Regularity Theory for the Obstacle Problem

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Declaration of Authorship

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Abstract

In this work, we introduce and study a classical **free boundary problem** known as the *obstacle problem*. This problem serves as a foundational example in the broader theory of free boundary problems, where part of the solution involves determining an unknown interface or region. We begin by presenting the classical formulation of the obstacle problem, which arises naturally in various physical and geometric contexts, such as elasticity, fluid dynamics, and potential theory.

We then explore the key theoretical aspects of the problem, focusing on the existence, uniqueness, and regularity of solutions. Special attention is given to the structure and behavior of the free boundary, the interface separating the active and inactive regions which plays a central role in understanding the qualitative features of the solution.

To analyze the behavior of the solution near the free boundary, we employ the method of *blow-ups*, a powerful technique that allows for the study of local properties by rescaling the problem around singular points. This approach provides deep insights into the regularity and classification of free boundary points, distinguishing between regular and singular behavior and leading to a better understanding of the geometry of the free boundary.

Overall, this work offers a rigorous introduction to the obstacle problem, combining classical theory with modern analytical tools to examine one of the most important and illustrative problems in the study of partial differential equations and variational inequalities.

Keywords: Obstacle Problem, Free boundary Problem, Regularity

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Introduction

« Qualche tempo dopo Stampacchia, partendo sempre dalla sua disequazione variazionale, aperse un nuovo campo di ricerche che si rivelò importante e fecondo. Si tratta di quello che oggi è chiamato il problema dell'ostacolo. »[1]

"Some time after Stampacchia—always starting from his variational inequality—he opened up a new field of research which proved both important and fruitful. This is what today is called the obstacle problem."

— Sandro Faedo

This work focuses on the study of a fundamental problem in the theory of nonlinear elliptic partial differential equations, known as the *obstacle problem*. This problem serves as a canonical model within the broader class of *free boundary problems*, which naturally arise in situations where both the solution and the domain in which it is defined must be determined simultaneously.

Free Boundary Problems and the Obstacle Problem

Free boundary problems occur in a variety of physical and engineering contexts where an interface separates regions with distinct behaviors or phases. Typical examples include the interface between ice and water, or the edge of a membrane constrained by a surface. What characterizes these problems is that part of the boundary of the domain is initially unknown and must be found as part of the solution.

One of the most classical and well studied instances of a free boundary problem is the obstacle problem, which can be interpreted as finding the equilibrium configuration of an elastic membrane whose boundary is fixed and which is constrained to lie above a given obstacle.

[1] [Fae86, p. 107]

1.1 The obstacle problem

Physical Motivation: The Elastic Membrane

Consider an elastic membrane such as a thin fabric sheet, rigidly clamped along its boundary. When the membrane is under uniform tension, gravitational effects are negligible compared to the stretching forces, so in the absence of any obstacle the membrane remains flat. If, however, a rigid object is placed beneath the membrane, the fabric must deform around the obstacle, lifting off the frame to accommodate its shape. Our goal is to determine the resulting equilibrium profile u of the membrane.

From a mechanical standpoint, the membrane seeks the configuration of least elastic energy. Neglecting bending stiffness and gravity, the dominant contribution is the *surface energy*, proportional to the area of the deformed membrane. Let

$$D \subset \mathbb{R}^2$$

be the fixed domain over which the membrane is stretched, and let

$$u : D \longrightarrow \mathbb{R}$$

denote its vertical displacement. Then the exact surface area of the membrane is

$$A[u] = \int_D \sqrt{1 + |\nabla u(x)|^2}\, dx,$$

since each infinitesimal patch in the base is stretched by the factor $\sqrt{1 + |\nabla u|^2}$. The membrane's equilibrium shape minimizes this area subject to the clamped boundary condition and the requirement that it does not penetrate the obstacle:

$$\min_u A[u] \quad \text{subject to} \quad u = 0 \text{ on } \partial D, \quad u(x) \geq \varphi(x)\ \forall\, x \in D,$$

where $\varphi : D \longrightarrow \mathbb{R}$ describes the obstacle's profile.

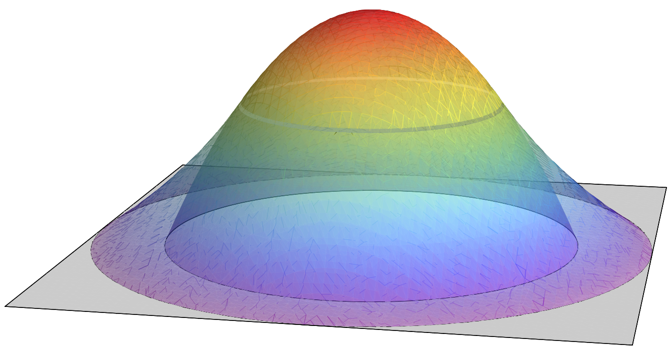

Figure 1.1: *An elastic membrane deformed by an obstacle [Edq10].*

For small slopes, $|\nabla u| \ll 1$, one may expand the integrand:

$$A[u] = \int_D \sqrt{1 + |\nabla u|^2}\, dx \;\approx\; |D| + \tfrac{1}{2}\int_D |\nabla u|^2\, dx.$$

Ignoring the constant area $|D|$, this leads to the *Dirichlet energy*

$$\mathcal{E}(u) = \frac{1}{2}\int_D |\nabla u(x)|^2\, dx,$$

and the classical variational formulation of the obstacle problem:

$$\min_{u \in \mathcal{K}} \mathcal{E}(u), \qquad \mathcal{K} = \{\, u \in H^1(D) \mid u = 0 \text{ on } \partial D,\ u \geq \varphi\}.$$

Mathematical Formulation

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and fix boundary data f and obstacle ϕ. We set

$$\mathcal{K} := \left\{ v \in H^1(\Omega) \;\middle|\; v|_{\partial\Omega} = f,\ v \geq \phi \text{ in } \Omega \right\},$$

and consider the Dirichlet energy

$$\mathcal{E}(v) := \frac{1}{2}\int_\Omega |\nabla v(x)|^2\, dx.$$

The *classical obstacle problem* is the variational problem

$$\min_{v \in \mathcal{K}} \mathcal{E}(v). \tag{1.1}$$

Its unique minimizer $u \in \mathcal{K}$ represents the equilibrium configuration of an elastic membrane clamped to f on $\partial\Omega$ and constrained to lie above the obstacle ϕ in Ω.

Remark. We have not specified any smoothness assumptions on f or ϕ at this stage; the precise regularity requirements and their impact on the solution will be discussed in the following chapter devoted to regularity theory.

Variational Inequality Formulation

An equivalent formulation of the obstacle problem is via a *variational inequality*. Recall the admissible set

$$\mathcal{K} = \left\{ v \in H^1(\Omega) \mid v|_{\partial\Omega} = f,\ v \geq \phi \text{ in } \Omega \right\}.$$

A function $u \in \mathcal{K}$ minimizes the Dirichlet energy $\mathcal{E}(v) = \frac{1}{2}\int_\Omega |\nabla v|^2$ if and only if

$$\int_\Omega \nabla u \cdot \nabla(v - u)\, dx \;\geq 0 \quad \forall v \in \mathcal{K}.$$

Indeed, for any $v \in \mathcal{K}$ and $\varepsilon \in (0,1)$, the perturbed $u_\varepsilon = u + \varepsilon(v - u)$ lies in $\mathcal{K}$, and since it's a minimum

$$0 \le \frac{d}{d\varepsilon}\mathcal{E}(u_\varepsilon)\Big|_{\varepsilon=0^+} = \int_\Omega \nabla u \cdot \nabla(v - u)\, dx.$$

Euler–Lagrange Equations and Complementarity

From the variational inequality one deduces:

(i) Taking $v = u - \eta$ with $\eta \in C_c^\infty(\Omega)$, and $u - \eta \ge \phi$ wich is equivalent to $\eta \ge 0$, yields

$$\int_\Omega \nabla u \cdot \nabla\eta\, dx \le 0 \implies -\int_\Omega (\Delta u)\, \eta\, dx \le 0 \implies \Delta u \le 0 \text{ in } \mathcal{D}'(\Omega).$$

Hence u is *superharmonic*.

(ii) If $\omega \subset \{u > \phi\}$ is any open set on which $u - \varepsilon\eta \ge \phi$ for small ε and all $\eta \in C_c^\infty(\omega)$, then using both $v = u \pm \varepsilon\eta$ in the variational inequality gives $\int \nabla u \cdot \nabla\eta = 0$ for all η, and thus

$$\Delta u = 0 \quad \text{in } \{u > \phi\}.$$

That is, u is *harmonic* in the non–contact region.

In summary, the variational inequality simultaneously enforces

$$\Delta u \le 0 \quad \text{in the whole domain}, \qquad \Delta u = 0 \quad \text{in the open set } \{u > \phi\},$$

while of course $u \ge \phi$ everywhere. These three properties form the Euler–Lagrange conditions for the obstacle problem and will be the starting point for our regularity analysis in the chapters that follow.

Thus, the minimizer v of the constrained energy problem satisfies the following Euler–Lagrange system (in the weak sense), together with the boundary condition $v|_{\partial\Omega} = g$:

$$\begin{cases} v \ \ge \ \varphi, & \text{in } \Omega, \\ \Delta v \ \le \ 0, & \text{in } \Omega, \\ \Delta v \ = \ 0, & \text{in } \{ v > \varphi\}. \end{cases} \tag{1.2}$$

Reformulation via the gap function.

Setting the gap function as

$$u = v - \varphi,$$

the obstacle problem becomes equivalent to finding $u \in H^1(\Omega)$ with $u \geq 0$ and $u|_{\partial\Omega} = g - \varphi$ such that

$$\begin{cases} u \geq 0, & \text{in } \Omega, \\ \Delta u \leq f, & \text{in } \Omega, \\ \Delta u = f, & \text{in } \{u > 0\}, \end{cases} \tag{1.3}$$

where $f := -\Delta\varphi$. Equivalently, u minimizes the energy

$$\int_\Omega \left(\tfrac{1}{2}|\nabla u|^2 + f\,u\right) dx \quad \text{over } \{u \in H^1(\Omega) : u \geq 0,\ u|_{\partial\Omega} = g - \varphi\}. \tag{1.4}$$

In this formulation, the complementarity condition is expressed by

$$\min\{-\Delta u + f,\ u\} = 0 \quad \text{in } \Omega,$$

or equivalently

$$(\Delta u - f)\,u = 0, \quad \Delta u \leq f, \quad u \geq 0.$$

This gap formulation often simplifies the analysis of the free boundary $\partial\{u > 0\}$.

The Free Boundary and the Contact Set

In the obstacle problem the solution u and the interface on which it detaches from the obstacle are both unknown. To describe these, we introduce three fundamental sets:

Definition 1.1 (Contact set)**.** The *contact set* is the region where the membrane touches the obstacle:

$$\mathcal{C}(u) \ := \ \{\, x \in \Omega : u(x) = \varphi(x)\}.$$

Definition 1.2 (Non-contact region)**.** The *non-contact region* is its complement in Ω, where the obstacle is inactive and u is harmonic:

$$\mathcal{N}(u) \ := \ \{\, x \in \Omega : u(x) > \varphi(x)\}.$$

Definition 1.3 (Free boundary)**.** The *free boundary* is the internal boundary between contact and non-contact:

$$\mathcal{F}(u) \ := \ \partial\mathcal{N}(u) \ \cap \ \Omega \ = \ \partial\mathcal{C}(u) \ \cap \ \Omega.$$

Together with u, the set $\mathcal{F}(u)$ is part of the solution; this feature is what makes the obstacle problem a true *free boundary problem*.

On the non-contact region $\mathcal{N}(u)$, the solution satisfies

$$\Delta u = 0 \quad \text{in } \mathcal{N}(u),$$

whereas on the contact set $C(u)$ one has

$$u = \varphi.$$

Jointly with u, the free boundary it's and unknown for the problem, this is why it's categorized as a free boundary problem.

The problem can be rearanged in order to involve the freee boundary as part of the problem, this is that ther's $v \geq \phi$:

$$\begin{cases} \Delta v = 0 & \text{in } \{v > \phi\}, \\ v = \phi & \text{on } \mathcal{F}, \\ \nabla u = 0 & \text{on } \mathcal{F}, \end{cases} \tag{1.5}$$

Also for the gap formulation (1.3):

$$\begin{cases} \Delta u = f & \text{in } \{u > 0\}, \\ u = 0 & \text{on } \mathcal{F}, \\ \nabla u = 0 & \text{on } \mathcal{F}, \end{cases} \tag{1.6}$$

Then truns out that the free boundary its part of the solution, so understanding the structure and regularity of this free boundary is one of the main challenges in the theory of obstacle problems.

Historical Development of the Theory

The obstacle problem has been central to the development of modern free boundary theory. The foundational existence and uniqueness results were established in the 1960s through the development of the theory of *variational inequalities*, with key contributions by Guido Stampacchia and Gaetano Fichera.

In the 1970s, Kinderlehrer and Nirenberg proved that, under certain conditions, the free boundary is an analytic surface. A major conceptual breakthrough came with the work of Luis Caffarelli (1977), who introduced the method of *blow-up analysis*, allowing for a detailed local study of the free boundary's geometry.

Since then, powerful analytical tools such as *monotonicity formulas* (e.g., Alt-Caffarelli-Friedman, Weiss, Monneau) have enabled a deeper understanding of the classification of free boundary points into regular and singular types, and the precise geometric structure of each.

Objectives of This Work

This work is based on the presentation in the book [FRRO22], where the obstacle problem is treated in the classical *constant* setting wich is comon in all the literature. Our goal is to extend that framework to a more general context, allowing for a nonconstant source term f under suitable regularity assumptions.

Despite its deceptively simple formulation, the obstacle problem exhibits a rich mathematical structure and remains a cornerstone of free boundary theory. In this spirit, the objectives of the present work are:

- To review and expand the classical obstacle problem, in the spirit of [FRRO22].
- To develop existence and uniqueness results for solutions when the source term f is nonconstant but satisfies appropriate growth and continuity conditions.
- To analyze the regularity of both the solution and the free boundary in this more general setting.
- To introduce and adapt key techniques blow-up analysis, classification of global profiles, and monotonicity formulas to handle the variable f case.

1.2 Applications

The obstacle and free-boundary problems introduced above arise in a wide variety of settings. Below we list some of the most important examples:

- **Fluid filtration and dam problems.** In models of groundwater flow through a porous medium, the free boundary represents the moving interface between the saturated and unsaturated regions. One obtains a variational inequality for the hydraulic head that is exactly of obstacle-type, with the obstacle given by the impermeable bedrock or dam structure.
- **American option pricing.** The early-exercise feature of an American derivative leads to a free-boundary problem for the option price. Writing the underlying asset as a diffusion process and the payoff as the obstacle, one arrives at the classical obstacle problem (a linear complementarity problem) whose solution yields both the price and the optimal exercise boundary.
- **The Stefan problem.** In phase-change phenomena (e.g. melting ice or solidification of a liquid), the unknown interface between phases moves according to latent-heat balance. The temperature solves a heat-equation with a moving boundary, which can be reformulated as an obstacle-type condition on the enthalpy.
- **Interacting particle systems.** In the hydrodynamic limit of certain lattice-gas models with exclusion or aggregation, one obtains continuum PDEs with density constraints. These lead to variational inequalities describing crowding effects, clogging, or congestion, and are closely related to obstacle-type free-boundary formulations.
- **Control theory.** Stochastic control and differential games yield fully nonlinear Hamilton–Jacobi–Bellman or Isaacs equations. In particular, optimal stopping and impulse-control problems give rise to obstacle and double-obstacle problems for these nonlinear operators, encoding the optimal intervention strategies.

One of the examples that we will present a little bit deeper is the one related to probability, for more details on this topic see [FRRO22] and [Eva13].

Probabilistic Interpretation of Elliptic PDEs

Below we give a detailed account of how various probabilistic constructions, ranging from pure Brownian motion to controlled and stopped processes, lead naturally to (possibly fully nonlinear) elliptic PDEs.

Brownian motion and the Laplace equation

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and X_t^x a standard Brownian motion starting at $x \in \Omega$. Denote by

$$\tau = \inf\{ t > 0 : X_t^x \in \partial\Omega\}$$

its first exit time from Ω. Given a boundary payoff

$$g : \partial\Omega \to \mathbb{R},$$

one considers the "game":

$$\text{Run the Brownian path until it hits } \partial\Omega; \quad \text{pay } g(X_\tau^x).$$

The *expected payoff*

$$u(x) = \mathbb{E}\big[g\big(X_\tau^x\big)\big]$$

satisfies the classical Dirichlet problem

$$\begin{cases} \Delta u = 0, & x \in \Omega, \\ u = g, & x \in \partial\Omega. \end{cases}$$

This follows from the mean-value property of harmonic functions and the isotropy of Brownian motion.

Infinitesimal generator and general elliptic operators

More generally, let X_t be a Markov process in $\mathbb{R}^n$ with

- independent, stationary increments,
- continuous paths,
- (optionally) rotational symmetry.

Its *infinitesimal generator* L acts on smooth $u : \mathbb{R}^n \to \mathbb{R}$ by

$$Lu(x) = \lim_{t \downarrow 0} \frac{\mathbb{E}[\, u(x + X_t)\,] - u(x)}{t}\,. \tag{1.7}$$

Under mild conditions one shows

$$Lu(x) = \sum_{i,j=1}^{n} a_{ij}(x)\,\partial_{ij}u(x) + \sum_{i=1}^{n} b_i(x)\,\partial_i u(x) + c(x)\,u(x),$$

a second-order elliptic operator. Moreover the fundamental relation

$$\mathbb{E}\big[u(x + X_t)\big] = u(x) \; + \; \mathbb{E}\Big[\int_0^t L\,u(x + X_s)\,ds\Big] \tag{1.8}$$

plays the role of the Fundamental Theorem of Calculus.

Exit-time problems

Applying (1.8) to the expected exit-time function

$$\tau_x \;=\; \mathbb{E}\big[\tau \;\big|\; X_0 = x\big]$$

one finds

$$\begin{cases} -L\,\tau_x = 1, & x \in \Omega, \\ \tau_x = 0, & x \in \partial\Omega, \end{cases}$$

a Poisson problem giving the mean time to exit Ω.

Optimal stopping and the obstacle problem

Allow stopping at any $\theta \ge 0$ to collect payoff $\varphi(x + X_\theta)$. The *value function*

$$u(x) \;=\; \sup_{\theta} \mathbb{E}\big[\varphi(x + X_\theta)\big]$$

satisfies the variational inequality

$$\min\big\{-L\,u,\; u - \varphi\big\} = 0 \quad \text{in } \mathbb{R}^n,$$

equivalently

$$\begin{cases} u \ge \varphi, \\ -L\,u \ge 0, \\ L\,u = 0 \quad \text{on } \{u > \varphi\}, \end{cases}$$

which is the classical obstacle (free-boundary) problem.

Stochastic control and nonlinear PDEs

Finally, if one allows a controller to choose at each time a drift or volatility within a set of admissible controls, the dynamic programming principle leads to a fully nonlinear

Hamilton–Jacobi–Bellman equation

$$\sup_{\alpha \in A} \Big\{ L^{\alpha} u(x) \Big\} = 0,$$

and in two-player zero-sum games to the Isaacs equation

$$\inf_{\beta} \sup_{\alpha} \{ L^{\alpha,\beta} u \} = 0.$$

These are the prototypical fully nonlinear elliptic PDEs with deep connections to probability.

In summary, from Brownian motion to stopping and control, each probabilistic "game" or payoff-maximization problem is encoded by an elliptic (possibly nonlinear) PDE whose solution precisely captures the optimal expected outcome.

2

Preliminaries

In this chapter we assemble the fundamental tools and results needed for the obstacle problem. After recalling key concepts from functional analysis, Sobolev spaces, variational inequalities, and partial differential equations, we state the classical theorems without proof when they are standard, and give full proofs only for those results that play a critical role later. For a more comprehensive treatment and further developments we refer to [Eva98], [Bre11], [Jos12], [Neč12], [Tri10], [Gri11], and [FRRO22].

Before advancing to the main theory, we begin with several preliminary lemmas and propositions. These foundational results, though elementary, will be used repeatedly throughout the sequel chpaters.

2.1 Function Spaces and Variational Formulation

2.1.1 Sobolev Spaces and Fundamental Results

Let $\Omega \subset \mathbb{R}^n$ be an open set. For $1 \leq p \leq \infty$ and integer $k \geq 0$, the Sobolev space $W^{k,p}(\Omega)$ is defined by

$$W^{k,p}(\Omega) = \left\{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \text{ for } |\alpha| \leq k\right\},$$

with norm

$$\|u\|_{W^{k,p}(\Omega)} = \Big(\sum_{|\alpha|\leq k} \|D^\alpha u\|^p_{L^p(\Omega)}\Big)^{1/p}.$$

In particular, $H^k(\Omega) = W^{k,2}(\Omega)$ is a Hilbert space.

Theorem 2.1 (Poincaré Inequality)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, and let $p \in [1, \infty)$. Then, for every $u \in W^{1,p}(\Omega)$,*

$$\int_\Omega |u - u_\Omega|^p \, dx \leq C_{\Omega,p} \int_\Omega |\nabla u|^p \, dx,$$

where $u_\Omega := \frac{1}{|\Omega|}\int_\Omega u\,dx$. *Furthermore,*

$$\int_\Omega |u|^p\,dx \le C'_{\Omega,p}\left(\int_\Omega |\nabla u|^p\,dx + \int_{\partial\Omega} |u|^p\,d\sigma\right),$$

for constants $C_{\Omega,p}$, $C'_{\Omega,p}$ *depending only on* n, p, *and* Ω.

Theorem 2.2 (Morrey's Inequality). *Let* $p > n$. *Then,*

$$\sup_{x\neq y}\frac{|u(x)-u(y)|}{|x-y|^\alpha} \le C\left(\int_{\mathbb{R}^n}|\nabla u|^p\,dx\right)^{1/p}, \quad \text{where } \alpha = 1-\frac{n}{p},$$

for some constant C *depending only on* n *and* p.

In particular, when $p > n$, any function in $W^{1,p}$ is continuous (possibly after being redefined on a set of measure zero).

Theorem 2.3 (Rellich–Kondrachov Compactness Theorem). *Let* $\Omega \subset \mathbb{R}^n$ *be a bounded Lipschitz domain. Then the embedding*

$$H^1(\Omega) \hookrightarrow L^2(\Omega)$$

is compact.

Theorem 2.4 (Trace Theorem). *Let* Ω *be any bounded Lipschitz domain, and* $1 \le p \le \infty$. *Then, there exists a continuous (and compact for* $p > 1$*) trace operator from* $W^{1,p}(\Omega)$ *to* $L^p(\partial\Omega)$. *For* C^0 *functions, this trace operator is simply* $u \mapsto u|_{\partial\Omega}$.

As a result, for any function $u \in H^1(\Omega)$, *we will still denote by* $u|_{\partial\Omega}$ *its trace on* $\partial\Omega$.

2.1.2 Weak Formulation of Elliptic Problems

Consider the Dirichlet problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Multiplying by a test function $v \in H_0^1(\Omega)$ and integrating by parts leads to the bilinear form

$$a(u,v) = \int_\Omega \nabla u\cdot\nabla v\,dx, \quad L(v) = \int_\Omega f\,v\,dx.$$

A function $u \in H_0^1(\Omega)$ is a *weak solution* if

$$a(u,v) = L(v) \quad \forall v \in H_0^1(\Omega).$$

Theorem 2.5 (Lax–Milgram Theorem). *Let* V *be a Hilbert space, let*

$$a : V\times V \to \mathbb{R}$$

be a continuous, coercive bilinear form (i.e. there exist $M, \alpha > 0$ such that

$$|a(u,v)| \leq M\|u\|_V\|v\|_V, \quad a(v,v) \geq \alpha\|v\|_V^2,$$

for all $u, v \in V$), and let

$$L : V \to \mathbb{R}$$

be a continuous linear functional. Then there exists a unique $u \in V$ such that

$$a(u,v) = L(v) \quad \forall v \in V.$$

2.2 Qualitative Properties

Theorem 2.6 (Weak maximum principle)**.** *Assume $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ and $c \equiv 0$ in Ω.*

(*i*) *If $p(x,D)u \leq 0$ in Ω (subsolution), then*

$$\max_{\overline{\Omega}} u \;=\; \max_{\partial\Omega} u.$$

(*ii*) *If $p(x,D)u \geq 0$ in Ω (supersolution), then*

$$\min_{\overline{\Omega}} u \;=\; \min_{\partial\Omega} u.$$

In particular, for a smooth solution u of $p(x,D)u = 0$ in Ω, we have

$$\max_{\Omega}|u| \;=\; \max_{\partial\Omega}|u|.$$

Theorem 2.7 (Comparison principle)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and let $u, v \in C^2(\Omega) \cap C^1(\overline{\Omega})$ satisfy $c \equiv 0$ in Ω. If*

$$p(x,D)u \;\leq\; p(x,D)v \quad \text{in } \Omega,$$

and

$$u \;\leq\; v \quad \text{on } \partial\Omega,$$

then

$$u \;\leq\; v \quad \text{throughout } \Omega.$$

Remark 2.8. Given any function u, define

$$u^+ = \max\{u, 0\} \quad \text{and} \quad u^- = \max\{-u, 0\},$$

so that

$$u = u^+ - u^-.$$

Then, for any $u \in W^{1,p}(\Omega)$, we have $u^+, u^- \in W^{1,p}(\Omega)$, and

$$\nabla u = \nabla u^+ - \nabla u^- \quad \text{a.e. in } \Omega.$$

In particular, the gradient of Sobolev functions vanishes almost everywhere on level sets:

$$\nabla u(x) = 0 \quad \text{for a.e. } x \in \{u = 0\}.$$

Theorem 2.9 (Harnack's Inequality)**.** *Let $u \in H^1(B_1)$ be a non-negative, harmonic function in B_1. Then the infimum and supremum of u over the smaller ball $B_{1/2}$ are comparable. That is,*

$$\begin{cases} \Delta u = 0 & \text{in } B_1, \\ u \geq 0 & \text{in } B_1 \end{cases} \quad \Rightarrow \quad \sup_{B_{1/2}} u \leq C \inf_{B_{1/2}} u,$$

for some constant $C > 0$ depending only on the dimension n.

Proof. The result follows from the mean value property of harmonic functions. Alternatively, it can be derived using the Poisson kernel representation:

$$u(x) = c_n \int_{\partial B_1} \frac{(1 - |x|^2)\, u(z)}{|x - z|^n}\, dz,$$

for $x \in B_1$, where c_n is a dimensional constant.

For $x \in B_{1/2}$ and $z \in \partial B_1$, observe that $|x - z|^n$ is bounded above and below:

$$2^{-n} \leq |x - z|^n \leq (3/2)^n, \quad \text{and} \quad \frac{3}{4} \leq 1 - |x|^2 \leq 1.$$

Thus, since $u \geq 0$ in B_1, we obtain the bounds

$$C^{-1} \int_{\partial B_1} u(z)\, dz \leq u(x) \leq C \int_{\partial B_1} u(z)\, dz, \quad \text{for all } x \in B_{1/2},$$

for some constant $C > 0$ depending only on n.

In particular, for any $x_1, x_2 \in B_{1/2}$, we have $u(x_1) \leq C^2 u(x_2)$. Taking the supremum over $x_1 \in B_{1/2}$ and the infimum over $x_2 \in B_{1/2}$ yields

$$\sup_{B_{1/2}} u \leq \tilde{C} \inf_{B_{1/2}} u,$$

for some constant $\tilde{C} > 0$ depending only on n, as claimed. □

Lemma 2.10. *Let u be any weak solution to*

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Then,

$$\|u\|_{L^\infty(\Omega)} \leq C \left(\|f\|_{L^\infty(\Omega)} + \|g\|_{L^\infty(\partial\Omega)} \right),$$

for a constant C depending only on the diameter of Ω.

Proof. Define the normalized function

$$\tilde{u}(x) := \frac{u(x)}{\|f\|_{L^\infty(\Omega)} + \|g\|_{L^\infty(\partial\Omega)}}.$$

We aim to show that $|\tilde{u}(x)| \leq C$ in Ω, for some constant C depending only on the diameter of Ω.

Note that $\tilde{u}$ satisfies

$$\begin{cases} \Delta \tilde{u} = \tilde{f} & \text{in } \Omega, \\ \tilde{u} = \tilde{g} & \text{on } \partial\Omega, \end{cases}$$

with $|\tilde{f}| \leq 1$ and $|\tilde{g}| \leq 1$.

Choose R large enough so that $B_R \supset \Omega$; after a translation, we may assume $R = \frac{1}{2}\operatorname{diam}(\Omega)$. Consider the function

$$w(x) := \frac{R^2 - x_1^2}{2} + 1$$

defined in B_R. This function satisfies

$$\begin{cases} \Delta w = -1 & \text{in } \Omega, \\ w \geq 1 & \text{on } \partial\Omega. \end{cases}$$

By the comparison principle, we deduce that

$$\tilde{u} \leq w \quad \text{in } \Omega,$$

and similarly $\tilde{u} \geq -w$, completing the proof. □

Theorem 2.11. *Let* $f \in L^\infty(B_1)$*, and* $u \in H^1(B_1)$ *satisfy*

$$\begin{cases} \Delta u = f & \text{in } B_1, \\ u \geq 0 & \text{in } B_1. \end{cases}$$

Then,

$$\sup_{B_{1/2}} u \leq C \left(\inf_{B_{1/2}} u + \|f\|_{L^\infty(B_1)} \right),$$

for some constant C depending only on the dimension n.

Proof. We decompose u as $u = v + w$, where

$$\begin{cases} \Delta v = 0 & \text{in } B_1, \\ v = u & \text{on } \partial B_1, \end{cases} \quad \text{and} \quad \begin{cases} \Delta w = f & \text{in } B_1, \\ w = 0 & \text{on } \partial B_1. \end{cases}$$

By Theorem 2.9 and Lemma 2.10, we obtain

$$\sup_{B_{1/2}} v \le C \inf_{B_{1/2}} v, \quad \text{and} \quad \|w\|_{L^\infty(B_1)} \le C\|f\|_{L^\infty(B_1)}.$$

Therefore,

$$\sup_{B_{1/2}} u \le \sup_{B_{1/2}} v + \sup_{B_{1/2}} w \le C \inf_{B_{1/2}} v + C\|f\|_{L^\infty(B_1)}.$$

Finally, noting that $\inf_{B_{1/2}} v \le \inf_{B_{1/2}} u$, we conclude

$$\sup_{B_{1/2}} u \le C \left(\inf_{B_{1/2}} u + \|f\|_{L^\infty(B_1)} \right),$$

possibly increasing C if needed. □

Thus, as before, we also obtain an oscillation decay estimate, now including an error term of size $\|f\|_{L^\infty}$.

Lemma 2.12 (Hopf Lemma). *Let $\Omega \subset \mathbb{R}^n$ be any domain satisfying the interior ball condition. Let $u \in C(\overline{\Omega})$ be any positive harmonic function in $\Omega \cap B_2$, with $u \ge 0$ on $\partial\Omega \cap B_2$.*

Then, $u \ge c_0 d$ in $\Omega \cap B_1$ for some $c_0 > 0$, where $d(x) := \mathrm{dist}(x, \Omega^c)$.

Proof. Since u is positive and continuous in $\Omega \cap B_2$, we have that $u \ge c_1 > 0$ in $\{d \ge \rho_0/2\} \cap B_{3/2}$ for some $c_1 > 0$.

Let us consider the solution of $\Delta w = 0$ in $B_{\rho_0} \setminus B_{\rho_0/2}$, with $w = 0$ on ∂B_{ρ_0} and $w = 1$ on $\partial B_{\rho_0/2}$. Such function w is explicit it is simply a truncated and rescaled version of the fundamental solution Φ defined as:

$$\Phi(x) := \begin{cases} \kappa_n |x|^{2-n}, & n \ge 3, \\ \kappa_2 \log \dfrac{1}{|x|}, & n = 2. \end{cases}$$

In particular, it is immediate to check that

$$w \ge c_2(\rho_0 - |x|) \quad \text{in } B_{\rho_0}$$

for some $c_2 > 0$.

By using the function $c_1 w(x_0 + x)$ as a subsolution in any ball $B_{\rho_0}(x_0) \subset \Omega \cap B_{3/2}$, we deduce that

$$u(x) \ge c_1 w(x_0 + x) \ge c_1 c_2(\rho_0 - |x - x_0|) \ge c_1 c_2 d(x) \quad \text{in } B_{\rho_0}(x_0).$$

Setting $c_0 = c_1 c_2$ and using the previous inequality for every ball $B_{\rho_0}(x_0) \subset \Omega \cap B_{3/2}$, the result follows. □

Theorem 2.13 (Mean Value Property). *A function $u \in C^2(\Omega)$ is harmonic ($\Delta u = 0$) if and only if for every ball $B_r(x) \subset \Omega$,*

$$u(x) = \fint_{B_r(x)} u(y)\, dy.$$

Theorem 2.14 (Liouville Theorem). *Any bounded harmonic function on $\mathbb{R}^n$ is constant.*

2.3 Super-Sub Harmonicity

A locally integrable function u is called

$$u \text{ is } \begin{cases} \text{superharmonic,} & \text{if } r \mapsto \fint_{B_r(x)} u(y)\, dy \text{ is monotone nonincreasing,} \\ \text{subharmonic,} & \text{if } r \mapsto \fint_{B_r(x)} u(y)\, dy \text{ is monotone nondecreasing,} \end{cases} \tag{2.1}$$

for all $0 < r < \operatorname{dist}(x, \partial\Omega)$.

2.3.1 Convergence of Monotone Sequences

Lemma 2.15. *Let w_n be a sequence of uniformly bounded functions on Ω satisfying the above monotonicity condition and converging pointwise to w. Then w also satisfies the same monotonicity.*

Proof. Let $\varphi_{x,n}(r) = \fint_{B_r(x)} w_n(y), dy$, which is nonincreasing in r. Passing to the limit and using dominated convergence shows the same for $\varphi_{x,\infty}(r)$. □

2.3.2 Lower Semicontinuity

Lemma 2.16. *Under the previous monotonicity, the function w is, up to a null set, lower semicontinuous in Ω.*

Proof. Define $w_0(x) = \lim_{r \downarrow 0} \fint_{B_r(x)} w(y), dy$. Since the averages are nonincreasing, w_0 is lower semicontinuous and equals w almost everywhere. □

Definition 2.17 (Weak Convergence). Let X be a Hilbert space with inner product $(\cdot, \cdot)_X$. A sequence $\{u_k\} \subset X$ is said to *converge weakly* to $u \in X$, written $u_k \rightharpoonup u$ in X, if

$$(u_k, v)_X \longrightarrow (u, v)_X \quad \forall v \in X.$$

Theorem 2.18 (Weak Compactness in $H^1(\Omega)$). *Let $\Omega \subset \mathbb{R}^n$ be a bounded C^1 domain. Then any bounded sequence $\{u_k\} \subset H^1(\Omega)$ admits a subsequence $\{u_{k_j}\}$ and a limit $u \in H^1(\Omega)$ such that*

$$u_{k_j} \rightharpoonup u \quad \textit{in } H^1(\Omega), \tag{2.2}$$

i.e.

$$\left(u_{k_j}, v\right)_{H^1(\Omega)} \longrightarrow (u, v)_{H^1(\Omega)} \quad \forall v \in H^1(\Omega).$$

Lemma 2.19 (Weak Lower Semicontinuity of the Dirichlet Energy)**.** *Under the same hypotheses as Theorem 2.18, if $u_{k_j} \rightharpoonup u$ in $H^1(\Omega)$ then*

$$\|u\|^2_{H^1(\Omega)} \;\leq\; \liminf_{j\to\infty} \|u_{k_j}\|^2_{H^1(\Omega)}. \tag{2.3}$$

Lemma 2.20 (Compact Embedding into L^2)**.** *If Ω is as above then the embedding*

$$H^1(\Omega) \xhookrightarrow{\text{compact}} L^2(\Omega)$$

is compact. Hence any sequence converging weakly in $H^1(\Omega)$ converges strongly in $L^2(\Omega)$. In particular, for the subsequence u_{k_j} of Theorem 2.18 one has

$$\|u_{k_j}\|_{L^2(\Omega)} \;\longrightarrow\; \|u\|_{L^2(\Omega)}. \tag{2.4}$$

By the weak lower semicontinuity of the Dirichlet energy, given by Theorem 2.18, specifically equations (2.3) and (2.4), it follows that the weak limit u is in fact a minimizer of the energy functional.

3

Properties of the solutions

In this chapter, we address key questions related to the obstacle problem, including the existence of solutions, their regularity, and non-degeneracy properties.

One of the first questions that we will explore regarding the obstacle problem is the existence of solutions.

Throughout this chapter, Ω will denote a bounded Lipschitz domain in $\mathbb{R}^n$, φ a smooth obstacle, and $v \in H^1(\Omega)$ the unique minimizer of

$$\mathcal{E}(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\,dx \quad \text{subject to } v|_{\partial\Omega} = g,\ v \geq \varphi. \tag{3.1}$$

We will frequently draw on the interplay between variational methods and classical PDE techniques, relying on some material presented in Chapter 2.

3.1 Existence and uniqueness

This is a straightforward consequence of the convexity of both the functional $\mathcal{E}(u)$ and the set $\mathcal{K}$.

Proposition 3.1 (Existence and uniqueness)**.** *Let $\Omega \subset \mathbb{R}^n$ be any bounded Lipschitz domain, and let $g : \partial\Omega \to \mathbb{R}$ and $\phi \in H^1(\Omega)$ be such that*

$$\mathcal{K} = \{w \in H^1(\Omega) : w \geq \phi \text{ in } \Omega,\ w|_{\partial\Omega} = g\} \neq \emptyset.$$

Then, there exists a unique minimizer of $\mathcal{E}(v) = \frac{1}{2}\int_\Omega |\nabla v|^2\,dx$ among all functions $v \in \mathcal{K}$.

Proof. Let

$$\theta_0 := \inf\{\mathcal{E}(w) : w \in C\},$$

and let $\{v_k\} \subset \mathcal{K}$ be a minimizing sequence of the functional, i.e., $\mathcal{E}(v_k) \to \theta_0$.

By the Poincaré inequality (Theorem 2.1), the sequence $\{v_k\}$ is uniformly bounded in $H^1(\Omega)$, and therefore a subsequence $\{v_{k_j}\}$ will converge to a certain function v strongly in $L^2(\Omega)$ and weakly in $H^1(\Omega)$. Moreover, by compactness of the trace operator

(Theorem 2.4), we will have $v_{k_j}|_{\partial\Omega} \to v|_{\partial\Omega}$ in $L^2(\partial\Omega)$, so that $v|_{\partial\Omega} = g$. Furthermore, such function v will satisfy

$$\mathcal{E}(v) \leq \liminf_{k\to\infty} \mathcal{E}(v_k) = \theta_0,$$

By the weak lower semicontinuity of the Dirichlet energy, given by Theorem 2.18, especifically equations 2.3 and 2.4 it will be a minimizer of the energy functional.

Since $v_{k_j} \geq \varphi$ in Ω and $v_{k_j} \to v$ almost in $L^2(\Omega)$.Thus, $v \in C$ and it achieves the minimum.

To show uniqueness, Suppose u is any weak solution to (3.1). Then for every $v \in H_0^1(\Omega)$,

$$\mathcal{E}(u+v) = \frac{1}{2}\int_\Omega |\nabla u + \nabla v|^2\,dx = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \int_\Omega \nabla u\cdot\nabla v\,dx + \frac{1}{2}\int_\Omega |\nabla v|^2\,dx.$$

Since $\int_\Omega \nabla u \cdot \nabla v\,dx = 0$. Hence

$$\mathcal{E}(u+v) = \mathcal{E}(u) + \frac{1}{2}\int_\Omega |\nabla v|^2\,dx \;\geq\; \mathcal{E}(u),$$

with strict inequality whenever $v \not\equiv 0$. It follows that no two distinct functions in $u + H_0^1(\Omega)$ can both minimize $\mathcal{E}$, and therefore u is unique. □

Least Supersolution and Connection to Viscosity Theory

Another elegant approach to establishing the existence of solutions to the obstacle problem, especially useful in nonlinear or degenerate settings is via the concept of a *least supersolution*. This method is closely related to the theory of viscosity solutions and resembles Perron's method used in classical potential theory.

We define the function $v : \Omega \to \mathbb{R}$ by

$$v(x) := \inf\left\{ w(x) \,\middle|\, w \in C(\overline{\Omega}),\ -\Delta w \geq 0 \text{ in } \Omega,\ w \geq \varphi \text{ in } \Omega,\ w|_{\partial\Omega} \geq g \right\}.$$

In this formulation:

- The inequality $-\Delta w \geq 0$ is understood in the *viscosity sense*—that is, as a comparison principle holding with respect to smooth test functions from below.
- The set of admissible supersolutions consists of continuous functions that lie above the obstacle φ in Ω, satisfy a supersolution condition for the Laplace operator, and obey the boundary constraint $w \geq g$ on $\partial\Omega$.

The infimum over this class of supersolutions is itself a well-defined, upper semicontinuous function. Under mild assumptions on the data (e.g., continuity of φ and g), one can show that v is actually continuous, satisfies the obstacle condition $v \geq \varphi$, and is a supersolution to $-\Delta v \geq 0$ in the viscosity sense.

Moreover, a crucial observation is that v satisfies $\Delta v = 0$ in the open set $\{v > \varphi\}$. That is, wherever the function is not constrained by the obstacle, it behaves like a classical harmonic function. This combination of properties implies that v is not only a viscosity solution but also a weak solution to the variational inequality, and hence coincides with the minimizer of the energy functional.

In conclusion, we have the equivalence:

$$\left\{ \begin{array}{c} \text{least supersolution (viscosity)} \\ \text{with constraints } w \geq \varphi,\ w|_{\partial\Omega} \geq g \end{array} \right\} \iff \left\{ \begin{array}{c} \text{unique minimizer of} \\ \text{the Dirichlet energy} \end{array} \right\}.$$

For more information about this method we refer to [FRRO22].

Penalization Method

An alternative approach to solving the obstacle problem is to approximate the variational inequality using a family of penalized functionals. The idea is to replace the constraint $u \geq \varphi$ with a penalty term that becomes increasingly singular as the constraint is violated. This yields a sequence of smooth unconstrained problems whose solutions converge to the true solution of the obstacle problem.

Let $\beta_\varepsilon : \mathbb{R} \to \mathbb{R}$ be a family of smooth, convex, non-increasing functions satisfying

$$\beta_\varepsilon(t) \to \beta_0(t) := \begin{cases} 0 & \text{if } t \geq 0, \\ \infty & \text{if } t < 0, \end{cases} \quad \text{as } \varepsilon \to 0.$$

A concrete example is $\beta_\varepsilon(t) = e^{-t/\varepsilon}$. The role of β_ε is to penalize any violation of the constraint $v \geq \varphi$, and the idea is that as $\varepsilon \to 0$, this penalty enforces the constraint more strictly.

We consider the functional

$$J_\varepsilon(v) := \frac{1}{2} \int_\Omega |\nabla v|^2 \, dx + \int_\Omega \beta_\varepsilon(v - \varphi) \, dx,$$

subject to the Dirichlet boundary condition $v = f$ on $\partial\Omega$, and define the admissible class

$$\mathcal{A} := \{v \in H^1(\Omega) \mid v = f \text{ on } \partial\Omega\}.$$

Since J_ε is strictly convex, coercive, and weakly lower semi-continuous on $\mathcal{A}$, the direct method of the calculus of variations guarantees the existence of a unique minimizer $v_\varepsilon \in \mathcal{A}$. Moreover, because β_ε is smooth and the problem is unconstrained, classical elliptic regularity implies $v_\varepsilon \in C^\infty(\Omega) \cap C(\overline{\Omega})$.

The minimizer v_ε satisfies the Euler–Lagrange equation

$$\Delta v_\varepsilon = \beta'_\varepsilon(v_\varepsilon - \varphi) \quad \text{in } \Omega.$$

Because $\beta_\varepsilon' \leq 0$, and $\beta_\varepsilon'(t) = 0$ when $t \geq 0$, we deduce the following properties:

$$\begin{cases} -\Delta v_\varepsilon \geq 0 & \text{in } \Omega, \\ \Delta v_\varepsilon = 0 & \text{in the set } \{v_\varepsilon > \varphi\}. \end{cases}$$

Thus, v_ε is superharmonic in Ω, and harmonic in the non-contact region. For more details and result regarding this method we refer to [PSU12] (Section 1.3) wich results and derivation are no relevant for the scope of this work.

3.2 Properties of solutions

Let us next prove that any minimizer v of $\mathcal{E}(u)$ is actually continuous and solves (3.2). From now on we will assume the obstacle to be smooth ie. $\varphi \in C^\infty(\Omega)$. Note that analogous results hold under weaker hypotheses on φ.

Proposition 3.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $\varphi \in C^\infty(\Omega)$, and let $v \in H^1(\Omega)$ be the minimizer of (1.1) with $v|_{\partial\Omega} = g$. Then $v \in C(\Omega)$ and satisfies*

$$\begin{cases} v \geq \varphi, & \text{in } \Omega, \\ \Delta v \leq 0, & \text{in } \Omega, \\ \Delta v = 0, & \text{in the open set } \{\, v > \varphi\}. \end{cases} \tag{3.2}$$

Proof. By the construction of the minimizer, we already have $v \geq \varphi$ almost everywhere in Ω. Also we have alredy seen that implies $-\Delta v \geq 0$ in $\mathcal{D}'(\Omega)$, so v is (weakly) superharmonic in Ω. Up to modifying v on a set of measure zero, we may also assume v is lower–semicontinuous. It remains to show that $\Delta v = 0$ in $\{v > \varphi\} \cap \Omega$ and that v is continuous.

First we prove that the non–contact set $\{v > \varphi\} \cap \Omega$ is open. Let $x_0 \in \{v > \varphi\} \cap \Omega$, so $v(x_0) - \varphi(x_0) = \varepsilon_0 > 0$. By lower–semicontinuity of v and continuity of φ, there exists $\delta > 0$ such that

$$v(x) - \varphi(x) \geq \frac{\varepsilon_0}{2} \quad \text{for all } x \in B_\delta(x_0).$$

Hence $B_\delta(x_0) \subset \{v > \varphi\} \cap \Omega$, proving openness.

Next, for any $x_0 \in \{v > \varphi\} \cap \Omega$ and any $\eta \in C_c^\infty(B_\delta(x_0))$ with $|\eta| \leq 1$, the functions

$$v \pm \varepsilon\eta$$

remain in C for all $|\varepsilon| < \varepsilon_0/2$. Since v minimizes $\mathcal{E}$, the usual Taylor expansion argument gives

$$\int_\Omega \nabla v \cdot \nabla \eta \, dx = 0 \quad \forall \eta \in C_c^\infty(B_\delta(x_0)),$$

and hence $\Delta v = 0$ in $B_\delta(x_0)$. As x_0 was arbitrary, $\Delta v = 0$ in $\{v > \varphi\} \cap \Omega$.

Finally, we show v is continuous across the free boundary. Inside $\{v > \varphi\} \cap \Omega$, v is harmonic and therefore smooth. On the contact set $\{v = \varphi\} \cap \Omega$, v is superharmonic and lower–semicontinuous. Let $y_0 \in \{v = \varphi\} \cap \Omega$ and suppose to the contrary that v is not continuous at y_0. Then there exists a sequence $y_k \to y_0$ with

$$v(y_k) \;\geq\; v(y_0) + \varepsilon_0 \;=\; \varphi(y_0) + \varepsilon_0$$

for some $\varepsilon_0 > 0$. After passing to a subsequence we may assume $y_k \in \{v > \varphi\}$. Let z_k be the projection of y_k onto the contact set $\{v = \varphi\}$, so that $z_k \to y_0$. Since v is superharmonic, the mean–value inequality gives for each k and small radius r_k:

$$v(z_k) \;\geq\; \fint_{\partial B_{r_k}(z_k)} v = \;(1 - 2^{-n}) \fint_{B_{r_k}(z_k)\setminus B_{r_k/2}(z_k)} v \;+\; 2^{-n} \fint_{B_{r_k/2}(z_k)} v = I_1 + I_2.$$

For large k, lower–semicontinuity yields $v \geq \varphi(y_0) - 2^{-n}\varepsilon_0$ on $B_{r_k}(z_k)$, so

$$I_1 \geq (1 - 2^{-n})(\varphi(y_0) - 2^{-n}\varepsilon_0).$$

Meanwhile, v is harmonic on $B_{r_k/2}(z_k)$ containing y_k, so the mean–value property gives

$$I_2 = 2^{-n} v(y_k) \geq 2^{-n}(\varphi(y_0) + \varepsilon_0).$$

Combining,

$$v(z_k) \geq (1 - 2^{-n})\varphi(y_0) - (1 - 2^{-n})2^{-n}\varepsilon_0 + 2^{-n}\varphi(y_0) + 2^{-n}\varepsilon_0 > \varphi(y_0),$$

contradicting $v(z_k) = \varphi(z_k) \to \varphi(y_0)$. Hence v is continuous at y_0, and the proof is complete. □

We next prove the following result, which says that v can be characterized as the least supersolution above the obstacle

Proposition 3.3 (Least supersolution)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $\varphi \in H^1(\Omega)$, and let $v \in H^1(\Omega)$ be the minimizer of (1.1) subject to the boundary condition $v|_{\partial\Omega} = g$. Then for any function w satisfying*

$$-\Delta w \geq 0 \quad \text{in } \Omega, \qquad w \geq \varphi \quad \text{in } \Omega, \qquad w|_{\partial\Omega} \geq g,$$

we have $w \geq v$ in Ω. In other words, any supersolution above the obstacle φ dominates the minimizer v.

Proof. Suppose w satisfies $-\Delta w \geq 0$ in Ω, $w \geq \varphi$ in Ω, and $w|_{\partial\Omega} \geq g = v|_{\partial\Omega}$. Consider the open set $\{v < w\} \cap \Omega$. On this set,

$$-\Delta w \;\geq\; 0 \quad \text{and} \quad -\Delta v \;\leq\; 0,$$

so $-\Delta(w - v) \geq 0$. Moreover, on the boundary of $\{v < w\} \cap \Omega$, either $v = w$ or $v = \varphi \leq w$.

By the weak maximum principle (Proposition 3.4), it follows that $w-v \geq 0$ throughout $\{v < w\} \cap \Omega$, hence $w \geq v$ on that set. Since trivially $w \geq v$ on its complement, the inequality holds everywhere in Ω. □

Proposition 3.4. *Let $\Omega \subset \mathbb{R}^n$ be any bounded open set. Assume that $u \in H^1(\Omega)$ satisfies, in the weak sense,*

$$\begin{cases} -\Delta u \geq 0 & \text{in } \Omega \\ u \geq 0 & \text{on } \partial\Omega. \end{cases}$$

Then, $u \geq 0$ in Ω.

Proof. Notice that $-\Delta u \geq 0$ in Ω if and only if

$$\int_\Omega \nabla u \cdot \nabla v \, dx \geq 0 \quad \text{for all } v \geq 0,\ v \in H_0^1(\Omega). \tag{3.3}$$

Let us consider $u^- := \max\{-u, 0\}$ and $u^+ := \max\{u, 0\}$, so that $u = u^+ - u^-$. By 2.8 we have that $u^\pm \in H^1(\Omega)$ whenever $u \in H^1(\Omega)$, and thus we can choose $v = u^- \geq 0$ in (3.3). Namely, using that $u^+u^- = 0$ and $\nabla u = \nabla u^+ - \nabla u^-$, we get

$$0 \leq \int_\Omega \nabla u \cdot \nabla u^- \, dx = -\int_\Omega |\nabla u^-|^2 \, dx.$$

Since $u^-|_{\partial\Omega} \equiv 0$, this implies $u^- \equiv 0$ in Ω, that is, $u \geq 0$ in Ω.

□

Optimal Regularity of Solutions

Thanks to Proposition 3.2, we know that any minimizer of the Dirichlet energy functional (1.1) is continuous and satisfies the Euler–Lagrange system (1.5). To investigate finer properties of the solution, we now localize the problem and study its regularity in a standard setting.

Without loss of generality, we consider the localized formulation on the unit ball $B_1 \subset \Omega$, where the solution v satisfies:

$$\begin{cases} v \geq \varphi, & \text{in } B_1, \\ \Delta v \leq 0, & \text{in } B_1, \\ \Delta v = 0, & \text{in } \{\, v > \varphi \,\} \cap B_1. \end{cases} \tag{3.4}$$

Remarks. The obstacle problem leads to a PDE that behaves differently in two distinct regions:

- In the *non-contact set* $\{v > \varphi\}$, the solution is harmonic: $\Delta v = 0$.
- In the *contact set* $\{v = \varphi\}$, we have $v = \varphi$, and formally $\Delta v = \Delta\varphi$ holds wherever $\varphi \in C^2$.

As a consequence, the Laplacian Δv generally exhibits a discontinuity across the free boundary $\partial\{v > \varphi\}$. This implies that the second derivatives of v may not be continuous across the interface, and thus we cannot expect the solution to belong to $C^2(B_1)$ in general. In fact, $C^{1,1}$ is the optimal regularity one can hope for in the general case.

Theorem 3.5 (Optimal regularity). *Let $\varphi \in C^\infty(\overline{B_1})$ and let v solve (3.4). Then $v \in C^{1,1}(B_{1/2})$ and satisfies*

$$\|v\|_{C^{1,1}(B_{1/2})} \leq C\big(\|\varphi\|_{C^{1,1}(B_1)} + \|v\|_{L^\infty(B_1)}\big),$$

where C depends only on the dimension n.

The key ingredient is the following quadratic growth estimate near the free boundary.

Lemma 3.6 (Quadratic growth). *Assume $\varphi \in C^{1,1}(B_1)$ and let v solve (3.4). For any point $x_0 \in \overline{B_{1/2}} \cap \{v = \varphi\}$ and any radius $r \in (0, 1/4]$, one has*

$$0 \leq \sup_{B_r(x_0)} \big(v - \varphi\big) \leq C\, r^2,$$

where C depends only on n and $\|\varphi\|_{C^{1,1}(B_1)}$.

Proof. We begin by normalizing the function φ. Since we are only concerned with upper bounds on $v - \varphi$, we may divide v by a constant if necessary to assume:

$$\|\varphi\|_{C^{1,1}(B_1)} \leq 1.$$

Let $\ell(x) := \varphi(x_0) + \nabla\varphi(x_0)\cdot(x - x_0)$ be the first-order Taylor expansion (i.e., the affine approximation) of φ at the point x_0. Fix a small radius $r \in \big(0, \frac{1}{4}\big)$. The $C^{1,1}$ regularity of φ ensures that for all $x \in B_r(x_0)$, we have:

$$\ell(x) - r^2 \leq \varphi(x).$$

Moreover, we are assuming that $v(x) \geq \varphi(x)$, so we also have:

$$\ell(x) - r^2 \leq \varphi(x) \leq v(x).$$

Our goal is to show that v does not grow much faster than φ within the small ball $B_r(x_0)$. More precisely, we aim to prove:

$$v(x) \leq \ell(x) + C r^2 \quad \text{in } B_r(x_0).$$

To do this, define an auxiliary function w as:

$$w(x) := v(x) - \left[\ell(x) - r^2\right].$$

This function measures how far v is above the "shifted" affine function $\ell(x) - r^2$, which lies below φ. Note that $w(x) \geq 0$ in $B_r(x_0)$, since $v(x) \geq \varphi(x) \geq \ell(x) - r^2$.

Observe that:

$$-\Delta w = -\Delta v \geq 0 \quad \text{in } B_r(x_0),$$

so w is *subharmonic*.

To estimate w, we split it into two parts:

$$w = w_1 + w_2,$$

where

- w_1 is the harmonic replacement of w in $B_r(x_0)$, defined by

$$\begin{cases} \Delta w_1 = 0, & \text{in } B_r(x_0), \\ w_1 = w, & \text{on } \partial B_r(x_0). \end{cases}$$

- $w_2 := w - w_1$, so that

$$\begin{cases} -\Delta w_2 \geq 0, & \text{in } B_r(x_0), \\ w_2 = 0, & \text{on } \partial B_r(x_0). \end{cases}$$

Both w_1 and w_2 are nonnegative in $B_r(x_0)$:

$$0 \leq w_1 \leq w, \qquad 0 \leq w_2 \leq w.$$

Since w_1 is harmonic and non-negative, we apply the Harnack inequality in the smaller ball $B_{r/2}(x_0)$:

$$\|w_1\|_{L^\infty(B_{r/2}(x_0))} \leq C w_1(x_0).$$

But we know:

$$w_1(x_0) \leq w(x_0) = v(x_0) - [\ell(x_0) - r^2] = r^2,$$

so we conclude:

$$\|w_1\|_{L^\infty(B_{r/2}(x_0))} \leq C r^2.$$

Next, observe that:

$$\Delta w_2 = \Delta v \quad \text{in } B_r(x_0).$$

In the region where $v > \varphi$, we know that v is a solution of the equation, so $\Delta w_2 = \Delta v = 0$ there.

Hence, w_2 is subharmonic, vanishes on the boundary, and attains its maximum inside the set $\{v = \varphi\}$.

In this set, we estimate:

$$w_2 = w - w_1 = v - (\ell - r^2) - w_1 = \varphi - (\ell - r^2) - w_1 \leq \varphi - (\ell - r^2).$$

Since $\varphi(x) \le \ell(x) + r^2$ by $C^{1,1}$ regularity, we find:

$$w_2 \le (\ell + r^2) - (\ell - r^2) = 2r^2 \le Cr^2.$$

Thus:

$$\|w_2\|_{L^\infty(B_r(x_0))} \le Cr^2.$$

Putting the bounds together:

$$\|w\|_{L^\infty(B_{r/2}(x_0))} \le \|w_1\|_{L^\infty(B_{r/2}(x_0))} + \|w_2\|_{L^\infty(B_r(x_0))} \le Cr^2.$$

Recalling that $w(x) = v(x) - (\ell(x) - r^2)$, this implies:

$$v(x) \le \ell(x) + Cr^2 \quad \text{in } B_{r/2}(x_0),$$

which proves the desired upper bound.

□

This proves the claimed quadratic separation from the obstacle:

> *At every free boundary point x_0, the solution v separates from φ at most quadratically.*

As we will see next, this is a key step toward establishing the optimal $C^{1,1}$ regularity of solutions.

Proof of Theorem 3.5. To simplify constants, we may normalize v and φ so that

$$\|v\|_{L^\infty(B_1)} + \|\varphi\|_{C^{1,1}(B_1)} \le 1.$$

We begin by recalling known regularity properties:

- Since v is harmonic in the set $\{v > \varphi\}$, it follows that

$$v \in C^\infty\big(\{v > \varphi\}\big).$$

- Inside the contact set $\{v = \varphi\}$, v inherits regularity from φ, which is assumed to be smooth, so

$$v \in C^\infty\big(\text{int}\{v = \varphi\}\big).$$

The critical behavior occurs near the *free boundary* $\Gamma := \partial\{v > \varphi\}$, where the solution transitions from being strictly above the obstacle to coinciding with it. From previous results (specifically Lemma 3.5), we have the *quadratic growth estimate* at any point $x_0 \in \Gamma$:

$$\sup_{B_r(x_0)} (v - \varphi) \le Cr^2.$$

Our goal is to use this growth estimate to establish that $v \in C^{1,1}(B_{1/2})$, i.e., that $D^2 v$ is bounded.

Let us fix a point $x_1 \in \{v > \varphi\} \cap B_{1/2}$, and let $x_0 \in \Gamma$ be a closest free boundary point to x_1. Define $\rho := |x_1 - x_0|$ so that $B_\rho(x_1)$ lies inside the non-contact region $\{v > \varphi\}$.

Since v is harmonic in $\{v > \varphi\}$, we have:

$$\Delta v = 0 \quad \text{in } B_\rho(x_1).$$

Let $\ell(x)$ be the linear approximation of φ at x_0, i.e., the first-order Taylor polynomial of φ at x_0:

$$\ell(x) := \varphi(x_0) + \nabla\varphi(x_0) \cdot (x - x_0).$$

Because ℓ is affine, we also have $\Delta(v - \ell) = 0$ in $B_\rho(x_1)$.

Now, applying standart interior estimates for harmonic functions, we obtain:

$$\|D^2 v\|_{L^\infty(B_{\rho/2}(x_1))} = \|D^2(v - \ell)\|_{L^\infty(B_{\rho/2}(x_1))} \le \frac{C}{\rho^2}\|v - \ell\|_{L^\infty(B_\rho(x_1))}.$$

To estimate $\|v - \ell\|_{L^\infty(B_\rho(x_1))}$, we use the quadratic growth near the free boundary. Since $x_0 \in \Gamma$ and $x_1 \in B_\rho(x_0)$, Lemma 3.5 gives:

$$\|v - \ell\|_{L^\infty(B_\rho(x_1))} \le C\rho^2.$$

Plugging this into our estimate above, we find:

$$\|D^2 v\|_{L^\infty(B_{\rho/2}(x_1))} \le \frac{C}{\rho^2}\rho^2 = C.$$

In particular, this gives $|D^2 v(x_1)| \le C$. Since $x_1 \in \{v > \varphi\} \cap B_{1/2}$ was arbitrary, this bound holds throughout the non-contact region inside $B_{1/2}$. On the other hand, on $\partial\{v > \varphi\}$, the quadratic growth from Lemma 3.5 ensures that v approaches φ with the correct second-order rate. Hence, the second derivatives of v are bounded up to the free boundary.

We conclude that:

$$\|v\|_{C^{1,1}(B_{1/2})} \le C,$$

as desired.

□

3.2.1 Nondegeneracy

We now show that at every free boundary point $x_0 \in \partial\{v > \varphi\}$, the solution separates from the obstacle at least quadratically. Specifically, there exist constants $c, C > 0$ such that for all sufficiently small $r > 0$,

$$0 < cr^2 \leq \sup_{B_r(x_0)} (v - \varphi) \leq Cr^2. \tag{3.5}$$

This nondegeneracy estimate is crucial for the finer analysis of the free boundary in later chapters.

Remark 3.7. Since $-\Delta v \geq 0$ everywhere in Ω, if $x_0 \in \partial\{v > \varphi\}$ is a free boundary point, then necessarily

$$-\Delta\varphi(x_0) \geq 0,$$

for otherwise we would have $-\Delta\varphi(x_0) < 0$, which would contradict $-\Delta v(x_0) \geq 0$ as v touches φ from above at x_0.

Moreover, one can show that if $\Delta\varphi$ and $\nabla\Delta\varphi$ do *not* vanish simultaneously, then in fact

$$-\Delta\varphi > 0$$

in a neighborhood of every free boundary point. This motivates the following structural assumption on the obstacle:

$$-\Delta\varphi \geq c_0 > 0 \quad \text{in } B_1.$$

This lower bound on the Laplacian is not restrictive. As discussed in Remark 3.7, it can typically be ensured, possibly after shrinking the domain.

This nondegeneracy condition is both natural and essential. Without it, estimate (3.5) fails, and the behavior of the solution near the free boundary may become irregular. In fact, if the condition is violated, one can construct examples where the free boundary lacks regularity and may even exhibit fractal behavior with infinite perimeter.

Example 3.8 (Fractal Boundary Example)**.** Consider the obstacle problem in the unit disk $B_1 \subset \mathbb{R}^2$, take $C \subset [-1, 1]$ be the Cantor set, and define

$$\psi(x, y) = x^2 - y^2, \qquad \varphi(x, y) = \begin{cases} \psi(x, y), & x \in C, \\ \psi(x, y) - \varepsilon, & x \notin C, \end{cases} \quad (x, y) \in B_1,$$

with some fixed $\varepsilon > 0$.

We claim in fact that the problem

$$\begin{cases} v \geq \varphi, & \text{in } B_1, \\ \Delta v \leq 0, & \text{in } B_1, \\ \Delta v = 0, & \text{in } \{ v > \varphi \} \cap B_1, \end{cases} \tag{3.6}$$

has the unique solution

$$v(x, y) := \psi(x, y).$$

Verification of (3.6):

(*i*) *Obstacle constraint.* If $x \in C$, then $\varphi = \psi$, so $v = \psi = \varphi$. If $x \notin C$, then $\varphi = \psi - \varepsilon$, so $v = \psi > \psi - \varepsilon = \varphi$. Thus $v \geq \varphi$ throughout B_1.

(*ii*) *Subharmonicity.* Since

$$\Delta v = \Delta\psi = (\partial_{xx} + \partial_{yy})(x^2 - y^2) = 2 - 2 = 0,$$

we have $\Delta v = 0 \leq 0$ in all of B_1.

(*iii*) *Harmonicity off the contact set.* On the non-contact region $\{v > \varphi\}$ we still have $\Delta v = 0$, verifying the third line.

Finally, observe that

$$\Delta\varphi(x, y) = \begin{cases} 0, & x \in C, \\ 0, & x \notin C, \end{cases}$$

so $-\Delta\varphi \equiv 0$ in B_1. Hence the usual non-degeneracy condition $-\Delta\varphi \geq c_0 > 0$ fails in this example. And finally we have that $\psi = \varphi$ in C.

To complement the upper quadratic growth established above, we next show that under a uniform nondegeneracy assumption on the obstacle, the solution must also detach at least quadratically from the obstacle near every free boundary point:

Proposition 3.9 (Nondegeneracy). *Let $\varphi \in C^\infty(\overline{B_1})$, and let v solve the obstacle problem (3.4). Assume further that $-\Delta\varphi \geq c_0 > 0$ in B_1. Then for every free boundary point $x_0 \in \partial\{v > \varphi\} \cap B_{1/2}$ and for all $r \in (0, 1/4]$,*

$$0 < cr^2 \leq \sup_{B_r(x_0)} (v - \varphi) \leq Cr^2,$$

where $c > 0$ depends only on n and c_0, and C depends on $\|\varphi\|_{C^{1,1}(B_1)}$.

Proof. Let $x_1 \in \{v > \varphi\} \cap B_{1/2}$ be a point arbitrarily close to the free boundary point $x_0 \in \partial\{v > \varphi\}$. We let $x_1 \to x_0$ at the end of the proof.

Define the auxiliary function

$$w(x) := v(x) - \varphi(x) - \frac{c_0}{2n}|x - x_1|^2.$$

We analyze this function in the region $\{v > \varphi\} \cap B_r(x_1)$ for a small $r > 0$. By assumption, $\Delta v = 0$ in $\{v > \varphi\}$, and $-\Delta\varphi \geq c_0 > 0$ in B_1. Hence,

$$\Delta w = \Delta v - \Delta\varphi - c_0 = -\Delta\varphi - c_0 \geq 0,$$

so w is subharmonic in $\{v > \varphi\} \cap B_r(x_1)$.

Note that:

- $w(x_1) = v(x_1) - \varphi(x_1) > 0$ since $x_1 \in \{v > \varphi\}$,

- $w < 0$ on the free boundary part of $\partial\{v > \varphi\}$, because $v = \varphi$ there and the quadratic term is negative,
- w satisfies the maximum principle, so it attains its maximum on $\partial B_r(x_1)$.

Therefore,

$$\sup_{\partial B_r(x_1)} (v - \varphi) \geq \sup_{\partial B_r(x_1)} w + \frac{c_0}{2n} r^2 > \frac{c_0}{2n} r^2.$$

Letting $x_1 \to x_0$, we deduce that for any free boundary point $x_0 \in \partial\{v > \varphi\} \cap B_{1/2}$,

$$\sup_{B_r(x_0)} (v - \varphi) \geq c r^2,$$

for some constant $c > 0$ depending only on c_0 and n. □

3.3 Regularity of free boundaries

In this sectionn, we focus on the study of the obstacle problem in the following form:

$$\begin{cases} u \in C^{1,1}(B_1), \\ u \geq 0 \quad \text{in } B_1, \\ \Delta u = f \quad \text{in } u > 0, \end{cases} \tag{3.7}$$

we impose the following conditions on the obstacle function f:

$$f \in C^\infty(B_1) \quad \text{and} \quad f \geq c\_0 > 0. \tag{3.8}$$

These assumptions ensure that the problem is well-posed enought and to extrat some results. Notice that on the free boundary we have that

$$u = 0 \quad \text{on } \mathcal{F}, \quad \nabla u = 0 \quad \text{on } \mathcal{F}.$$

The central mathematical challenge in the obstacle problem is to *understand the geometry/regularity of the free boundary* $\mathcal{F}$.

It is important to note that, despite knowing the optimal regularity of the solution u (namely, $u \in C^{1,1}$), this alone gives us no information about the nature of the free boundary $\mathcal{F}$. In principle, $\mathcal{F}$ could exhibit highly irregular behavior, it may even be a fractal set with infinite perimeter.

However, as we will soon establish, the natural condition $f \geq c_0 > 0$ ensures that the free boundary possesses significant regularity: it is smooth in a large portion of the domain, except possibly on a small singular set where irregularities may concentrate.

Blow-up Analysis and the Structure of the Free Boundary

- In this chapter, we investigate the **quadratic behavior** of solutions near the free boundary, a key ingredient in understanding its fine structure.
- The free boundary $\mathcal{F}$ admits a fundamental decomposition into two disjoint classes:
  - **Regular points**, where the free boundary is smooth,
  - **Singular points**, where smoothness breaks down.
- The set of regular points forms an open subset of the free boundary. Moreover, in a neighborhood of any regular point, the free boundary $\mathcal{F}$ is C^∞.
- Singular points are those at which the contact set $\{u = 0\}$ has zero density. Although they mark a breakdown in regularity, singular points are not pathological: if they occur, they are contained in a $(n-1)$-dimensional C^1 manifold.

To analyze the behavior of solutions near $\mathcal{F}$, we adopt the method of *blow-up analysis*. This technique allows us to zoom in on the solution at a small scale near a given free boundary point and identify limiting profiles that characterize the local geometry of $\mathcal{F}$.

Let $x_0 \in \mathcal{F}$ be a point on the free boundary. We define the rescaled family of functions:

$$u_r(x) := \frac{u(x_0 + rx)}{r^2}, \qquad r > 0. \tag{3.9}$$

This particular scaling reflects the known quadratic growth of the solution near the free boundary:

$$cr^2 \leq \sup_{B_r(x_0)} u \leq Cr^2.$$

As such, u_r captures the geometry of the solution at the scale r, and the prefactor r^{-2} ensures a nontrivial limit as $r \to 0$.

We say that a function u_0 is a **blow-up limit** of u at x_0 if

$$u_r \to u_0 \quad \text{locally uniformly in } \mathbb{R}^n,$$

The key result of the blow-up analysis is the classification of all possible blow-up limits. Any blow-up u_0 at a free boundary point $x_0 \in \mathcal{F}$ must fall into one of two canonical types:

- **Regular blow-ups:** There exists a unit vector $e \in \mathbb{S}^{n-1}$ such that

  $$u_0(x) = \frac{f(0)}{2}(x \cdot e)_+^2.$$

  These correspond to regular points of the free boundary. In this case, the free boundary is smooth in a neighborhood of x_0, and the solution resembles a half-space solution near the contact.
- **Singular blow-ups:** There exists a symmetric, positive semi-definite matrix $A \in \mathbb{R}^{n\times n}$ with $\operatorname{tr} A = 1$, such that

  $$u_0(x) = \frac{f(0)}{2} x^T A x.$$

  These blow-ups characterize singular points, where the geometry of the free boundary is more degenerate.

This dichotomy is central to the modern theory of free boundaries. The regularity theory of $\mathcal{F}$ ultimately hinges on classifying and understanding the behavior of such blow-up solutions, which in turn dictate the smoothness or singularity of the boundary at different scales.

3.3.1 Homogeneity of Blow-ups

We will see that the Blow-ups are homogenuous with the following proof, due to Spruck [Spr83].

Proposition 3.10 (Homogenety of Blow-ups). *Let u be any solution of (3.7) where 0 it's a free boundary point. Then, any blow-up of u at 0 is homogenius of degree 2.*

Proof. Spruck's homogeneity: Let u_0 be a blow-up given by the limit along a sequence $r_k \downarrow 0$,

$$u_0(x) := \lim_{k\to\infty} r_k^{-2} u(r_k x).$$

By taking polar coordinates $(\varrho, \theta) \in [0, +\infty) \times \mathbb{S}^{n-1}$ with $x = \varrho\theta$, and by denoting $\tilde{u}_0(\varrho, \theta) = u_0(\varrho\theta) = u_0(x)$, we will prove that

$$u_0(x) = \varrho^2 \tilde{u}_0(1, \theta) = |x|^2 u_0(x/|x|).$$

Let us define $\tau := -\log \varrho$, $\tilde{u}(\varrho, \theta) := u(x)$, and $\psi = \psi(\tau, \theta)$ as

$$\psi(\tau, \theta) := \varrho^{-2} \tilde{u}(\varrho, \theta) = e^{2\tau} u(e^{-\tau}\theta)$$

for $\tau \geq 0$.

We observe that, since $\|u\|_{L^\infty(B_r)} \leq Cr^2$, ψ is bounded. Moreover,

$$\psi \in C^1((0, \infty) \times \mathbb{S}^{n-1}) \cap C^2\left(\{\psi > 0\}\right)$$

from the regularity of u; and $\partial_\tau \psi$ and $\nabla_\theta \psi$ are not only continuous, but also uniformly bounded in $[0, \infty) \times \mathbb{S}^{n-1}$.

Indeed,

$$|\nabla_\theta \psi(\tau, \theta)| \leq e^\tau |\nabla u(e^{-\tau}\theta)| \leq C,$$

since $\|\nabla u\|_{L^\infty(B_r)} \leq Cr$ by $C^{1,1}$ regularity and the fact that $\nabla u(0) = 0$.

For the same reason we also obtain

$$|\partial_\tau \psi(\tau, \theta)| \leq 2\psi(\tau, \theta) + e^\tau |\nabla u(e^{-\tau}\theta)| \leq C.$$

Observe that, by assumption, if we denote $\tau_k := -\log r_k$,

$$\psi(\tau_k, \theta) \to \tilde{u}_0(1, \theta) \quad \text{uniformly on } \mathbb{S}^{n-1}, \text{ as } k \to \infty. \tag{3.10}$$

Let us now write an equation for ψ. In order to do that, since we know that $\Delta u = \chi_{\{u>0\}}$ and $\chi_{\{u>0\}} = \chi_{\{\psi>0\}}$, we have

$$\Delta(\varrho^2 \psi(-\log \varrho, \theta)) = \chi_{\{\psi>0\}} f(\varrho\theta).$$

By expanding the Laplacian in polar coordinates,

$$\Delta = \partial_{\varrho\varrho} + \frac{n-1}{\varrho}\partial_\varrho + \varrho^{-2}\Delta_{\mathbb{S}^{n-1}}$$

(where $\Delta_{\mathbb{S}^{n-1}}$ denotes the spherical Laplacian, i.e., the Laplace–Beltrami operator on $\mathbb{S}^{n-1}$), we obtain

$$2n\psi - (n+2)\partial_\tau\psi + \partial_{\tau\tau}\psi + \Delta_{\mathbb{S}^{n-1}}\psi = f(e^{-\tau}\theta)\chi_{\{\psi>0\}}. \tag{3.11}$$

We multiply the previous equality by $\partial_\tau\psi$ and integrate in $[0,\tau]\times\mathbb{S}^{n-1}$. We can consider the terms separately, integrating in τ first:

$$2n\int_{\mathbb{S}^{n-1}}\int_0^\tau \psi\partial_\tau\psi = n\int_{\mathbb{S}^{n-1}}\left(\psi^2(\tau,\theta) - \psi^2(0,\theta)\right)d\theta$$

and

$$\int_{\mathbb{S}^{n-1}}\int_0^\tau \partial_{\tau\tau}\psi\,\partial_\tau\psi = \frac{1}{2}\int_{\mathbb{S}^{n-1}}\left((\partial_\tau\psi)^2(\tau,\theta) - (\partial_\tau\psi)^2(0,\theta)\right)d\theta$$

and then integrating by parts in θ first, to integrate in τ afterward:

$$\int_0^\tau\int_{\mathbb{S}^{n-1}}\Delta_{\mathbb{S}^{n-1}}\psi\,\partial_\tau\psi = -\frac{1}{2}\int_0^\tau\int_{\mathbb{S}^{n-1}}\partial_\tau|\nabla_\theta\psi|^2 = \frac{1}{2}\int_{\mathbb{S}^{n-1}}\left(|\nabla_\theta\psi|^2(0,\theta) - |\nabla_\theta\psi|^2(\tau,\theta)\right)d\theta.$$

Finally, since $\partial_\tau\psi = 0$ whenever $\psi = 0$, we have $\chi_{\{\psi>0\}}\partial_\tau\psi = \partial_\tau\psi$ and taking

$$\int_{\mathbb{S}^{n-1}}\int_0^\tau f(e^{-\tau}\theta)\chi_{\{\psi>0\}}\partial_\tau\psi = \int_{\mathbb{S}^{n-1}}\int_0^\tau f(e^{-\tau}\theta)\partial_\tau\psi + (f(0)-f(0))\partial_\tau\psi =$$

$$\int_{\mathbb{S}^{n-1}}\int_0^\tau (f(e^{-\tau}\theta) - f(0))\partial_\tau\psi + \int_{\mathbb{S}^{n-1}}\int_0^\tau f(0)\partial_\tau\psi \tag{3.12}$$

Using that $f\in C^{0,\alpha}$ ie. $|f(e^{-\tau}\theta) - f(0)| \leq C_f e^{-\tau\alpha}$, we can obtain

$$\int_{\mathbb{S}^{n-1}}\int_0^\tau (f(e^{-\tau}\theta) - f(0))\partial_\tau\psi \leq C_2e^{-\alpha}\int_{\mathbb{S}^{n-1}}\int_0^\tau \partial_\tau\psi = C_2e^{-\alpha}\int_{\mathbb{S}^{n-1}}\left(\psi(\tau,\theta) - \psi(0,\theta)\right)d\theta$$

wich implies that (3.12) is below

$$(C_2e^{-\alpha} + f(0))\int_{\mathbb{S}^{n-1}}\left(\psi(\tau,\theta) - \psi(0,\theta)\right)d\theta$$

By plugging back in (3.11) the previous expressions, and using that $\partial_\tau\psi$ and $\nabla_\theta\psi$ are uniformly bounded in $[0,\infty)\times\mathbb{S}^{n-1}$, we deduce that

$$\int_0^\infty\int_{\mathbb{S}^{n-1}}(\partial_\tau\psi)^2 = \int_0^\infty \|\partial_\tau\psi\|^2_{L^2(\mathbb{S}^{n-1})} \leq C < \infty. \tag{3.13}$$

To finish, now observe that for any $|s| \le C_*$ fixed and for a sufficiently large k (such that $\tau_k + s \ge 0$),

$$\|\psi(\tau_k+s,\cdot)-\tilde{u}_0(1,\cdot)\|_{L^2(\mathbb{S}^{n-1})} \le \|\psi(\tau_k+s,\cdot)-\psi(\tau_k,\cdot)\|_{L^2(\mathbb{S}^{n-1})}+\|\psi(\tau_k,\cdot)-\tilde{u}_0(1,\cdot)\|_{L^2(\mathbb{S}^{n-1})}.$$

The last term goes to zero, by (3.10). On the other hand, for the first term and by Hölder's inequality

$$\|\psi(\tau_k+s,\cdot)-\psi(\tau_k,\cdot)\|^2_{L^2(\mathbb{S}^{n-1})} \le \left\|\int_0^s \partial_\tau\psi(\tau_k+\tau,\cdot)\,d\tau\right\|^2_{L^2(\mathbb{S}^{n-1})} \le C_* \int_{\tau_k}^{\tau_k+s} \|\partial_\tau\psi\|^2_{L^2(\mathbb{S}^{n-1})} \to 0,$$

as $k \to \infty$, where we are using (3.13). Hence, $\psi(\tau_k + s, \cdot) \to \tilde{u}_0(1,\cdot)$ in $L^2(\mathbb{S}^{n-1})$ as $k \to \infty$, for any fixed $s \in \mathbb{R}$.

On the other hand,

$$\psi(\tau_k + s, \theta) = e^{2s} r_k^{-2} u(e^{-2} r_k\theta) \to e^{2s} u_0(e^{-s}\theta) = e^{2s}\tilde{u}_0(e^{-s},\theta).$$

That is, for any $\rho = e^{-s} > 0$,

$$\tilde{u}_0(1,\cdot) = \rho^{-2}\tilde{u}_0(\rho,\cdot),$$

as we wanted to see. □

3.3.2 Convexity of Blow-Ups

Using the fact that blow-ups are 2-homogeneous, we now provide a proof that they are also convex. More precisely, we will show that any global 2-homogeneous solution to the obstacle problem is convex. In particular, by Proposition 3.10, this implies that blow-ups are convex.

Let's state some general result that we will need in the followings. We consider the operator

$$\mathcal{L}v := -\mathrm{div}(A(x)\nabla v), \tag{3.14}$$

where $A(x)$ is a uniformly elliptic matrix with ellipticity constants $0 < \lambda \le \Lambda$.

Lemma 3.11. *Let $\mathcal{L}$ be the operator defined in (3.14), and let $v \in H^1(B_1)$ satisfy $\mathcal{L}v \le 0$ in B_1. Then $\mathcal{L}(v_+) \le 0$ in the weak sense.*

Proof. We proceed by approximation. Let $F \in C^\infty(\mathbb{R})$ be a smooth, non-decreasing, convex function with globally bounded first derivatives. Our goal is to show that $\mathcal{L}(F(v)) \le 0$ in B_1.

Since $v \in W^{1,2}(B_1)$, it follows that $F(v) \in W^{1,2}(B_1)$. Given that $\mathcal{L}v \le 0$, we have:

$$\int_{B_1} \nabla\eta \cdot A\nabla v\,dx \le 0 \quad \text{for all } \eta \in H_0^1(B_1),\ \eta \ge 0.$$

Now, for any $\eta \in H_0^1(B_1)$ with $\eta \geq 0$, we compute:

$$\int_{B_1} \nabla\eta \cdot A\nabla F(v)\,dx = \int_{B_1} F'(v)\nabla\eta \cdot A\nabla v\,dx$$
$$= \int_{B_1} \nabla(F'(v)\eta)\cdot A\nabla v\,dx - \int_{B_1} \eta F''(v)\nabla v \cdot A\nabla v\,dx.$$

The first term is non-positive since $F'(v)\eta \in H_0^1(B_1)$ and $F'(v) \geq 0$, making it an admissible test function. The second term is also non-positive because $\eta F''(v) \geq 0$ and $\nabla v \cdot A\nabla v \geq 0$ due to ellipticity. Moreover, the integral is well-defined since $\eta F''(v)$ can be assumed bounded, and $\int_{B_1} \nabla v \cdot A\nabla v \leq \Lambda \|\nabla v\|^2_{L^2(B_1)}$. Therefore,

$$\int_{B_1} \nabla\eta \cdot A\nabla F(v)\,dx \leq 0.$$

To conclude, we take a family of smooth approximations F_ε converging uniformly on compact sets to the function $F(x) = \max\{x, 0\}$. These approximations can be constructed so that $\|F_\varepsilon(v)\|_{W^{1,2}(B_1)} \leq C$, for some constant C independent of $\varepsilon > 0$, which yields the desired result. □

Lemma 3.12. *Let $\Lambda \subset B_1$ be a closed set, and suppose that $w \in H^1(B_1)$ satisfies $w \geq 0$ on Λ, and that w is superharmonic in the weak sense in $B_1 \setminus \Lambda$. Then $\min\{w, 0\}$ is superharmonic in the weak sense in B_1.*

Proof. We first assume that w is continuous. In this case, define $w_\varepsilon := \min\{w, -\varepsilon\} \in H^1(B_1)$. In a neighborhood of the set $\{w = -\varepsilon\}$, w is superharmonic. Applying Lemma 3.11 to $v = -w - \varepsilon$, we conclude that w_ε is superharmonic in the weak sense.

Furthermore, the functions w_ε are uniformly bounded in H^1, so (up to a subsequence) they converge weakly to $\min\{w, 0\}$. Since the weak limit of weakly superharmonic functions is superharmonic, we obtain the desired conclusion.

To remove the continuity assumption, we repeat the argument as in Lemma 3.11. The only point to verify is that $F'(v)\eta \in H_0^1(B_1 \setminus \Lambda)$, which holds because the function belongs to $H^1(B_1)$ and vanishes on Λ. The proof of that fact can be found on [AH96] (Theorem 9.13) □

Theorem 3.13. *Let $u_0 \in C^{1,1}$ be any 2-homogeneous global solution to*

$$\begin{cases} u_0 \geq 0 & \text{in } \mathbb{R}^n \\ \Delta u_0 = f(0) & \text{in } \{u_0 > 0\} \\ \text{0 is a free boundary point.} \end{cases}$$

Then, u_0 is convex.

Proof. Fix any direction $e \in S^{n-1}$ and set

$$w_0(x) := \min\{\partial_{ee} u_0(x), 0\}.$$

We will show w_0 is superharmonic in $\mathbb{R}^n$ in the weak sense, which will force $w_0 \equiv 0$ and hence $\partial_{ee} u_0 \geq 0$.

For $t > 0$ define the second difference

$$\delta_t^2 u_0(x) := \frac{u_0(x+te) + u_0(x-te) - 2u_0(x)}{t^2}.$$

Since $\Delta u_0 = f_0\, \chi_{\{u_0>0\}}$, one computes in the weak sense

$$\Delta(\delta_t^2 u_0) = f(0)\, \frac{\chi_{\{u_0(\cdot+te)>0\}} + \chi_{\{u_0(\cdot-te)>0\}} - 2}{t^2} \;\leq\; 0 \quad \text{in } \{u_0 > 0\}.$$

On the other hand $\delta_t^2 u_0 \geq 0$ on $\{u_0 = 0\}$ and $\delta_t^2 u_0 \in C^{1,1}$. By Lemma 3.12 the function

$$w_t(x) \;:=\; \min\{\delta_t^2 u_0(x), 0\}$$

is weakly superharmonic and hence satisfies (2.1). Moreover, since $u_0 \in C^{1,1}$ the family $\{\delta_t^2 u_0\}$ is uniformly bounded and converges pointwise to w_0 as $t \downarrow 0$. By Lemma 2.15 it follows that w_0 is superharmonic in the weak sense .

Finally, up to a set of measure zero w_0 is lower semi-continuous (Lemma 2.16) and 0–homogeneous, so it attains its minimum on B_1. But the average $⨍_{B_r} w_0$ is non-increasing in r, forcing $w_0 \equiv$ constant. Since w_0 vanishes on the free boundary, $w_0 \equiv 0$. As this holds for every $e \in S^{n-1}$, we conclude $\partial_{ee} u_0 \geq 0$ everywhere, i.e. u_0 is convex. □

3.3.3 Classification of blow-ups

We next want to classify all possible blow-ups for solutions to the obstacle problem (3.7) (again considering $0 \in \mathcal{F}$), in the case where the equation becomes $\Delta u = f(x)$.

The first step is to establish the existence and basic properties of blow-up limits. The next proposition guarantees that any blow-up limit of a solution at the origin satisfies strong regularity properties, solves a constant-coefficient obstacle problem, and exhibits both convexity and homogeneity. This justifies focusing our classification on a restricted family of functions.

Proposition 3.14. *Let u be any solution to (3.7) and consider the function u_r as defined in (3.9). Then, for any sequence $r_k \to 0$ there is a subsequence $r_{k_j} \to 0$ such that*

$$u_{r_{k_j}} \to u_0 \quad \text{in } C^1_{loc}(\mathbb{R}^n)$$

as $k_j \to \infty$, *for some function* u_0 *satisfying*

$$\begin{cases} u_0 \in C^{1,1}_{loc}(\mathbb{R}^n) \\ u_0 \geq 0 \quad \text{in } B_1 \\ \Delta u_0 = f(0) \quad \text{in } \{u_0 > 0\} \\ 0 \text{ is a free boundary point} \\ u_0 \text{ is convex} \\ u_0 \text{ is homogeneous of degree } 2 \end{cases}$$

Proof. By $C^{1,1}$ regularity of u, and by nondegeneracy, we have that

$$\frac{1}{C} \leq \sup_{B_1} u_r \leq C$$

for some $C > 0$. Moreover, again by $C^{1,1}$ regularity of u, we have

$$\|D^2 u_r\|_{L^\infty(B_{1/(2r)})} \leq C.$$

Since the sequence $\{u_{r_k}\}$, for $r_k \to 0$, is uniformly bounded in $C^{1,1}(K)$ for each compact set $K \subset \mathbb{R}^n$, there is a subsequence $r_{k_j} \to 0$ such that

$$u_{r_{k_j}} \to u_0 \quad \text{in } C^1_{\text{loc}}(\mathbb{R}^n).$$

for some $u_0 \in C^{1,1}(K)$. Moreover, such function u_0 satisfies $\|D^2 u_0\|_{L^\infty(K)} \leq C$, with C independent of K, and clearly $u_0 \geq 0$ in K.

To verify that $\Delta u_0 = f(0)$ in $\{u_0 > 0\} \cap K$, observe that for any smooth function $\eta \in C_c^\infty(\{u_0 > 0\} \cap K)$, we have that for large k_j, $u_{r_{k_j}} > 0$ on the support of η. Then:

$$\int_{\mathbb{R}^n} \nabla u_{r_{k_j}} \cdot \nabla \eta \, dx = -\int_{\mathbb{R}^n} f(r_{k_j} x) \eta(x) \, dx.$$

As $r_{k_j} \to 0$, we have $f(r_{k_j} x) \to f(0)$ uniformly on compact sets. Hence,

$$\int_{\mathbb{R}^n} \nabla u_0 \cdot \nabla \eta \, dx = -\int_{\mathbb{R}^n} f(0) \eta(x) \, dx.$$

This shows that $\Delta u_0 = f(0)$ in $\{u_0 > 0\}$.

That 0 is a free boundary point for u_0 follows by passing to the limit in the blow-up sequence. Since $u_{r_k}(x) = \frac{u(r_k x)}{r_k^2}$ and $u(0) = 0$, it follows that $u_{r_k}(0) = 0$ for all k. In addition, by the nondegeneracy of u, we have the uniform estimate $\|u_{r_k}\|_{L^\infty(B_\rho)} \approx \rho^2$ for every $\rho \in (0,1)$, independent of k. Passing to the limit, we obtain $u_0(0) = 0$ and $\sup_{B_\rho} u_0 \approx \rho^2$, which implies that $u_0 > 0$ in every neighborhood of the origin. Therefore, $0 \in \partial\{u_0 > 0\}$, and the origin is a free boundary point for u_0.

Finally, the homogeneity and convexity of u_0 follow from general structural results. Proposition 3.10 states that any blow-up limit at a free boundary point is homogeneous of degree two, meaning $u_0(\lambda x) = \lambda^2 u_0(x)$ for all $\lambda > 0$, a consequence of the scale invariance of the equation under the quadratic rescaling. Moreover, Theorem 3.13 ensures that such blow-up limits are convex functions, reflecting the geometric and regularity properties of the original solution near the free boundary.

□

Having established these structural properties, we now ask: **what are all the possible forms such a blow-up limit u_0 can take?** Despite the flexibility in the original problem, the answer is remarkably rigid. As we now show, all blow-up limits fall into one of two explicit classes, either a half-space quadratic profile or a full quadratic form with positive semidefinite matrix structure. This dichotomy is fundamental in distinguishing regular and singular free boundary points.

Theorem 3.15 (Classification of Blow-Ups)**.** *Let u be any solution to (3.7), and let u_0 be a blow-up of u at the origin. Then, exactly one of the following holds:*

(a) *There exists $e \in \mathbb{S}^{n-1}$ such that*

$$u_0(x) = \frac{f(0)}{2}(x \cdot e)_+^2.$$

(b) *There exists a symmetric matrix $A \geq 0$ with $\operatorname{tr}(A) = 1$ such that*

$$u_0(x) = \frac{f(0)}{2} x^T A x.$$

Remark. A priori, different subsequences may yield different blow-up limits u_0.

To establish Theorem 3.15, we will rely on several auxiliary results. These lemmas help reduce the classification to a small family of explicit profiles by exploiting some properties of the solutions.

Lemma 3.16. *Let $\Sigma \subset \mathbb{R}^n$ be a closed convex cone with nonempty interior and vertex at the origin. Suppose $w \in C(\mathbb{R}^n)$ satisfies*

$$\Delta w = 0 \quad \text{in } \Sigma^c, \qquad w > 0 \quad \text{in } \Sigma^c, \qquad w = 0 \quad \text{in } \Sigma,$$

and assume that w is homogeneous of degree 1. Then, Σ must be a half-space.

Proof. By the convexity of Σ, there exists a half-space $H = \{x \cdot e > 0\}$, with $e \in \mathbb{S}^{n-1}$, such that $H \subset \Sigma^c$.

Define $v(x) = (x \cdot e)_+$, which is harmonic and positive in H, and vanishes in H^c. By the Hopf Lemma 2.12), there exists $c_0 > 0$ such that

$$w \geq c_0 d_\Sigma \quad \text{in } \Sigma^c \cap B_1,$$

where $d_\Sigma(x) := \operatorname{dist}(x, \Sigma)$. Since both w and d_Σ are homogeneous of degree 1, the inequality extends to all of Σ^c.

Moreover, since $d_\Sigma \geq d_{H^c} = v$, we conclude that

$$w \geq c_0 v \quad \text{in } \Sigma^c.$$

We now consider scaling v up until it "touches" w. Define

$$c_* := \sup\{c > 0 : w \geq cv \text{ in } \Sigma^c\}.$$

Note that $c_* \geq c_0 > 0$. Consider the function $w - c_* v \geq 0$. If $w - c_* v$ is not identically zero, then it is harmonic in H and strictly positive by the strong maximum principle:

$$w - c_* v > 0 \quad \text{in } H.$$

Applying the Hopf Lemma again, we find

$$w - c_* v \geq c_0 d_{H^c} = c_0 v,$$

so that

$$w - (c_* + c_0)v \geq 0,$$

which contradicts the definition of c_*. Therefore, $w \equiv c_* v$ in Σ^c.

This implies that w is a multiple of the function $v = (x \cdot e)_+$, so the positivity set of w is the half-space H, and thus $\Sigma = H^c$. □

Lemma 3.17. *Assume that $\Delta u = c$ in $\mathbb{R}^n \setminus \partial H$ and $c \in \mathbb{R}$ where ∂H is a hyperplane, and that $f \in C(\mathbb{R}^n)$. If $u \in C^1(\mathbb{R}^n)$, then $\Delta u = c$ in all of $\mathbb{R}^n$.*

Proof. Assume without loss of generality that $\partial H = \{x_1 = 0\}$. Let $B_R \subset \mathbb{R}^n$ be a ball. Define w as the solution to

$$\Delta w = c \text{ in } B_R, \quad w = u \text{ on } \partial B_R,$$

and let $v = u - w$. Then, $v \in C^1(B_R)$, and satisfies

$$\Delta v = 0 \quad \text{in } B_R \setminus \partial H, \quad v = 0 \text{ on } \partial B_R.$$

We aim to show that $v \equiv 0$ in B_R, which would imply $\Delta u = c$ in all of B_R, and hence in $\mathbb{R}^n$ by the arbitrariness of the ball.

Since v is bounded and continuous in B_R, we can use the barrier function $\psi(x) = \kappa(2R - |x_1|)$, which is harmonic in $B_R \setminus \partial H$. Then, for sufficiently large $\kappa > 0$,

$$v(x) \leq \kappa(2R - |x_1|) \quad \text{in } B_R.$$

Define

$$\kappa^* := \inf\{\kappa \geq 0 : v(x) \leq \kappa(2R - |x_1|) \text{ in } B_R\}.$$

Suppose $\kappa^* > 0$. Since v and $2R - |x_1|$ are continuous, and $v = 0$ on ∂B_R, there exists a point $p \in B_R$ such that

$$v(p) = \kappa^*(2R - |p_1|).$$

As in the standard touching argument, this implies $p \notin \partial H$, since $2R - |x_1|$ has a corner along ∂H, while $v \in C^1$. But then v and the barrier touch tangentially at an interior point where both are harmonic, contradicting the strong maximum principle. Hence, $\kappa^* = 0$, and so $v \leq 0$ in B_R.

Applying the same argument to $-v$, we deduce $v \equiv 0$ in B_R, so $u = w$ and $\Delta u = c$ in B_R. Since B_R was arbitrary, we conclude that $\Delta u = c$ in all of $\mathbb{R}^n$. □

Lemma 3.18. *Let $u : \mathbb{R}^n \to \mathbb{R}$ be a convex function such that the set $\{u = 0\}$ contains the entire line $\{te' : t \in \mathbb{R}\}$, for some direction $e' \in \mathbb{S}^{n-1}$. Then,*

$$u(x + te') = u(x) \quad \textit{for all } x \in \mathbb{R}^n \textit{ and all } t \in \mathbb{R}.$$

Proof. Without loss of generality, we may assume (after a rotation) that $e' = e_n$. Writing $x = (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$, we are given that $u(0, x_n) = 0$ for all $x_n \in \mathbb{R}$, and we aim to prove that

$$u(x', x_n) = u(x', 0) \quad \text{for all } x' \in \mathbb{R}^{n-1},\ x_n \in \mathbb{R}.$$

By convexity, for any $\varepsilon > 0$ and any $M \in \mathbb{R}$, we have the inequality

$$(1 - \varepsilon)u(x', x_n) + \varepsilon u(0, x_n + M) \geq u((1 - \varepsilon)x', x_n + \varepsilon M).$$

Since $u(0, x_n + M) = 0$ by hypothesis, the inequality simplifies to

$$(1 - \varepsilon)u(x', x_n) \geq u((1 - \varepsilon)x', x_n + \varepsilon M).$$

Now, choosing $M = \lambda/\varepsilon$ for some fixed $\lambda \in \mathbb{R}$ and letting $\varepsilon \to 0$, we obtain

$$u(x', x_n) \geq u(x', x_n + \lambda).$$

Applying the same argument with λ replaced by $-\lambda$ yields the reverse inequality, so we conclude that

$$u(x', x_n) = u(x', x_n + \lambda) \quad \text{for all } \lambda \in \mathbb{R}.$$

In particular, taking $\lambda = -x_n$ shows that $u(x', x_n) = u(x', 0)$, and the proof is complete. □

With all these preparatory results in place, we can now give the proof of Theorem 3.15.

Proof of Theorem 3.15. Let u_0 be any blow–up of u at 0. By the same scaling argument as before, one shows that

$$\Delta u_0 \;=\; f(0) \quad \text{in } \{u_0 > 0\},$$

and that u_0 is convex and 2–homogeneous. We again split into two cases.

Case 1. $\{u_0 = 0\}$ has nonempty interior. Then $\Sigma := \{u_0 = 0\}$ is a closed convex cone with interior. Pick any direction $\tau \in \mathbb{S}^{n-1}$ with $-\tau \in \Sigma$. By the same "one–sided" convexity argument, $\partial_\tau u_0 \geq 0$ everywhere, and for at least one such τ it is not identically zero. Set

$$w := \partial_\tau u_0 \geq 0.$$

Since $\Delta(\partial_\tau u_0) = \partial_\tau(\Delta u_0) = \partial_\tau f(0) = 0$ in Σ^c, w is harmonic there and 1–homogeneous. Lemma 3.16 then forces Σ to be a half–space. By convexity of u_0 and Lemma 3.18, u_0 is one dimensional,

$$u_0(x) = U(x \cdot e),$$

with $U \in C^{1,1}$, $U'' = f(0)$ on $(0, \infty)$ and $U \equiv 0$ on $(-\infty, 0]$. Hence

$$U(t) = \frac{f(0)}{2}\, t_+^2,$$

and

$$u_0(x) = \frac{f(0)}{2}(x \cdot e)_+^2\,.$$

Case 2. Assume $\{u_0 = 0\}$ has empty interior. Convexity then forces $\{u_0 = 0\}$ to lie in a hyperplane ∂H. We have $\Delta u_0 = f(0)$ in $\mathbb{R}^n \setminus \partial H$ and $u_0 \in C^{1,1}$. By Lemma 3.17 this extends across ∂H, so $\Delta u_0 = f(0)$ in all of $\mathbb{R}^n$. But then each second derivative $\partial_{ij} u_0$ is global harmonic and bounded, hence constant. Therefore u_0 is a quadratic form,

$$u_0(x) = \frac{f(0)}{2}\, x^T A x,$$

with $A \geq 0$ and $\operatorname{tr} A = 1$, and $u_0(0) = 0$, $\nabla u_0(0) = 0$ force no lower–order terms. □

3.3.4 Regularity of the free boundary

If u is any solution of (3.7) satisfying

$$\limsup_{r\to 0} \frac{\big|\{u=0\}\cap B_r\big|}{|B_r|} > 0, \tag{3.15}$$

that is, if the contact set has positive density at the origin, then the free boundary $\partial\{u>0\}$ is of class C^∞ in a neighborhood of the origin.

To this end, we will leverage the blow-up classification from the previous section together with key results from Ros-Oton and Torres-Labore's work [ROTL21]. We begin by recalling several definitions and stating the principal theorems whose proofs appear in the original article that extend the boundary Harnack inequality to the more general right-hand sides under consideration here.

In those results $\mathcal{L}$ for one of two uniformly elliptic operators:

$$\mathcal{L}u \;=\; \operatorname{Tr}\big(A(x)\,D^2u\big). \quad A\in L^\infty\big(B_1;\mathbb{R}^{n\times n}\big),\ \lambda I\le A(x)\le \Lambda I, \tag{3.16}$$

a non-divergence-form operator with bounded measurable coefficients, or

$$\mathcal{L}u \;=\; \operatorname{div}\big(A(x)\,\nabla u\big), \quad A\in C^0\big(B_1;\mathbb{R}^{n\times n}\big),\ \lambda I\le A(x)\le \Lambda I, \tag{3.17}$$

a divergence-form operator with continuous coefficients.

Definition 3.19 (Lipschitz domain)**.** We say Ω is a Lipschitz domain with Lipschitz constant L if Ω is of the form

$$\Omega \;=\; \big\{(x',x_n)\in B_1'\times\mathbb{R}:\ x_n>g(x')\big\},$$

where $g\colon B_1'\to\mathbb{R}$ is a Lipschitz function satisfying

$$g(0)=0, \qquad \|g\|_{C^{0,1}(B_1')}=L.$$

Additionally, we recall the Alexandroff–Bakelman–Pucci (ABP) estimate, a cornerstone of regularity theory. A proof can be found in Theorem 3.2 of [CC95] and in Proposition 3.3 of [CCK□96].

Theorem 3.20 (ABP estimate)**.** *Assume that $\Omega\subset\mathbb{R}^n$ is a bounded domain. Let $\mathcal{L}$ be a non-divergence–form operator as in (3.16), and let*

$$u\in C(\overline{\Omega})$$

satisfy

$$\mathcal{L}u \;\ge\; f \quad \text{in the } L^n\text{–viscosity sense},$$

with $f \in L^n(\Omega)$. Assume also that u is bounded on $\partial\Omega$. Then

$$\sup_{\Omega} u \;\le\; \sup_{\partial\Omega} u \;+\; C\,\mathrm{diam}(\Omega)\,\|f\|_{L^n(\Omega)},$$

where the constant C depends only on the dimension n and the ellipticity constants λ, Λ.

With this notation in hand, we can now state our boundary Harnack and comparison theorems (Theorem 3.21 and Corollary 3.22).

Theorem 3.21 (Boundary Harnack principle)**.** *Let $q > n$ and $\mathcal{L}$ be as in (3.16) or (3.17). There exist constants $c_0 > 0$ and $L_0 > 0$ such that the following holds. Let Ω be a Lipschitz domain as in Definition 3.19, with Lipschitz constant $L < L_0$. Let $u, v > 0$ be solutions of*

$$\begin{cases} \mathcal{L}u = f, & \text{in } \Omega \cap B_1, \\ u = 0, & \text{on } \partial\Omega \cap B_1, \end{cases} \quad \text{and} \quad \begin{cases} \mathcal{L}v = g, & \text{in } \Omega \cap B_1, \\ v = 0, & \text{on } \partial\Omega \cap B_1, \end{cases}$$

in the L^n–viscosity or weak sense, with

$$\|f\|_{L^q(B_1)} \le c_0, \quad \|g\|_{L^q(B_1)} \le c_0. \tag{3.18}$$

Additionally, assume that

$$v(e_n/2) \ge 1 \quad \text{and either} \quad \big(u > 0 \text{ and } u(e_n/2) \le 1\big) \quad \text{or} \quad \|u\|_{L^p(B_1)} \le 1$$

for some $p > 0$. Then there is a constant $C > 0$ and an exponent $\alpha \in (0,1)$ such that

$$u \le C\,v \quad \text{in } B_{1/2}, \qquad \left\| \tfrac{u}{v} \right\|_{C^{0,\alpha}(\overline{\Omega} \cap B_{1/2})} \le C.$$

The constants C, c_0, L_0, α depend only on n, q, λ, Λ, and when applicable on p and σ. In the case $u, v > 0$ throughout $\Omega \cap B_1$, this recovers the classical symmetric formulation of the boundary Harnack inequality.

Corollary 3.22 (Normalized Boundary Harnack)**.** *Let $q > n$ and L be as in (3.16) or (3.17). There exist small constants $c_0 > 0$ and $L_0 > 0$ such that the following holds. Let Ω be a Lipschitz domain as in Definition 3.19, with Lipschitz constant $L < L_0$. Let $u, v > 0$ be solutions of*

$$\begin{cases} \mathcal{L}u = f, & \text{in } \Omega \cap B_1, \\ u = 0, & \text{on } \partial\Omega \cap B_1, \end{cases} \qquad \begin{cases} \mathcal{L}v = g, & \text{in } \Omega \cap B_1, \\ v = 0, & \text{on } \partial\Omega \cap B_1, \end{cases}$$

in the L^n–viscosity or weak sense, with f and g satisfying (3.18). Assume u and v are normalized so that

$$u(e_n/2) = v(e_n/2) = 1.$$

Then there is a constant $C > 0$ and exponent $\alpha \in (0,1)$ such that

$$C^{-1} \;\le\; \frac{u}{v} \;\le\; C \quad \text{in } B_{1/2},$$

and

$$\left\|\frac{u}{v}\right\|_{C^{0,\alpha}\left(\overline{\Omega}\cap B_{1/2}\right)} \leq C.$$

The constants C, c_0, L_0, α depend only on the dimension n, the exponent q, the ellipticity constants λ, Λ, and when applicable the parameter σ.

With all this we can continue with the $C^{1,\alpha}$ **regularity of the free boundary.** The first step is to translate the density condition (3.15) into information on a blow-up limit u_0. More precisely, one shows that any nontrivial blow-up at the origin satisfies a uniform flatness estimate, which in turn yields $C^{1,\alpha}$ regularity of $\partial\{u > 0\}$ near 0.

We next show that the density condition (3.15) forces the blow-up to have a nonempty interior in its contact set.

Lemma 3.23. *Let u be any solution of (3.7), and assume that (3.15) holds. Then there exists a blow-up u_0 of u at the origin such that $\{u_0 = 0\}$ has nonempty interior.*

Proof. By (3.15) there is a sequence $r_k \to 0$ along which

$$\lim_{k\to\infty} \frac{\left|\{u = 0\} \cap B_{r_k}\right|}{|B_{r_k}|} \geq \theta > 0.$$

By Proposition 3.14 we may extract a subsequence (still denoted r_k) so that

$$u_{r_k}(x) = \frac{u(r_k x)}{r_k^2} \longrightarrow u_0(x) \quad \text{uniformly on compact sets,}$$

and u_0 is convex.

Suppose for contradiction that $\{u_0 = 0\}$ has empty interior. Convexity implies that is contained in a hiperplane, lets take

$$\{u_0 = 0\} \subset \{x_1 = 0\}.$$

Since $u_0 > 0$ on $\{x_1 \neq 0\}$ and is continuous, for each $\delta > 0$ there is $\varepsilon > 0$ such that

$$u_0(x) \geq \varepsilon \quad \text{on } \{|x_1| > \delta\} \cap B_1.$$

Uniform convergence implies that for all large k,

$$u_{r_k}(x) \geq \frac{\varepsilon}{2} \quad \text{on } \{|x_1| > \delta\} \cap B_1.$$

Hence

$$\{u_{r_k} = 0\} \cap B_1 \subset \{|x_1| \leq \delta\} \cap B_1,$$

and therefore

$$\frac{\left|\{u_{r_k} = 0\} \cap B_1\right|}{|B_1|} \leq \frac{\left|\{|x_1| \leq \delta\} \cap B_1\right|}{|B_1|} \leq C\,\delta.$$

Rescaling back to u gives

$$\frac{\left|\{u=0\}\cap B_{r_k}\right|}{|B_{r_k}|} = \frac{\left|\{u_{r_k}=0\}\cap B_1\right|}{|B_1|} < C\,\delta.$$

Since $\delta > 0$ was arbitrary, this contradicts the lower bound $\theta > 0$. Thus $\{u_0 = 0\}$ must have nonempty interior. □

Combining Lemma 3.23 with the blow-up classification of Theorem 3.15, we immediately obtain:

Corollary 3.24. *Let u be any solution of (3.7), and assume that (3.15) holds. Then there is at least one blow-up of u at 0 of the form*

$$u_0(x) = \frac{f(0)}{2}(x\cdot e)_+^2, \qquad e \in \mathbb{S}^{n-1}.$$

Proof. The result follows immediately from Lemma 3.23 and Theorem 3.15. □

We now want to use this information to show that the free boundary is smooth in a neighborhood of 0. For this, we begin with:

Proposition 3.25. *Let u be any solution of (3.7), and assume that (3.15) holds. Fix any $\varepsilon > 0$. Then there exist a direction $e \in \mathbb{S}^{n-1}$ and a radius $r_0 > 0$ such that*

$$\left|u_{r_0}(x) - \tfrac{f(0)}{2}(x\cdot e)_+^2\right| \le \varepsilon \quad \text{in } B_1,$$

and, for every $\tau \in \mathbb{S}^{n-1}$,

$$\left|\partial_\tau u_{r_0}(x) - f(0)\,(x\cdot e)_+\,(\tau\cdot e)\right| \le \varepsilon \quad \text{in } B_1.$$

Proof. By Corollary 3.24 and Proposition 3.14, we know that there is a subsequence $r_j \to 0$ for which

$$u_{r_j} \longrightarrow \frac{f(0)}{2}(x\cdot e)_+^2 \quad \text{in } C^1_{\mathrm{loc}}(\mathbb{R}^n),$$

for some $e \in S^{n-1}$. In particular, for every $\tau \in S^{n-1}$ we have

$$u_{r_j} \longrightarrow \frac{f(0)}{2}(x\cdot e)_+^2, \quad \partial_\tau u_{r_j} \longrightarrow \partial_\tau\Big[\frac{f(0)}{2}(x\cdot e)_+^2\Big] \quad \text{uniformly in } B_1.$$

Thus given $\varepsilon > 0$, there exists j_0 such that

$$\Big|u_{r_{j_0}}(x) - \frac{f(0)}{2}(x\cdot e)_+^2\Big| \le \varepsilon \quad \text{and} \quad \Big|\partial_\tau u_{r_{j_0}}(x) - \partial_\tau\Big[\frac{f(0)}{2}(x\cdot e)_+^2\Big]\Big| \le \varepsilon \quad \text{in } B_1.$$

Since

$$\partial_\tau\Big[\frac{f(0)}{2}(x\cdot e)_+^2\Big] = f(0)(x\cdot e)_+(\tau\cdot e),$$

the proposition follows. □

Note that if $(\tau \cdot e) > 0$, then

$$\partial_\tau u_0(x) = f(0)(x \cdot e)_+ (\tau \cdot e)$$

is nonnegative and strictly positive in the set $\{x \cdot e > 0\}$.

We now want to transfer this information to u_{r_0}, and prove that $\partial_\tau u_{r_0} \geq 0$ in B_1 for all $\tau \in \mathbb{S}^{n-1}$ satisfying $\tau \cdot e \geq \frac{1}{2}$. For this, we need the following.

Proposition 3.26. *Let u be a solution of (3.7), with $f \in W^{1,n}$ and $f \geq \tau_0 > 0$. Assume that 0 is a regular free boundary point. Then, for every $L_0 > 0$ there exists $r > 0$ such that the free boundary in B_r is the graph of a Lipschitz function with Lipschitz constant $L < L_0$.*

To prove that the free boundary is Lipschitz, we will use the interior and exterior cone conditions, and for this we must show

$$\partial_\nu u_r \geq 0 \quad \text{whenever } \nu \cdot e > c(L),$$

where $c(L)$ is the positive constant for which the cone $\{x \in \mathbb{R}^n : x \cdot e = |x|\, c(L)\}$ has Lipschitz constant L. We achieve this via a suitable positivity lemma.

Lemma 3.27. *Let u solve (3.7) with $f \in W^{1,n}(B_1)$. For $r > 0$ set*

$$u_r(x) = \frac{u(rx)}{r^2}, \quad \Omega = \{u_r > 0\}, \quad w(x) = \partial_\nu u_r(x),$$

where ν is a fixed unit direction. Then w satisfies

$$\Delta w = g, \quad w = 0 \text{ on } \partial\Omega, \qquad g(x) = r\,\partial_\nu f(rx).$$

If there exist $\delta > 0$ and constants $\varepsilon, M > 0$ so that

$$N_\delta = \{x \in B_1 : \operatorname{dist}(x, \partial\Omega) < \delta\}, \quad w > -\varepsilon \text{ in } N_\delta, \quad w > M \text{ in } \Omega \setminus N_\delta, \tag{3.19}$$

then for ε, r, and δ sufficiently small one has $w \geq 0$ in $\Omega \cap B_{1/2}$.

Proof. First, it is clear that $w > 0$ in $\Omega \setminus N_\delta$. Suppose there exists $x_0 \in B_{1/2} \cap N_\delta$ such that $w(x_0) < 0$. We will arrive at a contradiction using the maximum principle, combined with the ABP estimate 3.20, with the function

$$v(x) = w(x) - \eta\Big(u_r(x) - \frac{f(x_0)}{2n}|x - x_0|^2\Big).$$

Consider the set $\Omega \cap B_{1/4}(x_0)$. On $\partial\Omega$, $u_r = 0$, hence $v \geq 0$. On $\partial B_{1/4}(x_0) \cap N_\delta$,

$$v(x) \;\geq\; -\varepsilon - \eta\,\delta \|u_r\|_{C^1} + \frac{\eta}{32n}.$$

On $\partial B_{1/4}(x_0) \cap (\Omega \setminus N_\delta)$,

$$v(x) \;\geq\; M - \eta \|u_r\|_{C^1}.$$

Notice that $\|u_r\|_{C^1}$ is uniformly bounded as $r \to 0$. Hence, choosing η small enough, the second inequality implies $v \geq M/2$. For the first inequality, choosing ε and δ small enough, we obtain

$$v \;\geq\; \frac{\eta}{64n}.$$

This function satisfies

$$\Delta v(x) = g(x) \;-\; \eta\big(f(x) - f(x_0)\big),$$

hence by the ABP 3.20 estimate,

$$v(x_0) \;\geq\; \min\Big\{\tfrac{M}{2},\, \tfrac{\eta}{64n}\Big\} \;-\; C\,\big\| g(x) - \eta\big(f(rx) - f(rx_0)\big)\big\|_{L^n\left(B_{1/4}(x_0)\right)}.$$

We estimate g and $f - f(x_0)$ separately. Using the scaling of the L^n–norm and letting $r \to 0$,

$$\|g\|_{L^n(B_{1/4}(x_0))} = \big\|\partial_\nu f\big\|_{L^n(B_{r/4}(rx_0))} \;\longrightarrow\; 0.$$

On the other hand, by the Poincaré inequality,

$$\|f(rx) - f(rx_0)\|_{L^n(B_{1/4}(x_0))} = \frac{\|f - f(x_0)\|_{L^n(B_{r/4}(x_0))}}{r} \;\leq\; C\,\|\nabla f\|_{L^n(B_{r/4}(rx_0))} \;\longrightarrow\; 0.$$

Hence, choosing r small enough, we get

$$v(x_0) \;\geq\; \frac{1}{2}\,\min\Big\{\tfrac{M}{2},\, \tfrac{\eta}{64n}\Big\},$$

which contradicts $v(x_0) < 0$.

□

Using the lemma, we then prove that there is an arbitrarily wide cone of directions in which $\partial_\nu u_r \geq 0$, for all sufficiently small $r > 0$.

Proof of Proposition 3.26. We only need to check that, for any $\nu \in S^{n-1}$ with $\nu \cdot e > c(L)$, the hypotheses of the lemma are satisfied. By construction, it suffices to verify that (3.19) holds for all sufficiently small $r > 0$.

Set

$$\delta \;=\; \varepsilon^{1/8}.$$

By the blow-up argument, there exists $r > 0$ such that

$$\big|u_r(x) \;-\; \tfrac{\gamma}{2}(x\cdot e)_+^2\big| \;<\; \varepsilon, \qquad \big|\partial_\nu u_r(x) \;-\; \gamma\,(x\cdot e)_+\,(x\cdot\nu)\big| \;<\; \varepsilon$$

for all x in the domain under consideration. Hence, if $\varepsilon > 0$ is chosen sufficiently small, then on the set $\{x\cdot e > \delta^2\}$ we have

$$u_r(x) > \frac{\gamma}{2}(x\cdot e)_+^2 \;-\; \varepsilon \;\geq\; \frac{\gamma}{2}\,\delta^4 \;-\; \varepsilon \;=\; \frac{\gamma}{2}\,\varepsilon^{1/2} \;-\; \varepsilon \;>\; 0.$$

Moreover, we claim

$$u_r(x) = 0 \quad \text{for } x \cdot e < -\delta^2.$$

Indeed, if there were a point y with $y \cdot e < -\delta^2$ and $u_r(y) > 0$, this would contradict the nondegeneracy estimate and the fact that u_r vanishes to second order near the contact set.

See, supposing $u_r(y) > 0$ we have:

$$\sup_{B_{\delta^2}(y)} u_r \ \geq \ c\,\delta^4 \ = \ c\,\varepsilon^{1/2},$$

but since $B_{\delta^2}(y) \subset \{x \cdot e < 0\}$ we have $u_r < \varepsilon$, which is impossible for ε small. Hence the free boundary is contained in the strip $\{|x \cdot e| < \delta^2\}$.

Now let $\nu \in S^{n-1}$ satisfy $\nu \cdot e > c(L)$. In the inner region N_δ, the blow-up convergence gives

$$\partial_\nu u_r \ > \ \gamma\, c(L)\,(x \cdot e)_+ \ - \ \varepsilon \ \geq \ -\,\varepsilon.$$

On the other hand, if $z \in \Omega \setminus N_\delta$, then the free boundary lies within distance δ^2 of the hyperplane $\{x \cdot e = 0\}$, so $z \cdot e > \delta - \delta^2$. Therefore

$$\partial_\nu u_r(z) \ > \ \gamma\, c(L)\,(z \cdot e)_+ \ - \ \varepsilon \ > \ \gamma\, c(L)\,(\delta - \delta^2) \ - \ \varepsilon \ = \ \gamma\, c(L)\,(\varepsilon^{1/8} - \varepsilon^{1/4}) \ - \ \varepsilon =: M,$$

where $M > 0$ for ε sufficiently small.

Since r and ε (hence δ) were chosen independently of ν, we may apply Lemma 3.27: for every unit ν with $\nu \cdot e > c(L)$ we have $\partial_\nu u_r \geq 0$, equivalently $\partial_\nu u \geq 0$ in B_r. Now if $x_0 \in B_r$ is a free boundary point, then $u(x_0) = 0$, and along the ray $x_0 - t\nu \in B_r$ we have $u(x_0 - t\nu) \leq 0$, hence $u(x_0 - t\nu) = 0$. It follows that

$$u = 0 \quad \text{in} \quad B_r \cap \{(x - x_0) \cdot e < -\,c(L)\,|x|\},$$

which is the *interior* cone condition.

To obtain the exterior cone, suppose by contradiction that there is another free boundary point

$$x_1 \in B_r \cap \{x_0 + t\nu \,:\, \nu \cdot e > c(L),\ t > 0\}.$$

Then applying the interior cone condition at x_1 shows x_0 cannot be free, a contradiction. Hence in B_r the free boundary is a Lipschitz graph in direction e with constant L.

□

Now we can use the modified Boundary Harnack inequality 3.21 to prove the $C^{1,\alpha}$ regularity of the free boundary at regular points *à la* Caffarelli. To this end, we assume that the right–hand side f belongs to the Sobolev space $W^{1,q}(\Omega)$ with

$$q > n,$$

a hypothesis which is slightly more restrictive than before and in particular implies that f is Hölder continuous.

Corollary 3.28. *Let u be a solution of (3.7) with*

$$f \geq c_0 > 0 \quad in\ W^{1,q}(B_1),$$

where $q > n$, and assume that the origin is a regular free boundary point in the sense of Definition 1.5. Then the free boundary

$$\Gamma = \partial\{u > 0\}$$

is, in a neighborhood of the origin, a $C^{1,\alpha}$ graph.

Proof. As is customary in this kind of argument, we apply the boundary Harnack inequality to the derivatives of u_r. Set

$$L = \frac{L_0(q, n, 1, 1)}{2},$$

where L_0 is as in Corollary 3.22. By Proposition 3.26 there exists $r > 0$ such that the free boundary in B_r is a Lipschitz graph of constant $L < 1$, which we may assume without loss of generality is oriented in the e_n–direction.

For $i = 1, \ldots, n-1$ define

$$w_1 = \partial_i u_r, \qquad w_2 = \partial_n u_r.$$

Then each w_j satisfies

$$\Delta w_j = g_j,$$

where

$$g_1(x) = r\,\partial_i f(rx), \qquad g_2(x) = r\,\partial_n f(rx).$$

Moreover $w_2 > 0$ in the positivity set of u_r, so to apply the boundary Harnack we must show the right–hand sides g_j are small. Indeed, as $r \to 0$,

$$\|g_j\|_{L^q(B_1)} \leq \left\|r\,\nabla f(rx)\right\|_{L^q(B_1)} = 2\,r^{1-\frac{n}{q}}\,\|\nabla f\|_{L^q(B_r)} \longrightarrow 0.$$

Finally, by the blow-up convergence we have

$$w_j(e_n/2) > \tfrac{\gamma}{2} - \varepsilon > \tfrac{\gamma}{4}, \qquad w_j(e_n/2) < \tfrac{\gamma}{2} + \varepsilon < \gamma.$$

Hence we may normalize each w_j by dividing by $w_j(e_n/2)$, and the right–hand side still tends to zero in L^q.

Let $\Omega_r = \{u_r > 0\}$. By the boundary Harnack inequality with right–hand side (Theorem 3.21),

$$\frac{w_1}{w_2} \in C^{0,\alpha}(B_{1/2} \cap \Omega_r) \implies \frac{\partial_i u_r}{\partial_n u_r} \in C^{0,\alpha}(B_{1/2} \cap \Omega_r).$$

The unit normal vector to the level set $\{u_r = t\}$, $t > 0$, is given componentwise by

$$\hat{n}^i = \frac{\partial_i u_r}{|\nabla u_r|} = \frac{\partial_i u_r/\partial_n u_r}{\sqrt{1 + \sum_{j=1}^{n-1}(\partial_j u_r/\partial_n u_r)^2}} \in C^{0,\alpha}(B_{1/2} \cap \Omega_r).$$

Since this expression is $C^{0,\alpha}$ up to the boundary, it follows that the normal vector to the free boundary is $C^{0,\alpha}$, and therefore the free boundary itself is $C^{1,\alpha}$.

□

For now we have seen that:

$$\begin{pmatrix} \{u=0\} \text{ has positive} \\ \text{density at the origin} \end{pmatrix} \Longrightarrow \begin{pmatrix} \text{any blow-up is} \\ u_0 = \frac{f(0)}{2}(x\cdot e)_+^2 \end{pmatrix} \Longrightarrow \begin{pmatrix} \text{free boundary} \\ \text{is } C^{1,\alpha} \text{ near } 0 \end{pmatrix}$$

Higher-order regularity

We now upgrade our $C^{1,\alpha}$ regularity result to full smoothness. To this end, we invoke the following higher-order boundary Harnack estimate (stated without proof):

Theorem 3.29 (Higher-order Boundary Harnack, [Kuk21, Thm. 1.2])**.** *Let $\Omega \subset \mathbb{R}^n$ be a C^1_p domain in Q_1, and let*

$$Lu = \operatorname{div}\big(A\nabla u\big) + b\cdot\nabla u$$

be as in (3.16)–(3.17), with $A \in C(\overline{\Omega})$ uniformly elliptic and $b \in L^\infty(\Omega)$. Suppose $u_1, u_2 \geq 0$ solve

$$Lu_i = f_i \quad \text{in } \Omega \cap Q_1,$$

with $f_i \in L^\infty(\Omega \cap Q_1)$. Assume moreover that

$$u_2 \geq c_2 d \quad \text{in } \Omega \cap Q_1, \qquad \|f_2\|_{L^\infty(\Omega\cap Q_1)} + \|u_2\|_{L^\infty(\Omega\cap Q_1)} \leq C_2,$$

for some constants $c_2 > 0$, $C_2 < \infty$. Then for any $\varepsilon \in (0,1)$ there exists $C > 0$, depending only on $n, \varepsilon, c_2, C_2, \Omega$, the modulus of continuity of A, and the ellipticity constants, such that

$$\left\|\frac{u_1}{u_2}\right\|_{C^{1-\varepsilon}(\overline{\Omega\cap Q_{1/2}})} \leq C\Big(\|f_1\|_{L^\infty(\Omega\cap Q_1)} + \|u_1\|_{L^\infty(\Omega\cap Q_1)}\Big).$$

Where C^β_p is the parabolic Hölder space defined in the text. This result is even more general than the case we are treating here, also, in the original text the smothness of free boundary is proven for the parabolic case.

Corollary 3.30 ([Kuk21, Cor. 1.4])**.** *Let $v\colon Q_1 \to \mathbb{R}$ solve the parabolic obstacle problem*

$$\min\{\partial_t v - \Delta v - f(x),\ v\} = 0 \quad \text{in } Q_1,$$

with $f \geq c_0 > 0$ and $f \in C^\theta(B_1)$ for some $\theta > 1$. Assume that $(0,0) \in \{v > 0\}$ and that

$\partial_t v > 0$ is of class $C^{\theta,1}_{t,x}$ in Q_1. Then the free boundary $\partial\{v > 0\}$ is $C^{\theta+1}$ in a neighborhood of $(0, 0)$.

In particular, if $f \in C^\infty(B_1)$ then the free boundary is C^∞ near the origin in both space and time.

The key point is that the same ratio-argument used to obtain the $C^{1,\alpha}$ regularity is applied inductively. Concretely, once you know the normal vector is $C^{1-\varepsilon}_p(\partial\Omega \cap Q_{r/2})$ you have that the free boundary is $C^{2-\varepsilon}_p$, iterating the same argument we get the smoothnes.

3.4 Singular points

We finally study the behavior of the free boundary at singular points, i.e. those x_0 for which

$$\lim_{r\to 0} \frac{\left|\{u = 0\} \cap B_r(x_0)\right|}{|B_r|} = 0. \tag{3.20}$$

As a first consequence of the results in the previous section, we obtain the following dichotomy:

Theorem 3.31 (Classification Dichotomy)**.** *Let u be any solution of (3.7) at the origin. Then exactly one of the following occurs:*

(*a*) *Either the density condition (3.15) holds and all blow-ups of u at 0 are of the form*

$$u_0(x) = \frac{f(0)}{2}(x \cdot e)^2_+, \quad e \in \mathbb{S}^{n-1}.$$

(*b*) *Or the vanishing density condition (3.20) holds and all blow-ups of u at 0 are of the form*

$$u_0(x) = \frac{f(0)}{2}x^T Ax, \quad A \geq 0,\ \mathrm{tr}A = 1.$$

Points in case (a) where alredy estudied, while now we wil focus on points in case (b), called *singular points*, wich happens to have 0 density.

To prove this, we begin with the following lemma.

Lemma 3.32. *Let u be a solution of (3.7), and assume (3.20) holds. Then every blow-up u_0 of u at 0 satisfies $|\{u_0 = 0\}| = 0$.*

Proof. Let $u_{r_k}(x) := \frac{u(r_k x)}{r_k^2}$, and suppose $u_{r_k} \to u_0$ in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$ for some blow-up u_0. Each u_{r_k} satisfies, in the weak sense,

$$\Delta u_{r_k} = f(r_k x)\chi_{\{u_{r_k}>0\}} \quad \text{in } B_1.$$

In weak form:

$$\int_{B_1} \nabla u_{r_k} \cdot \nabla \eta = \int_{B_1} f(r_k x)\chi_{\{u_{r_k}>0\}}\,\eta \quad \forall \eta \in C^\infty_c(B_1). \tag{3.21}$$

Since $f \in C^0(B_1)$, we have $f(r_k x) \to f(0)$ uniformly on compact subsets. Also, $\chi_{\{u_{r_k}>0\}} \to \chi_{\{u_0>0\}}$ almost everywhere, and is uniformly bounded. So, by dominated convergence,

$$\int_{B_1} f(r_k x)\chi_{\{u_{r_k}>0\}}\,\eta \to f(0)\int_{B_1}\chi_{\{u_0>0\}}\eta.$$

Passing to the limit in (3.21) gives

$$\int_{B_1} \nabla u_0\cdot\nabla\eta = f(0)\int_{B_1}\chi_{\{u_0>0\}}\eta, \tag{3.22}$$

so

$$\Delta u_0 = f(0)\chi_{\{u_0>0\}} \quad \text{in the weak sense .}$$

Moreover, by assumption (3.20), we have $\left|\{u_r = 0\}\cap B_1\right| \longrightarrow 0$ and thus, taking limits $r_k \to 0$ in (3.22), we deduce that

$$\Delta u_0 = f(0) \quad \text{in } B_1.$$

Since we know that u_0 is convex, nonnegative, and homogeneous, this implies

$$\left|\{u_0 = 0\}\right| = 0.$$

□

With this being said, we can now state the proof of the theorem

Proof of Theorem 3.31. By the blow-up classification (Theorem 3.15), any blow-up at 0 is either of the half-space form or the full quadratic form. If (3.15) holds for at least one blow-up, then by regularity of the free boundary it holds for all blow-ups, and Corollary 3.24 gives

$$u_0(x) = \frac{f(0)}{2}(x\cdot e)_+^2,$$

with uniqueness of the blow-up. Otherwise (3.20) must hold, and then Lemma 3.32 shows that any blow-up has $|\{u_0 = 0\}| = 0$, forcing the form

$$u_0(x) = \frac{f(0)}{2}x^T A x, \quad A \geq 0,\ \mathrm{tr}A = 1,$$

as in case (b). □

The next question we will address is the uniqueness of these convergence polynomials.

Lemma 3.33 (Local non-degeneracy and $C^{1,1}$ and regularity estimates)**.** *Let u be the solution of* (3.7) *here is a universal constant C such that for all $r \in (0, \frac{1}{2})$,*

$$r^2 \leq C\sup_{\partial B_r} u, \quad \|u\|_{L^\infty(B_r)} \leq C r^2, \quad \|Du\|_{L^\infty(B_r)} \leq C r, \quad \|D^2 u\|_{L^\infty(B_r)} \leq C. \tag{3.23}$$

See [Caf98][(Theorem 2, Lemma 5) for the derivation.

Also we will define the following functionals: For $w \in C^{1,1}_{\mathrm{loc}}(B_1)$ and $r \in (0,1)$, define the rescaled function

$$w_r(x) = r^{-2} w(rx),$$

and the following dimensionless energy-type functionals:

$$\begin{aligned} D(r,w) &:= r^{2-n} \int_{B_r} |\nabla w|^2 = \int_{B_1} |\nabla w_r|^2, \\ H(r,w) &:= r^{1-n} \int_{\partial B_r} w^2 = \int_{\partial B_1} w_r^2. \end{aligned} \tag{3.24}$$

Now we show that the blow-ups are unique using Weiss's monotonicity formula for the adjusted energy (see [Wei99]). We set for any $v \in C^{1,1}_{\mathrm{loc}}(B_1)$ and any homogeneity parameter $\lambda \in \mathbb{R}$, define the *Weiss functional*

$$W_\lambda(r,v) := r^{-2\lambda}\{D(r,v) - \lambda H(r,v)\},$$

where D and H are as in (3.24).

Let

$$\mathcal{P} := \{p(x) \in \mathbb{R}[x_1,\dots,x_n] : \deg p = 2, \ \Delta p \equiv 1\}.$$

In other words,

$$\mathcal{P} = \left\{p(x) = \tfrac{1}{2} x^T A x + \ell \cdot x + c : A \in \mathrm{Sym}_n, \ \mathrm{tr}\, A = 1, \ \ell \in \mathbb{R}^n, \ c \in \mathbb{R}\right\}.$$

We are now able to state the following lemmas.

Lemma 3.34 (A.2)**.** *Let u solve the obstacle problem in B_1 and suppose $0 \in \Sigma(u)$. Fix any sequence $r_k \downarrow 0$. Then, up to a subsequence,*

$$r_k^{-2} u(r_k \cdot) \longrightarrow f(0)\, p_2 \quad \text{in } C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n),$$

where p_2 is a nonnegative, 2-homogeneous polynomial satisfying $\Delta p_2 = 1$. We denote by $\mathcal{P}$ the family of all such polynomials.

Proof. Set

$$v_k(x) := r_k^{-2} u(r_k x) \in C^{1,1}(B_{1/r_k}).$$

By the uniform $C^{1,1}$–bounds on u (Lemma 3.33) the family $\{v_k\}$ is equi-bounded in $C^{1,1}(B_R)$ for every fixed $R > 0$. Hence, by weak*–compactness, a subsequence converges in

$$C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n) \quad \text{to some} \quad v \in C^{1,1}(\mathbb{R}^n).$$

Moreover $v \geq 0$, $v(0) = 0$, and

$$\|\nabla^2 v\|_{L^\infty(\mathbb{R}^n)} \leq \liminf_{k\to\infty} \|\nabla^2 v_k\|_{L^\infty(B_{1/(2r_k)})} \leq \|D^2 u\|_{L^\infty(B_{1/2})} \leq C.$$

Since $0 \in \Sigma(u)$, one also has

$$f(r_k\,\cdot)\,\chi_{\{v_k=0\}} \;\longrightarrow\; f(0) \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^n).$$

In the limit $\Delta v = f(0)$ throughout $\mathbb{R}^n$. By Liouville's theorem for entire functions with bounded Hessian, v must be a quadratic polynomial of the form $f(0)\,p_2$ with $p_2 \in \mathcal{P}$. This completes the proof. □

Lemma 3.35. *There is a universal constant C such that for all $p \in \mathcal{P}$ and all $r \in (0,1)$,*

$$\frac{d}{dr}W_2\big(r, u - f(0)p\big) \;\geq\; -C\,r^{\delta-1}. \tag{3.25}$$

Proof. Set $v := u - f(0)p$. A direct computation gives

$$\frac{d}{dr}W_2(r,v) \;\geq\; \frac{2}{r^5}\int_{B_1}\big(2v_r - x\cdot\nabla v_r\big)\,\Delta v_r.$$

Since

$$\big|\Delta v_r + r^2 f_r\,\chi_{\{v_r=0\}}\big| \;\leq\; r^2\sup_{B_r}\big|f - f(0)\big|,$$

we estimate

$$\begin{aligned}\int_{B_1}(2v_r - x\cdot\nabla v_r)\,\Delta v_r \;&\geq\; -r^2\int_{B_1\cap\{u=0\}}(2v_r - x\cdot\nabla v_r)\,f_r \;-\; C\int_{B_1}\big|2v_r - x\cdot\nabla v_r\big|\,r^{2+\delta}\\ &\geq\; r^2\int_{B_1\cap\{u=0\}}\underbrace{\big(2p_r - x\cdot\nabla p_r\big)}_{=0}\,f_r \;-\; C\,r^{4+\delta} \;\geq\; -C\,r^{4+\delta}.\end{aligned}$$

Plugging back into the derivative of W_2 yields (3.25). □

Lemma 3.36. *For every $p \in \mathcal{P}$ one has*

$$W_2\big(0^+,\,u - f(0)p\big) \;=\; 0, \qquad \frac{d}{dr}\Big(r^{-4}H\big(r, u - f(0)p\big)\Big) \;\geq\; -C\,r^{\delta-1} \quad \textit{for } r \in (0,1),$$

with C universal. In particular, at any singular point the blow-up is unique, and there exists a universal modulus of continuity $\omega : (0,1) \to \mathbb{R}$, $\omega(0^+) = 0$, such that

$$r^{-4}H\big(r, u - f(0)p_2\big) \;\leq\; \omega(r) \quad \textit{for all } r \in (0,1),$$

provided p_2 is the blow-up.

Proof. Choose a subsequence $r_k \downarrow 0$ and $p \in \mathcal{P}$ with $r_k^{-2}u_{r_k} \to p$. Then by Lemmas 3.35 and 3.34,

$$W_2\big(0^+, u - f(0)p\big) = \lim_{k\to\infty} W_2\big(r_k, u - f(0)q\big) = \lim_{k\to\infty}\Big[D\big(1, r_k^{-2}u_{r_k} - f(0)q\big) - 2\,H\big(1, r_k^{-2}u_{r_k} - f(0)q\big)\Big].$$

Since p, q are 2-homogeneous with $\Delta p = \Delta q = 1$, one checks

$$\int_{B_1} |\nabla(p-q)|^2 \; - \; 2\int_{\partial B_1} (p-q)^2 \; = \; 0,$$

and hence $W_2(0^+, u - f(0)p) = 0$.

Integrating equation (3.25) we get $W_2(r, v) \geq -C\, r^{\delta}$, so by a direct computation:

$$\begin{aligned} \frac{d}{dr}\big(r^{-4}H(r, u - f(0)p)\big) &= \frac{2}{r}\Big\{W_2(r,v) + \frac{1}{r^4}\int_{B_1} v_r\,\Delta v_r\Big\} \\ &\geq \frac{2}{r}\Big\{-C\,r^{\delta} \; + \; \int_{B_1\cap\{u_r=0\}} f(0)\,p\,f(r\cdot) \; - \; C\,r^{\delta}\Big\} \; \geq \; -C\,r^{\delta-1}. \end{aligned}$$

This immediately gives uniqueness of the blow-ups. Let us prove existence of a universal rate of convergence of u to such blow-ups. Arguing by contradiction one finds $\varepsilon > 0$ and a sequence u_k such that

$$r_k^{-4}H\big(r_k, u_k - f(0)p_{2,k}\big) \;\geq\; \varepsilon.$$

Setting $v_k := r_k^{-2}u_k(r_k\cdot)$ and arguing as in Lemma 3.34 one finds $q \in \mathcal{P}$ such that $v_k \to f(0)q$ in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$. Now we get a contradiction using Monneau's monotonicity on u_k and q:

$$\begin{aligned} \varepsilon \leq H\big(1, v_k - f(0)p_{2,k}\big) &\lesssim H\big(1, v_k - f(0)q\big) + H\big(1, f(0)q - f(0)p_{2,k}\big) \\ &\leq H\big(1, v_k - f(0)q\big) + r_k^{-4}H\big(r_k, u_k - f(0)q\big) + Cr_k^{\delta} \\ &\leq 2\,H\big(1, v_k - f(0)q\big) + Cr_k^{\delta} \;\longrightarrow\; 0, \quad k\to\infty. \end{aligned}$$

□

Hence, uniqueness is established.

Summary. We have seen that at a free-boundary point x_0 exactly one of two scenarios occurs:

(i) The contact set $\{u = 0\}$ has positive density at x_0, in which case all blow-ups are half-space quadratics $\frac{f(x_0)}{2}(x\cdot e)_+^2$ and $\partial\{u > 0\}$ is C^∞ near x_0.

(ii) The contact set has zero density at x_0, in which case the unique blow-up is a full quadratic $f(x_0)\,p(x)$ with $\Delta p = 1$, and $\{u = 0\}$ remains "thin" at x_0.

With these dichotomy and uniqueness results in hand, we are ready to state the main theorem of this section.

Theorem 3.37 (Dichotomy at the origin)**.** *Let u be any solution of the obstacle problem*

$$u \in C^{1,1}(B_1), \quad u \geq 0, \quad \Delta u = f(x) \text{ in } \{u > 0\},$$

and assume $f \in C^0(B_1)$, $f(0) > 0$. Then exactly one of the following two alternatives holds:

1. Regular blow-up. *Every blow-up at* 0*,*

$$u_{r_k}(x) = \frac{u(r_k x)}{r_k^2}, \qquad r_k \downarrow 0,$$

converges (up to a subsequence) to

$$u_0(x) \; = \; \frac{f(0)}{2}\,(x \cdot e)_+^2$$

for some unit vector $e \in \mathcal{S}^{n-1}$*. In this case the free boundary* $\partial\{u > 0\}$ *is* C^∞ *in a neighborhood of the origin.*

2. Singular blow-up. *There exists a single quadratic polynomial*

$$p(x) \in \mathcal{P}, \quad p \geq 0, \quad \Delta p = 1,$$

such that

$$\lim_{r \to 0} \big\| u_r - f(0)\, p \big\|_{L^\infty(B_1)} \; = \; 0,$$

equivalently

$$\| \, u - r^2 f(0)\, p(r^{-1} \cdot) \|_{L^\infty(B_r)} = o(r^2) \quad \text{as } r \to 0.$$

In particular,

$$\lim_{r \to 0} \frac{\big|\{ \, u = 0\} \cap B_r\big|}{|B_r|} \; = \; 0.$$

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