UNIVERSITAT DE BARCELONA

Contributions to the Theory of Large Cardinals Beyond Choice

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Doctoral Thesis

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A thesis submitted in fulfillment of the requirements for the degree of Doctor of Philosophy in the program of Mathematics and Computer Science

Departament de Matemàtiques i Informàtica

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Abstract

This thesis investigates large cardinals that are inconsistent with the Axiom of Choice. First, we characterize Berkeley cardinals in terms of a restricted form of Vopěnka's Principle, and determine the consistency strength of several related theories. Next, we present a method for producing elementary embeddings from homomorphisms, which is then used to show that the Strongly Rigid Relation Principle is a weak Choice principle. We also provide a characterization of rank-Berkeley cardinals in terms of a strong failure of this principle.

We then explore the connection between elementary embeddings from the universe into itself and eventually dominating functions, culminating in an alternative proof of Kunen's Inconsistency Theorem. Finally, using the method of forcing, we establish the consistency (relative to large cardinals) of the successor of the first singular cardinal being supercompact in the transitive model of Hereditarily Ordinal Definable sets.

Keywords: Set Theory, Large Cardinals, Axiom of Choice, Forcing.

Contribucions a la teoria dels cardinals grans més enllà de l'elecció

per Marwan Salam MOHAMMD

Resum

Aquesta tesi investiga els cardinals grans que són incompatibles amb l'Axioma d'Elecció. En primer lloc, caracteritzem els cardinals de Berkeley en termes d'una forma restringida del Principi de Vopěnka i determinem la força, en termes de consistència, de diverses teories que hi estan relacionades. A continuació, presentem un mètode per produir aplicacions elementals a partir d'homomorfismes, que s'utilitza per mostrar que el Principi de la Relació Fortament Rígida és un principi feble d'elecció. També proporcionem una caracterització dels cardinals rank-Berkeley en termes d'una negació forta d'aquest principi.

Tot seguit, explorem la connexió entre les aplicacions elementals de l'univers en si mateix i les funcions eventualment dominants, culminant en una demostració alternativa del Teorema de la Inconsistència de Kunen. Finalment, utilitzant el mètode del forcing, establim la consistència (relativa a l'existència de cardinals grans) del fet que el successor del primer cardinal singular sigui supercompacte en el model transitiu dels conjunts definibles ordinalment de manera hereditària.

Paraules clau: Teoria de conjunts, Cardinals grans, Axioma de l'elecció, Forçament.

Acknowledgements

I want to thank my supervisor, Professor Joan Bagaria, for his generous mentorship, patience, and for giving me the freedom to explore my own research directions. His encouragement, especially during moments of setback, made a meaningful difference.

I am deeply grateful to my parents for their incredible patience and support throughout my years of studying abroad.

I am grateful to my friend and colleague Catalina Torres for going over some of my work and for being there during tough times.

I want to thank Dr. Alejandro Poveda for hosting me during my stay at Harvard University, for his kindness, and for teaching me a technically demanding topic in a short time with great clarity.

I am thankful to Professor W. Hugh Woodin for taking the time to meet with me during my visit to Harvard. Those meetings gave me a lot to think about and plenty of material for future work.

I also thank Professor Gabriel Goldberg for a stimulating mathematical discussion during my time at Harvard.

I am grateful to Professor David Asperó for reading and providing helpful comments on my first paper.

I want to thank Professor Vladimir Tolstykh, whose rigorous and inspiring lectures first sparked my interest in mathematics. His encouragement and guidance during my transitional period after undergrad played a pivotal role in my academic path.

I am especially grateful to Idris Nechirvan Barzani, founder and president of the Rwanga Foundation, whose generous support funded the majority of my studies. I also thank the Generalitat de Catalunya (Catalan Government) under grant 2021 SGR 00348 and the Spanish Government under grants MTM-PID2020-116773GBI00, PID2023-147428NB-I00, and EUR2022-134032 for their support.

Contents

CHAPTER 0

Introduction

The study of large cardinals lies at the heart of modern set theory. They provide a hierarchy of increasingly powerful axioms that extend Zermelo-Fraenkel set theory (ZF) and explore the nature of infinity. Remarkably, the various well-known large cardinal notions can be characterized by the existence of elementary embeddings $j\colon V \to M$, where V is the universe of sets and $M \subset V$ is a transitive inner model. When such an embedding is nontrivial—that is, not the identity—they imply the existence of a least cardinal κ such that $j(\kappa) > \kappa$; this cardinal is known as the *critical point of j* and is denoted by $\mathrm{crit}(j)$. As pointed out in [Kan03], the closer the target model M resembles the universe V, the larger the critical point must be.

Traditionally, large cardinal axioms have been studied within the framework of ZFC, which includes the Axiom of Choice (AC). In 1967, William N. Reinhardt pushed the resemblance paradigm to its natural limit by postulating the existence of an elementary embedding $j\colon V \to V$ [Rei67]. Such embeddings are now commonly referred to as *Reinhardt embeddings*, and the critical point of such an embedding is called a *Reinhardt cardinal*. However, not long after their introduction, Kenneth Kunen [Kun71] found an inconsistency of such embeddings with ZFC, using AC in his proof. The question remains open to this day whether AC is absolutely necessary for this refutation.

Theorem 0.0.1 (ZFC, Kunen's Inconsistency Theorem [Kun71])**.** *There is no nontrivial elementary embedding $j\colon V \to V$.*

W. Hugh Woodin, in his Berkeley set theory graduate course around 1990, introduced a notion stronger than a Reinhardt cardinal known as a "Berkeley cardinal" as an exercise for his students to explore potential inconsistencies within ZF only. But, despite the passage of almost half a century, no inconsistencies have been found. Reinhardt cardinals, Berkeley cardinals, and a few other variations on these two large cardinal notions are the topics of the article "Large Cardinals Beyond Choice" [BKW19] by Joan Bagaria, Peter Koellner, and W. Hugh Woodin.

The study of large cardinals beyond Choice has gained wide interest recently in light of new results by Woodin in [Woo10] and [Woo11]. Of particular importance is the following theorem. Recall that HOD is the transitive inner model consisting of hereditarily ordinal definable sets.

Theorem 0.0.2 (Woodin's HOD Dichotomy Theorem [Woo17])**.** *Suppose that κ is an extendible cardinal. Then exactly one of the following holds:*

(1) *For every singular cardinal $\gamma > \kappa$, γ is singular in* HOD *and* $(\gamma^+)^{\mathrm{HOD}} = \gamma^+$.
(2) *Every regular cardinal $\gamma \geq \kappa$ is measurable in* HOD.

This theorem states that, assuming the existence of an extendible cardinal, the universe V is, loosely speaking, either (1) close to HOD or (2) far from it. It is argued in [BKW19] that large cardinals beyond Choice imply the second alternative.

On the other hand, the program of Inner Model Theory aims to establish the first alternative. The official version of the HOD Dichotomy replaces "measurable in HOD" with the stronger "ω-strongly measurable in HOD" in clause (2). Supporting the first alternative of the dichotomy, Woodin formulated the following hypothesis:

Definition 0.0.3 (Woodin's HOD Hypothesis [Woo17])**.** There exists a proper class of regular cardinals γ which are not ω-strongly measurable in HOD.

Even more recently, in as-yet-unpublished joint work by Joan Bagaria, Philipp Lücke, and Juan P. Aguilera, strong evidence has been presented for the failure of the HOD Hypothesis, using large cardinals that are not known to be inconsistent with AC. This development further emphasizes the significance of large cardinals beyond Choice, highlighting the need for a clearer understanding of their hierarchy in order to better grasp the structure of the set-theoretic universe V.

This thesis contributes to the theory of large cardinals beyond Choice by developing new connections, refining existing hierarchies, and clarifying their consistency strength relative to classical principles. Additionally, it investigates the potential divergence between the universe V and the inner model HOD. The results presented here aim to deepen our understanding of the Choiceless large cardinal landscape and its implications for the broader foundations of set theory.

Outline

This thesis is divided into four chapters, each dealing with a separate topic. With the exception of the last chapter, all the results are established in ZF alone, and any use of AC is explicitly noted. Basic knowledge of set theory is assumed, and for background material not covered here, the reader is referred to [Jec03], [Kan03], and [Kun14].

We now present a brief outline of the contents of each chapter.

CHAPTER 1: Exploring the relations between large cardinals beyond Choice and principles of structural reflection, we establish the following characterization of Berkeley cardinals in terms of a certain restricted form of Vopěnka's Principle:

Corollary 1.5.3. *For $n \geq 1$, if $\mathrm{VP}(\mathbf{\Pi}_{n+1})$ fails, then*

$$\sup\{\mu \mid \mathrm{VP}^{\mu}(\mathbf{\Pi}_{n+1}) \textit{ holds}\} = \sup\{\delta \mid \delta \textit{ is a Berkeley cardinal}\}.$$

A similar characterization for rank-Berkeley cardinals is used to prove the following in ZFC:

Corollary 1.5.11 (ZFC)**.** $\mathbb{VP}$ *restricted to definable, without parameters, classes of structures of the same finite type implies (and hence is equivalent to)* $\mathbb{VP}$.

Additionally, the consistency strength of some relevant theories that arise is determined.

CHAPTER 2: We present a method for producing elementary embeddings from homomorphisms, proving the following:

Theorem 2.3.2. *If there exists a set $G \subset \mathbb{R} \times \mathrm{OR}$ for which there is no strongly rigid relation, then for some ordinal α there exists a nontrivial elementary embedding $j\colon V_\alpha^L \to V_\alpha^L$.*

Using this, we prove that the “Strongly Rigid Relation Principle” (SRR) is a weak Choice principle:

Corollary 2.4.2. SRR *is independent from* ZF, *follows from* AC, *but is not equivalent to* AC.

Finally, we characterize proto Berkeley cardinals in terms of a strong failure of SRR.

Theorem 2.5.1. *A cardinal δ is proto Berkeley iff for any graph (G, E) with an injection $f\colon \delta \to G$, there exists an endomorphism $h\colon (G, E) \to (G, E)$ such that* $h|_{\mathrm{ran}(f)} \neq \mathrm{id}$.

<u>CHAPTER 3</u>: We explore some of the consequences of the existence of an elementary embedding $j\colon V \to V$. First, we prove a result concerning j and functions on ordinals that “eventually dominate” j.

Theorem 3.2.1. *If $\delta > \mathrm{crit}(j)$ is a regular cardinal such that $j(\delta) = \delta$, then there is no function $g\colon \delta \to \delta$ in the range of j such that $j|_\delta \leq^* g$.*

Using that, we prove the existence of elementary embeddings reminiscent of extendibility in a more local setting, and provide additional perspective on Goldberg’s “almost” supercompact cardinals.

The chapter ends with an alternative proof of Kunen’s Inconsistency Theorem:

Theorem 3.4.1 (Kunen’s Inconsistency Theorem)**.** *Suppose $j\colon V \to V$ is a Reinhardt embedding. Then* AC *implies* $0 = 1$.

<u>CHAPTER 4</u>: In this final chapter, we use forcing to establish the consistency relative to large cardinals of a significant instance of divergence between V and HOD, notably occurring at the level of small cardinals. In particular, we prove the following:

Theorem 4.1.1. *It is consistent relative to the existence of two supercompact cardinals that $\aleph_\omega$ is a strong limit and $\aleph_{\omega+1}$ is supercompact in* HOD.

Moreover, in the above model, Woodin’s HOD *hypothesis holds.*

The contents of this chapter represent my contributions to an ongoing project with Alejandro Poveda. The overarching idea of the project, including the general approach of using Supercompact Prikry forcings interleaved with collapses, is due to Alejandro Poveda.

CHAPTER 1

Berkeley Cardinals and Vopěnka's Principle

1.1 Introduction

In this chapter, we relate Berkeley cardinals to a very well-known principle of structural reflection called Vopěnka's Principle. Vopěnka's Principle, $\mathbb{VP}$, states that for any proper class of structures of the same type, there exist two distinct members in the class such that one is elementarily embeddable into the other. The precise formulation of this notion will be given in Section 1.3.

Let $\mathrm{VP}(\mathbf{\Pi}_n)$ denote $\mathbb{VP}$ restricted to Π_n-definable, with parameters, classes, and let $\mathrm{VP}^\alpha(\mathbf{\Pi}_n)$ denote $\mathrm{VP}(\mathbf{\Pi}_n)$ restricted to classes with type $\tau \in V_\alpha$. The main results of this chapter are the following:

Theorem 1.5.2. *For $n \geq 1$, if $\mathrm{VP}^\omega(\mathbf{\Pi}_{n+1})$ holds while $\mathrm{VP}(\mathbf{\Pi}_{n+1})$ fails, then $\delta = \sup\{\mu \mid \mathrm{VP}^\mu(\mathbf{\Pi}_{n+1})$ holds$\}$ is a Berkeley cardinal.*

Corollary 1.5.3. *For $n \geq 1$, if $\mathrm{VP}(\mathbf{\Pi}_{n+1})$ fails, then*

$$\sup\{\mu \mid \mathrm{VP}^\mu(\mathbf{\Pi}_{n+1}) \textit{ holds}\} = \sup\{\delta \mid \delta \textit{ is a Berkeley cardinal}\}.$$

These results raise questions about the consistency strength of $\mathrm{ZF} + \mathrm{VP}^\omega(\mathbf{\Pi}_{n+1}) + \neg\mathrm{VP}(\mathbf{\Pi}_{n+1})$. In regards to this, we prove the proposition below. Let BC denote the assertion that there exists a Berkeley cardinal, let $\mathbb{VP}^\omega$ denote $\mathbb{VP}$ restricted to classes of finite type, and let $\mathrm{VP}(\Pi_n(\mathrm{OR}))$ denote $\mathrm{VP}(\mathbf{\Pi}_n)$ restricted to classes definable with ordinal parameters only.

Proposition 1.5.13. *The following theories are equiconsistent:*

(1) $\mathrm{ZF} + \mathrm{VP}^\omega(\mathbf{\Pi}_{n+1}) + \neg\mathrm{VP}(\mathbf{\Pi}_{n+1})$, *for some $n \geq 1$.*
(2) $\mathrm{ZF} + \mathrm{BC}$.
(3) $\mathrm{ZF} + \mathbb{VP}^\omega + \neg\mathrm{VP}(\Pi_1(\mathrm{OR}))$.

Rank-Berkeley cardinals (Definition 1.4.4) are a natural weakening of Berkeley cardinals first discovered by Farmer Schlutzenberg and W. Hugh Woodin, independently, when they realized that their existence follows from the existence of a Reinhardt cardinal. We will establish analogues of 1.5.2, 1.5.3, and 1.5.13 for rank-Berkeley cardinals as well. As an application of that, we get the following corollary:

Corollary 1.5.11 (ZFC)**.** *$\mathbb{VP}$ restricted to definable, without parameters, classes of structures of the same finite type implies (and hence is equivalent to) $\mathbb{VP}$.*

That is, under AC, the weakest possible form of $\mathbb{VP}$ is equivalent to its strongest form. We are not aware if there is a more direct proof of this result (see Remark 1.5.12).

In [Bag12], Bagaria establishes an exact relation between $\mathbb{VP}$ and what he calls $C^{(n)}$-extendible cardinals (see Definition 1.6.6 for an equivalent definition).

Theorem 1.1.1 (ZFC, Bagaria [Bag12]). *For $n \geq 1$, the following are equivalent:*

(1) $\mathrm{VP}(\mathbf{\Pi}_{n+1})$.
(2) *There is a proper class of $C^{(n)}$-extendible cardinals.*

The proof uses the following alternative form of Kunen's inconsistency:

Theorem 1.1.2 (ZFC, [Kan03, Corollary 23.14(a)]). *For any δ, there is no nontrivial elementary embedding $j\colon V_{\delta+2} \to V_{\delta+2}$.*

Theorem 1.5.2 comes from the struggle of bringing Theorem 1.1.1 into the Choiceless context. Thus, we introduce the n-choiceless extendible cardinals to play the role of $C^{(n)}$-extendible cardinals in Bagaria's work, without relying on AC. We establish a characterization parallel to Theorem 1.1.1, stated as follows:

Theorem 1.3.1. *For $n \geq 1$, the following are equivalent:*

(1) $\mathrm{VP}(\Pi_{n+1}(\mathrm{OR}))$
(2) *There is a proper class of n-choiceless extendible cardinals.*
(3) $\mathrm{VP}(\mathbf{\Pi}_{n+1})$.

We also consider the consistency strength of the failure of Theorem 1.1.1 in the Choiceless context.

Corollary 1.6.8. *For $n \geq 1$, the following theories are equiconsistent:*

(1) $\mathrm{ZF} + \mathbb{VP} +$ "$\forall\kappa$ (κ is not $C^{(0)}$-extendible)"
(2) $\mathrm{ZF} + \mathrm{VP}(\mathbf{\Pi}_{n+1}) +$ "$\forall\kappa$ (κ is not $C^{(n)}$-extendible)"
(3) $\mathrm{ZF} + \mathrm{VP}(\mathbf{\Pi}_{n+1}) +$ "$\exists\xi\ \forall\kappa > \xi$ (κ is not $C^{(n)}$-extendible)"
(4) ZF + "There are unboundedly many rank-Berkeley cardinals"

Another interesting theory is $\mathrm{ZF} + \mathrm{VP}(\mathbf{\Pi}_{n+1}) +$ "OR is not $\mathbf{\Pi}_{n+1}$-Mahlo" (Definition 1.6.9). Assuming AC, since $C^{(n)}$-extendible cardinals are inaccessible and by Theorem 1.1.1, it follows that $\mathrm{VP}(\mathbf{\Pi}_{n+1})$ implies OR is $\mathbf{\Pi}_{n+1}$-Mahlo. In the choiceless context, we have the following:

Corollary 1.6.11. *The following theories are equiconsistent:*

(1) $\mathrm{ZF} + \mathrm{VP}(\mathbf{\Pi}_{n+1}) +$ "OR is not $\mathbf{\Pi}_{n+1}$-Mahlo", *for some $n \geq 1$.*
(2) ZF + "There are unboundedly many rank-Berkeley cardinals"
(3) $\mathrm{ZF} + \mathbb{VP} +$ "OR is not $\mathbf{\Pi}_2$-Mahlo"

The outline of this chapter is as follows. In Section 1.2, we introduce "n-choiceless" extendible and supercompact cardinals, and prove relations between them similar to those of Bagaria's $C^{(n)}$-extendible and Σ_n-supercompact cardinals in [Bag12] and [BP23]. In Section 1.3, we prove Theorem 1.3.1. In Sections 1.4 and 1.5, we define the various notions of Berkeley cardinals, prove some results about them, and show how they relate to Vopěnka's Principle. Finally, in the last section, we prove Corollaries 1.6.8 and 1.6.11.

1.2 The Choiceless Cardinals

Recall that $C^{(n)}$ is the class of ordinals α that are Σ_n-correct, i.e., $V_\alpha \prec_{\Sigma_n} V$. Given any set X, the statement "$X \prec_{\Sigma_n} V$" is given by the following formula:

$$\forall\varphi \in \Sigma_n\ \forall x \in X^{<\omega}\ (V \models_n \varphi[x] \implies X \models \varphi[x]). \tag{1.1}$$

Now, the satisfaction relation $\models$ for sets is Δ_1, and the global satisfaction relation $\models_n$ for Σ_n formulas is Σ_n when $n \geq 1$ (see [Kan03], Section 0). Hence, (1.1) is Π_n when $n \geq 1$.

The class $C^{(0)}$ is clearly the entire class OR of ordinals, and is therefore Δ_0-definable. For $n \geq 1$, the class $C^{(n)}$ is defined by:

$$\alpha \in C^{(n)} \iff \forall X\ (X = V_\alpha \implies X \prec_{\Sigma_n} V).$$

Since "$X = V_\alpha$" is Π_1, the defining formula for $C^{(n)}$ is Π_2 when $n = 1$, and Π_n when $n \geq 2$. But, $C^{(1)}$ is also definable using the following:

$$\alpha \in C^{(1)} \iff \exists X\ (X = V_\alpha\ \wedge\ X \prec_{\Sigma_1} V)$$

which is Σ_2. As a result, we have that $C^{(n)}$ is Δ_0 when $n = 0$, Δ_2 when $n = 1$, and Π_n when $n \geq 2$. We remark that if AC holds, $C^{(1)}$ becomes Π_1 [Bag12, Section 1].

Definition 1.2.1. For each $n \geq 0$, given ordinals $\alpha < \gamma < \mu$ with μ in $C^{(n)}$, we say that γ is (α, μ, n)-*choiceless extendible* iff there is ν in $C^{(n)}$ and an elementary embedding $j\colon V_\mu \to V_\nu$ such that $\mathrm{crit}(j) > \alpha$ and $j(\gamma) > \mu$. We say that γ is α-n-*choiceless extendible* iff γ is (α, μ, n)-choiceless extendible for all $\mu > \gamma$ in $C^{(n)}$, and we simply say that γ is n-*choiceless extendible* iff it is α-n-choiceless extendible for all $\alpha < \gamma$.

We shall say that γ is $({<}\alpha, \mu, n)$-(respectively $({\leq}\alpha, \mu, n)$-)choiceless extendible iff it is (β, μ, n)-choiceless extendible for all $\beta < \alpha$ (respectively $\beta \leq \alpha$). Similar remarks hold for μ. Furthermore, we will allow the occurrence of OR in the second coordinate. Hence, for example, a cardinal γ is n-choiceless extendible iff it is $(<\gamma, <\mathrm{OR}, n)$-choiceless extendible.

The definition above stems from Bagaria's $C^{(n)}$-extendible cardinals [Bag12, p. 12]. Notions somewhat similar to α-0-choiceless extendibility appear in the works of David Asperó [Asp] and Gabriel Goldberg [Gol24]. Notice that if γ is α-n-choiceless extendible, then every ordinal $\delta > \gamma$ is α-n-choiceless extendible. On the other hand, if γ is n-choiceless extendible, then it must be a cardinal.

In the context of AC, one can use the fact that there is no elementary embedding $j\colon V_{\delta+2} \to V_{\delta+2}$ to show that an n-choiceless extendible cardinal γ is either $C^{(n)}$-extendible or a limit of $C^{(n)}$-extendible cardinals. However, if there is such an embedding, then it is possible that this fails for γ. And when it fails, we can argue that the majority of the witnessing js (meaning all except for set many) have critical points strictly between α and γ and are such that $\{\beta \mid j(\beta) = \beta\} \cap (\gamma \setminus \mathrm{crit}(j)) \neq \emptyset$. This will be clear in the final section.

A lot of the important properties of $C^{(n)}$-extendible cardinals are still provable under this new more general definition, albeit sometimes with slightly more technical difficulties and restrictions. For example, the following is generalized from the case of $C^{(n)}$-extendible cardinals in [Bag12].

Proposition 1.2.2. *For each $n \geq 0$, every n-choiceless extendible cardinal is Σ_{n+2}-correct.*

Proof. The proof is by finite induction. Fix an n-choiceless extendible cardinal γ, and suppose that γ is Σ_m-correct for some $m < n+2$. Let $\exists x \psi(x, y)$ be a Σ_{m+1} formula, and assume that $V \models \exists x \psi(x, y_0)$ for some $y_0 \in V_\gamma$. Let x_0 be a witness in V. Using the n-choiceless extendibility of γ, take an elementary embedding $j\colon V_\mu \to V_\nu$ such that $\mathrm{crit}(j) > \mathrm{rank}(y_0)$, $j(\gamma) > \mu$, $x_0 \in V_\mu$, and $\mu, \nu \in C^{(n)}$. Since $\psi(x, y)$ is Π_m where m

is at most $n+1$, and $\nu \in C^{(n)}$, we have $V_\nu \models \psi(x_0, y_0)$. By the inductive hypothesis, $V_{j(\gamma)} \prec_{\Sigma_m} V_\nu$, hence $V_{j(\gamma)} \models \psi(x_0, y_0)$. Finally, by elementarity of j and the fact that $j(y_0) = y_0$, we get $V_\gamma \models \exists x \psi(x, y_0)$. □

Joan Bagaria and Alejandro Poveda [BP23] prove an equivalence between the notions of $C^{(n)}$-extendibility and Σ_{n+1}-supercompactness. An analogous equivalence will be important for our purposes. We are therefore led to the following definitions.

Definition 1.2.3. For each $n \geq 0$, given ordinals $\alpha < \gamma < \lambda$ with λ in $C^{(n)}$, and given a set $a \in V_\lambda$, we say that γ is (α, λ, a, n)*-choiceless supercompact* iff there exists $\bar{\lambda} < \gamma$ in $C^{(n)}$ and $\bar{a} \in V_{\bar{\lambda}}$ for which there is an elementary embedding $j \colon V_{\bar{\lambda}} \to V_\lambda$ with $\mathrm{crit}(j) > \alpha$ and $j(\bar{a}) = a$. We say that γ is α*-*n*-choiceless supercompact* iff γ is (α, λ, a, n)-choiceless supercompact for all $\lambda > \gamma$ in $C^{(n)}$ and for all $a \in V_\lambda$, and we simply say that γ is n-choiceless supercompact iff it is α-n-choiceless supercompact for all $\alpha < \gamma$.

Again, similar to the case of choiceless extendible cardinals, we allow the use of inequality symbols and OR in our notation for choiceless supercompact cardinals. Here, for the third coordinate, we will also use the notation $<X$, where it will mean that the set a can be any member of the set or class X.

Definition 1.2.4. For each $n \geq 0$, given ordinals $\alpha < \gamma < \mu$, we say that γ is (α, μ, n)-*choiceless extendible** iff there is a $\nu > \mu$ and an elementary embedding $j \colon V_\mu \to V_\nu$ such that $\mathrm{crit}(j) > \alpha$, $j(\gamma) > \mu$, and $j(\gamma) \in C^{(n)}$. We say that γ is α*-*n*-choiceless extendible** iff γ is (α, μ, n)-choiceless extendible* for all $\mu > \gamma$ in $C^{(n)}$, and we simply say that γ is n*-choiceless extendible** iff it is (α, μ, n)-choiceless extendible* for all $\alpha < \gamma$.

Remarks similar to those following the previous two definitions about the use of symbols such as $\leq$ and OR apply for the above definition as well. We will also need the fact that n-choiceless extendible* cardinals are Σ_{n+2}-correct.

Proposition 1.2.5. *For each $n \geq 0$, every n-choiceless extendible* cardinal is Σ_{n+2}-correct.*

Proof. Fix γ that is n-choiceless extendible*. Let $\exists x \psi(x, y)$ be a Σ_{n+2} formula, and assume that $V \models \exists x \psi(x, y_0)$ for some $y_0 \in V_\gamma$. Let x_0 be a witness in V. Take an elementary embedding $j \colon V_\mu \to V_\nu$ such that $\mathrm{crit}(j) > \mathrm{rank}(y_0)$, $j(\gamma) > \mu$, $x_0 \in V_\mu$, and $j(\gamma) \in C^{(n)}$. As $\psi(x, y)$ is Π_{n+1} and $j(\gamma) \in C^{(n)}$, we have $V_{j(\gamma)} \models \psi(x_0, y_0)$. By elementarity of j and the fact that $j(y_0) = y_0$, we conclude that $V_\gamma \models \exists x \psi(x, y_0)$. □

Lemma 1.2.6. *For $n \geq 0$, if γ is α-n-choiceless extendible* for some fixed $\alpha < \gamma$, then it is also α-$n{+}1$-choiceless supercompact.*

Proof. Suppose γ is α-n-choiceless extendible*. Fix $\lambda > \gamma$ in $C^{(n+1)}$ and a set $a \in V_\lambda$, and let us show that γ is $(\alpha, \lambda, a, n{+}1)$-choiceless supercompact. Let $\mu > \lambda$ be in $C^{(n+1)}$ and, using the fact that γ is α-n-choiceless extendible*, let $j \colon V_\mu \to V_\nu$ be such that $j(\gamma) > \mu$, where $j(\gamma)$ is in $C^{(n)}$, and $\mathrm{crit}(j) > \alpha$. Notice now that $j|_{V_\lambda}$ belongs to V_ν.

Claim 1.2.6.1. $V_\lambda \prec_{\Sigma_{n+1}} V_\nu$.

Proof of claim. On the one hand, $\lambda \in C^{(n+1)}$ and $j(\gamma) \in C^{(n)}$ imply $V_\lambda \prec_{\Sigma_{n+1}} V_{j(\gamma)}$. On the other hand, $V_\gamma \prec_{\Sigma_{n+1}} V_\mu$ by Proposition 1.2.5, hence elementarity of j gives $V_{j(\gamma)} \prec_{\Sigma_{n+1}} V_\nu$. Putting both together, the claim follows. □

Thus, $j|_{V_\lambda}$ witnesses in V_ν the $(\alpha, j(\lambda), j(a), n{+}1)$-choiceless supercompactness of $j(\gamma)$. By elementarity of j, there must be some k witnessing the $(\alpha, \lambda, a, n{+}1)$-choiceless supercompactness of γ in V_μ. But, since μ is correct enough, any such k will be a real witness for the $(\alpha, \lambda, a, n{+}1)$-choiceless supercompactness of γ. Since λ and a were arbitrary, we are done. □

Lemma 1.2.7. *For $n \geq 1$, if γ is α-$n{+}1$-choiceless supercompact for some fixed $\alpha < \gamma$, then it is also α-n-choiceless extendible.*

Proof. Suppose γ is α-$n{+}1$-choiceless supercompact, and fix $\mu > \gamma$ in $C^{(n)}$. We want to show that γ is (α, μ, n)-choiceless extendible. Let $\lambda > \mu$ be in $C^{(n+1)}$ and using the α-$n{+}1$-choiceless supercompactness of γ, let $j \colon V_{\bar\lambda} \to V_\lambda$ be an elementary embedding such that $\mathrm{crit}(j) > \alpha$, $j(\bar\gamma) = \gamma$, $j(\bar\mu) = \mu$, and $\bar\lambda \in C^{(n+1)}$. By elementarity $\bar\mu$ is in fact in $C^{(n)}$, therefore $j|_{V_{\bar\mu}}$ witnesses the $(\alpha, \bar\mu, n)$-choiceless extendibility of $\bar\gamma$. The existence of such witness is a Σ_{n+1} statement in the parameters $\alpha, \bar\mu, V_{\bar\mu}$, and $\bar\gamma$, as seen in the formula

$$\exists k, Y, \nu\, (Y = V_\nu \ \wedge\ Y \prec_{\Sigma_n} V \ \wedge\ k \colon V_{\bar\mu} \prec Y \ \wedge\ \mathrm{crit}(k) > \alpha \ \wedge\ k(\bar\gamma) > \bar\mu). \tag{1.2}$$

Since $\bar\lambda \in C^{(n+1)}$, we must have $k \in V_{\bar\lambda}$ that witnesses the $(\alpha, \bar\mu, n)$-choiceless extendibility of $\bar\gamma$ in $V_{\bar\lambda}$. Now elementarity of j tells us that $j(k)$ witnesses the (α, μ, n)-choiceless extendibility of γ in V_λ. Since λ is correct enough, this last statement is also true in V; and as λ was arbitrary, we are done. □

Note that, for $n = 0$, the proof above fails because the complexity of (1.2) is Σ_2 rather than Σ_1. In fact, it is easy to show that proving the case $n = 0$ will render supercompact cardinals inconsistent in ZFC.

Theorem 1.2.8. *For $n \geq 1$, the following are equivalent:*

(1) *γ is n-choiceless extendible.*
(2) *γ is n-choiceless extendible**.
(3) *γ is $n{+}1$-choiceless supercompact.*

Proof. (1) implies (2): Clear using Proposition 1.2.2. (2) implies (3): Lemma 1.2.6. (3) implies (1): Lemma 1.2.7. □

1.3 VP and Choiceless Extendible Cardinals

Recall that *Vopěnka's Principle* is the axiom schema stating that for every proper class $\mathcal{C}$ of structures of the same type that is definable, with parameters, there exist $A \neq B$ in $\mathcal{C}$ such that A is elementarily embeddable into B.

Say that a class $\mathcal{C}$ is $\Sigma_n(X)$ (respectively $\Pi_n(X)$) for some class (or set) X iff $\mathcal{C}$ is definable with parameters from X by a Σ_n (respectively Π_n) formula. The boldface symbols $\boldsymbol{\Sigma}_n$ and $\boldsymbol{\Pi}_n$ are used in place of $\Sigma_n(V)$ and $\Pi_n(V)$, respectively, and the lightface symbols Σ_n and Π_n are used in place of $\Sigma_n(\emptyset)$ and $\Pi_n(\emptyset)$, respectively.

Let Γ be a placeholder for the symbols $\boldsymbol{\Sigma}_n, \boldsymbol{\Pi}_n, \Sigma_n, \Pi_n, \Sigma_n(X), \Pi_n(X)$. We say that $\mathrm{VP}(\Gamma)$ holds iff Vopenka's Principle holds for any proper class $\mathcal{C}$ (of structures of the same type) that is Γ. Notice that $\mathrm{VP}(\Gamma)$ can be stated by a single axiom. The failure of $\mathrm{VP}(\Gamma)$ will be denoted by $\neg\mathrm{VP}(\Gamma)$. The notation $\mathrm{VP}(X)$ for a class of parameters X is used iff $\mathrm{VP}(\Pi_n(X))$ holds for all n. Notice that $\mathrm{VP}(X)$ is an axiom schema. We will also use VP and $\mathbb{VP}$ in place of $\mathrm{VP}(\emptyset)$ and $\mathrm{VP}(V)$, respectively.

Theorem 1.3.1. *For $n \geq 1$, the following are equivalent:*

(1) $\mathrm{VP}(\Pi_{n+1}(\mathrm{OR}))$.
(2) *There is a proper class of n-choiceless extendible cardinals.*
(3) $\mathrm{VP}(\boldsymbol{\Pi}_{n+1})$.

Proof. (1) implies (2): We will show that $\mathrm{VP}(\Pi_{n+1}(\mathrm{OR}))$ implies that, for any α, there is a $\gamma_\alpha > \alpha$ that is α-n-choiceless extendible. Given this, we can then construct the sequence $\langle \alpha_n \rangle_{n<\omega}$ by letting $\alpha_0 = \alpha$ and $\alpha_{n+1} = \gamma_{\alpha_n}$. Clearly then $\delta = \lim \alpha_n$ will be an n-choiceless extendible cardinal. Moreover, since $\delta > \alpha$ and α was arbitrary, it will follow that there is a proper class of n-choiceless extendible cardinals.

So, let α be an arbitrary fixed ordinal and suppose, towards a contradiction, that no ordinal $\gamma > \alpha$ is α-n-choiceless extendible. This means that for any ordinal $\gamma > \alpha$ there exists $\mu > \gamma$ in $C^{(n)}$ for which the following holds:

$$\neg \exists j \colon V_\mu \prec V_\nu \; (\mathrm{crit}(j) > \alpha \;\wedge\; j(\gamma) > \mu \;\wedge\; V_\nu \prec_{\Sigma_n} V).$$

Let $\psi(\alpha, \gamma, \mu)$ the displayed formula above. Its complexity, when $n \geq 1$, is Π_{n+1}. Define $D(\alpha, n)$ to be the class of all β such that β is a limit ordinal above α and for every γ strictly between α and β there already is a $\mu \in C^{(n)}$ also below β for which $\psi(\alpha, \gamma, \mu)$ holds. Formally,

$$\begin{aligned}\beta \in D(\alpha, n) \iff \; & \beta \in \mathrm{Lim}(\mathrm{OR}) \setminus (\alpha + 1) \\ & \wedge\; \forall \gamma \in (\alpha, \beta)\, \exists \mu < \beta \; (\mu \in C^{(n)} \;\wedge\; \psi(\alpha, \gamma, \mu)).\end{aligned}$$

It is easy to see that $D(\alpha, n)$ is a club class of ordinals that is Π_{n+1}-definable.

Let $\mathcal{C}$ be the class of structures of the form $(V_\beta, D(\alpha, n) \cap \beta, \xi)_{\xi \leq \alpha}$ such that $\beta \in D(\alpha, n) \cap C^{(n+1)}$ and $D(\alpha, n) \cap \beta \in V_\beta$ (this last condition ensures that $D(\alpha, n) \cap \beta$ is bounded below β so that β is not a limit of ordinals in $D(\alpha, n)$). Notice that $\beta \in C^{(n+1)}$ implies $D(\alpha, n) \cap \beta = D(\alpha, n)^{V_\beta}$. Therefore, the class $\mathcal{C}$ is $\Pi_{n+1}(\mathrm{OR})$.

Now, we can apply $\mathrm{VP}(\Pi_{n+1}(\mathrm{OR}))$ to $\mathcal{C}$ and get an elementary embedding

$$j \colon (V_{\beta_1}, D(\alpha, n) \cap \beta_1, \xi)_{\xi \leq \alpha} \to (V_{\beta_2}, D(\alpha, n) \cap \beta_2, \xi)_{\xi \leq \alpha}$$

where $\beta_1 \neq \beta_2$. Let $\sigma_i = \sup(D(\alpha, n) \cap \beta_i)$ for $i = 1, 2$. Notice that the σ_i are both in $D(\alpha, n)$ and that $j(\sigma_1) = \sigma_2$. Since each σ_i is uniquely identified by their respective β_i and since $\beta_1 \neq \beta_2$, we also have $\sigma_1 \neq \sigma_2$. In particular, j is not the identity. We have $j(\xi) = \xi$ for all $\xi \leq \alpha$ due to the constants ξ for all $\xi \leq \alpha$, so the critical point of j must be above α. As $\beta_1 \in C^{(n+1)}$, there are $\mu \in C^{(n)}$ arbitrarily high in β_1. For any such μ, $j(\mu)$ is in $C^{(n)}$ as well, by elementarity of $j \colon V_{\beta_1} \to V_{\beta_2}$ and the fact that the β_i are both themselves in $C^{(n)}$. Thus, the restriction of j to V_μ for any $\mu \in C^{(n)}$ with $\sigma_1 < \mu < \beta_1$ will give us an elementary embedding $j|_{V_\mu} \colon V_\mu \to V_{j(\mu)}$ with $\mathrm{crit}(j|_{V_\mu}) > \alpha$, $j|_{V_\mu}(\sigma_1) = \sigma_2 \geq \beta_1 > \mu$, and $j(\mu) \in C^{(n)}$. But such an embedding cannot exist since we know that as $\beta_1 \in D(\alpha, n)$, for some $\mu \in C^{(n)}$ with $\sigma_1 < \mu < \beta_1$, the formula $\psi(\alpha, \sigma_1, \mu)$ holds.

(2) implies (3): Let $\mathcal{C}$ be a proper class of structures of the same type τ that is $\boldsymbol{\Pi}_{n+1}$. Let γ be n-choiceless extendible and sufficiently large so that there exists some $\alpha < \gamma$ such that τ along with any parameters p of some defining Π_{n+1} formula for $\mathcal{C}$ are all in V_α. Fix such an α. Using Theorem 1.2.8, we know that γ is $n{+}1$-choiceless supercompact. Let $B \in \mathcal{C}$ have rank above γ and let $\lambda \in C^{(n+1)}$ be an ordinal above this rank. By $n{+}1$-choiceless supercompactness let $j \colon V_{\bar{\lambda}} \to V_\lambda$ be an elementary embedding with $\bar{\lambda} \in C^{(n+1)}$, $j(\bar{B}) = B$, and $\mathrm{crit}(j) > \alpha$. By correctness of $\bar{\lambda}$ and elementarity of j we must have $\bar{B} \in \mathcal{C}$, and obviously $\bar{B} \neq B$ by considering their

respective ranks. Hence, the restriction of j to $\bar{B}$ is an elementary embedding from $\bar{B}$ into B, and we are done.

(3) implies (1): Trivial. □

Corollary 1.3.2. *The following are equivalent:* VP, VP(OR), *and* $\mathbb{VP}$.

Proof. VP implies VP(OR): Let $\phi(x, \alpha)$ be a formula that defines, for some ordinal α, a class of structures for which VP(OR) fails. Notice that the least α for which this happens is definable without parameters. Thus, VP fails too.

VP(OR) implies $\mathbb{VP}$ by the previous theorem, and $\mathbb{VP}$ implies VP trivially. □

1.4 Berkeley Cardinals

We start by recalling the definition of and some basic facts about Berkeley cardinals from [BKW19].

Definition 1.4.1 ([BKW19])**.** A cardinal δ is *ζ-proto Berkeley,* for some ordinal $\zeta < \delta$, iff for all transitive M with $\delta \in M$ there is an elementary embedding $j \colon M \to M$ with $\zeta < \mathrm{crit}(j) < \delta$. A cardinal δ is *Berkeley* iff it is ζ-proto Berkeley for all $\zeta < \delta$.

Lemma 1.4.2 ([BKW19])**.** *A cardinal δ is ζ-proto Berkeley iff for all transitive M with $\delta \in M$ and all $b \in M$ there is an elementary embedding $j \colon M \to M$ with $\zeta < \mathrm{crit}(j) < \delta$ and $j(b) = b$.*

Proposition 1.4.3 ([BKW19])**.** *For any fixed ordinal ζ, the least ζ-proto Berkeley cardinal, if it exists, is also a Berkeley cardinal.*

For our purposes, we will also be interested in a somewhat weakened version of Berkeley cardinals. We will simply restrict the definitions to transitive sets M of the form V_λ for some λ.

Definition 1.4.4. A cardinal δ is *ζ-proto rank-Berkeley,* for some ordinal $\zeta < \delta$, iff for all $\lambda > \delta$ there is an elementary embedding $j \colon V_\lambda \to V_\lambda$ with $\zeta < \mathrm{crit}(j) < \delta$ and $j(\delta) = \delta$. A cardinal δ is *rank-Berkeley* iff it is ζ-proto rank-Berkeley for all $\zeta < \delta$.

Let $\mathcal{E}_\lambda$ denote the set of all nontrivial elementary embeddings $j \colon V_\lambda \to V_\lambda$, and let $\mathcal{E}_\lambda^\delta = \{j \in \mathcal{E}_\lambda \mid \mathrm{crit}(j) < \delta \wedge j(\delta) = \delta\}$. We want to have an analogue of Lemma 1.4.2 which will allow us to impose the extra condition of fixing an arbitrary ordinal $\alpha < \lambda$ on $j \colon V_\lambda \to V_\lambda$. For this, we first need to define an operation from $\mathcal{E}_\lambda^\delta \times \mathcal{E}_\lambda^\delta$ to $\mathcal{E}_\lambda^\delta$.

Definition 1.4.5. If λ is a limit ordinal, then for any $j, k \colon V_\lambda \to V_\lambda$ define the operation $j[k]$, the *application of j to k*, by setting $j[k] = \bigcup_{\gamma<\lambda} j(k|_{V_\gamma})$.

The following lemma is similar to [Deh10, Lemma 1.6].

Lemma 1.4.6. *If λ is a limit ordinal and $j, k \in \mathcal{E}_\lambda^\delta$, then $j[k]$ is also in $\mathcal{E}_\lambda^\delta$. Moreover,* $\mathrm{crit}(j[k]) = j(\mathrm{crit}(k))$.

Proof. First note that, for any $\gamma_1, \gamma_2 < \lambda$, the fact that the two functions $k|_{V_{\gamma_1}}$ and $k|_{V_{\gamma_2}}$ are compatible implies that $j(k|_{V_{\gamma_1}})$ and $j(k|_{V_{\gamma_2}})$ are compatible. Therefore, $j[k]$ is a function with domain and codomain V_λ. Also, it is injective since it is the union of a $\subset$-chain of injections.

To see that it is elementary, fix a formula $\phi(x)$ and an ordinal $\gamma < \lambda$. By elementarity of k, we have

$$\forall x \in V_\gamma \ (V_\lambda \models \phi(x) \iff V_\lambda \models \phi(k|_{V_\gamma}(x))).$$

Applying j to the above formula gives

$$\forall x \in V_{j(\gamma)} \ (V_\lambda \models \phi(x) \iff V_\lambda \models \phi(j(k|_{V_\gamma})(x))).$$

Since γ was arbitrary, the above must be correct for all x in V_λ.

The fact that $\text{crit}(j[k]) = j(\text{crit}(k))$ follows from two facts: $j[k](j(\text{crit}(k))) > j(\text{crit}(k))$, which follows from $k(\text{crit}(k)) > \text{crit}(k)$, and $\forall \gamma < j(\text{crit}(k))\ (j[k](\gamma) = \gamma)$, which follows from $\forall \gamma < \text{crit}(k)\ (k(\gamma) = \gamma)$. Finally,

$$j[k](\delta) = j(k|_{V_{\delta+1}})(\delta) = j(k|_{V_{\delta+1}})(j(\delta)) = j(k|_{V_{\delta+1}}(\delta)) = j(\delta) = \delta. \qquad \square$$

Now, for limit ordinal λ and $j \in \mathcal{E}^\delta_\lambda$, define $j_0 = j$ and $j_{n+1} = j[j_n]$. By induction and the previous lemma, each j_n is in $\mathcal{E}^\delta_\lambda$. The lemma below will now work as an analogue of Lemma 1.4.2.

Lemma 1.4.7 ([GS24, Theorem 5.6(4)])**.** *Let λ be a limit ordinal, and let $j \in \mathcal{E}^\delta_\lambda$. Then for every $\alpha < \lambda$, there exists n such that for all $m \geq n$, $j_m(\alpha) = \alpha$.*

Proof. Suppose this is not the case for some fixed $j \in \mathcal{E}^\delta_\lambda$. Let $\alpha < \lambda$ be the least counterexample. For each n, let $A_n \subset \alpha$ be the set of ordinals below α that are fixed by j_m for all $m \geq n$. By minimality of α, it must be that $\alpha = \bigcup_n A_n$. Now, $j(A_n)$ is the set of ordinals below $j(\alpha)$ that are fixed by $j[j_m] = j_{m+1}$ for all $m \geq n$, and $j(\alpha) = j(\bigcup_n A_n) = \bigcup_n j(A_n)$. By assumption, $\alpha < j(\alpha)$, so there must be some n such that $\alpha \in j(A_n)$. But this means that α is a fixed point of j_m for all $m \geq n+1$, contradicting the choice of α. $\square$

Proposition 1.4.8. *For any ordinal ζ, the least ζ-proto rank-Berkeley cardinal, if it exists, is also a rank-Berkeley cardinal.*

Proof. Let δ be the least ζ-proto rank-Berkeley cardinal and suppose it is not rank-Berkeley. Fix $\alpha \in (\zeta, \delta)$ and $\lambda > \delta$ such that $\text{crit}(j) \leq \alpha$ for all $j \in \mathcal{E}^\delta_\lambda$. We will show that α must be a ζ-proto rank-Berkeley, contradicting the choice of δ.

Let $\mu > \alpha$ be arbitrary and let ν be an ordinal above $\max\{\mu, \lambda\}$. By ζ-proto rank-Berkeleyness of δ, fix a $j \in \mathcal{E}^\delta_\nu$ such that $\text{crit}(j) > \zeta$. By Lemma 1.4.7, there is n such that j_n fixes α, λ, and μ, and moreover, $\text{crit}(j_n) > \zeta$. Now, $j_n|_{V_\lambda} \in \mathcal{E}^\delta_\lambda$, hence $\text{crit}(j_n) \leq \alpha$. But also $j_n(\alpha) = \alpha$, so in fact $\text{crit}(j_n) < \alpha$. Finally, since j_n fixes μ, we can restrict j_n to V_μ so that $j_n|_{V_\mu}$ witnesses ζ-proto rank-Berkeleyness of α at the arbitrary ordinal μ. $\square$

1.5 VP and Berkeley Cardinals

Let $\text{VP}^\alpha(\Gamma)$ be $\text{VP}(\Gamma)$ restricted to structures of type $\tau \in V_\alpha$. It is easy to see that the class $\{\alpha \mid \text{VP}^\alpha(\Gamma) \text{ holds}\}$ is closed. The following result is similar to [Gol24, Corollary 2.3], but for Berkeley cardinals.

Proposition 1.5.1. *If δ is a Berkeley cardinal, then $\mathbb{VP}^\delta$ holds.*

Proof. Let $\mathcal{C}$ be a definable, with parameters, proper class of structures of the same type $\tau \in V_\delta$. Let $\text{rank}(x)$ denote the rank function and $\text{otp}(x)$ the order-type function on sets of ordinals. Consider the class function $F\colon \mathcal{C} \to \text{OR}$ defined by

$$F(A) = \text{otp}(\text{rank}(A) + 1 \cap \text{ran}(\text{rank}\,|_{\mathcal{C}})).$$

Since $\mathcal{C}$ is a proper class, $\text{ran}(F) = \text{OR}$.

Denote $\mathcal{C} \cap V_\alpha$ by $\mathcal{C}_\alpha$. Let $\lambda > \delta$ be large enough so that $\delta \subset \mathrm{ran}(F|_{\mathcal{C}_\lambda})$. By Berkeleyness, fix an elementary embedding $j\colon V_{\lambda+1} \to V_{\lambda+1}$ such that $j(\mathcal{C}_\lambda) = \mathcal{C}_\lambda$ and $\tau \in V_\kappa$, where κ is the critical point. We know that there is $A \in V_\lambda$ such that $F(A) = \kappa$. Now, since $F(j(A)) = j(F(A)) = j(\kappa) \neq \kappa$, we must have $A \neq j(A)$. Also, as j fixes $\mathcal{C}_\lambda$, $j(A)$ is in $\mathcal{C}_\lambda$ as well. So j restricted to the structure A of $\mathcal{C}$ elementarily embeds it into the different structure $j(A)$ of $\mathcal{C}$, and we are done. □

In the other direction, by building on the proof of Theorem 1.3.1, we get:

Theorem 1.5.2. *For $n \geq 1$, if $\mathrm{VP}^\omega(\mathbf{\Pi}_{n+1})$ holds while $\mathrm{VP}(\mathbf{\Pi}_{n+1})$ fails, then $\delta = \sup\{\mu \mid \mathrm{VP}^\mu(\mathbf{\Pi}_{n+1})$ holds$\}$ is a Berkeley cardinal.*

Proof. Since $\mathrm{VP}(\mathbf{\Pi}_{n+1})$ fails, by Theorem 1.3.1, there are only boundedly many n-choiceless extendible cardinals, if any. By the first paragraph of the proof of the same theorem, for some $\alpha > \delta$, there is no α-n-choiceless extendible ordinal. We will show that α is a ζ-proto Berkeley cardinal for all $\zeta < \delta$. Then, by Proposition 1.4.3, this will imply that for each $\zeta < \delta$, there is a Berkeley cardinal δ_ζ such that $\zeta < \delta_\zeta \leq \alpha$. By the definition of δ and Proposition 1.5.1, none of the δ_ζ can be greater than δ. In particular, this will mean that δ is either δ_ζ for some ζ or the limit of all the δ_ζ. In either case, δ will be a Berkeley cardinal.

So let us show that α is a ζ-proto Berkeley for all $\zeta < \delta$. Thus, fix any $\zeta < \delta$. Let M be any transitive set that contains α. Define the class $\mathcal{C}(\alpha, M, \zeta)$ to be the class of structures of the form $(V_\beta, D(\alpha, n) \cap \beta, \alpha, M, \zeta, \xi)_{\xi<\zeta}$ where $D(\alpha, n)$ is the same as in Theorem 1.3.1, $\beta \in D(\alpha, n) \cap C^{(n+1)}$, $D(\alpha, n) \cap \beta \in V_\beta$, and $M \in V_\beta$. Again, we have that $\mathcal{C}(\alpha, M, \zeta)$ is Π_{n+1}-definable from the parameters α, M, ζ.

Now, an application of $\mathrm{VP}^\delta(\mathbf{\Pi}_{n+1})$ to the class $\mathcal{C}(\alpha, M, \zeta)$ will give us an elementary embedding

$$j\colon (V_{\beta_1}, D(\alpha, n) \cap \beta_1, \alpha, M, \zeta, \xi)_{\xi<\zeta} \to (V_{\beta_2}, D(\alpha, n) \cap \beta_2, \alpha, M, \zeta, \xi)_{\xi<\zeta}$$

for some $\beta_1 \neq \beta_2$. j is not the identity, because $\beta_1 \neq \beta_2$ implies $j(\sigma_1) = \sigma_2 \neq \sigma_1$, where $\sigma_i = \sup(D(\alpha, n) \cap \beta_i)$, $i \in \{1, 2\}$. The critical point of j cannot be α as α is fixed by j, and it cannot be higher than α by the definition of $\mathcal{C}(\alpha, M, \zeta)$ and the choice of α. Moreover, the critical point has to be strictly greater than ζ due to the constants ζ and ξ for all $\xi < \zeta$. So, we are left with $\zeta < \mathrm{crit}(j) < \alpha$. Thus, the restriction of j to M is our desired elementary embedding. □

Corollary 1.5.3. *For $n \geq 1$, if $\mathrm{VP}(\mathbf{\Pi}_{n+1})$ fails, then*

$$\sup\{\mu \mid \mathrm{VP}^\mu(\mathbf{\Pi}_{n+1}) \textit{ holds}\} = \sup\{\delta \mid \delta \textit{ is a Berkeley cardinal}\}.$$

Proof. By Theorem 1.5.2 and Proposition 1.5.1. □

Corollary 1.5.4. *For $n \geq 1$, if $\mathrm{VP}(\mathbf{\Pi}_{n+1})$ fails, then the existence of a Berkeley cardinal is equivalent to the schema $\mathbb{VP}^\omega$.*

Corollary 1.5.5 (ZFC, folklore)**.** *$\mathbb{VP}^\omega$ implies $\mathbb{VP}$.*

Similar relations hold between rank-Berkeley cardinals and VP(OR). For example, in the proof of Proposition 1.5.1, if λ is chosen to be correct enough and j is so that it fixes the ordinal defining the class of structures, then we would have:

Proposition 1.5.6 ([Gol24, Cor. 2.3])**.** *If δ is a rank-Berkeley cardinal, then $\mathrm{VP}^\delta(\mathrm{OR})$ holds.* □

An analogue of Theorem 1.5.2 for rank-Berkeley cardinals is achieved in an essentially similar way. The only detail that is not outright obvious is the fact that a limit of rank-Berkeley cardinals is again a rank-Berkeley cardinal. In the case of Berkeley cardinals, this is straightforward from the definition. But, for rank-Berkeley cardinals, Lemma 1.4.7 is necessary. Thus, let δ be a limit of rank-Berkeley cardinals and let α, λ be arbitrary ordinals satisfying $\alpha < \delta < \lambda$. Fix a rank-Berkeley cardinal δ_0 such that $\alpha < \delta_0 < \delta$, and let $j \in \mathcal{E}_\lambda^{\delta_0}$ be such that $\mathrm{crit}(k) > \alpha$. Lemma 1.4.7 is now necessary to find a $j_n \in \mathcal{E}_\lambda^{\delta_0}$ that fixes δ, so that $j_n \in \mathcal{E}_\lambda^{\delta}$.

Theorem 1.5.7. *For $n \geq 1$, if $\mathrm{VP}^\omega(\Pi_{n+1}(\mathrm{OR}))$ holds while $\mathrm{VP}(\Pi_{n+1}(\mathrm{OR}))$ fails, then $\delta = \sup\{\mu \mid \mathrm{VP}^\mu(\Pi_{n+1}(\mathrm{OR}))$ holds$\}$ is a rank-Berkeley cardinal.* □

Corollary 1.5.8. *For $n \geq 1$, if $\mathrm{VP}(\Pi_{n+1}(\mathrm{OR}))$ fails, then*

$$\sup\{\mu \mid \mathrm{VP}^\mu(\Pi_{n+1}(\mathrm{OR}))\ \textit{holds}\} = \sup\{\delta \mid \delta\ \textit{is a rank-Berkeley cardinal}\}.$$

Corollary 1.5.9. *For $n \geq 1$, if $\mathrm{VP}(\Pi_{n+1}(\mathrm{OR}))$ fails, then the existence of a rank-Berkeley cardinal is equivalent to the schema $\mathrm{VP}^\omega(\mathrm{OR})$.*

Corollary 1.5.10 (ZFC)**.** $\mathrm{VP}^\omega(\mathrm{OR})$ *implies* $\mathrm{VP}(\mathrm{OR})$.

This last corollary is a much stronger result than the analogous Corollary 1.5.5 (see Remark 1.5.12). We already know that VP(OR) is equivalent to $\mathbb{VP}$ (Corollary 1.3.2). By using the proof of the case (1) implies (2) of 1.3.2, we can also show that VP^ω implies $\mathrm{VP}^\omega(\mathrm{OR})$. Putting everything together, we now have the following:

Corollary 1.5.11 (ZFC)**.** VP^ω *implies (and hence is equivalent to)* $\mathbb{VP}$. □

In other words, assuming AC, the weakest form of Vopěnka's Principle, where we only allow for definable, with no parameters, proper classes of structures of the same finite type, implies the strongest form of Vopěnka's Principle, where we allow for all definable, with parameters, proper classes of structures of any type.

Remark 1.5.12*.* The folklore result stated in Corollary 1.5.5 can be established directly using the well-known fact that any type can be coded by a binary relation, using AC. However, the same argument does not apply to the final corollary above, as parameters are not allowed in the weaker form VP^ω, rendering any AC-based coding ineffective.

Let BC and rBC denote the axioms asserting the existence of a Berkeley cardinal and a rank-Berkeley cardinal, respectively. We show next that the theories ZF + $\mathrm{VP}^\omega(\mathbf{\Pi}_{n+1}) + \neg\mathrm{VP}(\mathbf{\Pi}_{n+1})$, for $n \geq 1$, and ZF + BC are all equiconsistent. A cardinal κ is *inaccessible* iff there is no cofinal map $f\colon V_\alpha \to \kappa$, for any $\alpha < \kappa$. It is easy to show that if κ is inaccessible then $(V_\kappa, V_{\kappa+1}) \models \mathrm{ZF}_2$, where ZF_2 is the second-order Zermelo-Fraenkel set theory. Moreover, the critical point of any nontrivial elementary embedding $j\colon V_\mu \to V_\nu$ is an inaccessible cardinal.

Proposition 1.5.13. *The following theories are equiconsistent:*

(1) $\mathrm{ZF} + \mathrm{VP}^\omega(\mathbf{\Pi}_{n+1}) + \neg\mathrm{VP}(\mathbf{\Pi}_{n+1})$, *for some* $n \geq 1$.
(2) ZF + BC.
(3) $\mathrm{ZF} + \mathbb{VP}^\omega + \neg\mathrm{VP}(\Pi_1(\mathrm{OR}))$.

Proof. (1) to (2) is Theorem 1.5.2 and (3) to (1) is trivial. For (2) to (3): Working in (2), let δ be a Berkeley cardinal. We can assume that there are no inaccessible cardinals $\kappa > \delta$ by simply passing to V_κ for the least inaccessible cardinal $\kappa > \delta$ if necessary. Our claim then is that $\mathbb{VP}^\omega$ holds and $\mathrm{VP}(\Pi_1(\mathrm{OR}))$ fails. First, $\mathbb{VP}^\omega$

holds by Proposition 1.5.1. Next, the failure of $\mathrm{VP}(\Pi_1(\mathrm{OR}))$ is witnessed by the class $\mathcal{C} = \{(V_{\alpha+1}, \xi)_{\xi\leq\delta} \mid \alpha > \delta\}$ that is Π_1-definable from the parameter δ. For otherwise, we would have an elementary embedding $j\colon (V_{\alpha_1+1}, \xi)_{\xi\leq\delta} \to (V_{\alpha_2+1}, \xi)_{\xi\leq\delta}$, where $\alpha_1 \neq \alpha_2$. Any such embedding will have to be nontrivial since $j(\alpha_1) = \alpha_2$. Moreover, it has to fix ξ for all $\xi \leq \delta$, and so must have a critical point (an inaccessible cardinal) $\kappa > \delta$. □

The case of rank-Berkeley cardinals is as follows:

Proposition 1.5.14. *The following theories are equiconsistent:*

(1) $\mathrm{ZF} + \mathrm{VP}^{\omega}(\Pi_{n+1}(\mathrm{OR})) + \neg\mathrm{VP}(\Pi_{n+1}(\mathrm{OR}))$, *for some* $n \geq 1$.
(2) $\mathrm{ZF} + \mathrm{rBC}$
(3) $\mathrm{ZF} + \mathrm{VP}^{\omega}(\mathrm{OR}) + \neg\mathrm{VP}(\Pi_1(\mathrm{OR}))$ □

1.6 Class Many Rank-Berkeley Cardinals

In this section, we will consider the failure in ZF of two results; Theorem 1.1.1 and a consequence of it. Since a proper class of $C^{(n)}$-extendible cardinals always implies $\mathrm{VP}(\mathbf{\Pi}_{n+1})$ (Theorem 1.3.1), failure of Theorem 1.1.1 can only happen if $\mathrm{VP}(\mathbf{\Pi}_{n+1})$ holds while there are no $C^{(n)}$-extendible cardinals beyond some ordinal ξ. We will show that this failure implies the existence of unboundedly many rank-Berkeley cardinals, and then establish the equiconsistency of these two theories.

An interesting consequence of Theorem 1.1.1 is that $\mathbb{VP}$ implies OR is Mahlo (Definition 1.6.9) [BP23, Lemma 6.3]. Since this equivalence may fail in the context of ZF, and since there is no guarantee so far that n-choiceless extendible cardinals are inaccessible, one must wonder whether it is possible to have $\mathbb{VP}$ while OR fails to be Mahlo. We will show that this implies the existence of unboundedly many-rank Berkeley cardinals as well, and then prove the equiconsistency of the two theories.

We start by proving some intermediary results. Lemma 1.6.1 and Lemma 1.6.4 below are generalizations to n-choiceless supercompact cardinals of results by Menachem Magidor [Mag71, Lemmas 1 & 2, resp.], modulo slightly stronger assumptions.

Lemma 1.6.1. *For $n \geq 0$, given $\alpha < \kappa < \delta < \lambda$ such that $(\kappa, \delta) \cap C^{(n)} \neq \emptyset$, if κ is $(\alpha, <\delta, <V_\delta, n)$-choiceless supercompact and δ is $(\alpha, \lambda, <V_\lambda, n)$-choiceless supercompact, then κ is $(\alpha, \lambda, <V_\lambda, n)$-choiceless supercompact as well.*

Proof. Fix $a \in V_\lambda$. We need to show that κ is (α, λ, a, n)-choiceless supercompact. Since δ is $(\alpha, \lambda, <V_\lambda, n)$-choiceless supercompact, we get an elementary embedding $j\colon V_{\bar\lambda} \to V_\lambda$ such that $\mathrm{crit}(j) > \alpha$, $\bar\lambda < \delta$ and is in $C^{(n)}$, and there is some $\bar a \in V_{\bar\lambda}$ for which $j(\bar a) = a$. Now, if it so happens that $\bar\lambda < \kappa$, then this j witnesses that κ is indeed (α, λ, a, n)-choiceless supercompact.

Else, if $\bar\lambda = \kappa$, we proceed as follows. Fix a $\gamma \in (\kappa, \delta) \cap C^{(n)}$. By using $(\alpha, <\delta, <V_\delta, n)$-choiceless supercompactness of κ, fix an elementary embedding $k\colon V_{\bar\gamma} \to V_\gamma$ such that $\mathrm{crit}(k) > \alpha$, $\bar\gamma < \kappa$ and is in $C^{(n)}$, and there is $\bar{\bar a}, \bar{\bar\lambda} \in V_{\bar\gamma}$ for which $k(\bar{\bar a}) = \bar a$ and $k(\bar{\bar\lambda}) = \bar\lambda$. Since $\bar a \in V_{\bar\lambda}$, we must have $\bar{\bar a} \in V_{\bar{\bar\lambda}}$ by elementarity. Now, the composite map $j \circ k|_{V_{\bar{\bar\lambda}}}\colon V_{\bar{\bar\lambda}} \to V_\lambda$ is an elementary embedding with critical point strictly above α and $j \circ k|_{V_{\bar{\bar\lambda}}}(\bar{\bar a}) = j(\bar a) = a$. It remains to show that $\bar{\bar\lambda} < \kappa$ and is in $C^{(n)}$. But this is clear, since $\bar{\bar\lambda} < \bar\gamma < \kappa$, while $\bar{\bar\lambda} \in C^{(n)}$ by elementarity of k and the fact that $\bar\gamma, \gamma, \bar\lambda \in C^{(n)}$.

Finally, we need to consider the case where $\bar\lambda > \kappa$. This is similar to the above, but simpler, because now we do not need a γ, and we can just use $\bar\lambda$ itself in place of γ in the above argument. □

Definition 1.6.2. Given a nontrivial elementary embedding $j\colon V_\mu \to V_\nu$ such that $\sup\{\beta \mid j(\beta) = \beta\} < \mu$, we define the *last sequence* of j to be the longest sequence $\langle\gamma_i\rangle_{i<m(j)}$ satisfying $\gamma_0 = \sup\{\beta \mid j(\beta) = \beta\}$ and $\gamma_i = j(\gamma_{i-1})$ for all nonzero $i < m(j)$, where $m(j) \le \omega$. The ordinal γ_0 will be called the *last point* of j.

The following lemma is easy.

Lemma 1.6.3. *Let $n \ge 0$, and suppose $\alpha < \beta < \gamma$ are ordinals such that $\gamma \in C^{(n)}$ and $\beta \in C^{(n+1)}$. If $V_\gamma \models$ "$\alpha \in C^{(n+1)}$", then $\alpha \in C^{(n+1)}$.*

Proof. $\gamma \in C^{(n)}$ and $\beta \in C^{(n+1)}$ imply $V_\gamma \models$ "$\beta \in C^{(n+1)}$". That, along with $V_\gamma \models$ "$\alpha \in C^{(n+1)}$", implies $V_\gamma \models$ "$V_\alpha \prec_{\Sigma_{n+1}} V_\beta$". The satisfaction relation $\models$ for set models is Δ_1, so actually $V_\alpha \prec_{\Sigma_{n+1}} V_\beta$. That is to say, $V_\beta \models$ "$\alpha \in C^{(n+1)}$". But $\beta \in C^{(n+1)}$, so it is correct in its identification of Σ_{n+1}-correct ordinals. Therefore, $\alpha \in C^{(n+1)}$. □

Lemma 1.6.4. *For $n \ge 0$, if $j\colon V_\mu \to V_\nu$, for some $\mu \in \mathrm{Lim}(C^{(n+1)})$ and $\nu \in C^{(n)}$, is an elementary embedding with critical point κ and last sequence $\langle\gamma_i\rangle_{i<m(j)}$, then γ_0, the last point of j, is $(<\kappa, <\mu, <V_\mu, n{+}1)$-choiceless supercompact.*

Proof. First, let us consider the interval $(\gamma_0, \gamma_1 \cap \mu)$. Fix some $\alpha < \kappa$, $\lambda \in (\gamma_0, \gamma_1 \cap \mu) \cap C^{(n+1)}$, and $a \in V_\lambda$. We must show that γ_0 is $(\alpha, \lambda, a, n{+}1)$-choiceless supercompact. Notice that $\mu, \lambda \in C^{(n+1)}$ implies $V_\mu \models$ "$\lambda \in C^{(n+1)}$", and so $V_\nu \models$ "$j(\lambda) \in C^{(n+1)}$" by elementarity. Also, $\nu \in C^{(n)}$ and $\lambda \in C^{(n+1)}$ imply $V_\nu \models$ "$\lambda \in C^{(n+1)}$". Therefore, the restricted map $j|_{V_\lambda}\colon V_\lambda \to V_{j(\lambda)}$ witnesses that $\gamma_1 = j(\gamma_0)$ is $(\alpha, j(\lambda), j(a), n{+}1)$-choiceless supercompact in V_ν.[1] Again by elementarity, γ_0 is $(\alpha, \lambda, a, n{+}1)$-choiceless supercompact in V_μ. But μ is correct enough so that any such witness is a witness in V as well. Thus we have shown that γ_0 is $(<\kappa, <\gamma_1 \cap \mu, <V_{\gamma_1\cap\mu}, n{+}1)$-choiceless supercompact.

If $m(j) = 2$, then $\gamma_1 \cap \mu = \mu$, and so the above argument finishes the proof. However, if $m(j) > 2$, then we just get that γ_0 is $(<\kappa, <\gamma_1, <V_{\gamma_1}, n{+}1)$-choiceless supercompact. This will serve as the base case for an inductive argument. For the inductive step, we will prove that, for γ_0, being $(<\kappa, <\gamma_m, <V_{\gamma_m}, n{+}1)$-choiceless supercompact implies being $(<\kappa, <\gamma_{m+1}, <V_{\gamma_{m+1}}, n{+}1)$-choiceless supercompact, whenever $0 < m < m(j) - 1$. Since $\sup\{\gamma_i \mid i < m(j)\} \cap \mu = \mu$ by definition, we will have shown that γ_0 is $(<\kappa, <\mu, <V_\mu, n{+}1)$-choiceless supercompact in the case $m(j) > 2$ as well.

So, assume γ_0 is $(<\kappa, <\gamma_m, <V_{\gamma_m}, n{+}1)$-choiceless supercompact, where $0 < m < m(j) - 1$. This last inequality means that $\gamma_m \in V_\mu$, so, by correctness of μ, V_μ satisfies that γ_0 is $(<\kappa, <\gamma_m, <V_{\gamma_m}, n{+}1)$-choiceless supercompact as well. By elementarity of j, V_ν satisfies that γ_1 is $(<j(\kappa), <\gamma_{m+1}, <V_{\gamma_{m+1}}, n{+}1)$-choiceless supercompact. Since $\kappa < j(\kappa)$, we actually have that

$$V_\nu \models \text{“}\gamma_1 \text{ is } (<\kappa, <\gamma_{m+1}, <V_{\gamma_{m+1}}, n{+}1)\text{-choiceless supercompact”}. \tag{1.3}$$

Let us show that V also satisfies the formula in (1.3). Fix $\alpha < \kappa$, $\lambda \in C^{(n+1)} \cap (\gamma_1.\gamma_{m+1})$, and $a \in V_\lambda$. As $\lambda \in (C^{(n+1)})^{V_\nu}$, by (1.3), we are given an elementary embedding $k\colon V_{\bar\lambda} \to V_\lambda$ such that $\mathrm{crit}(k) > \alpha$, $\bar\lambda < \gamma_1$ is in $(C^{(n+1)})^{V_\nu}$, and $j(\bar a) = a$ for some $\bar a \in V_{\bar\lambda}$. An application of Lemma 1.6.3 with $\bar\lambda < \lambda < \nu$ in place of $\alpha < \beta < \gamma$ yields $\bar\lambda \in C^{(n+1)}$. Hence, k witnesses that γ_1 is $(\alpha, \lambda, a, n{+}1)$-choiceless supercompact in V.

We have, by inductive assumption, that γ_0 is $(<\kappa, <\gamma_m, <V_{\gamma_m}, n{+}1)$-choiceless supercompact, and we just showed that γ_1 is $(<\kappa, <\gamma_{m+1}, <V_{\gamma_{m+1}}, n{+}1)$-choiceless

[1] $j|_{V_\lambda}$ is in V_ν, as ν is a limit ordinal by elementarity and the fact that $\mu \in \mathrm{Lim}(C^{(n+1)})$.

supercompact. The proof of the inductive step will be over by an application of Lemma 1.6.1, if we can show that $(\gamma_0, \gamma_1) \cap C^{(n+1)} \neq \emptyset$. We do this by showing $\gamma_1 \in \mathrm{Lim}(C^{(n+1)})$. First notice that since $\sup\{\gamma_i \mid i < m(j)\} \geq \mu$ and $\mu \in \mathrm{Lim}(C^{(n+1)})$, there exist $\gamma_s > \gamma_1$ and $\delta \in C^{(n+1)}$ such that $\gamma_1 < \delta < \gamma_s$. Fix any $\beta_0 \in \{\beta \mid j(\beta) = \beta\}$. In V, and hence also in V_ν, $(\beta_0, \gamma_s) \cap C^{(n+1)} \neq \emptyset$ (as witnessed by δ). By elementarity of j and correctness of μ, we get $(\beta_0, \gamma_{s-1}) \cap C^{(n+1)} \neq \emptyset$ in V. Repeating this argument finitely many times, we get $(\beta_0, \gamma_0) \cap C^{(n+1)} \neq \emptyset$. As β_0 was arbitrary, we have shown that $\gamma_0 \in \mathrm{Lim}(C^{(n+1)})$. By correctness of μ and elementarity of j, we have that $\gamma_1 \in (\mathrm{Lim}(C^{(n+1)}))^{V_\nu}$. Finally, since $\gamma_1 < \mu < \nu$, $\mu \in C^{(n+1)}$, and $\nu \in C^{(n)}$, we may apply Lemma 1.6.3 to show that in fact $\gamma_1 \in \mathrm{Lim}(C^{(n+1)})$. □

We can use the above lemmas to prove the following properties of the least α-n-choiceless extendible cardinal, for any fixed α.

Proposition 1.6.5. *For any $n \geq 1$, if α is an ordinal and $\gamma > \alpha$ is the least α-n-choiceless extendible ordinal, then γ satisfies the following:*

(1) *There exists μ_0 such that for all $\mu \geq \mu_0$ in $\mathrm{Lim}(C^{(n+1)})$, and for all elementary embeddings $j\colon V_\mu \to V_\nu$ with $\nu \in C^{(n)}$, if $\mathrm{crit}(j) > \alpha$ and $j(\gamma) > \mu$, then $\sup\{\beta \mid j(\beta) = \beta\} = \gamma$.*
(2) $\gamma \in \mathrm{Lim}(C^{(n+1)})$.
(3) *There is no cofinal map $f\colon V_\xi \to \gamma$ for any $\xi \leq \alpha$. In particular, $\mathrm{cof}(\gamma) > \alpha$.*

Proof. Part (1): Suppose that this is not the case. By α-n-choiceless extendibility of γ, we have a proper class of $\mu \in \mathrm{Lim}(C^{(n+1)})$ for which there are elementary embeddings $j\colon V_\mu \to V_\nu$ with $\nu \in C^{(n)}$, $\mathrm{crit}(j) > \alpha$, $j(\gamma) > \mu$, and $\sup\{\beta \mid j(\beta) = \beta\} < \gamma$. This means that there exist κ and δ, satisfying $\alpha < \kappa \leq \delta < \gamma$, and there is a proper class of $\mu \in \mathrm{Lim}(C^{(n+1)})$ for which there are elementary embeddings $j\colon V_\mu \to V_\nu$ with $\nu \in C^{(n)}$, $\mathrm{crit}(j) = \kappa$, $j(\gamma) > \mu$, and $\sup\{\beta \mid j(\beta) = \beta\} = \delta$. Thus, for any such j, the ordinal δ is the last point. Hence, by Lemma 1.6.4, we must have that δ is α-n+1-choiceless supercompact. By Lemma 1.2.7, we then have that $\delta < \gamma$ is α-n-choiceless extendible. But this cannot be by minimality of γ.

Part (2): Fix an elementary embedding $j\colon V_\mu \to V_\nu$ as in (1). Let $\xi < \gamma$ be arbitrary. Fix β such that $\xi < \beta < \gamma$ and $j(\beta) = \beta$. Notice that V_ν satisfies $(\beta, j(\gamma)) \cap C^{(n+1)} \neq \emptyset$ (as witnessed by μ). By elementarity of j and correctness of μ, we have that $(\beta, \gamma) \cap C^{(n+1)} \neq \emptyset$.

Part (3): Assume towards a contradiction that $f\colon V_\xi \to \gamma$ is cofinal for some $\xi \leq \alpha$. Fix j as in (1). Define $f^*\colon V_\xi \to \gamma$ so that $f^*(x)$ is the least β above $f(x)$ such that $j(\beta) = \beta$. Then, f^* must also be a cofinal map in γ. Hence, by elementarity of j, the map $j(f^*)\colon V_\xi \to j(\gamma)$ must be a cofinal map in $j(\gamma)$. But that cannot be since $j(\gamma) > \gamma$, while $j(f^*)(x) = j(f^*)(j(x)) = j(f^*(x)) = f^*(x)$ for all $x \in V_\xi$. □

Definition 1.6.6 ([Bag12])**.** For each $n \geq 0$, a cardinal κ is said to be *μ-$C^{(n)}$-extendible* for some ordinal $\mu > \kappa$ in $C^{(n)}$ iff there is a $\nu > \mu$ in $C^{(n)}$ and an elementary embedding $j\colon V_\mu \to V_\nu$ such that $\mathrm{crit}(j) = \kappa$ and $j(\kappa) > \mu$. κ is said to be *$<\delta$-$C^{(n)}$-extendible* iff it is μ-$C^{(n)}$-extendible for all $\kappa < \mu < \delta$ in $C^{(n)}$. Finally, κ is said to be *$C^{(n)}$-extendible* iff it is μ-$C^{(n)}$-extendible for all $\mu > \kappa$ in $C^{(n)}$.[2]

Theorem 1.6.7. *For $n \geq 1$, if $\mathrm{VP}(\mathbf{\Pi}_{n+1})$ holds while there are no $C^{(n)}$-extendible cardinals above some ordinal ξ, then there are unboundedly many rank-Berkeley cardinals.*

[2]It is easy to see that this notion is equivalent to Bagaria's $C^{(n)}$-extendibility [Bag12] by an argument similar to the one leading to Theorem 1.2.8.

Proof. Let $\alpha > \xi$ be an arbitrary ordinal, and let $\gamma > \alpha$ be the least α-n-choiceless extendible ordinal. By Proposition 1.6.5, part (1), there are elementary embeddings $j\colon V_\mu \to V_\nu$ satisfying $\mathrm{crit}(j) > \alpha$, $j(\gamma) > \mu$, and $\sup\{\beta \mid j(\beta) = \beta\} = \gamma$, for arbitrarily high $\mu, \nu \in C^{(n)}$. If all such j have critical points equal to γ, then clearly γ would be a $C^{(n)}$-extendible cardinal, contrary to the fact that there are no $C^{(n)}$-extendible cardinals above ξ. Hence, there must be some elementary embedding $j\colon V_\mu \to V_\nu$ satisfying $\alpha < \mathrm{crit}(j) < \gamma$, $j(\gamma) > \mu$, and $\sup\{\beta \mid j(\beta) = \beta\} = \gamma$, for some $\mu, \nu \in C^{(n)}$. Define $\lambda < \gamma$ as the first fixed point of j above $\mathrm{crit}(j)$.

By Proposition 1.6.5, part (3), cofinality of γ is greater than ω. Hence, the set $\{\beta \mid j(\beta) = \beta\}$ forms an ω-club below γ. Also, by part (2) of the same proposition, $C^{(n+1)} \cap \gamma$ must form an ω-club below γ too. Therefore, we see that $C^{(n+1)} \cap \{\beta \mid j(\beta) = \beta\}$ is nonempty and, in fact, unbounded in γ. Let $\delta > \lambda$ be an ordinal in this intersection and notice that we now have an elementary embedding $j|_{V_\delta}\colon V_\delta \to V_\delta$.

We claim that $V_\delta \models$ "λ is rank-Berkeley". Otherwise, there would be a least counterexample $\sigma \in V_\delta$. σ is definable from λ in V_δ, and since $j|_{V_\delta}$ fixes λ, it must also fix σ. But, the restriction of $j|_{V_\delta}$ to V_σ will give a contradiction, so the claim is correct. Now, as $\delta \in C^{(n+1)} \subset C^{(2)}$ and being rank-Berkeley is a Π_2 statement, the cardinal λ must be rank-Berkeley in V too. As α was chosen arbitrarily and $\lambda > \alpha$, we get that there are arbitrarily high rank-Berkeley cardinals. □

Corollary 1.6.8. *For $n \geq 1$, the following theories are equiconsistent:*

(1) ZF + $\mathbb{VP}$ + "$\forall\kappa$ (κ is not $C^{(0)}$-extendible)"
(2) ZF + VP($\mathbf{\Pi}_{n+1}$) + "$\forall\kappa$ (κ is not $C^{(n)}$-extendible)"
(3) ZF + VP($\mathbf{\Pi}_{n+1}$) + "$\exists\xi\ \forall\kappa > \xi$ (κ is not $C^{(n)}$-extendible)"
(4) ZF + "There are unboundedly many rank-Berkeley cardinals"

Proof. (1) to (2) and (2) to (3) are trivial. (3) to (4) is by Theorem 1.6.7.

From (4) to (1): Work in (4). We can assume that there is no inaccessible cardinal κ that is a limit of rank-Berkeley cardinals by simply passing to V_κ, where κ is the least such cardinal if it exists. It is easy to see that any $C^{(0)}$-extendible cardinal is both inaccessible and a limit of rank-Berkeley cardinals, and so cannot exist. Meanwhile, $\mathbb{VP}$ holds, by Proposition 1.5.6 and Corollary 1.3.2. □

Definition 1.6.9. By "OR is $\mathbf{\Pi}_n$-Mahlo" we mean that every club class of ordinals C that is $\mathbf{\Pi}_n$ has an inaccessible cardinal. For the negation of this we will say "OR is not $\mathbf{\Pi}_n$-Mahlo". If OR is $\mathbf{\Pi}_n$-Mahlo for all n, we will say "OR is Mahlo".

Theorem 1.6.10. *For $n \geq 1$, if* VP($\mathbf{\Pi}_{n+1}$) *holds while* OR *is not $\mathbf{\Pi}_{n+1}$-Mahlo, then there are unboundedly many rank-Berkeley cardinals.*

Proof. Fix a club class of ordinals C that is $\mathbf{\Pi}_{n+1}$ and has no inaccessible cardinals. Fix an arbitrary ordinal α such that all the parameters for C appear in V_α, and let γ be the least α-n-choiceless extendible ordinal. Using Proposition 1.6.5 and α-n-choiceless extendibility of γ, fix an elementary $j\colon V_\mu \to V_\nu$ with $\mu \in \mathrm{Lim}(C^{(n+1)}) \cap C$, $\nu \in C^{(n)}$, $\alpha < \mathrm{crit}(j)$, $j(\gamma) > \mu$, and $\sup\{\beta \mid j(\beta) = \beta\} = \gamma$. We will show that $C \cap \gamma$ is unbounded in γ, which will imply $\gamma \in C$. If we have that, then we can argue that $\mathrm{crit}(j) < \gamma$ as members of C are not inaccessible. Finally, letting $\lambda = \min\{\beta > \mathrm{crit}(j) \mid j(\beta) = \beta\}$, we can argue that λ is rank-Berkeley just as in paragraphs 2-3 of the proof of Theorem 1.6.7, and we will be done because α was arbitrary.

Thus, let us show that $C \cap \gamma$ is unbounded in γ. Fix an arbitrary $\xi < \gamma$. Let β be such that $\xi < \beta < \gamma$ and $j(\beta) = \beta$. We have that $\exists\sigma < j(\gamma)(\sigma \in C \ \wedge\ \sigma > \beta)$ holds as witnessed by μ. Since $\nu \in C^{(n)}$ and C is $\mathbf{\Pi}_{n+1}$, the same is true in V_ν. Taking this back along j, we get $\exists\sigma < \gamma(\sigma \in C \;\wedge\; \sigma > \beta)$ holds in V_μ. Finally, since $\mu \in C^{(n+1)}$, the same holds in V, and we are done as ξ was arbitrary. □

Corollary 1.6.11. *The following theories are equiconsistent:*

(1) $\mathrm{ZF} + \mathrm{VP}(\mathbf{\Pi}_{n+1})$ + "OR is not $\mathbf{\Pi}_{n+1}$-Mahlo", *for some* $n \geq 1$.
(2) ZF + "There are unboundedly many rank-Berkeley cardinals"
(3) $\mathrm{ZF} + \mathbb{VP}$ + "OR is not $\mathbf{\Pi}_2$-Mahlo"

Proof. (1) to (2) is Theorem 1.6.10 and (3) to (1) is trivial. For (2) to (1): Suppose there are unboundedly many rank-Berkeley cardinals. Just as in the proof of case (4) to (1) of Theorem 1.6.8, $\mathbb{VP}$ holds and we can assume that there is no inaccessible cardinal κ that is a limit of rank-Berkeley cardinals. Now, the club class C consisting of limits of rank-Berkeley cardinals contains no inaccessible cardinals. It is easy to see that this class is Π_2. □

CHAPTER 2

Strong Rigidity and Elementary Embeddings

2.1 Introduction

In this chapter, we will be dealing with structures endorsed with a single binary relation. Such structures are intuitively easier to visualize as directed graphs. A *directed graph* (G, E) is an ordered pair consisting of a set G of *vertices* and a set $E \subset G \times G$ of *arrows*. For the sake of being concise, we will drop the word "directed" in this definition. This should cause no confusion as we will be dealing only with directed graphs. When talking about some graph (G, E), we might use the binary relation notation to indicate arrows, that is, we might write $x \; E \; y$ to indicate $(x, y) \in E$.

Recall that a map $h \colon G_1 \to G_2$ is called a *homomorphism* from the graph (G_1, E_1) to the graph (G_2, E_2) iff $(x, y) \in E_1 \implies (h(x), h(y)) \in E_2$. The fact that h is a homomorphism from (G_1, E_1) to (G_2, E_2) will be indicated by writing $h \colon (G_1, E_1) \to (G_2, E_2)$. An *isomorphism* is a bijective map $h \colon G_1 \to G_2$ such that both h and h^{-1} are homomorphisms. A homomorphism from a graph to itself is called an *endomorphism* and an isomorphism from a graph to itself is called an *automorphism*.

Definition 2.1.1. A binary relation E on a set G is said to be *rigid* iff the graph (G, E) has no nontrivial automorphisms. If, furthermore, (G, E) has no nontrivial endomorphisms, then E is said to be *strongly rigid*. If E is a (strongly) rigid relation on E, then we say that the graph (G, E) is *(strongly) rigid*.

Definition 2.1.2 ([HP11])**.** The *Rigid Relation Principle*, RR, asserts that every set G has a rigid relation E. The *Strongly Rigid Relation Principle*, SRR, asserts that every set G has a strongly rigid relation E.

SRR implies RR by definition. Vopěnka, Pultr, and Hedrlín [VPH65] prove, using AC, that every set has a strongly rigid relation.[1] Thus, AC implies SRR, and hence also RR. Joel David Hamkins and Justin Palumbo [HP11] prove that RR does not imply AC and also establish the independence of RR from ZF. We therefore have the following theorem:

Theorem 2.1.3 ([VPH65] and [HP11])**.** RR *is independent from* ZF, *follows from* AC, *but is not equivalent to* AC.

We will show that the same is true for SRR as well. That is,

Corollary 2.4.2. SRR *is independent from* ZF, *follows from* AC, *but is not equivalent to* AC.

[1] In graph theory literature, what we call "strongly rigid" (following [HP11]) is called just "rigid."

The independence of SRR from ZF already follows from the independence of RR and the fact that ¬RR $\implies$ ¬SRR. Also, [VPH65] already proves that SRR follows from AC, as mentioned above. So, we only need to prove that SRR does not imply AC. We do so by combining ideas from [HP11] and a method for producing elementary embeddings from homomorphisms. First, we will prove the following in section 3:

Theorem 2.3.2. *If there exists a set* $G \subset \mathbb{R} \times \mathrm{OR}$ *for which there is no strongly rigid relation, then for some ordinal* α *there exists a nontrivial elementary embedding* $j\colon V_\alpha^L \to V_\alpha^L$.

Using that, in section 4, we prove that SRR does not imply AC by building a model for ZF + ¬AC + SRR.

Theorem 2.4.1. *If* ZF *is consistent, then so is* ZF + ¬AC + SRR.

In the last section, we will use our method for getting elementary embeddings from homomorphisms to characterize proto Berkeley cardinals in terms of a strong failure of SRR. Recall that the *Hartog's number* of a set X, denoted by $\aleph(X)$, is the least ordinal α such that there is no injection from α into X.

Theorem 2.5.1. *A cardinal* δ *is proto Berkeley iff for any graph* (G, E) *such that* $\aleph(G) > \delta$ *and any injection* $f\colon \delta \to G$, *there is an endomorphism* $h\colon (G, E) \to (G, E)$ *such that* $h|_{\mathrm{ran}(f)} \neq \mathrm{id}$.

2.2 Countable Sets and Ordinal Numbers

By a *countable* set we mean a set that is in bijection with ω. A set that is either countable or finite is said to be *at most countable.* In this section, we first make the well-known observation that every at most countable set has a strongly rigid relation, and then we present our main method for producing elementary embeddings from homomorphisms. In particular, we will prove that if for some ordinal β there is no strongly rigid relation (so that there are plenty of homomorphisms), then there exists an elementary embedding $j\colon V_\alpha^L \to V_\alpha^L$ for some ordinal α, where V_α^L is V_α as computed in L.

An *uncountable* set is one that is not at most countable. Although in the context of ZF it might not be true that every set has a cardinality, we can still define the *cardinality* of a set x to be the least ordinal that is in bijection with x, if it exists, denoting it by $|x|$.

We will also need some basic graph theoretic terminology before we can continue. Let (G, E) be some graph. A *loop* is an arrow $(u, v) \in E$ such that $u = v$. For $n \geq 2$, a *n-cycle* is a sequence $\langle v_0, \dots, v_n\rangle$ of vertices with $(v_i, v_{i+1}) \in E$, for all $i < n$, and $v_0 = v_n$. For any $n < \omega$ and vertex v, the *outdegree* of v is said to be n iff there are exactly n arrows outgoing from v, that is, $|\{(v, w) \in E \mid w \in G\}| = n$. Given $H \subset G$, the *subgraph induced by* H is the graph (H, F) where $F = \{(u, v) \in E \mid u, v \in H\}$.

Proposition 2.2.1. *Every at most countable set has a strongly rigid relation.*

Proof. Let G be a set that is at most countable. If G is countable, then let E be such that (G, E) is the graph in Figure 2.1. Fix an endomorphism $h\colon (G, E) \to (G, E)$. First, notice that u_0 has outdegree 2 due to the two arrows connecting it with u_1 and u_2. Since u_1 and u_2 are connected by an arrow, and since (G, E) is free of loops $h(u_1) \neq h(u_2)$. Thus, $h(u_0)$ must have outdegree at least 2. But, u_0 is the only vertex with outdegree at least 2, and therefore $h(u_0) = u_0$. Now, h can either fix both u_1

and u_2 or swap them. However, the arrow (u_1, u_2) ensures that swapping them is not possible, so that h fixes u_1 and u_2 too. It is now easy to see that h must fix every other vertex as well, meaning that $h = \mathrm{id}$.

If G is finite, say $G = \{u_0, \ldots, u_n\}$, then simply take the induced subgraph of the graph in Figure 2.1. $\square$

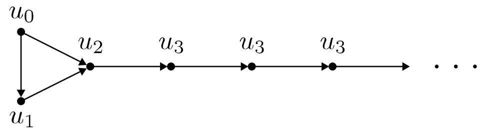

FIGURE 2.1: A strongly rigid graph on a countable set

The language of set theory, LST, is defined in V to be the first order language with the binary relation symbol $\in$. Denote by FORM the set of all formulas of LST. To distinguish metatheoretic formulas from those in V, whenever a formula appears in this chapter, we will mention that it belongs to the set FORM precisely when the formula is in V.

Theorem 2.2.2. *If there is an ordinal β for which there is no strongly rigid relation, then for some ordinal α there is a nontrivial elementary embedding $j\colon V^L_\alpha \to V^L_\alpha$.*

Proof. Let $\kappa = |\beta|$. Since κ is a cardinal in V, it is also a cardinal in L. The fact that GCH holds in L implies that for every cardinal λ in L, there is some ordinal α such that $|V^L_\alpha|^L = \lambda$. Fix α such that $|V^L_\alpha|^L = \kappa$. Our argument would be easier if α were a limit ordinal so that V^L_α is closed under taking ordered pairs and finite sequences. But, we do not know this for sure, so let $N \supset V^L_\alpha \cup \{V^L_\alpha\}$ be the smallest set closed under taking finite subsets, i.e., $\{x_1, \ldots, x_n\} \subset N \implies \{x_1, \ldots, x_n\} \in N$ for all $n < \omega$. Notice that $N \in L$ and has cardinality κ in L, and since κ is a cardinal in V too, we have that $|N| = \kappa$. Furthermore, N is transitive and closed under the operations

N1 $x, y \mapsto (x, y)$.
N2 $x_1, \ldots, x_n \mapsto \langle x_1, \ldots, x_n\rangle$, for all $n < \omega$.

Let N^c be a copy of N. Define $<^c_L \subset N^c \times N^c$ by $x^c <^c_L y^c \iff x <_L y$. Consider the graph (G, E) in Figure 2.2, where

G1 $(w_0, x), (x, x^c), (w_1, x^c) \in E$, for every $x \in N$.
G2 $(x, y) \in E$ iff $x \in y$, for every $x, y \in N$.
G3 $(x^c, y^c) \in E$ iff $x^c <^c_L y^c$, for every $x^c, y^c \in N^c$.
G4 $(p_n, x^c) \in E$ iff $|x| = n$, for every $n < \omega$ and every $x^c \in N^c$.
G5 For every $n < \omega$ and every $x^c \in N^c$, $(q_n, x^c) \in E$ iff one of the followings hold:

- (a) $x \in V^L_\alpha$, $n = \ulcorner\psi(a)\urcorner$ for some $\psi(a) \in$ FORM, and $V^L_\alpha \models \psi[x]$.
- (b) $x = \langle x_1, \ldots, x_m\rangle$ for some $m > 1$, $x_1, \ldots, x_m \in V^L_\alpha$, $n = \ulcorner\psi(a_1, \ldots, a_m)\urcorner$ for some $\psi(a_1, \ldots, a_m) \in$ FORM, and $V^L_\alpha \models \psi[x_1, \ldots, x_m]$.

The cardinality of G is $|A| + |B| + |N| + |N^c| = 4 + \omega + \kappa + \kappa = \kappa$. Thus, the graph (G, E) has a nontrivial endomorphism $h\colon (G, E) \to (G, E)$. We will show that the restriction $h|_{V^L_\alpha}$ is a nontrivial elementary embedding of V^L_α into itself.

We start by showing that h fixes every vertex in A. Observe that the 3-cycle $c = \langle u_0, u_1, u_2, u_0\rangle$ is the only cycle in the entire graph: None of the arrows outside of

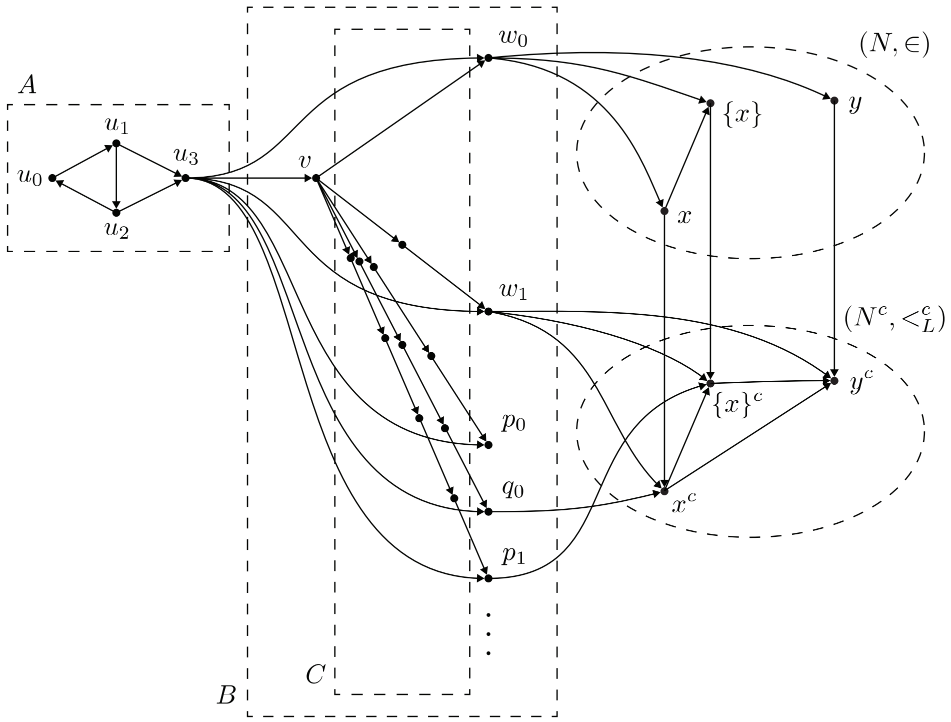

FIGURE 2.2

those in c, N, and N^c can belong to a cycle because they only flow from left to right, while inside of each of N and N^c, the wellfoundedness of $\in$ and $<_L$ prohibit cycles. As such, h must send c to itself. The two arrows (u_1, u_3) and (u_2, u_3) ensure that c is not rotated by h, and it is now straightforward to see that h fixes everything in A.

Next, we show that every vertex in B is fixed. Notice that the vertices in $B \setminus C$ are unique in that they are precisely those that have an incoming arrow from u_3. As h fixes u_3, we have that $B \setminus C$ is closed under h. Consider now the subgraphs of the form shown in Figure 2.3. Since the end vertices of such subgraphs are sent to the end vertices of similar subgraphs, and since h must maintain lengths of such graphs, we conclude that every vertex in B must be fixed.

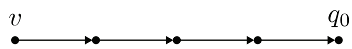

FIGURE 2.3

Similar to how $B \setminus C$ was closed under h, the two sets N and N^c are also closed under h. Furthermore, because $(x, y^c) \in E$ iff $x = y$ by G1, we must have $h(x)^c = h(x^c)$, for all $x \in N$. Thus, h behaves the same in N and N^c. Now, it is easy to see that $h|_N$ is injective because $h|_{N^c}$ is:

$$\begin{aligned} x^c \neq y^c &\implies (x^c, y^c) \in E \ \vee \ (y^c, x^c) \in E \\ &\implies (h(x^c), h(y^c)) \in E \ \vee \ (h(y^c), h(x^c)) \in E \\ &\implies h(x^c) \neq h(y^c). \end{aligned}$$

Let us show that, for all $n < \omega$ and all distinct $x_1, \ldots, x_n \in N$,

$$h(\{x_1, \ldots, x_n\}) = \{h(x_1), \ldots, h(x_n)\}. \tag{2.1}$$

Denote by S the set $\{x_1, \ldots, x_n\}$. We have $(x_i, S) \in E$, for all i, by G2. By endomorphism of h, we have $(h(x_i), h(S)) \in E$, for all i. This means that $h(x_i) \in h(S)$, for all i, by G2. We already know that h is injective on N, so we deduce that $h(S)$ has at least the n distinct elements $h(x_1), \ldots, h(x_n)$. We will be done if we can show that the cardinality of $h(S)$ is n. By G4, $(p_n, S^c) \in E$, and by applying h to this we get $(h(p_n), h(S^c)) \in E$. p_n is fixed by h and we saw in the previous paragraph that $h(S^c) = h(S)^c$, therefore, $(p_n, h(S)^c) \in E$. Again by G4, $|h(S)| = n$.

Using (2.1) above, we can show that, for all $(x, y) \in N$, $h((x, y)) = (h(x), h(y))$:

$$\begin{aligned} h((x,y)) &= h(\{\{x\},\{x,y\}\}) \\ &= \{h(\{x\}), h(\{x,y\})\} \\ &= \{\{h(x)\},\{h(x),h(y)\}\} \\ &= (h(x),h(y)). \end{aligned}$$

Observe that, by G5(a) and the fact that all $m < \omega$ are uniquely definable inside of V_α^L, we must have $h(m) = m$ for all $m < \omega$. We can now also show that, for all $\langle x_1, \ldots, x_n\rangle \in N$, $h(\langle x_1, \ldots, x_n\rangle) = \langle h(x_1), \ldots, h(x_n)\rangle$:

$$\begin{aligned} h(\langle x_1,\ldots,x_n\rangle) &= h(\{(0,x_1),\ldots,(n-1,x_n)\}) \\ &= \{h((0,x_1)),\ldots,h((n-1,x_n))\} \\ &= \{(h(0),h(x_1)),\ldots,(h(n-1),h(x_n))\} \\ &= \{(0,h(x_1)),\ldots,(n-1,h(x_n))\} \\ &= \langle h(x_1),\ldots,h(x_n)\rangle. \end{aligned}$$

There are two last facts that we will need before we present our final argument. The first one is the fact that $x \in V_\alpha^L \implies h(x) \in V_\alpha^L$. We show this by proving that h fixes V_α^L, so that $x \in V_\alpha^L \implies h(x) \in h(V_\alpha^L) = V_\alpha^L$. Observe that every set in $N \setminus V_\alpha^L$ is finite except for V_α^L, which is infinite because $|V_\alpha^L| = \kappa > \omega$ by Proposition 2.2.1. Also, $h(V_\alpha^L)$ must be an infinite set too by injectivity of h on $(N, \in)$, so we will be done if we can show that $h(V_\alpha^L) \notin V_\alpha^L$. But, $h(V_\alpha^L) \in V_\alpha^L$ is clearly not possible because repeated applications of h to this relation can give an infinitely descending chain $V_\alpha^L \ni h(V_\alpha^L) \ni h^2(V_\alpha^L) \ni \cdots$.

The other fact we need is that indeed h is nontrivial on V_α^L. Suppose that this is not the case. We have already shown that h fixes V_α^L as well as everything outside of N and N^c. A simple inductive argument using the fact that $h(\{x_1, \ldots, x_n\}) = \{h(x_1), \ldots, h(x_n)\}$, for any $n < \omega$, will easily demonstrate that h must fix everything in $N \setminus (V_\alpha^L \cup \{V_\alpha^L\})$ too. This contradicts the fact that $h\colon (G, E) \to (G, E)$ is nontrivial.

We are now ready to conclude our proof. Fix a formula $\psi(a_1, \ldots, a_m) \in$ FORM for some $m > 1$, the case where ψ is a single variable formula is the same. We have, for any $x_1, \ldots, x_m \in V_\alpha^L$,

$$\begin{aligned} V_\alpha^L \models \psi[x_1,\ldots,x_m] &\iff (q_{\ulcorner \psi(a_1,\ldots,a_m)\urcorner}, \langle x_1,\ldots,x_m\rangle^c) \in E \\ &\implies (h(q_{\ulcorner \psi(a_1,\ldots,a_m)\urcorner}), h(\langle x_1,\ldots,x_m\rangle^c)) \in E \\ &\iff (q_{\ulcorner \psi(a_1,\ldots,a_m)\urcorner}, \langle h(x_1),\ldots,h(x_m)\rangle^c) \in E \\ &\iff V_\alpha^L \models \psi(h(x_1),\ldots,h(x_m)). \end{aligned}$$

By using $\neg\psi$ in place of ψ in the above displayed argument, we can also reverse that single one way implication in the second line. Thus, for any $x_1, \ldots, x_m \in V_\alpha^L$,

$$V_\alpha^L \models \psi(x_1,\ldots,x_m) \iff V_\alpha^L \models \psi(h(x_1),\ldots,h(x_m)). \qquad \square$$

2.3 Subsets of $\mathbb{R} \times \mathrm{OR}$

In this section, we generalize the theorem from the previous section to sets $G \subset \mathbb{R} \times \mathrm{OR}$. We take $\mathbb{R}$ to be the set of all binary sequences of length ω, i.e., $\mathbb{R} = 2^\omega$. The relation $< \subset \mathbb{R} \times \mathbb{R}$ is the lex ordering. We will need the following lemma first:

Lemma 2.3.1. *Every set $G \subset \mathbb{R} \times \mathrm{OR}$ with no countable subset has a strongly rigid relation.*

Proof. Define the relation E on G by setting $(r_1, \xi_1)\ E\ (r_2, \xi_2)$ iff $\xi_1 < \xi_2$ or $\xi_1 = \xi_2 \ \wedge\ r_1 < r_2$. Suppose $h\colon (G, E) \to (G, E)$ is a nontrivial endomorphism, and let $(r, \xi) \in G$ be such that $h((r, \xi)) \neq (r, \xi)$. Without loss of generality, we may assume $(r, \xi)\ E\ h((r, \xi))$. Repeated applications of h to this relation will give $(r, \xi)\ E\ h((r, \xi))\ E\ h^2((r, \xi))\ E \cdots$. But this chain forms a countable subset of G, contrary to the assumption that G has no countable subsets. □

Let $\pi_2\colon V \to V$ be the operation defined by $(x_1, x_2) \mapsto x_2$. For any finite sequence $s = \langle x_0, \ldots, x_{n-1}\rangle \in V^{<\omega}$, define $\mathrm{len}(s) = \mathrm{dom}(s) = n$.

An ordinal is said to be *odd* iff it can be written in the form $\alpha + 2n + 1$ for some limit ordinal α and some $n < \omega$. An ordinal is *even* iff it is not odd. Denote by OR^{o} and OR^{e} the classes of odd and even ordinals, respectively.

Theorem 2.3.2. *If there exists a set $G \subset \mathbb{R} \times \mathrm{OR}$ for which there is no strongly rigid relation, then for some ordinal α there exists a nontrivial elementary embedding $j\colon V^L_\alpha \to V^L_\alpha$.*

Proof. Take such a set G. By Proposition 2.2.1, G must be uncountable, and by Lemma 2.3.1, it must have a countable subset. Fix $H \subset G$ that is countable, and let $K = G \setminus H$. By simply renaming ordinals, we may assume that π_2"$K = \kappa$, where κ is the cardinality of π_2"K.

Claim 2.3.2.1. $\kappa \geq \omega$.

Proof of claim. We will show that if $\kappa < \omega$, then G would have a strongly rigid relation, contrary to the choice of G. So, assume that $\kappa < \omega$. Notice that K cannot be empty by Proposition 2.2.1, hence $0 < \kappa < \omega$. Fix $n < \omega$ such that $\kappa = n + 1$. Partition K by defining, for each $i < n + 1$, $K_i = \{(r, \xi) \in K \mid \xi = i\}$. Consider the graph (G, E) shown in Figure 2.4, where

G1 For every $i < n + 1$: $(w_i, (r, i)) \in E$, for all $(r, i) \in K_i$.
G2 $(e_s, (r, \xi)) \in E$ iff $r|_{\mathrm{len}(s)} = s$, for all $s \in 2^{<\omega}$ and all $(r, \xi) \in K$.

Fix an endomorphism $h\colon (G, E) \to (G, E)$. Arguing just as in the proof of Theorem 2.2.2, all the vertices in H must be fixed by h and, moreover, each of the K_is is closed under h. Suppose, working towards a contradiction, that for some fixed $i < n + 1$, $h((r_1, i)) = (r_2, i)$ for distinct $(r_1, i), (r_2, i) \in K_i$. Fix $k < \omega$ such that $r_1|_k \neq r_2|_k$. By G2, we must have $(e_{r_1|_k}, (r_1, i)) \in E$. Applying h, we get $(h(e_{r_1|_k}), h((r_1, i))) = (e_{r_1|_k}, (r_2, i)) \in E$. By G2, this means that $r_2|_k = r_1|_k$, a contradiction. Thus, h must fix every vertex in each of the K_is as well, so that $h = \mathrm{id}$. As h was arbitrary, we conclude that E is a strongly rigid relation on G. □

Continuing with the proof of the theorem, we will use the following notation: Given any two sets X, Y of vertices, instead of saying, for all $a \in X$ and all $b \in Y$, $(a, b) \in E$, we will simply say $(X, Y) \in E^*$. In a figure, while a small black node represents a vertex, sets of vertices will be represented by a slightly bigger node that is white with

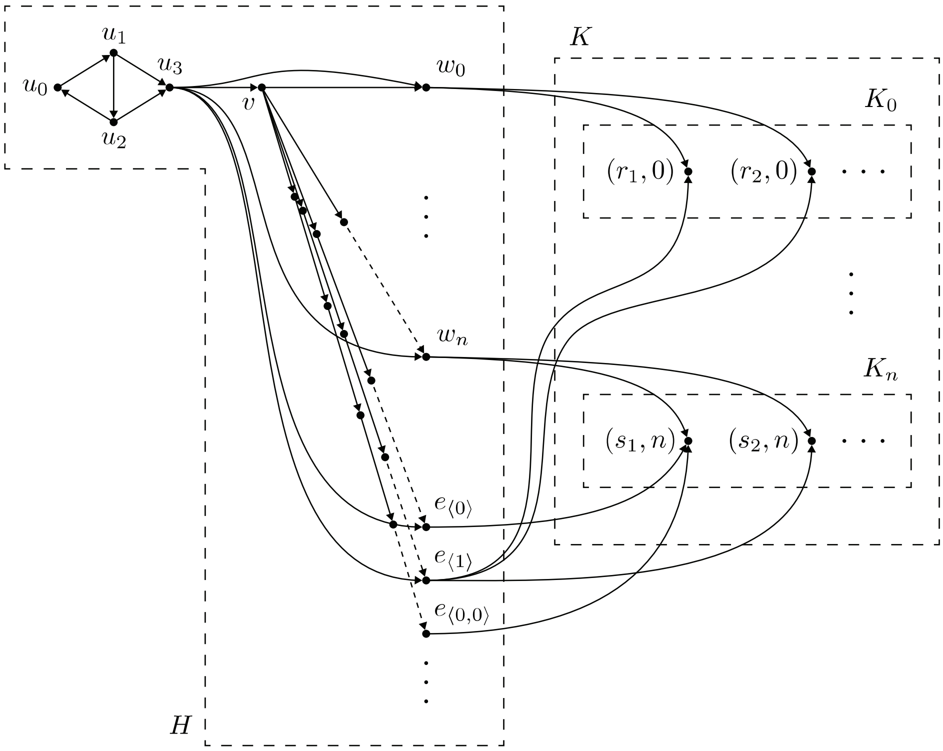

FIGURE 2.4

a black circumference (see Figure 2.5 for an example). The fact that $(X, Y) \in E^*$ will be represented by a dotted arrow from the node for X (or a if $X = \{a\}$) to the node for Y (or b if $Y = \{b\}$). Unlike a dotted arrow, an arrow with dots and dashes only indicates that there are some, but not necessarily all possible, arrows from the vertices at its tail to the vertices at its head.

Let $\kappa^{\mathrm{e}} = \kappa \cap \mathrm{OR}^{\mathrm{e}}$ and $\kappa^{\mathrm{o}} = \kappa \cap \mathrm{OR}^{\mathrm{o}}$, and observe that $\kappa = |\kappa^{\mathrm{e}}| = |\kappa^{\mathrm{o}}|$. Partition K into the two sets $K^{\mathrm{e}} = \{(r, \xi) \in K \mid \pi_2(\xi) \in \kappa^{\mathrm{e}}\}$ and $K^{\mathrm{o}} = \{(r, \xi) \in K \mid \pi_2(\xi) \in \kappa^{\mathrm{o}}\}$. Similar to the proof of Theorem 2.2.2, we can fix V^L_α such that $|V^L_\alpha| = \kappa$ and take N, and its copy N^c, such that $N \supset V^L_\alpha \cup \{V^L_\alpha\}$ is the smallest set closed under taking finite subsets. As $|N| = \kappa = |\kappa^{\mathrm{e}}|$, we can take a bijection $f \colon \kappa^{\mathrm{e}} \to N$, and use it to replace each $(r, \xi) \in K^{\mathrm{e}}$ with $(r, f(\xi))$. Thus, we may assume that $\pi_2"K^{\mathrm{e}} = N$. Similarly, we may assume that $\pi_2"K^{\mathrm{o}} = N^c$.

We partition each of K^{e} and K^{o} by the equivalence $(r, x) \sim (s, y)$ iff $x = y$, and denote the class of (r, x) by $[x]$. Let $\underline{N} = \{[x] \mid x \in N\}$ and $\underline{N^c} = \{[x^c] \mid x^c \in N^c\}$. Define the relation $\underline{\in} \subset \underline{N} \times \underline{N}$ by $[x] \mathrel{\underline{\in}} [y] \iff x \in y$, for all $x, y \in N$. Similarly, define $\underline{<^c_L} \subset \underline{N^c} \times \underline{N^c}$ and $<^c_L \subset N^c \times N^c$ by $[x^c] \mathrel{\underline{<^c_L}} [y^c] \iff x^c <^c_L y^c \iff x <_L y$, for all $x^c, y^c \in N^c$.

Consider the graph (G, E) in Figure 2.5, where

G1 $(\{w_0\}, [x]), ([x], [x^c]), (\{w_1\}, [x^c]) \in E^*$, for every $x \in N$.

G2 $([x], [y]) \in E^*$ iff $[x] \mathrel{\underline{\in}} [y]$, for every $[x], [y] \in \underline{N}$.

G3 $([x^c], [y^c]) \in E^*$ iff $[x^c] \mathrel{\underline{<^c_L}} [y^c]$, for every $[x^c], [y^c] \in \underline{N^c}$.

G4 $(\{p_n\}, [x^c]) \in E^*$ iff $|x| = n$, for every $n < \omega$ and every $[x^c] \in \underline{N^c}$.

G5 For every $n < \omega$ and every $[x^c] \in \underline{N^c}$, $(\{q_n\}, [x^c]) \in E^*$ iff one of the followings hold:

(a) $x \in V^L_\alpha$, $n = \ulcorner \psi(a) \urcorner$ for some $\psi(a) \in \mathrm{FORM}$, and $V^L_\alpha \models \psi[x]$.

(b) $x = \langle x_1, \ldots, x_m \rangle$ for some $m > 1$, $x_1, \ldots, x_m \in V_\alpha^L$, $n = \ulcorner \psi(a_1, \ldots, a_m) \urcorner$ for some $\psi(a_1, \ldots, a_m) \in$ FORM, and $V_\alpha^L \models \psi[x_1, \ldots, x_m]$.

G6 $(e_s, (r, \xi)) \in E$ iff $r|_{\text{len}(s)} = s$, for all $s \in 2^{<\omega}$ and all $(r, \xi) \in K$.

Figure 2.6 depicts what the dotted arrows represent by showing the subgraph of (G, E) induced by $\{w_0\} \cup [x] \cup [x^c]$.

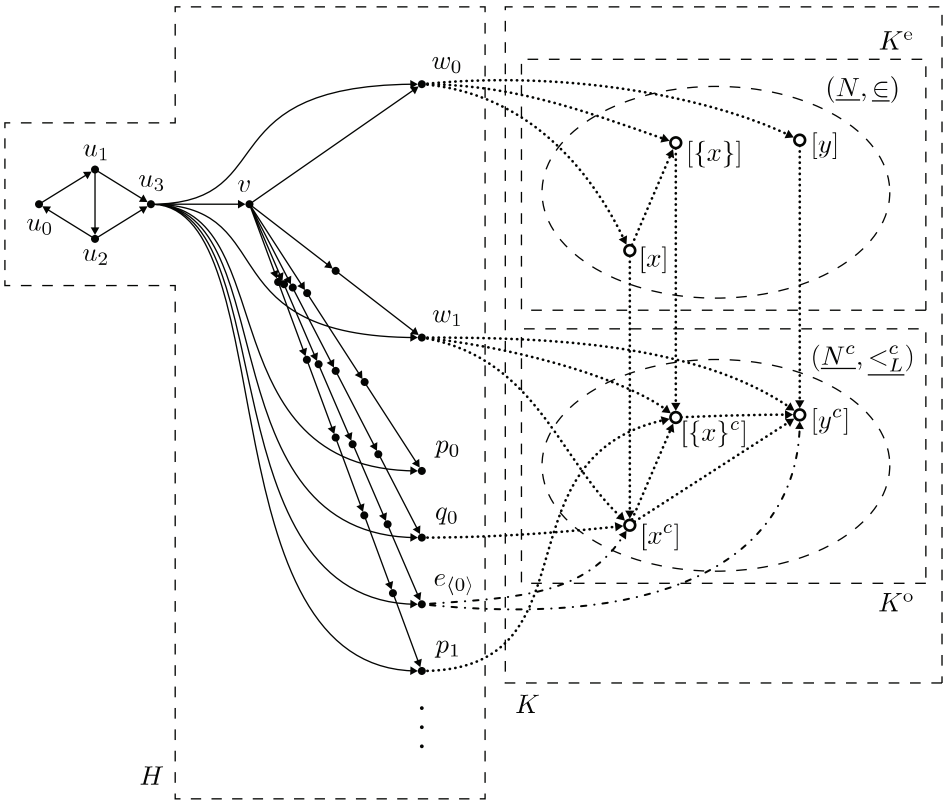

Figure 2.5

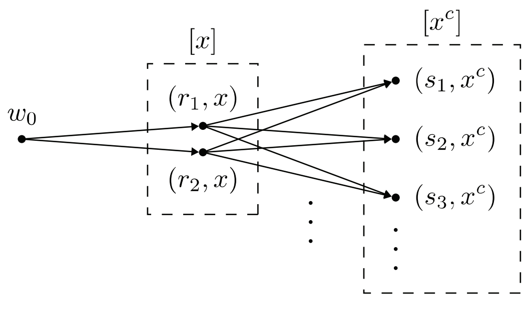

Figure 2.6

As G has no strongly rigid relation, we get a nontrivial endomorphism $h\colon (G, E) \to (G, E)$. Again, h must fix every vertex in H, and the two sets K^{e} and K^{o} are each closed under h by G1.

We want to prove that if a member of some class $[x]$ is sent by h to a member of some class $[y]$, then every member of $[x]$ is sent to a member of $[y]$. We will show this for the classes in $\underline{N}$, the argument for the classes in $\underline{N^c}$ is done similarly. To do this, we will prove that $h((r_1, x)) \in [y_1] \wedge h((r_2, x)) \in [y_2] \implies y_1 = y_2$, for every $x, y_1, y_2 \in N$

and every $(r_1, x), (r_2, x) \in [x]$. Suppose, towards a contradiction, that $h((r_1, x)) \in [y_1]$ and $h((r_2, x)) \in [y_2]$, but $y_1 \neq y_2$. We know that there exists some real number s such that $(s, x^c) \in K^o$, because $\pi_2[K^o] = N^c \ni x^c$. By G1, $((r_1, x), (s, x^c)) \in E$. Applying h to this, we get $(h((r_1, x)), h((s, x^c))) \in E$. By closure of K^e and K^o, $h((r_1, x)) \in K^e$ and $h((s, x^c)) \in K^o$. Therefore, by G1, $\pi_2(h((s, x^c))) = \pi_2(h((r_1, x)))^c = y_1^c$. But, by symmetry, we also get $\pi_2(h((s, x^c))) = \pi_2(h((r_2, x)))^c = y_2^c$. This is a contradiction since $y_1 \neq y_2$.

Using the above, we can define a function $j\colon N \cup N^c \to N \cup N^c$, by letting $j(x) = y$ iff $h((r, x)) = (s, y)$ for some $(r, x) \in [x]$ and $(s, y) \in [y]$. Now, by dropping the notations $\underline{\ }$ and $[-]$, we can argue for elementarity of $j|_{V_\alpha^L}\colon V_\alpha^L \to V_\alpha^L$ just as we argued for elementarity of $h|_{V_\alpha^L}$ in the proof of Theorem 2.2.2. There will be just one detail that will be different, and that is, for the injectivity argument, we will need the fact that there are no arrows between vertices belonging to the same class. Using that, we can argue that the members of two distinct $[x^c], [y^c]$ cannot be sent by h to the same class $[z^c]$.

Observe that, although we have established the elementarity of $j|_{V_\alpha^L}$, the argument in Theorem 2.2.2 is not sufficient to prove that $j|_{V_\alpha^L}$ is nontrivial. What could happen is that $j = \mathrm{id}$, but the nontriviality of h happens inside some class $[x]$, where $h((r_1, x)) = (r_2, x)$ for some distinct $(r_1, x), (r_2, x) \in [x]$. Condition G6 is there precisely to prohibit this scenario. The argument is the same as in the proof of Claim 2.3.2.1 with its G2. □

Since the critical point of any nontrivial elementary embedding $j\colon V_\alpha^L \to V_\alpha^L$ is an inaccessible cardinal in L, the following corollary is an immediate consequence of the previous theorem.

Corollary 2.3.3. *If there are no inaccessible cardinals in L, then every set $G \subset \mathbb{R} \times \mathrm{OR}$ has a strongly rigid relation.*

2.4 A Model for $\neg$AC + SRR

We are now ready to build a model for ZF + $\neg$AC + SRR, thus establishing that SRR does not imply AC.

Theorem 2.4.1. *If* ZF *is consistent, then so is* ZF + $\neg$AC + SRR.

Proof. Work inside V. We can assume that there are no inaccessible cardinals in L (if there are any, then simply work inside V_κ^L for κ the least inaccessible in L). Given this assumption and Corollary 2.3.3, the same model that worked for Hamkins and Palumbo [HP11] for the consistency of ZF + $\neg$AC + RR will work for us too. The model in question is the symmetric Cohen model M, built as follows: Let $\mathbb{P} = \mathrm{Add}(\omega, \omega)$ be the usual forcing notion that adds countably many Cohen reals. Thus, $\mathbb{P} = \{p\colon \omega \times \omega \to \{0, 1\} \mid p \text{ is finite}\}$ and $p \leq q \iff p \supset q$ for all $p, q \in \mathbb{P}$. Every permutation $\pi\colon \omega \to \omega$ induces an automorphism $\bar{\pi}\colon \mathbb{P} \to \mathbb{P}$ by letting

$$\bar{\pi}(p) = \{(\pi(n), m) \mid (n, m) \in p\}.$$

This automorphism, in turn, induces an automorphism $\hat{\pi}\colon V^{\mathbb{P}} \to V^{\mathbb{P}}$ of the class of $\mathbb{P}$-names by the recursive definition

$$\hat{\pi}(\tau) = \{(\hat{\pi}(\sigma), \bar{\pi}(p)) \mid (\sigma, p) \in \tau\}.$$

It is easy to prove, using induction on complexity of formula, that $p \Vdash_{\mathbb{P}} \psi[\tau] \iff \bar{\pi}(p) \Vdash_{\mathbb{P}} \psi[\hat{\pi}(\tau)]$. A $\mathbb{P}$-name τ is said to be *symmetric* iff there exists a finite set $e \subset \omega$

such that whenever $\pi\colon \omega \to \omega$ is a permutation that fixes every member of e, then $\hat{\pi}(\tau) = \tau$. The class of hereditarily symmetric names is denoted by HS. The symmetric Cohen model M is defined in an extension $V[G]$ to be the class of the interpretations of hereditarily symmetric names. That is, $M = \{i_G(\tau) \mid \tau \in \mathrm{HS}\}$.

In [Jec73, Lemma 5.25], it is established in M that there is a set of reals A such that every set can be injected into $A^{<\omega} \times \mathrm{OR}$. Working in M, as $\mathbb{R}$ and $\mathbb{R}^{<\omega}$ are in bijection, this means that every set is in bijection with a subset of $\mathbb{R} \times \mathrm{OR}$. But, since $L^M = L^V$, the assumption that there are no inaccessible cardinals in L^M and Corollary 2.3.3 imply that every subset of $\mathbb{R} \times \mathrm{OR}$, and hence every set, has a strongly rigid relation. □

Corollary 2.4.2. SRR *is independent from* ZF, *follows from* AC, *but is not equivalent to* AC.

2.5 Proto Berkeley Cardinals

The theorem below gives a characterization of proto Berkeley cardinals in terms of a strong failure of SRR.

Theorem 2.5.1. *A cardinal δ is proto Berkeley iff for any graph (G, E) such that $\aleph(G) > \delta$ and any injection $f\colon \delta \to G$, there is an endomorphism $h\colon (G, E) \to (G, E)$ such that $h|_{\mathrm{ran}(f)} \neq \mathrm{id}$.*

Proof. From left to right is easy, so we do that first. Fix any (G, E) and $f\colon \delta \to G$ as in the statement of the theorem. Let V_α be such that $\langle G, E, f\rangle \in V_\alpha$, and let $M = V_\alpha \cup \{V_\alpha, \{\langle G, E, f\rangle, V_\alpha\}\}$. Clearly, M is a transitive set and $\delta \in M$. By proto Berkeleyness, we can find an elementary embedding $j\colon M \to M$ that has critical point κ strictly below δ. The pair $\{\langle G, E, f\rangle, V_\alpha\}$ is definable in M as the unique set with exactly two members, one of which is a set that has every other set as a member. It easily follows from this that each of G, E, f is definable in M, and we deduce that j must fix each one of them. This shows that $j|_G\colon (G, E) \to (G, E)$ is an endomorphism. Also, $j(f(\kappa)) = j(f)(j(\kappa)) = f(j(\kappa)) \neq f(\kappa)$, so that $j|_G$ is not the identity on the range of f.

For the other direction, fix a transitive set M such that $\delta \in M$. Our aim is to show that there is a nontrivial elementary embedding $j\colon M \to M$ with critical point strictly below δ. The idea of the proof is similar to that of the proof of Theorem 2.2.2. Thus, let $N \supset M \cup \{M\}$ be the smallest transitive set closed under taking finite subsets, and let N^c be a copy of it.

(We need a countable set disjoint from $N \cup N^c$ that will be fixed by any endomorphism to play the role of the set $A \cup B$ of Theorem 2.2.2, and we also need a relation on N^c which can be used to argue for injectivity of any endomorphism on N. For the injectivity argument, we need to have an arrow between any two distinct $x^c, y^c \in N^c$. For this, we made use of $<_L$ in Theorem 2.2.2, but in our current situation, there is no guarantee that N is wellorderable. An obvious candidate for our case is $\neq$, but this causes another problem: In the proof of Theorem 2.2.2, when we argued that A was fixed, we needed the fact that there were no 3-cycles in the graph other than the one in A. However, if we use $\neq$, we will be adding cycles of all sizes to the graph, and the argument for A being fixed fails. One remedy to this is to replace A with another solution. That is where we will use the notion of Hartog's number.)

Let $A = \alpha + 1$, where $\alpha = \aleph(N \cup N^c)$. Consider the graph (G, E) in Figure 2.7, where

G1 $(w_0, x), (x, x^c), (w_1, x^c) \in E$, for every $x \in N$.
G2 $(x, y) \in E$ iff $x \in y$, for every $x, y \in N$.
G3 $(x^c, y^c) \in E$ iff $x^c \neq y^c$, for every $x^c, y^c \in N^c$.
G4 $(p_n, x^c) \in E$ iff $|x| = n$, for every $n < \omega$ and every $x^c \in N^c$.
G5 For every $n < \omega$ and every $x^c \in N^c$, $(q_n, x^c) \in E$ iff one of the followings hold:

(a) $x \in M$, $n = \ulcorner\psi(a)\urcorner$ for some $\psi(a) \in \mathrm{FORM}$, and $M \models \psi[x]$.
(b) $x = \langle x_1, \dots, x_m\rangle$ for some $m > 1$, $x_1, \dots, x_m \in M$, $n = \ulcorner\psi(a_1, \dots, a_m)\urcorner$ for some $\psi(a_1, \dots, a_m) \in \mathrm{FORM}$, and $M \models \psi[x_1, \dots, x_m]$.

G6 $(\xi_1, \xi_2) \in E$ iff $\xi_1 < \xi_2$, for every $\xi_1, \xi_2 \in \alpha + 1$.

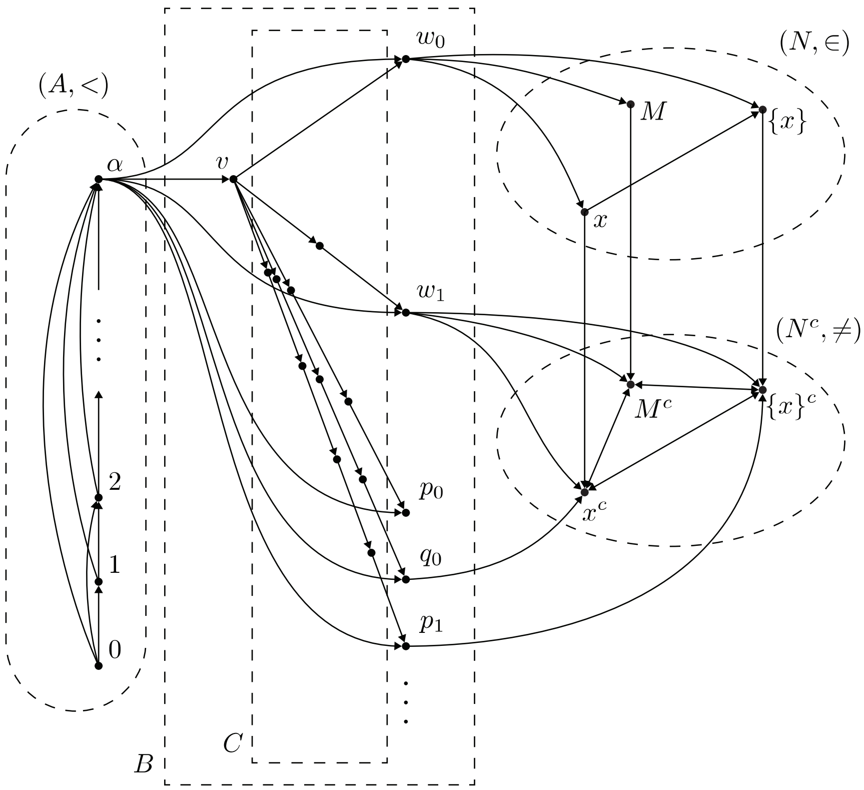

FIGURE 2.7

We remark that although A and N have some common ordinals, we really mean that they are disjoint and merely use their real names for convenience. Normally one would use a copy of, say, the set A, just as we did with N. We will aim to make it clear from the context which copy of a given ordinal we are talking about.

Since $\aleph(G) > \delta$, we can fix an endomorphism $h\colon (G, E) \to (G, E)$ that is nontrivial on $\delta \subset N$, if we take as f the injection $\mathrm{id}\colon \delta \to \delta \subset N$. That will take care of the critical point being strictly below $\delta \in M$, and we only need to show that $h|_M\colon M \to M$ is an elementary embedding. Although we cannot argue that every member of A is fixed by h, we can, nonetheless, argue that $\alpha \in A$ is fixed by h. Using that, we can proceed as in the proof of Theorem 2.2.2 and show that $h|_M\colon M \to M$ is indeed an elementary embedding, which will finish the proof.

So, let us show that α is fixed by h. Henceforth, every ordinal we name belongs to A, and not N. First, we argue that $h(\beta) \in A$, for all $\beta < \alpha$. Working towards a contradiction, fix $\beta < \alpha$ such that $h(\beta) \notin A$. If $h(\gamma) \in A$ for some $\gamma > \beta$, then the arrow $(h(\beta), h(\gamma)) \in E$ will be flowing from outside of A into A. But, no such arrows exist in our graph, so we must conclude that $h(\gamma) \notin A$, for all $\gamma > \beta$. We have established that $h|_{[\beta,\alpha]}$ has $G \setminus A$ as codomain. By choice of α, $h|_{[\beta,\alpha]}$ cannot be an injection. Therefore, there are $\beta \leq \xi_1 < \xi_2 < \alpha$ such that $h(\xi_1) = h(\xi_2)$. This is a contradiction, since $(\xi_1, \xi_2) \in E$ implies $(h(\xi_1), h(\xi_2)) \in E$, which is a loop, and our graph is free of loops.

Fix any $\beta_1 < \beta_2 < \alpha$. By the above and endomorphism of h, $h(\beta_1) < h(\beta_2) \leq \alpha$. Since $(h(\beta_1), h(\alpha)) \in E$ and there are no arrows flowing from $A \setminus \{\alpha\}$ to outside of A, we must have $h(\alpha) \in A$ too. Hence, either $h(\alpha) = \alpha$ or $h(\alpha) < \alpha$. Suppose, again towards a contradiction, that $h(\alpha) = \beta < \alpha$. Taking any $\gamma < \alpha$, we see that $h(\gamma) < h(\alpha) = \beta$. Hence, $h|_{[0,\alpha)}$ has $[0, \beta)$ as codomain. Since α is clearly a cardinal and $\beta < \alpha$, $h|_{[0,\alpha)}$ cannot be injective. Again, this implies the existence of a loop, which is a contradiction. The only option we are left with is $h(\alpha) = \alpha$. $\square$

There is another way to deal with the problem of the injectivity argument in the proof above that does not involve creating cycles. For this, we need two copies of N, call them N^{c_1} and N^{c_2}. Define $<_{\text{rk}}$ by $x <_{\text{rk}} y \iff \text{rank}(x) < \text{rank}(y)$, for all sets x, y. Define also $\notin_{\text{rk}}$ by $x \notin_{\text{rk}} y \iff x \notin y \wedge x <_{\text{rk}} y$, for all sets x, y. Let $<^{c_2}_{\text{rk}} \subset N^{c_2} \times N^{c_2}$ be defined by $x^{c_2} <^{c_2}_{\text{rk}} y^{c_2} \iff x <_{\text{rk}} y$, for all $x^{c_2}, y^{c_2} \in N^{c_2}$. Similarly, let $\notin^{c_1}_{\text{rk}} \subset N^{c_1} \times N^{c_1}$ be defined by $x^{c_1} \notin^{c_1}_{\text{rk}} y^{c_1} \iff x \notin_{\text{rk}} y$, for all $x^{c_1}, y^{c_1} \in N^{c_1}$.

Consider the graph (G, E) in Figure 2.8, where

G1 $(x, x^{c_1}), (x^{c_1}, x^{c_2}) \in E$, for every $x \in N$.
G2 $(x, y) \in E$ iff $x \in y$, for every $x, y \in N$.
G3 $(x^{c_1}, y^{c_1}) \in E$ iff $x^{c_1} \notin^{c_1}_{\text{rk}} y^{c_1}$, for every $x^{c_1}, y^{c_1} \in N^{c_1}$.
G4 $(x^{c_2}, y^{c_2}) \in E$ iff $x^{c_2} <^{c_2}_{\text{rk}} y^{c_2}$, for every $x^{c_2}, y^{c_2} \in N^{c_2}$.

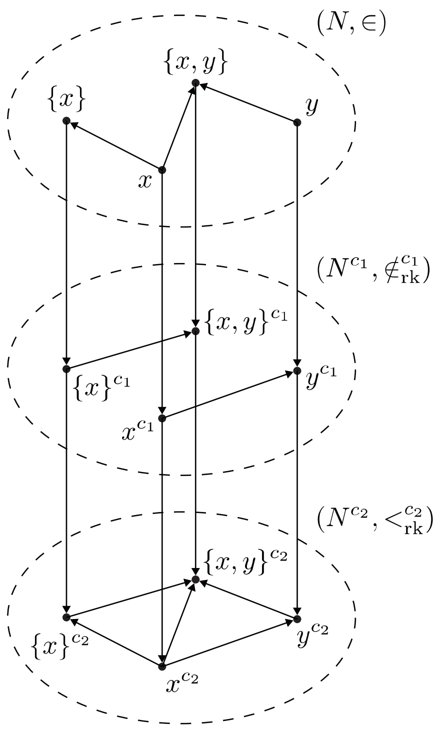

Figure 2.8

Clearly, this graph does not contain any cycles by wellfoundedness of $\in$ and $<_{\text{rk}}$. We can use this graph as part of a bigger graph which ensures that each of the copies of N are closed under any endomorphism h, and which also ensures elementarity of the appropriate restrictions of h. Here, we will only show the following:

Proposition 2.5.2. *If (G, E) is the graph in Figure 2.8 and $h\colon (G, E) \to (G, E)$ is any endomorphism, under which each of the copies of N are closed, then h is injective on N and behaves the same across all copies of N.*

Proof. Fix such an endomorphism h. By G1,

C1 $h(x^{c_1}) = h(x)^{c_1}$, for every $x \in N$.
C2 $h(x^{c_2}) = h(x)^{c_2}$, for every $x \in N$.

This shows that h behaves the same across all copies of N. Now, for injectivity, fix any $x, y \in N$ such that $x \neq y$. If $x <_{\mathrm{rk}} y$ or $y <_{\mathrm{rk}} x$, then $h(x) <_{\mathrm{rk}} h(y)$ or $h(y) <_{\mathrm{rk}} h(x)$ by G4, applying h, and C2. Thus, in such cases, $h(x) \neq h(y)$. Suppose now that $\mathrm{rank}(x) = \mathrm{rank}(y)$. Fix $a \in N$ such that, without loss of generality, $a \in x$ but $a \notin y$. $a \in x$ implies $\mathrm{rank}(a) < \mathrm{rank}(x) = \mathrm{rank}(y)$. That, along with $a \notin y$, imply that $a \notin_{\mathrm{rk}} y$. By G3, applying h, and C1, we get $h(a) \notin_{\mathrm{rk}} h(y)$. In particular, $h(a) \notin h(y)$. Meanwhile, we also have $h(a) \in h(x)$ by G2 and applying h. That is, $h(a) \in h(x)$ but $h(a) \notin h(y)$, which is to say $h(x) \neq h(y)$. As x, y were arbitrary, we deduce that h is injective on N. □

CHAPTER 3

Reinhardt Cardinals and Eventually Dominating Functions

3.1 Introduction

Let us supplement the usual first-order language of set theory with a functional symbol j, and let $\mathrm{ZF}(j)$ denote the theory consisting of the axioms of ZF, together with comprehension and Replacement for formulas in which j appears. Recall that a Reinhardt embedding is a nontrivial elementary embedding $j\colon V \to V$. Reinhardt embeddings are typically studied within the framework of $\mathrm{ZF}(j)$. Thus, the background theory of any result in this chapter is assumed to be $\mathrm{ZF}(j)$ whenever the hypothesis involves a Reinhardt embedding $j\colon V \to V$; otherwise, the background theory is simply ZF.

In Section 3.2, we prove a result concerning Reinhardt embeddings $j\colon V \to V$ and their relation to functions that "eventually dominate" them on regular cardinals.

Definition 3.1.1. Given a limit ordinal δ and two functions $f, g\colon \delta \to \delta$, we say that g *eventually dominates* f, and write $f \leq^* g$, iff there exists $\alpha < \delta$ such that $f(\beta) \leq g(\beta)$ for all $\beta \geq \alpha$.

Theorem 3.2.1. *Suppose $j\colon V \to V$ is a Reinhardt embedding. If $\delta > \mathrm{crit}(j)$ is a regular cardinal such that $j(\delta) = \delta$, then there is no function $g\colon \delta \to \delta$ in the range of j such that $j|_\delta \leq^* g$.*

For any elementary embedding $k\colon V_\delta \to V_\delta$, where δ is a limit ordinal, we will consider elementary embeddings $l\colon V_\delta \to V_\delta$ that are "roots" of k (Definition 3.2.4). Concerning this, in the same section, we prove the following result which shows the existence of an ordinal α that behaves somewhat like an extendible cardinal in V_δ.

Theorem 3.2.8. *Suppose $j\colon V \to V$ is a Reinhardt embedding. For all regular cardinals $\delta > \mathrm{crit}(j)$ with $j(\delta) = \delta$, there exists $\alpha < \delta$ such that for every $\beta \in (\alpha, \delta)$, there is a root k of $j|_{V_\delta}$ with $k(\alpha) > \beta$.*

Such extendibility-like behavior, in a more global form and under ZF alone, has already been considered by Goldberg [Gol24] and Asperó [Asp], as well as in Chapter 1 of this thesis. The ideas of that section motivate the following definition, from which the subsequent result becomes evident.

Definition 3.1.2. Suppose $j\colon V \to V$ is a Reinhardt embedding. Given a regular cardinal $\delta > \mathrm{crit}(j)$ with $j(\delta) = \delta$ and a set $X \subset V_\delta$, we say that X is (j, δ)*-small* iff $j|_{V_\delta} \in j(X)$ and $\sup\{k(\xi) \mid k \in X\} < \delta$, for all $\xi < \delta$.

Theorem 3.2.9. *Suppose $j\colon V \to V$ is a Reinhardt embedding. Then, there is no (j, δ)-small set for any regular cardinal $\delta > \mathrm{crit}(j)$ with $j(\delta) = \delta$.*

In [Gol24], Goldberg showed that the existence of a Reinhardt cardinal implies the existence of a proper class of cardinals that are "almost" supercompact (Definition 3.3.1). In the same paper, he shows that if η is almost supercompact then either η or η^+ is regular.[1] We improve slightly on this result by proving the following in Section 3.3:

Theorem 3.3.12. *Suppose $j\colon V \to V$ is a Reinhardt embedding, and suppose $\lambda > \mathrm{crit}(j)$ is the least ordinal such that $j(\lambda) = \lambda$. If $\eta > \lambda$ is an almost supercompact cardinal that is not a limit of almost supercompact cardinals, then it is not regular.*

Hence, in such cases, it is always η^+ that is a regular cardinal. Finally, in the last section, we present an alternative proof of Kunen's Inconsistency Theorem by showing that, under the assumption of the existence of a Reinhardt embedding $j\colon V \to V$, AC implies that there are (j, δ)-small sets.

Theorem 3.4.1 (Kunen's Inconsistency Theorem)**.** *Suppose $j\colon V \to V$ is a Reinhardt embedding. Then* AC *implies* $0 = 1$.

3.2 Extendibility Behavior

We begin by proving the theorem that relates Reinhardt embeddings and eventually dominating functions, before exploring the extendibility behavior.

Theorem 3.2.1. *Suppose $j\colon V \to V$ is a Reinhardt embedding. If $\delta > \mathrm{crit}(j)$ is a regular cardinal such that $j(\delta) = \delta$, then there is no function $g\colon \delta \to \delta$ in the range of j such that $j|_\delta \leq^* g$.*

Proof. Assume towards a contradiction that $j(f) = g$ where $g\colon \delta \to \delta$ and $j|_\delta \leq^* g$. Let $\kappa = \mathrm{crit}(j)$, and let $\alpha < \delta$ be such that $j(\beta) \leq g(\beta)$ for all β such that $\alpha \leq \beta < \delta$. Let γ be the κ-th ordinal ξ such that $\alpha < \xi < \delta$ and $f"\xi \subset \xi$ (this exists as δ is regular). Then γ has cofinality κ, so $j(\gamma) > \gamma$. But since $f"\gamma \subset \gamma$ and $j(f) = g$, we have $g"j(\gamma) \subset j(\gamma)$. Therefore $g(\gamma) < j(\gamma)$, although $\gamma > \alpha$, contradicting the choice of α.[2] □

Given a nontrivial elementary embedding $k\colon V_\delta \to V_\delta$ (allowing for $\delta = \mathrm{OR}$ here), the *critical sequence* $\langle \kappa_n(k) \mid n < \omega \rangle$ of k is defined recursively by setting $\kappa_0(k) = \mathrm{crit}(k)$ and $\kappa_{n+1}(k) = k(\kappa_n(k))$. The supremum of this sequence will be denoted by $\lambda(k)$. Notice that if $\lambda(k) < \delta$, then $\lambda(k)$ is the first fixed point of k above its critical point.

Recall that $\mathcal{E}_\delta$ is the set of all nontrivial elementary embeddings $k\colon V_\delta \to V_\delta$. It is an easy argument to show that if $j\colon V \to V$ is a Reinhardt embedding, then $\mathcal{E}_\delta$ is nonempty for all $\delta \geq \lambda(j)$: If not true, take the least δ_0 counterexample and notice that $j|_{V_{\delta_0}} \in \mathcal{E}_{\delta_0}$. The next lemma is an easy fact about the application operation (Definition 1.4.5).

Lemma 3.2.2. *For $k, l \in \mathcal{E}_\delta$ where δ is a limit ordinal, $k[l](k(a)) = k(l(a))$ for all $a \in V_\delta$.*

Proof. Fix some $\alpha < \delta$ such that $a \in V_\alpha$. Then, $k[l](k(a)) = k(l|_{V_\alpha})(k(a)) = k(l|_{V_\alpha}(a)) = k(l(a))$. □

[1] Successor cardinals in ZF need not be regular. See the discussion at the beginning of Section 3.3 for more details.

[2] We thank the anonymous reviewer for suggesting this simple argument.

The following lemma slightly improves Lemma 1.4.6.

Lemma 3.2.3. *If $k, l \in \mathcal{E}_\delta$ where δ is a limit ordinal, then $k[l]$ is also in $\mathcal{E}_\delta$. Moreover,* $\mathrm{crit}(k[l]) = k(\mathrm{crit}(l))$, *and if $\langle \gamma_n \mid n < \omega \rangle$ is the critical sequence of l, then $\langle k(\gamma_n) \mid n < \omega \rangle$ is the critical sequence of $k[l]$.*

Proof. Only the claim regarding the critical sequence is new compared to Lemma 1.4.6, so it suffices to prove that part alone:

$$\begin{aligned} k[l]^n(\mathrm{crit}(k[l])) = k[l]^n(k(\mathrm{crit}(l))) &= k[l]^{n-1}(k[l](k(\mathrm{crit}(l)))) \\ &= k[l]^{n-1}(k(l(\mathrm{crit}(l)))) = k(l^n(\mathrm{crit}(l))) \end{aligned}$$

by n applications of Lemma 3.2.2. □

Throughout the remainder of this chapter, we will assume that δ is a limit ordinal.

Definition 3.2.4. For any $k \in \mathcal{E}_\delta$, we can define the two sets $I(k) = \{k_n \mid n < \omega\}$, where $k_0 = k$ and $k_{n+1} = k_n[k_n]$,[3] and $R(k) = \{l \in \mathcal{E}_\delta \mid l[l] = k\}$. Whenever $l[l] = k$, we will call l a *root of* k and k the *square of* l.

Definition 3.2.5. For $k \in \mathcal{E}_\delta$ define the set $A(k)$ by the following recursion: Set $A_0 = I(k)$ and $A_{n+1} = A_n \cup \bigcup_{l \in A_n} R(l)$, and let $A(k) = \bigcup_n A_n$.

Notice that the set $A(k)$ is the smallest set containing k and closed under taking squares and roots.

Lemma 3.2.6. *If $\delta > \kappa$ is a limit ordinal such that $j(\delta) = \delta$, then $j(A(j|_{V_\delta})) = A(j|_{V_\delta})$.*

Proof. Denote $j|_{V_\delta}$ by j' for simplicity. First, by elementarity of j, we have $j(A(j')) = A(j(j'))$. Then, noticing that $j' = \bigcup_{\alpha<\delta} j'|_{V_\alpha}$, we get

$$j(j') = j\Big(\bigcup_{\alpha<\delta} j'|_{V_\alpha}\Big) = \bigcup_{\alpha<\delta} j(j'|_{V_\alpha}) = \bigcup_{\alpha<\delta} j'(j'|_{V_\alpha}) = j'[j'].$$

Hence, $A(j(j')) = A(j'[j'])$. Finally, we establish $A(j'[j']) = A(j') : I(j'[j']) \subset I(j')$ implies $A(j'[j']) \subset A(j')$ by definition. For the reverse inclusion, notice that $j' \in R(j'[j']) \subset A(j'[j'])$, and since $A(j')$ is the smallest set containing j' and closed under taking squares and roots, it must be that $A(j') \subset A(j'[j'])$. Putting everything together, we have

$$j(A(j')) = A(j(j')) = A(j'[j']) = A(j').$$ □

Theorem 3.2.7. *For all regular cardinals $\delta > \lambda$ such that $j(\delta) = \delta$, there exists $\alpha < \delta$ such that for every $\beta \in (\alpha, \delta)$, there is $k \in A(j|_{V_\delta})$ with $k(\alpha) > \beta$.*

Proof. Let A denote $A(j|_{V_\delta})$ for simplicity. Working towards a contradiction, fix any such δ and suppose there is no such $\alpha < \delta$. Define $f\colon \delta \to \mathrm{OR}$ by setting $f(\xi) = \sup\{k(\xi) \mid k \in A\}$. By assumption, $f(\xi) < \delta$, for all $\xi < \delta$. Since by Lemma 3.2.6 $j(A) = A$, we must have $j(f) = f$. But clearly $f \geq^* k|_\delta$ for all $k \in A$, and in particular, $f \geq^* j|_\delta$, contradicting Theorem 3.2.1. □

[3] In Section 1.4, we used $j_{n+1} = j[j_n]$. We have not overloaded the subscript notation here since it is actually easy to show that $k[k_n] = k_n[k_n]$ for all $n < \omega$ and all $k \in \mathcal{E}_\delta$. However, this fact will not be necessary for our purpose.

Thus α behaves somewhat similar to extendible cardinals inside V_δ. We can impose even further restrictions on the elementary embeddings k above while still getting the same result:

Theorem 3.2.8. *For all regular cardinals $\delta > \lambda$ such that $j(\delta) = \delta$, there exists $\alpha < \delta$ such that for every $\beta \in (\alpha, \delta)$, there is $k \in R(j|_{V_\delta})$ with $k(\alpha) > \beta$.*

Proof. Notice that $R(j|_{V_\delta})$ is not empty since $j(R(j|_{V_\delta})) = R(j(j|_{V_\delta})) = R(j|_{V_\delta}[j|_{V_\delta}])$ is not, as witnessed by $j|_{V_\delta}$. This time define $f\colon \delta \to \mathrm{OR}$ by setting $f(\xi) = \sup\{k(\xi) \mid k \in R(j|_{V_\delta})\}$. Again, if the theorem fails for δ, then $f(\xi) < \delta$ for all $\xi < \delta$. Clearly $f \geq^* k|_\delta$ for all $k \in R(j|_{V_\delta})$. By elementarity, $j(f) \geq^* k|_\delta$ for all $k \in j(R(j|_{V_\delta})) = R(j(j|_{V_\delta})) = R(j|_{V_\delta}[j|_{V_\delta}])$. In particular, $j(f) \geq^* j|_\delta$, contradicting Theorem 3.2.1. □

The above theorem remains valid when $R(j|_{V_\delta})$ is replaced by any set $X \subset \mathcal{E}_\delta$ satisfying $j|_{V_\delta} \in j(X)$. Thus, we have the following theorem about (j, δ)-small sets (Definition 3.1.2):

Theorem 3.2.9. *Suppose $j\colon V \to V$ is a Reinhardt embedding. Then, there is no (j, δ)-small set for any regular cardinal $\delta > \mathrm{crit}(j)$ with $j(\delta) = \delta$.* □

We will show in the last section that, under the assumption of the existence of a Reinhardt embedding $j\colon V \to V$, AC implies the existence of (j, δ)-small sets for unboundedly many δ, which will give us Kunen's Inconsistency Theorem.

3.3 Regular Cardinals

In the context of Choice, it is a basic set theoretic fact that every successor cardinal is regular. In the absence of AC, there is no guarantee that successor cardinals are regular. In fact, Moti Gitik has showed that it is consistent with ZF that there are no regular uncountable cardinals [Git80].

If $j\colon V \to V$ is a Reinhardt embedding, we already know that the $\kappa_n(j)$ are regular for all $n < \omega$. David Asperó asked whether there are regular cardinals above $\lambda(j)$, and Goldberg answered this question positively in [Gol24]. We will need a more detailed account of Goldberg's result, so let us start by recalling what is necessary from his paper.

Definition 3.3.1 ([Gol24])**.** A cardinal η is said to be (γ, ν, x)-*almost supercompact* for $\gamma < \eta < \nu$ and $x \in V_\nu$ iff there exists $\bar{\nu} < \eta$ and $\bar{x} \in V_{\bar{\nu}}$ for which there is an elementary embedding $k\colon V_{\bar{\nu}} \to V_\nu$ such that $k(\gamma) = \gamma$ and $k(\bar{x}) = x$. We say that η is $<\mu$-*almost supercompact* iff η is (γ, ν, x)-almost supercompact for all $\gamma < \eta < \nu < \mu$ and all $x \in V_\nu$, and we simply say that η is almost supercompact iff it is $<\mu$-almost supercompact for all $\mu > \eta$.

Definition 3.3.2 ([Gol24])**.** A cardinal η is said to be (γ, ν)-*almost extendible* for $\gamma < \eta < \nu$ iff there is an elementary embedding $k\colon V_\nu \to V_{\nu'}$ such that $k(\gamma) = \gamma$ and $k(\eta) > \nu$. We say that η is $<\mu$-*almost extendible* iff η is (γ, ν)-almost extendible for all $\gamma < \eta < \nu < \mu$, and we simply say that η is *almost extendible* iff it is $<\mu$-almost extendible for all $\mu > \eta$.

The definitions above are slightly finer than in [Gol24]. There, a cardinal η is first defined to be (γ, ∞)-supercompact (resp. (γ, ∞)-extendible) iff it is precisely $(\gamma, \nu, \emptyset)$-almost supercompact (resp. (γ, ν)-almost extendible) for all $\nu > \eta$. Then, η is called almost supercompact (resp. almost extendible) iff it is (γ, ∞)-supercompact (resp.

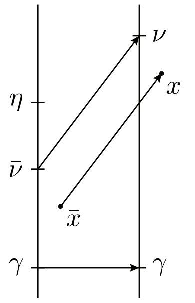

FIGURE 3.1: Almost supercompactness

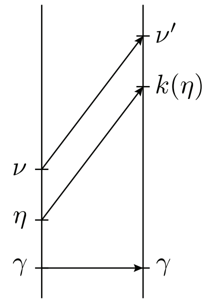

FIGURE 3.2: Almost extendibility

(γ, ∞)-extendible) for all $\gamma < \eta$. It is clear that we have not changed the meaning of almost extendibility. For almost supercompactness, we have decided to also incorporate the parameter x. This does not change the meaning of almost supercompactness as discussed in the paragraph preceding Lemma 2.6 of [Gol24].

Note that every limit of almost supercompact cardinals is almost supercompact, and likewise for almost extendible cardinals. Thus the two classes of almost supercompact cardinals and almost extendible cardinals are closed. Proposition 3.3.3 below follows by incorporating the parameter x into the proof of [Gol24, Lemma 2.6(1)], and the subsequent Proposition 3.3.4 follows from [Gol24, Proposition 2.1, Corollary 2.3, and Corollary 2.5].

Proposition 3.3.3 ([Gol24])**.** *If a cardinal η is $<\mu$-almost extendible, where μ is a limit ordinal, then it is also $<\mu$-almost supercompact.* □

Proposition 3.3.4 ([Gol24])**.** *If there is a Reinhardt embedding, then there is a club proper class of almost extendible cardinals.* □

Goldberg proves that if η is almost supercompact, then every successor cardinal greater than η has cofinality at least η [Gol24, Corollary 2.18]. This, in turn, implies that if η is almost supercompact, then either η or η^+ is regular; together with Propositions 3.3.3 and 3.3.4, this yields a proper class of regular cardinals in the presence of a Reinhardt embedding. Goldberg proves his result by the use of ultrafilters generalized to the Choiceless context. We give here a direct proof that is essentially Goldberg's idea without digressing into ultrafilters.

Recall that a prewellorder is a non-strict wellordering that is not necessarily antisymmetric. If (X, E) is a prewellorder, we can partition X by identifying symmetric objects with each other, i.e., $x \sim y$ iff $x \mathrel{E} y$ and $y \mathrel{E} x$. The resulting set of classes X^* and the induced strict ordering E^* together form a wellordering (X^*, E^*). For the purpose of this section, we will refer to the ordertype of E^* as the *rank of* E and denote it by $\operatorname{rank}(E)$. For $x \in X$, the *rank of x in E*, denoted by $\operatorname{rank}_E(x)$, is the rank of E restricted to $\{y \in X \mid y \mathrel{E} x \wedge \neg(x \mathrel{E} y)\}$.

The prewellorders of a set X are in one-to-one correspondence with surjective maps from X onto an ordinal. Given a family of sets Γ, let $\delta(\Gamma)$ be the supremum of the ranks of all prewellorders that are members of Γ.

For any set X, the *Lindenbaum number of X*, denoted by $\theta(X)$, is the least ordinal α such that there is no surjection $f : X \to \alpha$. The Lindenbaum number exists for every set X by the Axiom Schema of Replacement applied to the subset of $\mathcal{P}(X \times X)$ that consists of all the prewellorders of X, and the one-to-one correspondence between prewellorders of X and surjective maps from X onto an ordinal. It is easy to see that $\theta(X) = \delta(\mathcal{P}(X \times X))$.

Lemma 3.3.5 ([Gol24]). *Suppose α is a cardinal and let $\Gamma = \mathcal{P}(\alpha \times \alpha)$. If $\sigma \subset \Gamma$ is such that $\theta(\sigma) < \alpha$, then $\delta(\sigma) < \theta(\alpha)$.*

Proof. We need to define a partial surjection $G\colon \alpha \to \delta(\sigma)$. Since α is a cardinal, the canonical wellordering of $\mathrm{OR} \times \mathrm{OR}$ restricts to a bijection between $\alpha \times \alpha$ and α. Thus, we will be done if we instead just defined a partial surjection $G\colon \alpha \times \alpha \to \delta(\sigma)$. First, let us consider the map $F\colon \alpha \times \sigma \to \delta(\sigma)$ defined by $F(\xi, E) = \mathrm{rank}_E(\xi)$. This map is surjective by definition of $\delta(\sigma)$. For each $\xi < \alpha$, let $F_\xi\colon \sigma \to \delta(\sigma)$ be defined by $F_\xi(E) = F(\xi, E)$. Let $A_\xi = F_\xi``\sigma$, and let $\pi_\xi\colon A_\xi \to \alpha_\xi$ be its collapsing map. The map $\pi \circ F_\xi\colon \sigma \to \alpha_\xi$ is surjective, and therefore, $\alpha_\xi < \theta(\sigma) < \alpha$ for all $\xi < \alpha$. Now, the partial function $G\colon \alpha \times \alpha \to \delta(\sigma)$ defined by $G(x, y) = \pi_x^{-1}(y)$ whenever $y < \alpha_x$ is the desired partial surjection. □

Proposition 3.3.6 ([Gol24]). *If η is almost supercompact, then every successor cardinal greater than η has cofinality at least η.*

Proof. Fix a cardinal $\alpha \geq \eta$, and assume towards a contradiction that $\mathrm{cof}(\alpha^+) < \eta$. Let $\gamma = \mathrm{cof}(\alpha^+)$, and let $c = \langle c_\zeta \mid \zeta < \gamma\rangle$ be an increasing sequence cofinal in α^+. Let $\Gamma = \mathcal{P}(\alpha \times \alpha)$, and choose a limit ordinal $\nu > \theta(\Gamma)$. Using almost supercompactness of η, fix an elementary embedding $k\colon V_{\bar\nu} \to V_\nu$ such that $k(\gamma, \bar c, \bar\alpha, \bar\Gamma) = (\gamma, c, \alpha, \Gamma)$.

Clearly, $\delta(\bar\Gamma) = \delta(\mathcal{P}(\bar\alpha \times \bar\alpha)) = \theta(\bar\alpha) = \bar\alpha^+$. We want to show that $\delta(k``\bar\Gamma) = \alpha^+$. To do this, it suffices to show the equivalent statement: $\sup(k``\bar\alpha^+) = \alpha^+$. Thus, fix an arbitrary $\beta < \alpha^+$. Let $\zeta < \gamma$ be such that $c_\zeta > \beta$. Now, observe that $k(\bar c_\zeta) = c_{k(\zeta)} \geq c_\zeta > \beta$. Hence, $\sup(k``\bar\alpha^+) = \alpha^+$, and therefore $\delta(k``\bar\Gamma) = \alpha^+$.

On the other hand, we have $\theta(k``\bar\Gamma) = \theta(\bar\Gamma) = (\theta(\bar\Gamma))^{V_{\bar\nu}} < \bar\nu < \alpha$, which by Lemma 3.3.5 implies $\delta(k``\bar\Gamma) < \theta(\alpha) = \alpha^+$. This is a contradiction. □

Corollary 3.3.7 ([Gol24]). *If η is almost supercompact, then either η or η^+ is a regular cardinal.*

Let us call an almost supercompact cardinal that is not a limit of almost supercompact cardinals a *successor almost supercompact.* We will prove that if η is a successor almost supercompact cardinal above λ, then it cannot be regular. Thus, by Goldberg's result, we must have η^+ regular for all $\eta > \lambda$ that is a successor almost supercompact cardinal. We will need a few intermediary results first. The following lemma is easy.

Lemma 3.3.8. *If η_0 is $<\eta_1$-almost supercompact and η_1 is almost supercompact, then η_0 is also almost supercompact.*

Proof. Fix $\gamma < \eta_0$, some $\nu \geq \eta_1$, and $x \in V_\nu$. Let γ' code the pair $\langle\gamma, \eta_0\rangle$ in the canonical wellordering of $\mathrm{OR} \times \mathrm{OR}$. Notice that $\gamma' < \eta_0 < \eta_1$, so by almost supercompactness of η_1, we can find an elementary embedding $k\colon V_{\bar\nu+\omega} \to V_{\nu+\omega}$ such that $\bar\nu + \omega < \eta_1$, $k(\gamma') = \gamma'$, $k(\langle\bar\gamma, \bar\eta_0\rangle) = \langle\gamma, \eta_0\rangle$, and $k(\bar x) = x$, for some $\langle\bar\gamma, \bar\eta_0\rangle, \bar x \in V_{\bar\nu}$. Since the canonical wellordering of $\mathrm{OR} \times \mathrm{OR}$ is Δ_0, elementarity of k and the fact that $k(\gamma') = \gamma'$ imply that $\langle\bar\gamma, \bar\eta_0\rangle = \langle\gamma, \eta_0\rangle$. Hence, both γ and η_0 are fixed by k. Since $\eta_0 < \eta_1 \leq \nu$, we must also have $\eta_0 < \bar\nu$ by elementarity.

We already know that $\bar\nu < \eta_1$, so we can now use $<\eta_1$-almost supercompactness of η_0 to get another elementary embedding $l\colon V_{\bar{\bar\nu}} \to V_{\bar\nu}$ such that $\bar{\bar\nu} < \eta_0$, $l(\gamma) = \gamma$, and $l(\bar{\bar x}) = \bar x$. Finally, the composite elementary embedding $k \circ l\colon V_{\bar{\bar\nu}} \to V_\nu$ fixes γ and has x in its range, and is therefore our witness. □

Definition 1.4.5 and Lemma 3.2.3 also work with Reinhardt embeddings. Thus, given $j\colon V \to V$, we can consider $j_n, n < \omega$, as in Definition 3.2.4. The following lemma is similar to Lemma 1.4.7.

Lemma 3.3.9 ([GS24, Theorem 5.6(4)]). *Suppose $j\colon V \to V$ is a Reinhardt embedding. Then for all α, there exists n such that $j_n(\alpha) = \alpha$.*

Proof. Suppose this is not the case and let α be the least counterexample. Let $\gamma > \alpha$ be a limit ordinal fixed by j, and denote $j|_{V_\gamma}$ by j'. Clearly, $j'_n = j_n|_{V_\gamma}$ for all $n < \omega$. Now, consider the sequence $j(\langle j'_0, j'_1, j'_2, \ldots\rangle) = \langle j(j'_0), j(j'_1), j(j'_2), \ldots\rangle$. Since $j(j'_0) = j'_1$ by definition, it follows by elementarity of j that $j(j'_1) = j'_2$, $j(j'_2) = j'_3$, and so on. Thus, $j(j'_n) = j_{n+1}|_{V_\gamma}$ for all $n < \omega$. The rest is similar to the proof of Lemma 1.4.7. □

Using the lemma above and coding finite sequences of ordinals as single ordinals, we can always find some n such that j_n fixes any desired finite set of ordinals. Given an elementary embedding $k\colon V_\delta \to V_\delta$ and $\gamma < \delta$, let $R^\gamma(k) = \{l \in R(k) \mid l(\gamma) = \gamma\}$.

Theorem 3.3.10. *Suppose $j\colon V \to V$ is a Reinhardt embedding. Let $\delta > \lambda(j)$ be a regular cardinal, let $\gamma < \delta$, and let n be such that j_n fixes both γ and δ. Then there exists $\alpha < \delta$ such that for all $\beta > \alpha$, there exists $k \in R^\gamma(j_n|_{V_\delta})$ with $k(\alpha) > \beta$.*

Proof. Similar to the proof of Theorem 3.2.8 using j_n in place of j. □

Corollary 3.3.11. *Suppose $j\colon V \to V$ is a Reinhardt embedding. For any regular limit cardinal $\delta > \lambda(j)$, V_δ satisfies that there exists a club class of almost extendible cardinals.*

Proof. For each $\gamma < \delta$, let $\alpha_\gamma < \delta$ be the least ordinal α such that for all $\beta > \alpha$, there exists $k \in \mathcal{E}_\delta$ with $k(\gamma) = \gamma$ and $k(\alpha) > \beta$ (such an α exists by Theorem 3.3.10). Define $F\colon \delta \to \delta$ by setting $F(\xi) = \sup\{\alpha_\gamma \mid \gamma < \xi\}$. This is welldefined by the regularity of δ. Notice that F is monotone increasing and continuous, so the set of fixed points of F form a club $C \subset \delta$. If $D \subset \delta$ is the club of all cardinals below δ, then clearly any member of $C \cap D$ is an almost extendible cardinal in V_δ. □

We are now ready to prove the main theorem of this section.

Theorem 3.3.12. *Suppose $j\colon V \to V$ is a Reinhardt embedding. If $\eta > \lambda(j)$ is a successor almost supercompact cardinal, then it is not regular.*

Proof. Let $\alpha < \eta$ be such that there is no almost supercompact cardinal in the open interval (α, η) and assume towards a contradiction that η is regular. It is easy to see that any almost supercompact cardinal is a limit cardinal, so Corollary 3.3.11 applies to η, and we can fix a cardinal $\beta \in (\alpha, \eta)$ that is almost extendible in V_η. This means that β is $<\eta$-almost extendible, and hence $<\eta$-almost supercompact by Proposition 3.3.3. Now, β must be almost supercompact by Lemma 3.3.8, a contradiction. □

By Goldberg's result, Corollary 3.3.7, we obtain the following:

Corollary 3.3.13. *Suppose $j\colon V \to V$ is a Reinhardt embedding. If $\eta > \lambda(j)$ is a successor almost supercompact cardinal, then η^+ is regular.*

Goldberg proved these last two results in a stronger form in [Gol, Proposition 6.1 & Theorem 6.3], showing that successor supercompact cardinals above the least rank-Berkeley cardinal have cofinality ω.

3.4 Small Sets Under AC

In this section we present an alternative proof of Kunen's Inconsistency Theorem. In particular, we will prove that AC implies the existence of small sets, which contradicts Theorem 3.2.9. The proof follows the strategy of Solovay's theorem, as presented in [Jec03, Theorem 20.8], which establishes that the Singular Cardinal Hypothesis holds above a strongly compact cardinal.

Theorem 3.4.1 (Kunen's Inconsistency Theorem)**.** *Suppose $j\colon V \to V$ is a Reinhardt embedding. Then* AC *implies* $0 = 1$.

Proof. Let κ denote $\mathrm{crit}(j)$. We assume AC and arrive at a contradiction in a series of claims.

Claim 3.4.1.1. There exists an ordinal θ such that for every singular almost supercompact cardinal $\eta > \theta$, there exists an almost supercompact cardinal $\eta' < \eta$ and a collection $\{M_\alpha \subset \eta^+ \mid \alpha < \eta^+\}$ such that $|M_\alpha| < \eta'$ for all $\alpha < \eta^+$ and

$$[\eta^+]^\omega = \bigcup_{\alpha<\eta^+} [M_\alpha]^\omega.$$

Proof of claim. Suppose otherwise and let $\langle \eta_\xi \mid \xi < \mathrm{OR}\rangle$ be an increasing enumeration of singular almost supercompact cardinals above $\lambda(j)$ for which this fails. By Corollary 3.3.7, η_ξ^+ is regular for every ξ. Notice that

$$\eta_\kappa < \eta_{\kappa+1} < \eta_{\kappa+1}^+ < j(\eta_\kappa) = \eta_{j(\kappa)} < \sup j\text{"}\eta_{\kappa+1}^+ < j(\eta_{\kappa+1}^+).$$

Let $\eta' = \eta_\kappa$, $\eta = \eta_{\kappa+1}$, and $\sigma = \sup j\text{"}\eta_{\kappa+1}^+$. Define the κ-complete ultrafilter D on η^+ by setting $X \in D \iff \sigma \in j(X)$, for all $X \subset \eta^+$. Since $\mathrm{cof}(\sigma) = \eta^+ < j(\eta')$, the set $E = \{\alpha < \eta^+ \mid \mathrm{cof}(\alpha) < \eta'\}$ belongs to D.

Using AC, for each $\alpha \in E$, fix $A_\alpha \subset \alpha$ cofinal with $|A_\alpha| < \eta'$. If $\alpha < \eta^+$ is not in E, then set $A_\alpha = \emptyset$. Let $\langle B_\alpha \mid \alpha < j(\eta^+)\rangle = j(\langle A_\alpha \mid \alpha < \eta^+\rangle)$. Since B_σ is cofinal in $\sigma = \sup j\text{"}\eta^+$, for every $\mu < \eta^+$, there is $\mu' \in (\mu, \eta^+)$ such that $[j(\mu), j(\mu')) \cap B_\sigma \neq \emptyset$. Define the sequence $\langle \mu_\zeta \mid \zeta < \eta^+\rangle$ in η^+ recursively by setting $\mu_0 = 0$, taking limits at limit stages, and at successor stages taking $\mu_{\zeta+1} < \eta^+$ to be such that $[j(\mu_\zeta), j(\mu_{\zeta+1})) \cap B_\sigma \neq \emptyset$. For $\zeta < \eta^+$, set $I_\zeta = [\mu_\zeta, \mu_{\zeta+1})$. For each $\alpha < \eta^+$, define

$$M_\alpha = \{\zeta < \eta^+ \mid I_\zeta \cap A_\alpha \neq \emptyset\}.$$

Now, fix any $\zeta < \eta^+$. By construction $j(I_\zeta) = [j(\mu_\zeta), j(\mu_{\zeta+1}))$ intersects B_σ. Thus, the set of all $\alpha < \eta^+$ such that $\zeta \in M_\alpha$ belongs to D.

We will show that $\{M_\alpha \mid \alpha < \eta^+\}$ and η' witness the conclusion of the lemma for η, thereby arriving at a contradiction. First, for each α, we have $|M_\alpha| \leq |A_\alpha| < \eta'$ since the I_ζ are mutually disjoint. Next, fix $x \in [\eta^+]^\omega$. For each $\zeta \in x$, the set of α such that $\zeta \in M_\alpha$ belongs to D. By the κ-completeness of D, there exists some α such that $x \subset M_\alpha$. Hence, $x \in [M_\alpha]^\omega$, as desired. □

Fix θ as in the claim above.

Claim 3.4.1.2. For every singular almost supercompact cardinal $\eta > \theta$ with $\mathrm{cof}(\eta) = \omega$, we have $|V_{\eta+1}| = \eta^+$.

Proof of claim. Fix such an η, and let $\eta' < \eta$ and $\{M_\alpha \mid \alpha < \eta^+\}$ be as in the previous claim. Notice that $|V_\eta| = \eta$. We now have

$$\begin{aligned}|V_{\eta+1}| = 2^{|V_\eta|} = 2^\eta = \eta^\omega \le (\eta^+)^\omega = |[\eta^+]^\omega| = |\bigcup_{\alpha<\eta^+}[M_\alpha]^\omega| \\ \le \sum_{\alpha<\eta^+} |[M_\alpha]^\omega| = \eta^+ \cdot \sup_{\alpha<\eta^+} |[M_\alpha]^\omega| \le \eta^+ \cdot \eta = \eta^+,\end{aligned}$$

where the last inequality follows from the facts that η' is a strong limit and that $|M_\alpha| < \eta'$ for all $\alpha < \eta^+$. □

Claim 3.4.1.3. For every singular almost supercompact cardinal $\eta > \theta$ with $\mathrm{cof}(\eta) = \omega$ and $j(\eta) = \eta$, there exists a (j, η^+)-small set.

Proof of claim. Fix such an η. By the previous claim, we can fix a surjection $b\colon \eta^+ \to \mathcal{E}_\eta \subset V_{\eta+1}$. Let $\beta < \eta^+$ be such that $j|_{V_\eta} \in \mathrm{ran}(j(b|_\beta))$. Let $X \subset \mathcal{E}_{\eta^+}$ consist of all those k such that $k|_{V_\eta} \in \mathrm{ran}(b|_\beta)$. We will show that X is (j, η^+)-small.

First, since $j|_{V_\eta} \in \mathrm{ran}(j(b|_\beta))$, we have $j|_{V_{\eta^+}} \in j(X)$. Next, we need to show that $\sup\{k(\xi) \mid k \in X\} < \eta^+$ for all $\xi < \eta^+$. Observe that for $k, l \in \mathcal{E}_{\eta^+}$, if $k|_{V_\eta} = l|_{V_\eta}$, then it follows that $k|_{V_{\eta+1}} = l|_{V_{\eta+1}}$. This holds because for any $A \subset V_\eta$ and any $k \in \mathcal{E}_{\eta^+}$, we have $k(A) = \bigcup_{\alpha<\eta} k(A \cap V_\alpha)$. Also, $k|_{V_{\eta+1}} = l|_{V_{\eta+1}}$ implies $k|_{\eta^+} = l|_{\eta^+}$, since every $\alpha \in (\eta, \eta^+)$ can be identified with a wellordering of η. Therefore, for each $\xi < \eta^+$,

$$|\{k(\xi) \mid k \in X\}| \le |\{k|_{\eta^+} \mid k \in X\}| \le |\{k|_{V_\eta} \mid k \in X\}| \le |(b|_\beta)| < \eta^+. \qquad \square$$

This final claim contradicts Theorem 3.2.9. □

CHAPTER 4

Small Cardinals Can be Large in HOD

4.1 Introduction

The study of HOD, the class of hereditarily ordinal definable sets, has become central in recent investigations into the foundations of set theory, particularly in light of Woodin's HOD Dichotomy Theorem (Theorem 0.0.2). A natural question arising from this dichotomy concerns how different V and HOD can be, especially at small cardinals. Recent advances using the method of forcing have demonstrated striking potential divergences, producing a growing body of works (e.g., [CFH15], [CFG15], [Cum+18], [GM18], [BU17], [Ben+20], [BH23], [BP23], [Pov23], and, more recently, [GOP24]).

In this chapter, we continue that investigation by focusing on a specific instance of disagreement at small cardinals. Answering a question of Dima Sinapova [Far+17, p.553], we establish the following consistency result:

Theorem 4.1.1. *It is consistent relative to the existence of two supercompact cardinals that $\aleph_\omega$ is a strong limit and $\aleph_{\omega+1}$ is supercompact in* HOD.

Moreover, in the above model, Woodin's HOD *hypothesis holds.*

4.2 The Strategy

In this section, we describe several potential strategies to arrive at the model of Theorem 4.1.1, and discuss the challenges that each of these approaches entail.

In [GM18], starting with two supercompact cardinals $\kappa < \lambda$, Carmi Merimovich and Moti Gitik establish the consistency of κ being a strong limit cardinal of countable cofinality such that $\kappa^+ = \lambda$ is supercompact in $\mathrm{HOD}_{\{x\}}$ for every $x \subset \kappa$. They prove this by analyzing the homogeneity properties of the Supercompact extender-based Prikry forcing developed by Merimovich in [Mer11]. In their construction, however, $\kappa > \aleph_\omega$. Our original aim was to establish the same while also collapsing κ to $\aleph_\omega$. Note that this is stronger than Theorem 4.1.1 since we are considering $\mathrm{HOD}_{\{x\}}$ for all $x \subset \aleph_\omega$.

The most straightforward—though perhaps naive—strategy would be to start working in the extension $V^{\mathbb{P}}$ of Merimovich-Gitik, take an ω-sequence $\vec{s}$ cofinal in κ, and then force with the full-support product $\mathbb{Q}$ of the Lévy collapse of successive members of $\vec{s}$. The problem with this approach is that one can no longer argue that HOD of $V^{\mathbb{P}*\mathbb{Q}}$ is an extension of V by a small forcing, which is necessary to establish that λ remains supercompact in HOD of $V^{\mathbb{P}*\mathbb{Q}}$. To be more precise, it is no longer clear that there is some projection π from $\mathbb{P} * \mathbb{Q}$ to some small forcing $\mathbb{R}$ such that HOD of $V^{\mathbb{P}*\mathbb{Q}}$ is a subclass of the intermediate model $V^{\mathbb{R}}$ given by the projection.

To remedy this, one could consider computing $\mathbb{Q}$ in an intermediate model between V and $V^{\mathbb{P}}$ rather than in $V^{\mathbb{P}}$. In particular, one could take the sequence $\vec{s}$ to be the Prikry sequence that is added by the Prikry forcing $\mathbb{P}_{E(\kappa)} \subset \mathbb{P}$ (see [GM18]), and compute $\mathbb{Q}$ in $V^{\mathbb{P}_{E(\kappa)}}$. This solves the problem of the existence of projections, however, it creates a problem elsewhere: It is now not possible to argue that κ is not collapsed in the final extension $V^{\mathbb{P}*\mathbb{Q}}$. The argument for this uses closure of factors of $\mathbb{Q}$ in $V^{\mathbb{P}}$. In fact, it is possible that an argument similar to the one showing that the diagonal Prikry forcing adds a sequence "dominating" the sequence one starts with could be used to show that there is a condition in $\mathbb{Q}$ as computed in $V^{\mathbb{P}}$ that dominate every condition in $\mathbb{Q}$ as computed in $V^{\mathbb{P}_{E(\kappa)}}$. This condition can then be used to show that the closure argument fails.

Another possible strategy is to incorporate the collapsing conditions into the forcing $\mathbb{P}$ of Merimovich-Gitik. Thus, to each condition p, one adjoins a function H^p which tells us what collapsing condition to put between the newly added points of the Prikry sequence. The main problem with this approach is again the existence of projections. In particular, there might be two different vectors in the tree of a given condition p such that their H^p values are incompatible while their restrictions to $\{\kappa\}$ results in the same vector. This obstructs the attempt to define a projection onto $\mathbb{P}_{\{\kappa\}}$ with collapses. However, in this situation there is a projection onto $\mathbb{P}_{\{\kappa\}}$ alone. This is the limitation which forces us to consider only HOD of the extension.

To prove our main theorem, we define a simplified variant of Merimovich's supercompact extender-based forcing tailored to our needs, and incorporate collapses into that forcing. Let us denote this forcing by $\mathbb{P}$, and let G be $(V, \mathbb{P})$-generic. As λ is not collapsed in $V[G]$, one could argue that for any $x \subset \aleph_\omega$ in $V[G]$, there exists $\delta \in [\kappa, \lambda)$ such that the set x can be recovered from the set $G \cap \{p \in \mathbb{P} \mid \mathrm{dom}(f^p) \leq \delta\}$. The problem of the projections means that we cannot argue that $G \cap \{p \in \mathbb{P} \mid \mathrm{dom}(f^p) \leq \delta\}$ is generic for $\{p \in \mathbb{P} \mid \mathrm{dom}(f^p) \leq \delta\}$.

This tension between maintaining projections for the homogeneity argument and collapsing λ to $\aleph_{\omega+1}$ reoccurs in various other attempts based on different kinds of forcings including a diagonal Supercompact forcing. We are not sure if this is a limitation of the technique or if there is a deeper theorem preventing this conclusion.

4.3 Projections and Homogeneity

In this section, we gather a few well-known facts about projections and homogeneity. Throughout this chapter, we assume familiarity with forcing as presented in [Kun14] and [Jec03], and the reader can refer to these books for any omitted definitions or technical background.

4.3.1 Projections

Let $\mathbb{P} = \langle \mathbb{P}, \leq_{\mathbb{P}}, 1_{\mathbb{P}} \rangle$ and $\mathbb{Q} = \langle \mathbb{Q}, \leq_{\mathbb{Q}}, 1_{\mathbb{Q}} \rangle$ be two forcing notions.[1]

Definition 4.3.1. A map $\pi \colon \mathbb{P} \to \mathbb{Q}$ is called a *projection* iff

(P1) $\pi(1_{\mathbb{P}}) = 1_{\mathbb{Q}}$.
(P2) $\forall p_1, p_2 \in \mathbb{P} \colon p_1 \leq_{\mathbb{P}} p_2 \implies \pi(p_1) \leq_{\mathbb{Q}} \pi(p_2)$.
(P3) $\forall p \in \mathbb{P}\, \forall q \leq_{\mathbb{Q}} \pi(p)\, \exists p' \leq_{\mathbb{P}} p \colon \pi(p') \leq_{\mathbb{Q}} q$.

[1] In a standard abuse of notation, we use the same symbol to represent the triple and the underlying set.

Given $J \subset \mathbb{Q}$, let $\mathrm{upcl}_{\mathbb{Q}}(J) = \{q \in \mathbb{Q} \mid \exists q' \in J : q \geq q'\} \subset \mathbb{Q}$ be the upward closure of J in $\mathbb{Q}$ and let $\pi^{-1}[J] = \{p \in \mathbb{P} \mid \pi(p) \in J\} \subset \mathbb{P}$.

Lemma 4.3.2. *If $\pi \colon \mathbb{P} \to \mathbb{Q}$ is a projection and $D \subset \mathbb{Q}$ is dense open, then $\pi^{-1}[D]$ is dense in $\mathbb{P}$.*

Proof. Fix any $p \in \mathbb{P}$. By denseness, let $q \leq_{\mathbb{Q}} \pi(p)$ be in D. By (P3), fix some $p' \leq_{\mathbb{P}} p$ such that $\pi(p') \leq_{\mathbb{Q}} q$. Then, by openness, $\pi(p') \in D$, and therefore, $p' \in \pi^{-1}[D]$. □

Proposition 4.3.3. *If $\pi \colon \mathbb{P} \to \mathbb{Q}$ is a projection and G is $(V, \mathbb{P})$-generic, then $H = \mathrm{upcl}_{\mathbb{Q}}(\pi"G)$ is $(V, \mathbb{Q})$-generic.*

Proof. Upward closure: Any output of the function upcl is upward closed. Compatibility: This easily follows from compatibility in G and (P2). Intersection with dense open sets: Fix any dense open set $D \subset \mathbb{Q}$. By Lemma 4.3.2, $\pi^{-1}[D]$ is dense in $\mathbb{P}$, so we can take $p \in G \cap \pi^{-1}[D]$. Then, $\pi(p) \in \pi"G \cap D \subset H \cap D$. □

There is some sort of a converse to the above proposition. Given a projection $\pi \colon \mathbb{P} \to \mathbb{Q}$ and a $(V, \mathbb{Q})$-generic H, we consider the set $\pi^{-1}[H]$. By (P1), $1_{\mathbb{P}} \in \pi^{-1}[H]$. So, we can consider the forcing $\pi^{-1}[H] = \langle \pi^{-1}[H], \leq_{\mathbb{P}}, 1_{\mathbb{P}} \rangle$.

Lemma 4.3.4. *If $\pi \colon \mathbb{P} \to \mathbb{Q}$ is a projection, $p \in \mathbb{P}$, and $E \subset \mathbb{P}$ is dense below p, then $\pi"E$ is dense below $\pi(p)$.*

Proof. Fix any $q \leq_{\mathbb{Q}} \pi(p)$. By (P3), let $p' \leq_{\mathbb{P}} p$ be such that $\pi(p') \leq_{\mathbb{Q}} q$. By denseness, fix $p^* \leq_{\mathbb{P}} p'$ in E. Now, $\pi(p^*) \leq_{\mathbb{Q}} \pi(p') \leq_{\mathbb{Q}} q$ and $\pi(p^*) \in \pi"E$. □

Lemma 4.3.5 ([Kun14, Lemma 7.10])**.** *If H_1, H_2 are $(V, \mathbb{Q})$-generic and $H_1 \subset H_2$, then $H_1 = H_2$.*

Proof. Suppose this is not the case, and fix $q \in H_2 \backslash H_1$. Consider the set $D = \{q' \in \mathbb{Q} \mid (q' \leq_{\mathbb{Q}} q) \ \vee \ (q' \perp_{\mathbb{Q}} q)\}$. This set is dense in $\mathbb{Q}$: Take any $r \in \mathbb{Q}$. If $r \notin D$, then $r \not\perp_{\mathbb{Q}} q$. So, there exists $s \leq_{\mathbb{Q}} r, q$. Now, $s \leq_{\mathbb{Q}} q$ means $s \in D$. Take $q' \in D \cap H_1$. Either $q' \leq_{\mathbb{Q}} q$ or $q' \perp_{\mathbb{Q}} q$. But, the former is not possible since $q \notin H_1$ and the latter is not possible since $q, q' \in H_2$ and conditions in any filter must be compatible. □

Given $I \subset \mathbb{P}$ and $p \in \mathbb{P}$, let $I/p = \{p' \in I \mid p' \leq_{\mathbb{P}} p\}$.

Proposition 4.3.6. *If $\pi \colon \mathbb{P} \to \mathbb{Q}$ is a projection, H is $(V, \mathbb{Q})$-generic, and G is $(V[H], \pi^{-1}[H])$-generic, then G is $(V, \mathbb{P})$-generic. Moreover, $\mathrm{upcl}_{\mathbb{Q}}(\pi"G) = H$.*

Proof. Compatibility is trivial. For upward closure, fix $p \in \mathbb{P}$ and $p' \in G \subset \pi^{-1}[H]$ such that $p' \leq_{\mathbb{P}} p$. By (P2), $\pi(p') \leq_{\mathbb{Q}} \pi(p)$, and since $\pi(p') \in H$, we must have $\pi(p) \in H$. Then, $p \in \pi^{-1}[H]$, and by upward closure of G in $\pi^{-1}[H]$, we must have $p \in G$. Now, fix any $D \in \mathcal{P}(\mathbb{P}) \cap V$ dense open in $\mathbb{P}$, and let us show that $G \cap D \neq \emptyset$. Consider $D \cap \pi^{-1}[H] \in V[H]$. We will be done if we can show that this set is dense in $\pi^{-1}[H]$. So, fix any $p \in \pi^{-1}[H]$. By Lemma 4.3.4, the set $\pi"D/p \in V$ is dense below $\pi(p)$. Since $\pi(p) \in H$, we can take some $q \in H \cap \pi"D/p$. As $q \in \pi"D/p$, there is $p' \in D/p$ such that $\pi(p') = q$. Then, $p' \in D \cap \pi^{-1}[H]$.

For the moreover part, first notice that $\mathrm{upcl}_{\mathbb{Q}}(\pi"G)$ is $(V, \mathbb{Q})$-generic by Proposition 4.3.3. Since $G \subset \pi^{-1}[H]$ implies $\mathrm{upcl}_{\mathbb{Q}}(\pi"G) \subset H$, the equality follows by Lemma 4.3.5. □

4.3.2 Homogeneous projections

Fix a projection $\pi\colon \mathbb{P} \to \mathbb{Q}$.

Definition 4.3.7 (Cone isomorphism property)**.** We say that $\pi\colon \mathbb{P} \to \mathbb{Q}$ satisfies the *cone isomorphism property* iff for every $p_1, p_2 \in \mathbb{P}$ with $\pi(p_1) \parallel \pi(p_2)$, there exists $p_1' \leq p_1$ and $p_2' \leq p_2$ such that $\mathbb{P}/p_1'$ and $\mathbb{P}/p_2'$ are isomorphic.

Lemma 4.3.8. *Suppose that $\pi : \mathbb{P} \to \mathbb{Q}$ satisfies the cone isomorphism property. Let φ be a formula in the $\mathbb{P}$-forcing language, let $a \in V$, and let $D \subset \mathbb{P}$ be the set of conditions that decide $\varphi(\check{a})$. Then, for any $p_1, p_2 \in D$ with $\pi(p_1) \parallel \pi(p_2)$, we have $p_1 \Vdash_{\mathbb{P}} \varphi(\check{a})$ iff $p_2 \Vdash_{\mathbb{P}} \varphi(\check{a})$.*

Proof. Suppose, WLOG, that $p_1 \Vdash_{\mathbb{P}} \varphi(\check{a})$ while $p_2 \Vdash_{\mathbb{P}} \neg\varphi(\check{a})$. The cone isomorphism property gives $p_1' \leq p_1$ and $p_2' \leq p_2$ such that $\mathbb{P}/p_1'$ and $\mathbb{P}/p_2'$ are isomorphic. But, this cannot be since $p_1' \Vdash_{\mathbb{P}} \varphi(\check{a})$ while $p_2' \Vdash_{\mathbb{P}} \neg\varphi(\check{a})$. □

Proposition 4.3.9. *Suppose that $\pi\colon \mathbb{P} \to \mathbb{Q}$ satisfies the cone isomorphism property. Let G be $(V, \mathbb{P})$-generic and let $H = \mathrm{upcl}_{\mathbb{Q}}(\pi\text{"}G)$. Then, we have $\mathrm{HOD}^{V[G]} \subset V[H]$.*

Proof. By AC, we only need to consider $A \in \mathrm{HOD}^{V[G]}$ that are subsets of ordinals. Thus, fix a set of ordinals $A \in \mathrm{HOD}^{V[G]}$. Let $\varphi(x, y)$ and β be such that

$$\alpha \in A \iff V[G] \models \varphi(\alpha, \beta).$$

Fix γ such that $A \subset \gamma$. For every $\alpha < \gamma$, let $D_\alpha \subset \mathbb{P}$ be the dense open set of conditions that decide $\varphi(\check{\alpha}, \check{\beta})$. By Lemma 4.3.4, $\pi\text{"}D_\alpha$ is dense in $\mathbb{Q}$, for every α. Hence, for every $\alpha < \gamma$, there exists $p_\alpha \in D_\alpha$ such that $\pi(p_\alpha) \in H$. By Lemma 4.3.8, any two such p_α must decide $\varphi(\check{\alpha}, \check{\beta})$ the same way. Thus, A is definable in $V[H]$ using

$$\alpha \in A \iff \exists p \in \mathbb{P}\ (V \models \text{“}p \Vdash_{\mathbb{P}} \varphi(\check{\alpha}, \check{\beta})\text{”} \ \wedge\ \pi(p) \in H). \qquad □$$

4.4 Supercompact Prikry with Collapses

Before turning to the main forcing construction in the next section, we explore a closely related attempt. While not sufficient for our final goal, this forcing illuminates some key ideas and techniques for the main argument. This forcing interleaves the strongly compact Prikry forcing [Git10] with collapses in the style of W. Hugh Woodin and Mathew Foreman [Cum10, Example 8.6]; using a “guiding generic.” The main challenge is coming up with the correct definition for the collapses so that one can carry out stabilization arguments (Lemmas 4.4.11 and 4.4.12) towards proving the Prikry property.

Let κ be a supercompact cardinal, and let $\lambda \geq \kappa$ be an inaccessible cardinal such that $2^\lambda = \lambda^+$. Recall that, for any cardinal α and set S, $\mathcal{P}_\alpha(S) = \{x \subset S \mid |x| < \alpha\}$. Let $\mathcal{A}(\kappa, \lambda) \subset \mathcal{P}_\kappa(\lambda)$ be the collection of all those x such that $x \cap \kappa$ and $\mathrm{otp}(x)$ are both inaccessible cardinals with $x \cap \kappa < \mathrm{otp}(x)$. For $x \in \mathcal{A}(\kappa, \lambda)$, let $\kappa(x) = x \cap \kappa$ and $\lambda(x) = \mathrm{otp}(x)$. For $x, y \in \mathcal{A}(\kappa, \lambda)$, we write $x \prec y$ iff $x \subset y$ and $\lambda(x) < \kappa(y)$.

Let $\mathcal{U}$ be a normal fine measure on $\mathcal{A}(\kappa, \lambda)$, and let $j\colon V \to M$ be the associated elementary embedding. Notice that $\lambda^+ < j(\kappa) < j(\lambda) < \lambda^{++}$ [Cum10, Proposition 4.5]. So, we can fix a *guiding generic* $K \in V$ that is $(M, (\mathrm{Col}(\lambda^+, {<}j(\lambda)))^M)$-generic (see [Cum10], Section 8 for details).

Definition 4.4.1. Suppose $T \subset \mathcal{A}(\kappa, \lambda)^{<\omega}$ and $H\colon T \to V_\lambda$.

- For each $\vec{x} \in T$, $\mathrm{Suc}_T(\vec{x}) = \{y \in \mathcal{A}(\kappa, \lambda) \mid \vec{x}^\frown\langle y\rangle \in T\}$.
- T ordered by end extension is called a *tree* iff it is closed under under initial segments.
- $T \neq \emptyset$ is called a $\mathcal{U}$*-tree* iff it is a tree and, for all $\langle x_0, \dots, x_{k-1}\rangle \in T$, we have
  (1) $\mathrm{Suc}_T(\langle x_0, \dots, x_{k-1}\rangle) \in \mathcal{U}$.
  (2) $\forall y \in \mathrm{Suc}_T(\langle x_0, \dots, x_{k-1}\rangle)\colon\ x_{k-1} \prec y$.
- The pair $\langle T, H\rangle$ is called a $\mathcal{U}$*-tree with guiding generic collapses* iff
  (1) T is a $\mathcal{U}$-tree.
  (2) $H(\langle\rangle) \in \mathrm{Col}(\alpha^+, {<}\lambda)$ for some $\alpha < \kappa$.
  (3) $\forall \vec{x}^\frown\langle y\rangle \in T\colon H(\vec{x}^\frown\langle y\rangle) \in \mathrm{Col}(\lambda(y)^+, {<}\lambda)$.
  (4) $\forall \vec{x}^\frown\langle y\rangle \in T\colon\ y \supset \mathrm{supp}(H(\vec{x}))$.
  (5) $\forall \vec{x} \in T\colon\ [y \mapsto H(\vec{x}^\frown\langle y\rangle)]_{\mathcal{U}} \in K$.

4.4.1 The collapses

The usual Lévy collapse, $\mathrm{Col}(\alpha, {<}\beta)$, as defined in [Jec03], collapses all the cardinals below some inaccessible cardinal β to some regular cardinal $\alpha < \beta$. The conditions are essentially partial functions $c\colon \beta \times \alpha \to \beta$. In [Kan03], the definition is generalized by restricting the domain of the conditions to $x \times \alpha$, where x is some subset of β, and the resulting forcing is denoted by $\mathrm{Col}(\alpha, x)$. The effect is that only the cardinals in x are collapsed to α.

Our forcing notion will, of course, incorporate such collapses. However, in order to prove the Prikry property, we will need to slightly modify the definition of $\mathrm{Col}(\alpha, x)$. In particular, we will also require that the codomain of the conditions be x, rather than all of β. The definition below makes this precise:

Definition 4.4.2. Given a regular cardinal $\alpha < \kappa$ and a set $x \in \mathcal{A}(\kappa, \lambda)$ such that $\kappa(x) > \alpha$, define the forcing $\mathrm{Col}(\alpha, {<}x)$ to be the poset whose conditions are c such that

(1) c is a function with cardinality less than α and $\mathrm{dom}(c) \subset x \times \alpha$.
(2) $\forall\langle \xi, \zeta\rangle \in \mathrm{dom}(x)\colon c(\xi, \zeta) = 0$ or $c(\xi, \zeta) \in \xi \cap x$.

The order is defined by setting $d \leq c$ iff $d \supset c$.

Remark 4.4.3. When $x = \beta$ is an inaccessible cardinal, our notation $\mathrm{Col}(\alpha, < \beta)$ agrees with that of [Jec03]. Furthermore, it is straightforward to verify the fact that $\mathrm{Col}(\alpha, {<}x) \cong \mathrm{Col}(\alpha, {<}\mathrm{otp}(x))$, by identifying each $\xi \in x$ with $\mathrm{otp}(\xi \cap x)$.

Given $c \in \mathrm{Col}(\alpha, {<}\, x)$, let $\mathrm{dom}_1(c) = \{\xi \mid \exists\zeta\colon \langle\xi, \zeta\rangle \in \mathrm{dom}(c)\}$, $\mathrm{dom}_2(c) = \{\zeta \mid \exists\xi\colon \langle\xi, \zeta\rangle \in \mathrm{dom}(c)\}$, and

$$\mathrm{supp}(c) = \mathrm{dom}_1(c) \cup \mathrm{dom}_2(c) \cup \mathrm{ran}(c).$$

If $\langle c_0, \dots, c_{n-1}\rangle$ is a finite sequence such that, for all i, $c_i \in \mathrm{Col}(\alpha_i, {<}x_i)$ for some α_i and x_i, then we also define $\mathrm{supp}(\langle c_0, \dots, c_{n-1}\rangle) = \bigcup_i \mathrm{supp}(c_i)$.

4.4.2 The forcing

We are now ready to introduce the forcing construction of this section.

Definition 4.4.4 (The forcing)**.** Define $\mathbb{P} = \mathbb{P}(\kappa, \lambda, \mathcal{U}, K)$ to be the poset whose conditions are

$$p = \langle a^p, l^p, T^p, H^p \rangle$$

where:

(1) $a^p = \langle x_0^p, \dots, x_{n-1}^p \rangle \in \mathcal{A}(\kappa, \lambda)^{<\omega}$ is $\prec$-increasing.
(2) $l^p = \langle c_{-1}^p, c_0^p, \dots, c_{n-1}^p \rangle$, where
   (a) $c_{-1}^p \in \mathrm{Col}(\omega_1, {<}x_0^p)$;
   (b) $c_i^p \in \mathrm{Col}(\lambda(x_i^p)^+, {<}x_{i+1}^p)$, for $0 \le i < n-1$;
   (c) $c_{n-1}^p \in \mathrm{Col}(\lambda(x_{n-1}^p)^+, {<}\lambda)$.

   If $n = 0$, then $c_{-1}^p \in \mathrm{Col}(\omega_1, {<}\lambda)$.
(3) $\langle T^p, H^p \rangle$ is a $\mathcal{U}$-tree with guiding generic collapses such that $H^p(\langle\rangle) = c_{n-1}^p$ and $x_{n-1}^p \prec y$ for all $y \in \mathrm{Suc}_{T^p}(\langle\rangle)$.

We also define $\mathrm{stem}(p) = \langle a^p, l^p \rangle$ and $\mathrm{len}(p) = n$.

If $l^p = \langle c_{-1}^p, c_0^p, \dots, c_{n-1}^p \rangle$ is as in the definition above, we will abuse the domain restriction operation on l^p by always including the element indexed with -1, that is, for $m < n$, we take $l^p|_m = \langle c_{-1}^p, \dots, c_{m-1}^p \rangle$.

The following three consecutive definitions specify the ordering on $\mathbb{P}$.

Definition 4.4.5 (n-point extensions)**.** Given a condition $p = \langle a, l, T, H \rangle$ and $\vec{x} = \langle x_0, \dots, x_{k-1} \rangle \in T$, the *extension of p by $\vec{x}$* is $p_{\vec{x}} = \langle a_{\vec{x}}, l_{\vec{x},H}, T_{\vec{x}}, H_{\vec{x}} \rangle$, where

(1) $a_{\vec{x}} = a^\frown \langle x_0, x_1, \dots, x_{k-1} \rangle$.
(2) $l_{\vec{x},H} = l^\frown \langle H(\langle x_0 \rangle), H(\langle x_0, x_1 \rangle), \dots, H(\langle x_0, \dots, x_{k-1} \rangle) \rangle$.
(3) $T_{\vec{x}} = \{\vec{y} \mid \vec{x}^\frown \vec{y} \in T\}$.
(4) $H_{\vec{x}} \colon T_{\vec{x}} \to V_\lambda$ is defined by $\vec{y} \mapsto H(\vec{x}^\frown \vec{y})$.

Definition 4.4.6 (Direct extensions)**.** Given two conditions $p = \langle a^p, l^p, T^p, H^p \rangle$ and $q = \langle a^q, l^q, T^q, H^q \rangle$, we write $q \le^* p$ iff

(1) $a^q = a^p$.
(2) $l^q \le l^p$, with the order defined coordinate-wise.
(3) $T^q \subset T^p$.
(4) $\forall \vec{x} \in T^q\colon\ H^q(\vec{x}) \le H^p(\vec{x})$.

Definition 4.4.7 (The main order)**.** Given conditions $p, q \in \mathbb{P}$, we write $q \le p$ provided there exists $\vec{x} \in T^p$ such that $q \le^* p_{\vec{x}}$.

4.4.3 General properties

In this subsection, we begin by proving a series of lemmas, the last of which establishes the Prikry property. These culminate in a final proposition that enumerates the facts that hold in every extension by $\mathbb{P}$.

Lemma 4.4.8. *If $p, q \in \mathbb{P}$ are such that* $\mathrm{stem}(p) = \mathrm{stem}(q)$*, then there exists $r \le^* p, q$ such that* $\mathrm{stem}(r) = \mathrm{stem}(p) = \mathrm{stem}(q)$.

Proof. Let $p = \langle a, l, T^p, H^p\rangle$ and $q = \langle a, l, T^q, H^q\rangle$. Using the compatibility property of the guiding generic K, for every $\vec{x} \in T^p \cap T^q$, let $[f_{\vec{x}}]_{\mathcal{U}} \in K$ be such that $[f_{\vec{x}}]_{\mathcal{U}} \leq [y \mapsto H^p(\vec{x}^\frown\langle y\rangle)]_{\mathcal{U}}, [y \mapsto H^q(\vec{x}^\frown\langle y\rangle)]_{\mathcal{U}}$. Let $S_{\vec{x}} \subset \mathrm{Suc}_{T^p}(\vec{x}) \cap \mathrm{Suc}_{T^q}(\vec{x})$ be a $\mathcal{U}$-large set such that, for all $y \in S_{\vec{x}}$, $f_{\vec{x}}(y) \leq H^p(\vec{x}^\frown\langle y\rangle), H^q(\vec{x}^\frown\langle y\rangle)$. Define $r = \langle a, l, T, H\rangle$ so that:

(1) T is the smallest set such that $\langle\rangle \in T$, $\{\langle y\rangle \mid y \in S_{\langle\rangle}\} \subset T$, and, for all $\vec{x}^\frown\langle y\rangle \in T$, $\{\vec{x}^\frown\langle y, z\rangle \mid z \in S_{\vec{x}^\frown\langle y\rangle} \ \wedge\ z \supset \mathrm{supp}(f_{\vec{x}}(y))\} \subset T$.
(2) $H(\langle\rangle) = H^p(\langle\rangle) = H^q(\langle\rangle)$ and $H(\vec{x}^\frown\langle y\rangle) = f_{\vec{x}}(y)$, for all $\vec{x}^\frown\langle y\rangle \in T$.

Then, clearly $r \in \mathbb{P}$ and $r \leq^* p, q$. □

Lemma 4.4.9. $\mathbb{P}$ *satisfies the* λ^+*-Knaster property.*

Proof. This follows from Lemma 4.4.8 and the fact that there are only λ many possible stems. □

The following lemma is an easy technical fact about the ultrapower embedding j.

Lemma 4.4.10. *If* $f\colon \mathcal{A}(\kappa,\lambda) \to V$ *is such that* $|[f]_{\mathcal{U}}| < \kappa$ *and* $[f]_{\mathcal{U}} \subset j``V$, *then* f *is constant on a* $\mathcal{U}$*-large set.*

Proof. Let $X = \{x \mid j(x) \in [f]_{\mathcal{U}}\}$. Then, $|X| < \kappa$ and $j(X) = j``X = [f]_{\mathcal{U}}$. But also, $j(X) = [c_X]_{\mathcal{U}}$, where $c_X\colon \mathcal{A}(\kappa,\lambda) \to V$ is constantly X. □

Given $p \in \mathbb{P}$, $D \subset \mathbb{P}$ dense open below p, $\langle y\rangle \in T^p$, $e \leq l^p$ such that $\mathrm{supp}(e) \subset y$, and $d \leq H^p(\langle y\rangle)$, let $\Psi^{p,D,y,e}(d)$ be the following statement:

$$\exists q\ (q \leq^* p_{\langle y\rangle} \ \wedge\ l^q = e^\frown\langle d\rangle \ \wedge\ q \in D).$$

Lemma 4.4.11. *For every* $p = \langle a, l, T, H\rangle$ *and* $D \subset \mathbb{P}$ *dense open below* p, *there is* $p^* = \langle a, l, T^*, H^*\rangle \leq^* p$ *satisfying the following: For all* $\vec{x}^\frown\langle y\rangle \in T^*$ *and all* $e \leq l_{\vec{x},H^*}$ *with* $\mathrm{supp}(e) \subset y$,

$$\exists d \leq H^*_{\vec{x}}(\langle y\rangle)\ (\Psi^{p^*_{\vec{x}},D,y,e}(d)) \implies \Psi^{p^*_{\vec{x}},D,y,e}(H^*_{\vec{x}}(\langle y\rangle)).$$

Proof. Fix any $\vec{x} \in T$ and work in M. Consider the condition $j(p_{\vec{x}})$. For every $e \leq j(l_{\vec{x},H})$ with $\mathrm{supp}(e) \subset j``\lambda$, let $E^e_{\vec{x}} \subset \mathrm{Col}(\lambda^+, <j(\lambda))/j(H_{\vec{x}})(\langle j``\lambda\rangle)$ be the set of conditions d satisfying the following:

$$\Psi^{j(p_{\vec{x}}),j(D),j``\lambda,e}(d) \ \vee\ \forall d' \leq d\ (\neg\Psi^{j(p_{\vec{x}}),j(D),j``\lambda,e}(d')).$$

Clearly, each $E^e_{\vec{x}}$ is dense open below $j(H_{\vec{x}})(\langle j``\lambda\rangle)$. Notice that there are only $\lambda$ many possible $e$'s. So, since $\mathrm{Col}(\lambda^+, <j(\lambda))$ is $\lambda^+$-closed, the set $E_{\vec{x}} = \bigcap_e E^e_{\vec{x}}$ is also dense open below $j(H_{\vec{x}})(\langle j``\lambda\rangle)$. Stepping outside M, we know that $j(H_{\vec{x}})(\langle j``\lambda\rangle) = [y \mapsto H_{\vec{x}}(\langle y\rangle)]_{\mathcal{U}} \in K$ and $K$ is generic over $M$, so we can take $[f_{\vec{x}}]_{\mathcal{U}} \leq j(H_{\vec{x}})(\langle j``\lambda\rangle)$ in $K \cap E_{\vec{x}}$. Notice now that, in M, $[f_{\vec{x}}]_{\mathcal{U}}$ satisfies that, for all $e \leq j(l_{\vec{x},H})$ with $\mathrm{supp}(e) \subset j``\lambda$, we have

$$\exists d' \leq [f_{\vec{x}}]_{\mathcal{U}}\ (\Psi^{j(p_{\vec{x}}),j(D),j``\lambda,e}(d')) \implies \Psi^{j(p_{\vec{x}}),j(D),j``\lambda,e}([f_{\vec{x}}]_{\mathcal{U}}).$$

Back in V, let $S_{\vec{x}} \subset \mathrm{Suc}_{T_{\vec{x}}}(\langle\rangle)$ be a $\mathcal{U}$-large set such that, for all $y \in S_{\vec{x}}$, $f_{\vec{x}}(y) \leq H_{\vec{x}}(\langle y\rangle)$ and, for all $e \leq l_{\vec{x},H}$ with $\mathrm{supp}(e) \subset y$, we have

$$\exists d' \leq f_{\vec{x}}(y)\ (\Psi^{p_{\vec{x}},D,y,e}(d')) \implies \Psi^{p_{\vec{x}},D,y,e}(f_{\vec{x}}(y)). \tag{4.1}$$

Define $T^* \subset T$ to be the smallest set such that $\langle\rangle \in T^*$, $\{\langle y\rangle \mid y \in S_{\langle\rangle}\} \subset T^*$, and, for all $\vec{x}^\frown\langle y\rangle \in T^*$, $\{\vec{x}^\frown\langle y, z\rangle \mid z \in S_{\vec{x}^\frown\langle y\rangle} \wedge z \supset \mathrm{supp}(f_{\vec{x}}(y))\} \subset T^*$. Define $H^*\colon T^* \to V_\lambda$ by setting $H^*(\langle\rangle) = H(\langle\rangle)$ and $H^*(\vec{x}^\frown\langle y\rangle) = f_{\vec{x}}(y)$. Clearly, $p^* = \langle a, l, T^*, H^*\rangle$ is a valid condition and $p^* \leq^* p$.

Now, take any $\vec{x}^\frown\langle y\rangle \in T^*$ and $e \leq l_{\vec{x},H^*}$ with $\mathrm{supp}(e) \subset y$. Let $d \leq H^*_{\vec{x}}(\langle y\rangle)$ be such that $\Psi^{p^*_{\vec{x}},D,y,e}(d)$ holds. This means that

$$\exists q\ (q \leq^* p^*_{\vec{x}^\frown\langle y\rangle}\ \wedge\ l^q = e^\frown\langle d\rangle\ \wedge\ q \in D).$$

Since $p^*_{\vec{x}^\frown\langle y\rangle} \leq^* p_{\vec{x}^\frown\langle y\rangle}$, the same q will also witness $\Psi^{p_{\vec{x}},D,y,e}(d)$. $H^*_{\vec{x}}(\langle y\rangle) = f_{\vec{x}}(y)$ by definition, so we must have $\Psi^{p_{\vec{x}},D,y,e}(H^*_{\vec{x}}(\langle y\rangle))$ by (4.1). Thus,

$$\exists r\ (r \leq^* p_{\vec{x}^\frown\langle y\rangle}\ \wedge\ l^r = e^\frown\langle H^*_{\vec{x}}(\langle y\rangle)\rangle\ \wedge\ r \in D).$$

We already have $e \leq l_{\vec{x},H^*}$, so by Lemma 4.4.8 we can take $r' \leq^* r, p^*_{\vec{x}^\frown\langle y\rangle}$ with $\mathrm{stem}(r') = \langle a_{\vec{x}}^\frown\langle y\rangle, e^\frown\langle H^*_{\vec{x}}(\langle y\rangle)\rangle\rangle$. This r' witnesses $\Psi^{p^*_{\vec{x}},D,y,e}(H^*_{\vec{x}}(\langle y\rangle))$, and we are done. □

Lemma 4.4.12. *Suppose φ is a formula in the forcing language and $p \in \mathbb{P}$ satisfies the property in Lemma 4.4.11 for the dense open set $D \subset \mathbb{P}$ of conditions deciding φ. Then, if for $\mathcal{U}$-large many $y \in \mathrm{Suc}_{T^p}(\langle\rangle)$, there is $q \leq^* p_{\langle y\rangle}$ in D, then there is $r \leq^* p$ in D.*

Proof. Let $n = \mathrm{len}(p)$. Working in M, by assumption, there exists a condition $q \leq^* j(p)_{\langle j``\lambda\rangle}$ deciding $j(\varphi)$. By the property in Lemma 4.4.11, we can assume that $c^q_n = j(H^p)(\langle j``\lambda\rangle)$. WLOG, suppose that $q \Vdash j(\varphi)$. Back in V, take $g_{c^q_{-1}}, \ldots, g_{c^q_{n-1}}$ such that $[g_{c^q_{-1}}]_{\mathcal{U}} = c^q_{-1}, \ldots, [g_{c^q_{n-1}}]_{\mathcal{U}} = c^q_{n-1}$. For any $y \in \mathcal{A}(\kappa, \lambda)$, let

$$l(y) = \langle g_{c^q_{-1}}(y), \ldots, g_{c^q_{n-1}}(y)\rangle.$$

Now, for $\mathcal{U}$-large many $y \in \mathrm{Suc}_{T^p}(\langle\rangle)$, there exists $T(y), H(y)$ such that

$$q(y) = \langle a^{p\frown}\langle y\rangle, l(y)^\frown\langle H^p(\langle y\rangle)\rangle, T(y), H(y)\rangle \leq^* p_{\langle y\rangle}\ \wedge\ q(y) \Vdash \varphi.$$

For every $i \in [-1, \ldots, n-1]$, since $\mathrm{supp}(c^q_i) \subset j``\lambda$ and $|c^q_i| < \kappa$, we can use Lemma 4.4.10 to take c^*_i such that $f_{c^q_i}$ is c^*_i on a $\mathcal{U}$-large set. Let $l^* = \langle c^*_{-1}, \ldots, c^*_{n-1}\rangle$. Let $S \subset \mathrm{Suc}_{T^p}(\langle\rangle)$ be a $\mathcal{U}$-large set such that for all $y \in S$, we have

$$q(y) = \langle a^{p\frown}\langle y\rangle, l^{*\frown}\langle H^p(\langle y\rangle)\rangle, T(y), H(y)\rangle \leq^* p_{\langle y\rangle}\ \wedge\ q(y) \Vdash \varphi.$$

Define T^* to be the smallest set so that $\langle\rangle \in T^*$, $\{\langle y\rangle \mid y \in S\} \subset T^*$, and, for all $\langle y\rangle \in T^*$, $\{\langle y\rangle^\frown\vec{z} \mid \vec{z} \in T(y)\} \subset T^*$. Define $H^*\colon T^* \to V_\lambda$ by $H^*(\langle\rangle) = c^*_{n-1}$, $H^*(\langle y\rangle) = H^p(\langle y\rangle)$, for all $\langle y\rangle \in T^*$, and $H^*(\langle y\rangle^\frown\vec{z}) = H(y)(\vec{z})$, for all $\langle y\rangle^\frown\vec{z} \in T^*$. Let $r = \langle a^p, l^*, T^*, H^*\rangle$. Clearly, $r \in \mathbb{P}$ and $r \leq^* p$. Also, $r_{\langle y\rangle} = q(y) \Vdash \varphi$, for all $\langle y\rangle \in T^*$. Finally, since $\{r_{\langle y\rangle} \mid \langle y\rangle \in T^*\}$ is a maximal antichain below r, we must have $r \Vdash \varphi$. □

Lemma 4.4.13 (Prikry property)**.** *For every formula φ in the forcing language and every condition p, there is $p^* \leq^* p$ that decides φ.*

Proof. By $\leq^*$-extending p, we may assume that p satisfies the property given in Lemma 4.4.11 for the dense open set $D \subset \mathbb{P}$ of conditions deciding φ. Notice that this means, for all $\vec{x} \in T^p$, $p_{\vec{x}}$ also satisfies the same property for D. Working towards a contradiction, suppose that the lemma is false. For every $\vec{x} \in T^p$ such that there is no $q \leq^* p_{\vec{x}}$ in D, define the set

$$A^{\vec{x}} = \{y \in \mathrm{Suc}_{T^p}(\vec{x}) \mid \neg(\exists q \leq^* p_{\vec{x}^\frown\langle y\rangle} \colon q \in D)\}.$$

By Lemma 4.4.12 applied to $p_{\vec{x}}$, it must be that $A^{\vec{x}} \in \mathcal{U}$. Define $B^{\vec{x}} = \{\vec{x}^\frown\langle y\rangle \mid y \in A^{\vec{x}}\}$. Let $T^* \subset T^p$ be the smallest set such that $\langle\rangle \in T^*$ and, for each $\vec{x} \in T^*$, $B^{\vec{x}} \subset T^*$. By our construction, T^* is a $\mathcal{U}$-tree. Consider the condition $r = \langle a^p, l^p, T^*, H^p|_{T^*}\rangle \leq^* p$. Since D is dense open, we can find $q \leq r$ in D. Fix $\vec{x} \in T^*$ such that $q \leq^* r_{\vec{x}}$. Since $r_{\vec{x}} \leq^* p_{\vec{x}}$, we also have $q \leq^* p_{\vec{x}}$, but this contradicts the fact that $\vec{x} \in T^*$. □

Suppose G is $(V, \mathbb{P})$-generic. For each $n < \omega$, define x_n^G to be x_n^p for any $p \in G$ such that $\mathrm{len}(p) > n$. For each $-1 \leq n < \omega$, define $c_n^G = \bigcup\{c_n^p \mid p \in G\}$.

Proposition 4.4.14. *Suppose G is $(V, \mathbb{P})$-generic. Then, in $V[G]$:*

(1) *Every V-regular cardinal in the interval $[\kappa, \lambda]$ has $V[G]$-cofinality ω.*
(2) *Every V-cardinal in the interval $(\kappa, \lambda]$ is collapsed.*
(3) *Every bounded subset of κ in $V[G]$ is in $V[c_{-1}^G \times \cdots \times c_{n-1}^G]$ for some $n < \omega$.*
(4) $\aleph_{2n+2} = \lambda(x_n^G)$ *and* $\aleph_{2n+3} = (\lambda(x_n^G)^+)^V$, *for all $n < \omega$.*
(5) $\aleph_\omega = \kappa$.
(6) $\aleph_\omega$ *is a strong limit.*
(7) $\aleph_{\omega+1} = (\lambda^+)^V$.

Proof. (1): A standard density argument using the fineness of $\mathcal{U}$ shows that for any V-regular cardinal $\delta \in [\kappa, \lambda]$, the sequence $\langle \sup(x_n \cap \delta) \mid n < \omega\rangle$ is cofinal in δ.

(2): If $\delta \in (\kappa, \lambda]$ is the least V-cardinal not collapsed, then it must be a successor cardinal in $V[G]$. Therefore, δ would be regular in $V[G]$, and hence also in V. But, by (1), the $V[G]$-cofinality of δ must then be ω, contradicting its $V[G]$-regularity.

(3): Suppose otherwise and fix $p \in G$, $\dot{A}$, and $\gamma < \kappa$ such that $\mathrm{len}(p) = n+1$ for some $n < \omega$, $\lambda(x_n^p) > \gamma$, and $p \Vdash$ "$\dot{A} \subset \check{\gamma} \wedge \dot{A} \notin \check{V}[c_{-1}^{\dot{G}} \times \cdots \times c_{n-1}^{\dot{G}}]$". We work in V. Set $l = \langle c_{-1}^p, \ldots, c_{n-1}^p\rangle$,[2] $\rho = \lambda(x_n^p)$, and $\mathbb{C} = \mathrm{Col}(\omega_1, {<}x_0^p) \times \cdots \times \mathrm{Col}(\lambda(x_{n-1}^p)^+, {<}x_n^p)$. For any $q \leq p$ and $e \in \mathbb{C}$, we use the notation $q+e$ to denote $\langle a^q, e^\frown\langle c_n^q, \ldots, c_{m-1}^q\rangle, T^q, H^q\rangle$, where $m = \mathrm{len}(q)$. We also use the notation $r \leq^\dagger q$ for $q, r \leq p$ to mean that $r \leq^* q$ and $l^r|_n = l^q|_n$. It is routine to check that $\leq^\dagger$ is ρ^+-closed below p.

Fix an enumeration $\langle e^{(\xi)} \mid \xi < \rho\rangle$ of $\mathbb{C}/l$. We will assume that $\mathrm{OR} \times \mathrm{OR}$ is ordered lexicographically. Define recursively a $\leq^\dagger$-decreasing sequence $\langle p^{(\alpha,\xi)} \mid (\alpha, \xi) < (\gamma, \rho)\rangle$ below p as follows: Having defined $\langle p^{(\alpha,\xi)} \mid (\alpha, \xi) < (\beta, \zeta)\rangle$, first take p' to be $\leq^\dagger$-below $p^{(\alpha,\xi)}$, for every $(\alpha, \xi) < (\beta, \zeta)$ (for the base case $(\beta, \zeta) = (0, 0)$, take $p' = p$). Then, if there exists $q \leq^* p'$ such that $q \parallel \check{\beta} \in \dot{A}$ and $l^q|_n = e^{(\zeta)}$, then set $p^{(\beta,\zeta)} = q + l$. Else, set $p^{(\beta,\zeta)} = p'$. Finally, set $p^\dagger$ to be $\leq^\dagger$-below $p^{(\alpha,\xi)}$, for every $(\alpha, \xi) < (\gamma, \rho)$.

Now, fix some $\alpha < \gamma$. For any $e \in \mathbb{C}/l$, using the Prikry property, we can find $p^* \leq^* p^\dagger + e$ such that $p^* \parallel \check{\alpha} \in \dot{A}$. Let $\xi < \rho$ be such that $l^{p^*}|_n = e^{(\xi)}$. By our construction, this means that $p^{(\alpha,\xi)}$ is such that $(p^{(\alpha,\xi)} + e^{(\xi)}) \parallel \check{\alpha} \in \dot{A}$. Since $p^\dagger + e^{(\xi)} \leq^\dagger p^{(\alpha,\xi)} + e^{(\xi)}$, we have $(p^\dagger + e^{(\xi)}) \parallel \check{\alpha} \in \dot{A}$. Since $e^{(\xi)} \leq e$, we have established that the set $E_\alpha \subset \mathbb{C}/l$ of all e such that $(p^\dagger + e) \parallel \check{\alpha} \in \dot{A}$ is dense open in $\mathbb{C}/l$.

By genericity, it is easy to see that for any dense open set $E \subset \mathbb{C}/l$, $p^\dagger \Vdash \exists e \in \check{E}(p^\dagger + e \in \dot{G})$. Let $E_\alpha^+ \subset E_\alpha$ be the set of all e such that $(p^\dagger + e) \Vdash \check{\alpha} \in \dot{A}$. It is now easy to see that

$$p^\dagger \Vdash \text{“}\forall\alpha < \check{\gamma}\ (\alpha \in \dot{A} \iff \exists e \in (c_{-1}^{\dot{G}} \times \cdots \times c_{n-1}^{\dot{G}}) \cap \check{E}_\alpha^+)\text{”}.$$

[2] Notice that l is shorter than l^p by one coordinate.

Hence, $p^\dagger \Vdash \dot{A} \in \check{V}[c_{-1}^{\dot{G}} \times \cdots \times c_{n-1}^{\dot{G}}]$. But, this contradicts the fact that $p \Vdash \dot{A} \notin \check{V}[c_{-1}^{\dot{G}} \times \cdots \times c_{n-1}^{\dot{G}}]$.

(4): In V, we have $\mathrm{Col}(\omega_1, {<}x_0^G) \cong \mathrm{Col}(\omega_1, {<}\lambda(x_0^G))$, by Remark 4.4.3. Similarly, for all $n < \omega$, $\mathrm{Col}(\lambda(x_n^G)^+, {<}x_{n+1}^G) \cong \mathrm{Col}(\lambda(x_n^G)^+, {<}\lambda(x_{n+1}^G))$. Hence, in $V[G]$, every cardinal between ω_1^V and $\lambda(x_0^G)$, as well as every cardinal between $(\lambda(x_n^G)^+)^V$ and $\lambda(x_{n+1}^G)$ for $n < \omega$, is collapsed. The fact that no other cardinal below κ is collapsed follows from (3) and well-known closure and chain condition properties of Lévy collapse forcings.

(5): This follows from (4) and the fact that $\sup_n \lambda(x_n^G) = \kappa$.

(6): Fix any $\gamma < \kappa$. Let $n < \omega$ be such that $\lambda(x_n^G) > \gamma$. By (3) and closure properties of Lévy collapse forcings, we get $\mathcal{P}(\gamma) \cap V[G] = \mathcal{P}(\gamma) \cap V[c_{-1}^G \times \cdots \times c_{n-1}^G]$. As κ is still inaccessible in $V[c_{-1}^G \times \cdots \times c_{n-1}^G]$, we have $V[c_{-1}^G \times \cdots \times c_{n-1}^G] \models |\mathcal{P}(\gamma)| < \kappa$. Finally, the fact that κ is not collapsed in $V[G]$ gives $V[G] \models |\mathcal{P}(\gamma)| < \kappa$.

(7): This follows from (5), (2), and the fact that $\mathbb{P}$ has the λ^+-Knaster property (Lemma 4.4.9). □

4.5 Down to $\aleph_{\omega+1}$

This section is dedicated to proving the main theorem of this chapter. Some notation and conventions in this section are independent of those in the previous section. Let κ be a supercompact cardinal, and let $\lambda > \kappa$ be an inaccessible cardinal. For $\delta \in [\kappa, \lambda]$, let $\mathcal{A}(\kappa, \delta) \subset \mathcal{P}_\kappa(\delta)$ be the collection of all those x such that $\kappa(x) = x \cap \kappa$ is an inaccessible cardinal.

Let $\mathcal{U}$ be a normal fine measure on $\mathcal{A}(\kappa, \lambda)$. For any $X \subset \mathcal{A}(\kappa, \lambda)$ and $\delta < \lambda$, let $X \restriction \delta = \{x \cap \delta \mid x \in X\}$. For $\delta \in [\kappa, \lambda)$, define the normal fine measure $\mathcal{U}_\delta$ on $\mathcal{A}(\kappa, \delta)$ to be the set of all $X \restriction \delta$ where $X \in \mathcal{U}$. For $\vec{x} = \langle x_0, \ldots, x_{n-1} \rangle \in \mathcal{A}(\kappa, \lambda)^{<\omega}$ and $\delta < \lambda$, we also define $\vec{x} \restriction \delta = \langle x_0 \cap \delta, \ldots, x_{n-1} \cap \delta \rangle$.

Definition 4.5.1. Suppose $\delta \in [\kappa, \lambda)$, $T \subset \mathcal{A}(\kappa, \delta)^{<\omega}$, and $H \colon T \to V_\kappa$.

- For each $\delta' < \delta$, $T \restriction \delta' = \{\vec{x} \restriction \delta' \mid \vec{x} \in T\}$.
- $T \neq \emptyset$ is a *weak $\mathcal{U}_\delta$-tree* iff it is a tree and, for all $\langle x_0, \ldots, x_{k-1} \rangle \in T$, we have
  (1) $\mathrm{Suc}_T(\langle x_0, \ldots, x_{k-1} \rangle) \in \mathcal{U}_\delta$.
  (2) $\forall y \in \mathrm{Suc}_T(\langle x_0, \ldots, x_{k-1} \rangle) \colon y \supset x_{k-1}$ and $\kappa(y) > \kappa(x_{k-1})$.
- The pair $\langle T, H \rangle$ is called a *weak $\mathcal{U}_\delta$-tree with collapses* iff
  (1) T is a weak $\mathcal{U}_\delta$-tree.
  (2) $H(\langle\rangle) \in \mathrm{Col}(\alpha^+, {<}\kappa)$ for some $\alpha < \kappa$.
  (3) $\forall \vec{x}^\frown\langle y \rangle \in T \colon H(\vec{x}^\frown\langle y \rangle) \in \mathrm{Col}(\kappa(y)^+, {<}\kappa)$.
  (4) $\forall \vec{x}^\frown\langle y \rangle \in T \colon\ \kappa(y) > \sup(\mathrm{supp}(H(\vec{x})))$.

4.5.1 The main forcing

The following four definitions establish the forcing and its ordering.

Definition 4.5.2 (The forcing)**.** Define $\mathbb{P} = \mathbb{P}(\kappa, \lambda, \langle \mathcal{U}_\delta \rangle_{\delta \in [\kappa, \lambda)})$ as the poset whose conditions are

$$p = \langle s^p, l^p, f^p, T^p, H^p \rangle$$

where:

(1) $s^p = \langle \alpha^p_0, \dots, \alpha^p_{n-1} \rangle$ is an increasing sequence of inaccessible cardinals below κ.
(2) $l^p = \langle c^p_{-1}, c^p_0, \dots, c^p_{n-1} \rangle$, where
   (a) $c^p_{-1} \in \mathrm{Col}(\omega_1, {<}\alpha^p_0)$;
   (b) $c^p_i \in \mathrm{Col}((\alpha^p_i)^+, {<}\alpha^p_{i+1})$, for $0 \leq i < n-1$;
   (c) $c^p_{n-1} \in \mathrm{Col}((\alpha^p_{n-1})^+, {<}\kappa)$.

   If $n = 0$, then $c^p_{-1} \in \mathrm{Col}(\omega_1, {<}\kappa)$.
(3) $f^p \colon \mathrm{dom}(f^p) \to \omega$ and $\mathrm{dom}(f^p) \in [\kappa, \lambda)$.
(4) $\langle T^p, H^p \rangle$ is a weak $\mathcal{U}_{\mathrm{dom}(f^p)}$-tree with collapses such that $H^p(\langle\rangle) = c^p_{n-1}$ and $\kappa(y) > \alpha^p_{n-1}$ for all $y \in \mathrm{Suc}_{T^p}(\langle\rangle)$.

The motivation for f^p will be explained in Remark 4.5.7.

Definition 4.5.3 (n-point extensions)**.** Given a condition $p = \langle s, l, f, T, H \rangle$ and $\vec{x} = \langle x_0, \dots, x_{k-1} \rangle \in T$, the *extension of p by $\vec{x}$* is given by $p_{\vec{x}} = \langle s_{\vec{x}}, l_{\vec{x},H}, f_{\vec{x}}, T_{\vec{x}}, H_{\vec{x}} \rangle$, where

(1) $s_{\vec{x}} = s^\frown \langle \kappa(x_0), \dots, \kappa(x_{k-1}) \rangle$.
(2) $l_{\vec{x},H} = l^\frown \langle H(\langle x_0 \rangle), H(\langle x_0, x_1 \rangle), \dots, H(\langle x_0, \dots, x_{k-1} \rangle) \rangle$.
(3) $f_{\vec{x}} \colon \mathrm{dom}(f) \to \omega$ is defined by

$$f_{\vec{x}}(\alpha) = f(\alpha) + |\{i < k \mid \alpha \in x_i\}|.$$

(4) $T_{\vec{x}} = \{\vec{y} \mid \vec{x}^\frown \vec{y} \in T\}$.
(5) $H_{\vec{x}} \colon T_{\vec{x}} \to V_\kappa$ is defined by $\vec{y} \mapsto H(\vec{x}^\frown \vec{y})$.

Definition 4.5.4 (Direct extensions)**.** Given two conditions $p = \langle s^p, l^p, f^p, T^p, H^p \rangle$ and $q = \langle s^q, l^q, f^q, T^q, H^q \rangle$, we write $q \leq^* p$ iff

(1) $s^q = s^p$.
(2) $l^q \leq l^p$, with the order defined coordinate-wise.
(3) $f^q \supset f^p$.
(4) $T^q \restriction \mathrm{dom}(f^p) \subset T^p$.
(5) $\forall \vec{x} \in T^q \colon\ H^q(\vec{x}) \leq H^p(\vec{x} \restriction \mathrm{dom}(f^p))$.

Definition 4.5.5 (The main order)**.** Given conditions $p, q \in \mathbb{P}$, we write $q \leq p$ provided there exists $\vec{x} \in T^p$ such that $q \leq^* p_{\vec{x}}$.

Remark 4.5.6. Let $p = \langle \langle\rangle, \langle\emptyset\rangle, f, T, H \rangle$ be the condition where $f \colon \kappa \to \{0\}$, T is the set of all finite increasing sequences of inaccessible cardinals below κ, and $H \colon T \to \{\emptyset\}$. We restrict $\mathbb{P}$ to only those conditions q such that $q \leq p$, effectively taking p as the top element $1_{\mathbb{P}}$.

Remark 4.5.7. The role of f^p is to ensure that $p_{\langle x \rangle} \perp p_{\langle y \rangle}$ whenever $x \neq y$. One might wonder why we do not simply take f^p to be a finite sequence of subsets of λ, and define $f^p_{\vec{x}} = f^{p\frown} \vec{x}$. Under such a definition f^p would be more like a $\delta \times n$ matrix with entries in $\{0, 1\}$ that can expand in both dimensions. However, carrying out the argument in Lemma 4.5.8 under that definition, it becomes apparent that f^p needs more flexibility in order for the argument to go through. In particular, the rows of f^p should be allowed to have different lengths. But then the cells with entry 0 become irrelevant. Thus, the definition of f^p given in Definition 4.5.2 naturally emerges.

4.5.2 General properties

The main facts that hold in every extension by $\mathbb{P}$ are established in this subsection.

Lemma 4.5.8. *For every $p = \langle s, l, f, T, H\rangle$ and $D \subset \mathbb{P}$ dense open below p, there is $p^* = \langle s, l, f^*, T^*, H^*\rangle \leq^* p$ satisfying the following: For all $\vec{x} \in T^*$,*

$$\exists q\ (q \leq^* p^*_{\vec{x}} \ \wedge\ q \in D) \implies \exists q', T', H'\ (q' = \langle s_{\vec{x}}, l^q, f^*_{\vec{x}}, T', H'\rangle \ \wedge\ q' \leq^* p^*_{\vec{x}} \ \wedge\ q' \in D).$$

Proof. Fix p and D. Let H_χ be large enough. Let $\mathbb{P}^f = \langle\{f^p \mid p \in \mathbb{P}\}, \supset\rangle$. We define a sequence $\langle \mathcal{M}_\zeta; f_\zeta \mid \zeta < \kappa\rangle$, with $\langle f_\zeta \mid \zeta < \kappa\rangle \in (\mathbb{P}^f)^\kappa$ a $\supset$-decreasing sequence, as follows: Let $\mathcal{M}_0 \prec H_\chi$ be such that $|\mathcal{M}_0| < \lambda$, $\mathcal{M}_0 \cap \lambda \in \lambda$, $\mathcal{M}_0^{<\kappa} \subset \mathcal{M}_0$, and $\{\mathbb{P}, D, p\} \cup V_\kappa \subset \mathcal{M}_0$. Since the degree of closure of $\mathbb{P}^f$ is larger than the size of $\mathcal{M}_0$, we can take

$$f_0 \in \bigcap\{E \in \mathcal{M}_0 \mid E \subset \mathbb{P}^f \text{ is dense open below } f\}.$$

Suppose that $\langle \mathcal{M}_\zeta; f_\zeta \mid \zeta < \eta\rangle$ has been defined. Let $\bigcup_{\zeta<\eta} \mathcal{M}_\zeta \subset \mathcal{M}_\eta \prec H_\chi$ be such that $|\mathcal{M}_\eta| < \lambda$, $\mathcal{M}_\eta \cap \lambda \in \lambda$, $\mathcal{M}_\eta^{<\kappa} \subset \mathcal{M}_\eta$, and $\bigcup_{\zeta<\eta} f_\zeta \in \mathcal{M}_\eta$. Choose $f_\eta \supset \bigcup_{\zeta<\eta} f_\zeta$ such that

$$f_\eta \in \bigcap\{E \in \mathcal{M}_\eta \mid E \subset \mathbb{P}^f \text{ is dense open below } f\}.$$

Finally, set $f^* = \bigcup_{\zeta<\kappa} f_\zeta$ and $\mathcal{M} = \bigcup_{\zeta<\kappa} \mathcal{M}_\zeta$.

Let $\delta = \mathrm{dom}(f)$ and $\delta^* = \mathrm{dom}(f^*)$. Let T^* be the $\mathcal{U}_{\delta^*}$-tree that is the union of all $\mathcal{U}_{\delta^*}$-trees T' such that $T' \restriction \delta \subset T$. Let $H^* \colon T^* \to V_\kappa$ be defined by $H^*(\vec{x}) = H(\vec{x}\restriction\delta)$. Set $p^* = \langle s, l, f^*, T^*, H^*\rangle$.

Now, fix some $\vec{x} \in T^*$ and $q = \langle s_{\vec{x}}, l^q, f^q, T^q, H^q\rangle \leq^* p^*_{\vec{x}}$ in D. Let $\vec{y} = \vec{x}\restriction\delta$, and notice that $q \leq^* p_{\vec{y}}$. Consider the set

$$E_{\vec{x},l^q} = \{g \supset f \mid \exists T', H'\ (\langle s_{\vec{x}}, l^q, g_{\vec{x}}, T', H'\rangle \in D/p_{\vec{y}}) \text{ or } \forall g' \supset g\ \forall T', H'\ (\langle s_{\vec{x}}, l^q, g'_{\vec{x}}, T', H'\rangle \notin D/p_{\vec{y}})\}.$$

Notice that $\delta^* \subset \mathcal{M}$, so $\vec{x}, \vec{y} \in \mathcal{M}$ by closure of $\mathcal{M}$. Also, $\{f, s, l^q, D, p\} \subset \mathcal{M}$, and so $E_{\vec{x},l^q} \in \mathcal{M}$. It is easy to see that $E_{\vec{x},l^q}$ is dense open below f. We conclude that $f^* \in E_{\vec{x},l^q}$. Therefore, either $\exists T', H'\ (\langle s_{\vec{x}}, l^q, f^*_{\vec{x}}, T', H'\rangle \in D/p_{\vec{y}})$ or $\forall g' \supset f^*\ \forall T', H'\ (\langle s_{\vec{x}}, l^q, g'_{\vec{x}}, T', H'\rangle \notin D/p_{\vec{y}})$. But the latter is not possible, as witnessed by the counterexample given by $T' = T^q$, $H' = H^q$, and the function $g' \colon \mathrm{dom}(f^q) \to \omega$ defined by

$$g'(\alpha) = \begin{cases} f^*(\alpha) & \alpha \in \delta^*, \\ f^q(\alpha) & \alpha \in \mathrm{dom}(f^q) \setminus \delta^*. \end{cases}$$

Since in that case $g'_{\vec{x}} = f^q$. Hence, there is T', H' such that $q' = \langle s_{\vec{x}}, l^q, f^*_{\vec{x}}, T', H'\rangle \in D/p_{\vec{y}}$. Since $\mathrm{len}(p_{\vec{y}}) = \mathrm{len}(q')$, we must have $q' \leq^* p_{\vec{y}}$. Therefore, $T'\restriction\delta \subset T_{\vec{y}}$ and, for all $\vec{z} \in T'$, $H'(\vec{z}) \leq H_{\vec{y}}(\vec{z}\restriction\delta)$. By our choice for T^* and H^*, we must have $T' \subset T^*_{\vec{x}}$ and, for all $\vec{z} \in T'$, $H'(\vec{z}) \leq H^*_{\vec{x}}(\vec{z})$. This means that $q' \leq^* p^*_{\vec{x}}$, and we are done. □

Lemma 4.5.9. *Suppose φ is a formula in the forcing language and $p \in \mathbb{P}$ satisfies the property in Lemma 4.5.8 for the dense open set $D \subset \mathbb{P}$ of conditions deciding φ. Then, if for $\mathcal{U}_{\mathrm{dom}(f)}$-large many $y \in \mathrm{Suc}_{T^p}(\langle\rangle)$, there is $q \leq^* p_{\langle y\rangle}$ in D, then there is $r \leq^* p$ in D.*

Proof. The proof is similar to that of Lemma 4.4.12. Thus, let $n = \mathrm{len}(p)$ and $\delta = \mathrm{dom}(f^p)$. Let $j_\delta \colon V \to M_\delta$ be the ultrapower embedding associated with $\mathcal{U}_\delta$. Working in M_δ, by assumption, there exists a condition $q \leq^* j_\delta(p)_{\langle j_\delta``\delta\rangle}$ deciding $j_\delta(\varphi)$. By the property in Lemma 4.5.8, we can assume that $f^q = j_\delta(f^p)_{\langle j_\delta``\delta\rangle}$. WLOG, suppose that $q \Vdash j_\delta(\varphi)$. Back in V, let $h \colon \mathcal{A}(\kappa, \delta) \to V_\kappa$ be such that $[h]_{\mathcal{U}_\delta} = c^q_n$, and let $l^* = \langle c^q_{-1}, \ldots, c^q_{n-1}\rangle$. Notice that $j_\delta(l^*) = l^*$ because $\mathrm{crit}(j_\delta) = \kappa = (\kappa(j_\delta``\delta))^{M_\delta} > \sup(\mathrm{supp}(l^*))$. Now, we can take a $\mathcal{U}_\delta$-large set $S \subset \mathrm{Suc}_{T^p}(\langle\rangle)$ such that, for every $y \in S$, there exists $T(y), H(y)$ such that

$$q(y) = \langle s^p{}^\frown\langle\kappa(y)\rangle, l^*{}^\frown\langle h(y)\rangle, f^p_{\vec{y}}, T(y), H(y)\rangle \;\wedge\; q(y) \leq^* p_{\langle y\rangle} \;\wedge\; q(y) \Vdash \varphi.$$

Define T^* to be the smallest set so that $\langle\rangle \in T^*$, $\{\langle y\rangle \mid y \in S\} \subset T^*$, and, for all $\langle y\rangle \in T^*$, $\{\langle y\rangle^\frown\vec{z} \mid \vec{z} \in T(y)\} \subset T^*$. Define $H^*\colon T^* \to V_\kappa$ by $H^*(\langle\rangle) = c^q_{n-1}$, $H^*(\langle y\rangle) = h(y)$ for all $\langle y\rangle \in T^*$, and $H^*(\langle y\rangle^\frown\vec{z}) = H(y)(\vec{z})$ for all $\langle y\rangle^\frown\vec{z} \in T^*$. Let $r = \langle s^p, l^*, f^p, T^*, H^*\rangle$. Clearly, $r \in \mathbb{P}$ and $r \leq^* p$. Also, $r_{\langle y\rangle} = q(y) \Vdash \varphi$, for all $\langle y\rangle \in T^*$. Finally, since $\{r_{\langle y\rangle} \mid \langle y\rangle \in T^*\}$ is a maximal antichain below r, we must have $r \Vdash \varphi$. □

Lemma 4.5.10 (Prikry property). *For every formula φ in the forcing language and every condition p, there is $p^* \leq^* p$ that decides φ.*

Proof. The proof is identical to that of Lemma 4.4.13, and can be repeated verbatim with Lemmas 4.4.11 and 4.4.12 replaced by Lemmas 4.5.8 and 4.5.9, respectively. □

To argue for the preservation of λ as a cardinal, we will make use of a lemma similar to the Prikry property. Suppose $p \Vdash \dot{F}\colon \check{\gamma} \to \check{\lambda}$ for some $\gamma < \kappa$ and let $\alpha < \gamma$. We will say that $q \leq p$ bounds the value of $\dot{F}(\check{\alpha})$ iff there is $\theta < \lambda$ such that $q \Vdash \dot{F}(\check{\alpha}) < \check{\theta}$. Notice that the set $D \subset \mathbb{P}$ of conditions that bound the value of $\dot{F}(\check{\alpha})$ is dense open below p, because it contains the dense open set of conditions below p that decide the value $\dot{F}(\check{\alpha})$. The following lemma is similar to Lemma 4.5.9.

Lemma 4.5.11. *Suppose $p \Vdash \dot{F}\colon \check{\gamma} \to \check{\lambda}$ for some $\gamma < \kappa$ and let $\alpha < \gamma$. Suppose also that p satisfies the property in Lemma 4.5.8 for the set $D \subset \mathbb{P}$ of conditions that bound the value of $\dot{F}(\check{\alpha})$. Then, if for $\mathcal{U}_{\mathrm{dom}(f)}$-large many $y \in \mathrm{Suc}_{T^p}(\langle\rangle)$, there is $q \leq^* p_{\langle y\rangle}$ in D, then there is $r \leq^* p$ in D.*

Proof. The proof is similar to that of Lemma 4.5.9. We only need to add the following changes: First, we notice that j_δ fixes α and λ, and that the set $\{\beta < \lambda \mid j_\delta(\beta) = \beta\}$ is unbounded in λ. This allows us to take $q \leq^* j_\delta(p)_{\langle j_\delta``\delta\rangle}$ such that $q \Vdash j_\delta(\dot{F})(\check{\alpha}) < \check{\theta}$, where $\theta < \lambda$ satisfies $j_\delta(\theta) = \theta$. The rest is similar. □

Lemma 4.5.12. *Suppose $p \Vdash \dot{F}\colon \check{\gamma} \to \check{\lambda}$ for some $\gamma < \kappa$ and let $\alpha < \gamma$. Then, there exists $p^* \leq^* p$ and $\theta < \lambda$ such that $p^* \Vdash \dot{F}(\check{\alpha}) < \check{\theta}$.*

Proof. The proof is a verbatim repetition of that of Lemma 4.4.13 with the following minor changes: Lemmas 4.4.11 and 4.4.12 are replaced by Lemmas 4.5.8 and 4.5.11, respectively, and $D \subset \mathbb{P}$ is taken to be the set of conditions that bound the value of $\dot{F}(\check{\alpha})$. □

Suppose G is $(V, \mathbb{P})$-generic. Define $s^G = \bigcup\{s^p \mid p \in G\}$. For each $-1 \leq n < \omega$, define $c^G_n = \bigcup\{c^p_n \mid p \in G\}$.

Proposition 4.5.13. *Suppose G is $(V, \mathbb{P})$-generic. Then, in $V[G]$:*

1. *Every V-regular cardinal in the interval $[\kappa, \lambda)$ has $V[G]$-cofinality ω.*
2. *Every V-cardinal in the interval (κ, λ) is collapsed.*
3. *λ is not collapsed.*
4. *Every bounded subset of κ in $V[G]$ is in $V[c^G_{-1} \times \cdots \times c^G_{n-1}]$ for some $n < \omega$.*
5. *$\aleph_{2n+2} = s^G(n)$ and $\aleph_{2n+3} = (s^G(n)^+)^V$, for all $n < \omega$.*
6. *$\aleph_\omega = \kappa$.*

(7) $\aleph_\omega$ *is a strong limit.*
(8) $\aleph_{\omega+1} = \lambda$.

Proof. (1) Fix a V-regular cardinal $\gamma \in [\kappa, \lambda)$. By density, there is $p \in G$ such that $\mathrm{dom}(f^p) \geq \gamma$. By condition (2) in the definition of weak $\mathcal{U}_\delta$-trees in Definition 4.5.1, it is easy to see that the set $\{p_{\langle x \rangle} \mid x \in \mathrm{Suc}_{T^p}(\langle\rangle)\}$ is a maximal antichain below p. Hence, there exists a unique $x_0 \in \mathrm{Suc}_{T^p}(\langle\rangle)$ such that $p_{\langle x_0 \rangle} \in G$. Repeating the same argument for $p_{\langle x_0 \rangle}$, we get $x_1 \in \mathrm{Suc}_{T^p}(\langle x_0 \rangle)$ such that $p_{\langle x_0, x_1 \rangle} \in G$. Proceeding like this, we define a sequence $\langle x_n \mid n < \omega \rangle$ such that $p_{\langle x_0, \ldots, x_{k-1} \rangle} \in G$ for all $k < \omega$.

Now, let $\alpha < \gamma$ be arbitrary. The set $D_\alpha = \{q \leq p \mid f^q(\alpha) > f^p(\alpha)\}$ is dense below p by the fineness of the ultrafilters, so we can fix $q \in G \cap D_\alpha$. Let $\vec{y} = \langle y_0, \ldots, y_{k-1} \rangle \in T^p$ be such that $q \leq^* p_{\vec{y}}$. Clearly, $p_{\vec{y}} \in G$ and $\alpha \in y_i$ for some $i < k$. By the previous paragraph, $\langle y_0, \ldots, y_{k-1} \rangle = \langle x_0, \ldots, x_{k-1} \rangle$. In particular, $\alpha \in x_i$. We have established that, for each $\alpha < \gamma$, there exists $i < \omega$ such that $\alpha \in x_i$. Thus, $\bigcup_n (x_n \cap \gamma) = \gamma$. Finally, the fact that $|x_n|^V < \kappa$ for all $n < \omega$, implies that the sequence $\langle \sup(x_n \cap \gamma) \mid n < \omega \rangle$ is cofinal in γ.

(2): This follows easily from (1) and is similar to Proposition 4.4.14 (2).

(3): The idea is similar to that of Proposition 4.4.14 (3). Thus, suppose otherwise that λ is collapsed. By (1), $\mathrm{cof}(\lambda)^{V[G]} < \kappa$. Fix $p \in G, \dot{F}$, and $\gamma < \kappa$ such that $\mathrm{len}(p) = n+1$ for some $n < \omega$, $\alpha^p_n > \gamma$, and $p \Vdash$ "$\dot{F} \colon \check{\gamma} \to \check{\lambda} \ \wedge \ \sup(\mathrm{ran}(\dot{F})) = \check{\lambda}$". We work in V. Set $l = \langle c^p_{-1}, \ldots, c^p_{n-1} \rangle$,[3] $\rho = \alpha^p_n$, and $\mathbb{C} = \mathrm{Col}(\omega_1, < \alpha^p_0) \times \cdots \times \mathrm{Col}((\alpha^p_{n-1})^+, < \alpha^p_n)$. For any $q \leq p$ and $e \in \mathbb{C}$, we use the notation $q + e$ to denote $\langle s^q, e^\frown \langle c^q_n, \ldots, c^q_{m-1} \rangle, f^q, T^q, H^q \rangle$, where $m = \mathrm{len}(q)$. We also use the notation $r \leq^\dagger q$ for $q, r \leq p$ to mean that $r \leq^* q$ and $l^r|_n = l^q|_n$. Notice that $\leq^\dagger$ is ρ^+-closed below p.

Fix an enumeration $\langle e^{(\xi)} \mid \xi < \rho \rangle$ of $\mathbb{C}/l$. We will assume that $\mathrm{OR} \times \mathrm{OR}$ is ordered lexicographically. Define recursively a $\leq^\dagger$-decreasing sequence $\langle p^{(\alpha,\xi)} \mid (\alpha, \xi) < (\gamma, \rho) \rangle$ below p along with an increasing sequence of ordinals $\langle \theta^{(\alpha,\xi)} \mid (\alpha,\xi) < (\gamma,\rho) \rangle$ below λ as follows: Having defined $\langle p^{(\alpha,\xi)} \mid (\alpha,\xi) < (\beta,\zeta) \rangle$ and $\langle \theta^{(\alpha,\xi)} \mid (\alpha,\xi) < (\beta,\zeta) \rangle$, first take p' to be $\leq^\dagger$-below $p^{(\alpha,\xi)}$, for every $(\alpha,\xi) < (\beta,\zeta)$, and $\theta' = \sup\{\theta^{(\alpha,\xi)} \mid (\alpha,\xi) < (\beta,\zeta)\}$ (for the base case $(\beta,\zeta) = (0,0)$, take $p' = p$ and $\theta' = 0$). Then, if there exists $q \leq^* p'$ and $\theta < \lambda$ such that $q \Vdash \dot{F}(\check{\beta}) < \check{\theta}$ and $l^q|_n = e^{(\zeta)}$, then set $p^{(\beta,\zeta)} = q + l$ and $\theta^{(\beta,\zeta)} = \max\{\theta', \theta\}$. Else, set $p^{(\beta,\zeta)} = p'$ and $\theta^{(\beta,\zeta)} = \theta'$. Finally, set $p^\dagger$ to be $\leq^\dagger$-below $p^{(\alpha,\xi)}$, for every $(\alpha,\xi) < (\gamma,\rho)$, and set $\theta^\dagger = \sup\{\theta^{(\alpha,\xi)} \mid (\alpha,\xi) < (\gamma,\rho)\} < \lambda$.

Now, fix some $\alpha < \gamma$. For any $e \in \mathbb{C}/l$, using Lemma 4.5.12, we can find $p^* \leq^* p^\dagger + e$ and $\theta < \lambda$ such that $p^* \Vdash \dot{F}(\check{\alpha}) < \check{\theta}$. Let $\xi < \rho$ be such that $l^{p^*}|_n = e^{(\xi)}$. By our construction, this means that $p^{(\alpha,\xi)}$ and $\theta^{(\alpha,\xi)}$ are such that $(p^{(\alpha,\xi)} + e^{(\xi)}) \Vdash \dot{F}(\check{\alpha}) < \check{\theta}^{(\alpha,\xi)}$. Since $p^\dagger + e^{(\xi)} \leq^\dagger p^{(\alpha,\xi)} + e^{(\xi)}$ and $\theta^\dagger \geq \theta^{(\alpha,\xi)}$, we have $(p^\dagger + e^{(\xi)}) \Vdash \dot{F}(\check{\alpha}) < \check{\theta}^\dagger$. Since $e^{(\xi)} \leq e$, we have established that the set $E_\alpha \subset \mathbb{C}/l$ of all e such that $(p^\dagger + e) \Vdash \dot{F}(\check{\alpha}) < \check{\theta}^\dagger$ is dense open in $\mathbb{C}/l$. By genericity, it is easy to see that for any dense open set $E \subset \mathbb{C}/l$, $p^\dagger \Vdash \exists e \in \check{E}\ (p^\dagger + e \in \dot{G})$. Therefore, $p^\dagger \Vdash \dot{F}(\check{\alpha}) < \check{\theta}^\dagger$. As α was arbitrary, we get $p^\dagger \Vdash \forall \alpha < \check{\gamma}\ (\dot{F}(\alpha) < \check{\theta}^\dagger)$. But, $\theta^\dagger < \lambda$, so this contradicts the fact that $p \Vdash \sup(\mathrm{ran}(\dot{F})) = \check{\lambda}$.

(4): The proof is exactly the same as that of Proposition 4.4.14 (3) with the following minor changes: $\lambda(x^p_n)$ is replaced by α^p_n, and $\mathbb{C}, q+e$, and $\leq^\dagger$ are taken as in (3) of the current proposition.

(5): Clearly, every cardinal between ω_1^V and $s^G(0)$, as well as every cardinal between $(s^G(n)^+)^V$ and $s^G(n+1)$ for $n < \omega$, is collapsed. The fact that no other cardinal below κ is collapsed follows from (4) and well-known closure and chain condition properties of Lévy collapse forcings.

[3] Notice that l is shorter than l^p by one coordinate.

(6): This follows from (5) and the standard fact that s^G is cofinal in κ.
(7): This is similar to (6) of Proposition 4.4.14.
(8): This follows from (2) and (3). □

4.5.3 HOD analysis

In this final subsection, we show that there is a projection with the cone isomorphism property from $\mathbb{P}$ to the usual tree Prikry forcing. That fact is used to finalize the proof of the main theorem.

Given a condition $p = \langle s, l, f, T, H\rangle$, define $\pi(p) = \langle s, T\restriction\kappa\rangle$. Let $\mathbb{U} = \{\pi(p) \mid p \in \mathbb{P}\}$ and $1_{\mathbb{U}} = \pi(1_{\mathbb{P}})$. Given $u = \langle s^u, T^u\rangle \in \mathbb{U}$ and $\vec{x} \in T^u$,[4] let $u_{\vec{x}} = \langle s^u_{\vec{x}}, T^u_{\vec{x}}\rangle$. For $u, v \in \mathbb{U}$, set $v \leq^*_{\mathbb{U}} u$ iff $s^v = s^u$ and $T^v \subset T^u$. The order $\leq_{\mathbb{U}}$ on $\mathbb{U}$ is defined by setting $v \leq u$ iff there exists $\vec{x} \in T^u$ such that $v \leq^* u_{\vec{x}}$. Thus, $\mathbb{U}$ is just the usual tree Prikry forcing with the ultrafilter $\mathcal{U}_\kappa$, and it is straightforward to see that $\pi\colon \mathbb{P} \to \mathbb{U}$ is a projection.

Lemma 4.5.14. *For every $p, q \in \mathbb{P}$ such that $\pi(p) \parallel \pi(q)$, there exists $p' \leq p$ and $q' \leq q$ such that $s^{p'} = s^{q'}$, $\mathrm{dom}(f^{p'}) = \mathrm{dom}(f^{q'})$, and $T^{p'} = T^{q'}$.*

Proof. By $\leq^*$-extending either p or q, if necessary, we may assume that $\mathrm{dom}(f^p) = \mathrm{dom}(f^q) = \delta$ for some $\delta \in [\kappa, \lambda)$. Fix $u \leq_{\mathbb{U}} \pi(p), \pi(q)$, and let $\vec{x} \in T^p$ and $\vec{y} \in T^q$ be such that

$$u \leq^*_{\mathbb{U}} \pi(p)_{\vec{x}\restriction\kappa}, \pi(q)_{\vec{y}\restriction\kappa}.$$

Notice now that $s^p_{\vec{x}} = s^u = s^q_{\vec{y}}$. Let T be any weak $\mathcal{U}_\delta$-tree such that $T\restriction\kappa = T^u$. Set $T' = T \cap T^p_{\vec{x}} \cap T^q_{\vec{y}}$. Finally, take $p' = \langle s^p_{\vec{x}}, l^p_{\vec{x},H^p}, f^p_{\vec{x}}, T', H^p_{\vec{x}}\rangle$ and $q' = \langle s^q_{\vec{y}}, l^q_{\vec{y},H^q}, f^q_{\vec{y}}, T', H^q_{\vec{y}}\rangle$. □

Suppose α is a regular cardinal and $\beta > \alpha$ is an inaccessible cardinal. Let $c, d \in \mathrm{Col}(\alpha, <\beta)$ be such that $\mathrm{dom}(c) = \mathrm{dom}(d)$. It is a well known fact that the map $\Psi\colon \mathrm{Col}(\alpha, <\beta)/c \to \mathrm{Col}(\alpha, <\beta)/d$ defined by $r \mapsto d \cup r|_{\mathrm{dom}(r)\setminus\mathrm{dom}(d)}$ is an isomorphism. We will call Ψ the *standard isomorphism.*

Lemma 4.5.15. *$\pi\colon \mathbb{P} \to \mathbb{U}$ satisfies the cone isomorphism property. That is, for every $p, q \in \mathbb{P}$ such that $\pi(p) \parallel \pi(q)$, there exists $p' \leq p$ and $q' \leq q$ such that $\mathbb{P}/p'$ and $\mathbb{P}/q'$ are isomorphic.*

Proof. Fix any $p, q \in \mathbb{P}$ such that $\pi(p) \parallel \pi(q)$. By Lemma 4.5.14, we may assume that $s^p = s^q = s$ for some s, $\mathrm{dom}(f^p) = \mathrm{dom}(f^q) = \delta$ for some $\delta \in [\kappa, \lambda)$, and $T^p = T^q = T$ for some weak $\mathcal{U}_\delta$-tree T. Set $n = \mathrm{len}(p) = \mathrm{len}(q)$ and $s = \langle \alpha_0, \ldots, \alpha_{n-1}\rangle$.

For every $i \in [-1, n-1]$, fix $c^{p'}_i \leq c^p_i$ and $c^{q'}_i \leq c^q_i$ such that $\mathrm{dom}(c^{p'}_i) = \mathrm{dom}(c^{q'}_i)$, as well as the standard isomorphism

$$\Psi_i\colon \mathrm{Col}(\alpha_i^+, <\alpha_{i+1})/c^{p'}_i \to \mathrm{Col}(\alpha_i^+, <\alpha_{i+1})/c^{q'}_i,$$

where we take $\alpha_{-1} = \omega$ and $\alpha_n = \kappa$. Similarly, for each nonempty $\vec{x} = \langle x_0, \ldots, x_{k-1}\rangle \in T$, fix $H^{p'}(\vec{x}) \leq H^p(\vec{x})$ and $H^{q'}(\vec{x}) \leq H^q(\vec{x})$ such that $\mathrm{dom}(H^{p'}(\vec{x})) = \mathrm{dom}(H^{p'}(\vec{x}))$, as well as the standard isomorphism

$$\Psi_{\vec{x}}\colon \mathrm{Col}(\kappa(x_{k-1})^+, <\kappa)/H^{p'}(\vec{x}) \to \mathrm{Col}(\kappa(x_{k-1})^+, <\kappa)/H^{q'}(\vec{x}).$$

Take $H^{p'}(\langle\rangle) = c^{p'}_{n-1}$ and $H^{q'}(\langle\rangle) = c^{q'}_{n-1}$. Define $l^{p'} = \langle c^{p'}_{-1}, \ldots, c^{p'}_{n-1}\rangle$ and $l^{q'} = \langle c^{q'}_{-1}, \ldots, c^{q'}_{n-1}\rangle$. Finally, set $p' = \langle s, l^{p'}, f^p, T, H^{p'}\rangle$ and $q' = \langle s, l^{q'}, f^q, T, H^{q'}\rangle$.

[4] Notice that $\vec{x}$ is really just an increasing sequence of inaccessible cardinals below κ.

We are now ready to define an isomorphism $\Phi\colon \mathbb{P}/p' \to \mathbb{P}/q'$. Given a vector $\vec{x} = \langle x_0, \dots, x_{k-1}\rangle \in T$ and $r \in \mathbb{P}/p'$ such that $r \leq^* p'_{\vec{x}}$, we let $\Phi(r)$ be the following condition:

(1) $s^{\Phi(r)} = s_{\vec{x}}$.
(2) $l^{\Phi(r)} = \langle \Psi_{-1}(c^r_{-1}), \dots, \Psi_{n-1}(c^r_{n-1})\rangle^\frown\langle \Psi_{\vec{x}|_1}(c^r_n), \Psi_{\vec{x}|_2}(c^r_{n+1}), \dots, \Psi_{\vec{x}}(c^r_{(n-1)+k})\rangle$.
(3) $f^{\Phi(r)} = f^q_{\vec{x}} \cup f^r|_{\mathrm{dom}(f^r)\setminus\delta}$.
(4) $T^{\Phi(r)} = T^r$.
(5) $H^{\Phi(r)}$ satisfies $H^{\Phi(r)}(\langle\rangle) = c^{\Phi(r)}_{(n-1)+k}$ and, for nonempty $\vec{y} \in T^{\Phi(r)}$, $H^{\Phi(r)}(\vec{y}) = \Psi_{\vec{x}^\frown\vec{y}\restriction\delta}(H^r(\vec{y}))$.

It is routine to check that Φ is an isomorphism. □

The *generic* HOD, denoted by gHOD, was introduced in [Fuc08]. It is defined as the intersection of the HODs of all set-forcing extensions of V. In other words,

$$\mathrm{gHOD} = \{x \mid \forall\mathbb{P}\ 1_{\mathbb{P}} \Vdash_{\mathbb{P}} \check{x} \in \mathrm{HOD}\}.$$

Proposition 4.5.16. *Assume that λ is supercompact and $V = \mathrm{gHOD}$. If G is $(V, \mathbb{P})$-generic, then in $V[G]$:*

(1) *$\aleph_\omega$ is a strong limit.*
(2) *$\aleph_{\omega+1}$ is supercompact in* HOD.
(3) *The* HOD *hypothesis holds.*

Proof. (1): This is just (7) of Proposition 4.5.13.

(2): Let $H = \mathrm{upcl}_{\mathbb{Q}}(\pi''G)$. By Lemma 4.5.15 and Proposition 4.3.9, we have

$$V = \mathrm{gHOD} \subset \mathrm{HOD}^{V[G]} \subset V[H] \subset V[G]. \tag{4.2}$$

By the Intermediate Model Theorem [Kan03, Proposition 10.10] and the fact that $\mathbb{U} \in V^V_\lambda$, we have that $\mathrm{HOD}^{V[G]}$ is a generic extension of V by a forcing in V^V_λ. Since small forcings do not destroy supercompactness [Jec03, Theorem 21.2], we conclude that λ remains supercompact in $\mathrm{HOD}^{V[G]}$. Finally, $\lambda = (\aleph_{\omega+1})^{V[G]}$ by Proposition 4.5.13.

(3): The λ^+-c.c. of $\mathbb{P}$ in V implies that all the cardinals above λ are preserved between V and $V[G]$. By (4.2), these cardinals are also preserved in $\mathrm{HOD}^{V[G]}$. Hence, the class of all successor cardinals in $V[G]$ above λ forms a proper class of $V[G]$-regular cardinals that are not measurable in $\mathrm{HOD}^{V[G]}$. □

The standard class forcing notion that codes the universe of sets into the powerset function pattern due to McAloon [McA71] already gives a model for $V = \mathrm{gHOD}$. Ad hoc arguments based on [BP23] can be used to show that this forcing preserves supercompactness. Thus, starting with two supercompact cardinals, one can always assume that $V = \mathrm{gHOD}$ holds, and this finalizes the proof of Theorem 4.1.1.

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