we can perform a continuation with respect to the parameter. In the following, we describe a method for performing the continuation by considering the variational equation in the parameter. We know that the torus depends on the parameter, i.e., $K_\varepsilon(\theta, \varphi)$, the first variational derivative in ε is

$$K_{\varepsilon+\delta_\varepsilon}(\theta, \varphi) = K_\varepsilon(\theta, \varphi) + \frac{\partial}{\partial \varepsilon} K_\varepsilon(\theta, \varphi) \delta_\varepsilon + \mathcal{O}(\delta_\varepsilon^2). \quad (4.1.9)$$

Therefore, we need to calculate $\frac{\partial}{\partial \varepsilon} K_\varepsilon(\theta, \varphi)$, invoking the invariance equation (1.2.6) and defined $\Delta K_\varepsilon(\theta, \varphi) := \frac{\partial}{\partial \varepsilon} K_\varepsilon(\theta, \varphi)$, we obtain,

$$D_z Z_h(K_\varepsilon(\theta, \varphi), \varphi) \Delta K_\varepsilon(\theta, \varphi) + \mathfrak{L}_{\omega, \alpha}(\Delta K_\varepsilon(\theta, \varphi)) = -\frac{\partial}{\partial \varepsilon} Z_h(K_\varepsilon(\theta, \varphi), \varphi), \quad (4.1.10)$$

we proceed in a manner analogous to the *numerical method*, that is, we introduce a new $\xi_\varepsilon : \mathbb{T}^n \times \mathbb{T}^\ell \rightarrow \mathbb{R}^{2n}$, such that $\Delta K_\varepsilon(\theta, \varphi) = P(\theta, \varphi) \xi_\varepsilon(\theta, \varphi)$, to obtain

$$\begin{aligned} \Lambda(\theta, \varphi) \xi_\varepsilon(\theta, \varphi) + \mathfrak{L}_{\omega, \alpha} \xi_\varepsilon(\theta, \varphi) \\ = \Omega_0 P(\theta, \varphi)^\top \Omega(K_\varepsilon(\theta, \varphi)) \frac{\partial}{\partial \varepsilon} Z_{h_\varepsilon}(K_\varepsilon(\theta, \varphi), \varphi) \\ = \begin{pmatrix} -N(\theta, \varphi)^\top \Omega(K_\varepsilon(\theta, \varphi)) \frac{\partial}{\partial \varepsilon} Z_{h_\varepsilon}(K_\varepsilon(\theta, \varphi), \varphi) \\ L(\theta, \varphi)^\top \Omega(K_\varepsilon(\theta, \varphi)) \frac{\partial}{\partial \varepsilon} Z_{h_\varepsilon}(K_\varepsilon(\theta, \varphi), \varphi) \end{pmatrix} =: \begin{pmatrix} \eta_\varepsilon^L(\theta, \varphi) \\ \eta_\varepsilon^N(\theta, \varphi) \end{pmatrix}. \end{aligned} \quad (4.1.11)$$

Therefore, to find ΔK_ε , it is enough to solve a cohomological equation and $\langle \eta_\varepsilon^N \rangle = O_n$. For the latter, we have that

$$\begin{aligned} \eta_\varepsilon^N(\theta, \varphi) &= L(\theta, \varphi)^\top \Omega(K_\varepsilon(\theta, \varphi)) \frac{\partial Z_{h_\varepsilon}}{\partial \varepsilon}(K_\varepsilon(\theta, \varphi), \varphi) \\ &= \left(D_\theta \left(\frac{\partial h_\varepsilon}{\partial \varepsilon}(K_\varepsilon(\theta, \varphi), \varphi) \right) \right)^\top. \end{aligned} \quad (4.1.12)$$

Hence, η_ε^N is the derivative of periodic functions, and we obtain $\langle \eta_\varepsilon^N \rangle = O_n$.

Algorithm 4.1.4 (Computation of the First Order Correction). *Let $K_\varepsilon(\theta, \varphi)$ satisfy the invariance equation approximately. To perform the continuation with respect to ε , compute the first variation ΔK_ε by following these steps:*

1. Calculate the derivative of $Z_{h_\varepsilon}(K(\theta, \varphi), \varphi)$ with respect to ε .
2. Evaluate $\eta_\varepsilon(\theta, \varphi)$, the right-hand side of (4.1.12).
3. Solve the cohomological equation for ξ_ε^N and ξ_ε^L .
4. Compute $P(\theta, \varphi)$ using Algorithm 4.1.2.

5. Improve $K_{\varepsilon+\delta\varepsilon}(\theta, \varphi) \leftarrow K_\varepsilon(\theta, \varphi) + P(\theta, \varphi)\xi_\varepsilon(\theta, \varphi)$.
6. Save the $K_{\varepsilon+\delta\varepsilon}(\theta, \varphi)$ to be used in the continuation step.

Algorithm 4.1.5 (Continuation of the Torus family). *Let K approximately satisfy the invariance equation. Follow these steps to do a continuation respect to ε :*

1. Construct an approximate invariant parametrization.
2. Improve the parametrization, $K_\varepsilon(\theta, \varphi)$, using Algorithm 4.1.3.
3. Save the updated torus $K_\varepsilon(\theta, \varphi)$.
4. Define the continuation step δ .
5. Compute $K_{\varepsilon+\delta\varepsilon}(\theta, \varphi)$ using Algorithm 4.1.4.
6. Repeat from step (2), until the desired value of ε is attained.

Remark 4.1.6. The algorithms require to perform basic operations $(+, -, /, *)$, compositions, derivatives, and computing the $\mathfrak{R}_{\omega, \alpha}$ operator. They do not require numerical integrations as in [FMHM24a] and can be applied as long as the family of tori exists given we have enough computational resources available.

Remark 4.1.7. Another approach to performing continuation with respect to parameters is the pseudo-arclength continuation method, as described in [GHdL22]. This work also provides a detailed implementation of the predictor used in the natural continuation method.

4.1.2 Comments on the implementations

In this section, we present the numerical implementation of the algorithms derived in the previous sections. While the theorem provides a rigorous mathematical foundation, its proof leads to an algorithm whose numerical implementation requires careful consideration of discretization differentiation, equation-solving, and other numerical tools. To achieve this, we have defined three key objects: **Matrix**, **Grid**, and **Fourier**, as in [HCF⁺16] each playing a distinct role in the overall implementation.

Numerically, for the parameterization, we consider a set of sample points on a regular grid of size $N_\theta = (N_{\theta,1}, \dots, N_{\theta,n}) \in \mathbb{N}^n$ and $N_\varphi = (N_{\varphi,1}, \dots, N_{\varphi,\ell}) \in \mathbb{N}^\ell$, such that,

$$\begin{aligned}\theta_i &:= (\theta_{i_1}, \dots, \theta_{i_n}) = \left(\frac{i_1}{N_{\theta,1}}, \dots, \frac{i_n}{N_{\theta,n}} \right), \\ \varphi_j &:= (\varphi_{j_1}, \dots, \varphi_{j_\ell}) = \left(\frac{j_1}{N_{\varphi,1}}, \dots, \frac{j_\ell}{N_{\varphi,\ell}} \right),\end{aligned}$$

where $i = (i_1, \dots, i_n)$, $j = (j_1, \dots, j_\ell)$, with $0 \leq i_r \leq N_{\theta,r}$, $0 \leq j_s \leq N_{\varphi,s}$, further $1 \leq r \leq n$ and $1 \leq s \leq \ell$. The total number of points is $N_D = N_{\theta,1} \cdots N_{\theta,n} N_{\varphi,1} \cdots N_{\varphi,\ell}$, where $N_{\theta,i} = 2^{q_i}$, $N_{\varphi,j} = 2^{q_j}$, with $q_i \in \mathbb{N}$, $q_j \in \mathbb{N}$ for $i = 1, \dots, n$ and $j = 1, \dots, \ell$.

We use the **Matrix** object, for functions, like; $K, E, \eta, \xi : \mathbb{T}^n \times \mathbb{T}^\ell \rightarrow \mathbb{R}^{2n}$ or like; $L, N, T : \mathbb{T}^n \times \mathbb{T}^\ell \rightarrow \mathbb{R}^{2n}$. We can use the **Grid** or **Fourier** objects for function components like; $K_x, K_y : \mathbb{T}^n \times \mathbb{T}^\ell \rightarrow \mathbb{R}^n$ or $\xi^N, \xi^L : \mathbb{T}^n \times \mathbb{T}^\ell \rightarrow \mathbb{R}^n$. The **Grid** object is useful and efficient to do multiplications, sum, etc, or evaluating the vector field, and we employ the **Fourier** object to calculate the derivative, or solving cohomological equations. We switch between both representations with the aid of the Fast Fourier Transform (FFT), which has a complexity of $\mathcal{O}(N_D \log(N_D))$.

One advantage of defining the torsion as in (1.4.14), see [CHP25a] for details, instead of $T(\theta, \varphi) = N(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) (D_z Z(K(\theta, \varphi), \varphi) + \mathfrak{L}_{\omega, \alpha} K(\theta, \varphi))$ is that we avoid the need to compute derivatives in Fourier space, as [HL19]. In other words, we do not generate a **Fourier** object, which would require additional memory and computational time for the derivative calculation. Instead, it suffices to perform grid-based multiplications.

4.2 Numerical study of KAM Theory in a Tokamak model

In this section, we apply the KAM algorithm to a Tokamak model with one and a half degrees of freedom, following the background established in [CVC⁺05, VL21]. As in, [Abd06] we calculate the Poincaré section in the (ψ, θ) plane. From this section, and using Birkhoff averages, we obtain the internal frequency with the methods developed in [DSSY17]. With the internal frequency and the Poincaré section, we impose the invariance equation to construct a two-dimensional approximately invariant torus with an external frequency equal to 1. This torus serves as the initial guess for the method, to which we apply the KAM procedure.

Once the method converges, we perform a continuation with respect to the parameter and observe the behavior of some Sobolev norms. This provides estimates for the breakdown of analyticity of the invariant torus.

4.2.1 A Tokamak model

Tokamaks are devices for confining plasma in nuclear fusion, see [Art72]. They exhibit complicated dynamics that can be described using periodic non-autonomous Hamiltonian systems. In these systems, charged particles move along paths determined by magnetic fields, where the topological geometry is essential for maintaining plasma stability. KAM theory offers a theoretical framework for understanding how quasi-periodic orbits persist despite disturbances, thus contributing to the stability of invariant tori in phase space and controlling the plasma.

Reference [Abd06] presents a non-autonomous Hamiltonian system that models the magnetic field lines of a Tokamak for plasma confinement, similar to equation (4.2.1), but without the term ε^2 . Where, φ represents the toroidal angle acting as the time, ψ denotes the normalized toroidal flux, and H stands for the poloidal flux. The poloidal angle θ is the conjugate variable to the action H , as depicted in figure 4.1.

In [CVC⁺05], the authors apply, to the same model, a method they developed, introducing a control technique aimed at creating barriers between two chaotic regions by employing a control

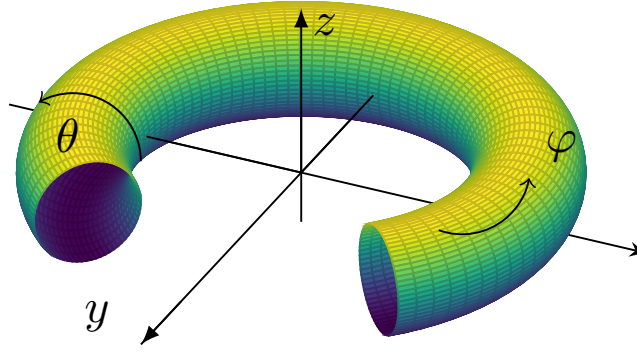


Figure 4.1: Where θ and φ are poloidal and toroidal angles of a system of coordinates.

term proportional to ε^2 , as depicted in figure (4.2). We can observe that in [VL21], the authors use the same model and, through frequency analysis combined with a rigorous (Computer-Assisted), demonstrate that in the phase space of the magnetic field, the control term generates a set of invariant tori that act as transport barriers.

$$\begin{aligned}
 H(\theta, \psi; \varphi) &= \int \frac{d\psi}{q(\psi)} + \varepsilon H_1(\theta, \varphi) + \varepsilon^2 f_2(\theta, \varphi), \\
 q(\psi) &= \frac{4}{(2 - \psi)(2 - 2\psi + \psi^2)}, \\
 H_1(\theta, \varphi) &= \cos(2\theta - \varphi) + \cos(3\theta - 2\varphi), \\
 f_2(\theta, \varphi) &= \left(-\frac{1}{2} \frac{d}{d\psi} \left(\frac{1}{q(\psi)} \right) \right) \bigg|_{\psi_0} \left(\frac{2 \cos(2\theta - \varphi)}{2w - 1} + \frac{3 \cos(3\theta - 2\varphi)}{3w - 2} \right)^2, \\
 w &= \frac{1}{q(\psi_0)}, \\
 q_0 &= 0.35,
 \end{aligned} \tag{4.2.1}$$

where ψ_0 is the initial normalized toroidal flux on the Poincare section.

Remark 4.2.1. In this case, the dimension are $n_1 = 1, n_2 = 1, \ell = 1$, and the degree map is $\mathfrak{D} = 1$. Then, the torus takes the form

$$\tilde{K}(\theta, \varphi) = \begin{pmatrix} K(\theta, \varphi) \\ \varphi \end{pmatrix} = \begin{pmatrix} \theta \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} K_x^p(\theta, \varphi) \\ K_y^p(\theta, \varphi) \\ \varphi \end{pmatrix}.$$

4.2.2 Invariant torus

As previously mentioned in this work and in [CHP25a], the method requires the construction of an initial torus with a small invariance error. To achieve this, we calculate numerically the stroboscopic map, by integrating with Newton's method, using the Julia package [PHB21] developed by Luis Benet and Jorge Pérez, figure (4.2), from the model described in equation (4.2.1), corresponding to $\varepsilon = 0.004$. To compute the internal frequency $\omega \approx 0.57981245427252670451$, we used Birkhoff

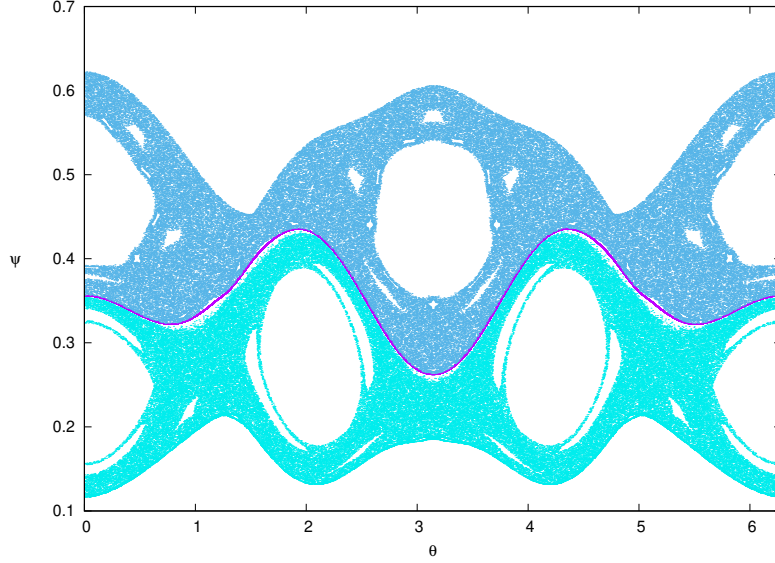


Figure 4.2: The colors Cyan, Magenta and Teal correspond to $\varphi = 0.35, 0.3566248878338341665, 0.36$, respectively.

averages on stroboscopic map, following [DSSY17]. The external frequency α , corresponding to the time-dependent part, is 1. On the other hand, we construct the two-dimensional initial torus from this curve, corresponding to $K(\theta, 0)$, denoted by $K_0(\theta)$. To do so, we construct the initial guess, we define $(\bar{\theta}_i, \bar{\varphi}_j)$ such that, $\theta_i = \bar{\theta}_i + \omega t$ and $\varphi_j = \bar{\varphi}_j + \alpha t$, and we impose that $K(\bar{\theta}_i, \bar{\varphi}_j)$ parametrized an invariant torus, i.e., the invariance equation is satisfied, equation (1.2.3), such that,

$$\Phi_t(K(\theta_i - \omega t, \varphi_j - \alpha t), \varphi_j - \alpha t) = K(\theta_i, \varphi_j),$$

since we want to construct the torus from $K_0(\theta)$, we impose $\varphi_j - \alpha t = 0$, with $t^* = \frac{\varphi_{j1}}{\alpha_1}$ such that solve the last equation, so we obtain,

$$\Phi_{t^*}(K_0(\theta_i - \omega t^*), 0) = K(\theta_i, \varphi_j). \quad (4.2.2)$$

In other words, to obtain $K(\theta_i, \varphi_j)$ it is enough to integrate the flow for a time t^* with initial condition $K_0(\theta_i - \omega t^*)$. To evaluate the initial condition required to satisfy (4.2.2), we evaluate K_0 using splines.

We use a regular grid of points $N_\theta = (2^{n_\theta})$ and $N_\varphi = (2^{n_\varphi})$, with

$$(\theta_i, \varphi_j) = (\theta_{i_1}, \varphi_{j_1}) = \left(\frac{i_1}{2^{n_\theta}}, \frac{j_1}{2^{n_\varphi}} \right),$$

with $1 \leq i_1 \leq 2^{n_\theta}$ and $1 \leq j_1 \leq 2^{n_\varphi}$. Once we construct the invariant torus, with an initial invariance error approximate of 9.6866×10^{-2} and $2^9 \times 2^9$, Fourier coefficients, it will serve as the initial guess of the quasi-Newton method. We obtained an approximately invariant torus with an invariance error of 3.96005×10^{-16} , figures (4.3) and (4.4), and $2^{12} \times 2^{11}$ Fourier coefficients in the θ and φ directions.

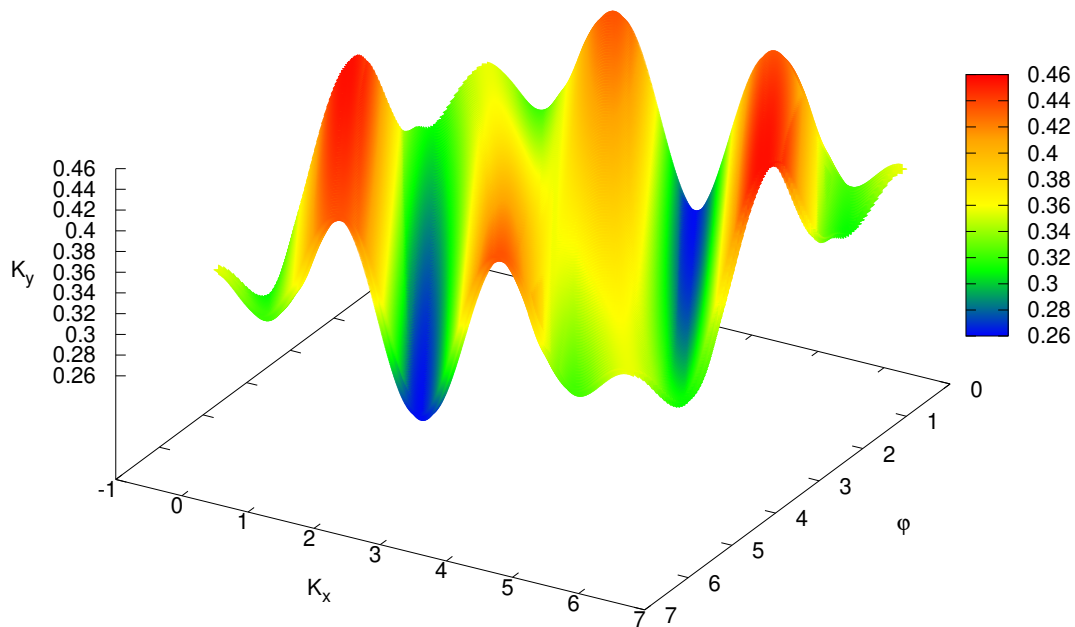


Figure 4.3: Parametrized surface of the invariant torus, displaying the components $K_x(\theta, \varphi)$, $K_y(\theta, \varphi)$, and φ , represented with 2^{23} Fourier modes and an invariance error equal to 3.96005×10^{-16} . Where blue represents the lowest height and red the highest, based on the $K_y(\theta, \varphi)$ component.

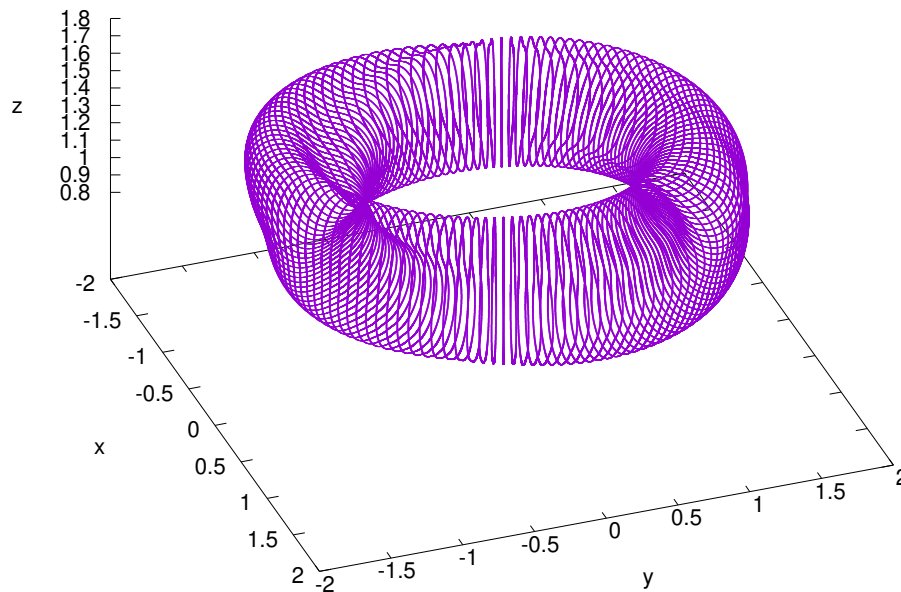


Figure 4.4: Invariant torus in Cartesian coordinates.

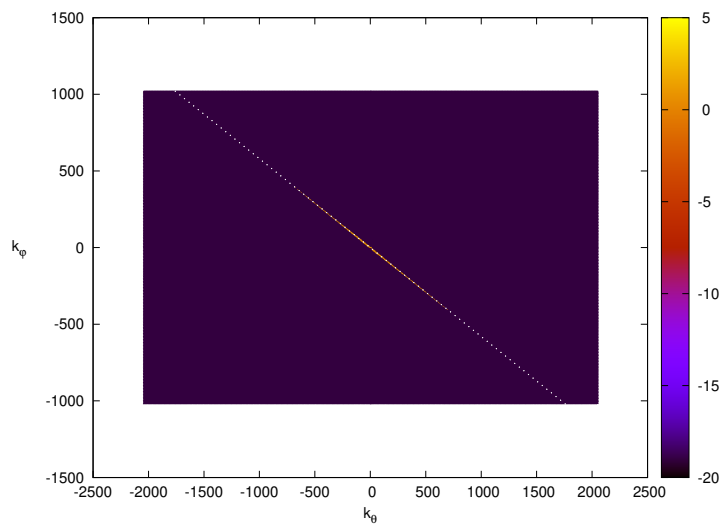


Figure 4.5: Norm of the Fourier coefficients on a logarithmic scale. The coefficients with the largest contributions, the 'significant' ones, align with the straight line shown in yellow in the image, which fits the equation $y = -0.58013x$.

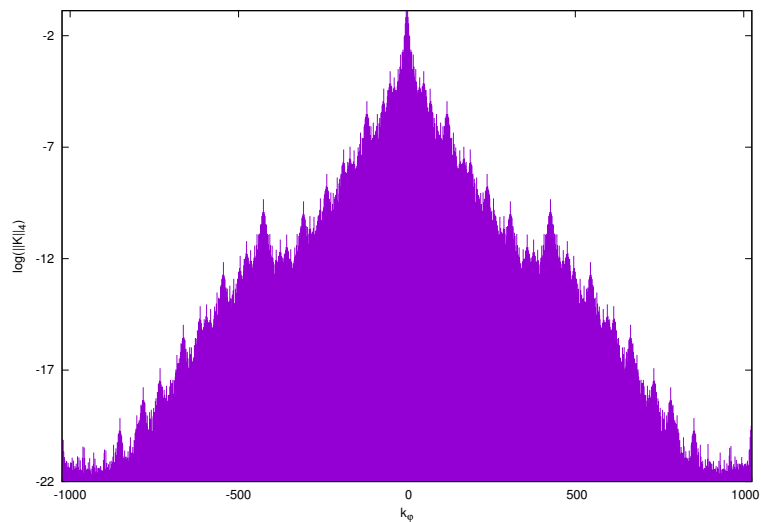


Figure 4.6: For each $k_φ$, we take the logarithm of the maximum of the H^4 Sobolev norm, equation (4.2.3), of the Fourier coefficients for the torus with $\varepsilon = 0.004$.

4.2.3 Continuation of the invariant torus

As we saw in Section 4.2.2, it is possible to apply the method without the need to perform a continuation over the perturbations starting from the integrable system. However, the strength of our approach lies in its ability to also perform a continuation over the perturbation parameter, either starting from the integrable system or from one already computed, as in Section 4.2.2. To achieve this, we employ Algorithm 4.1.5, using the case $\varepsilon = 0.004$ as the initial guess. The construction of the two-dimensional object follows an analogous process to that described in Section 4.1.1. By performing the continuation as outlined in Algorithm 4.1.4, i.e., with a prediction for the following torus, we can compute the subsequent torus in four iterations of the Quasi-Newton step, compared to the six iterations required when improving the prediction.

While this work aims to exemplify the suggested algorithms from [CHP25a], we also seek to show the method's effectiveness by estimating the breakdown. We, now analyze the behavior of the of the H^4 Sobolev norm of K defined as

$$\|K\|_4 := \sqrt{\|K_x^p\|_4^2 + \|K_y^p\|_4^2}. \quad (4.2.3)$$

We observe, that in figure 4.7 Sobolev norm has a exponential behavior [CDIL10], we obtain an estimate for the blow-up exponent (see equation (12) in [CL10]) in figure 4.8. The fit to estimate the exponent also shows an oscillatory behavior in the Sobolev norm, figure 4.8. These oscillations are not predicted by renormalization group, which is similar to the spin-orbit problem of Celestial Mechanics, [CCGdIL24].

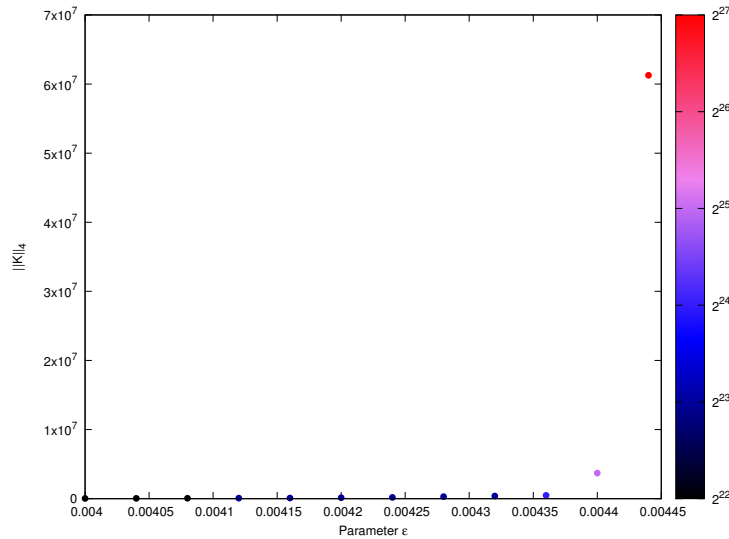


Figure 4.7: We observed that near the parameter value $\varepsilon = 0.00445$, there is an exponential growth, indicating signs of the breakdown of the torus.

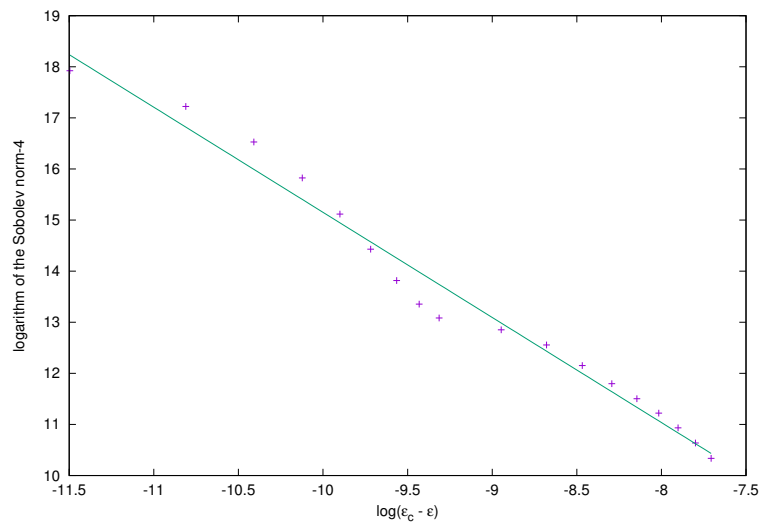


Figure 4.8: We observe that it aligns with a linear fit $y = -2.35233844056405x - 7.75624914299206$, with a correlation coefficient of $r = -0.987679863894756$ and a critical $\varepsilon_c = 0.0044615$.

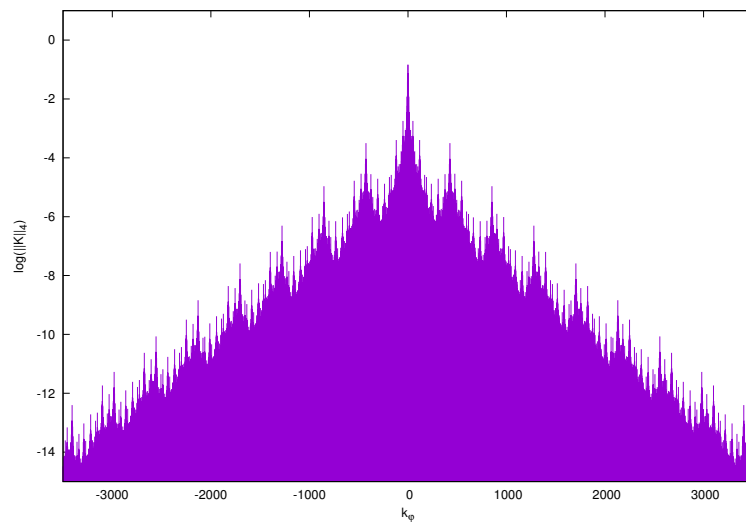


Figure 4.9: For each k_φ , we take the logarithm of the maximum of the H4-norm, equation (4.2.3), of the Fourier coefficients for the torus with $\varepsilon = 0.00445$.

4.3 Numerical study of KAM Theory in fluid and plasma transport models

In this section, we present the following model of a quasi-periodically forced pendulum with two driving frequencies,

$$h = \frac{p^2}{2} + \varepsilon_1 \cos(q) - (\varepsilon_2 + \varepsilon_3 \cos(\alpha_1 t)) \cos(q - \alpha_2 t), \quad (4.3.1)$$

where, $\varepsilon_1, \varepsilon_2$ and ε_3 are parameters, with $\alpha_1 = \sqrt{3}$, and $\alpha_2 = \frac{1+\sqrt{5}}{2}$. This pendulum model is motivated by the work in [dCN00]. The main objective of this section is to demonstrate that, using the KAM algorithm, it is possible to perform a continuation with respect to parameters in the pendulum. To illustrate this, we present three distinct examples.

1. The *single-wave model* simplifies the vorticity mixing problem, ultimately leading to the stream function. A comprehensive derivation is available in References [dCN98b, dCN98a], which provide a detailed analysis for interested readers. The continuation is carried out by initially constructing a planar pendulum as an initial guess and then gradually adapting it to the solitary wave model, i.e., $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 0$ to $\varepsilon_1 = 0$, $\varepsilon_2 \neq 0$ and $\varepsilon_3 \neq 0$.
2. Rotational tori. In this example, we compute invariant tori corresponding to the libration tori of the pendulum, starting from rotational torus of the simple pendulum, and use continuation to transform it into the quasi-periodic case. We can construct an initial guess torus by setting $\varepsilon_1 \neq 0$ and $\varepsilon_2 = \varepsilon_3 = 0$, and then continue varying the parameter ε_2 and ε_3 .
3. Lastly, Librational tori we perform a similar analysis for the rotational tori of the simple pendulum. This case is particularly interesting since the topology of librational tori differs from the others, warranting detailed analysis. In this case, the continuation is analogous to the rotational case.

Unlike the previous section, these three-dimensional objects require significant effort in numerical implementation. Special attention must be paid to optimizing resources, such as RAM usage, as well as to code efficiency. These considerations are critical for ensuring accurate and efficient computations during the continuation of the tori. Through these examples, we not only demonstrate the capability of the KAM method to handle complex quasiperiodic systems but also highlight the importance of a robust numerical implementation.

4.3.1 The single-wave model

In his work, del-Castillo (2000), [dCN00], presents the *single-wave model* as a further simplification of the vorticity mixing problem. Specifically, he introduces a pendulum-like Hamiltonian with a time-dependent amplitude, described by,

$$\mathcal{H} = \sum_{j=1}^N \left(\frac{p_j^2}{2\Gamma_j} - 2\Gamma_j \sqrt{J} \cos(q_j - \theta) \right) - UJ, \quad (4.3.2)$$

with Γ_j and U constants, and J, θ functions of time. Following [dCN00, MdRdCNOC15], we adopt

$$2\sqrt{J} = \varepsilon_2 + \varepsilon_3 \cos(\alpha_1 t), \quad \theta = \alpha_2 t,$$

Let us consider the model by setting $j = 1$, $\Gamma_1 = 1$ and $U = 0$, then we obtain

$$h = \frac{p^2}{2} - (\varepsilon_2 + \varepsilon_3 \cos(\alpha_1 t)) \cos(q_j - \alpha_2 t). \quad (4.3.3)$$

We observe that (4.3.3) coincides with equation (4.3.1), when $\varepsilon_1 = 0$. Therefore the vector field is

$$\tilde{Z}_h(z, \varphi_1, \varphi_2) = \begin{pmatrix} Z_h(z, \varphi_1, \varphi_2) \\ \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{pmatrix} = \begin{pmatrix} p \\ (\varepsilon_2 + \varepsilon_3 \cos(\varphi_1)) \cos(\varphi_2 - q) \\ \alpha_1 \\ \alpha_2 \end{pmatrix} \quad (4.3.4)$$

In this case the torus is rotational, and the parameterization is of the form

$$\tilde{K}(\theta, (\varphi_1, \varphi_2)) = \begin{pmatrix} K(\theta, (\varphi_1, \varphi_2)) \\ \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} \theta + K_x^p(\theta, (\varphi_1, \varphi_2)) \\ K_y^p(\theta, (\varphi_1, \varphi_2)) \\ \varphi_1 \\ \varphi_2 \end{pmatrix}.$$

To construct the initial torus, we set parameters as $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 0$, with the initial $q(0) = 0$ and $p(0) = 2$. This implies that the internal frequency is $\omega = 2$, while the torus components are $K_x(\theta, \varphi_1, \varphi_2) = \theta$ and $K_y(\theta, \varphi_1, \varphi_2) = p_0$. Under these conditions, the parameterization of the torus is flat. For the continuation, we keep $\varepsilon_1 = 0$ and $\varepsilon_2 = \varepsilon_3$, up to $\varepsilon_2 = 0.320$ and $\varepsilon_3 = 0.0155$, figure 4.10.

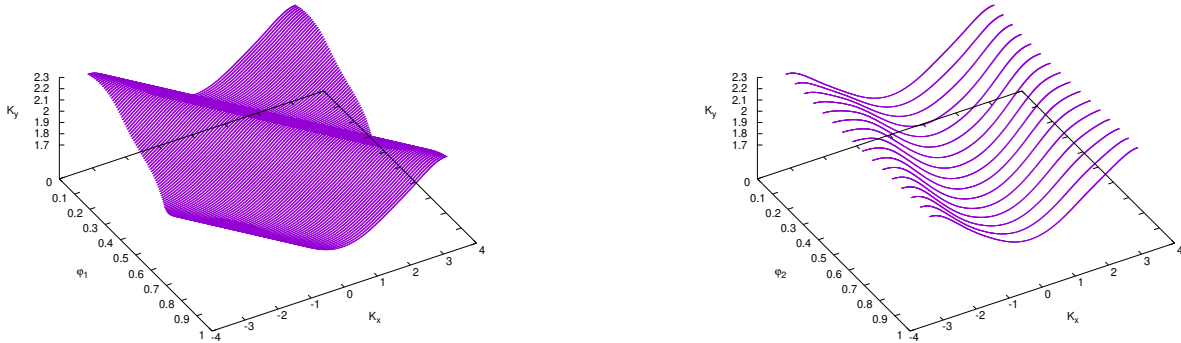


Figure 4.10: The parameters are $\varepsilon_2 = 0.320$ and $\varepsilon_3 = 0.155$. The coefficients in each direction are 2^9 , 2^9 , and 2^7 for θ , φ_1 , and φ_2 , respectively.

4.3.2 Quasi-periodic pendulum

In this section, we consider $\varepsilon_1 = 1$ in (4.3.1), i.e.,

$$h = \frac{p^2}{2} + \cos(q) - (\varepsilon_2 + \varepsilon_3 \cos(\alpha_1 t)) \cos(q_j - \alpha_2 t). \quad (4.3.5)$$

We explore numerically two different families of tori corresponding to rotational and librational movements, i.e., families that are topologically distinct. In both cases, we integrate the pendulum vector field with $\varepsilon_2 = \varepsilon_3 = 0$ to obtain $K_0(\theta) := K(\theta, 0)$. Afterward, we apply the method described in subsection 4.2.2 to construct the initial torus.

Rotational motions

To calculate the internal frequency in this case, we follow a procedure similar to that of the Tokamak in Section 4.2.2. We do the numerical integration of the vector field, with initial conditions $q(0) = -\pi$ and $p(0) = 2$, and apply Birkhoff average developed in [DSSY17], resulting in $\omega = 2.39509749976984749999$. The following selection of images shows, on the left, the components of the torus K with respect to the coordinate φ_1 , and on the right, the same components with respect to φ_2 .

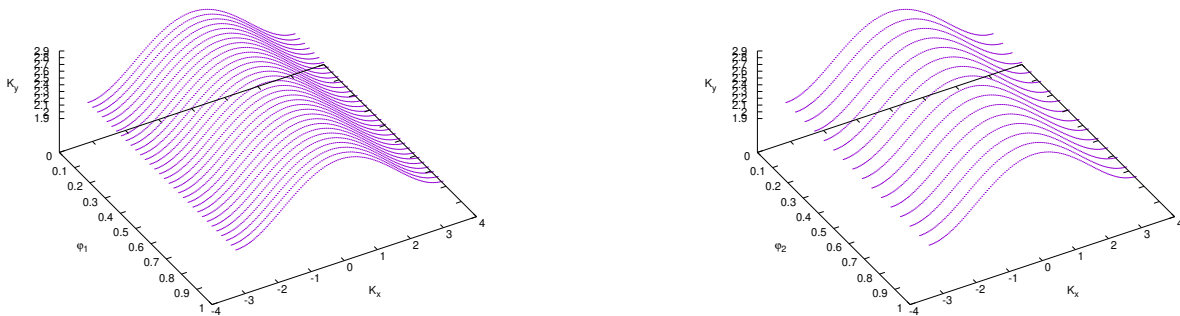


Figure 4.11: The parameters are $\varepsilon_1 = 1.000$ $\varepsilon_2 = 0.000$ and $\varepsilon_3 = 0.000$. The coefficients in each direction are 2^7 , 2^5 , and 2^4 for θ , φ_1 , and φ_2 , respectively.

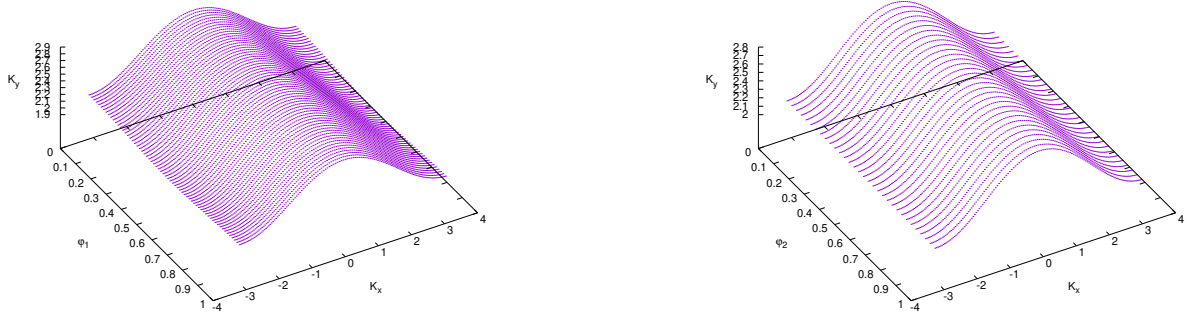


Figure 4.12: The parameters are $\varepsilon_1 = 0.096$ $\varepsilon_2 = 0.040$ and $\varepsilon_3 = 0.040$. The coefficients in each direction are 2^7 , 2^6 , and 2^5 for θ , φ_1 , and φ_2 , respectively.

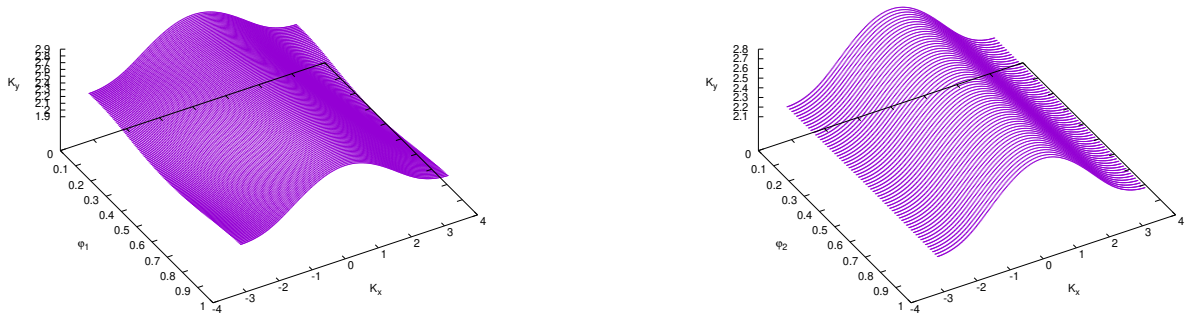


Figure 4.13: The parameters are $\varepsilon_1 = 0.092$ $\varepsilon_2 = 0.080$ and $\varepsilon_3 = 0.040$. The coefficients in each direction are 2^8 , 2^5 , and 2^4 for θ , φ_1 , and φ_2 , respectively.

Librational motions

For the librational tori, we consider the initial conditions $q(0) = \frac{\pi}{2}$ and $p(0) = 0$. From these, using Jacobi elliptic integrals, internal frequency is $\omega = 0.84721308479397908659$. Similar to the previous section, the following set of images provides a comparative view of the torus K . On the left, the components are plotted with respect to the coordinate φ_1 , while on the right, the same components are shown relative to φ_2 .

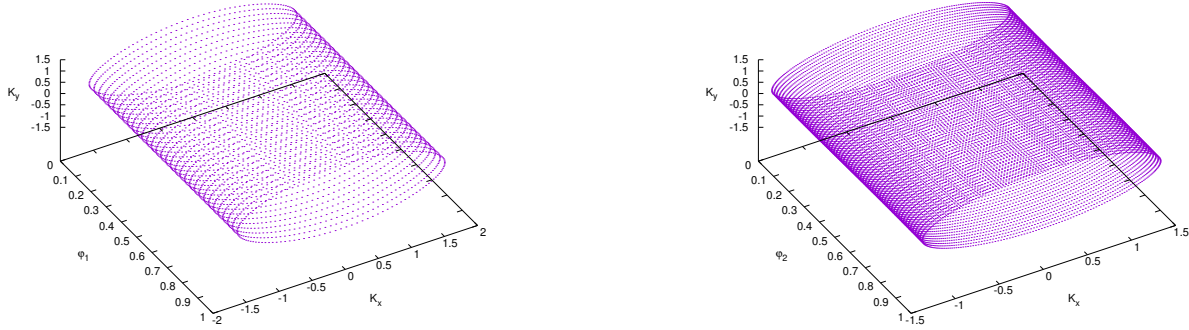


Figure 4.14: The parameters are $\varepsilon_1 = 1.000$, $\varepsilon_2 = 0.000$ and $\varepsilon_3 = 0.000$. The coefficients in each direction are 2^7 , 2^5 , and 2^4 for θ , φ_1 , and φ_2 , respectively.

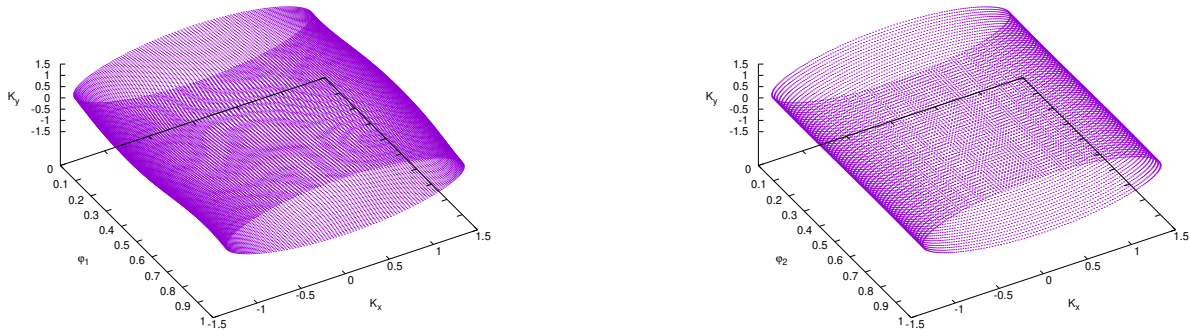


Figure 4.15: The parameters are $\varepsilon_1 = 0.930$, $\varepsilon_2 = 0.035$ and $\varepsilon_3 = 0.007$. The coefficients in each direction are 2^8 , 2^7 , and 2^6 for θ , φ_1 , and φ_2 , respectively.

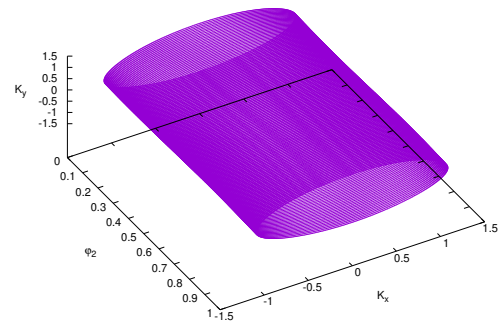
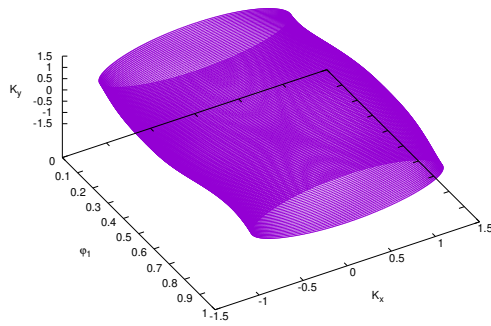


Figure 4.16: The parameters are $\varepsilon_1 = 0.860$ $\varepsilon_2 = 0.070$ and $\varepsilon_3 = 0.014$. The coefficients in each direction are 2^9 , 2^8 , and 2^7 for θ , φ_1 , and φ_2 , respectively.

Conclusions

In this work, we have presented two applications of the method for the existence and stability of invariant tori in time-dependent quasi-periodic Hamiltonian systems. Unlike other approaches, our method is applied directly to the vector field of the system, avoiding the need to construct a quasi-periodic map, as is commonly done in studies based on the standard map. This formulation allows for a more direct and flexible implementation in different physical contexts, facilitating its adaptation to problems with various dynamical structures.

Additionally, we exploit the fibered structure of the system to reduce the problem's dimension from $2n + \ell$ to $2n$, where the external frequencies are naturally incorporated into the formulation. This not only simplifies the theoretical analysis but also improves computational efficiency. In particular, the torsion expression used in this work avoids the computation of the Lie derivative, as required in other methods, significantly reducing computational complexity. Instead, our method only requires evaluating the torus parameterization, in the derivative of the vector field, and the product of geometric objects, leading to faster and more precise calculations. This reduction in computational cost is especially relevant for applications that require high numerical precision.

Furthermore, our approach enables more efficient Computer-Assisted Proofs (CAPs) compared to methods based on the system's flow. Since we work directly with the vector field rather than integrating the dynamics, we avoid the need to validate numerical integration through a CAP, which can be computationally expensive and numerically delicate. This aspect makes our method particularly suitable for applications requiring rigorous validation with lower computational effort.

In cases where the method converges, the number of Fourier nodes doubles at each iteration, leading to an increasing demand for computational resources. In the case of the Tokamak, the expansion is first attempted in the θ direction; if this does not work, it is tried in φ , and if it still does not converge, it is performed in both directions simultaneously. A similar procedure is followed for coupled pendulums. This behavior makes RAM management a crucial aspect of the method's implementation. In the future, when Computer-Assisted Proofs are carried out, it will be essential to perform calculations in multiple precision, further increasing memory demand and

requiring advanced resource management strategies. This challenge represents an interesting problem that is worth addressing in future research.

Our results strengthen the applicability of KAM methods in time-dependent Hamiltonian systems and open new perspectives for their implementation in concrete physical problems, such as the dynamics of confined plasmas and geometric transport models. Future work could explore extending this approach to systems with a larger number of degrees of freedom and integrating it with advanced numerical techniques for high-dimensional problems. Additionally, combining this method with computational analysis approaches and computer-assisted proofs could provide new tools for the rigorous study of the stability and persistence of invariant tori in more general scenarios.

The method is designed to compute tori of any dimension. The only restriction comes from the computational resources. Based on the information presented in figures 4.6 and 4.9, we can estimate the bandwidth ρ . Furthermore, we can compute the Diophantine constants, as referenced in [FH25]. Also, we can estimate all the constants listed in tables (1) and (2) from [CHP25a], to calculate the constant $\mathfrak{C}_1$ in equation (3.3), which is essential for computing the smallness condition and to do CAPs .

Appendix A

Extended Symplectic Formulation for QP Hamiltonian

A different approach to handling time-dependent quasiperiodic Hamiltonian systems involves extending the symplectic structure by introducing auxiliary variables. This method effectively transforms the system into an autonomous form, allowing the use of standard symplectic techniques. By augmenting the phase space with fictitious variables, one can preserve the Hamiltonian framework while incorporating time dependence naturally. In this appendix, we outline the construction of this extended formulation and discuss its advantages in the context of KAM theory.

We consider a qp-time-dependent Hamiltonian system of the form

$$\begin{aligned} h : \mathbb{R}^{2n} \times \mathbb{T}^\ell &\longrightarrow \mathbb{R}, \\ (z, \varphi) &\longmapsto h(z, \varphi), \end{aligned}$$

is to extend the symplectic formalism by introducing fictitious variables, $\mathbf{I}$, associated with momentum, which are conjugate to the quasi-periodic time variables, i.e.

$$\begin{aligned} \hat{h} : \mathbb{R}^{2n} \times \mathbb{T}^\ell \times \mathbb{R}^\ell &\longrightarrow \mathbb{R}, \\ (z, \varphi, \mathbf{I}) &\longmapsto \hat{h}(z, \varphi, \mathbf{I}) = h(z, \varphi) + \mathbf{I} \cdot \alpha, \end{aligned}$$

with $\alpha \in \mathbb{R}^\ell$ external frequencies. The corresponding symplectic vector field is then given by:

$$Z_{\hat{h}}(z, \varphi, \mathbf{I}) = \hat{\Omega}(z)^{-1} (D\hat{h}(z, \varphi, \mathbf{I}))^\top,$$

with,

$$\hat{\Omega}(z)^{-1} = \begin{pmatrix} \Omega(z)^{-1} & O_{2n \times 2\ell} \\ O_{2\ell \times 2n} & \Omega_\ell^{-1} \end{pmatrix},$$

where $\Omega(z)$ denotes a $2n$ symplectic matrix, while $\Omega_\ell = \begin{pmatrix} O_\ell & -I_\ell \\ I_\ell & O_\ell \end{pmatrix}$ is the 2ℓ standar symplectic matrix. Moreover, the matrix representations of the Riemannian metric and the linear isomorphism are expressed as follows.

$$\hat{G}(z) = \begin{pmatrix} G(z) & O_{2n \times 2\ell} \\ O_{2\ell \times 2n} & I_{2\ell} \end{pmatrix}, \quad \hat{J}(z) = \begin{pmatrix} J(z) & O_{2n \times 2\ell} \\ O_{2\ell \times 2n} & \Omega_\ell \end{pmatrix}.$$

Analogous to [HCF⁺16, HL19], we could to look a parameterization $\hat{K} : \mathbb{T}^n \times \mathbb{T}^\ell \rightarrow \mathcal{M} \times \mathbb{T}^\ell \times \mathbb{R}^\ell$

$$\hat{K}(\theta, \varphi) = \begin{pmatrix} K(\theta, \varphi) \\ \Phi(\theta, \varphi) \\ \mathbf{I}(\theta, \varphi) \end{pmatrix}.$$

That satisfies the follow inifinitesimal invariance equation, with internal dynamics is given by the constant vector field (ω, α) ,

$$Z_{\hat{h}}(K(\theta, \varphi), \Phi(\theta, \varphi), \mathbf{I}(\theta, \varphi)) = D(K(\theta, \varphi), \Phi(\theta, \varphi), \mathbf{I}(\theta, \varphi))(\omega, \alpha)^\top.$$

Where we obtain the follow equations,

$$Z_h(\theta, \varphi) =: \Omega(z)^{-1} (D_z h(\theta, \varphi))^\top = -\mathfrak{L}_{\omega, \alpha} K(\theta, \varphi), \quad (\text{A.1})$$

$$\alpha = -\mathfrak{L}_{\omega, \alpha} \Phi(\theta, \varphi), \quad (\text{A.2})$$

$$-(D_\varphi h(\theta, \varphi))^\top = -\mathfrak{L}_{\omega, \alpha} \mathbf{I}(\theta, \varphi). \quad (\text{A.3})$$

By solving these equations, we can obtain a parameterization of an invariant torus of dimension $2(n + \ell)$. Since we assume that $\varphi \in \mathbb{T}^\ell$, we obtain that,

$$\Phi(\theta, \varphi) = \varphi,$$

i.e.,

$$\hat{K}(\theta, \varphi) = \begin{pmatrix} K(\theta, \varphi) \\ \varphi \\ \mathbf{I}(\theta, \varphi) \end{pmatrix}.$$

Therefore, the tangent bundle, $\hat{L} : \mathbb{T}^n \times \mathbb{T}^\ell \rightarrow \mathbb{R}^{2(n+\ell) \times (n+\ell)}$ takes the following form,

$$\hat{L}(\theta, \varphi) = \begin{pmatrix} D_\theta K(\theta, \varphi) & D_\varphi K(\theta, \varphi) \\ O_{\ell \times n} & I_\ell \\ D_\theta \mathbf{I}(\theta, \varphi) & D_\varphi \mathbf{I}(\theta, \varphi) \end{pmatrix}.$$

In comparison with the approach in this work, by assuming $\varphi \in \mathbb{T}^\ell$, therefore We not only reduce the dimension of the torus, from $2(n + \ell)$ to $2n$, also the number of equations to solve, thereby lowering the computational cost of the resulting algorithms derived from the proof.

Appendix B

Computation of a Novel Torsion Expression

In this appendix, we derive an expression for the torsion without involving Lie derivatives, explicitly obtaining equation (1.4.4) from (1.4.14).

Lema 4.3.1. *Let $T : \mathbb{T}^n \times \mathbb{T}^\ell \rightarrow \mathbb{R}^n$ be the torsion matrix, as defined in equation (1.4.14). Then, it satisfies the expression given in equation (1.4.4).*

Proof. Let us consider

$$\tilde{N}(\theta, \varphi) = N(\theta, \varphi)^0 B(\theta, \varphi) = J(K((\theta, \varphi))) L(\theta, \varphi) B(\theta, \varphi) .$$

Through the substitution of this expression into the equations, (1.4.9) and (1.4.13), we obtain:

$$N(\theta, \varphi) = L(\theta, \varphi) A(\theta, \varphi) + \tilde{N}(\theta, \varphi) , \tag{B.1}$$

$$A(\theta, \varphi) = -\frac{1}{2} \tilde{N}(\theta, \varphi)^\top \Omega(K(\theta, \varphi), \varphi) \tilde{N}(\theta, \varphi) . \tag{B.2}$$

Now, we calculate the following derivatives; $\mathfrak{L}_{\omega, \alpha} L(\theta, \varphi)$, $\mathfrak{L}_{\omega, \alpha} B(\theta, \varphi)$, $\mathfrak{L}_{\omega, \alpha} \tilde{N}(\theta, \varphi)$ and $\mathfrak{L}_{\omega, \alpha} A(\theta, \varphi)$. Since they are necessary to calculate $\mathfrak{L}_{\omega, \alpha} N(\theta, \varphi)$. For the first expression, it is sufficient to differentiate the invariance equation, (1.2.6), with respect to θ , that is,

$$\mathfrak{L}_{\omega, \alpha} L(\theta, \varphi) = -D_z Z_h(K(\theta, \varphi), \varphi) L(\theta, \varphi) , \tag{B.3}$$

For the second one, let us take the Lie derivative of $I_d = B^{-1}B$, we have obtain that $\mathfrak{L}_{\omega, \alpha} B =$

$$-B\mathfrak{L}_{\omega,\alpha}B^{-1}B = -B\mathfrak{L}_{\omega,\alpha}(L^\top(G \circ K)L)B, \text{ i.e.,}$$

$$\begin{aligned} \mathfrak{L}_{\omega,\alpha}B(\theta, \varphi) &= B(\theta, \varphi)^\top L(\theta, \varphi)^\top \left(\Omega(K(\theta, \varphi), \varphi) D_z Z_h(K(\theta, \varphi), \varphi) \right. \\ &\quad - \Omega(K(\theta, \varphi)) D_z J(K(\theta, \varphi)) [Z_h(K(\theta, \varphi), \varphi)] J(K(\theta, \varphi))^{-1} \\ &\quad \left. - \Omega(K(\theta, \varphi)) J(K(\theta, \varphi)) D_z Z_h(K(\theta, \varphi), \varphi) J(K(\theta, \varphi))^{-1} \right) \tilde{N}(\theta, \varphi) \\ &= \tilde{N}(\theta, \varphi)^\top J(K(\theta, \varphi))^{-\top} T_h(K(\theta, \varphi), \varphi) \tilde{N}(\theta, \varphi). \end{aligned} \quad (\text{B.4})$$

Equations (B.3), (1.3.9) and

$$T_h(z, \varphi) = \Omega(z) \left(D_z Z_h(z, \varphi) - D_z J(z) [z, \varphi] J(z)^{-1} - J(z) D_z Z_h(z, \varphi) J(z)^{-1} \right),$$

were used for this calculation. For the next one, we consider, the definition of $\tilde{N}$ and the equations (B.3) and (B.4).

$$\begin{aligned} \mathfrak{L}_{\omega,\alpha}\tilde{N}(\theta, \varphi) &= -D_z J(K(\theta, \varphi)) [Z_h(K(\theta, \varphi), \varphi)] L(\theta, \varphi) B(\theta, \varphi) \\ &\quad - J(K(\theta, \varphi)) D_z Z_h(K(\theta, \varphi), \varphi) L(\theta, \varphi) B(\theta, \varphi) \\ &\quad + J(K(\theta, \varphi)) L(\theta, \varphi) \tilde{N}(\theta, \varphi)^\top J(K(\theta, \varphi))^{-\top} T_h(K(\theta, \varphi), \varphi) \tilde{N}(\theta, \varphi). \end{aligned} \quad (\text{B.5})$$

Therefore the Lie derivative of A , is:

$$\begin{aligned} \mathfrak{L}_{\omega,\alpha}A(\theta, \varphi) &= -\frac{1}{2} \mathfrak{L}_{\omega,\alpha} \tilde{N}(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) \tilde{N}(\theta, \varphi) \\ &\quad - \frac{1}{2} \tilde{N}(\theta, \varphi)^\top \mathfrak{L}_{\omega,\alpha} \Omega(K(\theta, \varphi)) \tilde{N}(\theta, \varphi) \\ &\quad - \frac{1}{2} \tilde{N}(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) \mathfrak{L}_{\omega,\alpha} \tilde{N}(\theta, \varphi) \\ &= \frac{1}{2} \tilde{N}(\theta, \varphi)^\top J(K(\theta, \varphi))^{-\top} D_z J(K(\theta, \varphi)) [Z_h(K(\theta, \varphi), \varphi)]^\top \Omega(K(\theta, \varphi)) \tilde{N}(\theta, \varphi) \\ &\quad + \frac{1}{2} \tilde{N}(\theta, \varphi)^\top J(K(\theta, \varphi))^{-\top} (D_z Z_h(K(\theta, \varphi), \varphi))^\top J(K(\theta, \varphi))^\top \Omega(K(\theta, \varphi)) \tilde{N}(\theta, \varphi) \\ &\quad - \frac{1}{2} \tilde{N}(\theta, \varphi)^\top T_h(K(\theta, \varphi), \varphi)^\top J(K(\theta, \varphi))^{-1} \tilde{N}(\theta, \varphi) L(\theta, \varphi)^\top J(K(\theta, \varphi))^\top \Omega(K(\theta, \varphi)) \tilde{N}(\theta, \varphi) \\ &\quad - \frac{1}{2} \tilde{N}(\theta, \varphi)^\top (D_z Z_h(K(\theta, \varphi), \varphi))^\top \Omega(K(\theta, \varphi)) \tilde{N}(\theta, \varphi) \\ &\quad - \frac{1}{2} \tilde{N}(\theta, \varphi)^\top T_h(K(\theta, \varphi), \varphi) \tilde{N}(\theta, \varphi) \\ &\quad - \frac{1}{2} \tilde{N}(\theta, \varphi)^\top \Omega(K(\theta, \varphi)) J(K(\theta, \varphi)) L(\theta, \varphi) \tilde{N}(\theta, \varphi)^\top J(K(\theta, \varphi))^{-\top} T_h(K(\theta, \varphi), \varphi) \tilde{N}(\theta, \varphi), \end{aligned} \quad (\text{B.6})$$

Using (B.1), (B.3),(B.4), (B.5), (B.6) in (1.4.14), we obtain

$$\begin{aligned} T(K(\theta, \varphi), \varphi) = & \frac{1}{2} \tilde{N}(\theta, \varphi)^\top \left(T_h(K(\theta, \varphi), \varphi) + T_h(K(\theta, \varphi), \varphi)^\top \right) \tilde{N}(\theta, \varphi) \\ & + \tilde{N}(\theta, \varphi)^\top T_h(K(\theta, \varphi), \varphi)^\top L(\theta, \varphi) A(\theta, \varphi) \\ & - A(\theta, \varphi) L(\theta, \varphi)^\top T_h(K(\theta, \varphi), \varphi) \tilde{N}(\theta, \varphi). \end{aligned}$$

□

Appendix C

Compendium of Constants in the KAM Theory

In this appendix we collect the recipes to compute all constants involved in the different estimates presented in the thesis. Keeping track of these constants is crucial to apply the KAM theorem in particular problems and for concrete values of parameters. Let us remark that, as it becomes clear in the proof, the numbers a_1 , a_2 and a_3 are independent parameters, that can be selected in order to optimize the applicability of the theorems depending on the particular problem at hand. Table 4.1 corresponds to the geometric construction of the theorem. The constants needed for a Newton step are outlined in Table 4.2, while Table 4.3 contains the constants related to the method's convergence.

Table 4.1: Constants introduced in Chapter 2. Constants with * in the label correspond to **Case III**.

Object	Constant	Label	Result
$\mathfrak{L}_\omega \Omega_L$	$C_{\mathfrak{L}\Omega_L} = 2nc_{\Omega,0}\sigma_L + \sigma_{L^\top} c_{\Omega,1} \sigma_L \delta + n\sigma_{L^\top} c_{\Omega,0}$	(2.2.3)	Lemma 2.2.2
Ω_L	$C_{\Omega_L} = c_R C_{\mathfrak{L}\Omega_L}$	(2.2.4)	Lemma 2.2.2
L	$C_L = \sigma_L$	(2.2.8)	Lemma 2.2.4
$L^\top$	$C_{L^\top} = \sigma_{L^\top}$	(2.2.9)	Lemma 2.2.4
G_L	$C_{G_L} = C_{L^\top} c_{G,0} C_L$	(2.2.10)	Lemma 2.2.4
N^0	$C_{N^0} = c_{J,0} C_L$	(2.2.14)	Lemma 2.2.5
$(N^0)^\top$	$C_{N^{0,\top}} = C_{L^\top} c_{J^\top,0}$	(2.2.15)	Lemma 2.2.5
$\tilde{N}$	$C_{\tilde{N}} = C_{N^0} \sigma_B$	(2.2.16)	Lemma 2.2.5
$\tilde{N}^\top$	$C_{\tilde{N}^\top} = \sigma_B C_{N^{0,\top}}$	(2.2.17)	Lemma 2.2.5
A	$C_A = \frac{1}{2} C_{\tilde{N}^\top} c_{\Omega,0} C_{\tilde{N}}$, * $C_A = 0$	(2.2.18)*	Lemma 2.2.5
N	$C_N = C_L C_A + C_{\tilde{N}}$, * $C_N = C_{\tilde{N}}$	(2.2.19)	Lemma 2.2.5
$N^\top$	$C_{N^\top} = C_A C_{L^\top} + C_{\tilde{N}^\top}$, * $C_{N^\top} = C_{\tilde{N}^\top}$	(2.2.20)	Lemma 2.2.5
E_{sym}	$C_{\text{sym}} = (1 + C_A) \max\{1, C_A\} C_{\Omega_L}$, * $C_{\text{sym}} = C_{\Omega_L} \max\{1, (\sigma_B)^2\}$	(2.2.24)	Lemma 2.2.5

T	$C_T = \frac{1}{2}(C_{\tilde{N}^\top} c_{T_h,0} C_{\tilde{N}} + C_{\tilde{N}^\top} c_{T_h^\top,0} C_{\tilde{N}})$ $+ C_{\tilde{N}^\top} c_{T_h^\top,0} C_N + C_{N^\top} c_{T_h,0} C_{\tilde{N}}$ $^* C_T = C_{N^\top} c_{T_h,0} C_N$	(2.2.25)	Lemma 2.2.7
T_E	$C_{T_E} = c_{\Omega,1} \delta + c_{\Omega,0} c_{J,1} c_{J^{-1},0} \delta$ $+ 2n c_{J^\top,0} C_L \sigma_B c_{\Omega,0}^2 + n c_{J^\top,0} \sigma_B C_L^\top c_{\Omega,0}^2$	(2.2.43)	Lemma 2.2.10
$T_E^\top$	$C_{T_E^\top} = c_{\Omega,1} \delta + c_{J^{-\top},0} c_{J^\top,1} c_{\Omega,0} \delta$ $+ n \sigma_B C_L^\top c_{J,0} c_{\Omega,0}^2 + 2n C_L \sigma_B c_{J,0} c_{\Omega,0}^2$	(2.2.44)	Lemma 2.2.10
$E_{\mathfrak{L}B}$	$C_{E_{\mathfrak{L}B}} = C_{\tilde{N}^\top} c_{J^{-\top},0} C_{T_E} C_{\tilde{N}}$	(2.2.45)	Lemma 2.2.10
$E_{\mathfrak{L}B}^\top$	$C_{E_{\mathfrak{L}B}^\top} = C_{\tilde{N}^\top} C_{T_E^\top} c_{J^{-1},0} C_{\tilde{N}}$	(2.2.46)	Lemma 2.2.10
$E_{\mathfrak{L}\tilde{N}}$	$C_{E_{\mathfrak{L}\tilde{N}}} = \delta c_{J,1} C_L \sigma_B + n c_{J,0} \sigma_B + c_{J,0} C_L C_{E_{\mathfrak{L}B}}$	(2.2.47)	Lemma 2.2.10
$E_{\mathfrak{L}\tilde{N}}^\top$	$C_{E_{\mathfrak{L}\tilde{N}}^\top} = \delta \sigma_B C_L^\top c_{J^\top,1} + 2n \sigma_B c_{J^\top,0} + C_{E_{\mathfrak{L}B}^\top} C_L^\top c_{J^\top,0}$	(2.2.48)	Lemma 2.2.10
$E_{\mathfrak{L}A}$	$C_{E_{\mathfrak{L}A}} = \frac{1}{2}(C_{E_{\mathfrak{L}\tilde{N}}^\top} c_{\Omega,0} C_{\tilde{N}} + \delta C_{\tilde{N}^\top} c_{\Omega,1} C_{\tilde{N}} + C_{\tilde{N}^\top} c_{\Omega,0} C_{E_{\mathfrak{L}\tilde{N}}})$	(2.2.49)	Lemma 2.2.10
$E_{\mathfrak{L}N}$	$C_{E_{\mathfrak{L}N}} = C_{E_{\mathfrak{L}\tilde{N}}} + C_L C_{E_{\mathfrak{L}A}} + n C_A$	(2.2.50)	Lemma 2.2.10
E_T	$C_{E_T} = \frac{1}{2}(C_A C_{\Omega_L} C_{\tilde{N}^\top} c_{T_h^\top,0} C_{\tilde{N}} + C_A C_{\Omega_L} C_{\tilde{N}^\top} c_{T_h,0} C_{\tilde{N}})$ $+ C_A C_{\Omega_L} C_A c_{T_h,0} C_{\tilde{N}} + C_A C_{\Omega_L} C_{\tilde{N}^\top} c_{T_h^\top,0} C_{\tilde{N}} C_L C_A$ $+ \gamma \delta^\top C_A C_L^\top c_{\Omega,0} C_{E_{\mathfrak{L}N}} + \gamma \delta^\top C_{\tilde{N}^\top} c_{\Omega,0} C_{E_{\mathfrak{L}N}}$	(2.2.51)	Lemma 2.2.10
$\mathfrak{X}_L$	$C_{\mathfrak{X}_L} = n$	(2.2.61)	Lemma 2.2.11
$\mathfrak{X}_L^\top$	$C_{\mathfrak{X}_L^\top} = 2n$	(2.2.62)	Lemma 2.2.11
$\mathfrak{L}\bar{A}$	$C_{\mathfrak{L}A} = \frac{1}{2}(C_{\tilde{N}^\top} c_{T_h^\top,0} C_{\tilde{N}} + C_{\tilde{N}^\top} c_{T_h^\top,0} c_{J^{-1},0} C_{\tilde{N}} C_L^\top c_{J^\top,0} c_{\Omega,0} C_{\tilde{N}})$ $+ C_{\tilde{N}^\top} c_{T_h,0} C_{\tilde{N}} + C_{\tilde{N}^\top} c_{\Omega,0} c_{J,0} C_L C_{\tilde{N}^\top} c_{J^{-\top},0} c_{T_h,0} C_{\tilde{N}})$ $^* C_{\mathfrak{L}A} = 0$	(2.2.65)	Lemma 2.2.11
$E_{\text{red}}^{1,1}$	$C_{\text{red}}^{1,1} = C_{N^\top} c_{\Omega,0} C_{\mathfrak{X}_L}$	(2.2.67)	Lemma 2.2.11
$E_{\text{red}}^{1,2}$	$C_{\text{red}}^{1,2} = C_{E_T}$	(2.2.68)	Lemma 2.2.11
$E_{\text{red}}^{2,1}$	$C_{\text{red}}^{2,1} = C_L^\top c_{\Omega,0} C_{\mathfrak{X}_L}$	(2.2.69)	Lemma 2.2.11
$E_{\text{red}}^{2,2}$	$C_{\text{red}}^{2,2} = (C_L^\top c_{\Omega,1} C_N \delta + C_{\mathfrak{X}_L^\top} c_{\Omega,0} C_N + C_{\mathfrak{L}\tilde{N}} C_A + \sigma_L^\top c_{\Omega,0} \sigma_L C_{E_{\mathfrak{L}A}}) \gamma \delta^\top$ $+ C_{\Omega_L} C_{\mathfrak{L}A}$	(2.2.70)	Lemma 2.2.11
E_{red}	$C_{\text{red}} = \max\{C_{\text{red}}^{1,1} \gamma \delta^\top + C_{\text{red}}^{1,2}, C_{\text{red}}^{2,1} \gamma \delta^\top + C_{\text{red}}^{2,2}\}$	(2.2.71)	Lemma 2.2.11

Table 4.2: Constants introduced in Lemma 3.2.1. Given an object Z_h , we use the notation $\Delta \mathfrak{L} Z_h = \mathfrak{L}_{\omega,\alpha} \bar{Z}_h - \mathfrak{L}_{\omega,\alpha} Z_h$. Constants with * in the label are 0 in **Case III**.

Object	Constant	Label
ξ_0^N	$C_{\xi_0^N} = \sigma_T(C_{N^\top} c_{\Omega,0} \gamma \delta^\top + c_R C_T C_L^\top c_{\Omega,0})$	(3.2.12)
ξ^N	$C_{\xi^N} = C_{\xi_0^N} + c_R C_L^\top c_{\Omega,0}$	(3.2.13)
ξ^L	$C_{\xi^L} = c_R(C_{N^\top} c_{\Omega,0} \gamma \delta^\top + C_T C_{\xi^N})$	(3.2.14)
ξ	$C_\xi = \max\{C_{\xi^L}, C_{\xi^N} \gamma \delta^\top\}$	(3.2.15)
ΔK	$C_{\Delta K} = C_L C_{\xi^L} + C_N C_{\xi^N} \gamma \delta^\top$	(3.2.16)
$\mathfrak{L}_{\omega} \xi^N$	$C_{\mathfrak{L}_{\omega} \xi^N} = C_L^\top c_{\Omega,0}$	(3.2.19)
$\mathfrak{L}_{\omega} \xi^L$	$C_{\mathfrak{L}_{\omega} \xi^L} = C_{N^\top} c_{\Omega,0} \gamma \delta^\top + C_T C_{\xi^N}$	(3.2.20)

$\mathfrak{L}_\omega \xi$	$C_{\mathfrak{L}\xi} = \max\{C_{\mathfrak{L}\xi L}, C_{\mathfrak{L}\xi N} \gamma \delta^\tau\}$	(3.2.21)
E_{lin}	$C_{\text{lin}} = C_{\text{red}} C_\xi + c_{\Omega,0} C_{\text{sym}} C_{\mathfrak{L}\xi} \gamma \delta^\tau$	(3.2.22)
$\bar{E}$	$C_E = 2(C_L + C_N) C_{\text{lin}} \gamma \delta^{\tau-1} + \frac{1}{2} c_{Z_h,2} (C_{\Delta K})^2$	(3.2.26)
ΔL	$C_{\Delta L} = n C_{\Delta K}$	(3.2.29)
$\Delta L^\top$	$C_{\Delta L^\top} = 2n C_{\Delta K}$	(3.2.30)
ΔG	$C_{\Delta G} = c_{G,1} C_{\Delta K}$	(3.2.31)
ΔG_L	$C_{\Delta G_L} = C_L^\top c_{G,0} C_{\Delta L} + C_L^\top C_{\Delta G} C_L \delta + C_{\Delta L}^\top c_{G,0} C_L$	(3.2.32)
ΔB	$C_{\Delta B} = (\sigma_B)^2 C_{\Delta G_L}$	(3.2.34)
$\Delta \Omega$	$C_{\Delta \Omega} = c_{\Omega,1} C_{\Delta K}$	(3.2.35)
ΔJ	$C_{\Delta J} = c_{J,1} C_{\Delta K}$	(3.2.36)
$\Delta J^\top$	$C_{\Delta J^\top} = c_{J^\top,1} C_{\Delta K}$	(3.2.37)
ΔN^0	$C_{\Delta N^0} = c_{J,0} C_{\Delta L} + C_{\Delta J} C_L \delta$	(3.2.38)
$\Delta N^{0\top}$	$C_{\Delta N^{0\top}} = C_{\Delta L}^\top c_{J^\top,0} + C_L^\top C_{\Delta J^\top} \delta$	(3.2.39)
$\Delta \tilde{N}$	$C_{\Delta \tilde{N}} = C_{\Delta N^0} \sigma_B + C_{N^0} C_{\Delta B}$	(3.2.40)
$\Delta \tilde{N}^\top$	$C_{\Delta \tilde{N}^\top} = \sigma_B C_{\Delta N^{0\top}} + C_{\Delta B} C_{N^{0,\top}}$	(3.2.41)
ΔA	$C_{\Delta A} = \frac{1}{2} (C_{\Delta \tilde{N}^\top} c_{\Omega,0} C_{\tilde{N}} + C_{\tilde{N}^\top} C_{\Delta \Omega} C_{\tilde{N}} \delta + C_{\tilde{N}^\top} c_{\Omega,0} C_{\Delta \tilde{N}})$ * $C_{\Delta A} = 0$	(3.2.42)
ΔN	$C_{\Delta N} = C_A C_{\Delta L}^\top + C_{\Delta A} C_L^\top + C_{\Delta \tilde{N}}$	(3.2.43)
$\Delta N^\top$	$C_{\Delta N^\top} = C_A C_{\Delta L}^\top + C_{\Delta A} C_L^\top + C_{\Delta \tilde{N}^\top}$	(3.2.44)
ΔT_h	$C_{\Delta T_h} = c_{T_h,1} C_{\Delta K}$	(3.2.47)
$\Delta T_h^\top$	$C_{\Delta T_h^\top} = c_{T_h^\top,1} C_{\Delta K}$	(3.2.48)
ΔT	$C_{\Delta T} = \frac{1}{2} C_{\Delta \tilde{N}^\top} c_{T_h,0} C_{\tilde{N}} + C_{\tilde{N}^\top} C_{\Delta T_h} C_{\tilde{N}} \delta + C_{\tilde{N}^\top} c_{T_h,0} C_{\Delta \tilde{N}}$ + $\frac{1}{2} C_{\Delta \tilde{N}^\top} c_{T_h^\top,0} C_{\tilde{N}} + C_{\tilde{N}^\top} C_{\Delta T_h^\top} C_{\tilde{N}} \delta + C_{\tilde{N}^\top} c_{T_h^\top,0} C_{\Delta \tilde{N}}$ + $C_{\Delta \tilde{N}^\top} c_{T_h^\top,0} C_N + C_{\tilde{N}^\top} C_{\Delta T_h^\top} C_N \delta + C_{\tilde{N}^\top} c_{T_h^\top,0} C_{\Delta N}$ + $C_{\Delta N^\top} c_{T_h,0} C_{\tilde{N}} + C_{N^\top} C_{\Delta T_h} C_{\tilde{N}} \delta + C_{N^\top} c_{T_h,0} C_{\Delta \tilde{N}}$	(3.2.49)
$\Delta \langle T \rangle^{-1}$	$C_{\Delta \langle T \rangle^{-1}} = (\sigma_T)^2 C_{\Delta T}$	(3.2.51)

Table 4.3: Constants introduced in Section 3.3 associated with the convergence of que quasi-Newton method that yields Theorem 2.1.1, i.e., the ordinary formulation.

Constant	Label
$\mathfrak{C}_1 = \max\{(a_1 a_3)^{4\tau} C_E, (a_3)^{2\tau+1} \gamma^2 \rho^{2\tau-1} C_\Delta\}$	(3.3.6)
$C_{\Delta,1} = \max\left\{\frac{n C_{\Delta K}}{\sigma_L - \ DK\ _\rho}, \frac{2n C_{\Delta K}}{\sigma_L^\top - \ DK^\top\ _\rho}, \frac{C_{\Delta B}}{\sigma_B - \ B\ _\rho}, \frac{C_{\Delta \langle T \rangle^{-1}}}{\sigma_T - \langle T \rangle^{-1} }\right\}$	(3.3.7)
$C_{\Delta,2} = \frac{C_{\Delta K} \delta}{\text{dist}(K(\mathbb{T}_\rho^d), \partial \mathcal{B})}$	(3.3.8)
$C_\Delta = \max\left\{C_{\text{sym}} \gamma \delta^\tau, \frac{C_{\Delta,1}}{1-a_1^{1-2\tau}}, \frac{C_{\Delta,2}}{1-a_1^{-2\tau}}\right\}$	(3.3.9)
$\mathfrak{C}_2 = a_3^{2\tau} C_{\Delta K} / (1 - a_1^{-2\tau})$	(3.3.10)

Appendix D

Table of time and memory for Quasi-Newton method to Tokamak

In this appendix, we present a table of time and memory for Quasi-Newton method to Tokamak. The grid size is $2^{12} \times 2^{11}$. The numerical simulations were executed on a computer equipped with an **Intel(R) Xeon(R) E5-2665 CPU**, operating at **2.40 GHz** with a maximum boost frequency of **3.10 GHz**. This processor features a **16-core** architecture, allowing for **16 threads** due to its hyper-threading capability, with **8 cores per socket**. The system architecture is **x86_64**, supporting both **32-bit and 64-bit** modes. The CPU's cache hierarchy includes **256 KiB** of L1 data cache per core, **256 KiB** of L1 instruction cache per core, **2 MiB** of L2 cache per core, and **20 MiB** of shared L3 cache. The machine operates on a single NUMA node with all CPUs (0-15) accessible, and it supports virtualization through **VT-x** technology. The system is running on a **Linux** operating system. The machine is equipped with **32 GiB** of total RAM, Additionally, the system has a **32 GiB** swap space, The calculations were performed using **C++** with **long double**, providing a precision of up to 20 significant digits.

The following table shows the time and memory usage percentage at each step of the iterations of the quasi-Newton method. The steps are as follows: 1. Construction of the tangent frame, 2. Construction of torsion, 3. Correction in the symplectic frame, and 4. Computation of the new parameterization.

Iteration 1				
Step	Execution Time	Memory Usage	Objects	
1	26.9559 (s)	2.0 %	$\ L\ = 2.602235\text{e}+00$	$\ \Omega_L\ = 0.000000$
2	43.6666 (s)	3.2 %	$\ \mathcal{L}L\ = 0.32544$	$\ T\ = 1.15074036\text{e}+01$
3	18.5556 (s)	3.2 %	$\langle T \rangle = -2.35551120\text{e}+00$	$\langle T^{-1} \rangle = -4.24536295\text{e}-01$
4	0.776569 (s)	3.2 %	$\ \Delta K\ = 8.997\text{e}-03$	$\ E\ = 2.71804\text{e}-05$
Iteration 2				
Step	Execution Time	Memory Usage	Objects	
1	26.9567 (s)	2.0 %	$\ L\ = 2.757424\text{e}+00$	$\ \Omega_L\ = 0.000000$
2	43.6666 (s)	3.2 %	$\ \mathcal{L}L\ = 0.322738$	$\ T\ = 1.13239957\text{e}+01$
3	18.5332 (s)	3.2 %	$\langle T \rangle = -2.35326725\text{e}+00$	$\langle T^{-1} \rangle = -4.24941112\text{e}-01$
4	0.775557 (s)	3.2 %	$\ \Delta K\ = 7.180\text{e}-04$	$\ E\ = 5.35071\text{e}-06$
Iteration 3				
Step	Execution Time	Memory Usage	Objects	
1	26.978 (s)	2.0 %	$\ L\ = 2.789292\text{e}+00$	$\ \Omega_L\ = 0.000000$
2	43.7547 (s)	3.2 %	$\ \mathcal{L}L\ = 0.325661$	$\ T\ = 1.13989675\text{e}+01$
3	18.5258 (s)	3.2 %	$\langle T \rangle = -2.35312392\text{e}+00$	$\langle T^{-1} \rangle = -4.24966994\text{e}-01$
4	0.780214 (s)	3.2 %	$\ \Delta K\ = 1.940\text{e}-05$	$\ E\ = 4.22474\text{e}-08$
Iteration 4				
Step	Execution Time	Memory Usage	Objects	
1	27.0355 (s)	2.0 %	$\ L\ = 2.785728\text{e}+00$	$\ \Omega_L\ = 0.000000$
2	43.8932 (s)	3.2 %	$\ \mathcal{L}L\ = 0.325439$	$\ T\ = 1.13621447\text{e}+01$
3	18.5478 (s)	3.2 %	$\langle T \rangle = -2.35313653\text{e}+00$	$\langle T^{-1} \rangle = -4.24964718\text{e}-01$
4	0.778876 (s)	3.2 %	$\ \Delta K\ = 1.895\text{e}-08$	$\ E\ = 2.70577\text{e}-11$
Iteration 5				
Step	Execution Time	Memory Usage	Objects	
1	26.9686 (s)	2.0 %	$\ L\ = 2.785741\text{e}+00$	$\ \Omega_L\ = 0.000000$
2	56.311 (s)	3.2 %	$\ \mathcal{L}L\ = 0.32544$	$\ T\ = 1.136222460\text{e}+01$
3	18.4804 (s)	3.2 %	$\langle T \rangle = -2.35313653\text{e}+00$	$\langle T^{-1} \rangle = -4.24964718\text{e}-01$
4	0.778876 (s)	3.2 %	$\ \Delta K\ = 2.706\text{e}-08$	$\ E\ = 9.63249\text{e}-16$

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