

On the role of the fast oscillations in the secular dynamics of the lunar coplanar perturbation on Galileo satellites

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ABSTRACT

Motivated by the practical interest in the third-body perturbation as a natural cleaning mechanism for high-altitude Earth orbits, we investigate the dynamics stemming from the secular Hamiltonian associated with the lunar perturbation, assuming that the Moon lies on the ecliptic plane. The secular Hamiltonian defined in that way is characterized by two timescales. We compare the location and stability of the fixed points associated with the secular Hamiltonian averaged with respect to the fast variable with the corresponding periodic orbits of the full system. Focusing on the orbit of the Galileo satellites, it turns out that the two dynamics cannot be confused, as the relative difference depends on the ratio between the semi-major axis of Galileo and the one of the Moon, that is not negligible. The result is relevant to construct rigorously the Arnold diffusion mechanism that can drive a natural growth in eccentricity that allows a satellite initially on a circular orbit in Medium Earth Orbit to reenter into the Earth's atmosphere.

1. Introduction

The third-body gravitational perturbation on a bounded orbit at the Earth is known to yield a long-term variation in eccentricity (e.g., [1]). In the past years, a specific interest on this phenomenon has arisen, as a possible mechanism to facilitate the disposal of artificial satellites at the end-of-life and dilute the probability of collision (e.g., [2]). Special emphasis in this regard is given to high-altitude orbits, in particular to Medium Earth Orbits (MEO), where the Global Navigation Satellite Systems (i.e., GPS, Galileo, Beidou and GLONASS) are located.

Considering an equatorial geocentric reference system, the eccentricity growth has been investigated mostly associated with resonances that involve the argument of pericenter and the longitude of ascending node of the orbit of the satellite and the longitude of the ascending node of the orbit of the third body, namely, Moon or Sun. Pillar works on the identification of these resonances, the corresponding phase space and their practical role on high-altitude orbits are [3–8]. Several numerical studies (e.g., [9–11]) have highlighted that indeed there exist regions in the phase space such that a circular orbit can become very eccentric in the limit

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to reenter to the Earth's atmosphere without the need of any propulsion system. However, from the theoretical point of view, a firm conclusion on how to explain analytically the numerical results is still missing.

Motivated by this need, in [12] we show how an Arnold diffusion mechanism can be built to explain the phenomenon. That work lies in the same context of a series of works that are based on the analysis of the Kaula's expansion of the third-body perturbation [13], that is averaged over the mean anomaly of the orbit of the spacecraft and the one of the orbit of the third body in order to bring out the secular dynamics. We can say that these studies follow two main different, but related, concepts: (1.) the eccentricity growth is due to a chaotic behavior because more than one resonance is important in a given region [8,14]; (2.) the growth is due to a single resonance, whose phase space is modified by other terms of the Kaula's expansion [15–17].

In [12] we follow the latter, by considering the full quadrupolar secular expansion of the lunar perturbation. The aim is to show how to construct an Arnold diffusion mechanism, starting from the analysis of an autonomous reduced model. That is, we analyze first what we call the “full coplanar Hamiltonian” (namely, assuming that the Moon lies in the ecliptic plane) and use it as a first order for the full model. Then, we analyze the secular Hamiltonian perturbatively taking the inclination of the Moon as a small parameter. Note that Arnold diffusion can only take place for Hamiltonians systems of at least 3 degrees of freedom and therefore the coplanar Hamiltonian has too low dimension. However, it is a good first order to detect the hyperbolic structures that enhance the drift of eccentricity for the full secular Hamiltonian.

The main difference between our work in [12] and previous ones is that the first order that we consider does not neglect the fast oscillations of the coplanar dynamics. Let us explain what we mean by that.

As already mentioned, the mechanism considered to achieve eccentricity growth is to drift along a resonance. More concretely we consider the 2 : 1 resonance between the argument of the pericenter and the longitude of ascending node of the orbit of the satellite. Then, the coplanar model has two timescales: the slow one of the resonant angle and the fast one which is given by the oscillations of the argument of the pericenter or of the longitude of the ascending node.

In order to gain a better understanding of the full coplanar model, one can consider as an effective model its average over the fast oscillations along the resonance, which is integrable. We refer to this model as the “ h -averaged Hamiltonian”. This averaged model has been widely analyzed in literature (see [16–18]). It already presents hyperbolic orbits and invariant manifolds which are the “highways” to achieve eccentricity growth.

The purpose of the present paper is to analyze “how good” this h -averaged model is, by comparing it with the full coplanar model. That is, to understand how the dynamics and, more specifically, the hyperbolic invariant objects in phase space of the full coplanar Hamiltonian deviate from those of the h -averaged model. In other words, how the fast oscillations in the full coplanar Hamiltonian affect its global dynamics.

More concretely:

1. We analyze rigorously the dynamics of the h -averaged Hamiltonian. It turns out that one can describe analytically in a rather precise way the location and character of *all the critical points of the Hamiltonian for a rather large range of values of semi-major axis (which includes that of Galileo)*. Of major importance for [12] are the critical points with eccentricity equal to 0.² Therefore, we also analyze the eigenvalues of this critical points and show that they are hyperbolic in a small interval of energies, parabolic at the endpoints of this interval and elliptic otherwise. See Section 3.1.
2. We explain the need of overcoming the h -averaged approximation when studying the full coplanar model. The main result that we will present in this direction is to show that, at a fixed energy level, the distance between hyperbolic periodic circular orbits of the h -averaged system and that of the full coplanar Hamiltonian is proportional to a^3/a_M^3 with a the semi-major axis of the orbit and a_M the semi-major axis of the Moon. This is done through a perturbative analysis, that is, assuming a^3/a_M^3 small enough. Note, however, that for the semi-major axis of Galileo, a^3/a_M^3 is not so small. That is, for these values one should expect that the dynamics of the h -averaged Hamiltonian are rather “far” from that of the full coplanar model even at short timescales. In particular, the inclination of the periodic orbits presents a significant difference. See Section 3.2, in particular Corollary 3.12 and Remark 3.13.
3. We analyze the eigenvalues of the circular periodic orbits (i.e. periodic orbit with eccentricity equal to 0) of the full coplanar model. In particular, we perform a perturbative analysis in a^3/a_M^3 of the eigenvalues of the periodic orbits emanating from the origin and we show that, at a given energy level, the eigenvalues of the periodic orbit of the h -averaged and the full coplanar present a significant difference of order $(a/a_M)^{3/2}$. See Section 3.3.

The results in the present paper confirm the heuristic analysis done in [16–18] which already explains the necessity of overcoming the h -averaged model. However, we want to emphasize that in the present paper we do not provide heuristic arguments but rigorous results.

The development presented is focused on the orbit of the Galileo constellation, but the results are general and can be applied to other cases.

The paper is structured as follows. In Section 2 we introduce the models that we consider and a suitable set of coordinates to analyze it. We follow closely the exposition of the companion paper [12]. In Section 3, we state the main results of the paper. Sections 4 and 5 are devoted to prove, respectively, the results concerning to the circular and non-circular (eccentric) equilibrium points of the h -averaged system. Finally, in Section 6 we prove the results related with the periodic orbits for the full coplanar Hamiltonian system.

² In the following they will be referred as circular critical points

2. From the secular to the averaged Hamiltonians

Let us consider a spacecraft that moves under the gravitational attraction of the Earth, the perturbation due to the Earth's oblateness and the lunar gravitational perturbation. Assuming a geocentric equatorial reference system, the so-called orbital elements, namely, semi-major axis a , eccentricity e , inclination i , longitude of the ascending node Ω and argument of pericenter ω , that define the ellipse where the motion occurs, change in time because of the perturbations. The orbit of the Moon is defined in the geocentric ecliptic reference system by the corresponding orbital elements $(a_M, e_M, i_M, \Omega_M, \omega_M)$, where

$$a_M = 384400 \text{ km}, \quad e_M = 0.0549006, \quad i_M = 5.15^\circ,$$

while the longitude of the ascending node of the Moon with respect to the ecliptic plane varies approximately linearly with time in a period T_{Ω_M} of 1 Saros (about 6585.321347 days [19,20]) due to the solar gravitational perturbation, namely,

$$\Omega_M(t) = \Omega_{M,0} + n_{\Omega_M} t, \quad n_{\Omega_M} = 2\pi/T_{\Omega_M}. \quad (2.1)$$

In Delaunay action-angle variables (L, G, H, ℓ, g, h) , see, for instance, [21], considering as small parameter of the problem

$$\alpha = \frac{a}{a_M},$$

which characterizes the distance of the Moon with respect to the orbit of the satellite, the secular dynamics, see [8], is described by the non-autonomous (recall (2.1)) Hamiltonian

$$\mathcal{H}(L, G, H, g, h, \Omega_M; i_M) = \mathcal{H}_K(L) + \tilde{\mathcal{H}}_0(L, G, H) + \alpha^3 \tilde{\mathcal{H}}_1(L, G, H, g, h, \Omega_M; i_M), \quad (2.2)$$

where each term is defined as follows.

The first term,

$$\mathcal{H}_K(L) = -\frac{1}{2} \frac{\mu^2}{L^2},$$

is the constant term associated with the Earth's monopole, being

$$\mu = 398600.44 \text{ km}^3/\text{s}^2 \quad (2.3)$$

the mass parameter of the Earth.

The second term,

$$\tilde{\mathcal{H}}_0(L, G, H) = \frac{1}{4} \frac{\rho_0}{L^3} \frac{G^2 - 3H^2}{G^5}, \quad (2.4)$$

is the perturbative term associated with the Earth's oblateness, averaged over the orbital period of the spacecraft, being $\rho_0 = \mu^4 J_2 R^2$, with $J_2 = 1.08 \times 10^{-3}$ the coefficient of the second zonal harmonic in the geopotential and $R = 6378.14 \text{ km}$ the mean equatorial radius of the Earth.

Finally, the (truncated) secular perturbative term due to the Moon (see [22,23]) corresponds to

$$\begin{aligned} & \tilde{\mathcal{H}}_1(L, G, H, g, h, \Omega_M; i_M) \\ &= -\frac{\rho_1}{L^2} \sum_{m=0}^2 \sum_{p=0}^2 \tilde{D}_{m,p}(L, G, H) \sum_{s=0}^2 c_{m,s} F_{2,s,1}(i_M) \\ & \quad \times \left[U_2^{m,s}(\epsilon) \cos(\tilde{\psi}_{m,p,s}(g, h, \Omega_M)) + U_2^{m,s}(\epsilon) \cos(\tilde{\psi}_{m,p,-s}(g, h, \Omega_M)) \right]. \end{aligned} \quad (2.5)$$

In the expression above,

- the function $U_2^{m,\mp s}(\epsilon)$ corresponds to the Giacaglia function (see Table 7 in Appendix) being $\epsilon = 23.44^\circ$ the obliquity of the ecliptic with respect to the equatorial plane.
- One has that

$$\tilde{D}_{m,p}(L, G, H) = \tilde{F}_{m,p}(G, H) \tilde{X}_p(L, G), \quad (2.6)$$

with

$$\tilde{F}_{m,p}(G, H) = F_{2,m,p}\left(\arccos \frac{H}{G}\right), \quad \tilde{X}_p(L, G) = X_0^{2,2-2p}\left(\sqrt{1 - \frac{G^2}{L^2}}\right),$$

where $F_{2,m,p}(i)$ are the Kaula's inclination functions (see [22]), $X_0^{2,2-2p}(e)$ are the zero-order Hansen coefficients (see [23]) and we have used the definitions of inclination and eccentricity in terms of Delaunay coordinates, that is

$$H = G \cos i \quad \text{and} \quad G = L \sqrt{1 - e^2}. \quad (2.7)$$

- The angle

$$\tilde{\psi}_{m,p,s}(g, h, \Omega_M) = 2(1-p)g + mh + s\left(\Omega_M - \frac{\pi}{2}\right) - y_{|s|}\pi \quad (2.8)$$

defines the relative orientation satellite-Moon.

- The other constants are defined as

$$\begin{aligned} \rho_1 &= \frac{\mu\mu_M}{(1-e_M^2)^{3/2}}, & \epsilon_n &= \begin{cases} 1 & \text{if } n = 0 \\ 2 & \text{if } n \neq 0 \end{cases}, \\ c_{m,s} &= (-1)^{\lfloor m/2 \rfloor} \frac{\epsilon_m \epsilon_s}{2} \frac{(2-s)!}{(2+m)!}, & y_s &= \begin{cases} 0 & \text{if } s \text{ is even} \\ 1/2 & \text{if } s \text{ is odd} \end{cases}, \end{aligned} \quad (2.9)$$

where $\mu_M = 4902.87 \text{ km}^3/\text{s}^2$ is the mass parameter of the Moon.

Remark 2.1. By Kaula's inclination functions (see [22]), one obtains that

$$F_{2,s,1}(0) = \begin{cases} -\frac{1}{2} & \text{if } s = 0, \\ 0 & \text{if } s = 1, 2. \end{cases}$$

That is, assuming that the Moon lies on the ecliptic plane (i.e., $i_M = 0$), the Hamiltonian H in (2.2) is autonomous, because the angle $\tilde{\psi}_{m,p,s}(g, h, \Omega_M)$ does not depend on Ω_M for $s = 0$ (recall (2.1)).

In what follows, we will omit the Keplerian term of the Hamiltonian H_K , since it does not contribute to the variation of the orbital elements. Similarly, we omit the dependence of the Hamiltonian on the variable L , since it is a constant of motion.

2.1. A good system of coordinates

The Galileo constellation orbits at an inclination, $i \approx 56^\circ$, such that the dominant term in the lunar perturbation is the $2g + h$ resonance. This resonance is defined by the unperturbed Hamiltonian $\tilde{H}_0$ in (2.4) (i.e., $\alpha = 0$) and the condition $2\dot{g} + \dot{h} = 0$. Namely, $2\partial_g \tilde{H}_0 + \partial_h \tilde{H}_0 = 0$. This is satisfied provided that

$$5H^2 - G^2 - HG = 0,$$

that is, for all $G \neq 0$ (i.e., $e < 1$) and $H = G \cos i_\star$ that satisfy

$$5X^2 - 1 - X = 0, \quad X = \cos i_\star.$$

Hence, we can distinguish two situations:

- the prograde case for $i_\star = \arccos\left(\frac{1+\sqrt{21}}{10}\right) \simeq 56,06^\circ$,
- the retrograde case for $i_\star = \arccos\left(\frac{1-\sqrt{21}}{10}\right) \simeq 110,99^\circ$.

In what follows, $i_\star$ will be mentioned as the inclination of the “exact $2g + h$ ”-resonance and we will focus on the prograde case.

The unperturbed Hamiltonian highlights that $x = 2g + h$ is constant when $i = i_\star$, while it circulates for $i \neq i_\star$. Moreover, in a small enough neighborhood of the exact resonance the angular variables evolve at different rates: g and h are “fast” angles compared to x which undergoes a “slow” drift of order $\mathcal{O}(i - i_\star)$.

Slow-fast delaunay coordinates (y, x) . Instead of using the Delaunay action-angle variables, in order to take advantage of the timescales separation, we introduce the symplectic transformation $(G, H, g, h) = Y^{\text{Del}}(y, \Gamma, x, h)$ where

$$y = \frac{G}{2}, \quad \Gamma = H - \frac{G}{2}, \quad x = 2g + h. \quad (2.10)$$

Note that, the resonant action y does not depend on the inclination. Hence, the action-angle variables (y, x) are associated with the variation in eccentricity.

In slow-fast Delaunay variables, the secular Hamiltonian can be written as

$$H(y, \Gamma, x, h, \Omega_M; i_M) = H_0(y, \Gamma) + \alpha^3 H_1(y, \Gamma, x, h, \Omega_M; i_M)$$

where

$$H_0(y, \Gamma) = (\tilde{H}_0 \circ Y^{\text{Del}})(y, \Gamma) = \frac{\rho_0}{128} \frac{y^2 - 6y\Gamma - 3\Gamma^2}{L^3 y^5} \quad (2.11)$$

$$H_1(y, \Gamma, x, h, \Omega_M; i_M) = (\tilde{H}_1 \circ Y^{\text{Del}})(y, \Gamma, x, h, \Omega_M; i_M).$$

In the coordinates just defined, the $2g + h$ -resonance becomes the x -resonance, which is defined by

$$\omega(y, \Gamma) = \frac{\partial H_0(y, \Gamma)}{\partial y} = \frac{3}{128} \frac{\rho_0}{L^3} \frac{5\Gamma^2 + 8y\Gamma - y^2}{y^6} = 0.$$

This resonance takes place at the two lines

$$\Gamma = y \frac{-4 + \sqrt{21}}{5} \quad \text{and} \quad \Gamma = y \frac{-4 - \sqrt{21}}{5}, \quad (2.12)$$

with $(y, \Gamma) \neq (0, 0)$, that are associated with prograde and retrograde orbits, respectively.

Poincaré coordinates (η, ξ) . The slow-fast variables (2.10) derived from the Delaunay variables (as happens with the original Delaunay variables) are singular at $e = 0$. Thus, in order to study the dynamics of the secular Hamiltonian in a neighborhood of circular orbits, i.e., $0 < e \ll 1$, we introduce the Poincaré coordinates $(y, \Gamma, x, h) = Y^{\text{Poi}}(\eta, \Gamma, \xi, h)$ where

$$\xi = \sqrt{2L - 4y} \cos\left(\frac{x}{2}\right), \quad \eta = \sqrt{2L - 4y} \sin\left(\frac{x}{2}\right),$$

which are symplectic. Notice that ξ and η are respectively equivalent to $e \cos(x/2)$ and $e \sin(x/2)$ for quasi-circular orbits.

In these coordinates, the secular Hamiltonian becomes

$$\mathcal{H}(\eta, \Gamma, \xi, h, \Omega_M; i_M) = \mathcal{H}_0(\eta, \Gamma, \xi) + \alpha^3 \mathcal{H}_1(\eta, \Gamma, \xi, h, \Omega_M; i_M) \quad (2.13)$$

where

$$\begin{aligned} \mathcal{H}_0(\eta, \Gamma, \xi) &= (\mathcal{H}_0 \circ Y^{\text{Poi}})(\eta, \Gamma, \xi) \\ &= \frac{\rho_0}{2} \frac{(2L - \xi^2 - \eta^2)^2 - 24(2L - \xi^2 - \eta^2)\Gamma - 48\Gamma^2}{L^3(2L - \xi^2 - \eta^2)^5} \end{aligned} \quad (2.14)$$

$$\mathcal{H}_1(\eta, \Gamma, \xi, h, \Omega_M; i_M) = (\mathcal{H}_1 \circ Y^{\text{Poi}})(\eta, \Gamma, \xi, h, \Omega_M; i_M). \quad (2.15)$$

2.2. The hierarchy of models

In the present paper, we consider a hierarchy of models which stems from the secular Hamiltonian (2.13). They are what we call the *full coplanar (secular) Hamiltonian* and the *h-averaged (coplanar secular) Hamiltonian*. To construct these “intermediate models”, we rely on the fact that the model depends on two parameters: the inclination of the Moon with respect to the ecliptic plane, i_M , and the semi-major axis ratio between the satellite and the Moon, α .

In the companion paper [12], using the same hierarchy of models, we construct Arnold diffusion orbits for the secular Hamiltonian (2.13), that can lead to a drastic increase of the eccentricity of the satellite. A crucial point in our construction is the analysis of certain hyperbolic structures (normally hyperbolic invariant manifolds, stable and unstable invariant manifolds) of the coplanar secular Hamiltonian ($i_M = 0$) which are persistent in (2.13). Comparing the full coplanar model with the full inclined one ($i_M > 0$) allows us to construct the drifting orbits.

On the contrary, in the present paper we focus on the comparison between the full coplanar and the *h*-averaged Hamiltonians. There is now an extensive literature (see, e.g., [16,18]) analyzing the hyperbolic critical points of the *h*-averaged Hamiltonian and its invariant manifolds as a sign of existence of unstable motions for the full Hamiltonian (2.13). In the present paper, we compare analytically certain dynamics of the *h*-averaged and full coplanar models focusing both on what dynamics persist and which “deviations” has one model with respect to the other.

First reduction: the full coplanar model. Since the inclination of the Moon with respect to the ecliptic plane is relatively small (that is, $i_M = 5.15^\circ$), the first reduction that one can do to have an intermediate model is to take $i_M = 0$, which corresponds to assume that the orbit of the Moon is coplanar to that of the Earth.

When doing this reduction, the Hamiltonian $\mathcal{H}$ in (2.13) becomes Ω_M independent and thus autonomous (see Remark 2.1). Starting from (2.13), we define the 2 degrees of freedom Hamiltonian

$$\mathcal{H}_{\text{CP}}(\eta, \Gamma, \xi, h) = \mathcal{H}(\eta, \Gamma, \xi, h, \Omega_M; 0), \quad (2.16)$$

where the subscript CP stands for coplanar. Since this Hamiltonian is autonomous, $\mathcal{H}_{\text{CP}}$ is a first integral of the system. Moreover, it can be written as

$$\mathcal{H}_{\text{CP}}(\eta, \Gamma, \xi, h) = \mathcal{H}_0(\eta, \Gamma, \xi) + \alpha^3 \mathcal{H}_{\text{CP},1}(\eta, \Gamma, \xi, h),$$

where $\mathcal{H}_0$ is the Hamiltonian introduced in (2.14) and $\mathcal{H}_{\text{CP},1}$ is the Hamiltonian $\mathcal{H}_1$ in (2.15) with $i_M = 0$, that is,

$$\mathcal{H}_{\text{CP},1}(\eta, \Gamma, \xi, h) = \mathcal{H}_1(\eta, \Gamma, \xi, h, \Omega_M; 0). \quad (2.17)$$

See Appendix, and in particular (A.1), for the explicit expression of $\mathcal{H}_{\text{CP},1}$.

Second reduction: the h-averaged problem. When α is also taken as a small parameter, the autonomous Hamiltonian $\mathcal{H}_{\text{CP}}$ in (2.16) has a timescale separation between the slow and fast angles. Indeed,

$$\dot{\eta}, \dot{\xi} \sim \alpha^3 \quad \text{whereas} \quad \dot{h} \sim 1.$$

A classical way to exploit this feature is to simplify the Hamiltonian $\mathcal{H}_{\text{CP}}$ by averaging out the fast oscillations with respect to the longitude of the ascending node h . That is,

$$\mathcal{H}_{\text{AV}}(\eta, \Gamma, \xi) = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{H}_{\text{CP}}(\eta, \Gamma, \xi, h) dh.$$

This Hamiltonian coincides (after a canonical transformation) with the Hamiltonian $\mathcal{H}_R$ considered in [16] (see equation (26)).

Note that, since $\mathcal{H}_0$ in (2.14) is *h*-independent, $\mathcal{H}_{\text{AV}}$ can be written as

$$\mathcal{H}_{\text{AV}}(\eta, \Gamma, \xi) = \mathcal{H}_0(\eta, \Gamma, \xi) + \alpha^3 \mathcal{H}_{\text{AV},1}(\eta, \Gamma, \xi), \quad (2.18)$$

with

$$\mathcal{H}_{AV,1}(\eta, \Gamma, \xi) = \frac{1}{2\pi} \int_0^{2\pi} \mathcal{H}_{CP,1}(\eta, \Gamma, \xi, h) dh.$$

More specifically, the particular form of $\mathcal{H}_{CP,1}$ in (A.1) implies that

$$\begin{aligned} \mathcal{H}_{AV,1}(\eta, \Gamma, \xi) = \frac{\rho_1}{L^2} & \left[\frac{1}{2} U_2^{0,0} D_{0,1}(\eta, \Gamma, \xi) (8L^2 + 12L(\xi^2 + \eta^2) - 3(\xi^2 + \eta^2)^2) \right. \\ & \left. + \frac{1}{6} U_2^{1,0} D_{1,0}(\eta, \Gamma, \xi)(\xi^2 - \eta^2) \right], \end{aligned} \quad (2.19)$$

where the constants $U_2^{m,0}$ and the functions $D_{m,p}$ are given in Tables 7 and 9, respectively, in Appendix. See also this appendix for the deduction of this expression.

To analyze the dynamics of the h -averaged Hamiltonian, it will be also convenient to consider it in the slow-fast Delaunay coordinates (2.10). In these variables, it reads

$$\mathcal{H}_{AV}(y, \Gamma, x) = \mathcal{H}_0(y, \Gamma) + \alpha^3 \mathcal{H}_{AV,1}(y, \Gamma, x) \quad (2.20)$$

where $\mathcal{H}_0$ is the Hamiltonian introduced in (2.11),

$$\mathcal{H}_{AV,1}(y, \Gamma, x) = \frac{\rho_1}{L^2} \left[\frac{1}{2} U_2^{0,0} D_{0,1}(y, \Gamma) + \frac{1}{3} U_2^{1,0} D_{1,0}(y, \Gamma) \cos x \right],$$

and the functions $D_{m,p}$ are given in Table 8. We also refer to Appendix, for the deduction of the expression of $\mathcal{H}_{AV,1}$.

3. Main results

In this section, we present the main results of the paper, which deal with both the h -averaged Hamiltonian and the full coplanar Hamiltonian.

Remark 3.1. Along the paper we also present some numerical results. As a rule, the variables will be taken in non-dimensional units defined in such a way that the semi-major axis of the satellite (equal to 29 600 km) is the unit of distance and the corresponding orbital period is 2π .

However, in some places, with the purpose of keeping the exposition clear, we have kept the standard unit system. It will be pointed out along the text.

It is important to mention that the purpose of these numerical computations are presented just to illustrate the theoretical results along this work.

The results for the h -averaged Hamiltonian, in Section 3.1, are related with the existence and stability character of the stationary points. To be more precise,

- We study the stability character of the origin of the h -averaged Hamiltonian in Theorem 3.2. Then, in Proposition 3.4, we provide some properties of the eigenvalues of the linearization of $\mathcal{H}_{AV}$ at the origin (in Poincaré variables).
- Theorem 3.7 deals with the existence and the linear stability behavior of the eccentric critical points of the h -averaged Hamiltonian $\mathcal{H}_{AV}$. We state this result in Delaunay coordinates.

These results are proven for a specific range of the semi-major axis a , namely, we prove the results for values of $a \in [a_{\min}, a_{\max}]$ with

$$a_{\min} = 6378.14 \text{ km} \quad \text{and} \quad a_{\max} = 30000 \text{ km}. \quad (3.1)$$

The minimum value corresponds to a collision orbit with the Earth assuming $e = 0$. Note also that this interval contains the semi-major axis of Galileo $a = 29600$ km.

The full coplanar Hamiltonian $\mathcal{H}_{CP}$ can be seen as a perturbation of the h -averaged Hamiltonian $\mathcal{H}_{AV}$. From this point of view and recalling that the origin $\xi = \eta = 0$ is a fixed point of $\mathcal{H}_{AV}$, in Sections 3.2 and 3.3, we state the results for the Hamiltonian $\mathcal{H}_{CP}$ related to the periodic orbits emanating from the origin. Some of these results are perturbative results with respect to the parameter δ defined as

$$\delta = \rho \alpha^3 L^4, \quad \text{with} \quad \alpha = \frac{a}{a_M}, \quad \rho = \frac{\rho_1}{\rho_0}, \quad L = \sqrt{\mu a} \quad (3.2)$$

with ρ_0, ρ_1 introduced in Section 2.

In particular, in Section 3.2, we study the existence and main properties of the periodic orbits of the coplanar Hamiltonian:

- In Theorem 3.9, we characterize the energy levels E such that the coplanar Hamiltonian $\mathcal{H}_{CP}$ contains a periodic orbit lying in $\{\xi = \eta = 0\}$.

- For these values of energy, **E**, [Corollary 3.12](#) relates the Γ value of the point of the periodic orbit at the energy level $\{\mathcal{H}_{\text{CP}} = \mathbf{E}\}$ at the transverse section $\{h = 0\}$ with the corresponding parameter Γ such that the h -averaged Hamiltonian has the same energy level, namely $\{\mathcal{H}_{\text{AV}} = \mathbf{E}\}$. We see that, for small enough δ , the two values are order δ separated. Since for Galileo $\delta \sim 0.11$ (computed in the unities specified in [Remark 3.1](#)), this shows that the h -averaged model differs significantly from the full coplanar one. At circular motions, deviations in Γ are equivalent to deviations in (the cosine of the) inclination.
- [Proposition 3.14](#) computes a first order approximation with respect to δ of the periodic orbits and their corresponding periods.

Even when there is no physical meaning for values of $a < a_{\min}$, these results hold true for values of $a \in (0, a_{\max}]$.

In [Section 3.3](#), we perform the analysis of the stability character of the periodic orbits. This is the only study that requires the parameter δ to be small enough. In particular:

- [Theorem 3.17](#), and its refinement [Theorem 3.21](#), give an approximation, at least of order $\mathcal{O}(\delta^{3/2})$, of the eigenvalues of the monodromy matrix associated to the periodic orbits by means of the eigenvalues of the linearized part at the origin for the h -averaged Hamiltonian. As a consequence we are able to elucidate, for small values of δ , the character of the periodic orbits.

3.1. The critical points of the h -averaged system

We now present the results about the critical points of the h -averaged Hamiltonian. We notice that, to simplify the exposition, we state some results referred to the h -averaged Hamiltonian in Poincaré variables (see [\(2.18\)](#)), and others referred to the Hamiltonian expressed in slow-fast Delaunay variables (see [\(2.20\)](#)).

The specific range of the semi-major axis $[a_{\min}, a_{\max}]$ is given in [\(3.1\)](#). We also introduce the notation

$$L_{\min} = \sqrt{\mu a_{\min}}, \quad L_{\max} = \sqrt{\mu a_{\max}}, \quad \alpha_{\max} = \frac{a_{\max}}{a_M}, \quad \delta_{\max} = \rho \alpha_{\max}^3 L_{\max}^4, \quad (3.3)$$

where μ has been introduced in [\(2.3\)](#). Finally, we define

$$m = \frac{-4 + \sqrt{21}}{5} \sim 0.11651513891168, \quad (3.4)$$

the slope of the resonance prograde line, which is given by $\Gamma = my$ (see [\(2.12\)](#)).

In addition, we notice that, since we want to study the prograde resonance we keep our analysis to $\Gamma > 0$ and, from the expressions of $D_{m,p}$ in [Table 9](#), we deduce that

$$0 \leq \Gamma \leq \frac{L}{2}.$$

The first result deals with the character of the origin in Hamiltonian [\(2.18\)](#). Its proof is carried out in [Section 4.2](#).

Theorem 3.2. *The origin is a critical point of the h -averaged Hamiltonian $\mathcal{H}_{\text{AV}}$, see [\(2.18\)](#). Moreover, there exist two functions $\Gamma_-, \Gamma_+ : [L_{\min}, L_{\max}] \rightarrow (0, \frac{L}{2})$, such that the character of the critical point is described as follows for $L \in [L_{\min}, L_{\max}]$:*

- if either $\Gamma \in (0, \Gamma_-(L))$ or $\Gamma \in (\Gamma_+(L), \frac{L}{2})$, then $(0, 0)$ is a center.
- If $\Gamma \in (\Gamma_-(L), \Gamma_+(L))$, then $(0, 0)$ is a saddle.
- If $\Gamma = \Gamma_-(L)$ or $\Gamma = \Gamma_+(L)$, the origin is a degenerated equilibrium point with nilpotent linearization.

In addition, $L^{-1} \Gamma_{\pm}(L)$ can be expressed as

$$\frac{\Gamma_{\pm}(L)}{L} = \hat{\Gamma}_{\pm}(\delta), \quad \delta = \rho \alpha^3 L^4,$$

where $\hat{\Gamma}_{\pm} : [0, \delta_{\max}] \rightarrow \mathbb{R}$, see [\(3.2\)](#) and [\(3.3\)](#), and there exists a constant C_* such that for all $L \in [L_{\min}, L_{\max}]$,

$$\left| \hat{\Gamma}_{\pm}(\delta) - \frac{m}{2} \mp \frac{5U_2^{1,0}(m+3)\sqrt{3-2m-4m^2}}{3(16+20m)} \delta \right| \leq C_* \delta^2. \quad (3.5)$$

Remark 3.3. The values of $\hat{\Gamma}_{\pm}(\delta)$ are consistent with those of $i_{\min}(a)$, $i_{\max}(a)$ in [\[16\]](#). Indeed, recall that $H = G \cos i$ and $G = L\sqrt{1-e^2}$ (see [\(2.7\)](#)). Therefore, at circular motion one has $G = L$, then equation [\(2.10\)](#) and $\Gamma = L\hat{\Gamma}$ give $\Gamma L^{-1} =: \hat{\Gamma} = \cos i - \frac{1}{2}$.

For $\hat{\Gamma}_{\pm}(0) = \frac{m}{2}$, the corresponding inclination is $i_{\star} \approx 56.06^\circ$. That implies that the inclination is a decreasing function with respect to $\hat{\Gamma}$, when $\hat{\Gamma}$ is close to $\frac{m}{2}$. Therefore, the range of inclinations when the origin is unstable corresponds to $[i_+(\delta), i_-(\delta)]$ where $i_{\pm}(\delta)$ are the corresponding inclinations for $\hat{\Gamma}_{\pm}(\delta)$. In [\[16\]](#), using a truncated version of the h -averaged system, such values for the maximum and minimum inclination were computed numerically and also it was given an approximation of the form

$$i_{\min} \approx 56.06 - 0.00134 \left(\frac{a}{R} \right)^5, \quad i_{\max} \approx 56.06 + 0.00134 \left(\frac{a}{R} \right)^5. \quad (3.6)$$

Using [Theorem 3.2](#), we can also provide an estimation for $i_{\pm}(\delta)$ as

$$i_{\pm}(\delta) = i_{\star} - \frac{\partial_{\delta} \hat{\Gamma}_{\pm}(0)}{\sin i_{\star}} \frac{360}{2\pi} \delta + \mathcal{O}(\delta^2). \quad (3.7)$$

Table 1

The values i_+, i_- are the ones obtained with the perturbative scheme proposed in this work (the truncated up to $\mathcal{O}(\delta^2)$ formula in (3.7)), meanwhile $i_{\min}, i_{\max}$ are the ones obtained in [16].

semi-major axis (km)	δ	i_+	$i_{\min}$	i_-	$i_{\max}$
19000	0.012205013268684	55.84	55.75	56.28	56.38
24000	0.012205013268684	55.37	55.05	56.75	57.06
25450	0.052627069157991	55.13	54.76	56.99	57.41
29600	0.112002781052611	54.08	53.44	58.04	58.87

We obtain $i_{\pm}(\delta) \approx 56.06 \pm 17.6452\delta$ and comparing with the estimate (3.6)

$$i_{\pm}(\delta) \approx 56.06 \mp 0.00091805 \left(\frac{a}{R} \right)^5.$$

The small discrepancy is because the model used in [16] is a truncation of the h -averaged model. To compare our results with the ones in [16], in Table 1 is shown our approximations (discarding the $\mathcal{O}(\delta^2)$ terms of (3.7)) for the maximum and minimum inclination at some values of the semi-major axis a .

We denote by $\pm\lambda_{\text{AV}}(\Gamma; L)$ the eigenvalues of the linearization around the origin of the Hamiltonian system associated to (2.18). We assume that, in the saddle case, $\lambda_{\text{AV}}(\Gamma; L) > 0$ and in the center case, $-i\lambda_{\text{AV}}(\Gamma; L) > 0$. Notice that $\lambda_{\text{AV}}(\Gamma_-(L); L) = \lambda_{\text{AV}}(\Gamma_+(L); L) = 0$ where $\Gamma_-(L)$ and $\Gamma_+(L)$ are the functions defined in Theorem 3.2.

Proposition 3.4. For any $L \in [L_{\min}, L_{\max}]$ we have the following.

- There exists a unique $\Gamma_*(L) \in (\Gamma_-(L), \Gamma_+(L))$ such that for all $\Gamma \in (\Gamma_-(L), \Gamma_+(L))$, $\lambda_{\text{AV}}(\Gamma; L) \leq \lambda_{\text{AV}}(\Gamma_*(L); L)$ and $\Gamma_*(L)$ can be expressed as $\Gamma_*(L) = L\hat{\Gamma}_*(\delta)$ with

$$\left| \hat{\Gamma}_*(\delta) - \frac{m}{2} + \delta^2 \frac{25(U_2^{1,0})^2}{36} \frac{m(m+3)^2}{(m+4)^2} \right| \leq C_* \delta^3,$$

for some constant C_* depending only on $L_{\min}, L_{\max}$. In addition,

$$\lambda_{\text{AV}}(\Gamma_*(L); L) = \delta \frac{5\rho_0 U_2^{1,0}}{16L^7} (m+3) \sqrt{3-2m-m^2} + \delta^3 \frac{125\rho_0 (U_2^{1,0})^3 m^2 (m+3)^{\frac{5}{2}}}{288L^7 (1-m)^{\frac{1}{2}} (m+4)^2} + \mathcal{O}(\delta^4),$$

with $\mathcal{O}(\delta^4)$ uniform with respect to L , and

$$\frac{5\rho_0}{16L^7} U_2^{1,0} \delta (m+3) \sqrt{3-2m-m^2} \leq \lambda_{\text{AV}}(\Gamma_*(L); L) \leq \frac{15\sqrt{3}\rho_0}{16L^7} U_2^{1,0} \delta.$$

- For $\Gamma \in (0, \Gamma_-(L))$ the function $|\lambda_{\text{AV}}(\Gamma; L)|$ is smooth and decreasing with respect to Γ (and L fixed).
- For $\Gamma \in (\Gamma_+(L), \frac{L}{2})$, the function $|\lambda_{\text{AV}}(\Gamma; L)|$ is smooth and increasing with respect to Γ .

Finally, there exists a function $\hat{\lambda}_{\text{AV}} : \left[0, \frac{1}{2}\right] \times [0, \delta_{\max}] \rightarrow \mathbb{C}$ such that

$$\lambda_{\text{AV}}(\Gamma; L) = \frac{3\rho_0}{4L^7} \hat{\lambda}_{\text{AV}}\left(\frac{\Gamma}{L}; \delta\right), \quad \delta = \rho a^3 L^4.$$

This result is proven in Section 4.3 where we also provide some numerical computations to illustrate the results.

Remark 3.5. The eigenvalues $\lambda_{\text{AV}}(\Gamma; L)$ can be explicitly computed, see Remark 4.1, as function of Γ, L whilst $\Gamma_*(L)$ has to be computed numerically. In Fig. 1 it is depicted $\lambda_{\text{AV}}(\Gamma(L); L)$ as a function of the inclination i for some values of the semi-major axis a . Notice that, as claimed in Remark 3.3, $\Gamma = L \cos i - \frac{L}{2}$.

Remark 3.6. Proposition 3.4 provides a proof of the heuristic analysis performed in [16] (see, in particular, formulas (48)–(50)). Note however that their analysis is carried out for a truncated version of the Hamiltonian $\mathcal{H}_{\text{AV}}$ (called $\mathcal{H}_R$ in [16]) and therefore the location of the maximum of the eigenvalue is slightly different in these works. Indeed, in [16] the maximum eigenvalue is located at i_* , meanwhile in our analysis the difference between $i^*(\delta)$ and i_* is of $\mathcal{O}(\delta^2)$, namely

$$i^*(\delta) = i_* - \frac{\partial_{\delta} \hat{\Gamma}_*(0)}{\sin i_*} \frac{360}{2\pi} \delta^2 + \mathcal{O}(\delta^3) = i_* + 0.255496336000489 \delta^2 + \mathcal{O}(\delta^3).$$

In Table 2 are computed the first order approximation of the values of $i^*(\delta)$ for some values of the semi-major axis.

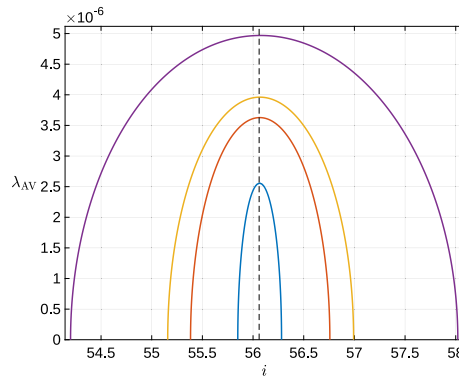


Fig. 1. In the figure, the eigenvalues $\lambda_{AV}(i; L)$ corresponding to the semi-major axis $a = 19000$ (in blue), $a = 24000$ (in red), $a = 25450$ (in orange) and $a = 29600$ (in magenta) km. The black straight line corresponds to $i_* = 56.06^\circ$. The unities are the ones specified in Remark 3.1. Notice that, through a change in the unities, it resembles Figure 2(c) in [16]. See also Fig. 9.

Table 2

The values of $i^*(\delta)$ (inclination having the maximum positive eigenvalue) for several values of the semi-major axis. They have been numerically computed just to illustrate the variation of $i^*(\delta) - i_*$ with respect to δ^2 .

semi-major axis (km)	δ^2	$i^*(\delta)$
19000	0.000148962348889	56.06466
24000	0.001540468844889	56.06501
25450	0.002769608408160	56.06533
29600	0.012544622963519	56.06782

To finish with the results related to the equilibrium points of the h -averaged system, we state the following theorem concerning “non-circular” critical points, namely, those critical points of the h -averaged system (2.20) (in slow-fast Delaunay coordinates) satisfying $e \neq 0$.

Note that we are interested in a region of the phase space satisfying the following conditions:

- We want the satellite to be in a prograde “elliptic regime”. Equivalently, we want the Delaunay coordinates to be well-defined and, therefore, we consider

$$(x, y) \in \mathbb{T} \times (0, L/2). \quad (3.8)$$

- We want the actions y and Γ to be close to the resonance (see, for instance, the first resonance (2.12)), namely the prograde resonance, so that the h -averaged Hamiltonian is a good approximation of the coplanar model. For this reason, we consider

$$\frac{\Gamma}{y} \in \left(0, \frac{1}{2}\right). \quad (3.9)$$

We recall that the resonance prograde line is $\Gamma = my$ with m defined in (3.4) and we notice that, from Table 8 the values of Γ, y are restricted to $\Gamma < y$.

- To avoid collision, the semi-major axis a and the eccentricity e have to satisfy $a(1 - e) \geq R = a_{\min}$. Then $e \leq 1 - \frac{R}{a}$. Moreover, since $\frac{y}{L} = \frac{1}{2}\sqrt{1 - e^2}$,

$$\frac{y}{L} \geq \frac{1}{2}\sqrt{2\frac{R}{a} - \frac{R^2}{a^2}} \geq \frac{1}{2}\sqrt{2\frac{R}{a_{\max}} - \frac{R^2}{a_{\max}^2}} =: \hat{y}_{\text{col}} = 0.308224183446436. \quad (3.10)$$

However, to understand the dynamics of H_{AV} , we go further in our analysis and we consider values of $\hat{y} \geq \hat{y}_{\min}$ with $\hat{y}_{\text{col}} > \hat{y}_{\min} > 0$, such that $\hat{y}_{\min}$ is the unique value satisfying that, for $\hat{\Gamma} = m\hat{y}_{\min}$, the h -averaged Hamiltonian H_{AV} has an equilibrium point at $(\cdot, \hat{y}_{\min})$. Such value can be explicitly computed as (see (5.6))

$$\hat{y}_{\min} = \sqrt{\frac{30U_2^{0,0}m(1+m)\sqrt{3-2m-m^2} + 20U_2^{1,0}(3+m)m^2}{24U_2^{0,0}(1-3m)\sqrt{3-2m-m^2} + 20U_2^{1,0}(3+m)(12-8m)}} \quad (3.11)$$

$$= 0.116589071022807.$$

The complete proof of Theorem 3.7 below relies on elementary analytic tools. However, due to the complexity of the expression of the functions involved in the definition of the Hamiltonian (2.18), it is not straightforward at all and it is postponed to Section 5

Table 3
Summary of the results in [Theorem 3.7](#).

$L \in [L_{\min}, L_*)$				
$\Gamma/L \in$	$[\hat{f}_1, \hat{f}_0)$	$[\hat{f}_0, \hat{f}_2]$	$(\hat{f}_2, \hat{f}_3]$	
Saddle	One at $x = \pi$			
Center	One at $x = 0$			

$L = L_*$				
$\Gamma/L \in$	$[\hat{f}_1, \hat{f}_0)$	$[\hat{f}_0, \hat{f}_2)$	$\hat{f}_2$	$(\hat{f}_2, \hat{f}_3]$
Saddle	One at $x = \pi$		Par. at $x = \pi$	
Center	One at $x = 0$			

$L \in (L_*, L_{\max}]$				
$\Gamma/L \in$	$[\hat{f}_1, \hat{f}_0)$	$[\hat{f}_0, \hat{f}_2)$	$[\hat{f}_2, \hat{f}_*)$	$(\hat{f}_*, \hat{f}_3]$
Saddle	One at $x = \pi$			Par. at $x = \pi$
Center		One at $x = 0$	Two at $x = 0, \pi$	One at $x = 0$

Theorem 3.7. *There exist two constants³*

$$L_* \sim 97590.90325766560, \quad \hat{F}_0 \sim 0.013584391815073,$$

two decreasing functions $\hat{F}_1(L)$, $\hat{F}_2(L)$ and one increasing function $\hat{F}_3(L)$ defined for $L \in [L_{\min}, L_{\max}]$ such that the Hamiltonian (2.20) has the following critical points satisfying (3.8), (3.9), (3.11):

- For $L \in [L_{\min}, L_*]$ and $\Gamma \in (0, L/2]$,
 1. if $\Gamma \notin [L\hat{F}_1(L), L\hat{F}_3(L)]$, has no critical points.
 2. If $\Gamma \in [L\hat{F}_1(L), L\hat{F}_0]$, has a unique critical point that is a saddle and of the form (π, y) .
 3. If $\Gamma \in [L\hat{F}_0, L\hat{F}_2(L)]$, has only two critical points $(0, y_1), (\pi, y_2)$, which are of center and saddle type respectively, except when $L = L_*$ and $\Gamma = L_*\hat{F}_2(L_*)$ that (π, y_2) is parabolic.
 4. If $\Gamma \in (L\hat{F}_2(L), L\hat{F}_3(L)]$, has a unique critical point that is a center of the form $(0, y)$.
- If $L \in (L_*, L_{\max}]$ and $\Gamma \in (0, L/2]$, there exists a function $\hat{F}_*(L) \in [\hat{F}_2(L), \hat{F}_3(L)]$ such that,
 1. if $\Gamma \notin [L\hat{F}_1(L), L\hat{F}_3(L)]$, has no critical points.
 2. If $\Gamma \in [L\hat{F}_1(L), L\hat{F}_0]$, has a unique critical point that is a saddle and of the form (π, y) .
 3. If $\Gamma \in [L\hat{F}_0, L\hat{F}_2(L)]$, has only two critical points $(0, y_1), (\pi, y_2)$, of center and saddle type respectively.
 4. If $\Gamma \in [L\hat{F}_2(L), L\hat{F}_*(L)]$, has only three critical points $(0, y), (\pi, y_{1,2})$ where $y_2 > y_1 > 0$ with $(0, y), (\pi, y_2)$ centers and (π, y_1) a saddle.
 5. If $\Gamma = L\hat{F}_*(L)$, has only two critical points $(0, y), (\pi, y_*)$ with $(0, y)$ a center and (π, y_*) parabolic.
 6. If $\Gamma \in (L\hat{F}_*(L), L\hat{F}_3(L)]$, has a unique critical point that is a center and of the form $(0, y)$.

In addition, since in a one degree Hamiltonian system two saddles cannot coexist without a center, $\Gamma_-(L) = L\hat{F}_2(L)$ and $\Gamma_+(L) = L\hat{F}_3(L)$ with $\Gamma_{\pm}(L)$ in [Theorem 3.2](#).

This result classifies the number and character of the eccentric fixed points ($e \neq 0$) for the h -averaged Hamiltonian. Before stating the results for the full coplanar Hamiltonian, we provide some clarifications of [Theorem 3.7](#):

- In it is written an schematic summary of the result.
- For different values of L (or a) and Γ , the families of fixed points $(0, y)$, (π, y_1) , (π, y_2) and their stability can be computed numerically. The numerical results are in agreement with [Theorem 3.7](#) and are depicted in [Figs. 2](#). To be more concrete, in [Figs. 2\(a\), 2\(b\)](#) [2\(c\)](#) there are two families of fixed points of the form $(0, y)$ and there are no fixed points (π, y) of center type (in green) because the corresponding L is less than L_* . Compare with [Figs. 2\(d\), 2\(e\)](#) and [2\(f\)](#) where such family of fixed points appears.
- See also [Fig. 3](#) for a more accurate picture where it can be seen that the red and blue curves are different in concordance with the case $\Gamma \in [L\hat{F}_0, L\hat{F}_2(L)]$ and moreover, there are values $\Gamma \in [L\hat{F}_1(L), L\hat{F}_0]$ having a unique saddle fixed point located at (π, y) as in [Theorem 3.7](#).
- In [Fig. 4](#) we show the phase portraits of the h -averaged Hamiltonian for different values of Γ/L and $a = 29600$ km (the semi-major axis corresponding to Galileo). They correspond to some of the different cases in [Theorem 3.7](#) and are also in

³ The value L_* (in international unities) corresponds to a semi-major axis $a_* \sim 23893.56218133389$ km. This value is not used along the proof of [Theorem 3.7](#); it has been computed numerically just for completeness.

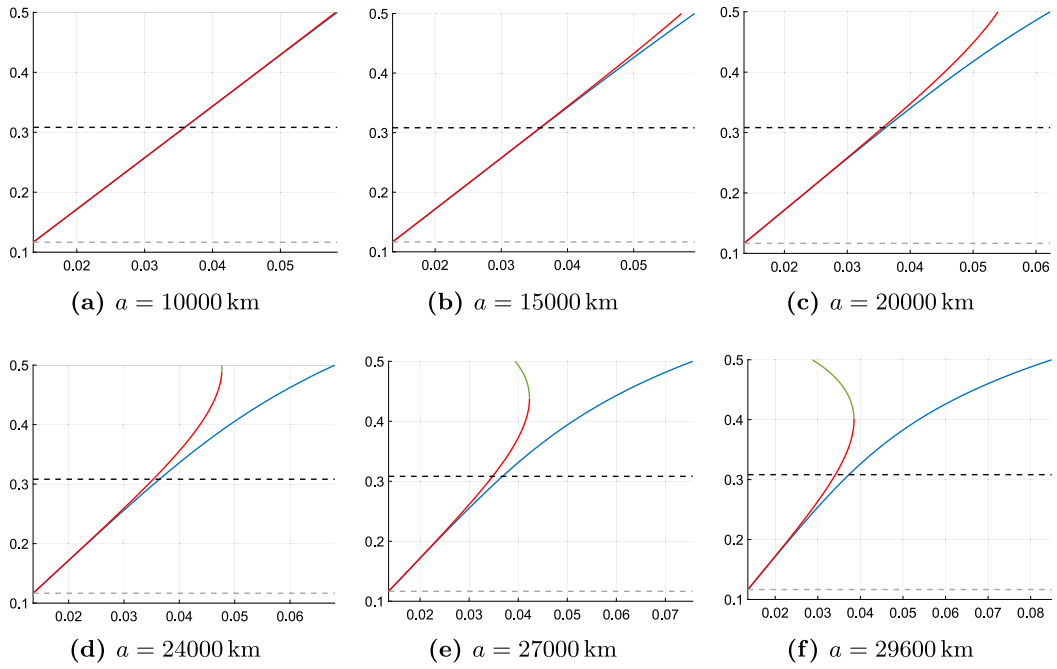


Fig. 2. Evolution in the $(\Gamma/L, y/L)$ plane of the three families of fixed points in [Theorem 3.7](#) for different values of semi-major axis, a . In red, the family of fixed points of the form (π, y) that are saddle points, in green the points of the form (π, y) that are of center type and in blue the fixed points of the form $(0, y)$ of center type. The black and gray dotted lines correspond to $\hat{y}_{\text{col}}$ and $\hat{y}_{\text{min}}$ respectively.

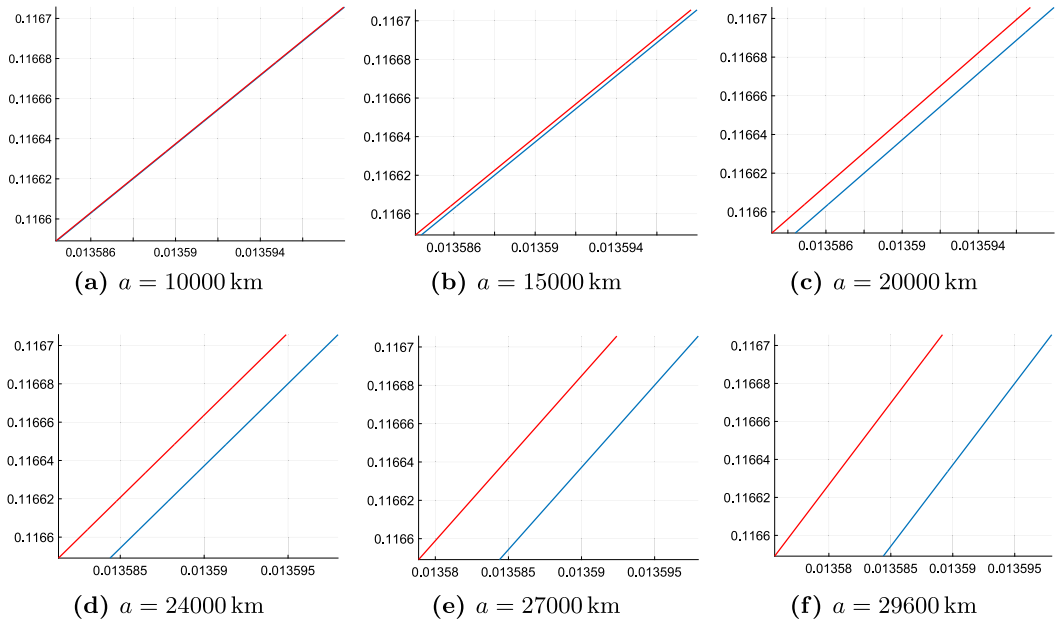


Fig. 3. A zoom in the $(\Gamma/L, y/L)$ plane of [Fig. 2](#) for the fixed point location (in red the saddle fixed points (π, y) and in blue the center fixed points $(0, y)$).

agreement with [Theorem 3.2](#). We have plotted the phase portrait in coordinates $(e \cos x, e \sin x)$ which are a good choice when e is not close to 1. However, when $e \sim 1$ the phase portrait is unclear. In [Figs. 4\(a\)](#) and [4\(f\)](#) the origin is the unique fixed point (see [Theorem 3.2](#)). In [Fig. 4\(b\)](#) there are three fixed points, two centers at $(0, 0)$ and $(0, y)$, and one saddle (at (π, y)). In this case the fixed points $(\cdot, y)$ are both close to $e = 1$. When $\Gamma/L = 0.037$ ([Fig. 4\(c\)](#)), there are four equilibrium points, two saddles and two centers. Finally, in [Figs. 4\(d\)](#) and [4\(e\)](#) there are one saddle fixed point at the origin and one center at $(0, y)$.

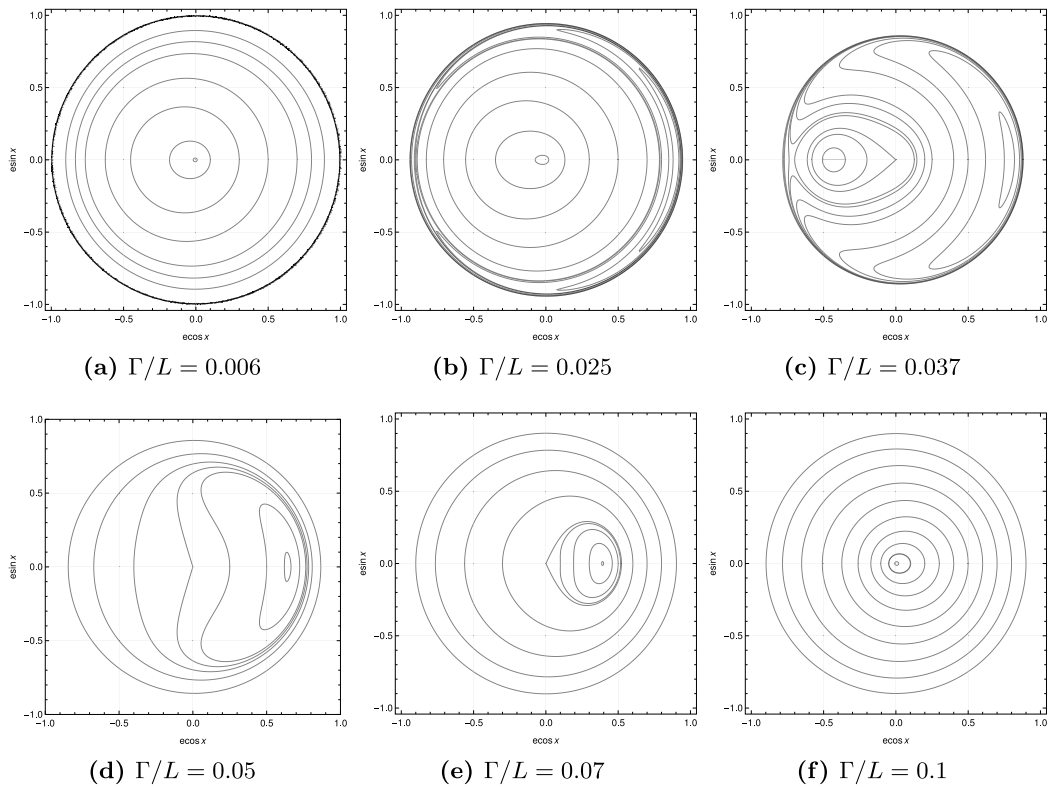


Fig. 4. Phase portrait of the h -averaged Hamiltonian for different values of Γ/L in the $(e \cos x, e \sin x)$ plane.

Remark 3.8. Along the proof of [Theorem 3.7](#), we check some extra properties of the functions involved. In particular, we deduce that

$$\hat{r}_1(L) \in \left[\frac{2}{5} \hat{r}_0, \hat{r}_0 \right], \quad \hat{r}_2(L) \in \left[\frac{1}{5} m, \frac{1}{2} m \right], \quad \hat{r}_3(L) \in \left[\frac{1}{2} m, \frac{3}{4} m \right].$$

The constant $\hat{r}_0$ is defined as $\hat{r}_0 = m \hat{y}_{\min}$ with $\hat{y}_{\min}$ defined in [\(3.11\)](#).

3.2. Periodic orbits of the full coplanar hamiltonian

The full coplanar Hamiltonian $\mathcal{H}_{\text{CP}} = \mathcal{H}_0 + \alpha^3 \mathcal{H}_{\text{CP},1}$ (see [\(2.16\)](#)) can be seen, when α^3 is small enough, as a perturbation of the h -averaged one. Enlarging the dimension, one can ask if, in some way, the equilibrium points of the h -averaged system, survive as periodic orbits in the coplanar system and, in this case, if they conserve the character (saddle or elliptic type) of the equilibrium point.

From expressions of $\mathcal{H}_0$ in [\(2.14\)](#) and $\mathcal{H}_{\text{CP},1}$ in [\(A.1\)](#) in [Appendix](#), $\Pi = \{\eta = \xi = 0\}$ is invariant by the flow associated to the full coplanar Hamiltonian $\mathcal{H}_{\text{CP}}$. From now on, we focus our analysis on these types of periodic orbits, which are the ones used in [\[12\]](#) to obtain drift in eccentricity in the secular Hamiltonian [\(2.13\)](#).

The results presented in this section attempt to give a theoretical framework of the previous numerical study in [\[12\]](#) by means of perturbative arguments with respect to the (small) parameter $\delta = \rho \alpha^3 L^4$ defined in [\(3.2\)](#).

We emphasize that on the invariant plane $\Pi = \{\xi = \eta = 0\}$, the eccentricity is $e = 0$. Therefore, $G = L$ (see [\(2.7\)](#)) and taking [\(2.10\)](#) into account, one has that

$$\Gamma = H - \frac{L}{2} = L \cos i - \frac{L}{2}. \quad (3.12)$$

Then, since L is a constant for the secular Hamiltonian, one can recover the inclination from the variable Γ . In particular, deviations in Γ correspond to deviations of $\cos i$. We express our results (and figures) by means of the variable Γ in general, because this is the variable used in the theoretical proofs.

The following result rigorously proves the existence of periodic orbits for a uniform set of initial conditions and also describes the suitable energy level of the Hamiltonian $\mathcal{H}_{\text{CP}}$ where they lie. It is proven in [Section 6.1](#).

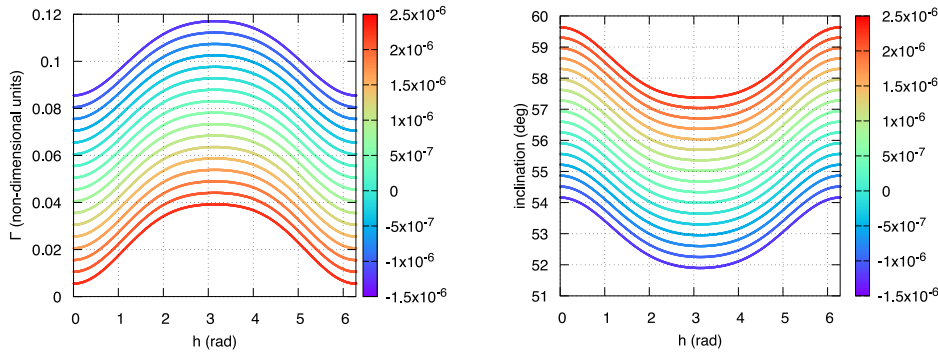


Fig. 5. Examples of periodic orbits for $\mathcal{H}_{CP}$ in the case $a = 29\,600$ km (Galileo semi-major axis) in non-dimensional units. On the left $\hat{r} = L^{-1}\Gamma$ as a function of h and on the right, the inclination along the period orbit, as a function of h . The colorbar reports the value of $\mathcal{H}_{CP}$.

Theorem 3.9. For $L \in (0, L_{\max}]$, the functions $\mathbf{E}_{\min, \max}$ defined by

$$\mathbf{E}_{\max}(L) = \mathcal{H}_{CP}(0, 0, 0, 0), \quad \mathbf{E}_{\min}(L) = \mathcal{H}_{CP}(0, 0.49L, 0, \pi)$$

can be written as

$$\mathbf{E}_{\max}(L) = \frac{\rho_0}{16L^6} \hat{\mathbf{E}}_{\max}(\delta), \quad \mathbf{E}_{\min}(L) = \frac{\rho_0}{16L^6} \hat{\mathbf{E}}_{\min}(\delta),$$

with $\delta = \rho\alpha^3 L^4$ introduced in (3.2), and $\hat{\mathbf{E}}_{\min, \max} : [0, \delta_{\max}] \rightarrow \mathbb{R}$.

Then, for any energy level $\mathbf{E}$ such that $\mathbf{E}_{\min}(L) \leq \mathbf{E} \leq \mathbf{E}_{\max}(L)$, there exists a periodic orbit $(0, \Gamma(t; L), 0, h(t; L))$ satisfying that $h(0; L) = 0$, $\Gamma(t; L) \in [0, 0.49L]$ for all $t \geq 0$,

$$\mathcal{H}_{CP}(0, \Gamma(t; L), 0, h(t; L)) = \mathbf{E}$$

and the differential equations

$$\begin{aligned} \dot{h} &= \frac{dh}{dt} = -\frac{3\rho_0(L+2\Gamma)}{4L^8} - \alpha^3 \frac{3\rho_1}{8L^4} \left(U_2^{0,0}(L+2\Gamma) + \frac{4}{3} U_2^{1,0} \frac{L^2 - 4L\Gamma - 4\Gamma^2}{\sqrt{(L-2\Gamma)(3L+2\Gamma)}} \cos h \right. \\ &\quad \left. - \frac{1}{3} U_2^{2,0}(L+2\Gamma) \cos(2h) \right) \\ \dot{\Gamma} &= \frac{d\Gamma}{dt} = -\alpha^3 \frac{\rho_1}{16L^4} \left(2U_2^{1,0} \sqrt{(L-2\Gamma)(3L+2\Gamma)}(2\Gamma+L) \sin h \right. \\ &\quad \left. + U_2^{2,0}(L-2\Gamma)(3L+2\Gamma) \sin(2h) \right). \end{aligned}$$

Remark 3.10. The values $\mathbf{E}_{\min}(L), \mathbf{E}_{\max}(L)$ can be easily computed. For the case $a = 29\,600$ km these are

$$\mathbf{E}_{\min}(1) = -2.558100888960067 \cdot 10^{-5}, \quad \mathbf{E}_{\max}(1) = 2.477266122798186 \cdot 10^{-6}.$$

Recall that the units are the ones in Remark 3.1, namely the semi-major axis of Galileo is $a = 1$ and the period of Galileo is 2π (this implies $L = 1$). Compare with the numerical results in [12] for hyperbolic periodic orbits in Fig. 5.

We emphasize that, by Theorem 3.9, the initial conditions $\Gamma(0; L) = \Gamma_0$ and $h(0; L) = 0$ for a given value $L \in (0, L_{\max}]$ of a periodic orbit are the ones satisfying

$$\mathbf{E}_{\min}(L) \leq \mathcal{H}_{CP}(0, \Gamma_0, 0, 0) \leq \mathbf{E}_{\max}(L), \quad (3.13)$$

with $\mathbf{E}_{\min, \max}(L)$ defined in Theorem 3.9.

For $\sigma \in \left[0, \frac{1}{2}\right)$, we define

$$\begin{aligned} \hat{a}_0(\sigma; \delta) &= -1 - 2\sigma - \delta \frac{1}{2} U_2^{0,0}(1+2\sigma), \\ \hat{e}_0(\sigma; \delta) &= -\frac{1}{24\hat{a}_0(\sigma; \delta)} \left[4U_2^{1,0} \sqrt{(1-2\sigma)(3+2\sigma)}(2\sigma+1) + U_2^{2,0}(1-2\sigma)(3+2\sigma) \right]. \end{aligned} \quad (3.14)$$

Remark 3.11. After some tedious computations (see Section 6.1.1) one can check that

$$\hat{a}_0\left(\frac{\Gamma}{L}; \rho\alpha^3 L^4\right) = \frac{4L^7}{3\rho_0} \partial_{\Gamma} \mathcal{H}_{AV}(0, \Gamma, 0),$$

$$\hat{c}_0\left(\frac{\Gamma}{L}; \rho\alpha^3 L^4\right) = \frac{\rho}{L^5} \frac{\mathcal{H}_{\text{CP},1}(0, \Gamma, 0, 0) - \mathcal{H}_{\text{AV},1}(0, \Gamma, 0)}{\partial_\Gamma \mathcal{H}_{\text{AV}}(0, \Gamma, 0)}.$$

The next result is a corollary of [Theorem 3.9](#).

Corollary 3.12. *There exists a constant $C_* > 0$ such that for a given $L \in (0, L_{\max}]$ and an energy level $E \in [E_{\min}(L), E_{\max}(L)]$, if $\Gamma_{\text{AV}}^E, \Gamma_{\text{CP}}^E$ satisfy*

$$E = \mathcal{H}_{\text{AV}}(0, \Gamma_{\text{AV}}^E, 0) = \mathcal{H}_{\text{CP}}(0, \Gamma_{\text{CP}}^E, 0, 0)$$

then there exists a function $\gamma_2 : \left(0, \frac{1}{2}\right) \times [0, \delta_{\max}] \rightarrow \mathbb{R}$ such that,

$$\Gamma_{\text{AV}}^E = \Gamma_{\text{CP}}^E + L\delta\hat{c}_0(\hat{\Gamma}_{\text{CP}}^E; \delta) + L\delta^2\gamma_2(\hat{\Gamma}_{\text{CP}}^E; \delta), \quad \hat{\Gamma}_{\text{CP}}^E = L^{-1}\Gamma_{\text{CP}}^E, \quad \delta = \rho\alpha^3 L^4,$$

with $\hat{c}_0$ in (3.14) and $|\gamma_2(\hat{\Gamma}_{\text{CP}}^E; \delta)| \leq C_$.*

The proof of this result is postponed to Section 6.1.1.

Remark 3.13. [Corollary 3.12](#) gives us a first order in δ correction of the constant value of Γ to consider in the h -averaged Hamiltonian with respect to the initial condition of the periodic orbit in order to conserve the same energy level. Such a correction will have a relevant role when the stability of the periodic orbit is analyzed.

As already mentioned, for Galileo $\delta \sim 0.11$. Then, [Corollary 3.12](#) shows that the two periodic orbits have a significant separation. Namely, [Corollary 3.12](#) implies that the inclination between the two periodic orbits is significantly different, see (3.12).

Our next goal is to provide a first order approximation, with respect to the parameter δ , defined in (3.2), of the periodic orbits $(0, \Gamma(t; L), 0, h(t; L))$ of the full coplanar Hamiltonian system $\mathcal{H}_{\text{CP}}$ emanating from the origin of the h -averaged Hamiltonian (2.18).

To keep our analysis as concrete as possible, before stating the result, we introduce the functions defined for $\sigma \in \left[0, \frac{1}{2}\right)$

$$\begin{aligned} \hat{c}_1(\sigma; \delta) &= \frac{1}{6\hat{a}_0(\sigma; \delta)} U_2^{1,0} \sqrt{(1-2\sigma)(3+2\sigma)(2\sigma+1)}, \\ \hat{c}_2(\sigma; \delta) &= \frac{1}{24\hat{a}_0(\sigma; \delta)} U_2^{2,0} (1-2\sigma)(3+2\sigma), \\ \hat{d}_1(\sigma; \delta) &= \frac{1}{3\hat{a}_0(\sigma; \delta)} U_2^{1,0} \frac{5-12\sigma-12\sigma^2}{\sqrt{(1-2\sigma)(3+2\sigma)}}, \\ \hat{d}_2(\sigma; \delta) &= \frac{1}{24\hat{a}_0(\sigma; \delta)} U_2^{2,0} \frac{1-12\sigma-12\sigma^2}{1+2\sigma}. \end{aligned} \quad (3.15)$$

Proposition 3.14. *There exists a constant $C_* > 0$ such that if $L \in (0, L_{\max}]$ and Γ_0 satisfy (3.13), then the periodic orbit $(0, \Gamma(t; L), 0, h(t; L))$ of $\mathcal{H}_{\text{CP}}$ with initial condition $\Gamma(0; L) = \Gamma_0 := L\hat{\Gamma}_0$ and $h(0; L) = 0$ is of the form*

$$\Gamma(t; L) = L\hat{\Gamma}\left(\frac{3\rho_0}{4L^7}t; \delta\right), \quad h(t; L) = \hat{h}\left(\frac{3\rho_0}{4L^7}t; \delta\right)$$

where $\delta = \rho\alpha^3 L^4$ as in (3.2) and the functions $\hat{\Gamma}, \hat{h} : [0, \delta_{\max}] \rightarrow \mathbb{R}$ are of the form

$$\hat{\Gamma}(s; \delta) = \hat{\Gamma}_0 + \delta\hat{\Gamma}_1(s; \delta) + \delta^2\hat{\Gamma}_2(s; \delta), \quad \hat{h}(s) = \hat{h}_0(s; \delta) + \delta\hat{h}_1(s; \delta) + \delta^2\hat{h}_2(s; \delta)$$

with $|\hat{\Gamma}_2(s; \delta)|, |\hat{h}_2(s; \delta)| \leq C_$ and*

$$\begin{aligned} \hat{\Gamma}_1(s; \delta) &= \hat{c}_0(\hat{\Gamma}_0; \delta) + \hat{c}_1(\hat{\Gamma}_0; \delta) \cos(\hat{a}_0(\hat{\Gamma}_0; \delta)s) + \hat{c}_2(\hat{\Gamma}_0; \delta) \cos(2\hat{a}_0(\hat{\Gamma}_0; \delta)s), \\ \hat{h}_0(s; \delta) &= \hat{a}_0(\hat{\Gamma}_0; \delta)s, \\ \hat{h}_1(s; \delta) &= \partial_{\hat{\Gamma}}\hat{a}_0(\hat{\Gamma}_0; \delta)\hat{c}_0(\hat{\Gamma}_0; \delta)s + \hat{d}_1(\hat{\Gamma}_0; \delta) \sin(\hat{a}_0(\hat{\Gamma}_0; \delta)s) + \hat{d}_2(\hat{\Gamma}_0; \delta) \sin(2\hat{a}_0(\hat{\Gamma}_0; \delta)s). \end{aligned}$$

The constants $\hat{a}_0, \hat{c}_{0,1,2}$ and $\hat{d}_{1,2}$ are defined in (3.14) and (3.15).

The period $\mathcal{T}(\Gamma_0; L)$ of the periodic orbit satisfies that

$$\mathcal{T}(\Gamma_0; L) = \frac{4L^7}{3\rho_0} \hat{\mathcal{T}}(\hat{\Gamma}_0; \delta), \quad \hat{\mathcal{T}}(\hat{\Gamma}_0; \delta) = \frac{2\pi + \delta^2\hat{\mathcal{T}}_2(\hat{\Gamma}_0; \delta)}{\left|\hat{a}_0(\hat{\Gamma}_0; \delta) + \delta\partial_{\hat{\Gamma}}\hat{a}_0(\hat{\Gamma}_0; \delta)\hat{c}_0(\hat{\Gamma}_0; \delta)\right|} \quad (3.16)$$

with $\hat{\mathcal{T}} : \left(0, \frac{1}{2}\right) \times [0, \delta_{\max}] \rightarrow \mathbb{R}$ and $|\hat{\mathcal{T}}_2(\hat{\Gamma}_0; \delta)| \leq M$.

This proposition is proven in Section 6.2.1. Notice that the inclination $i(h)$ along the periodic orbit with initial condition $(\Gamma, h) = (\Gamma_0, 0)$, $\Gamma_0 = L\hat{\Gamma}_0$, satisfies

$$\cos i(h) = \frac{1}{2} + \hat{\Gamma}_0 + \delta\hat{c}_0(\hat{\Gamma}_0; \delta) - \delta\frac{U_2^{1,0}\sqrt{3}}{6} \cos h - \delta\frac{U_2^{2,0}}{8} \cos(2h) + \mathcal{O}(\delta^2, \delta L^{-1}\Gamma_0).$$

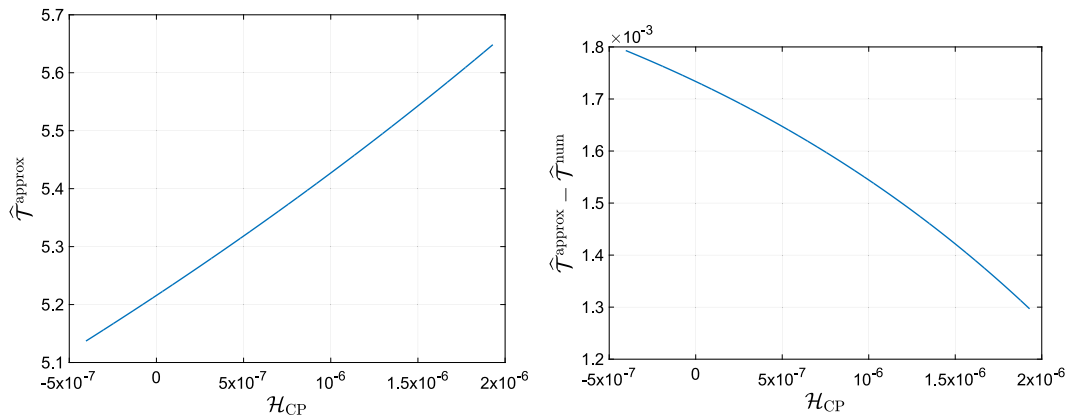


Fig. 6. On the left, for $a = 29600$, km, the approximated period $\hat{T}^{\text{approx}}$ in (3.17) of the hyperbolic periodic orbits as a function of $\mathcal{H}_{\text{CP}}$, indeed, $\mathcal{H}_{\text{CP}}(0, \Gamma_0, 0, 0) = \mathbf{E}$ implies that $(\Gamma_0, 0)$ is the initial condition for the hyperbolic periodic orbit lying in that energy level. On the right, the absolute value between the approximated period, $\hat{T}^{\text{approx}}$, and the period computed numerically $\hat{T}^{\text{num}}$ (with the obvious scaling). The quantities are in the unities of Remark 3.1. Notice that for Galileo, $\delta = 0.112002781052610$ and the maximum error computed in the left panel is $\sim 1.8 \cdot 10^{-3}$ which is $\mathcal{O}(\delta^2)$. The maximum error when the period is computed in years is about 3 days.

Letting i_0 be such that $\cos i_0 = \frac{1}{2} + \hat{F}_0 + \delta \hat{c}_0(\hat{F}_0; \delta)$, that is, $i_0 = 60^\circ + \mathcal{O}(\hat{F}_0, \delta)$, one deduces that

$$\begin{aligned} i(h) &= i_0 + \delta \frac{U_2^{1,0}}{3} \cos h + \delta \frac{U_2^{2,0} \sqrt{3}}{12} \cos(2h) + \mathcal{O}(\delta^2, \delta L^{-1} \Gamma_0) \\ &= i_0 + 5.4398 \cdot 10^{-4} \left(\frac{a}{M}\right)^5 \cos h + 3.0638 \cdot 10^{-4} \left(\frac{a}{M}\right)^5 \cos(2h) + \mathcal{O}(\delta^2, \delta L^{-1} \Gamma_0). \end{aligned}$$

Compare with (66) in [16].

Remark 3.15. We notice that the dependence on L of the periodic orbit comes through the dependence on suitable functions with respect to the “small” parameter δ . In [12], the period of the hyperbolic periodic orbits are numerically computed by some values of Γ/L for Galileo ($L = 1$). Indeed, in Fig. 6 there is the comparison between the numerically computed value of the period and the approximated value, $\hat{T}^{\text{approx}}$, given by formula (3.16) with $\hat{T}_2 \equiv 0$, namely

$$\hat{T}^{\text{approx}}(\Gamma_0; L) = \frac{2\pi}{\left| \hat{a}_0(\hat{F}_0; \delta) + \delta \partial_{\hat{F}} \hat{a}_0(\hat{F}_0; \delta) \hat{c}_0(\hat{F}_0; \delta) \right|} \quad (3.17)$$

Remark 3.16. From definition (3.14) of $\hat{a}_0$, we obtain that, if δ is small enough

$$\hat{T}(\hat{F}_0; \delta) = \frac{2\pi}{2\hat{F}_0 + 1} + \mathcal{O}(\delta) \in (\pi + \mathcal{O}(\delta), 2\pi + \mathcal{O}(\delta)).$$

3.3. Character of the periodic orbits of the full coplanar Hamiltonian

Our last result is the comparison between the stability of a periodic orbit of the coplanar Hamiltonian $\mathcal{H}_{\text{CP}}$ and the stability of the origin in the h -averaged Hamiltonian belonging to the same energy level (see Corollary 3.12).

In order to analyze the character of the periodic orbits, we denote by X_{CP} the vector field of the Hamiltonian $\mathcal{H}_{\text{CP}}$. For $(0, \Gamma(t; L), 0, h(t; L))$, a periodic orbit with initial conditions $\Gamma(0; L) = \Gamma_0$ and $h(0; L) = 0$, the variational equation around the periodic orbit is given by

$$\dot{z} = DX_{\text{CP}}(0, \Gamma(t; L), 0, h(t; L))z.$$

Let $\Phi(t; \Gamma_0, L)$ be the fundamental matrix satisfying the initial condition $\Phi(0; \Gamma_0, L) = \text{Id}$. The monodromy matrix is defined just by

$$\Phi(\mathcal{T}(\Gamma_0; L); \Gamma_0, L) \quad (3.18)$$

with $\Gamma_0 = \Gamma(0; L)$ and $\mathcal{T}(\Gamma_0; L)$, the period of the considered periodic orbit. Since $\mathcal{H}_{\text{CP}}$ is a 2-degrees-of-freedom Hamiltonian, the eigenvalues of the monodromy matrix are of the form

$$(\mu_{\text{CP}}(\Gamma_0; L), (\mu_{\text{CP}}(\Gamma_0; L))^{-1}, 1, 1). \quad (3.19)$$

We are interested now in comparing the eigenvalues of the monodromy matrix with the ones coming from the constant linear part around the origin of $\mathcal{H}_{\text{AV}}$, the h -averaged Hamiltonian (2.18). That is, we want to compare μ_{CP} with $e^{\lambda_{\text{AV}} \mathcal{T}}$ where λ_{AV} has been analyzed in Proposition 3.4.

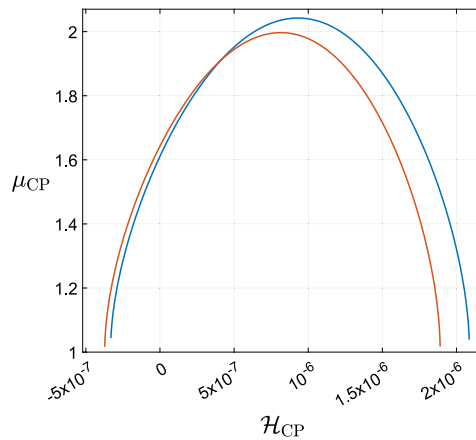


Fig. 7. The value of μ_{CP} (the eigenvalue greater than 1) as a function of $\mathcal{H}_{CP}$ for the hyperbolic periodic orbits. In orange, the numerical computations performed in [12], and in blue, the corresponding approximated value $\mu_{CP}^{(0)}$ in Theorem 3.17.

In the following result, we provide a first order (up to an error of order $\mathcal{O}(\delta^{3/2})$) approximation of the eigenvalues of the monodromy matrix.

Theorem 3.17. For $L \in (0, L_{\max}]$ and $\Gamma \in (0, \frac{L}{2})$, we define

$$\Gamma_{CP}^{(0)}(\Gamma; L) = L \hat{\Gamma}_{CP}^{(0)}(\hat{\Gamma}; \delta) := L(\hat{\Gamma} + \delta \hat{c}_0(\hat{\Gamma}; \delta)), \quad (3.20)$$

with $\hat{c}_0$ defined in (3.14) and $\delta = \rho \alpha^3 L^4$. We also introduce

$$\mu_{CP}^{(0)}(\Gamma; L) = \exp(\mathcal{T}^{(0)}(\Gamma; L) \lambda_{AV}(\Gamma; L)), \quad \mathcal{T}^{(0)}(\Gamma; L) = \frac{3\rho_0}{4L^7} \frac{2\pi}{|\hat{a}_0(\hat{\Gamma}; \delta)|}, \quad (3.21)$$

where $\hat{a}_0$ is defined in (3.14) and $\lambda_{AV}(\Gamma; L)$ (see Proposition 3.4) is an eigenvalue of the linearization around the origin of the h -averaged Hamiltonian $\mathcal{H}_{AV}$ for constant values Γ, L .

Then, there exist $\delta_0 > 0$ small enough and a constant $C_* > 0$ such that, if $L \in (0, L_{\max}]$, $\Gamma_0 = L \hat{\Gamma}_0$ satisfy (3.13) and $\delta = \rho \alpha^3 L^4 \in [0, \delta_0]$, we have that

$$|\mu_{CP}(\Gamma_0; L) - \mu_{CP}^{(0)}(\Gamma_0; L)| \leq C_* \delta^{3/2},$$

where $\mu_{CP}(\Gamma_0; L)$ is an eigenvalue of the monodromy matrix in (3.18) (see also (3.19)) corresponding to the linearized system around $(0, \Gamma(t; L), 0, h(t; L))$, the periodic orbit of $\mathcal{H}_{CP}$ with initial conditions $\Gamma(0; L) = \Gamma_0$ and $h(0; L) = 0$, given in Theorem 3.9.

The proof of this result is postponed to Section 6.2.2 and relies on a first order perturbation analysis.

Corollary 3.18. For any $\hat{\Gamma}_0 \in (0, \frac{1}{2})$ there exists $\delta_0 > 0$ such that if $L, \Gamma_0 = \hat{\Gamma}_0 L$ satisfy (3.13) and $\delta = \rho \alpha^3 L^4 \in [0, \delta_0]$, then the periodic orbit with initial condition $\Gamma_0 = \hat{\Gamma}_0 L$ is of saddle type if $\hat{\Gamma}_0 \in (\hat{\Gamma}_-(\delta), \hat{\Gamma}_+(\delta))$, with $\hat{\Gamma}_{\pm}(\delta)$ defined in Theorem 3.2, and of elliptic type otherwise.

In addition, there exists $\delta_0 > 0$, uniform for $\hat{\Gamma}_0 \in \left\{ \frac{m}{2} \right\} \cup \left(0, \frac{m}{4} \right) \cup \left[m, \frac{1}{2} \right)$, such that the periodic orbit with initial condition $\Gamma_0 = L \hat{\Gamma}_0$ is of saddle type if $\hat{\Gamma}_0 = \frac{m}{2}$ and of elliptic type if $\hat{\Gamma}_0 \in \left(0, \frac{m}{4} \right) \cup \left[m, \frac{1}{2} \right)$.

This result is a direct consequence of Theorems 3.17 and 3.2.

The theoretical approximation result in Theorem 3.17, agrees with the ones computed numerically in [12]. Indeed, in Fig. 7, we present the comparison between the first order approximated eigenvalue $\mu_{CP}^{(0)}$ and the numerical computation of μ_{CP} performed in [12]. The analysis is restricted to values of $\hat{\Gamma}_0$ close to $\frac{m}{2} \sim 0.058257569495584$ so that the periodic orbit could be (and from the numerical point of view is) hyperbolic.

Remark 3.19. By Corollary 3.12, we obtain that

$$\mathcal{H}_{CP}(0, \Gamma_0, 0, 0) = \mathcal{H}_{AV}(0, \Gamma_{CP}^{(0)}(\Gamma_0; L), 0) + \mathcal{O}(\delta^2)$$

with $\Gamma_{CP}^{(0)}$ defined in (3.20). Therefore, as a straightforward consequence of Theorem 3.17 we conclude that the eigenvalues of the monodromy matrix of a periodic orbit of period $\mathcal{T}_E$ lying in an energy level E are well approximated by $e^{\mathcal{T}_E \lambda_E}$ with λ_E an eigenvalue associated to the origin as critical point of the h -averaged Hamiltonian in the energy level $\{\mathcal{H}_{AV} = E\}$.

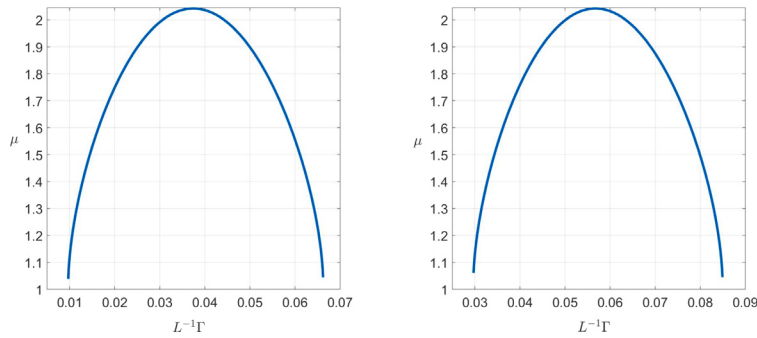


Fig. 8. Approximated value of the eigenvalue greater than 1 associated to the monodromy matrix as a function of the value $L^{-1}\Gamma$ of the periodic orbit at the section $h = 0$. On the left, for the Hamiltonian $\mathcal{H}_{CP}$ at the energy level $\mathcal{H}_{CP}(0, \Gamma, 0, 0)$ and on the right the corresponding one for the h -averaged Hamiltonian at the energy level $\mathcal{H}_{AV}(0, \Gamma, 0)$. Note that the range of $L^{-1}\Gamma$ in the two cases is different.

Table 4

The values of $\hat{\Gamma}_0^{(k)}$. For $k = 3$, $\hat{\Gamma}_0^{(3)} = 1/2$ and for $k \geq 4$, the values of $\hat{\Gamma}_0^{(k)}$ are out of the interval $(0, \frac{1}{2})$.

k	$\hat{\Gamma}_0^{(k)}$
1	0.189897948556636
2	0.338516480713450

Note, however, that, in terms of the initial condition of the periodic orbit, namely $\Gamma(0; L) = \Gamma_0$, the information about the eigenvalues of the monodromy matrix associated to a periodic orbit with initial condition Γ_0 of the full coplanar Hamiltonian comes from the h -averaged Hamiltonian corresponding to the *corrected* value of the parameter Γ given by $\Gamma_{CP}^{(0)}(\Gamma_0; L)$ in (3.20). To show this discrepancy, for $L = 1$, in Fig. 8 they are represented the eigenvalues of the monodromy matrix associated to the periodic orbit with respect to the initial condition Γ/L as well as $e^{\mathcal{T}(\Gamma; L)\lambda_{AV}(\Gamma; L)}$ the eigenvalues of the linearization of $\mathcal{H}_{AV}$ corresponding to the same parameter Γ/L .

For Galileo, which corresponds to $\mathcal{H}_{CP} = 0$ and semi-major axis $a = 1$ (in the normalized units). The value $\hat{\Gamma}_0 = 0.058788134221194$ corresponding to the initial condition $\mathcal{H}_{CP}(0, L\hat{\Gamma}_0, 0, 0) = 0$, can be numerically computed. Then the corrected value for Γ is $\Gamma_{CP}^{(0)}(\Gamma_0; L) = 0.079342370619096$. Therefore, we have that the approximated eigenvalue is

$$\mu_{CP}^{(0)}(\Gamma_{CP}^{(0)}(\Gamma_0; L); L) = 1.610955038638576.$$

However, if the correction in Γ is not considered in the h -averaged Hamiltonian and the period is only approximated by the averaged Hamiltonian, we obtain

$$e^{\lambda_{AV}(\Gamma_0; L)\mathcal{T}(\Gamma_0; L)} = 2.038395173606618.$$

Therefore, the difference between the eigenvalues of the monodromy and the corresponding orbit in the h -averaged system, namely with the same initial condition Γ_0 , cannot be *a priori* neglected.

The last result of this work can be seen as a refinement of Theorem 3.17. It gives a more accurate bound for the eigenvalues of the monodromy matrix for the most part of the values of L , Γ_0 . In order to state such a result, we introduce the values $\hat{\Gamma}_0^{(k)} \in (0, \frac{1}{2})$ satisfying that

$$\frac{2\pi}{\hat{\alpha}_0(\hat{\Gamma}_0^{(k)}; 0)} |1 - 16\hat{\Gamma}_0^{(k)} - 20(\hat{\Gamma}_0^{(k)})^2| = k\pi, \quad k = 1, 2. \quad (3.22)$$

That is $|1 - 16\hat{\Gamma}_0^{(k)} - 20(\hat{\Gamma}_0^{(k)})^2| = 2(1 + 2\hat{\Gamma}_0^{(k)})k$. These values, $\hat{\Gamma}_0^{(k)}$, can be analytically computed (see Table 4 for the numerical value) because they are the zeros of suitable polynomial of degree 2.

We also introduce the constant C_0 defined through (3.5) in Theorem 3.2, as

$$C_0 = \frac{5U_2^{1,0}(m+3)\sqrt{3-2m-4m^2}}{3(16+20m)} \quad (3.23)$$

in such a way that $\Gamma_{\pm}(L)L^{-1} = \frac{m}{2} \mp C_0\delta + \mathcal{O}(\delta^2)$. We recall that $\lambda_{AV}(\Gamma_{\pm}(L); L) = 0$.

Lemma 3.20. *There exists $\delta_0 > 0$ such that if L satisfies that $\delta = \rho\alpha^3 L^4 \in [0, \delta_0]$ then, there exist $\Gamma^{(k)}(L)$, $k = 1, 2$ such that*

$$\mu_{CP}^{(0)}(\Gamma^{(k)}(L); L) = (-1)^k,$$

where $\Gamma^{(k)}(L) = L\hat{\Gamma}^{(k)}(\delta)$ (with $\delta = \rho\alpha^3 L^4$) satisfy

$$\left| \hat{\Gamma}^{(k)}(\delta) - \hat{\Gamma}_0^{(k)} \right| \leq M\delta^2,$$

with the value $\hat{\Gamma}_0^{(k)}$ defined by (3.22) (see also Table 4) and the constant M is independent of L .

In addition, if either $\Gamma \neq \Gamma^{(k)}(L)$ or $\Gamma \neq \Gamma_{\pm}(L)$, then $\mu_{\text{CP}}^{(0)}(\Gamma; L) \neq \pm 1$.

This lemma is proven in Section 6.2.3.

Theorem 3.21. Assume that we are under the conditions of Theorem 3.17 and denote $\Gamma_{\text{CP}}^{(0)}(\Gamma_0; L) = L\hat{\Gamma}_{\text{CP}}^{(0)}(\hat{\Gamma}_0; \delta)$, as in (3.20).

Fix $C \in (0, C_0)$ (see (3.23)), $v \in [1, 2]$ and consider the sets, for $k = 1, 2$, defined for $\delta > 0$ as

$$I_{\pm, v}^{(0)}(\delta) = \left\{ \sigma \in \mathbb{R}, \left| \sigma - \frac{m}{2} \mp C_0 \delta \right| \leq C\delta^v \right\}, \quad I_v^{(k)}(\delta) = \left\{ \sigma \in \mathbb{R}, \left| \sigma - \hat{\Gamma}_0^{(k)} \right| \leq C\delta^{\frac{1+v}{2}} \right\}.$$

There exists a constant $C_* > 0$ and $\delta_0 > 0$ small enough, such that, for $\delta \in [0, \delta_0]$, if $\hat{\Gamma}_{\text{CP}}^{(0)}(\hat{\Gamma}_0; \delta) \notin I_{\pm, v}^{(0)}(\delta) \cup I_v^{(1)}(\delta) \cup I_v^{(2)}(\delta)$ then

$$\left| \mu_{\text{CP}}(\Gamma_0; L) - \mu_{\text{CP}}^{(0)}(\Gamma_{\text{CP}}^{(0)}(\Gamma_0; L); L) \right| \leq C_* \delta^{\frac{5-v}{2}}.$$

As a consequence when $\hat{\Gamma}_{\text{CP}}^{(0)}(\hat{\Gamma}_0; \delta) \notin I_{\pm, 1}^{(0)}(\delta) \cup I_1^{(1)}(\delta) \cup I_1^{(2)}(\delta)$, then

$$\left| \mu_{\text{CP}}(\Gamma_0; L) - \mu_{\text{CP}}^{(0)}(\Gamma_{\text{CP}}^{(0)}(\Gamma_0; L); L) \right| \leq C_* \delta^2.$$

This result is proven in Section 6.2.4.

The remaining part of this work is devoted to prove all the results stated in this section.

4. The circular critical points of the h -averaged Hamiltonian

In Section 4.1, we rewrite the averaged system in a more suitable way to analyze the critical points. This new expression will be used both to study the circular critical points and the eccentric ones. After that, in Sections 4.2 and 4.3, we prove Theorem 3.2 and Proposition 3.4 respectively.

4.1. Rewriting the h -averaged system

We start with the h -averaged Hamiltonian in slow-fast Delaunay variables (y, x) . That is, we consider (see (2.20))

$$\begin{aligned} H_{\text{AV}}(y, \Gamma, x; L) &= H_0(y, \Gamma; L) + \alpha^3 H_{\text{AV},1}(y, \Gamma, x; L) \\ &= \frac{\rho_0}{128} \frac{y^2 - 6y\Gamma - 3\Gamma^2}{L^3 y^5} + \alpha^3 \frac{\rho_1}{L^2} \left[\frac{1}{2} U_2^{0,0} D_{0,1}(y, \Gamma) + \frac{1}{3} U_2^{1,0} D_{1,0}(y, \Gamma) \cos x \right]. \end{aligned}$$

with $D_{0,1}$ and $D_{1,0}$ as given in Table 8. In order to rewrite it in a more concise form, we introduce the following polynomials in (y, Γ) , which depend implicitly on the parameter L ,

$$\begin{aligned} P_0(y, \Gamma) &= y^2 - 6y\Gamma - 3\Gamma^2, & Q_0(y; L) &= (5L^2 - 12y^2)y^3, \\ P_1(y, \Gamma) &= (y - \Gamma)(3y + \Gamma), & Q_1(y, \Gamma; L) &= (3y + \Gamma)(L - 2y)(L + 2y)y^3, \end{aligned}$$

and their derivatives with respect to y ,

$$\begin{aligned} P'_0(y, \Gamma) &= 2(y - 3\Gamma), & Q'_0(y; L) &= 15(L - 2y)(L + 2y)y^2, \\ P'_1(y, \Gamma) &= 2(3y - \Gamma), & Q'_1(y, \Gamma; L) &= y^2 \left[3(L - 2y)(L + 2y)(4y + \Gamma) - 8y^2(3y + \Gamma) \right]. \end{aligned}$$

Hence, the h -averaged Hamiltonian as well as its derived equations of motion can be written as follows

$$H_{\text{AV}}(y, \Gamma, x; L) = \frac{1}{128} \frac{\rho_0}{L^4 y^5} \left(\mathcal{A}_{0,\alpha}(y, \Gamma) + \alpha^3 \sqrt{P_1(y, \Gamma)} \mathcal{A}_1(y, \Gamma; L) \cos x \right)$$

and

$$\begin{aligned} \dot{x} &= -\frac{1}{256} \frac{\rho_0}{L^4 y^6} \frac{1}{\sqrt{P_1(y, \Gamma)}} \left[\sqrt{P_1(y, \Gamma)} B_{0,\alpha}(y, \Gamma; L) + \alpha^3 B_1(y, \Gamma; L) \cos x \right] \\ \dot{y} &= \frac{1}{128} \frac{\rho_0 \alpha^3}{L^4 y^5} \sqrt{P_1(y, \Gamma)} \mathcal{A}_1(y, \Gamma; L) \sin x, \end{aligned} \tag{4.1}$$

where the functions

$$\begin{aligned} \mathcal{A}_{0,\alpha} &= P_0(L + \alpha^3 d_0 Q_0), \\ \mathcal{A}_1 &= d_1 Q_1, \\ B_{0,\alpha} &= 10P_0(L + \alpha^3 d_0 Q_0) - 2y(L + \alpha^3 d_0 Q_0)P'_0 - 2\alpha^3 y d_0 P_0 Q'_0, \\ B_1 &= d_1(10P_1 Q_1 - y Q_1 P'_1 - 2y P_1 Q'_1) \end{aligned}$$

are polynomial in (y, Γ) , with

$$d_0 = 2U_2^{0,0} \frac{\rho_1}{\rho_0}, \quad d_1 = 20U_2^{1,0} \frac{\rho_1}{\rho_0} \quad (4.2)$$

and we recall that

$$U_2^{0,0} = 0.762646, \quad U_2^{1,0} = 0.547442, \quad \rho := \frac{\rho_1}{\rho_0} = 1.762157978551987 \cdot 10^{-18}. \quad (4.3)$$

To analyze the circular critical points (i.e., at $e = 0$), we work with Poincaré variables. In this case, the h -averaged Hamiltonian (2.18) as well as its derived equations of motion can be written as follows

$$H_{AV}(\xi, \eta) = \frac{1}{128} \frac{\rho_0}{L^4 y^5} \left(\mathcal{A}_{0,\alpha} + \alpha^3 d_1 \sqrt{P_1} \tilde{Q}_1 (\xi^2 - \eta^2) \right)$$

and

$$\begin{aligned} \dot{\xi} &= -\frac{1}{512} \frac{\rho_0}{L^4 y^6} \frac{\eta}{\sqrt{P_1}} \left[\sqrt{P_1} B_{0,\alpha} - 8y\alpha^3 d_1 P_1 \tilde{Q}_1 + \alpha^3 \tilde{B}_1 \times (\xi^2 - \eta^2) \right] \\ \dot{\eta} &= \frac{1}{512} \frac{\rho_0}{L^4 y^6} \frac{\xi}{\sqrt{P_1}} \left[\sqrt{P_1} B_{0,\alpha} + 8y\alpha^3 d_1 P_1 \tilde{Q}_1 + \alpha^3 \tilde{B}_1 \times (\xi^2 - \eta^2) \right] \end{aligned} \quad (4.4)$$

with

$$y = \frac{L}{2} - \frac{\xi^2 + \eta^2}{4},$$

and

$$\begin{aligned} \tilde{Q}_1 &= \frac{Q_1}{2L - 4y} = \frac{1}{2} (3y + \Gamma)(L + 2y)y^3, \\ \tilde{Q}_1' &= \frac{1}{2} y^2 \left[3L(\Gamma + 4y) + 2y(15y + 4\Gamma) \right], \\ \tilde{B}_1 &= d_1 (10P_1 \tilde{Q}_1 - y\tilde{Q}_1 P_1' - 2yP_1 \tilde{Q}_1'). \end{aligned}$$

For future purposes, we also decompose

$$B_{0,\alpha} = B_0 + \alpha^3 B_1 \quad (4.5)$$

with

$$B_0 = L(10P_0 - 2yP_0'), \quad B_1 = d_0(10P_0Q_0 - 2yQ_0P_0' - 2yP_0Q_0').$$

We recall that $a_{\max} = 30000$ km is the maximum semi-major axis we are going to consider and that $a_{\min} = 6378.14$ km (the radius of the Earth) is the minimum one.

Along the proof of the results this notation will be used extensively without an explicit mention.

4.2. Circular critical points: Proof of Theorem 3.2

The origin, $(\xi, \eta) = (0, 0)$, is clearly an equilibrium point of system (4.4). To prove the statements in Theorem 3.2, we only need to study the linear part of the averaged system at the origin and elucidate the values of the parameters L, Γ for which the origin is either a saddle, an elliptic point or a degenerated (parabolic) fixed point.

We observe that, since

$$y = y(\xi, \eta) = \frac{L}{2} - \frac{1}{4}(\xi^2 + \eta^2)$$

satisfies $y(0, 0) = L/2$ and $\partial_\xi y(0, 0) = \partial_\eta y(0, 0) = 0$, the variational equation of the averaged system around $(\xi, \eta) = (0, 0)$ is $\dot{z} = M(\Gamma; L)z$ where

$$M(\Gamma; L) = g(L) \begin{pmatrix} 0 & -X^\eta(\Gamma; L) \\ X^\xi(\Gamma; L) & 0 \end{pmatrix} \quad (4.6)$$

and

$$\begin{aligned} g(L) &= \frac{\rho_0}{8L^{10}} \\ X^\xi(\Gamma; L) &= B_0 \left(\frac{L}{2}, \Gamma; L \right) + \alpha^3 B_1 \left(\frac{L}{2}, \Gamma; L \right) + 4\alpha^3 d_1 L \sqrt{P_1 \left(\frac{L}{2}, \Gamma \right)} \tilde{Q}_1 \left(\frac{L}{2}, \Gamma; L \right) \\ X^\eta(\Gamma; L) &= B_0 \left(\frac{L}{2}, \Gamma; L \right) + \alpha^3 B_1 \left(\frac{L}{2}, \Gamma; L \right) - 4\alpha^3 d_1 L \sqrt{P_1 \left(\frac{L}{2}, \Gamma \right)} \tilde{Q}_1 \left(\frac{L}{2}, \Gamma; L \right). \end{aligned}$$

Therefore the eigenvalues λ of $M(\Gamma; L)$ satisfy

$$\lambda^2 = -g^2(L) X^\xi(\Gamma; L) \cdot X^\eta(\Gamma; L) \quad (4.7)$$

so that, to determine the character of $(\xi, \eta) = (0, 0)$, we need to study the sign of the product $X^\xi(\Gamma; L) \cdot X^\eta(\Gamma; L)$ with respect to the parameters Γ, L . To this end, we perform the scaling

$$\Gamma = L\sigma, \quad \text{with} \quad 0 < \sigma < \frac{1}{2}$$

and compute $X^\xi(L\sigma; L) \cdot X^\eta(L\sigma; L)$. One can check (after some tedious computations) that

$$\begin{aligned} X^\xi(L\sigma; L) &= \frac{3}{2}L^3 \left[\beta(L)\mathbf{a}(\sigma) + \frac{1}{12}d_1\alpha^3L^4\mathbf{b}(\sigma) \right] \\ X^\eta(L\sigma; L) &= \frac{3}{2}L^3 \left[\beta(L)\mathbf{a}(\sigma) - \frac{1}{12}d_1\alpha^3L^4\mathbf{b}(\sigma) \right] \end{aligned} \quad (4.8)$$

with

$$\mathbf{a}(\sigma) = 1 - 16\sigma - 20\sigma^2, \quad \mathbf{b}(\sigma) = (2\sigma + 3)\sqrt{3 - 4\sigma - 4\sigma^2} \quad (4.9)$$

(note that $p(\sigma) := 3 - 4\sigma - 4\sigma^2 > 0$ if $0 < \sigma < \frac{1}{2}$ and $p\left(\frac{1}{2}\right) = 0$) and

$$\beta(L) = 1 + \frac{d_0\alpha^3L^4}{4}.$$

Remark 4.1. Notice that the eigenvalues $\lambda = \lambda(\sigma; L)$ can be explicitly computed. Indeed, using (4.7) and definitions (4.2) and (4.3) for d_0, d_1 and ρ respectively

$$\lambda^2 = -\frac{9\rho_0^2}{256L^{14}} \left[\mathbf{a}(\sigma) + \rho\alpha^3L^4\mathbf{c}_-(\sigma) \right] \left[\mathbf{a}(\sigma) + \rho\alpha^3L^4\mathbf{c}_+(\sigma) \right]$$

with

$$\mathbf{c}_\pm(\sigma) = \frac{U_2^{0,0}}{2}\mathbf{a}(\sigma) \pm \frac{5U_2^{1,0}}{3}\mathbf{b}(\sigma).$$

Lemma 4.2. Consider

$$\mathcal{X}_\pm(\sigma; L) := \beta(L)\mathbf{a}(\sigma) \pm \frac{1}{12}d_1\alpha^3L^4\mathbf{b}(\sigma).$$

For $L \in [0, L_{\max}]$, the function $\mathcal{X}_+(\cdot; L)$ is strictly decreasing for $\sigma \in \left(0, \frac{1}{2}\right)$ and $\mathcal{X}_-(\cdot; L)$ is strictly decreasing for $\sigma \in \left(0, \frac{m}{2}\right)$.

Moreover, there exist two functions $\sigma_\pm : [0, L_{\max}] \rightarrow (0, \frac{1}{2})$ such that for any $L \in [0, L_{\max}]$ $\mathcal{X}_\pm(\sigma; L) = 0$ if and only if $\sigma = \sigma_\pm(L)$. In addition, $\sigma_\pm(0) = \frac{m}{2}$ and for $L > 0$

$$\sigma_-(L) \in \left(0, \frac{m}{2}\right), \quad \sigma_+(L) \in \left(\frac{m}{2}, \frac{1}{4}\right),$$

where m , defined in (3.4), is the slope of the prograde resonance (see also (2.12)).

Proof. First we notice that $\mathbf{a}, \mathbf{b}$ are decreasing functions for $\sigma \in (0, 1/2)$. Indeed, we only need to compute

$$\partial_\sigma \mathbf{b}(\sigma) = -8 \frac{\sigma(2\sigma + 3)}{\sqrt{3 - 4\sigma - 4\sigma^2}} < 0$$

(for $\mathbf{a}$ is obvious). Therefore, $\mathcal{X}_+(\sigma; L)$ is an strictly decreasing function (with respect to σ) and has at most a unique zero. Using that $\mathbf{a}\left(\frac{m}{2}\right) = 0$ we obtain that

$$\mathcal{X}_+\left(\frac{m}{2}; L\right) = \frac{1}{12}d_1\alpha^3L^4\mathbf{b}\left(\frac{m}{2}\right) > 0$$

and

$$\mathcal{X}_+\left(\frac{1}{4}; L\right) \leq \mathbf{a}\left(\frac{1}{4}\right) + \frac{1}{12}d_1\alpha_{\max}^3L_{\max}^4\mathbf{b}\left(\frac{1}{4}\right) = -3.744000070111437 < 0,$$

with $\alpha_{\max} = a_{\max}/a_M$, and by Bolzano theorem we obtain the conclusion for $\mathcal{X}_+$.

Concerning $\mathcal{X}_-(\sigma; L)$, we have that

$$\mathcal{X}_-\left(\frac{m}{2}; L\right) = -\frac{1}{12}d_1\alpha^3L^4\mathbf{b}\left(\frac{m}{2}\right) < 0$$

and

$$\mathcal{X}_-(0; L) = \beta(L) - \frac{1}{12}d_1\alpha^3L^43\sqrt{3} = 1 + \frac{\alpha^3L^4}{4}(d_0 - d_1\sqrt{3}).$$

Therefore, since $d_0 - d_1\sqrt{3} \sim -17.4387\rho < 0$,

$$\mathcal{X}_-(0; L) \geq 1 + \frac{\alpha_{\max}^3L_{\max}^4}{4}(d_0 - d_1\sqrt{3}) \geq 0.477808830620940 > 0.$$

Again there exists $\sigma_-(L) \in (0, \frac{m}{2})$ such that $\mathcal{X}_-(\sigma_-(L); L) = 0$. In addition, one can easily check that $\partial_\sigma \mathcal{X}_-(\sigma; L) < 0$ if $\sigma \in (0, \frac{m}{2})$ so that $\sigma = \sigma_-(L)$ is the only solution of $\mathcal{X}_-(\sigma; L) = 0$ belonging to $(0, \frac{m}{2})$.

To finish with this analysis, we point out that, for $\sigma \in (\frac{m}{2}, \frac{1}{2})$,

$$\mathcal{X}_-(\sigma; L) = \beta(L)\mathbf{a}(\sigma) - \frac{1}{12}d_1\alpha^3L^4\mathbf{b}(\sigma) < 0$$

provided $\beta(L) > 0$, $\mathbf{a}(\sigma) < 0$ and $\mathbf{b}(\sigma) > 0$ if $\sigma \in (\frac{m}{2}, \frac{1}{2})$. $\square$

From formula (4.7) of the eigenvalues and the previous lemma, recalling that $\Gamma = L\sigma$, we have that

- If either $\Gamma \in (0, L\sigma_-(L))$ or $\Gamma \in (L\sigma_+(L), \frac{L}{2})$, then $(\xi, \eta) = (0, 0)$ is a center equilibrium point.
- If $\Gamma \in (L\sigma_-(L), L\sigma_+(L))$, then $(\xi, \eta) = (0, 0)$ is a saddle equilibrium point.
- Using that $\mathcal{X}_+(\cdot; L)$ and $\mathcal{X}_-(\cdot; L)$ are not zero simultaneously, we deduce that, when $\Gamma = L\sigma_-(L)$ or $\Gamma = L\sigma_+(L)$, then $(\xi, \eta) = (0, 0)$ is a degenerated equilibrium point with nilpotent linear part.

Therefore, in order to prove Theorem 3.2, we only need to prove the expansions of $\sigma_\pm(L)$ in (3.5). Indeed, $\sigma_\pm(L)$ satisfy

$$\mathcal{X}_\pm(\sigma_\pm(L); L) = \beta(L)\mathbf{a}(\sigma_\pm(L)) \pm \frac{1}{12}d_1\alpha^3L^4\mathbf{b}(\sigma_\pm(L)) = 0. \quad (4.10)$$

We rewrite condition (4.10) in a more suitable way. To do so, denoting $\delta = \rho\alpha^3L^4$, we introduce the functions

$$\mathbf{A}_\pm(\sigma; \delta) := \mathbf{a}(\sigma) \left(1 + \frac{U_2^{0,0}}{2}\delta \right) \pm \frac{5U_2^{1,0}}{3}\delta\mathbf{b}(\sigma),$$

which are smooth for $(\sigma, \delta) \in (0, \frac{1}{2}) \times (0, \infty)$. Then, by the definition of d_0, d_1 in (4.2) and also (4.3), we have that, $\mathcal{X}_\pm(\sigma; L) = \mathbf{A}_\pm(\sigma; \delta)$ so that expression (4.10) is equivalent to

$$\mathbf{A}_\pm(\sigma_\pm(L); \delta) = 0.$$

Since $\mathbf{A}_\pm(\frac{m}{2}; 0) = 0$ and

$$\partial_\sigma \mathbf{A}_\pm\left(\frac{m}{2}; 0\right) = \partial_\sigma \mathbf{a}\left(\frac{m}{2}\right) = -16 - 20m \neq 0,$$

by the Implicit Function Theorem, $\sigma_\pm(L)$ is, in fact, a smooth function of $\delta = \rho\alpha^3L^4$, namely $\sigma_\pm(L) = \tilde{\sigma}_\pm(\delta)$ with $\tilde{\sigma}(0) = \frac{m}{2}$. Therefore,

$$\sigma_\pm(L) = \tilde{\sigma}_\pm(\delta) = \frac{m}{2} + c_\pm\delta + \mathcal{O}(\delta^2)$$

with

$$\begin{aligned} c_\pm &= \partial_\delta \tilde{\sigma}_\pm(0) = -\frac{\partial_\delta \mathbf{A}_\pm\left(\frac{m}{2}; 0\right)}{\partial_\sigma \mathbf{A}_\pm\left(\frac{m}{2}; 0\right)} = \frac{1}{16 + 20m} \left(\mathbf{a}\left(\frac{m}{2}\right) \frac{U_2^{0,0}}{2} \pm \frac{5U_2^{1,0}}{3}\mathbf{b}\left(\frac{m}{2}\right) \right) \\ &= \pm \frac{1}{16 + 20m} \frac{5U_2^{1,0}}{3}\mathbf{b}\left(\frac{m}{2}\right). \end{aligned}$$

Using that, by Lemma 4.2, $\mathcal{X}_+$ is a strictly decreasing function (with respect to σ) and that $\mathcal{X}_-$ is also a strictly decreasing function if $\sigma \in (0, \frac{m}{2})$, we conclude that $\tilde{\sigma}_\pm$ are well defined for $\delta \in [0, \rho\alpha_{\max}^3L_{\max}^4]$ and the proof of Theorem 3.2 is complete.

4.3. Modulus of the eigenvalues: Proof of Proposition 3.4

We use, along this section, the notation introduced in Section 4.2. We first define, from (4.7) and (4.8),

$$\mathcal{M}(\sigma; L) := -g^2(L) \frac{9L^6}{4} \left[\beta^2(L)\mathbf{a}^2(\sigma) - \tilde{\beta}^2(L)\mathbf{b}^2(\sigma) \right], \quad (4.11)$$

with $g(L) = \frac{\rho_0}{8L^{10}}$, $\tilde{\beta}(L) = \frac{d_1\alpha^3L^4}{12}$ and $\mathcal{X}_\pm$ were introduced in Lemma 4.2. It is not difficult to check that

$$\begin{aligned} \partial_\sigma \mathcal{M}(\sigma; L) &= \frac{-9\rho_0^2}{256L^{14}} \left[2\beta^2(L)\mathbf{a}(\sigma)\mathbf{a}'(\sigma) - 2\tilde{\beta}^2(L)\mathbf{b}(\sigma)\mathbf{b}'(\sigma) \right] \\ &= \frac{-9\rho_0^2}{256L^{14}} \left[\beta^2(L)(1600\sigma^3 + 1920\sigma^2 + 432\sigma - 32) + \tilde{\beta}^2(L)(16\sigma(2\sigma + 3)^2) \right]. \end{aligned} \quad (4.12)$$

Therefore $\partial_\sigma \mathcal{M}$, for any fixed L , is a degree three polynomial such that its derivative has no positive zero (all its coefficients are positive). That implies that $\partial_\sigma \mathcal{M}(\cdot; L)$ can at most have one zero for any value of L . Since, by definition, $\mathcal{M}(\sigma_+(L); L) = \mathcal{M}(\sigma_-(L); L) = 0$, by Rolle's theorem, there exists $\sigma_*(L) \in (\sigma_-(L), \sigma_+(L))$ such that $\partial_\sigma \mathcal{M}(\sigma_*(L); L) = 0$. Moreover, since $\mathbf{a}\left(\frac{m}{2}\right) = 0$,

$$\partial_\sigma \mathcal{M}(0; L) > 0, \quad \partial_\sigma \mathcal{M}\left(\frac{m}{2}; L\right) < 0,$$

we conclude that, for any $L_{\min} \leq L \leq L_{\max}$, the function $\mathcal{M}(\sigma; L)$ has a maximum at $\sigma_*(L)$. Therefore, for a fixed value of L ,

- on the interval $[\sigma_-(L), \sigma_+(L)]$, the function $\lambda_+(\sigma; L) = \sqrt{\mathcal{M}(\sigma; L)}$ has a maximum at $\sigma = \sigma_*(L)$.
- When $\sigma \in (0, \sigma_-(L))$ the function $\sqrt{|\mathcal{M}(\sigma; L)|}$ is decreasing with respect to σ
- and when $\sigma \in (\sigma_+(L), \frac{1}{2})$, $\sqrt{|\mathcal{M}(\sigma; L)|}$ is an increasing function.

We notice that, from (4.11) and using that $\mathbf{b}$ is a decreasing positive function, we deduce that, if $\sigma \in (\sigma_-(L), \sigma_+(L))$, then the corresponding eigenvalues $\pm \lambda(\sigma; L)$ satisfy

$$0 < \lambda(\sigma; L) \leq g(L) \frac{3L^3}{24} d_1 \alpha^3 L^4 \mathbf{b}(0) \leq \frac{\rho_0}{8L^{10}} \frac{3L^3}{24} d_1 \alpha^3 L^4 3\sqrt{3} = \frac{15\sqrt{3}\rho_0}{16L^7} U_2^{1,0} \rho \alpha^3 L^4,$$

with $\rho = \rho_1/\rho_0$. In addition, since $\lambda(\sigma_*(L); L) > \lambda\left(\sigma\left(\frac{m}{2}\right); L\right)$,

$$\lambda(\sigma_*(L); L) \geq g(L) \frac{3L^3}{24} d_1 \alpha^3 L^4 \mathbf{b}\left(\frac{m}{2}\right) = \frac{5\rho_0}{16L^7} U_2^{1,0} \rho \alpha^3 L^4 (m+3) \sqrt{3-2m-m^2}.$$

To finish the proof of Proposition 3.4, we prove the first order approximation for $\sigma_*(L)$ and we compute an approximation of $\lambda_+(\sigma_*(L); L)$. From (4.12) and using that

$$\beta(L) = 1 + \delta \frac{U_2^{0,0}}{2}, \quad \tilde{\beta}(L) = \delta \frac{5}{3} U_2^{1,0}, \quad (4.13)$$

we deduce that $\sigma_* := \sigma_*(L)$ satisfies

$$\left(1 + \delta \frac{U_2^{0,0}}{2}\right)^2 \mathbf{a}(\sigma_*) \partial_\sigma \mathbf{a}(\sigma_*) - \frac{25}{9} (U_2^{1,0})^2 \delta^2 \mathbf{b}(\sigma_*) \partial_\sigma \mathbf{b}(\sigma_*) = 0.$$

We introduce the notation

$$\bar{\delta} = \frac{25}{9 \left(1 + \delta \frac{U_2^{0,0}}{2}\right)^2} (U_2^{1,0})^2 \delta^2, \quad F(\sigma, \bar{\delta}) = \mathbf{a}(\sigma) \partial_\sigma \mathbf{a}(\sigma) - \bar{\delta} \mathbf{b}(\sigma) \partial_\sigma \mathbf{b}(\sigma)$$

so that $F(\sigma_*, \bar{\delta}) = 0$. Since $\mathbf{a}\left(\frac{m}{2}\right) = 0$, we note that $F\left(\frac{m}{2}, 0\right) = 0$. Then, by the implicit function theorem, $\sigma_* := \bar{\sigma}_*(\bar{\delta})$ depending analytically on $\bar{\delta}$ and, in addition, $\bar{\sigma}_*(\bar{\delta}) = \frac{m}{2} + \bar{\sigma}'_*(0) \bar{\delta} + \mathcal{O}(\bar{\delta}^2)$ with

$$\bar{\sigma}'_*(0) = -\frac{\partial_{\bar{\delta}} F\left(\frac{m}{2}, 0\right)}{\partial_\sigma F\left(\frac{m}{2}, 0\right)} = \frac{\mathbf{b}\left(\frac{m}{2}\right) \partial_\sigma \mathbf{b}\left(\frac{m}{2}\right)}{\left[\partial_\sigma \mathbf{a}\left(\frac{m}{2}\right)\right]^2} = -\frac{m(m+3)^2}{4(m+4)^2}.$$

As a consequence $\sigma_*(L) = \bar{\sigma}_*(\bar{\delta})$, satisfies that

$$\sigma_*(L) = \hat{\sigma}_*(\delta) = \frac{m}{2} - \delta^2 \frac{25(U_2^{1,0})^2}{9} \frac{m(m+3)^2}{4(m+4)^2} + \mathcal{O}(\delta^3)$$

for some analytic function $\hat{\sigma}$. Therefore, using that $\mathbf{a}\left(\frac{m}{2}\right) = 0$, expressions (4.13) of $\beta(L)$, $\tilde{\beta}(L)$ and formula (4.11) for $(\lambda_+(\sigma_*; L))^2 = \mathcal{M}(\sigma_*; L)$ one can check that

$$\frac{4}{9L^6 g^2(L)} (\lambda_+(\sigma_*; L))^2 = \frac{25}{9} (U_{1,0}^2)^2 \delta^2 \mathbf{b}^2\left(\frac{m}{2}\right) \left(1 + \delta^2 \frac{25}{9} (U_{1,0}^2)^2 \frac{\left[\partial_\sigma \mathbf{b}\left(\frac{m}{2}\right)\right]^2}{\left[\partial_\sigma \mathbf{a}\left(\frac{m}{2}\right)\right]^2} + \mathcal{O}(\delta^5)\right).$$

Therefore, using that $g(L) = \frac{\rho_0}{8L^{10}}$

$$\lambda_+(\sigma_*; L) = \frac{5\rho_0 \delta}{16L^7} U_{1,0}^2 (m+3) \sqrt{3-2m-m^2} \left(1 + \delta^2 \frac{25}{18} (U_{1,0}^2)^2 \frac{m^2(m+3)}{(1-m)(m+4)^2} + \mathcal{O}(\delta^3)\right)$$

and the proof of Proposition 3.4 is finished.

Remark 4.3. We emphasize that the values of $\sigma_*(L)$ are close to $\frac{m}{2} = 0.058257569495584$ which is the only positive zero of the polynomial $\mathbf{a}(\sigma)\mathbf{a}'(\sigma) = 1600\sigma^3 + 1920\sigma^2 + 432\sigma - 32$. This is because $\beta(L) \sim 1$ for all the values of semi-major axis a considered, meanwhile the range of values for $\tilde{\beta}(L)$ goes from $\mathcal{O}(10^{-10})$ to $\mathcal{O}(10^{-7})$. Therefore, the zeros of $\partial_\sigma \mathcal{M}$ are close to the ones of $\mathbf{a}(\sigma)\mathbf{a}'(\sigma)$.

Remark 4.4. As a consequence of our study, if we want to analyze the value of $\mathcal{M}(\sigma; L)$ at some interval $\sigma \in [\sigma_{\min}, \sigma_{\max}] \subset [\sigma_-(L), \sigma_+(L)]$, its minimum value is located either at $\sigma = \sigma_{\min}$ or $\sigma = \sigma_{\max}$.

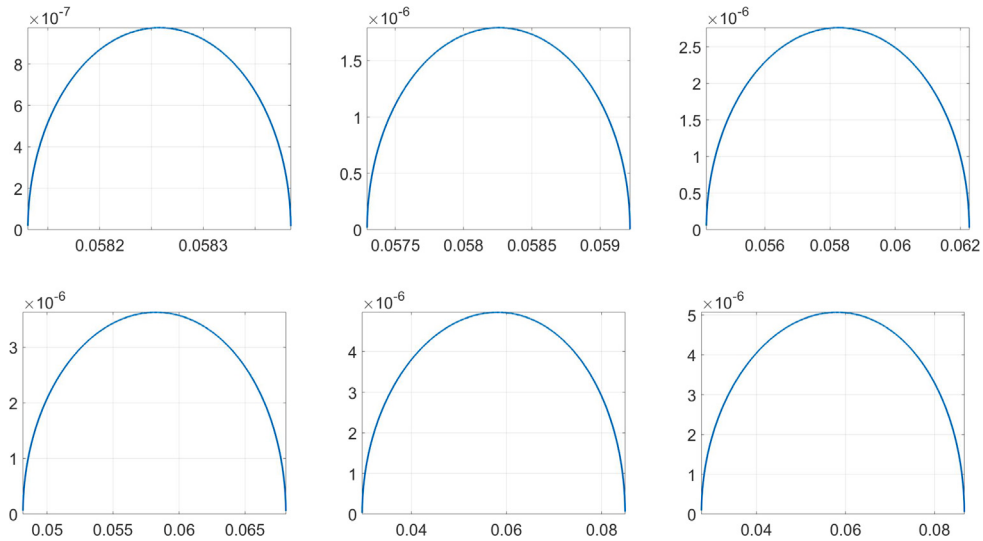


Fig. 9. These figures show $\lambda_+(\sigma; L) > 0$ as functions of σ for $a = \{10\,000, 15\,000, 20\,000, 24\,000, 29\,600, 30\,000\}$ km, when the origin is a saddle point (the units are the ones made explicit in Remark 3.1).

Table 5

Values of $\sigma_{\pm}$ and σ_* for some choices of semi-major axis a (km).

a	$\sigma_-(L)$	$\sigma_+(L)$	$\sigma_*(L)$
10000	0.058130693719535	0.058384404707503	0.058257569492108
15000	0.057294284878483	0.059218520762404	0.058257569455989
20000	0.054201084378365	0.062272995357407	0.058257569273126
24000	0.048177348389369	0.068087985206979	0.058257568831379
29600	0.029613649805289	0.084971418151141	0.058257567158286
30000	0.027639200647529	0.086680627913961	0.058257566962293

Remark 4.5. Using $L = \sqrt{\mu a}$ and $\alpha = \frac{a}{a_M}$, we write from (4.7), $\widetilde{\mathcal{M}}(\sigma; a) := \mathcal{M}(\sigma; \sqrt{\mu a})$ as

$$\begin{aligned}
 \widetilde{\mathcal{M}}(\sigma; a) &:= -\frac{9\rho_0^2}{256L^{14}} \left[\left(1 + \frac{d_0\alpha^3 L^4}{4} \right)^2 \mathbf{a}^2(\sigma) - \frac{1}{144} d_1^2 \alpha^6 L^8 \mathbf{b}^2(\sigma) \right] \\
 &= -\frac{9\rho_0^2}{256\mu^7 a^7} \left[\mathbf{a}^2(\sigma) \left(1 + \frac{d_0 a^5 \mu^2}{2a_M^3} + \frac{d_0^2 a^{10} \mu^4}{16a_M^6} \right) - \frac{1}{144} d_1^2 \mathbf{b}^2(\sigma) \frac{a^{10} \mu^4}{a_M^6} \right] \\
 &= -\frac{9\rho_0^2}{256\mu^7} \left[\mathbf{a}^2(\sigma) \left(\frac{1}{a^7} + \frac{d_0 \mu^2}{2a^2 a_M^3} + \frac{d_0^2 a^3 \mu^4}{16a_M^6} \right) - \frac{1}{144} d_1^2 \mathbf{b}^2(\sigma) \frac{a^3 \mu^4}{a_M^6} \right].
 \end{aligned}$$

We have that $\partial_a \widetilde{\mathcal{M}}(\sigma; a) > 0$ provided that, for $a \leq a_{\max}$,

$$-\frac{7}{a^8} - \frac{d_0 \mu^2}{a^3 a_M^3} + \frac{3d_0^2 a^2 \mu^4}{16a_M^6} = -\frac{1}{a^8} \left(7 + \frac{d_0 \mu^2}{a_M^3} a^5 - \frac{3d_0^2 \mu^4}{16a_M^6} (a^5)^2 \right) < 0$$

(the associated degree two polynomial has a unique positive zero which is greater than $a_{\max}$). Therefore, for $a_{\min} \leq a \leq a_{\max} = 30000$ we have that

$$0 < \widetilde{\mathcal{M}}(\sigma; a_{\min}) \leq \widetilde{\mathcal{M}}(\sigma; a) \leq \widetilde{\mathcal{M}}(\sigma; a_{\max}).$$

We present now some numerical results. In Fig. 9 it is depicted $\lambda_+(\sigma; L) > 0$ as a function of σ for some values of a .

We can also compute numerically $\sigma_{\pm}$ and σ_* . We compute $\sigma_{\pm}$ using that $\mathcal{X}_{\pm}(\sigma_{\pm}; L) = 0$ and σ_* as the unique positive zero of

$$\beta^2(L)(1600\sigma^3 + 1920\sigma^2 + 432\sigma - 32) + \widetilde{\beta}^2(L)(16\sigma(2\sigma + 3)^2).$$

We show its values in Table 5.

To finish, in Table 6, we present the maximum value of the positive eigenvalue in the saddle case, namely $\lambda_+(\sigma_*(L); L)$.

Table 6

Maximum value of the positive eigenvalue associated to the origin in the saddle case for some values of the semi-major axis a (km).

a	$\lambda_+(\sigma_*(L); L)$
10000	$0.97568857909377 \cdot 10^{-6}$
15000	$1.79245437499162 \cdot 10^{-6}$
20000	$2.75966404251617 \cdot 10^{-6}$
24000	$3.62767259360406 \cdot 10^{-6}$
29600	$4.96876878785117 \cdot 10^{-6}$
30000	$5.06982657623624 \cdot 10^{-6}$

5. Eccentric critical points of the h -averaged system

We devote this section to prove [Theorem 3.7](#) and analyze numerically the results obtained in this theorem. Along this section, we will use the notation introduced in Section 4.1 without any explicit mention.

The first step to compute the location of the equilibrium points of the h -averaged problem (4.1) is to analyze the equation $\dot{y} = 0$. It implies that a fixed point exists if it satisfies one of the following conditions:

$$x = k\pi, \quad k = 0, 1 \quad \text{or} \quad \sqrt{P_1}Q_1 = 0. \quad (5.1)$$

However, since we are assuming that $y \in (0, L/2)$ the latter condition cannot happen. That is, all critical points of (4.1) must be of the form

$$(k\pi, y) \quad \text{with} \quad y \in (0, L/2).$$

For given L and Γ , Eq. (5.1) implies that an eccentric orbit can be a critical point if it satisfies the condition $x = 0, \pi$. The location of the critical point (equivalently its associated eccentricity for a given Γ) is obtained by solving the equations

$$\begin{aligned} \sqrt{P_1(y, \Gamma)}B_{0,\alpha}(y, \Gamma; L) + \alpha^3 B_1(y, \Gamma; L) &= 0, & \text{for } x = 0, \\ \sqrt{P_1(y, \Gamma)}B_{0,\alpha}(y, \Gamma; L) - \alpha^3 B_1(y, \Gamma; L) &= 0, & \text{for } x = \pi, \end{aligned} \quad (5.2)$$

where P_1 , $B_{0,\alpha}$ and B_1 are polynomials. Notice that they are roots of the following polynomial in (y, Γ)

$$P_1(y, \Gamma)(B_{0,\alpha}(y, \Gamma; L))^2 - \alpha^6 (B_1(y, \Gamma; L))^2 = 0.$$

5.1. Existence of eccentric critical points: Formulation of the problem

To prove [Theorem 3.7](#), we have to analyze the existence and character of the fixed points of the form $(0, y)$ and (π, y) with $0 < y < \frac{L}{2}$.

First, let us now rewrite condition (5.2) on the fixed points. Using the decomposition in (4.5), the condition (5.2) can be rephrased as

$$F_{\pm}(y, \Gamma; L) := B_0(y, \Gamma; L) + \alpha^3 B_1(y, \Gamma; L) \pm \alpha^3 \frac{B_1(y, \Gamma; L)}{\sqrt{P_1(y, \Gamma)}} = 0 \quad (5.3)$$

where the sign $+$ corresponds to $x = 0$ and the $-$ sign corresponds to $x = \pi$.

We introduce the scaling

$$y = L\hat{y}, \quad \Gamma = y s = L\hat{y}s.$$

The values of the variables and parameters we are interested in are $0 < \hat{y} < \frac{1}{2}$, $a_{\min} \leq a \leq a_{\max}$ and $s \in (0, \frac{1}{2})$

We rewrite F in these new variables $(s, \hat{y})$, namely $\hat{F}_{\pm}(s, \hat{y}; L) := F_{\pm}(L\hat{y}, L\hat{y}s; L)$. Tedious but easy computations lead to

$$\begin{aligned} \hat{F}_{\pm}(s, \hat{y}; L) &= -6L^3 \hat{y}^2 (5s^2 + 8s - 1) + 12d_0 \alpha^3 L^7 \hat{y}^5 (4\hat{y}^2 - 5s - 12s\hat{y}^2 - 5s^2) \\ &\quad \pm 4d_1 \alpha^3 L^7 \hat{y}^5 \frac{(3+s)(12\hat{y}^2 - 8s\hat{y}^2 - s^2)}{\sqrt{3-2s-s^2}} \end{aligned}$$

where d_0 and d_1 were introduced in (4.2). We define

$$\begin{aligned} \mathcal{G}_{\pm}(s, \hat{y}; L) &:= 5s^2 + 8s - 1 + 2d_0 \alpha^3 L^4 \hat{y}^3 (5s^2 + 12s\hat{y}^2 + 5s - 4\hat{y}^2) \\ &\quad \pm 2d_1 \alpha^3 L^4 \hat{y}^3 \frac{(3+s)(s^2 + 8s\hat{y}^2 - 12\hat{y}^2)}{3\sqrt{3-2s-s^2}}. \end{aligned}$$

Then, recalling that $y = L\hat{y}$ and $\Gamma = y s = L\hat{y}s$,

$$F_{\pm}(y, \Gamma; L) = 0 \iff \hat{F}_{\pm}(s, \hat{y}; L) = 0 \iff \mathcal{G}_{\pm}(s, \hat{y}; L) = 0. \quad (5.4)$$

The function $G_{\pm}$ can be expressed as

$$G_{\pm}(s, \hat{y}; L) = \mathbf{a}(s) + 2\alpha^3 L^4 \rho[\mathbf{b}_{\pm}(s)\hat{y}^3 + \mathbf{c}_{\pm}(s)\hat{y}^5] \quad (5.5)$$

with

$$\begin{aligned} \mathbf{a}(s) &= 5s^2 + 8s - 1 \\ \mathbf{b}_{\pm}(s) &= 10U_2^{0,0}s(1+s) \pm \frac{20U_2^{1,0}(3+s)}{3\sqrt{3-2s-s^2}}s^2 \\ \mathbf{c}_{\pm}(s) &= 8U_2^{0,0}(3s-1) \pm \frac{20U_2^{1,0}(3+s)}{3\sqrt{3-2s-s^2}}(8s-12). \end{aligned}$$

Lemma 5.1. *The functions $\mathbf{a}$, $\mathbf{b}_{\pm}$, $\mathbf{c}_{\pm}$ are strictly increasing functions at $\left[0, \frac{1}{2}\right]$. As a consequence, for $\hat{y}, L > 0$, we have that $\partial_s G_{\pm}(s, \hat{y}; L) > 0$.*

In addition, $\mathbf{a}(m) = 0$, $\mathbf{a}(s) < 0$ if $s \in [0, m)$ and $\mathbf{a}(s) > 0$ if $s > m$.

Proof. The statements related to $\mathbf{a}$ are immediate. We compute now $\partial_s \mathbf{b}_{\pm}, \partial_s \mathbf{c}_{\pm}$,

$$\begin{aligned} \partial_s \mathbf{b}_{\pm}(s) &= 10U_2^{0,0}(1+2s) \pm 20U_2^{1,0} \frac{2s(9-4s^2-s^3)}{3(3-2s-s^2)^{3/2}} \\ \partial_s \mathbf{c}_{\pm}(s) &= 24U_2^{0,0} \mp 20U_2^{1,0} \frac{8s^2}{3(1-s)\sqrt{3-2s-s^2}}. \end{aligned}$$

The function $s(9-4s^2-s^3)$ is an increasing function for $s \in (0, 0.7)$ provided

$$9 - 12 \cdot 0.7^2 - 4 \cdot 0.7^3 = 1.748 > 0.$$

Therefore, denoting $s_* = 0.35$, if $s \in (0, s_*]$,

$$\begin{aligned} \left| \pm 20U_2^{1,0} \frac{2s(9-4s^2-s^3)}{3(3-2s-s^2)^{3/2}} \right| &\leq 20U_2^{1,0} \frac{2s_*(9-4s_*^2-s_*^3)}{3(3-2s_*-s_*^2)^{3/2}} \\ &= 6.731992451918273. \end{aligned}$$

Since $10U_2^{0,0} = 7.62646$, we conclude that $\partial_s \mathbf{b}_{\pm}(s) > 0$ if $s \in (0, s_*]$. When $s \in \left(s_*, \frac{1}{2}\right]$,

$$\left| \pm 20U_2^{1,0} \frac{2s(9-4s^2-s^3)}{3(3-2s-s^2)^{3/2}} \right| \leq 12.414817621987043$$

and $10U_2^{0,0}(1+2s) \geq 10U_2^{0,0}(1+2s_*) = 12.96498200000000$. Therefore, we conclude that $\partial_s \mathbf{b}_{\pm}(s) > 0$ if $s \in \left[0, \frac{1}{2}\right]$. On the other hand, if $0 < s \leq \frac{1}{2}$, then

$$\left| \mp 20U_2^{1,0} \frac{8s^2}{3(1-s)\sqrt{3-2s-s^2}} \right| \leq 20U_2^{1,0} \frac{2}{\frac{3}{2}\sqrt{\frac{7}{4}}} = 11.035393441766260$$

and $24U_2^{0,0} = 18.303504$. Therefore $\partial_s \mathbf{c}_{\pm}(s) > 0$.

When $s \geq 0$, $\partial_s \mathbf{a}(s) = 8 + 10s > 0$ and thus $\partial_s G_{\pm}(s, \hat{y}; L) > 0$ provided $\hat{y}, L > 0$. $\square$

5.2. A preliminary existence result

The next step to compute the eccentric critical points is to solve the equations in (5.4). We look for zeros of the functions $G_{\pm}$ in s .

Lemma 5.2. *Let L be satisfying $0 < L < L_{\max}$, $L = \sqrt{\mu a}$, and*

$$\hat{y}_{\min} = \sqrt{-\frac{\mathbf{b}_+(m)}{\mathbf{c}_+(m)}} = 0.116589071022807. \quad (5.6)$$

We have that $G_+(m, \hat{y}_{\min}; L) = 0$ and

1. *For any $0 < \hat{y} \leq \frac{1}{2}$, there exists a unique analytic function $s_+(\hat{y}; L)$, which belongs to $\left(0, \frac{3}{2}m\right)$, satisfying $G_+(s_+(\hat{y}; L), \hat{y}; L) = 0$. In addition, for $\hat{y}_{\min} < \hat{y} \leq \frac{1}{2}$ the function $s_+(\hat{y}; L)$ belongs to $\left(m, \frac{3}{2}m\right)$ and when $0 < \hat{y} < \hat{y}_{\min}$, $s_+(\hat{y}; L) \in (0, m)$.*
2. *For any $0 < \hat{y} \leq \frac{1}{2}$, there exists a unique analytic function $s_-(\hat{y}; L)$, which belongs to $\left(\frac{2}{3}m, m\right)$, satisfying $G_-(s_-(\hat{y}; L), \hat{y}; L) = 0$.*

Proof. To prove the existence of solutions of the equation $\mathcal{G}_\pm(s, \hat{y}; L) = 0$, we fix $\hat{y}, L$ (or a) in the ranges $0 < \hat{y} \leq \frac{1}{2}$ and $0 \leq a \leq a_{\max}$ and we use a Bolzano argument (with respect to the s variable).

We deal first with the $+$ case. Since $\mathbf{b}_+(0) = 0$,

$$\mathcal{G}_+(0, \hat{y}; L) = -1 + 2\alpha^3 L^4 \hat{y}^5 \mathbf{c}_+(0).$$

It is clear from the definition of $\mathbf{c}_+$ that $\mathbf{c}_+(0) < 0$ so that $\mathcal{G}_+(0, \hat{y}; L) < 0$. For fixed L , now we prove that the function $\mathcal{G}_+\left(\frac{3}{2}m, \hat{y}; L\right)$ is positive. We introduce

$$C_+(\hat{y}) := \mathcal{G}_+\left(\frac{3}{2}m, \hat{y}; L\right) = \hat{\mathbf{a}} + 2\alpha^3 L^4 \rho \left[\hat{\mathbf{b}}_+ \hat{y}^3 + \hat{\mathbf{c}}_+ \hat{y}^5 \right]$$

with

$$\hat{\mathbf{a}} = \mathbf{a}\left(\frac{3}{2}m\right) = 0.550909166052992, \quad \hat{\mathbf{b}}_+ = \mathbf{b}_+\left(\frac{3}{2}m\right) = 1.784508217583372, \quad (5.7)$$

$$\hat{\mathbf{c}}_+ = \mathbf{c}_+\left(\frac{3}{2}m\right) = -78.794344798569341. \quad (5.8)$$

For a fixed L , the function $C_+(\hat{y})$ has the minimum value either at the points $\hat{y} = 0, \hat{y} = \sqrt{\frac{-3\hat{\mathbf{b}}_+}{5\hat{\mathbf{c}}_+}} = 0.116570155823739$ or $\hat{y} = \frac{1}{2}$. Evaluating at these points, we obtain that $C_+(0) = \hat{\mathbf{a}} > 0$, from the fact that $\hat{\mathbf{c}}_+ < 0$:

$$C_+\left(\sqrt{\frac{-3\hat{\mathbf{b}}_+}{5\hat{\mathbf{c}}_+}}\right) = \hat{\mathbf{a}} - 2\alpha^3 L^4 \rho \frac{6\hat{\mathbf{b}}_+^2}{25\hat{\mathbf{c}}_+} \sqrt{\frac{-3\hat{\mathbf{b}}_+}{5\hat{\mathbf{c}}_+}} > 0$$

and finally, using that $4\hat{\mathbf{b}}_+ + \hat{\mathbf{c}}_+ < 0$,

$$\begin{aligned} C_+\left(\frac{1}{2}\right) &= \hat{\mathbf{a}} + \alpha^3 L^4 \rho \frac{1}{16} (4\hat{\mathbf{b}}_+ + \hat{\mathbf{c}}_+) \geq \hat{\mathbf{a}} + \rho \frac{a_{\max}^5 \mu^2}{16a_M^3} (4\hat{\mathbf{b}}_+ + \hat{\mathbf{c}}_+) \\ &= 0.014481715549738 > 0. \end{aligned} \quad (5.9)$$

The previous analysis proves that

$$\mathcal{G}_+(0, \hat{y}; L) \cdot \mathcal{G}_+\left(\frac{3}{2}m, \hat{y}; L\right) < 0$$

and therefore, there exists $s_+ = s_+(\hat{y}; L) \in \left(0, \frac{3}{2}m\right)$ such that $\mathcal{G}_+(s_+, \hat{y}; L) = 0$.

By Lemma 5.1, $\partial_s \mathcal{G}_+(s, \hat{y}; L) > 0$ so that, for fixed $\hat{y}, L$, $s_+ = s_+(\hat{y}; L)$ is the unique solution of $\mathcal{G}_+(s, \hat{y}; L) = 0$. By the implicit function theorem, the dependence on $\hat{y}, L$ of $s_+(\hat{y}; L)$ is analytic.

To finish the result related to the $+$ case, we consider now

$$\mathcal{G}_+(m, \hat{y}; L) = 2\alpha^3 L^4 \rho \left[\mathbf{b}_+(m) \hat{y}^3 + \mathbf{c}_+(m) \hat{y}^5 \right].$$

Notice that its sign does not depend on L . We have that

$$\mathbf{b}_+(m) = 1.085189654741836, \quad \mathbf{c}_+(m) = -79.834380790596938.$$

Therefore, for $0 < \hat{y} < \hat{y}_{\min}$, $\mathcal{G}_+(m, \hat{y}; L) > 0$ and when $\hat{y} > \hat{y}_{\min}$, then $\mathcal{G}_+(m, \hat{y}; L) < 0$. We conclude that for any $0 \leq a \leq a_{\max}$ and $\hat{y}_{\min} < \hat{y} \leq \frac{1}{2}$ the function, $s_+(\hat{y}; L) \in (m, \frac{3}{2}m)$ and when $0 < \hat{y} < \hat{y}_{\min}$, we have that $s_+(\hat{y}; L) \in (0, m)$.

For the $-$ case, we proceed analogously. We study the function $\mathcal{G}_-(m, \hat{y}; L)$. Using that $\mathbf{a}(m) = 0$,

$$\mathcal{G}_-(m, \hat{y}; L) = 2\alpha^3 L^4 \rho \left[\mathbf{b}_-(m) \hat{y}^3 + \mathbf{c}_-(m) \hat{y}^5 \right]$$

with

$$\mathbf{b}_-(m) = 0.899076688965464, \quad \mathbf{c}_-(m) = 71.897315415767750,$$

and hence $\mathcal{G}_-(m, \hat{y}; L) > 0$ provided $\hat{y}, L > 0$. Moreover, we have that

$$\mathbf{a}\left(\frac{2}{5}m\right) = -0.616290933136957, \quad \mathbf{b}_-\left(\frac{2}{5}m\right) = 0.357833740529904 \quad (5.10)$$

and

$$\mathbf{c}_-\left(\frac{2}{5}m\right) = 70.608271414680971. \quad (5.11)$$

From these values we have that

$$\mathcal{G}_-\left(\frac{2}{5}m, \hat{y}; L\right) \leq \mathcal{G}_-\left(\frac{2}{5}m, \frac{1}{2}; L_{\max}\right) = -0.076994095997652 < 0.$$

As a consequence, for any $0 < \hat{y} \leq \frac{1}{2}$ and $0 < a \leq a_{\max}$, there exists $s_- = s_-(\hat{y}; L) \in \left(\frac{2}{5}m, m\right)$ such that $\mathcal{G}_-(s_-, \hat{y}; L) = 0$. Then, for fixed $\hat{y}, L$, since by Lemma 5.1, $\partial_s \mathcal{G}_- > 0$, $s_-(\hat{y}; L)$ is the unique zero of $\mathcal{G}_-(s, \hat{y}; L)$ and by the Implicit Function Theorem s_- is an analytic function. $\square$

Remark 5.3. We stress that if the semi-major axis $a \rightarrow \infty$ (that is $L \rightarrow \infty$), $C_+ \left(\frac{1}{2} \right)$ in (5.9) is negative provided $4\hat{\mathbf{b}}_+ + \hat{\mathbf{c}}_+ < 0$. In order to be a negative quantity, the maximum semi-major axis has to satisfy

$$a^3 L^4 < \frac{16\hat{\mathbf{a}}}{|4\hat{\mathbf{b}}_+ + \hat{\mathbf{c}}_+|\rho} \Leftrightarrow a < \left(\frac{16\hat{\mathbf{a}} a_M^3}{|4\hat{\mathbf{b}}_+ + \hat{\mathbf{c}}_+|\rho\mu^2} \right)^{1/5} = 30160.25822948035 \text{ km},$$

that corresponds to $a < 1.018927642887850$ in the units specified in Remark 3.1. This fact leads us to think that $a_{\max}$ is (almost) the optimal value for our arguments to be true.

Remark 5.4. We have that $\mathbf{b}_-(\frac{1}{2}m), \mathbf{c}_-(\frac{1}{2}m) > 0$. Then for semi-major axis satisfying

$$a \leq \left(-\frac{16\mathbf{a}(\frac{1}{2}m)a_M^3}{\mu^2\rho(4\mathbf{b}_-(\frac{1}{2}m) + \mathbf{c}_-(\frac{1}{2}m))} \right)^{1/5} = 29700.20301662558 \text{ km}$$

(corresponding to $a \leq 1.003385237048162$ in the units in Remark 3.1) we have that $\mathcal{G}_-(\frac{m}{2}, \hat{y}; L) < 0$ and therefore $s_-(\hat{y}; L) \in \left(\frac{m}{2}, m \right)$.

A straightforward corollary of this preliminary analysis is the following result, see (5.4).

Corollary 5.5. Let a be a semi-major axis satisfying $0 < a \leq a_{\max}$. Then, given $L\hat{y}_{\min} < y \leq \frac{L}{2}$ with $L = \sqrt{\mu a} \in (0, L_{\max}]$, there exist two analytic functions $\Gamma_{\pm}(y; L)$ such that

$$\frac{\Gamma_+(y; L)}{my} \in \left(1, \frac{3}{2} \right), \quad \frac{\Gamma_-(y; L)}{my} \in \left(\frac{2}{5}, 1 \right), \quad F_{\pm}(y, \Gamma_{\pm}(y; L); L) = 0.$$

For y, L fixed, $\Gamma_{\pm} := \Gamma_{\pm}(y; L)$ are the unique solutions of $F_{\pm}(y, \Gamma_{\pm}; L) = 0$ belonging to $\left(0, \frac{y}{2} \right)$. We also have that $\partial_{\Gamma} F_{\pm}(y, \Gamma; L) < 0$ for $\Gamma \in \left(0, \frac{y}{2} \right)$.

Proof. For the existence and regularity of $\Gamma_{\pm}$, we only need to define $\Gamma_{\pm}(y; L) = ys_{\pm} \left(\frac{y}{L}; L \right)$ and to apply Lemma 5.1. In addition, using (5.4), we have that

$$F_{\pm}(y, \Gamma; L) = \hat{F}_{\pm} \left(\frac{\Gamma}{y}, \frac{y}{L}; L \right) = -6Ly^2 \mathcal{G}_{\pm} \left(\frac{\Gamma}{y}, \frac{y}{L}; L \right)$$

and then

$$\partial_{\Gamma} F_{\pm}(y, \Gamma; L) = -6Ly^2 \partial_s \mathcal{G}_{\pm} \left(\frac{\Gamma}{y}, \frac{y}{L}; L \right) < 0,$$

where we have used that, by Lemma 5.1, $\partial_s \mathcal{G}_{\pm}(s, \hat{y}; L) > 0$. $\square$

Corollary 5.5 proves the existence of critical points. However, we cannot elucidate neither the range of the parameters Γ, L for which they exist nor how many critical points the system possesses for given values of the parameters. To do so, let us study the functions

$$\hat{F}_{\pm}(\hat{y}; L) = \frac{\Gamma_{\pm}(L\hat{y}; L)}{L} = \hat{y}s_{\pm}(\hat{y}; L),$$

where $s_{\pm}(\hat{y}; L)$ are defined implicitly by the equations $\mathcal{G}_{\pm}(s, \hat{y}; L) = 0$ (see Lemma 5.2). That is $\mathcal{G}_{\pm}(s_{\pm}(\hat{y}; L), \hat{y}; L) = 0$. Therefore, writing $s_{\pm} = s_{\pm}(\hat{y}; L)$ and taking the derivative with respect to $\hat{y}$ in the equation $\mathcal{G}_{\pm}(s, \hat{y}; L) = 0$, we have that

$$\partial_{\hat{y}} \mathcal{G}_{\pm}(s_{\pm}, \hat{y}; L) + \partial_s \mathcal{G}_{\pm}(s_{\pm}, \hat{y}; L) \partial_{\hat{y}} s_{\pm}(\hat{y}; L) = 0.$$

We recall that by Lemma 5.1, $\partial_s \mathcal{G}_{\pm} > 0$. This fact allows computing $\partial_{\hat{y}} s_{\pm}(\hat{y}; L)$ in terms of $s_{\pm}$ and, consequently,

$$\begin{aligned} \partial_{\hat{y}} \hat{F}_{\pm}(\hat{y}; L) &= s_{\pm}(\hat{y}; L) + \hat{y} \partial_{\hat{y}} s_{\pm}(\hat{y}; L) \\ &= \frac{1}{\partial_s \mathcal{G}_{\pm}(s_{\pm}, \hat{y}; L)} \left[s_{\pm}(\hat{y}; L) \partial_s \mathcal{G}_{\pm}(s_{\pm}, \hat{y}; L) - \hat{y} \partial_{\hat{y}} \mathcal{G}_{\pm}(s_{\pm}, \hat{y}; L) \right]. \end{aligned} \quad (5.12)$$

We want to study the possible changes in the sign of $\partial_{\hat{y}} \hat{F}_{\pm}$. We notice that, following the same kind of computations,

$$\partial_L \hat{F}_{\pm}(\hat{y}; L) = -\frac{\partial_L \mathcal{G}_{\pm}(s_{\pm}, \hat{y}; L)}{\partial_s \mathcal{G}_{\pm}(s_{\pm}, \hat{y}; L)} \hat{y}. \quad (5.13)$$

Remark 5.6. Notice that the value $\hat{y}_{\min}$ defined in Lemma 5.2 satisfies that $\hat{y}_{\min} < \hat{y}_{\text{col}}$, the collision value $\hat{y}_{\text{col}}$ in (3.10). For that reason, from now on, we will restrict our analysis to values of $\hat{y} \in \left[\hat{y}_{\min}, \frac{1}{2} \right)$.

As we will see below, it turns out that the analysis of the functions $\hat{F}_+$ and $\hat{F}_-$ are quite different, being the corresponding to $\hat{F}_-$ more involved.

We start with the result for the fixed points of the form $(0, y)$, which corresponds with the $+$ case, in Section 5.3, and we postpone the study of the fixed points of the form (π, y) to Section 5.4.

5.3. The fixed points of the form $(0, y)$

For any $L_{\min} \leq L \leq L_{\max}$ we define

$$\begin{aligned}\hat{r}_{\min}^+ &= \hat{y}_{\min} m = 0.013584391815073, \\ \hat{r}_{\max}^+(L) &= \hat{r}_+\left(\frac{1}{2}; L\right) = \frac{1}{2} s_+\left(\frac{1}{2}; L\right).\end{aligned}$$

We emphasize that, by Lemma 5.2, $s_+\left(\frac{1}{2}; L\right) \in \left(m, \frac{3}{2}m\right)$ and therefore

$$\hat{r}_{\max}^+(L) \in \left[\frac{1}{2}m, \frac{3}{4}m\right] = [0.058257569495584, 0.087386354243376]. \quad (5.14)$$

Proposition 5.7. *The function $\hat{r}_{\max}^+(L)$ is strictly increasing on $[L_{\min}, L_{\max}]$ and therefore*

$$\hat{r}_{\max}^+(L) \in [\hat{r}_{\max}^+(L_{\min}), \hat{r}_{\max}^+(L_{\max})] = [0.058270961487710, 0.086680627913961]$$

where these values have been computed numerically.⁴ Moreover,

$$\hat{r}^+(\hat{y}; L) \in [\hat{r}_{\min}^+, \hat{r}_{\max}^+(L)], \quad \text{for all } \hat{y}_{\min} \leq \hat{y} \leq \frac{1}{2},$$

and $\hat{r}^+(\cdot; L)$ is an injective increasing function.

With respect to the equilibrium points, for any $L \in [L_{\min}, L_{\max}]$:

1. If $\hat{r} \notin [\hat{r}_{\min}^+, \hat{r}_{\max}^+(L)]$, system (4.1) has no fixed points of the form $(0, L\hat{y})$ satisfying $\hat{y}_{\min} < \hat{y} \leq \frac{1}{2}$ and $\hat{r} \in \left(0, \frac{1}{2}\hat{y}\right)$.
2. If $\hat{r} \in [\hat{r}_{\min}^+, \hat{r}_{\max}^+(L)]$, there exists a unique $\hat{y}_+ = \hat{y}_+(\hat{r}; L)$ such that $(0, L\hat{y}_+)$ is a fixed point of the system (4.1). In addition, $\hat{y}_+(\cdot; L)$ is a strictly increasing function.

Proof. We start with the statement related to $\hat{r}_{\max}^+$. Using (5.13), we have that

$$\partial_L \hat{r}_{\max}^+(L) = \partial_L \hat{r}^+\left(\frac{1}{2}; L\right) = -\frac{\partial_L G_+\left(s_+, \frac{1}{2}; L\right)}{2\partial_s G_+\left(s_+, \frac{1}{2}; L\right)}.$$

The sign of $\partial_L G_+\left(s_+, \frac{1}{2}; L\right)$ is the sign of $\mathbf{b}_+(s_+) + \frac{1}{4}\mathbf{c}_+(s_+)$ (see definition (5.5)). Then, since by Lemma 5.1, $\mathbf{b}_+, \mathbf{c}_+$ are increasing functions and by Lemma 5.2, $s_+ \in \left(m, \frac{3}{2}m\right)$

$$\mathbf{b}_+(s_+) + \frac{1}{4}\mathbf{c}_+(s_+) \leq \mathbf{b}_+\left(\frac{3}{2}m\right) + \frac{1}{4}\mathbf{c}_+\left(\frac{3}{2}m\right) < 0,$$

where we have used (5.7) and (5.8) to estimate $\mathbf{b}_+\left(\frac{3}{2}m\right) + \frac{1}{4}\mathbf{c}_+\left(\frac{3}{2}m\right)$. Therefore, $\partial_L G_+ < 0$ and, using that $\partial_s G_+ > 0$ by Lemma 5.1, $\partial_L \hat{r}_{\max}^+ > 0$.

Now we prove the rest of the properties. We write

$$\partial_{\hat{y}} G_+(s, \hat{y}; L) = 2\alpha^3 L^4 \rho \left[3\mathbf{b}_+(s)\hat{y}^2 + 5\mathbf{c}_+(s)\hat{y}^4 \right].$$

By Lemma 5.1, $\mathbf{b}_+, \mathbf{c}_+$ are increasing functions and, by Lemma 5.2, $s_+ \in \left(m, \frac{3}{2}m\right)$. Therefore, for $s \in \left(m, \frac{3}{2}m\right)$,

$$\partial_{\hat{y}} G_+(s, \hat{y}; L) \leq 2\alpha^3 L^4 \rho \left[3\mathbf{b}_+\left(\frac{3}{2}m\right)\hat{y}^2 + 5\mathbf{c}_+\left(\frac{3}{2}m\right)\hat{y}^4 \right].$$

Using the values of $\mathbf{b}_+\left(\frac{3}{2}m\right), \mathbf{c}_+\left(\frac{3}{2}m\right)$ in (5.7) and (5.8), if

$$\hat{y} \geq \sqrt{-\frac{3\mathbf{b}_+\left(\frac{3}{2}m\right)}{5\mathbf{c}_+\left(\frac{3}{2}m\right)}} = 0.116570155823739,$$

we have then $\partial_{\hat{y}} G_+(s, \hat{y}; L) \leq 0$. In particular, the same happens if $\hat{y} \geq \hat{y}_{\min}$ (see Lemma 5.2 for the exact value of $\hat{y}_{\min}$). As a consequence, from (5.12), $\partial_{\hat{y}} \hat{r}_+(\hat{y}; L) > 0$ if $\hat{y}_{\min} \leq \hat{y} \leq \frac{1}{2}$ and $L_{\min} \leq L \leq L_{\max}$ so that $\hat{r}_+(\cdot; L)$ is an injective (strictly increasing) function for any fixed $L \in [L_{\min}, L_{\max}]$. The range of values of $\hat{r}_+$ for a given L is then

$$\hat{r}^+(\hat{y}; L) \in \left[\hat{r}_+(\hat{y}_{\min}; L), \hat{r}_+\left(\frac{1}{2}; L\right)\right].$$

⁴ Notice that there is not a significant difference with the estimates in (5.14).

We notice that, since $G_+(m, \hat{y}_{\min}; L) = 0$ (see Lemma 5.1),

$$\hat{F}_+(\hat{y}_{\min}; L) = \hat{y}_{\min} s_+(\hat{y}_{\min}; L) = \hat{y}_{\min} m = \hat{F}_{\min}^+.$$

For a given L , let $\hat{y}_+(\hat{F}; L)$ be such that

$$\hat{F}_+(\hat{y}_+(\hat{F}; L); L) = \hat{F}.$$

It is clear that $\hat{y}_+(\cdot; L)$ is an increasing function and it is defined for $\hat{F} \in [\hat{F}_{\min}^+, \hat{F}_{\max}^+(L)]$. $\square$

Remark 5.8. If we want to control the range of $\hat{F}$ for $\hat{y}_{\text{col}} \leq \hat{y} \leq \frac{1}{2}$, we notice that, since $\hat{F}_+(\cdot; L)$ is an increasing function, the “new” $\hat{F}_{\min}^+ = \hat{F}_{\min}^+(L) = \hat{F}_+(y_{\text{col}}; L)$. Then, since $\hat{F}_+(y_{\text{col}}; L) = \hat{y}_{\text{col}} s_+(y_{\text{col}}; L)$,

$$\hat{F}_{\min}^+(L) \in \left[\hat{y}_{\text{col}} m, \hat{y}_{\text{col}} \frac{3}{2} m \right] = [0.035912783574701, 0.053869175362051].$$

We emphasize that $\hat{F}_+(y_{\text{col}}; \cdot)$ is also an increasing function. Indeed, writing $s_+ = s_+(y_{\text{col}}; L)$, the sign of $-\partial_L \hat{F}_+(y_{\text{col}}; \cdot)$ is the sign of

$$\mathbf{b}_+(s_+) + \hat{y}_{\text{col}}^2 \mathbf{c}_+(s_+) \leq \mathbf{b}_+\left(\frac{3}{2}m\right) + \hat{y}_{\text{col}}^2 \mathbf{c}_+\left(\frac{3}{2}m\right) = -5.701123730321832 < 0$$

where, again, we have used (5.7) and (5.8) for $\mathbf{b}_+\left(\frac{3}{2}m\right)$ and $\mathbf{c}_+\left(\frac{3}{2}m\right)$. Therefore, we numerically obtain that

$$\hat{F}_{\min}^+(L) \in [\hat{F}_+(y_{\text{col}}; L_{\min}), \hat{F}_+(y_{\text{col}}; L_{\max})] = [0.035913449550478, 0.037427016836718].$$

We notice that in this case, the numerical computation induces a more accurate range of values of $\hat{F}_{\min}^+(L)$.

5.4. The fixed points of the form (π, y)

Now we pay attention to the $-$ case. In this case, $\hat{F}_-$ is not an injective function and for that reason, for studying the behavior of $\hat{F}_-$, we must control the existence of critical points, namely the values of $\hat{y}, L$ such that $\partial_{\hat{y}} \hat{F}_-(\hat{y}; L) = 0$. To this end, we introduce

$$\mathcal{H}(s, \hat{y}; L) := s \partial_s \mathcal{G}_-(s, \hat{y}; L) - \hat{y} \partial_{\hat{y}} \mathcal{G}_-(s, \hat{y}; L), \quad \partial_{\hat{y}} \hat{F}_-(\hat{y}; L) = \frac{\mathcal{H}(s_-, \hat{y}; L)}{\partial_s \mathcal{G}_-(s_-, \hat{y}; L)} \quad (5.15)$$

with $s_- = s_-(\hat{y}; L)$. By the uniqueness statement in Lemma 5.1, we have that $\mathcal{G}_-(s, \hat{y}; L) = 0$ if and only if $s = s_-(\hat{y}; L)$. Moreover $s_-(\hat{y}; L) \in \left(\frac{2}{5}m, m\right)$. Therefore, the equation

$$\partial_{\hat{y}} \hat{F}_-(\hat{y}; L) = 0$$

is equivalent to the existence of solutions of

$$\mathcal{H}(s, \hat{y}; L) = 0, \quad \mathcal{G}_-(s, \hat{y}; L) = 0$$

under the restrictions $s \in \left(\frac{2}{5}m, m\right)$, $\hat{y} \in \left[\hat{y}_{\min}, \frac{1}{2}\right]$ (see Remark 5.6), $L \in [L_{\min}, L_{\max}]$.

From expression (5.5) of $\mathcal{G}_-$, we write the system in a more suitable way

$$\begin{aligned} \mathcal{H}(s, \hat{y}; L) &= \bar{\mathbf{a}}(s) + 2\alpha^3 L^4 \rho [\bar{\mathbf{b}}(s) \hat{y}^3 + \bar{\mathbf{c}}(s) \hat{y}^5] = 0 \\ \mathcal{G}_-(s, \hat{y}; L) &= \mathbf{a}(s) + 2\alpha^3 L^4 \rho [\mathbf{b}_-(s) \hat{y}^3 + \mathbf{c}_-(s) \hat{y}^5] = 0 \end{aligned} \quad (5.16)$$

with

$$\begin{aligned} \bar{\mathbf{a}}(s) &= s \partial_s \mathbf{a}(s) = 2s(5s + 4) \\ \bar{\mathbf{b}}(s) &= s \partial_s \mathbf{b}_-(s) - 3\mathbf{b}_-(s) = -10U_2^{0,0} s(s + 2) - \frac{20U_2^{1,0} s^2(s^2 + 4s - 3)}{3(1-s)\sqrt{3-2s-s^2}} \\ \bar{\mathbf{c}}(s) &= s \partial_s \mathbf{c}_-(s) - 5\mathbf{c}_-(s) = 8U_2^{0,0}(5 - 12s) - \frac{80U_2^{1,0}(8s^3 + 5s^2 - 60s + 45)}{3(1-s)\sqrt{3-2s-s^2}}. \end{aligned} \quad (5.17)$$

Hence, in order to solve system (5.16), we write from the second equation

$$2\alpha^3 L^4 \rho = -\frac{\mathbf{a}(s)}{\mathbf{b}_-(s) \hat{y}^3 + \mathbf{c}_-(s) \hat{y}^5}$$

and substituting this value into the first equation, we obtain that

$$\mathcal{H}(s, \hat{y}; L) = \frac{\bar{\mathbf{a}}(s) \mathbf{b}_-(s) - \mathbf{a}(s) \bar{\mathbf{b}}(s) + \hat{y}^2 (\bar{\mathbf{a}}(s) \mathbf{c}_-(s) - \mathbf{a}(s) \bar{\mathbf{c}}(s))}{\mathbf{b}_-(s) + \mathbf{c}_-(s) \hat{y}^2} = 0$$

so that

$$\hat{y}^2 = \frac{\mathbf{a}(s) \bar{\mathbf{b}}(s) - \bar{\mathbf{a}}(s) \mathbf{b}_-(s)}{\bar{\mathbf{a}}(s) \mathbf{c}_-(s) - \mathbf{a}(s) \bar{\mathbf{c}}(s)}.$$

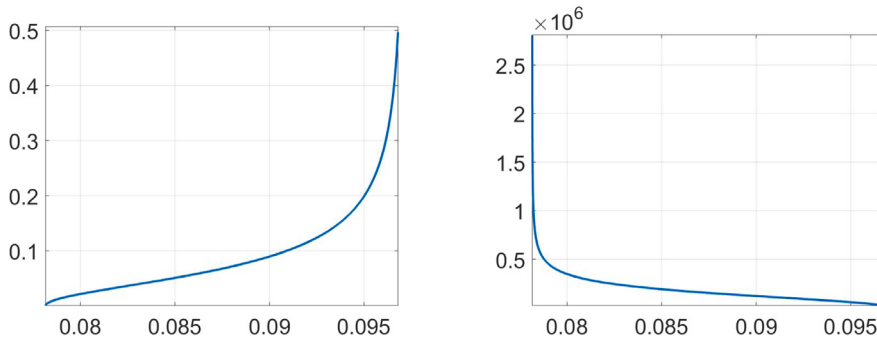


Fig. 10. On the left, $\hat{y}_*(s)$ and on the right $a_*(s)$. Only the values of s such that $\hat{y}_*(s) \in \left[0, \frac{1}{2}\right]$ are considered. In this figure, the semi-major axis a is measured in km.

Therefore system (5.16) has a unique solution for those values of $s \in \left(\frac{2}{5}m, m\right)$ such that the following restrictions hold

$$\begin{aligned} \hat{y}_{\min} \leq \hat{y} = \hat{y}_*(s) &:= \left(\frac{\mathbf{a}(s)\bar{\mathbf{b}}(s) - \bar{\mathbf{a}}(s)\mathbf{b}_-(s)}{\bar{\mathbf{a}}(s)\mathbf{c}_-(s) - \mathbf{a}(s)\bar{\mathbf{c}}(s)} \right)^{\frac{1}{2}} \leq \frac{1}{2}, \\ a_{\min} \leq a = a_*(s) &:= \left(-\frac{\mathbf{a}(s)a_M^3}{2\mu^2\rho(\mathbf{b}_-(s)\hat{y}_*^3(s) + \mathbf{c}_-(s)\hat{y}_*^5(s))} \right)^{\frac{1}{5}} \leq a_{\max}. \end{aligned} \quad (5.18)$$

Lemma 5.9. For those values of s such that $\hat{y}_*(s) \in \left(\hat{y}_{\min}, \frac{1}{2}\right]$, $\hat{y}_*(s)$ is an increasing function and $a_*(s)$ is decreasing. Defining implicitly $s_{\min}$ and $s_{\max}$ such that

$$a_*(s_{\min}) = a_{\max}, \quad \hat{y}_*(s_{\max}) = \frac{1}{2},$$

we have that, for $s \in [s_{\min}, s_{\max}]$, $\hat{I}_-(\hat{y}; L)$ has a unique critical point at $(\hat{y}, L) = (\hat{y}_*(s), L_*(s))$ with $L_*(s) = \sqrt{\mu a_*(s)}$. The values of $s_{\min}$ and $s_{\max}$ and $L_*(s_{\max}), \hat{y}_*(s_{\min})$ can be computed numerically:

$$s_{\min} = 0.096577225237580, \quad s_{\max} = 0.096796816334740,$$

and then $a_*(s_{\max}) = 23893.56218133389$ km,

$$L_*(s_{\max}) = 97590.90325766560 \text{ km}^2/\text{s}, \quad \hat{y}_*(s_{\min}) = 0.397273020602216.$$

In addition, if $L \in [L_{\min}, L_*(s_{\max})]$ there are no critical points of $\hat{I}_-(\cdot; L)$ belonging to $\left[\hat{y}_{\min}, \frac{1}{2}\right)$.

Proof. One can perform an analytic thorough study of $\hat{y}_*(s), a_*(s)$ as a function of s . However, we have preferred just to draw the explicit functions $\hat{y}_*(s), a_*(s)$ in order to convince the reader about this result. For those values of s such that $\hat{y}_*(s) \in \mathbb{R}$ with $0 < \hat{y}_*(s) \leq \frac{1}{2}$, we draw the corresponding semi-major axis a , see Fig. 10.

These figures illustrate that indeed $\hat{y}_*(s)$ is an increasing function of s meanwhile $a_*(s)$ is decreasing.

If we only consider the values of s such that $\hat{y}_*(s) \in \left[\hat{y}_{\min}, \frac{1}{2}\right]$ and $a_*(s) \in [a_{\min}, a_{\max}]$ (which are the ones we are interested in) we observe the behavior in Fig. 11.

Summarizing, for the values of $s_{\min}, s_{\max}$ in the lemma we have that for any $s \in [s_{\min}, s_{\max}]$,

$$\partial_{\hat{y}} \hat{I}_-(\hat{y}_*(s); L_*(s)) = 0$$

with $L_*(s) = \sqrt{\mu a_*(s)}$ and $\hat{y}_*(s), a_*(s)$ defined in (5.18). In addition, these are the only possible critical points of $\hat{I}_-$. $\square$

We introduce now the boundary values of $\hat{I}$. We first define, for $L \in [L_*(s_{\max}), L_{\max}]$,

$$\hat{I}_*^-(L) = \hat{y}_*(s), \quad \text{with } s = s(L) \quad \text{such that } L = L_*(s).$$

Then we introduce, for $L \in [L_{\min}, L_{\max}]$

$$\hat{I}_{\min}^-(L) = \hat{I}_-(\hat{y}_{\min}; L), \quad \hat{I}_{\max}^-(L) = \hat{I}_-\left(\frac{1}{2}; L\right) \quad (5.19)$$

that, by Corollary 5.5, satisfy

$$\begin{aligned} \hat{I}_{\min}^-(L) &\in \left[\hat{y}_{\min} \frac{2}{5}m, \hat{y}_{\min}m\right] = [0.005433756726029, 0.013584391815074] \\ \hat{I}_{\max}^-(L) &\in \left[\frac{1}{5}m, \frac{1}{2}m\right] = [0.023303027798234, 0.058257569495584]. \end{aligned} \quad (5.20)$$

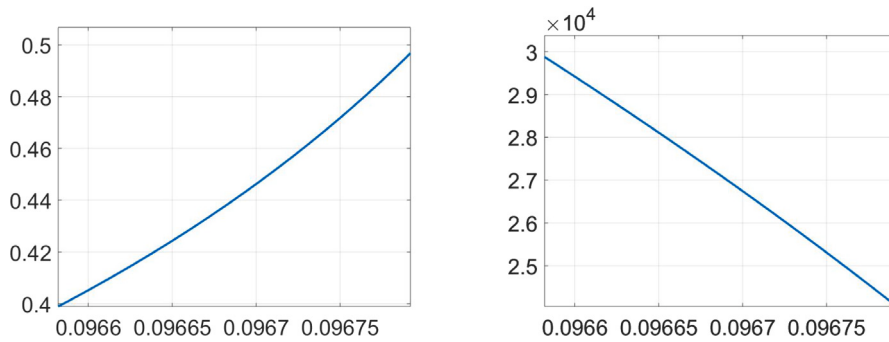


Fig. 11. As in Fig. 10, $\hat{y}_*(s)$ (left) and $a_*(s)$ (right). The values s considered as the ones such that $\hat{y}_{\min} \leq \hat{y}_*(s) \leq \frac{1}{2}$ and $a_{\min} \leq a \leq a_{\max}$ in km.

By Lemma 5.9, $\hat{y}_*(s)$ is an increasing function whereas $L_*(s)$ is decreasing. Let L_*^{-1} be its inverse. We have that $\hat{F}_*^-(L) = \hat{y}_*(L_*^{-1}(L))L_*^{-1}(L)$ is a decreasing function. Therefore

$$\hat{F}_*^-(L) \in [\hat{F}_*^-(L_{\max}), \hat{F}_*^-(L_*(s_{\max}))]$$

and, by definition of $s_{\min}, s_{\max}$ in Lemma 5.9,

$$\hat{F}_*^-(L) \in \left[\hat{y}_*(s_{\min})s_{\min}, \frac{1}{2} \right] = [0.038367525991514, 0.048398408167370].$$

Next lemma studies the monotonicity properties of the function $\hat{F}_-$.

Lemma 5.10. For $\hat{y}_{\min} \leq \hat{y} \leq \frac{1}{2}$, the function $\hat{F}_-(\hat{y}; \cdot)$ is strictly decreasing (with respect to L) on $[L_{\min}, L_{\max}]$. Therefore, $\hat{F}_{\min, \max}^-(L)$ (see definition (5.19)) are also decreasing and moreover

$$\hat{F}_{\min}^-(L) \in [\hat{F}_{\min}^-(L_{\max}), \hat{F}_{\min}^-(L_{\min})] = [0.013575332545548, 0.013584387878826]$$

and

$$\hat{F}_{\max}^-(L) \in [\hat{F}_{\max}^-(L_{\max}), \hat{F}_{\max}^-(L_{\min})] = [0.027639200647529, 0.058244177051364]$$

where the values have been computed numerically.

The function $\hat{F}_-(\cdot; L)$ satisfies, for $\hat{y}_{\min} \leq \hat{y} \leq \frac{1}{2}$, that

1. For $L \in [L_{\min}, L_*(s_{\max})]$, the function $\hat{F}_-(\cdot; L)$ is strictly increasing and moreover $\hat{F}_-(\hat{y}; L) \in [\hat{F}_{\min}^-(L), \hat{F}_{\max}^-(L)]$.
2. For $L \in (L_*(s_{\max}), L_{\max}]$, $\hat{F}_-(\cdot, L)$ has a maximum at $\hat{y}_{\max}(L) = \hat{y}_*(L_*^{-1}(L))$. In this case,

$$\hat{F}_-(\hat{y}; L) \in [\hat{F}_{\min}^-(L), \hat{F}_*^-(L)].$$

Remark 5.11. The statement of Lemma 5.10 can be graphically represented as in Fig. 12.

Proof. From formula (5.13) of $\partial_L \hat{F}_-$ and using that, by Lemma 5.1, $\partial_s \mathcal{G} > 0$, we have that the sign of $\partial_L \hat{F}_-$, is the same as for $-\partial_L \mathcal{G}_-(s_-, \hat{y}, L)$ or, in other words, the same as $-\mathbf{b}_-(s_-) - \hat{y}^2 \mathbf{c}_-(s_-)$ with $s_- = s_-(\hat{y}, L)$. Again, from Lemma 5.1, we have that $\mathbf{b}_-, \mathbf{c}_-$ are increasing functions and using their values (5.10) and (5.11) at $s = \frac{2}{5}m$, we conclude that $\mathbf{b}_-(s_-) + \hat{y}^2 \mathbf{c}_-(s_-) > 0$ and so $\hat{F}_-(\hat{y}; \cdot)$ is decreasing.

By Lemma 5.9, when $L \in [L_{\min}, L_*(s_{\max})]$ there are no critical points of $\hat{F}_-$ in the interval $\left[\hat{y}_{\min}, \frac{1}{2} \right)$. That implies that $\hat{F}_-(\cdot; L)$ is either increasing or decreasing. Notice that, by definition (5.19) of $\hat{F}_{\min, \max}^-$ and bounds (5.20),

$$\hat{F}_-(\hat{y}_{\min}; L) = \hat{F}_{\min}^-(L) \leq \hat{F}_{\max}^-(L) = \hat{F}_-\left(\frac{1}{2}; L\right)$$

and therefore $\hat{F}_-(\cdot; L)$ is a strictly increasing function in $[L_{\min}, L_*(s_{\max})]$.

When $L = L_*(s_{\max})$ we have that the corresponding critical point is $\hat{y} = y_*(s_{\max}) = \frac{1}{2}$. Therefore, the same argument as before allows to conclude that also in this case $\hat{F}_-(\cdot; L)$ is strictly increasing for $\hat{y}_{\min} \leq \hat{y} < \frac{1}{2}$.

Take now $L \in (L_*(s_{\max}), L_{\max}]$ and let $s = L_*^{-1}(L) \in [s_{\min}, s_{\max}]$ be such that $L = L_*(s)$. By Lemma 5.9, we already know that $\hat{y}_*(s)$ is a critical point of $\hat{F}_-(\cdot; L)$ and that $\hat{F}_-(\hat{y}_*(s); L_*(s)) = \hat{F}_*^-(L)$. So we only need to check that

$$\partial_{\hat{y}}^2 \hat{F}_-(\hat{y}_*(L_*^{-1}(L)); L) = \partial_{\hat{y}}^2 \hat{F}_-(\hat{y}_*(s); L_*(s)) < 0, \quad s = L_*^{-1}(L).$$

We first recall expression (5.15) of $\hat{F}_-$ in (5.15):

$$\partial_{\hat{y}} \hat{F}_-(\hat{y}; L) = \frac{\mathcal{H}(s_-(\hat{y}; L), \hat{y}; L)}{\partial_s \mathcal{G}_-(s_-(\hat{y}; L), \hat{y}; L)}.$$

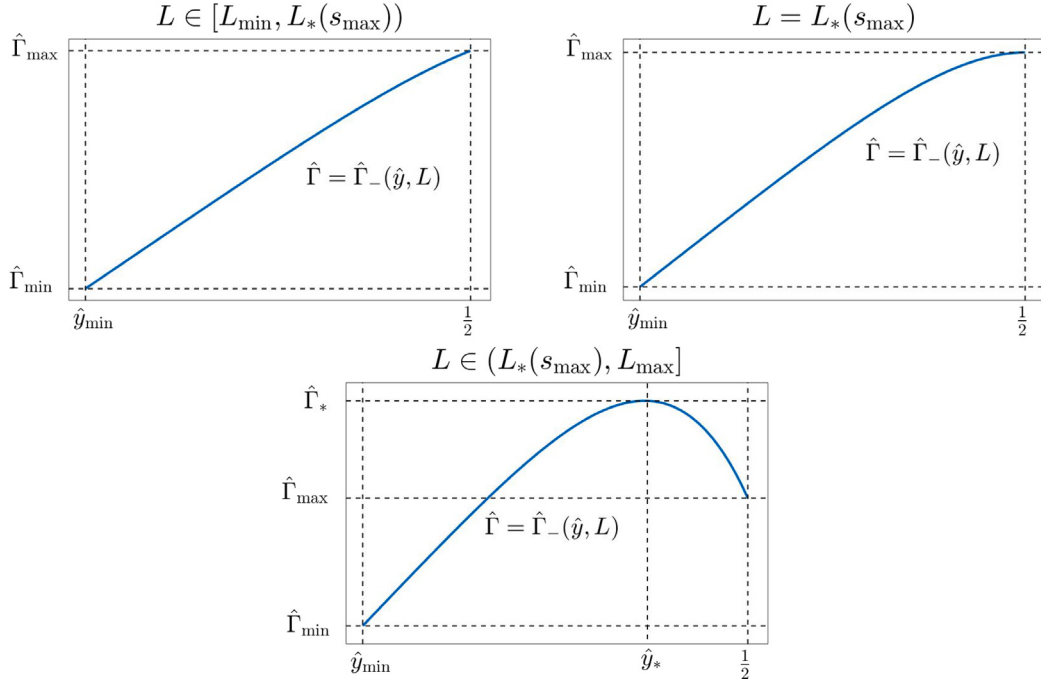


Fig. 12. For a given value of L , the figures represent qualitatively the function $\hat{\Gamma}_-(\hat{y}, L)$ (as a function of $\hat{y}$). There are depicted the three different behaviors depending on the value of L . See also Fig. 13 for numerical computations.

Then, using that, by construction, $\mathcal{H}(s, \hat{y}_*(s); L_*(s)) = 0$ and $s_-(\hat{y}_*(s); L_*(s)) = s$

$$\partial_{\hat{y}}^2 \hat{\Gamma}_-(\hat{y}_*(s); L_*(s)) = \frac{\partial_{\hat{y}} \mathcal{H}(s, \hat{y}_*(s); L_*(s))}{\partial_s \mathcal{G}_-(s, \hat{y}_*(s); L_*(s))},$$

and, since by Lemma 5.1, $\partial_s \mathcal{G} > 0$, we need to compute the sign of $\partial_{\hat{y}} \mathcal{H}(s, \hat{y}_*(s); L_*(s))$. From (5.16) we have that

$$\partial_{\hat{y}} \mathcal{H}(s, \hat{y}; L) = 2\alpha^3 L^4 \rho [3\bar{\mathbf{b}}(s)\hat{y}^2 + 5\bar{\mathbf{c}}(s)\hat{y}^4].$$

Then, using that $\mathcal{H}(s, \hat{y}_*(s); L_*(s)) = 0$, we deduce that

$$\partial_{\hat{y}} \mathcal{H}(s, \hat{y}_*(s); L_*(s)) = \frac{1}{\hat{y}_*(s)} (-3\bar{\mathbf{a}}(s) + 4\alpha^3 L^4 \rho \bar{\mathbf{c}}(s)(\hat{y}_*(s))^5).$$

On the other hand, from expression (5.17) for $\bar{\mathbf{c}}$,

$$\bar{\mathbf{c}}(s) \leq 8U_2^{0,0}(5 - 12s)$$

where we have used that $45 - 60s + 5s^2 + 8s^3 \geq 45 - 60s \geq 45 - 60m > 0$. Therefore, using again that $s \in (\frac{2}{5}m, m)$, $0 < \hat{y}_*(s) \leq \frac{1}{2}$ and $a_{\min} \leq a \leq a_{\max}$, we obtain

$$\begin{aligned} \hat{y}_*(s) \partial_{\hat{y}} \mathcal{H}(s, \hat{y}; L) &\leq -6s(5s + 4) + 32\alpha^3 L^4 \rho (\hat{y}_*(s))^5 U_2^{0,0}(5 - 12s) \\ &\leq -6 \cdot \frac{2}{5}m(2m + 4) + \frac{a_{\max}^3}{a_M^3} L_{\max}^4 \rho U_2^{0,0} \left(5 - \frac{24}{5}m\right) \\ &= -0.778057059724233 < 0. \end{aligned}$$

As a conclusion $\partial_{\hat{y}} \mathcal{H}(s, \hat{y}; L) < 0$ if $(\hat{y}, L) = (\hat{y}_*(s), L_*(s))$ and that implies that $\partial_{\hat{y}}^2 \hat{\Gamma}_-(\hat{y}_*(s); L_*(s)) < 0$ so that $\hat{y}_*(s)$ is a maximum of $\hat{\Gamma}(\cdot; L_*(s))$. $\square$

Remark 5.12. The values of $\hat{\Gamma}_-(\hat{y}; L)$ can be numerically computed for any fixed $\hat{y}$, L as $\hat{\Gamma}_-(\hat{y}; L) = s_-(\hat{y}; L)\hat{y}$ with s_- the zero of the function $\mathcal{G}_-(s, \hat{y}; L)$. In Fig. 13 we present some representative values of a (recall that $L = \sqrt{\mu a}$) where we can find the different behavior described in Lemma 5.10.

Remark 5.13. We first notice that $\hat{\Gamma}_*(L_*(s_{\max})) = \hat{\Gamma}_{\max}^-(L_*(s_{\max}))$. Then, using that by Lemma 5.10, $\hat{\Gamma}_{\max}^-$ is decreasing (in its variable L), we also have that for $L \in (L_*(s_{\max}), L_{\max}]$,

$$\hat{\Gamma}_{\max}^-(L) < \hat{\Gamma}_*(L).$$

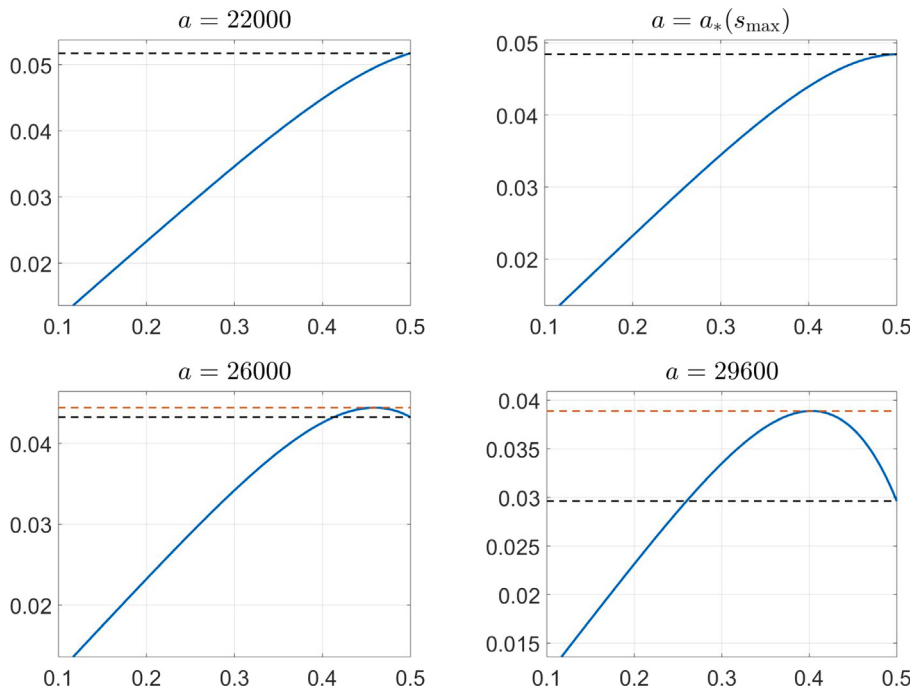


Fig. 13. The function $\hat{F}_-(\hat{y}; L)$ for $a = 22000, a_*(s_{\max}), 26000, 29600$, km. Notice that for $a = 22000, a_*(s_{\max})$ the function is strictly increasing but when $a = a_*(s_{\max})$ $\hat{F}_-(\hat{y}; L_*(s_{\max}))$ has a critical point at $\hat{y} = \frac{1}{2}$. The black dotted line is for $\hat{F}_{\max}^-(L)$ meanwhile the orange one is for $\hat{F}_*^-(L)$.

Moreover, since $s_{\max} = s_- \left(\frac{1}{2}, L_*(s_{\max}) \right)$, we have that $s_{\max} \leq m$. Therefore, using that by (5.14), $\hat{F}_{\max}^+(L) \geq \frac{1}{2}m$, we have that

$$\hat{F}_*^-(L) \leq \frac{1}{2}s_{\max} \leq \frac{1}{2}m \leq \hat{F}_{\max}^+(L).$$

From (5.20) it is also clear that the constant $\hat{F}_{\min}^+ = \hat{y}_{\min}m$ satisfies that

$$\hat{F}_{\min}^-(L) \leq \hat{F}_{\min}^+ \leq \hat{F}_{\max}^-(L).$$

From this analysis, it is straightforward to deduce the following result about the existence of equilibrium points of the form (π, y) (that is, the existence results of Theorem 3.7).

Proposition 5.14. *Let $L \in [L_{\min}, L_{\max}]$. If $\hat{F} \in [\hat{F}_{\min}^-(L), \hat{F}_{\max}^-(L)]$, there exists a unique $\hat{y}_-(\hat{F}; L)$ such that system (4.1) has a fixed point of the form $(\pi, y) = (\pi, L\hat{y}_-(\hat{F}; L))$. The function $\hat{y}_-(\cdot; L)$ is strictly increasing.*

In addition, for $L \in [L_{\min}, L_(s_{\max})]$:*

1. *If $\hat{F} \notin [\hat{F}_{\min}^-(L), \hat{F}_{\max}^-(L)]$, system (4.1) has no fixed points of the form $(\pi, L\hat{y})$ with $\hat{y}_{\min} \leq \hat{y} \leq \frac{1}{2}$ and $\hat{F} \in \left(0, \frac{1}{2}\hat{y}\right)$.*

and when $L \in (L_(s_{\max}), L_{\max}]$, we have that*

1. *If $\hat{F} \in [\hat{F}_{\max}^-(L), \hat{F}_*(L)]$, there exist only two functions $\hat{y}_{\pm}^{1,2}(\hat{F}; L)$ satisfying*

$$\hat{y}_{\pm}^1(\hat{F}; L) \leq \hat{y}_{\max}(L) \leq \hat{y}_{\pm}^2(\hat{F}; L)$$

with $\hat{y}_{\max}(L)$ defined in Lemma 5.10, such that system (4.1) has two fixed point of the form $(\pi, y) = (\pi, L\hat{y}_{\pm}^{1,2}(\hat{F}; L))$. In addition, $\hat{y}_{\pm}^1(\cdot; L)$ is strictly increasing and $\hat{y}_{\pm}^2(\cdot; L)$ is strictly decreasing.

2. *If $\hat{F} = \hat{F}_*(L)$, there exists only one fixed point $(\pi, \hat{y}_{\max}(L))$ with $\hat{y}_{\max}(L)$ a decreasing function.*
3. *If $\hat{F} \notin [\hat{F}_{\min}^-(L), \hat{F}_*(L)]$, system (4.1) has no fixed points of the form $(\pi, L\hat{y})$ with $\hat{y}_{\min} \leq \hat{y} \leq \frac{1}{2}$ and $\hat{F} \in \left(0, \frac{1}{2}\hat{y}\right)$.*

5.5. Linearization around the critical points

To complete the proof of Theorem 3.7, we are going to study the character of the fixed points of the form $(0, y), (\pi, y)$. Using the decomposition of $B_{0,\alpha}$ in (4.5), we write system (4.1) as

$$\dot{x} = -g(y; L)[X_1(y, F; L) + X_2(y, F; L)\cos x]$$

$$\dot{y} = h(y, \Gamma; L) \sin x$$

with

$$g(y; L) = \frac{1}{256} \frac{\rho_0}{L^4 y^6}, \quad h(y, \Gamma; L) = \frac{1}{128} \frac{\rho \alpha^3}{L^4 y^5} \sqrt{P_1(y, \Gamma)} \mathcal{A}_1(y, \Gamma; L)$$

and

$$X_1(y, \Gamma; L) = B_0(y, \Gamma; L) + \alpha^3 B_1(y, \Gamma; L), \quad X_2(y, \Gamma; L) = \alpha^3 \frac{B_1(y, \Gamma; L)}{\sqrt{P_1(y, \Gamma)}}.$$

By [Corollary 5.5](#), we can characterize the fixed points as the sets

$$\{(0, y, \Gamma_+(y; L), L)\}, \quad \{(\pi, y, \Gamma_-(y; L), L)\}$$

with $L \in [L_{\min}, L_{\max}]$ and $\hat{y}_{\min} L \leq y < \frac{L}{2}$. They satisfy $F_{\pm}(y, \Gamma_{\pm}(y; L); L) = 0$, where $F_{\pm}$ are defined in (5.3), the sign + corresponds to $(0, y)$ and – otherwise. Equivalently we have that

$$F_{\pm}(y, \Gamma_{\pm}(y; L); L) = X_1(y, \Gamma_{\pm}(y; L); L) \pm X_2(y, \Gamma_{\pm}(y; L); L) = 0.$$

The variational equation at the fixed points $(0, y, \Gamma_+(y; L), L)$ or $(\pi, y, \Gamma_-(y; L), L)$, $\dot{z} = M^L(y)z$, is given by

$$M^L(y) := \begin{pmatrix} 0 & -g(y; L)[\partial_y X_1(y, \Gamma_{\pm}; L) \pm \partial_y X_2(y, \Gamma_{\pm}; L)] \\ \pm h(y, \Gamma_{\pm}; L) & 0 \end{pmatrix},$$

where $\Gamma_{\pm} = \Gamma_{\pm}(y; L)$. The eigenvalues of M^L then satisfy

$$\lambda^2 = -g(y; L)h(y, \Gamma_{\pm}; L)[\partial_y X_2(y, \Gamma_{\pm}; L) \pm \partial_y X_1(y, \Gamma_{\pm}; L)].$$

Since clearly $g(y; L)h(y, \Gamma; L) > 0$, we need to study the sign of

$$\mathcal{E}_{\pm}(y; L) = -(\partial_y X_2(y, \Gamma_{\pm}(y; L); L) \pm \partial_y X_1(y, \Gamma_{\pm}(y; L); L)) = \mp \partial_y F_{\pm}(y, \Gamma_{\pm}; L)$$

for $L \in [L_{\min}, L_{\max}]$ and $\hat{y}_{\min} L \leq y \leq \frac{L}{2}$.

On the other hand, from $F_{\pm}(y, \Gamma_{\pm}(y; L); L) = 0$, we have that

$$\partial_y F_{\pm}(y, \Gamma_{\pm}(y; L); L) + \partial_{\Gamma} F_{\pm}(y, \Gamma_{\pm}(y; L); L) \partial_y \Gamma_{\pm}(y; L) = 0$$

and then

$$\mathcal{E}_{\pm}(y; L) = \pm \partial_{\Gamma} F_{\pm}(y, \Gamma_{\pm}(y; L); L) \partial_y \Gamma_{\pm}(y; L). \quad (5.21)$$

Proposition 5.15. *We have that*

1. For $L \in [L_{\min}, L_{\max}]$ and $\Gamma \in [L\hat{\Gamma}_{\min}^+, L\hat{\Gamma}_{\max}^+(L)]$, the unique fixed point of the form $(0, y)$ satisfies that $y = y_+(\Gamma; L) := L\hat{y}_+ \left(\frac{\Gamma}{L}; L \right)$ and it is a center.
2. For $L \in [L_{\min}, L_{\max}]$ and $\Gamma \in [L\hat{\Gamma}_{\min}^-(L), L\hat{\Gamma}_{\max}^-(L)]$, the unique fixed point of the form (π, y) satisfies that $y = y_-(\Gamma; L) := L\hat{y}_- \left(\frac{\Gamma}{L}; L \right)$ and it is a saddle.
3. For $L \in (L_*(s_{\max}), L_{\max}]$ and $\Gamma \in [L\hat{\Gamma}_{\max}^-(L), L\hat{\Gamma}_*^-(L)]$ there are two fixed points of the form (π, y) with $y = y_{-}^{1,2}(\Gamma; L) := L\hat{y}_{-}^{1,2} \left(\frac{\Gamma}{L}; L \right)$. Assume that $y^1 < y_{\max} < y^2$. Then $(\pi, y_{-}^1(\Gamma; L))$ is a saddle whereas $(\pi, y_{-}^2(\Gamma; L))$ is a center.
4. When $L \in [L_*(s_{\max}), L_{\max}]$ and $\Gamma = L\hat{\Gamma}_*^-(L)$ the unique fixed point $(\pi, y_-(\Gamma; L))$ is parabolic.

Proof. It is clear that if $\mathcal{E}_{\pm} > 0$, the corresponding fixed point is a saddle and if $\mathcal{E}_{\pm} < 0$, it is a center.

Note that (5.21) implies $\mathcal{E}_{\pm} = \mp \partial_{\Gamma} F_{\pm} \partial_y \Gamma_{\pm}$ and [Corollary 5.5](#) implies $\partial_{\Gamma} F_{\pm} < 0$. Then, when $\partial_y \Gamma_+ > 0$ the fixed point will be a center and if $\partial_y \Gamma_+ < 0$ the fixed point will be a saddle. Conversely, when $\partial_y \Gamma_- > 0$ the fixed point will be a saddle and if $\partial_y \Gamma_- < 0$ the fixed point will be a center.

From [Proposition 5.7](#), we have that $\partial_y \Gamma_+ > 0$ which proves the first item of the lemma.

For the values of L, Γ in the second item, by [Lemma 5.10](#) (see also [Fig. 12](#)), $\partial_y \Gamma_- = L\partial_{\hat{y}} \hat{\Gamma}_- > 0$ that implies that $\mathcal{E}_- > 0$ and the result follows.

With respect to the third item, we have that $\hat{\Gamma}_-(\cdot; L)$ has a maximum at $\hat{y}_{\max}$ so that $\partial_{\hat{y}} \hat{\Gamma}_-(\hat{y}; L) > 0$ for $\hat{y} < \hat{y}_{\max}$ and negative if $\hat{y} > \hat{y}_{\max}$ and the result holds true since $\hat{y}_-^1 < \hat{y}_{\max}$ and $\hat{y}_-^2 > \hat{y}_{\max}$.

The last item follows from the fact that, if $L = L_*(s)$ and $\hat{\Gamma} = \hat{\Gamma}_*(L)$, then $\partial_{\hat{y}} \hat{\Gamma}_-(\hat{y}_*(s); L_*(s)) = 0$ where we recall that $\hat{y}_*(s) = \hat{y}_{\max}$ and $\hat{\Gamma}_*(L) = \hat{y}_*(s)s$. $\square$

[Theorem 3.7](#) is a straightforward consequence of [Proposition 5.15](#) keeping track of the range of values of $\hat{\Gamma}_{\min, \max}^{\pm}$ and $\hat{\Gamma}_*$. Indeed, we only need to rename $\hat{\Gamma}_1 = \hat{\Gamma}_{\min}^+$, $\hat{\Gamma}_2 = \hat{\Gamma}_{\max}^+$, $\hat{\Gamma}_0 = \hat{\Gamma}_{\min}^+$, $\hat{\Gamma}_3 = \hat{\Gamma}_{\max}^+$, $\hat{\Gamma}_* = \hat{\Gamma}_*$, $L_* = L_*(s_{\max})$ and to rewrite [Proposition 5.15](#) in the terms of [Theorem 3.7](#), that is fixing the values of L . Notice that by [Remark 5.13](#) we have that

$$\hat{\Gamma}_1(L) \leq \hat{\Gamma}_0 \leq \hat{\Gamma}_2(L) \leq \hat{\Gamma}_*(L) \leq \hat{\Gamma}_3(L),$$

with the convection that $\hat{\Gamma}_*(L) = \hat{\Gamma}_2(L)$ when $L \in [L_{\min}, L_*(s_{\max})]$. To finish, we note that by [Proposition 5.7](#), $\hat{\Gamma}_3$ is increasing and by [Lemma 5.10](#) $\hat{\Gamma}_{1,2}$ are decreasing.

6. Periodic orbits of the full coplanar hamiltonian

6.1. Existence of periodic orbits. Proof of Theorem 3.9

Consider the full coplanar Hamiltonian $\mathcal{H}_{\text{CP}} = \mathcal{H}_0 + \alpha^3 \mathcal{H}_{\text{CP},1}$. We first emphasize that, by expressions of $\mathcal{H}_0$ in (2.14) and $\mathcal{H}_{\text{CP},1}$ in (A.1) in Appendix, $(\eta, \xi) = (0, 0)$ is invariant by the flow of $\mathcal{H}_{\text{CP}}$ and every orbit of the form $(0, \Gamma(t; L), 0, h(t; L))$ has to be in an energy level $\mathcal{H}_{\text{CP}}(0, \Gamma, 0, h) = E$ for some energy E with $\Gamma(t), h(t)$ satisfying the differential equations in Theorem 3.9.

We now prove that for a certain range of energies E (to be determined), the solutions $(0, \Gamma(t; L), 0, h(t; L))$ are periodic orbits which are a graph over the variable h .

Fix $E \in \mathbb{R}$ and $L \in (0, L_{\max}]$. Writing $\Gamma = L\hat{\Gamma}$,

$$\begin{aligned} E = \hat{\mathcal{H}}_{\text{CP}}(0, \hat{\Gamma}, 0, h) &:= -\frac{\rho_0}{16L^6}(1 - 12\hat{\Gamma} - 12\hat{\Gamma}^2) + \alpha^3 \frac{\rho_1 U_2^{0,0}}{32L^2}(1 - 12\hat{\Gamma} - 12\hat{\Gamma}^2) \\ &\quad - \alpha^3 \frac{\rho_1 U_2^{1,0}}{8L^2} \sqrt{(1 - 2\hat{\Gamma})(3 + 2\hat{\Gamma})} (2\hat{\Gamma} + 1) \cos h \\ &\quad - \alpha^3 \frac{\rho_1 U_2^{2,0}}{32L^2} (1 - 2\hat{\Gamma})(3 + 2\hat{\Gamma}) \cos 2h. \end{aligned} \quad (6.1)$$

We impose that $\partial_{\hat{\Gamma}} \hat{\mathcal{H}}_{\text{CP}}(0, \hat{\Gamma}, 0, h) \neq 0$ for all h , in other words $\hat{\Gamma} \neq 0$. This condition will give a set of possible values for $L, \hat{\Gamma}$. From the differential equations in Theorem 3.9, we need that

$$\begin{aligned} \dot{h} &= -\frac{3\rho_0(1 + 2\hat{\Gamma})}{4L^7} - \alpha^3 \frac{3\rho_1}{8L^3} U_2^{0,0}(1 + 2\hat{\Gamma}) \\ &\quad - \alpha^3 \frac{3\rho_1}{24L^3} \left(4U_2^{1,0} \frac{1 - 4\hat{\Gamma} - 4\hat{\Gamma}^2}{\sqrt{(1 - 2\hat{\Gamma})(3 + 2\hat{\Gamma})}} \cos h - U_2^{2,0}(1 + 2\hat{\Gamma}) \cos(2h) \right) < 0 \end{aligned}$$

for all $h \in [0, 2\pi]$ or equivalently

$$\begin{aligned} (1 + 2\hat{\Gamma}) + \frac{1}{2} \rho \alpha^3 L^4 U_2^{0,0} (1 + 2\hat{\Gamma}) + \frac{2}{3} \rho \alpha^3 L^4 U_2^{1,0} \frac{1 - 4\hat{\Gamma} - 4\hat{\Gamma}^2}{\sqrt{(1 - 2\hat{\Gamma})(3 + 2\hat{\Gamma})}} \cos h \\ - \frac{1}{6} \rho \alpha^3 L^4 U_2^{2,0} (1 + 2\hat{\Gamma}) \cos 2h > 0, \end{aligned} \quad (6.2)$$

where ρ has been introduced in (4.3).

To avoid cumbersome notations, we introduce

$$\begin{aligned} A &= A(\hat{\Gamma}; L) = (1 + 2\hat{\Gamma}) + \frac{1}{2} \rho \alpha^3 L^4 U_2^{0,0} (1 + 2\hat{\Gamma}) \\ B &= B(\hat{\Gamma}; L) = \frac{2}{3} \rho \alpha^3 L^4 U_2^{1,0} \frac{1 - 4\hat{\Gamma} - 4\hat{\Gamma}^2}{\sqrt{(1 - 2\hat{\Gamma})(3 + 2\hat{\Gamma})}} \\ C &= C(\hat{\Gamma}; L) = -\frac{1}{6} \rho \alpha^3 L^4 U_2^{2,0} (1 + 2\hat{\Gamma}), \end{aligned}$$

so that condition (6.2) reads as

$$f(h) = f(h; \hat{\Gamma}, L) := A + B \cos h + C \cos 2h > 0.$$

With respect to h , $f(h)$ has its global minimum either at $h = 0, \pi$ or, if $\left| \frac{B}{4C} \right| < 1$ at $h_{1,2}$ satisfying

$$\cos h_1 = \cos h_2 = -\frac{B}{4C}.$$

Using that $\cos 2h = 2\cos^2 h - 1$, we have that,

$$f(h) = A + B \cos h + C(2\cos^2 h - 1)$$

and then when $\left| \frac{B}{4C} \right| < 1$

$$f(h_1) = f(h_2) = A - \frac{B^2}{8C} - C.$$

Since $A > 0$ and $C < 0$, $f(h_1), f(h_2)$ are positive. Therefore we only need to impose $f(0), f(\pi) > 0$. Notice that $f(0) = A + B + C$ and $f(\pi) = A - B + C$ so that both conditions can be written as

$$A + C > |B|.$$

When $|B| < 4|C|$, this last condition is satisfied provided $a_{\max} = 30000$ km. Indeed, in this case

$$5|C| \leq \frac{5}{6} \rho \frac{a_{\max}^3}{a_M^3} L_{\max}^4 U_2^{2,0} (1 + 2\hat{\Gamma}) = 0.023691360650697(1 + 2\hat{\Gamma}) < A$$

and then $|B| < 4|C| \leq 5|C| + C < A + C$. Summarizing we only need to impose

$$A + C - |B| > 0 \quad \text{if} \quad \left| \frac{B}{4C} \right| \geq 1 \quad (6.3)$$

Lemma 6.1. For $\hat{r} \in [0, 0.49]$ and $L \in [L_{\min}, L_{\max}]$, we have that $\frac{5}{4}|B(\hat{r}, L)| \leq 1 < A$.

Proof. It is clear that

$$\frac{5}{4}|B(\hat{r}, L)| \leq \frac{5}{6} \rho \frac{a_{\max}^3}{a_M^3} L_{\max}^4 U_2^{1,0} \frac{|1 - 4\hat{r} - 4\hat{r}^2|}{\sqrt{(1 - 2\hat{r})(3 + 2\hat{r})}} \leq \frac{5}{3} \rho \frac{a_{\max}^3}{a_M^3} L_{\max}^4 U_2^{1,0} \frac{1}{\sqrt{3(1 - 2 \cdot 0.49)}}.$$

Computing this value, we have that $|B(\hat{r}, L)| \leq 0.446157052927936$ and we are done. $\square$

Lemma 6.1 implies that condition $A + C - |B| > 0$ is always satisfied if $\hat{r} \in [0, 0.49]$ and $|B| \geq 4|C|$ because $|B| - C \leq \frac{5}{4}|B| < A$.

As a consequence of the previous analysis,

$$\partial_{\hat{r}} \hat{H}_{\text{CP}} \neq 0 \quad \text{for} \quad L \in [L_{\min}, L_{\max}], \quad h \in [0, 2\pi], \quad \hat{r} \in [0, 0.49].$$

Therefore, any energy level $\hat{H}_{\text{CP}} = E$ which belongs to the cylinder $h \in [0, 2\pi]$, $\hat{r} \in [0, 0.49]$ is a closed curve, which is a graph over h and moreover its dynamics is periodic. Then, it only remains to characterize such energy levels.

We notice that, since $L\hat{r}(t; L)$, $h(t; L)$ satisfies the differential equations in [Theorem 3.9](#), one deduces that $\dot{\hat{r}} = 0$ if and only if $h = 0, \pi, 2\pi$. Indeed, $\dot{\hat{r}} = 0$ is equivalent to

$$\sin h \left(U_2^{1,0}(2\hat{r} + 1) + U_2^{2,0} \sqrt{(1 - 2\hat{r})(3 + 2\hat{r})} \cos h \right) = 0$$

and since

$$-\frac{U_2^{1,0}}{U_2^{2,0}} \frac{2\hat{r} + 1}{\sqrt{(1 - 2\hat{r})(3 + 2\hat{r})}} \leq -\frac{U_2^{1,0}}{U_2^{2,0} \sqrt{3}} = -1.331627517097770 < -1$$

the conclusion is obvious. Clearly $h = \pi$ corresponds to a maximum and $h = 0$ corresponds to a minimum of $\hat{r}$ as function of h .

We denote by $\hat{r}_{\min}(E)$, $\hat{r}_{\max}(E)$ the values of $\hat{r}$ such that

$$\hat{H}_{\text{CP}}(0, \hat{r}_{\min}(E), 0, 0) = E, \quad \hat{H}_{\text{CP}}(0, \hat{r}_{\max}(E), 0, \pi) = E.$$

The level curves $E = \hat{H}_{\text{CP}}(0, \hat{r}, 0, h)$ cannot intersect and, since $\partial_{\hat{r}} \hat{H}_{\text{CP}} \neq 0$, we have that $\partial_E \hat{r}_{\min}(E)$, $\partial_E \hat{r}_{\max}(E) \neq 0$. Therefore, in order to check the range of E allowed in our analysis, we compute

$$\mathbf{E}_{\max} = \mathbf{E}_{\max}(L) = \hat{H}_{\text{CP}}(0, 0, 0, 0), \quad \mathbf{E}_{\min} = \mathbf{E}_{\min}(L) = \hat{H}_{\text{CP}}(0, 0.49, 0, \pi)$$

and [Theorem 3.9](#) is proven, taking into account that, from (6.1), we easily deduce that

$$\mathbf{E}_{\max}(L) = \frac{\rho_0}{16L^6} \hat{\mathbf{E}}_{\max}(\delta), \quad \mathbf{E}_{\min}(L) = \frac{\rho_0}{16L^6} \hat{\mathbf{E}}_{\min}(\delta)$$

for some function $\hat{\mathbf{E}}_{\min, \max}$ defined for $\delta = \rho \alpha^3 L^4 \in [0, \delta_{\max}]$ (recall that $\delta_{\max}$ is defined in (3.3)).

To obtain the values in [Remark 3.10](#), it is enough to recall that, for Galileo, $a = 29600 \text{ km}$ (equivalently $L = 1$), and therefore $\mathbf{E}_{\min, \max}(1)$ with

$$\mathbf{E}_{\max}(1) = 2.477266122798186 \cdot 10^{-6}, \quad \mathbf{E}_{\min}(1) = -2.558100888960067 \cdot 10^{-5}.$$

6.1.1. Proof of [Corollary 3.12](#)

We fix $\mathbf{E}$, a given energy level, and let $\Gamma_{\text{CP}}^{\mathbf{E}}$ be such that $\mathcal{H}_{\text{CP}}(0, \Gamma_{\text{CP}}^{\mathbf{E}}, 0, 0) = \mathbf{E}$. Then, using formula (6.1) for $\hat{H}_{\text{CP}}(0, \hat{r}, 0, h) = \mathcal{H}_{\text{CP}}(0, L\hat{r}, 0, h)$ one has that

$$\frac{16L^6}{\rho_0} \mathbf{E} = \hat{H}_{\text{AV}}(0, \hat{r}_{\text{CP}}^{\mathbf{E}}, 0) + \delta \hat{H}_{\text{CP},1}^*(0, \hat{r}_{\text{CP}}^{\mathbf{E}}, 0, 0), \quad \hat{r}_{\text{CP}}^{\mathbf{E}} = L^{-1} \Gamma_{\text{CP}}^{\mathbf{E}},$$

with $\delta = \rho \alpha^3 L^4$,

$$\begin{aligned} \hat{H}_{\text{AV}}(0, \hat{r}, 0) &= \frac{16L^6}{\rho_0} \mathcal{H}_{\text{AV}}(0, L\hat{r}, 0) = (1 - 12\hat{r} - 12\hat{r}^2) \left(1 + \delta \frac{U_2^{0,0}}{2} \right), \\ \hat{H}_{\text{CP},1}^*(0, \hat{r}, 0, 0) &= \frac{16L^2}{\rho_1} (\mathcal{H}_{\text{CP},1}(0, L\hat{r}, 0, 0) - \mathcal{H}_{\text{AV},1}(0, L\hat{r}, 0)) \\ &= -2U_2^{1,0} \sqrt{(1 - 2\hat{r})(3 + 2\hat{r})} (2\hat{r} + 1) - \frac{U_2^{2,0}}{2} (1 - 2\hat{r})(3 + 2\hat{r}). \end{aligned}$$

Let $\hat{F}_{AV}^E$ be such that

$$\frac{16L^6}{\rho_0} \mathbf{E} = \hat{H}_{AV}(0, \hat{F}_{AV}^E, 0).$$

One has, from definition (3.14) of $\hat{a}_0, \hat{c}_0$, that

$$\hat{F}_{CP}^E - \hat{F}_{AV}^E = -\delta \frac{\hat{H}_{CP,1}^*(0, \hat{F}_{CP}^E, 0, 0)}{\partial_{\hat{F}} \hat{H}_{AV}(0, \hat{F}_{AV}^E, 0)} + \mathcal{O}(\delta^2) = -\delta \hat{c}_0(\hat{F}_{CP}^E; \delta) + \mathcal{O}(\delta^2)$$

and the proof is finished.

6.2. Perturbative analysis

We fix $L \in [L_{\min}, L_{\max}]$ and we consider the Hamiltonian $\mathcal{H}_{CP}$ in (2.16). Let $(\eta, \Gamma, \xi, h) = (0, \Gamma(t; L), 0, h(t; L))$ be a periodic orbit satisfying $\Gamma(0; L) = \Gamma_0$ and $h(0; L) = 0$ (see Theorem 3.9). We recall that in (3.2) we have introduced the parameter $\delta = \alpha^3 L^4 \rho$.

6.2.1. Proof of Proposition 3.14

The proof of Proposition 3.14 relies on a simple perturbative analysis. In order to make our analysis independent on L , we perform the change

$$\hat{F} = \Gamma/L, \quad s = 3\rho_0 t/(4L^7)$$

to the differential equation in Theorem 3.9 and we obtain the new system

$$\begin{aligned} \hat{h}' &= a_0(\hat{F}; L) + \delta a_1(\hat{F}, \hat{h}; L) \\ &:= -1 - 2\hat{F} - \alpha^3 L^4 \frac{\rho}{2} U_2^{0,0}(1 + 2\hat{F}) + \delta \frac{1}{6} \left(4U_2^{1,0} \frac{1 - 4\hat{F} - 4\hat{F}^2}{\sqrt{(1 - 2\hat{F})(3 + 2\hat{F})}} \cos \hat{h} \right. \\ &\quad \left. - U_2^{2,0}(1 + 2\hat{F}) \cos(2\hat{h}) \right) \\ \hat{F}' &= -\delta \frac{1}{12} \left(2U_2^{1,0} \sqrt{(1 - 2\hat{F})(3 + 2\hat{F})(2\hat{F} + 1)} \sin \hat{h} + U_2^{2,0}(1 - 2\hat{F})(3 + 2\hat{F}) \sin(2\hat{h}) \right). \end{aligned} \quad (6.4)$$

By the analyticity of system (6.4) with respect to δ , we have that

$$\hat{F}(s) = \hat{F}_0 + \delta \hat{F}_1(s) + \delta^2 \hat{F}_2(s, \delta), \quad \hat{h}(s) = \hat{h}_0(s) + \delta \hat{h}_1(s) + \delta^2 \hat{h}_2(s, \delta)$$

with $\hat{F}_0 \in \mathbb{R}$ the initial condition and

$$\begin{aligned} \hat{h}_0' &= a_0(\hat{F}_0; L), \\ \hat{F}_1' &= -\frac{1}{12} \left(2U_2^{1,0} \sqrt{(1 - 2\hat{F}_0)(3 + 2\hat{F}_0)(2\hat{F}_0 + 1)} \sin \hat{h}_0(s) \right. \\ &\quad \left. + U_2^{2,0}(1 - 2\hat{F}_0)(3 + 2\hat{F}_0) \sin(2\hat{h}_0(s)) \right), \\ \hat{h}_1' &= \partial_{\hat{F}} a_0(\hat{F}_0; L) \hat{F}_1(s) + a_1(\hat{F}_0, \hat{h}_0(s); L). \end{aligned}$$

Then, imposing that $\hat{h}(0) = 0$ and $\hat{F}(0) = \hat{F}_0$, that is $\hat{F}_1(0) = 0$, we have that

$$\begin{aligned} h_0(s) &= a_0(\hat{F}_0; L)s \\ \hat{F}_1(s) &= c_0(\hat{F}_0; L) + c_1(\hat{F}_0; L) \cos(a_0(\hat{F}_0; L)s) + c_2(\hat{F}_0; L) \cos(2a_0(\hat{F}_0; L)s) \\ h_1(s) &= \partial_{\hat{F}} a_0(\hat{F}_0; L) c_0(\hat{F}_0; L)s + d_1(\hat{F}_0; L) \sin(a_0(\hat{F}_0; L)s) + d_2(\hat{F}_0; L) \sin(2a_0(\hat{F}_0; L)s) \end{aligned}$$

where it is straightforward to check that the constants $a_0, c_{0,1,2}, d_{1,2}$ correspond to $\hat{a}_0, \hat{c}_{0,1,2}, \hat{d}_{1,2}$, the ones defined in (3.14) and (3.15), taking into account that the dependence with respect to the parameter L comes from the fact that

$$a_0(\hat{F}_0; L) = \hat{a}_0(\hat{F}_0; \rho \alpha^3 L^4).$$

With respect to the period of the periodic orbit, if $\hat{h}(\hat{\mathcal{T}}(\hat{F}_0; \delta)) = 2\pi$, that is $\hat{\mathcal{T}}(\hat{F}_0; \delta)$ is the period of the periodic orbit, then

$$\hat{\mathcal{T}}(\hat{F}_0; \delta) = \frac{2\pi + \mathcal{O}(\delta^2)}{|\hat{a}_0(\hat{F}_0; \delta) + \delta \partial_{\hat{F}} \hat{a}_0(\hat{F}_0; \delta) \hat{c}_0(\hat{F}_0; \delta)|} = \frac{2\pi + \mathcal{O}(\delta)}{|\hat{a}_0(\hat{F}_0; \delta)|}.$$

Undoing the change of coordinates we have that $\mathcal{T}(\Gamma_0; L) = \hat{\mathcal{T}}(\Gamma_0 L^{-1}; \delta) \frac{4L^7}{3\rho_0}$ and

$$\Gamma(t; L) = L \hat{\Gamma} \left(\frac{3\rho_0}{4L^7} t; \rho \alpha^3 L^4 \right), \quad h(t; L) = h \left(\frac{3\rho_0}{4L^7} t; \rho \alpha^3 L^4 \right).$$

This completes the proof of Proposition 3.14.

6.2.2. Proof of Theorem 3.17

Along this section, whenever there is no danger of confusion we will omit the dependence on $L \in [L_{\min}, L_{\max}]$.

It is convenient to introduce the (small) parameter $\epsilon = \alpha^3$ and $X(\xi, \eta, \Gamma, h; \epsilon)$,⁵ the vector field associated to $\mathcal{H}_\epsilon := \mathcal{H}_{\text{AV}} + \epsilon(\mathcal{H}_{\text{CP},1} - \mathcal{H}_{\text{AV},1})$.

The first step is to characterize the variational equation of X around the periodic orbit, $(0, 0, \Gamma(t), h(t))$. Note that

$$DX(0, 0, \Gamma(t), h(t); \epsilon) = \begin{pmatrix} \partial_{\xi, \eta} X_{\xi, \eta}(0, 0, \Gamma(t), h(t); \epsilon) & 0_{2 \times 2} \\ 0_{2 \times 2} & \partial_{\hat{\Gamma}, h} X_{\Gamma, h}(0, 0, \Gamma(t), h(t); \epsilon) \end{pmatrix}.$$

Then, if $\Phi(t; \epsilon)$ is the fundamental matrix of the linearized system

$$\dot{z} = DX(0, 0, \Gamma(t), h(t); \epsilon)z$$

satisfying $\Phi(0; \epsilon) = \text{Id}$, it has also the block form

$$\Phi(t; \epsilon) = \begin{pmatrix} \Psi(t; \epsilon) & 0_{2 \times 2} \\ 0_{2 \times 2} & \tilde{\Psi}(t; \epsilon) \end{pmatrix}, \quad \Psi(0; \epsilon) = \tilde{\Psi}(0; \epsilon) = \text{Id}. \quad (6.5)$$

Since the system is Hamiltonian, $\det \Phi(t; \epsilon) = 1$ for all t and therefore we also have that $\det \Psi(t; \epsilon) = \det \tilde{\Psi}(t; \epsilon) = 1$.

The character of the periodic orbit $(0, 0, \Gamma(t), h(t))$, is determined by the eigenvalues of the monodromy matrix $\Psi(\mathcal{T}; \epsilon)$, with $\mathcal{T} = \mathcal{T}(\Gamma(0); L)$ the period of the periodic orbit. To study the monodromy matrix, we first notice that

$$\partial_{\xi, \eta} X_{\xi, \eta}(0, 0, \Gamma(t), h(t); \epsilon) = \partial_{\xi, \eta} X_{\text{AV}}(0, 0, \Gamma(t)) + \epsilon \frac{\rho_1}{L^2} \tilde{B}(t)$$

where $X_{\text{AV}}(\xi, \eta, \Gamma)$ is the vector field associated to the h -averaged system $\mathcal{H}_{\text{AV}}$ and, using formulas (2.19) and (A.1) of $\mathcal{H}_{\text{AV},1}$ and $\mathcal{H}_{\text{CP},1}$ respectively, it is not difficult to see that the matrix $\tilde{B}$ has the form

$$\tilde{B}(t) = \begin{pmatrix} \sum_{j=1}^3 \tilde{b}_{11}^j(\Gamma(t), L) \sin jh(t) & \sum_{j=1}^3 \tilde{b}_{12}^j(\Gamma(t), L) \cos jh(t) \\ \sum_{j=1}^3 \tilde{b}_{21}^j(\Gamma(t), L) \cos jh(t) & -\sum_{j=1}^3 \tilde{b}_{11}^j(\Gamma(t), L) \sin jh(t) \end{pmatrix}.$$

The elements $\tilde{b}_{ij}$ depend on L, Γ through the functions $D_{l,k}(0, \Gamma, 0; L)$ (see Table 9). For future computations, we note that $\tilde{b}_{ik}^j$ are linear functions on $D_{i,k}(0, \Gamma, 0; L)$, namely

$$\tilde{b}_{ik}^j(\Gamma, L) = \mathbf{b}_{ik}^j(\{D_{i,k}(0, \Gamma, 0; L)\}_{i=0,1,2, k=0,2}, L\{D_{i,1}(0, \Gamma, 0; L)\}_{i=1,2}), \quad (6.6)$$

with $\mathbf{b}_{ik}^j$ linear functions. Therefore the matrix $\Psi(t; \epsilon)$ (see (6.5)) is the fundamental matrix of the two dimensional linear system

$$\dot{\zeta} = \left[\partial_{\xi, \eta} X_{\text{AV}}(0, 0, \Gamma(t)) + \epsilon \frac{\rho_1}{L^2} \tilde{B}(t) \right] \zeta, \quad \text{such that } \Psi(0, \epsilon) = \text{Id}, \det \Psi(t, \epsilon) = 1. \quad (6.7)$$

Lemma 6.2. Let $(0, 0, \Gamma(t), h(t))$ be a $\mathcal{T}$ -periodic orbit of the coplanar Hamiltonian $\mathcal{H}_{\text{CP}}$ with $\Gamma(0) = \Gamma_0, h(0) = 0$.

Consider the new variable $\hat{\Gamma} = \Gamma/L$, the new independent time $s = 3\rho_0 t/(4L^7)$, the initial condition $\hat{\Gamma}_0 = \Gamma_0/L$ and the parameters δ and σ_0 defined by

$$\delta = \epsilon L^4 \rho = \alpha^3 L^4 \frac{\rho_1}{\rho_0}, \quad \sigma_0 = \sigma_0(\hat{\Gamma}_0; \delta) = \hat{\Gamma}_0 + \delta \hat{c}_0(\hat{\Gamma}_0; \delta)$$

with $\hat{c}_0$ introduced in (3.14). Then

1. In the new variables, the variational equation (6.7) becomes

$$\zeta' = \left[\mathbf{X}(\sigma_0) + \delta \hat{B}(s) + \mathcal{O}(\delta^2) \right] \zeta, \quad \int_0^{\hat{\mathcal{T}}} \hat{B}(u) du = \mathcal{O}(\delta), \quad (6.8)$$

where $\mathbf{X}(\sigma_0)$ is defined as

$$\begin{aligned} \mathbf{X}(\sigma_0) &:= \frac{4L^7}{3\rho_0} \partial_{\xi, \eta} X_{\text{AV}}(0, 0, L\sigma_0) \\ &= \frac{1}{4} \begin{pmatrix} 0 & -(\mathbf{a}(\sigma_0) + \delta \mathbf{c}_-(\sigma_0)) \\ \mathbf{a}(\sigma_0) + \delta \mathbf{c}_+(\sigma_0) & 0 \end{pmatrix} \end{aligned} \quad (6.9)$$

where $\mathbf{a}(\sigma) = 1 - 16\sigma - 20\sigma^2$ has been defined previously in (4.9) and $\mathbf{c}_\pm$ have been introduced in Remark 4.1:

$$\mathbf{c}_\pm(\sigma) = \frac{U_2^{0,0}}{2} \mathbf{a}(\sigma) \pm \frac{5U_2^{1,0}}{3} \mathbf{b}(\sigma), \quad \mathbf{b}(\sigma) = (2\sigma + 3) \sqrt{3 - 4\sigma - 4\sigma^2}. \quad (6.10)$$

2. The fundamental matrix $\hat{\Psi}(s, \delta)$ of (6.8) satisfies that

$$\hat{\Psi}(s, \delta) = \Psi \left(\frac{3\rho_0}{4L^7} s, \frac{\delta\rho_0}{\rho_1 L^4} \right), \quad \hat{\Psi}(0, \delta) = \text{Id}, \quad \det \hat{\Psi}(s, \delta) = 1.$$

⁵ With this new order in the variables the linearized vector field around the periodic because has block structure.

Remark 6.3. The change of variables, presented in the previous lemma, allows to keep our perturbative analysis uniform in $L \in [L_{\min}, L_{\max}]$.

Proof. Performing the change $\hat{t} = \Gamma/L$ and $s = 4L^7 t/(3\rho_0)$, system (6.7) becomes

$$\zeta' = \frac{4L^7}{3\rho_0} \partial_{\xi,\eta} X_{AV}(0, 0, L\hat{t}(s))\zeta + \delta \frac{4L}{3} \hat{B} \left(\frac{4L^7}{3\rho_0} s \right) \zeta. \quad (6.11)$$

Using Table 9 for the explicit expressions of $D_{l,k}$, we obtain that

$$\begin{aligned} D_{l,k}(0, L\hat{t}, 0; L) &= \frac{1}{L} \mathbf{d}_{l,k}(\hat{t}), & l = 0, 1, 2, k = 0, 2 \\ D_{l,1}(0, L\hat{t}, 0; L) &= \frac{1}{L^2} \mathbf{d}_{l,1}(\hat{t}), & l = 0, 1, 2, \end{aligned}$$

for smooth functions $\mathbf{d}_{l,k}$. Then, by formula (6.6) of $\tilde{b}_{ij}$ (we recall that the functions $\mathbf{b}_{ij}$ are linear), we conclude that Eq. (6.11) can be rewritten as

$$\zeta' = \frac{4L^7}{3\rho_0} \partial_{\xi,\eta} X_{AV}(0, 0, L\hat{t}(s))\zeta + \delta B(s)\zeta \quad (6.12)$$

with

$$B(s) = \begin{pmatrix} \sum_{j=1}^3 b_{11}^j(\hat{t}(s)) \sin j\hat{h}(s) & \sum_{j=1}^3 b_{12}^j(\hat{t}(s)) \cos j\hat{h}(s) \\ \sum_{j=1}^3 b_{21}^j(\hat{t}(s)) \cos j\hat{h}(s) & -\sum_{j=1}^3 b_{11}^j(\hat{t}(s)) \sin j\hat{h}(s) \end{pmatrix}.$$

As a consequence, the fundamental matrix $\hat{\Psi}(s, \delta)$ of (6.12) satisfies that

$$\hat{\Psi}(s, \delta) = \Psi \left(\frac{4L^7}{3\rho_0} s, \frac{\delta\rho_0}{\rho_1 L^4} \right), \quad \hat{\Psi}(0, \delta) = \text{Id}, \quad \det \hat{\Psi}(s, \delta) = 1.$$

This proves the second item in the lemma, provided that system (6.12) coincides with (6.8).

Now we study $\partial_{\xi,\eta} X_{AV}$. By (4.6),

$$\partial_{\xi,\eta} X_{AV}(0, 0, L\hat{t}) = \frac{\rho_0}{8L^{10}} \begin{pmatrix} 0 & -X^\eta(L\hat{t}; L) \\ X^\xi(L\hat{t}; L) & 0 \end{pmatrix}.$$

Then, using the expressions (4.8) for $X^{\xi,\eta}(L\hat{t}; L)$ and the formulas for d_0, d_1 in (4.2), we conclude that

$$\frac{4L^7}{3\rho_0} \partial_{\xi,\eta} X_{AV}(0, 0, L\hat{t}) = \frac{1}{4} \begin{pmatrix} 0 & -(\mathbf{a}(L\hat{t}) + \delta \mathbf{c}_-(L\hat{t})) \\ \mathbf{a}(L\hat{t}) + \delta \mathbf{c}_+(L\hat{t}) & 0 \end{pmatrix} \quad (6.13)$$

with $\mathbf{a}(\sigma) = 1 - 16\sigma - 20\sigma^2$ (and $\mathbf{b}(\sigma) = (2\sigma + 3)\sqrt{3 - 4\sigma - 4\sigma^2}$) defined in (4.9) and

$$\mathbf{c}_\pm(\sigma) = \frac{U_2^{0,0}}{2} \mathbf{a}(\sigma) \pm \frac{5U_2^{1,0}}{3} \mathbf{b}(\sigma).$$

On the other hand, since Proposition 3.14 implies that $\hat{t}(s) = \hat{t}_0 + \delta \hat{c}_0(\hat{t}_0) + \delta b(s) + \mathcal{O}(\delta^2)$, with

$$\int_0^{\hat{t}} b(s) ds = \int_0^{\frac{2\pi}{|\hat{t}_0|}} b(s) ds + \mathcal{O}(\delta) = \mathcal{O}(\delta),$$

we have that the variational equation for $\hat{\Psi}$ in (6.12) is of the form

$$\zeta' = \left[\frac{4L^7}{3\rho_0} \partial_{\xi,\eta} X_{AV}(0, 0, L(\hat{t}_0 + \delta \hat{c}_0(\hat{t}_0))) + \delta \hat{B}(s) + \mathcal{O}(\delta^2) \right] \zeta, \quad \int_0^{\hat{t}} \hat{B}(s) dt = \mathcal{O}(\delta)$$

where $\partial_{\xi,\eta} X_{AV}(0, 0, L(\hat{t}_0 + \delta \hat{c}_0(\hat{t}_0)))$ is a constant matrix (see (6.13)) studied in Section 4.2, and the $\mathcal{O}(\delta^k)$ are uniform in $\hat{t}_0, L$. Taking $\sigma_0 = \sigma_0(\hat{t}_0; \delta) = \hat{t}_0 + \delta \hat{c}_0(\hat{t}_0)$ and using again (6.13) the result holds true. $\square$

We write the fundamental matrix $\hat{\Psi}$ as

$$\hat{\Psi}(s; \delta) = \hat{\Psi}_0(s; \sigma_0) + \delta \hat{\Psi}_1(s; \sigma_0) + \delta^2 \hat{\Psi}_2(s; \sigma_0, \delta),$$

where, $\hat{\Psi}_0(s; \sigma_0), \hat{\Psi}_1(s; \sigma_0)$ depend on δ through the parameter σ_0 and $\hat{\Psi}_2$ is bounded.

The matrix $\hat{\Psi}_0(s; \sigma_0)$ is the fundamental matrix of the constant coefficients linear system

$$\zeta' = \mathbf{X}(\sigma_0)\zeta, \quad \text{with} \quad \hat{\Psi}_0(0; \sigma_0) = \text{Id},$$

with eigenvalues $e^{\pm \hat{\lambda}_0(\sigma_0)}$ satisfying, from expression (6.9) of $\mathbf{X}(\sigma_0)$,

$$|\hat{\lambda}_0(\sigma_0)| = \frac{1}{4} \sqrt{|\mathbf{a}(\sigma_0) + \delta \mathbf{c}_+(\sigma_0)| |\mathbf{a}(\sigma_0) + \delta \mathbf{c}_-(\sigma_0)|}. \quad (6.14)$$

The matrix $\hat{\Psi}_1(s; \sigma_0)$ satisfies

$$\hat{\Psi}'_1(s; \sigma_0) = \mathbf{X}(\sigma_0)\hat{\Psi}_1(s; \sigma_0) + \hat{B}(s)\hat{\Psi}_0(s; \sigma_0), \quad \hat{\Psi}_1(0; \sigma_0) = 0$$

and therefore

$$\hat{\Psi}_1(s; \sigma_0) = \hat{\Psi}_0(s; \sigma_0) \int_0^s (\hat{\Psi}_0(u; \sigma_0))^{-1} \hat{B}(u) \hat{\Psi}_0(u; \sigma_0) du. \quad (6.15)$$

Whenever there is no danger of confusion, we will omit the dependence on $\hat{I}_0, \sigma_0, \delta$. In particular, we write $\hat{\mathcal{T}} = \hat{\mathcal{T}}(\hat{I}_0, \delta)$, $\hat{\Psi}_0 = \hat{\Psi}_0(\hat{\mathcal{T}}, \sigma_0)$, $\hat{\Psi}_1 = \hat{\Psi}_1(\hat{\mathcal{T}}, \sigma_0)$ and $\hat{\Psi} = \hat{\Psi}(\hat{\mathcal{T}}, \delta)$. In addition, we will denote by M a generic constant, independent on $\hat{I}_0, \delta$ that, along the proof, can change its value.

In order to prove [Theorem 3.17](#), we need to compare the eigenvalues of $\hat{\Psi}$ with the ones of $\hat{\Psi}_0$. To do so, we first compare the trace of $\hat{\Psi}$ with the one of $\hat{\Psi}_0$.

Lemma 6.4. *Let $\hat{\lambda}_0 = \hat{\lambda}_0(\sigma_0)$ be such that the eigenvalues of $\mathbf{X}(\sigma_0)$ are $\pm \hat{\lambda}_0$. Then, there exists $\delta_0 > 0$ such that, if $\delta \in [0, \delta_0]$,*

$$\text{tr}(\hat{\Psi}) = \text{tr}(\hat{\Psi}_0) + \mathcal{O}(\mathbf{L}\delta^2) + \delta^3 T(\delta)$$

with $T(\delta)$ bounded on $[0, \delta_0]$ and

$$\mathbf{L} := \min_{k \in \mathbb{N}} \{|\hat{\mathcal{T}}|\hat{\lambda}_0| - \pi k|\}.$$

Remark 6.5. The quantity $\mathbf{L}$ is well defined and $0 \leq \mathbf{L} \leq \frac{\pi}{2}$. Indeed, for a given $\hat{\mathcal{T}}|\hat{\lambda}_0|$, the integer part $k' = \left\lfloor \frac{\hat{\mathcal{T}}|\hat{\lambda}_0|}{\pi} \right\rfloor \in \mathbb{N}$ satisfies $\hat{\mathcal{T}}|\hat{\lambda}_0| - k'\pi \in [0, \pi)$. Therefore if

$$\hat{\mathcal{T}}|\hat{\lambda}_0| - k'\pi \in \left[0, \frac{\pi}{2}\right], \quad \text{implies} \quad \mathbf{L} = \hat{\mathcal{T}}|\hat{\lambda}_0| - k'\pi \leq \frac{\pi}{2}.$$

On the contrary, when $\hat{\mathcal{T}}|\hat{\lambda}_0| - k'\pi \in \left[\frac{\pi}{2}, \pi\right)$, $\mathbf{L} = |\hat{\mathcal{T}}|\hat{\lambda}_0| - (k' + 1)\pi \leq \frac{\pi}{2}$.

Proof. Let $\kappa > 1$ big enough (its value will be determined later). We distinguish two cases, namely $\mathbf{L} \geq \kappa\delta$ and $\mathbf{L} \leq \kappa\delta$.

When $|\hat{\mathcal{T}}|\hat{\lambda}_0| \leq \kappa\delta$, from (6.14) and [Remark 3.16](#), we deduce that $|\mathbf{a}(\sigma_0)| \leq M\kappa\delta$. Then, since $\mathbf{a}(\sigma) = 1 - 16\sigma - 20\sigma^2$, its unique positive zero is $\frac{m}{2}$, with m defined in (3.4) and therefore the condition $|\hat{\mathcal{T}}|\hat{\lambda}_0| \leq \kappa\delta$ implies that

$$\left| \sigma_0 - \frac{m}{2} \right| \leq M\kappa\delta$$

for some constant M . Therefore, using that

$$\mathbf{X}(\sigma_0) = \mathbf{X}\left(\frac{m}{2}\right) + \mathcal{O}(\delta) = \mathcal{O}(\delta)$$

we obtain that

$$\hat{\Psi}_0(\hat{\mathcal{T}}; \sigma_0) = \hat{\Psi}_0\left(\hat{\mathcal{T}}; \frac{m}{2}\right) + \mathcal{O}(\delta) = \text{Id} + \mathcal{O}(\delta).$$

If $|\hat{\mathcal{T}}|\hat{\lambda}_0| - k\pi \leq \kappa\delta$ for some $k \in \mathbb{N} \setminus \{0\}$, by [Theorem 3.2](#) the eigenvalues of $\mathbf{X}(\sigma_0)$ are of the form $\pm i|\hat{\lambda}_0|$, if δ is small enough. Therefore, if P is such that $P\mathbf{X}(\sigma_0)P^{-1}$ is a diagonal matrix,

$$\begin{aligned} \hat{\Psi}_0 &= P \begin{pmatrix} e^{i|\hat{\lambda}_0|\hat{\mathcal{T}}} & 0 \\ 0 & e^{-i|\hat{\lambda}_0|\hat{\mathcal{T}}} \end{pmatrix} P^{-1} = P \begin{pmatrix} (-1)^k + \mathcal{O}(\delta) & 0 \\ 0 & (-1)^k + \mathcal{O}(\delta) \end{pmatrix} P^{-1} \\ &= (-1)^k \text{Id} + \mathcal{O}(\delta). \end{aligned}$$

Therefore, if k is such that $\mathbf{L} = |\hat{\mathcal{T}}|\hat{\lambda}_0| - k\pi$, $\hat{\Psi}_0 = (-1)^k \text{Id} + \mathcal{O}(\delta)$. Then, from (6.8) and (6.15),

$$\begin{aligned} \hat{\Psi}_1 &= ((-1)^k \text{Id} + \mathcal{O}(\delta)) \int_0^{\hat{\mathcal{T}}} ((-1)^k \text{Id} + \mathcal{O}(\delta))^{-1} \hat{B}(u) ((-1)^k \text{Id} + \mathcal{O}(\delta)) du \\ &= ((-1)^k \text{Id} + \mathcal{O}(\delta)) \left(\int_0^{\hat{\mathcal{T}}} (-1)^k \hat{B}(u) du + \mathcal{O}(\delta) \right) = \mathcal{O}(\delta). \end{aligned}$$

Therefore $\hat{\Psi} = \hat{\Psi}_0 + \mathcal{O}(\delta^2)$. We write

$$\hat{\Psi} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a_0 + \delta^2 a_1 + \mathcal{O}(\delta^3) & b_0 + \delta^2 b_1 + \mathcal{O}(\delta^3) \\ c_0 + \delta^2 c_1 + \mathcal{O}(\delta^3) & d_0 + \delta^2 d_1 + \mathcal{O}(\delta^3) \end{pmatrix}$$

where the elements a_j, b_j, c_j, d_j are the corresponding ones for $\hat{\Psi}_j$, $j = 0, 1$. Imposing that $\det(\hat{\Psi}) = \det(\hat{\Psi}_0) = 1$, we have

$$\delta^2(a_0 d_1 + a_1 d_0 - b_0 c_1 - b_1 c_0) + \mathcal{O}(\delta^3) = 0.$$

Then, since $a_0, d_0 = 1 + \mathcal{O}(\delta)$ (in fact $1 + \mathcal{O}(\delta^2)$) and $b_0, c_0 = \mathcal{O}(\delta)$, we obtain that $d_1 + a_1 = \mathcal{O}(\delta)$. As a consequence

$$\text{tr}(\hat{\Psi}) = \text{tr}(\hat{\Psi}_0) + \mathcal{O}(\delta^3)$$

and, because we are assuming that $\mathbf{L} \leq \kappa\delta$, the lemma is proven for this case.

Now we deal with the case $|\hat{\mathcal{T}}\hat{\lambda}_0| \geq \mathbf{L} \geq \kappa\delta$ with κ a large enough constant. If δ is small enough, by [Theorem 3.2](#), the eigenvalues of $\mathbf{X}(\sigma_0)$ are of the form $\pm i|\hat{\lambda}_0|$ with $|\lambda_0|$ in [\(6.14\)](#) and moreover, again by [Remark 3.16](#),

$$\left| \sigma_0 - \frac{m}{2} \right| \geq M\kappa\delta$$

for some constant M . As a consequence, recalling that $\mathbf{a}\left(\frac{m}{2}\right) = 0$ and $\partial_\sigma \mathbf{a}(\sigma) < 0$, for $\sigma > 0$, we have that

$$|\mathbf{a}(\sigma_0)| \geq \left| \mathbf{a}\left(\frac{m}{2} \pm \kappa\delta\right) \right| \geq M\kappa\delta.$$

In addition, it is clear from the definition of $\mathbf{c}_\pm$ in the second item of [Lemma 6.2](#) that, $|\mathbf{c}_\pm(\sigma_0)| \leq M$. Then,

$$\frac{\mathbf{a}(\sigma_0) + \delta \mathbf{c}_\pm(\sigma_0)}{\mathbf{a}(\sigma_0) + \delta \mathbf{c}_\mp(\sigma_0)} \geq \frac{1 - \delta \frac{|\mathbf{c}_\pm(\sigma_0)|}{|\mathbf{a}(\sigma_0)|}}{1 + \delta \frac{|\mathbf{c}_\mp(\sigma_0)|}{|\mathbf{a}(\sigma_0)|}} \geq 1 - \frac{M}{\kappa}, \quad \frac{\mathbf{a}(\sigma_0) + \delta \mathbf{c}_\pm(\sigma_0)}{\mathbf{a}(\sigma_0) + \delta \mathbf{c}_\mp(\sigma_0)} \leq 1 + \frac{M'}{\kappa}$$

for some constants M, M' and κ large enough. This trivial fact implies that, $P(\sigma_0; \delta)$, the constant linear change of variables that transforms $\mathbf{X}(\sigma_0)$ in its diagonal form satisfies that $\|P(\sigma_0; \delta)\|, \|P^{-1}(\sigma_0; \delta)\|$ are bounded uniformly in δ . After this (constant) linear change of variables, $\tilde{\Psi} = P\hat{\Psi}P^{-1} = \tilde{\Psi}_0 + \delta\tilde{\Psi}_1 + \mathcal{O}(\delta^2)$ with $\tilde{\Psi}_0$

$$\tilde{\Psi}_0 = \begin{pmatrix} e^{i|\hat{\lambda}_0|\hat{\mathcal{T}}} & 0 \\ 0 & e^{-i|\hat{\lambda}_0|\hat{\mathcal{T}}} \end{pmatrix} = (-1)^k \begin{pmatrix} e^{i\mathbf{L}} & 0 \\ 0 & e^{-i\mathbf{L}} \end{pmatrix}$$

and $\tilde{\Psi}_1 = \tilde{\Psi}_1(\tilde{\mathcal{T}}; \sigma_0)$ with

$$\tilde{\Psi}_1(s; \sigma_0) = \tilde{\Psi}_0(s; \sigma_0) \int_0^s (\tilde{\Psi}_0(u; \sigma_0))^{-1} P^{-1}(\sigma_0, \delta) \hat{B}(u) P(\sigma_0, \delta) \tilde{\Psi}_0(u; \sigma_0) du.$$

Using the bound for $\hat{B}$ in [\(6.8\)](#) and the fact that P, P^{-1} are uniformly bounded with respect to δ , we obtain that

$$\int_0^{\hat{\mathcal{T}}} P^{-1}(\sigma_0; \delta) \hat{B}(u) P(\sigma_0; \delta) du = P^{-1}(\sigma_0; \delta) \left[\int_0^{\hat{\mathcal{T}}} \hat{B}(u) du \right] P(\sigma_0; \delta) = \mathcal{O}(\delta).$$

Then, one easily checks that

$$\begin{aligned} \tilde{\Psi} &= \tilde{\Psi}_0 + \delta\tilde{\Psi}_1 + \mathcal{O}(\delta^2) \\ &= (-1)^k \begin{pmatrix} e^{i\mathbf{L}} & 0 \\ 0 & e^{-i\mathbf{L}} \end{pmatrix} \left[\text{Id} + \delta \begin{pmatrix} 0 & b_{12}(\hat{\mathcal{T}}_0; \delta) \\ b_{21}(\hat{\mathcal{T}}_0; \delta) & 0 \end{pmatrix} \right] + \mathcal{O}(\delta^2) \end{aligned}$$

with, writing $P^{-1}\hat{B}(u)P = (\hat{b}_{ij}(u))_{i,j}$,

$$b_{12}(\hat{\mathcal{T}}_0; \delta) = \int_0^{\hat{\mathcal{T}}} \hat{b}_{12}(u) e^{-2i\mathbf{L}u} du, \quad b_{21}(\hat{\mathcal{T}}_0; \delta) = \int_0^{\hat{\mathcal{T}}} \hat{b}_{12}(u) e^{2i\mathbf{L}u} du.$$

Therefore, using [\(6.8\)](#),

$$\tilde{\Psi} = \tilde{\Psi}_0 \left[\text{Id} + \mathbf{L}\delta \begin{pmatrix} 0 & \beta_1(\hat{\mathcal{T}}_0) \\ \beta_2(\hat{\mathcal{T}}_0) & 0 \end{pmatrix} + \mathcal{O}(\delta^2) \right]$$

for some constants β_1, β_2 . We emphasize that, when $\mathbf{L} \geq M$, namely is not small when δ is small, this expression still holds true. We write now

$$\tilde{\Psi} = (-1)^k \begin{pmatrix} e^{i\mathbf{L}} + a_2\delta^2 & \beta_1\mathbf{L}\delta + b_2\delta^2 \\ \beta_2\mathbf{L}\delta + c_2\delta^2 & e^{-i\mathbf{L}} + d_2\delta^2 \end{pmatrix}$$

Therefore $\det(\tilde{\Psi}) = 1$ implies that

$$1 = 1 + \delta^2(e^{i\mathbf{L}}d_2 + e^{-i\mathbf{L}}a_2) - \beta_1\beta_2\mathbf{L}^2\delta^2 + \mathcal{O}(\delta^3\mathbf{L}) + \mathcal{O}(\delta^4).$$

Then, since $e^{\pm i\mathbf{L}} = 1 + \mathcal{O}(\mathbf{L})$

$$a_2 + d_2 = \mathcal{O}(\mathbf{L}) + \mathcal{O}(\delta\mathbf{L}) + \mathcal{O}(\delta^2) = \mathcal{O}(\mathbf{L}),$$

where we have used that $\mathbf{L} \geq \kappa\delta$. As a consequence,

$$\text{tr}(\hat{\Psi}) = \text{tr}(\tilde{\Psi}) = \text{tr}(\tilde{\Psi}_0) + \mathcal{O}(\mathbf{L}\delta^2) = \text{tr}(\hat{\Psi}_0) + \mathcal{O}(\mathbf{L}\delta^2)$$

and the lemma is proven. $\square$

We denote by $\mu = e^{i\hat{\mathcal{T}}}$ one eigenvalue of $\hat{\Psi}$. Denoting $\tau = \text{tr}(\hat{\Psi})$ and using that $\det(\hat{\Psi}) = 1$, we can assume that

$$\mu = \frac{1}{2}\tau + \frac{1}{2}\sqrt{\tau^2 - 4}.$$

Let $\mu_0 = e^{i\hat{\mathcal{T}}_0}$ be such that

$$\mu_0 = \frac{1}{2}\tau_0 + \frac{1}{2}\sqrt{\tau_0^2 - 4}.$$

with $\tau_0 = \text{tr}(\hat{\Psi}_0)$. Then

$$2|\mu - \mu_0| \leq |\tau - \tau_0| + \left| \sqrt{\tau^2 - 4} - \sqrt{\tau_0^2 - 4} \right|.$$

By Lemma 6.4, $|\tau - \tau_0| \leq M\delta^2$, therefore, to prove Theorem 3.17 we have to analyze

$$\Delta\mu := \left| \sqrt{\tau^2 - 4} - \sqrt{\tau_0^2 - 4} \right|.$$

Lemma 6.6. Take $C_1, C_2 > 0$ two constants. There exist $\delta_0 > 0$ and a constant $C_* > 0$ such that, for $\delta \in [0, \delta_0]$, we have that

- When $\mathbf{L} \geq C_1\delta$, then $\Delta\mu \leq C_*\delta^2$.
- If $C_2\delta^{3/2} \leq \mathbf{L} \leq C_1\delta$, then $\Delta\mu \leq C_*\delta^3\mathbf{L}^{-1} \leq C_*\delta^{3/2}C_2^{-1}$.
- Otherwise, that is if $\mathbf{L} \leq C_2\delta^{3/2}$, then $\Delta\mu \leq C_*\delta^{3/2}$.

As a consequence $\Delta\mu \leq M\delta^{3/2}$ for some positive constant M .

Proof. We recall that, by Remark 6.5, $\mathbf{L} \leq \frac{\pi}{2}$. Assume first that $\mathbf{L} \geq \kappa\delta$ for some κ big enough. Then by Theorem 3.2, the eigenvalues of $\mathbf{X}(\hat{F}_0)$ in (6.9) are of the form $\pm i|\hat{\lambda}_0|$ with $|\hat{\lambda}_0|$ in (6.14). Then, $\tau_0 = e^{i|\hat{\lambda}_0|\hat{\mathcal{T}}} + e^{-i|\hat{\lambda}_0|\hat{\mathcal{T}}}$ satisfies

$$|\tau_0^2 - 4| = 4 - 4\cos^2(\hat{\mathcal{T}}\hat{\lambda}_0) = 4\sin^2(\hat{\mathcal{T}}\hat{\lambda}_0) = 4\sin^2\mathbf{L}.$$

In view of Lemma 6.4, $\tau = \tau_0 + \mathbf{L}\delta^2\tau_1$ with $|\tau_1| \leq M$. Then

$$\begin{aligned} \Delta\mu &= \left| \sqrt{\tau^2 - 4} \right| \left| 1 - \sqrt{1 + \frac{2\delta^2\mathbf{L}\tau_0\tau_1 + \delta^4\mathbf{L}^2\tau_1^2}{\tau_0^2 - 4}} \right| \leq \frac{2\delta^2\mathbf{L}|\tau_0||\tau_1|}{\sqrt{|\tau_0^2 - 4|}} + M\delta^2|\tau_1| \\ &\leq M\delta^2|\tau_1|. \end{aligned}$$

In the last bound we have used that $\sqrt{\tau_0^2 - 4} = 2\sin\mathbf{L} \geq M\mathbf{L}$, provided $0 \leq \mathbf{L} \leq \frac{\pi}{2}$.

Now assume that $\mathbf{L} \leq \kappa\delta$. By Lemma 6.4, $\tau = \tau_0 + \delta^3\tau_1$ with $|\tau_1| \leq M$. We notice that, using that either $\tau_0 = \pm(e^{\mathbf{L}} + e^{-\mathbf{L}})$ or $\tau_0 = \pm(e^{i\mathbf{L}} + e^{-i\mathbf{L}})$. Then

$$|\tau_0^2 - 4| = 2\mathbf{L}^2 + \mathcal{O}(\mathbf{L}^4).$$

When $\mathbf{L}^2 \geq M\delta^3$, for some suitable constant

$$\begin{aligned} \Delta\mu &= \left| \sqrt{\tau_0^2 - 4} \right| \left| 1 - \sqrt{1 + \frac{2\delta^3\tau_0\tau_1 + \delta^6\tau_1^2}{\tau_0^2 - 4}} \right| \leq M\delta^3 \frac{|\tau_1|}{\sqrt{|\tau_0^2 - 4|}} \\ &\leq M\delta^3\mathbf{L}^{-1}|\tau_1|. \end{aligned}$$

On the other hand, when $\mathbf{L}^2 \leq M\delta^3$, clearly,

$$\Delta\mu \leq 2\sqrt{|\tau_0^2 - 4|} + \delta^{3/2}\sqrt{2|\tau_0||\tau_1|} + \delta^3|\tau_1| \leq M\delta^{3/2}|\tau_1|^{1/2}. \quad \square$$

Theorem 3.17 is a straightforward consequence of Lemma 6.6 just taking into account that, using definitions (3.16), (3.20) and (3.21) of $\mathcal{T}$, $\mathcal{T}^{(0)}$ and $\Gamma_{\text{CP}}^{(0)}$ we have that, by Taylor's theorem,

$$\hat{\mathcal{T}}^{(0)}(\hat{F}_{\text{CP}}^{(0)}(\hat{F}_0; \delta); \delta) = \frac{4L^7}{3\rho_0}\mathcal{T}^{(0)}(\Gamma_{\text{CP}}^{(0)}(\Gamma_0; L); L) = \hat{\mathcal{T}}(\hat{F}_0; \delta) + \mathcal{O}(\delta^2).$$

Therefore,

$$e^{i|\hat{\lambda}_0|\hat{\mathcal{T}}^{(0)}} = e^{i|\hat{\lambda}_0|\hat{\mathcal{T}}} + \mathcal{O}(\delta^2).$$

6.2.3. Proof of Lemma 3.20

To prove this result, by expressions (3.21) and (6.14) of $|\hat{\lambda}_0|$ and $\mu_{\text{CP}}^{(0)}$ respectively, we need to solve the equation

$$\frac{2\pi}{|\hat{a}_0(\sigma; \delta)|} \frac{1}{4} \sqrt{|\mathbf{a}(\sigma) + \delta\mathbf{c}_+(\sigma)| |\mathbf{a}(\sigma) + \delta\mathbf{c}_-(\sigma)|} = k\pi,$$

with $\hat{a}_0$ defined in (3.14), for $k = 1, 2$. Using definition (6.10) of $\mathbf{c}_\pm$, we rewrite it as

$$H_k(\sigma; \delta) = \mathbf{a}^2(\sigma) + \delta \mathbf{a}^2(\sigma) U_2^{0,0} + \delta^2 \mathbf{c}_+(\sigma) \mathbf{c}_-(\sigma) - 4k^2 \hat{a}_0^2(\sigma; \delta).$$

Let $\hat{\Gamma}_k^{(0)}$ be the values such that this equation holds true for $\delta = 0$ (see Table 4 for the exact values of $\Gamma_k^{(0)}$). One can easily check that $\partial_\sigma H_k(\hat{\Gamma}_k^{(0)}; \delta) \neq 0$ and therefore, by the implicit function theorem, there exists $\hat{\Gamma}_k(\delta)$ such that $H_k(\hat{\Gamma}_k(\delta); \delta) = 0$ and the lemma holds true. In addition a simply computation proves that

$$\partial_\delta \hat{\Gamma}_k(0) = -\frac{\partial_\delta H_k(\hat{\Gamma}_k^{(0)}; 0)}{\partial_\sigma H_k(\hat{\Gamma}_k^{(0)}; 0)} = 0$$

and therefore $\hat{\Gamma}_k(\delta) = \hat{\Gamma}_k^{(0)} + \mathcal{O}(\delta^2)$.

6.2.4. Proof of Theorem 3.21

The proof of Theorem 3.21 is a nontrivial consequence of Lemma 6.6. Fix $\delta_0 > 0$ small enough (to be determined later) and consider $\delta \in [0, \delta_0]$. By Theorem 3.2 and Lemma 3.20, there exist $\sigma_\pm$ and σ_k , $k = 1, 2$ such that

$$\hat{\lambda}_0(\sigma_\pm) = 0, \quad \hat{\mathcal{T}}^{(0)}(\sigma_k; \delta) |\hat{\lambda}_0(\sigma_k)| = \frac{2\pi |\hat{\lambda}_0(\sigma_k)|}{|\hat{a}_0(\sigma_k; \delta)|} = k\pi, \quad k = 1, 2$$

with $\hat{a}_0$ defined in (3.14). Notice also that

$$\sigma_\pm = \frac{m}{2} \pm \delta C_0 + \mathcal{O}(\delta^2), \quad \sigma_k = \hat{\Gamma}_k^{(0)} + C_k \delta + \mathcal{O}(\delta^2).$$

Therefore, since $1 \leq \nu \leq 2$, we can assume that the parameter σ satisfies, taking δ small enough,

$$|\sigma - \sigma_\pm| \geq C \delta^\nu, \quad |\sigma - \sigma_k| \geq C \delta^{\frac{1+\nu}{2}}.$$

We now observe that, using the definition of $\mathbf{c}_\pm$ in (6.10)

$$16 |\hat{\lambda}_0(\sigma)|^2 = \mathbf{a}^2(\sigma) (1 + \delta U_2^{0,0}) + \mathcal{O}(\delta^2) \quad (6.16)$$

with $\mathbf{a} = 1 - 16\sigma - 20\sigma^2$. In addition

$$\hat{\mathcal{T}}^{(0)}(\sigma; \delta) := \frac{2\pi}{|\hat{a}_0(\sigma; \delta)|} = \hat{\mathcal{T}}(\sigma; \delta) + \mathcal{O}(\delta^2). \quad (6.17)$$

We distinguish the cases $|\mathbf{a}(\sigma)| \geq \delta |\log \delta|$ and $|\mathbf{a}(\sigma)| < \delta |\log \delta|$. We recall that $\mathbf{a}\left(\frac{m}{2}\right) = 0$. Then, the cases considered are equivalent to $\left|\sigma - \frac{m}{2}\right| \geq \kappa \delta |\log \delta|$ and $\left|\sigma - \frac{m}{2}\right| < \kappa \delta |\log \delta|$ for some constant κ .

We start with $\left|\sigma - \frac{m}{2}\right| \geq \kappa \delta |\log \delta|$. From (6.16) we have that

$$4 |\hat{\lambda}_0(\sigma)| = |\mathbf{a}(\sigma)| (1 + \mathcal{O}(|\log \delta|^{-1})). \quad (6.18)$$

- If $\sigma \in \left(0, \frac{m}{2} - \kappa \delta |\log \delta|\right)$, by Proposition 3.4, the function $|\hat{\lambda}_0|$ is decreasing. Therefore, taking into account (6.17) and definition (3.14) of $\hat{a}_0$, if δ is small enough,

$$\frac{1}{2} \left| \hat{\lambda}_0 \left(\frac{m}{2} - \delta |\log \delta| \right) \right| \hat{\mathcal{T}} \left(\frac{m}{2} - \delta |\log \delta|; \delta \right) \leq |\hat{\lambda}_0(\sigma)| \hat{\mathcal{T}}(\sigma; \delta) \leq \frac{3}{2} |\hat{\lambda}_0(0)| \hat{\mathcal{T}}(0; \delta)$$

Then $|\hat{\lambda}_0(\sigma)| \hat{\mathcal{T}}(\sigma; \delta) \leq \frac{3\pi}{4} (1 + M\delta)$ and $|\hat{\lambda}_0(\sigma)| \hat{\mathcal{T}}(\sigma; \delta) \geq M\delta |\log \delta|$. Those bounds imply that, for all k

$$\left| |\hat{\lambda}_0(\sigma)| \hat{\mathcal{T}}(\sigma; \delta) - k\pi \right| \geq M\delta |\log \delta|$$

and, in conclusion,

$$\mathbf{L} \geq M\delta |\log \delta|.$$

By Lemma 6.6, $|\mu - \mu_0| \leq M\delta^2$ and the result follows for this case.

- If $\sigma > \frac{m}{2} + \delta |\log \delta|$, we notice that, using (6.17) and (6.18),

$$\begin{aligned} \hat{\mathcal{T}}(\sigma; \delta) |\hat{\lambda}_0(\sigma)| - k\pi &= \hat{\mathcal{T}}(\sigma; \delta) |\hat{\lambda}_0(\sigma)| - \hat{\mathcal{T}}(\sigma_k; \delta) |\hat{\lambda}_0(\sigma_k)| \\ &= \left[\partial_\sigma \left(\frac{1}{4} \frac{|\mathbf{a}(s)|}{|\hat{a}_0(s; 0)|} \right) + \mathcal{O}(|\log \delta|^{-1}) \right] |\sigma - \sigma_k| \\ \hat{\mathcal{T}}(\sigma; \delta) |\hat{\lambda}_0(\sigma)| &= \left[\partial_\sigma \left(\frac{1}{4} \frac{|\mathbf{a}(t)|}{|\hat{a}_0(t; 0)|} \right) + \mathcal{O}(|\log \delta|^{-1}) \right] |\sigma - \sigma_+|, \end{aligned}$$

with $s \in \overline{\sigma, \sigma_k}$, $t \in \overline{\sigma, \sigma_+}$ and $\hat{a}_0$ defined in (3.14). Since $\sigma, \sigma_k, \sigma_+ > \frac{m}{2}$ (see Lemma 3.20),

$$\partial_\sigma \left(\frac{1}{4} \frac{|\mathbf{a}(s)|}{|\hat{a}_0(s; 0)|} \right) = \frac{1}{4 |\hat{a}_0(s; 0)|^2} (40s^2 + 40s + 18) \geq M.$$

Therefore $\mathbf{L} \geq M \min \left\{ \delta |\log \delta|, \delta^{\frac{1+\nu}{2}} \right\} = M \delta^{\frac{1+\nu}{2}} \geq M \delta^{\frac{3}{2}}$. Therefore, Lemma 6.6 assures that

$$|\mu - \mu_0| \leq M \delta^3 \mathbf{L}^{-1} \leq M \delta^{\frac{5-\nu}{2}}.$$

Now we deal with the case $\left| \sigma - \frac{m}{2} \right| \leq \kappa \delta |\log \delta|$, which in particular implies that $|\hat{\lambda}_0(\sigma)| \leq M \delta |\log \delta|$. Then, by continuity, in this case $\mathbf{L} = \hat{\mathcal{T}}|\hat{\lambda}_0(\sigma)|$, if δ is small enough. We first observe that, $\sigma_{\pm} = \Gamma_{\pm} L^{-1}$ with $\Gamma_{\pm}$ defined in Theorem 3.2 and $\sigma_{\pm} = \frac{m}{2} \pm C_0 \delta + \mathcal{O}(\delta^2)$ for some constant C_0 (see (3.23)). Therefore $|\sigma - \sigma_{\pm}| \leq \kappa \delta |\log \delta|$ for a (different) constant κ .

Assume now that $C \delta^{\nu} \leq |\sigma - \sigma_{-}| \leq |\sigma - \sigma_{+}| \leq \kappa \delta |\log \delta|$. We have that (see Theorem 3.2)

$$\mathbf{a}(\sigma_{-}) + \delta \mathbf{c}_{-}(\sigma_{-}) = 0.$$

Taking $\Delta \sigma = \sigma - \sigma_{-} = \mathcal{O}(\delta |\log \delta|)$, we have that

$$\begin{aligned} |\hat{\lambda}_0(\sigma)|^2 &= |\partial_{\sigma} \mathbf{a}(\sigma_{-}) \Delta \sigma + \Delta \sigma \mathcal{O}(\delta)| \left| \partial_{\sigma} \mathbf{a}(\sigma_{-}) \Delta \sigma + \delta (\mathbf{c}_{+}(\sigma_{-}) - \mathbf{c}_{-}(\sigma_{-})) + \Delta \sigma \mathcal{O}(\delta) \right| \\ &= \left| [\partial_{\sigma} \mathbf{a}(\sigma_{-})]^2 (\Delta \sigma)^2 + \partial_{\sigma} \mathbf{a}(\sigma_{-}) (\mathbf{c}_{+}(\sigma_{-}) - \mathbf{c}_{-}(\sigma_{-})) \Delta \sigma \delta + \Delta \sigma \mathcal{O}(\delta^2 |\log \delta|^2) \right|. \end{aligned}$$

We note that $|\partial_{\sigma} \mathbf{a}(\sigma)| = 16 + 40\sigma > 16$ and, therefore, we can write $|\hat{\lambda}_0(\sigma)|^2$ as

$$|\hat{\lambda}_0(\sigma)|^2 = [\partial_{\sigma} \mathbf{a}(\sigma_{-})]^2 \Lambda$$

where Λ , using formula (6.10) for $\mathbf{c}_{\pm}$, formula (3.23) of C_0 and that $\sigma_{-} = \frac{m}{2} + \mathcal{O}(\delta)$, is given by

$$\Lambda = \left| (\Delta \sigma)^2 - 2\delta \Delta \sigma C_0 + \Delta \sigma \mathcal{O}(\delta^2 |\log \delta|^2) \right| = |\Delta \sigma| \left| \Delta \sigma - 2\delta C_0 + \mathcal{O}(\delta^2 |\log \delta|^2) \right|.$$

Since $|\sigma - \sigma_{-}| \leq |\sigma - \sigma_{+}|$ and $\sigma_{\pm} = \frac{m}{2} \pm C_0 \delta + \mathcal{O}(\delta^2 |\log \delta|^2)$, $\Delta \sigma < 2\delta C_0$ and then, using that $|\Delta \sigma| \geq C \delta^{\nu}$ with $C < C_0$, for δ small enough

$$\Lambda = |\Delta \sigma| (2\delta C_0 - \Delta \sigma + \mathcal{O}(\delta^2 |\log \delta|^2)) \geq C \delta^{\nu} (2\delta C_0 - C \delta^{\nu} + \mathcal{O}(\delta^2 |\log \delta|^2)) \geq M \delta^{1+\nu}$$

for some constant $M > 0$. Therefore, since $1 + \nu \leq 3$, $\mathbf{L} = \hat{\mathcal{T}}|\hat{\lambda}_0| \geq M \delta^{\frac{1+\nu}{2}} \geq M \delta^{3/2}$ and by Lemma 6.6,

$$|\mu - \mu_0| \leq C_{*} \delta^{\frac{5-\nu}{2}}. \quad (6.19)$$

An analogous analysis can be done if $C \delta^{\nu} \leq |\sigma - \sigma_{+}| \leq |\sigma - \sigma_{-}| \leq \kappa \delta |\log \delta|$ and, as a consequence, (6.19) holds true for $C \delta^{\nu} \leq |\sigma - \sigma_{\pm}| \leq \kappa \delta |\log \delta|$.

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Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Table 7

The Giacaglia function $U_2^{m,0}(\epsilon)$ for the moon perturbation (see [24]) where $C = \cos \frac{\epsilon}{2}$ and $S = \sin \frac{\epsilon}{2}$ and its value for $\epsilon = 23.44^\circ$.

m	$U_2^{m,0}(\epsilon)$	$\approx$
0	$1 - 6C^2 + 6C^4$	0.762646
1	$-3CS^{-1}(2C^4 - 3C^2 + 1)$	0.547442
2	$6C^2S^{-2}(C^2 - 1)^2$	0.237353

Appendix. Expression of the Hamiltonian

This appendix is devoted to obtain the expressions for the full coplanar and h -averaged Hamiltonians introduced in Section 2.2.

Slow-fast delaunay coordinates (y, x) . Analogously to the definition of the coplanar Hamiltonian $H_{CP,1}$ in Poincaré coordinates (see (2.17)), we define

$$H_{CP,1}(y, \Gamma, x, h) = H_1(y, \Gamma, x, h, \Omega_M; 0).$$

Recall that, by Remark 2.1, H_1 is independent of Ω_M when $i_M = 0$. Then, by (2.5) and (2.11), one has that

$$H_{CP,1}(y, \Gamma, x, h) = \frac{\rho_1}{L^2} \sum_{m=0}^2 \sum_{p=0}^2 \hat{c}_m U_2^{m,0} D_{m,p}(y, \Gamma) \cos(\psi_{m,p,0}(x, h)),$$

where $D_{m,p} = \tilde{D}_{m,p} \circ Y^{\text{Del}}$, $\psi_{m,p,0} = \tilde{\psi}_{m,p,0} \circ Y^{\text{Del}}$, with $\tilde{D}_{m,p}$ and $\tilde{\psi}_{m,p,0}$ defined in (2.6) and (2.8) respectively, and

$$\hat{c}_0 = c_{0,s} = \frac{1}{2}, \quad \hat{c}_1 = c_{1,s} = \frac{1}{3}, \quad \hat{c}_2 = c_{2,s} = -\frac{1}{12}.$$

Notice that these last definitions are possible since, for $s, m \in \{0, 1, 2\}$, the constants $c_{m,s}$ as defined in (2.9) do not depend on s . In addition, $U_2^{m,0}(\epsilon)$ is the Giacaglia function given in Table 7.

Applying the slow-fast change of coordinates, one obtains that

$$\psi_{m,p,0}(x, h) = (1-p)x - (1-p-m)h.$$

See Table 8 for the explicit expressions of $\psi_{m,p,0}(x, h)$ and $D_{m,p}(y, \Gamma)$.

Now, we consider the h -average of $H_{CP,1}$,

$$H_{AV,1}(y, \Gamma, x) = \frac{1}{2\pi} \int_0^{2\pi} H_{CP,1}(y, \Gamma, x, h) dh$$

The definition of $\psi_{m,p,0}(x, h)$ implies that the only terms independent of h are for the couples $(m, p) = (0, 1)$ and $(m, p) = (1, 0)$ (see Table 8). As a consequence, we obtain the expression of the averaged Hamiltonian presented in (2.20).

Poincaré coordinates (η, ξ) . Taking into account that the Poincaré change of coordinates satisfies that

$$y = \frac{2L - \xi^2 - \eta^2}{4}, \quad (L - 2y) \cos x = \frac{\xi^2 - \eta^2}{2}, \quad (L - 2y) \sin x = \xi\eta,$$

one has that

$$\begin{aligned} H_{CP,1} = \frac{\rho_1}{L^2} \sum_{m=0}^2 \hat{c}_m U_2^{m,0} & \left[D_{m,0}(\eta, \Gamma, \xi) \left(\frac{\xi^2 - \eta^2}{2} \cos((1-m)h) + \xi\eta \sin((1-m)h) \right) \right. \\ & + D_{m,1}(\eta, \Gamma, \xi) (8L^2 + 12L(\xi^2 + \eta^2) - 3(\xi^2 + \eta^2)^2) \cos(mh) \\ & \left. + D_{m,2}(\eta, \Gamma, \xi) \left(\frac{\xi^2 - \eta^2}{2} \cos((1+m)h) + \xi\eta \sin((1+m)h) \right) \right], \end{aligned} \quad (\text{A.1})$$

where the functions $(D_{m,p})_{m,p \in \{0,1,2\}}$ are given in Table 9.

Let us now consider the h -averaged Hamiltonian $H_{AV,1}$ as defined in (2.18). Analogously to $H_{AV,1}$, one sees that the only terms independent of h are for the couples $(m, p) = (0, 1)$ and $(1, 0)$. As a consequence, one obtains the expression in (2.19).

Data availability

No data was used for the research described in the article.

Table 8

Computation of the functions $(D_{m,p})_{m,p \in \{0,1,2\}}$ and $(\psi_{m,p,0})_{m,p \in \{0,1,2\}}$ for the prograde case.

m	p	$D_{m,p}(y, \Gamma)$	$\psi_{m,p,0}$
0	0	$-\frac{15}{64}(Ly)^{-2}(y-\Gamma)(3y+\Gamma)(L-2y)(L+2y)$	$x-h$
0	1	$\frac{1}{32}(Ly)^{-2}(y^2-6y\Gamma-3\Gamma^2)(5L^2-12y^2)$	0
0	2	$-\frac{15}{64}(Ly)^{-2}(y-\Gamma)(3y+\Gamma)(L-2y)(L+2y)$	$-x-h$
1	0	$\frac{15}{32}(Ly)^{-2}\sqrt{(y-\Gamma)(3y+\Gamma)}(3y+\Gamma)(L-2y)(L+2y)$	x
1	1	$-\frac{3}{16}(Ly)^{-2}\sqrt{(y-\Gamma)(3y+\Gamma)}(\Gamma+y)(5L^2-12y^2)$	h
1	2	$-\frac{15}{32}(Ly)^{-2}\sqrt{(y-\Gamma)(3y+\Gamma)}(y-\Gamma)(L-2y)(L+2y)$	$-x+2h$
2	0	$\frac{15}{32}(Ly)^{-2}(3y+\Gamma)^2(L-2y)(L+2y)$	$x+h$
2	1	$\frac{3}{16}(Ly)^{-2}(y-\Gamma)(3y+\Gamma)(5L^2-12y^2)$	$2h$
2	2	$\frac{15}{32}(Ly)^{-2}(y-\Gamma)^2(L-2y)(L+2y)$	$-x+3h$

Table 9

Computation of the functions $(D_{m,p})_{m,p \in \{0,1,2\}}$ with $M := \xi^2 + \eta^2$.

m	p	$D_{m,p}(\eta, \Gamma, \xi)$
0	0	$-\frac{15}{128}L^{-2}(2L-M)^{-2}(2L-M-4\Gamma)(6L-3M+4\Gamma)(4L-M)$
0	1	$\frac{1}{128}L^{-2}(2L-M)^{-2}((2L-M)^2-24(2L-M)\Gamma-48\Gamma^2)$
0	2	$-\frac{15}{128}L^{-2}(2L-M)^{-2}(2L-M-4\Gamma)(6L-3M+4\Gamma)(4L-M)$
1	0	$\frac{15}{64}L^{-2}(2L-M)^{-2}\sqrt{(2L-M-4\Gamma)}(6L-3M+4\Gamma)^{3/2}(4L-M)$
1	1	$-\frac{3}{64}L^{-2}(2L-M)^{-2}\sqrt{(2L-M-4\Gamma)}(6L-3M+4\Gamma)(2L-M+4\Gamma)$
1	2	$-\frac{15}{64}L^{-2}(2L-M)^{-2}(2L-M-4\Gamma)^{3/2}\sqrt{6L-3M+4\Gamma}(4L-M)$
2	0	$\frac{15}{64}L^{-2}(2L-M)^{-2}(6L-3M+4\Gamma)^2(4L-M)$
2	1	$\frac{3}{64}L^{-2}(2L-M)^{-2}(2L-M-4\Gamma)(6L-3M+4\Gamma)$
2	2	$\frac{15}{64}L^{-2}(2L-M)^{-2}(2L-M-4\Gamma)^2(4L-M)$

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