UNIVERSITAT DE BARCELONA

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Irene Macías Tarrío.
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Certifico que la present memòria ha estat desenvolupada per Irene Macías Tarrío i dirigida per mi.

Dra. Laura Costa Farràs
Barcelona, 19 de gener de 2026

Para la abuela Piti

Airiños, airiños aires,
airiños da miña terra,
airiños, airiños aires,
airiños, leváime a ela.

...

Doces galleguiños aires,
quitadoiriños de penas,
encantadores das auguas,
amantes das arboredas,
música das verdes canas
do millo das nosas veigas,
alegres compañeiriños,
run-run de tódalas festas,
leváime nas vosas alas
como unha folliña seca.

— Rosalía de Castro

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Abstract

In this thesis, we primarily consider moduli spaces of slope-stable vector bundles with fixed rank and Chern classes, with a particular focus on Brill–Noether theory, and wall-and-chamber structures. As well, we will focus on the closely related class of prioritary vector bundles, which are characterized by specific cohomological properties.

The first main part of the thesis addresses Brill–Noether theory of stable vector bundles on ruled surfaces over a smooth irreducible curve. In this context, Brill–Noether loci are defined as subvarieties of the moduli space that parametrize stable bundles with a prescribed number of global sections. General results on the existence of these loci are known, but their geometry remains largely unexplored beyond the curve case. We obtain new results on the non-emptiness of Brill–Noether loci of stable rank two vector bundles on ruled surfaces.

A second line of investigation focuses on the construction of higher-rank prioritary vector bundles on ruled surfaces, which is the subject of Chapter 3. Using extension techniques, generalized Cayley–Bacharach properties and inductive constructions, we construct simple prioritary bundles of arbitrary rank with many global sections. These constructions provide explicit families of vector bundles that play an important role in the study of moduli spaces.

The third main theme of the thesis is the study of moduli spaces of stable bundles on varieties of dimension greater than or equal to three. In Chapter 4, we introduce a modified wall-and-chamber theory that overcomes the limitations that other theories have in higher dimension. This new framework allows us to compare moduli spaces associated with different polarizations and to study their decomposition across walls. Applying this theory in Chapter 5, we investigate moduli spaces of stable rank two vector bundles on ruled threefolds over $\mathbb{P}^2$. We describe the wall-and-chamber structure explicitly, analyze the decomposition of moduli spaces, and identify rational components in certain cases. Finally, we apply these results to prove the non-emptiness of several Brill–Noether loci.

Resum

L'objectiu d'aquesta tesi és estudiar l'espai de moduli de fibrats de rang 2 estables respecte d'un divisor ampli fixat. En particular, s'estudiarà la seva geometria en funció de les subvarietats anomenades llocs de Brill-Noether i, mitjançant la teoria de parets i camares, s'analitzarà com depèn l'espai de moduli de la polarització fixada per definir l'estabilitat dels fibrats. A part estudiem els fibrats prioritaris, que són fibrats definits a partir de condicions cohomològiques específiques.

La primera part de la tesi es centra en la teoria de Brill–Noether per a fibrats vectorials estables de rang 2 sobre superfícies reglades sobre una corba llisa i irreductible. En aquest context, els llocs de Brill–Noether es defineixen com subvarietats de l'espai de moduli que parametritzen fibrats estables amb un nombre determinat de seccions globals. Es coneixen resultats generals sobre l'existència d'aquestes subvarietats, però la seva geometria continua sent força inexplorada més enllà del cas en el que la varietat subjacent és una corba. Obtenim principalment resultats sobre quan aquests llocs de Brill–Noether són no buits i, a més, estudiem la seva dimensió, irreducibilitat i llisitut.

Una segona línia de recerca se centra en la construcció de fibrats vectorials prioritaris de rang superior sobre superfícies reglades, que és l'objectiu del Capítol 3. De la mateixa manera que restringim el conjunt de fibrats imposant una condició d'estabilitat, també es poden imposar condicions cohomològiques. Els fibrats prioritaris es defineixen a partir de condicions cohomològiques, però en el context de superfícies reglades, també estan relacionats amb els fibrats estables. Utilitzant extensions de fibrats, una generalització de la propietat de Cayley–Bacharach i construccions inductives, construïm fibrats prioritaris i simples de rang arbitrari amb moltes seccions globals. Aquestes construccions ens donen famílies explícites de fibrats prioritaris que a més tenen un paper important en l'estudi dels espais de moduli.

El tercer tema principal de la tesi és l'estudi dels espais de moduli de fibrats estables de rang 2 sobre varietats de dimensió major o igual a tres. En el Capítol

4, introduïm una teoria de parets i cambres que permet estudiar aquests espais de moduli quan la varietat base és de dimensió superior, cas que presentava obstacles aplicant altres teories existents en la literatura. Gràcies a la teoria de parets i cambres que desenvolupem, podem comparar espais de moduli diferents a partir de les polaritzacions sobre les quals es té l'estabilitat, així com estudiar la seva descomposició a través de l'estructura de parets. Aplicant aquesta teoria, en el Capítol 5 estudiem espais de moduli de fibrats vectorials estables de rang 2 sobre varietats reglades de dimensió 3 construïdes sobre $\mathbb{P}^2$. Donem una descripció explícita de l'estructura de parets i cambres en aquest context, amb la qual podem determinar la descomposició en components d'alguns espais de moduli. Finalment, com a conseqüència d'aquests resultats, obtenim aplicacions a la teoria de Brill–Noether en aquest context, determinant casos en els que aquestes subvarietats són no buides.

Contents

Notation

K	Algebraically closed field of characteristic 0
X	Smooth projective variety of dimension n
K_X	Canonical divisor of X
$\mathcal{C}_X$	Kähler cone expanded by polarizations on X
$\mathrm{Pic}(X)$	Picard group of X
$\mathrm{Num}(X)$	Group of divisors on X modulo numerical equivalence
$\bigwedge^i$	i-th exterior power
$\mathcal{F}^*$	Dual of a sheaf $\mathcal{F}$
V^*	Dual of a vector space V
I_Z	Ideal sheaf of a codimension 2 subscheme $Z \subset X$
$\mathcal{N}_{Z/X}$	Normal sheaf
$\mathcal{H}om$	Sheaf of homomorphisms of sheaves
Hom	Group of homomorphisms of sheaves
hom	Dimension of Hom
Ext^i	i-th Ext group
ext^i	Dimension of Ext^i

$\mathrm{H}^i(E)$	i-th cohomology group of a vector bundle E
$\mathrm{h}^i(E)$	Dimension of $\mathrm{H}^i(E)$
$c_i(E)$	i-th Chern class of a vector bundle E
$\chi(E)$	Euler characteristic of a vector bundle E
$\mu_H(E)$	H-slope of a vector bundle E
M_H	Moduli space of vector bundles stable with respect to an ample divisor H
W_H^k	k-th Brill–Noether locus of H-stable vector bundles
$\mathbb{P}(\mathcal{E})$	Projectivization of a vector bundle $\mathcal{E}$
$\pi : X = \mathbb{P}(\mathcal{E}) \to Y$	Ruled variety over a smooth projective variety Y constructed from a vector bundle $\mathcal{E}$ over Y

Introduction

The main objective of this thesis is to contribute to the study of moduli spaces of vector bundles. As will be explained in detail later, it is well known that in order to endow the set of vector bundles with the structure of an algebraic variety, it is necessary to impose suitable conditions. In this work, we focus primarily on moduli spaces of stable vector bundles, and thus stability conditions play a central role in our approach. In addition, we study the class of prioritary vector bundles, which are characterized by certain cohomological conditions.

More precisely, given X a smooth projective variety of dimension n, denote by $M_H(r; c_1, \ldots, c_s)$ the scheme parametrizing H-stable rank r vector bundles E on X with Chern classes $c_i(E) = c_i$, $1 \leq i \leq \min\{r, n\}$. The principal problems investigated in this thesis, and on which we obtain relevant new results, are:

Problem 1: Given X a smooth irreducible surface and $k \in \mathbb{Z}$, describe the Brill-Noether subvarieties $W^k_H(2; c_1, c_2)$ of the moduli space $M_H(2; c_1, c_2)$ defined such that

$$\operatorname{Supp}(W^k_H(2; c_1, c_2)) = \{E \in M_H(2; c_1, c_2) \mid \mathrm{h}^0 E \geq k\}.$$

Problem 2: Given X a smooth irreducible surface, prove the existence of rank r prioritary vector bundles on X with sections.

Problem 3: Given X a smooth projective variety of dimension greater than or equal to 3, describe the moduli space $M_H(r; c_1, \ldots, c_s)$ and how it varies when we move the polarization H inside the ample cone of X.

These questions are highly ambitious in their general formulation. The results presented in this thesis are obtained within the framework of ruled varieties as the underlying space and constitute a significant advancement in the understanding of these problems.

The main contributions addressing these questions can be found in:

- L. Costa and I. Macías Tarrío, *Brill–Noether theory of stable vector bundles on ruled surfaces*, Mediterr. J. Math. **21** (2024), no. 3, Art. 118.
- L. Costa and I. Macías Tarrío, *Higher rank prioritary bundles on ruled surfaces and their global sections*, Ann. Mat. Pura Appl. (4) **204** (2025), no. 4, 1633–1655.
- L. Costa and I. Macías Tarrío, *Moduli spaces of stable bundles on ruled 3-folds and Brill–Noether problems*, Preprint, https://arxiv.org/abs/2507.06882 (2025).

Since the early seventies, the theory of vector bundles on projective varieties has become one of the mainstreams in Algebraic Geometry. For instance, they offer a valuable point of view in order to better understand the geometry of algebraic varieties. Despite huge achievements in vector bundles theory, many naif questions remain still open.

Since the whole category of vector bundles on a projective variety is usually unwieldy, one has to impose some restrictions on the families of vector bundles under consideration. One possible restriction is to consider vector bundles satisfying a stability condition. In the literature, there are several notions of stability. In this work, we will consider the slope stability, also known as Mumford-Takemoto stability, which was first introduced by Mumford in the 1960's for higher rank torsion-free sheaves on smooth curves (see [52]). Later, in the early 1970's, Takemoto generalized this definition to higher rank vector bundles on smooth projective varieties of arbitrary dimension (see [62] and [63]). The set of H-stable vector bundles is parametrized by a moduli space, which was constructed by Maruyama in [45] and [46]. In fact, given X a smooth projective variety of dimension n, Maruyama proved the existence of a scheme $M_H(r; c_1, \ldots, c_s)$, which rougly speaking parametrizes H-stable rank r vector bundles E on X with Chern classes $c_i(E) = c_i$, $1 \leq i \leq \min\{r, n\}$. Once the existence is established, it is interesting to study them from different perspectives. We will be mainly interested in them from the point of view of algebraic geometry.

When the underlying variety is a surface, there are general results concerning, among others, the smoothness, irreducibility, and non-emptiness of the moduli space $M_H(2; c_1, c_2)$ and, in certain cases, a refined description depending on the position of the surface within the Kodaira classification (we refer to the introduction of Chapter 2 for a clearer overview). By contrast, when the underlying variety has dimension greater or equal than 3, there are no general results that describe the moduli space $M_H(r; c_1, \dots, c_s)$, and a variety of pathological phenomena may arise. Moreover, despite their intrinsic interest, only few results in the literature address the geometry of these moduli spaces.

A fundamental and highly interesting problem that arises while studying the geometry of the moduli space $M_H(r; c_1, \dots, c_s)$, is to understand how two different moduli spaces M_H and $M_{H'}$ are related when one varies the polarization, that is, when one considers two different ample divisors H and H'.

In [59], Qin introduced a general theory of walls and chambers asserting that the moduli space $M_H(2; c_1, c_2)$ only depends on the chamber containing the polarization. In general, when two polarizations lie in different chambers, the corresponding moduli spaces are not isomorphic. This wall–and–chamber theory has made it possible to describe the changes between moduli spaces, proving that if two polarizations lie in the same chamber then the corresponding moduli spaces are isomorphic. Moreover, it has allowed the description of moduli spaces of stable rank 2 vector bundles on projective surfaces and to uncover rich geometric structures on them.

If we want to move to a higher dimension, we may find difficulties in Qin's theory. This is where our work in [17] emerges, and we develop a slightly different theory of walls and chambers that we will use to describe the moduli space of stable vector bundles on varieties of dimension greater than or equal to 3. Applying this theory, we get results on moduli spaces of stable bundles on higher-dimensional varieties. In particular, we will study the decomposition of moduli spaces of stable bundles on ruled 3-folds.

Another way to face the study of the geometry of the moduli space is to describe some of its subvarieties. In particular, one can study the subvarieties defined by imposing cohomological conditions on the stable points inside the moduli space. This naturally leads us to consider Brill–Noether loci.

Classical Brill-Noether theory deals with line bundles on smooth curves and it was introduced by Brill and Noether by the end of the 19th century in [11]. In this context, the Brill-Noether locus was defined as the variety parametrizing line bundles of fixed degree on a smooth curve with a certain number of independent sections. Questions related to the non-emptiness, irreducibility, connectedness, smoothness, or dimension of these Brill-Noether locus, among others, have been answered when the curve C is generic on the moduli space of curves of genus g. The main results for the classical case can be found in [2].

Once these Brill-Noether loci were well understood, it was natural to look for generalizations. One way to generalize the Brill-Noether locus is to consider, instead of line bundles, higher rank bundles on smooth curves (see, for instance, [7], [9], [10], [12], [30], [36], [38] and [68]). The other way of generalizing this theory is by considering line bundles on varieties of arbitrary dimension, which corresponds to the study of its Picard group. Eventually, one can go in both directions and consider higher rank vector bundles on higher-dimensional varieties. That is, given an integer $k \geq 1$, consider subvarieties $W^k_H(r; c_1, \ldots, c_s)$ such that

$$\mathrm{Supp}(W^k_H(r; c_1, \ldots, c_s)) = \{E \in M_H(r; c_1, \ldots, c_s) | h^0E \geq k\}.$$

The existence of these subvarieties for the general case was first given in [18]. The result states that, under certain cohomological conditions, the Brill-Noether locus has the structure of a determinantal variety. In addition, from this determinantal structure, it is stated that any irreducible component of the Brill-Noether locus has dimension at least

$$\rho^k_H(r; c_1, \ldots, c_s) := \dim(M_H(r; c_1, \ldots, c_s)) - k(k - \chi(r; c_1, \ldots, c_s)),$$

which is known as the expected dimension.

Recently, an alternative proof for endowing the Brill-Noether locus with a scheme structure was given in [55]. The significance of these results is that they allow to avoid the cohomological assuptions needed in [18] and, hence, one can always ensure the existence of these loci for varieties of arbitrary dimension.

Once the existence of the Brill-Noether locus is defined, the questions enounced and partially solved for the classical case are naturally asked for the generalized one. They are the following:

Question A:

i) Can we determine the emptiness and non-emptiness of $W^k_H(r; c_1, \ldots, c_s)$?

ii) Does $\rho^k_H(r; c_1, \ldots, c_s) \geq 0$ imply that $W^k_H(r; c_1, \ldots, c_s) \neq \emptyset$?
Does $\rho^k_H(r; c_1, \ldots, c_s) < 0$ imply that $W^k_H(r; c_1, \ldots, c_s) = \emptyset$?

iii) Is $W^k_H(r; c_1, \ldots, c_s)$ smooth or irreducible?

iv) Can we compute the dimension of $W^k_H(r; c_1, \ldots, c_s)$?
If so, is it the expected one? That is, is $W^k_H(r; c_1, \ldots, c_s)$ of dimension $\rho^k_H(r; c_1, \ldots, c_s)$?

If the underlying variety has dimension greater than or equal to two, there are no general results answering these questions and only few results are known. For instance, if it is a surface, there are partial results if X is the projective plane $\mathbb{P}^2$, a $K3$ surface, an Enriques surface or a ruled surface (the reader may refer to [14], [19], [24], [39], [40], [44] or [54], and references quoted there). Our contribution when the underlying variety is a surface is given in [15], which corresponds to the results in Chapter 2. We study the non-emptiness of the Brill-Noether locus of stable bundles on ruled surfaces. If the underlying variety has dimension greater than or equal to 3, despite that one can guarantee the existence of Brill-Noether locus, few is known. We give results for this case in [17], which are collected in Chapter 3. We prove the non-emptiness of the Brill-Noether locus of rank 2 stable bundles on ruled 3-folds over $\mathbb{P}^2$.

Another approach that has recently been used to study the set of vector bundles deals with their cohomological properties. The idea is to impose some cohomological conditions on the bundles to restrict the class of bundles under study.

Nowadays, there is a substantial amount of work on vector bundles without intermediate cohomology (ACM bundles) and, in particular, many results focus on those with the maximal number of global sections, namely Ulrich bundles. Another class of vector bundles that share a strong cohomological property are the prioritary bundles, which were introduced in the early nineties by Hirschowitz and Laszlo in [33], to study vector bundles on $\mathbb{P}^2$. Later on, in [66], the notion of prioritary was generalized for bundles on rules surfaces by Walter.

This class of bundles is of particular interest to us because, apart from the fact that prioritary bundles have a nice cohomological description, they also play an important role in the study of the moduli space of H-stable vector bundles. In fact, under certain conditions on the polarization H, H-stable vector bundles on ruled surfaces are prioritary, and the moduli space of H-stable bundles is an open substack inside the moduli stack of prioritary bundles. This fact has led to numerous contributions in the literature. For example, prioritary bundles have been used to describe the Picard group of the moduli space of sheaves on quadric surfaces (see [58]) and the weak Brill-Noether property (see [14]). Hence, in some sense, prioritary bundles link both approaches. Moreover, it is interesting to study vector bundles with a certain number of global sections. In this work, we construct higher rank prioritary bundles on ruled surfaces, which are in addition indecomposable and have many sections. We provide different constructions in Chapter 3, which are the results in [16].

To end, let us outline the structure of the thesis.

First, Chapter 1 is devoted to provide the reader with a general background that we will need in the sequel. In the first section, we will introduce basic results and properties of vector bundles. Next, in Section 2, we will present relevant results about moduli spaces of stable vector bundles, as well as the ones concerning the Brill-Noether theory. To end the chapter, we will introduce generalities on ruled varieties, that are the underlying variety of the moduli spaces we will work on. In particular, we will be focused on ruled surfaces over a smooth curve and on ruled 3-folds over $\mathbb{P}^2$.

The aim of Chapter 2 is to develop results in the Brill–Noether theory of stable vector bundles on ruled surfaces. In particular, we address Problem 1 and several aspects of the questions formulated in Question A of this introduction, in line with the results presented in [15].

First of all, in Section 2.1, we will fix the setting following the notation previously introduced in Chapter 1. In Sections 2.2 and 2.3, we will study the non-emptiness of the Brill-Noether locus $W^k_H(2; c_1, c_2)$ with normalized first Chern class $c_1 \in \{S, S+f, f\}$. We will give a characterization of the non-emptiness according to k and, in addition, we will prove that it is no longer true that a negative value of the expected dimension determines the emptiness. The case $c_1 \in \{S, S+f\}$ is proved in Theorems 2.2.2 and 2.2.8, and the case $c_1 = f$ in Theorems 2.3.1 and 2.3.3. To end, in Section 2.4, we will study the locus $W^1_H(2; f, c_2)\backslash W^2_H(2; f, c_2)$. In Theorem 2.4.1, we will prove that it is irreducible, smooth and of the expected dimension.

The results in Chapter 3 are collected in [16], and concern the construction of indecomposable higher rank prioritary bundles on ruled surfaces with a certain number of sections. They contribute to the knowledge of Problem 2. Depending on the choice of the first Chern class c_1, the methods employed, and the rank of the prioritary bundles constructed, this chapter is divided into five sections.

First of all, in Section 3.1, we will introduce the definition and foundamental properties of prioritary bundles on ruled surfaces. In addition, we will be interested in simple prioritary bundles, property that guarantees that they do not decompose in lower rank bundles. In Section 3.2, in Theorem 3.2.1, we will introduce a method to construct rank r simple prioritary bundles by extensions of a rank $r-1$ bundle and a line bundle. Moreover, this method allows us to recursively construct prioritary bundles, and we prove that we can use it to construct arbitrary rank prioritary bundles from a rank 2 prioritary vector bundle. Later on, we will apply this method and explicitly construct prioritary bundles of arbitrary rank that, in addition, have a certain number of independent sections. In Section 3.3, we will construct rank 3 simple prioritary vector bundles with many sections (see Theorems 3.3.2, 3.3.5 and 3.3.7), using different constructions according to their normalized first Chern class. In Section 3.4, in Theorems 3.4.5 and 3.4.7, we will use the method in Theorem 3.2.1 and the constructions in Section 3.3 to construct rank 4 simple prioritary bundles with many sections. To end, in Section 3.5, in Theorems 3.5.2, 3.5.3, 3.5.4 and 3.5.5, we will construct rank r simple prioritary vector bundles using the generalized Cayley-Bacharach property. In addition, we will see that the bundles we construct have many global sections.

In Chapter 4, we will introduce a wall-and-chamber theory inspired by the one developed by Qin in [59]. As mentioned before, Qin's theory presents certain difficulties when the underlying variety has dimension greater than 2. Nevertheless, the study of walls and chambers remains a fruitful approach for understanding the geometry of moduli spaces and for comparing moduli spaces associated with different choices of polarization. In this chapter, we introduce a slightly modified wall-and-chamber theory that resolves the gaps present in Qin's original framework. The main results are collected in [17].

In Section 4.1, we will introduce the theory of walls and chambers and we will prove that, according to the definitions given, two polarizations in the same chambers are equivalent. This leads to the fact that the stability only depends on the chamber and thus we can compare two different moduli spaces by relating two polarizations. As a byproduct, the wall-and-chamber theory allows us to construct rank 2 stable vector bundles. In Section 4.2, we will use the Hartshorne-Serre correspondence to construct a family $E_\xi(c_1, c_2)$ of rank 2 vector bundles and, using the wall-and-chamber theory introduced in Section 4.1, we will prove that in fact it is a family of stable bundles. Finally, in Section 4.3, in Theorem 4.3.2, we will give a general result that states sufficient conditions to study the decomposition of a moduli space of stable bundles with respect to a polarization lying in a chamber, in terms of the families $E_\xi(c_1, c_2)$ and the moduli space of stable bundles with respect to a polarization lying in the adjacent chamber. All the results in this chapter are stated for any smooth projective variety and we will apply them in the next chapter.

We will end the thesis with Chapter 5, which results can be found in [17]. In this Chapter, we contribute to Problem 3. In fact, we will use the wall-and-chamber theory established in Chapter 4 to study the moduli space of rank 2 stable bundles on ruled 3-folds over $\mathbb{P}^2$. In addition, we will study Brill-Noether Theory in this context, given non-emptiness results of these subvarieties, which contribute to Question A.

In Section 5.1, we will show that, under suitable assumptions, we can provide a geometric interpretation as well as an explicit description of walls and chambers of type (c_1, c_2) on these ruled threefolds. Subsequently, in Section 5.2, we will determine the wall-and-chamber structure for the cases when the first Chern class is $c_1 \in \{S, S+L, L, 0\}$ and we will study the moduli spaces corresponding to polarizations lying in a fixed chamber. Section 5.2 is further divided into several subsections according to the value of the first Chern class under consideration. The procedure is as follows: first, we fix a class defining a wall of type (c_1, c_2) and analyze the wall it determines, together with the adjacent chambers separated by it. We then will apply Theorem 4.3.2 to obtain results concerning the decomposition of the moduli space, as well as a criteria for its emptiness and non-emptiness. Moreover, in certain cases, this decomposition allows us to describe rational components of the moduli space (see Theorems 5.2.7, 5.2.8, 5.2.9, and 5.2.10; Theorems 5.2.17, 5.2.18, and 5.2.21; Theorem 5.2.26 and Theorem 5.2.28; and Theorems 5.2.34 and 5.2.38). Finally, in Section 5.3, we will apply the results obtained in Section 5.2 to study the non-emptiness of the Brill–Noether loci in this setting (see Theorems 5.3.2, 5.3.3, 5.3.4, 5.3.5, 5.3.6, 5.3.7, 5.3.8, and 5.3.9). The guiding idea is that the family $E_\xi(c_1, c_2)$ constitutes a component of a Brill–Noether locus; consequently, in certain cases, we are able to prove the non-emptiness of these subvarieties.

Chapter 1

Preliminaries

As noted in the introduction, the main purpose of this first chapter is to review basic facts and results on vector bundles, moduli spaces of stable bundles and ruled varieties, which constitute the framework of our work. We will start by giving definitions and fundamental properties concerning vector bundles, such as their Chern classes, Serre duality or the Riemann-Roch theorem. Then, we will introduce the notion of stability with respect to an ample divisor, the construction of the moduli space of stable bundles with fixed invariants, and we will provide an overview of Brill–Noether theory. To end, we will present the construction of ruled varieties and we will describe its Picard group and the cohomology of line bundles on these varieties. We will be mainly interested in ruled surfaces over a smooth projective curve C and in ruled 3-folds over $\mathbb{P}^2$.

This chapter only contains a few proofs. For further details, we refer the reader to the books [2], [25], [31], [35] and [57].

1.1 Vector bundles on projective varieties

In this first section, we will give a general account of the basic properties of vector bundles on a smooth projective variety. We will recall some results and properties that will be used later on, as well as classical constructions of vector bundles.

Most of the definitions and properties can be stated for torsion-free sheaves in general but, since we will study vector bundles, we restrict ourselves to this setting.

Let X be a smooth projective variety of dimension n over an algebraically closed field K of characteristic 0 and consider a rank r vector bundle E on X.

We start by introducing the Chern classes of E.

Chern classes. Let us recall the definition and basic properties of the Chern classes. More details can be found in [29], [31, Appendix A] and [34].

By a cycle on X, we mean an element of the free abelian group generated by all closed irreducible subvarieties of X. If all the components of a cycle have codimension r, we say that the cycle has codimension r. We can introduce an equivalence relation, called rational equivalence, on codimension r cycles, which coincides with the linear equivalence of divisors if $r = 1$. We denote by $A^r(X)$ the group of equivalence classes. We can define an intersection pairing $A^r(X) \times A^s(X) \to A^{r+s}(X)$, which allows us to construct a commutative graded ring $A(X) = \bigoplus_{i=0}^n A^i(X)$. This ring is known as the Chow ring of X.

Example 1.1.1. The Chow ring of the n-dimensional projective space $\mathbb{P}^n$ is $A(\mathbb{P}^n) \cong \mathbb{Z}[h]/h^{n+1}$, where h is the class of an hyperplane of $\mathbb{P}^n$. It follows from the fact that any codimension i subvariety of $\mathbb{P}^n$ is rationally equivalent to d times a linear space of the same dimension ([31, Appendix A, Example 2.0.1]).

Given a rank r vector bundle E on X, we will define its Chern classes $c_i(E) \in A^i(X)$, for $1 \leq i \leq n$, such that $c_0(E) = 1$ and $c_i(E) = 0$ for any $i > r$.

Remark 1.1.2. Notice that, for $i = 1$, $c_1(E) \in A^1(X)$ and the rational equivalence coincides with the linear equivalence. Therefore, we can assume that $c_1(E)$ is an element of $\mathrm{Pic}(X)$.

Following this notation, we introduce the next definitions.

Definition 1.1.3. The total Chern class of E is defined as

$$c(E) = c_0(E) + c_1(E) + \cdots + c_n(E) \in A(X).$$

The Chern polynomial is

$$c_t(E) = 1 + c_1(E)t + \cdots + c_{n-1}(E)t^{n-1} + c_t(E)t^n.$$

Remark 1.1.4. Notice that, since $c_0(E) = 1$ and $c_i(E) = 0$ for any $i > r$, we only need to consider the Chern classes $c_i(E)$ for $1 \leq i \leq s := \min\{r, n\}$.

The Chern classes of a vector bundle are uniquely determined by the following properties.

i) If $\mathcal{L}$ is a line bundle on X, then it is of the form $\mathcal{L} = \mathcal{O}_X(D)$ for some divisor D on X and then $c_1(\mathcal{L})$ is the class of D in $A^1(X)$. In particular, $c_1(\mathcal{L}_1 \otimes \mathcal{L}_2) = c_1(\mathcal{L}_1) + c_1(\mathcal{L}_2)$ and $c_1(\mathcal{L}^*) = -c_1(\mathcal{L})$.

ii) If
$$0 \to E \to F \to G \to 0$$
is an exact sequence of vector bundles, then $c_t(F) = c_t(E) \cdot c_t(G)$.

iii) If $f\colon X \to Y$ is a morphism of projective varieties and E is a vector bundle on Y, then $c(f^*E) = f^*(c(E))$.

We also know how to compute the Chern class of a tensor product of a vector bundle by a line bundle. If $\mathcal{L}$ is a line bundle on X, then
$$c_k(E \otimes \mathcal{L}) = \sum_{i=0}^{k} \binom{r-i}{k-i} c_i(E) \cdot \mathcal{L}^{k-i}.$$
In general, if E and F are vector bundles of ranks r and s and Chern polynomials $c_t(E) = \prod_{i=1}(1 + a_i t)$ and $c_t(F) = \prod_{j=1}(1 + b_j t)$, respectively, then
$$c_t(E \otimes F) = \prod_{i,j}[1 + (a_i + b_j)t].$$
In addition, the Chern class of the exterior power of a rank r vector bundle E with Chern polynomial $c_t(E) = \prod_{i=1}(1 + a_i t)$, can be computed as
$$c_t(\wedge^p E) = \prod_{1 \le i_1 < \cdots < i_p \le r} [1 + (a_{i_1} + \cdots + a_{i_p})t].$$
Finally, if E is a rank 2 vector bundle with first Chern class c_1, then the dual bundle of E satisfies $E^* \cong E(-c_1)$.

Remark 1.1.5. In the context of complex geometry, the holomorphic vector bundles have invariants called characteristic classes, that are elements of the integer cohomology of the underlying variety. Their algebraic analogous are the Chern classes just defined. If X is defined over $\mathbb{C}$, there exists a natural homomorphism $A^i(X) \to \mathrm{H}^{2i}(X, \mathbb{Z})$, which sends a codimension i cycle to its foundamental class. Therefore, in our setting, we can assume that the Chern classes are elements $c_i(E) \in \mathrm{H}^{2i}(X, \mathbb{Z})$. See, for instance, [25, Chapter 2].

Euler characteristic, Riemann-Roch Theorem and Serre Duality. The Riemann-Roch theorem states that the Euler characteristic of a vector bundle can be expressed in terms of its Todd class, which is defined as follows.

Definition 1.1.6. If E is a rank r vector bundle on X with Chern polynomial $c_t(E) = \prod_{i=1}(1 + a_i t)$, its Todd class is defined as
$$\mathrm{td}(E) = \prod_{i=1}^{r} \frac{a_i}{1 - e^{-a_i}} = \prod_{i=1}^{r}(1 + \frac{1}{2}a_1 + \frac{1}{12}a_2^2 + \cdots).$$

Definition 1.1.7. The Euler characteristic of a vector bundle E on a smooth projective variety X of dimension n is defined as

$$\chi(E) = \sum_{i=0}^{n} (-1)^i \operatorname{h}^i(X, E).$$

According to this definition, the Riemann-Roch theorem states how to compute the Euler characteristic of a vector bundle from its Chern classes.

Theorem 1.1.8. *Let X be a smooth projective variety of dimension n and E a rank r vector bundle on X with Chern classes c_i for $1 \leq i \leq s := \min\{r, n\}$. Then,*

$$\chi(E) = \deg((c(E)\operatorname{td}(T_X))_n),$$

where T_X is the tangent bundle on X and $(.)_n$ is the component of degree n in $A(X) \otimes \mathbb{Q}$.

In particular, if X is a smooth curve and E a rank r vector bundle on X of degree d,

$$\chi(E) = \operatorname{h}^0(X, E) - \operatorname{h}^1(X, E) = r(1-g) + d,$$

where g is the genus of the curve.

If X is a surface and E a rank r vector bundle on X, with Chern classes c_1 and c_2,

$$\chi(E) = \operatorname{h}^0 E - \operatorname{h}^1 E + \operatorname{h}^2 E = r(p_a(X) + 1) - \frac{c_1 \cdot K_X}{2} + \frac{c_1^2 - 2c_2}{2},$$

where $\operatorname{h}^i E$ stands for $\operatorname{h}^i(X, E)$, $p_a(X) = \chi\mathcal{O}_X - 1$ is the arithmetic genus of X and K_X is the canonical divisor.

Notation 1.1.9. Let X be a smooth projective variety of dimension n, $r \in \mathbb{Z}$ and consider the Chern classes c_i for $1 \leq i \leq \min\{r, n\} := s$. Sometimes, it will be convenient to write $\chi(r; c_1, \ldots, c_s) := \chi(E)$, where E is a rank r vector bundle on X with Chern classes c_i.

We end this part by recalling the well-known Serre duality.

Theorem 1.1.10. *Let X be a smooth irreducible projective variety of dimension n, and let $\mathcal{F}$ and $\mathcal{G}$ be torsion-free sheaves on X. Then, there exists a canonical isomorphism*

$$\operatorname{Ext}^i(\mathcal{F}, \mathcal{G}) \cong \operatorname{Ext}^{n-i}(\mathcal{G}, \mathcal{F} \otimes K_X)^*.$$

In addition, if $\mathcal{F}$ is a vector bundle, then there exists a canonical isomorphism

$$\operatorname{H}^i(X, \mathcal{F}) \cong \operatorname{H}^{n-i}(X, \mathcal{F}^* \otimes K_X)^*.$$

Construction of vector bundles. Before introducing the construction of vector bundles of rank $r \geq 2$ on smooth projective varieties, let us recall some facts concerning line bundles and rank 1 torsion-free sheaves.

As we have pointed out before, a line bundle $\mathcal{L}$ on a projective variety X can be written as $\mathcal{O}_X(D)$ for some divisor $D \in \mathrm{Pic}(X)$ and its first Chern class is D. An ample divisor H is a divisor such that some power of the corresponding line bundle $\mathcal{O}_X(H)$ gives an immersion of X into a projective space. The Naikai-Moishezon criterion (see [31, Theorem 5.1, Appendix A]), gives a characterization of ample divisors. The statement in our context is the following.

Theorem 1.1.11. *Let H be a divisor on a smooth projective variety X of dimension n. Then, H is ample if and only if, for every effective divisor D on X, we have $D \cdot H^{n-1} > 0$.*

Proof. In [53], Nakai proved this statement in the case when X is a projective scheme and, independently, Moishezon proved it in [50] for abstract complete varieties. □

Remark 1.1.12. Since the above characterization depends only on the numerical equivalence class, it extends naturally to $H \in \mathrm{Num}(X) \otimes \mathbb{R}$ (see, for instance, [37, Proposition 1.3.13]).

Now, let us focus our attention on torsion-free sheaves on a surface X, that is, on ideal sheaves $I_Z(D)$ for $Z \subset X$ a 0-dimensional subscheme and $D \in \mathrm{Pic}(X)$. The resolution of I_Z corresponds to the following exact sequence

$$0 \to \mathcal{O}_Z \to \mathcal{O}_X \to I_Z \to 0. \tag{1.1}$$

As a consequence, we get

$$\mathrm{h}^2\, I_Z(D) = \mathrm{h}^2\, \mathcal{O}_X(D) \quad \text{and} \quad \mathrm{h}^0\, I_Z(D) \leq \mathrm{h}^0\, \mathcal{O}_X(D). \tag{1.2}$$

Moreover, if Z is a generic 0-dimensional subscheme (in the sense that it is supported in points in general position), we have the following result.

Proposition 1.1.13. *Let X be a smooth projective surface, D a divisor on X and $Z \subset X$ a generic 0-dimensional subscheme of X. If $|Z| \geq \mathrm{h}^0\, \mathcal{O}_X(D)$, then $\mathrm{h}^0\, I_Z(D) = 0$.*

Proof. Let us sketch the proof. Let us consider the short exact sequence

$$0 \to I_Z(D) \to \mathcal{O}_X(D) \to \mathcal{O}_Z(D) \to 0.$$

Taking global sections yields the exact sequence

$$0 \to \mathrm{H}^0(X, I_Z(D)) \to \mathrm{H}^0(X, \mathcal{O}_X(D)) \xrightarrow{\mathrm{ev}} \mathrm{H}^0(X, \mathcal{O}_Z(D)) \to \cdots .$$

Since Z consists of $l = |Z|$ reduced points, we have

$$\mathrm{H}^0(X, \mathcal{O}_Z(D)) \cong K^l.$$

The map ev is the evaluation map which sends a section of $\mathcal{O}_X(D)$ to its values at the points of Z. Because the points of Z are generic, they impose independent linear conditions on the sections of $\mathcal{O}_X(D)$. Consequently, the evaluation map ev has maximal rank, namely

$$\mathrm{rank}(\mathrm{ev}) = \min\{\mathrm{h}^0(X, \mathcal{O}_X(D)),\, l\}.$$

If $l \geq \mathrm{h}^0(X, \mathcal{O}_X(D))$, then ev is injective and, therefore,

$$\mathrm{H}^0(X, I_Z(D)) = 0,$$

which finishes the proof. □

Let E be a rank r vector bundle on a smooth projective variety X and $\mathcal{L} = \bigwedge^r E$ its determinant line bundle. If Z is the dependency locus of $r-1$ sections s_i of E, then E sits in the following exact sequence

$$0 \to \mathcal{O}_X^{\oplus r-1} \xrightarrow{s_1,\ldots,s_{r-1}} E \to I_Z \otimes \mathcal{L} \to 0.$$

In particular, this implies that $\bigwedge^2(\mathcal{N}_{Z/X}) \otimes \mathcal{L}^*_{|Z}$ is globally generated by $r-1$ sections, where $\mathcal{N}_{Z/X}$ is the normal bundle of Z in X. The Hartshorne-Serre correspondence consists of reversing this process. That is, to give sufficient conditions to construct a rank r vector bundle on X by an exact sequence of this type. The precise statement goes as follows.

Theorem 1.1.14. *Let X be a smooth projective variety and let Z be a locally complete intersection subscheme of codimension two in X. Let $\mathcal{N}_{Z/X}$ be the normal bundle of Z in X and let $\mathcal{L}$ be a line bundle on X with $H^2(X, \mathcal{L}^*) = 0$. Assume that $\bigwedge^2 \mathcal{N}_{Z/X} \otimes \mathcal{L}^*|_Z$ has $r-1$ generating global sections $s_1, \ldots, s_{r-1}$. Then, there exists a rank r vector bundle E on X given by a non-trivial extension*

$$0 \to \mathcal{O}_X^{\oplus r-1} \to E \to I_Z \otimes \mathcal{L} \to 0$$

such that

i) $\bigwedge^r E = \mathcal{L}$*;*

ii) E has $r-1$ global sections $s_1, \dots, s_{r-1}$ whose dependency locus is Z and such that

$$s_1|_Z + \cdots + s_{r-1}|_Z = 0.$$

Proof. See [3], [28] and [65]. □

If X is a smooth projective surface, the Hartshorne-Serre correspondence can be stated more explicitely and it is related to the Cayley-Bacharach property. Classical Cayley-Bacharach property deals with cubic curves in $\mathbb{P}^2$. In our context, the definition is the following.

Definition 1.1.15. Let X be a smooth projective surface and Z a 0-dimensional subscheme of X and L a line bundle on X. We say that Z satisfies the Cayley-Bacharach property with respect to the linear system $|L|$ if, for any $s \in \mathrm{H}^0\, \mathcal{O}_X(L)$ such that $s_{|Z'} = 0$ for $Z' \subset Z$ with $|Z'| = |Z| - 1$, we get $s_{|Z} = 0$.

Remark 1.1.16. Notice that, if $\mathrm{h}^0\, \mathcal{O}_X(L) = 0$ or $\mathrm{h}^0\, I_Z(L) = 0$, then the Cayley-Bacharach property is automatically satisfied.

According to this definition, we have the following equivalence.

Theorem 1.1.17. *Let $Z \subset X$ be a locally complete intersection subscheme of codimension two, and let D and D' be divisors on X. Then, there exists an extension of type*

$$e: \quad 0 \to \mathcal{O}_X(D') \to E \to I_Z(D) \to 0,$$

with E a rank 2 vector bundle if and only if the locally complete intersection subscheme Z satisfies the Cayley-Bacharach property with respect to the linear system $|D - D' + K_X|$.

Proof. See [35, Theorem 5.2.1]. □

Remark 1.1.18. Theorem 1.1.17 states that if Z satisfies the Cayley-Bacharach property with respect to the linear system $|D - D' + K_X|$, then E is a vector bundle, which in particular implies that the extension $e \in \mathrm{Ext}^1(I_Z(D), \mathcal{O}_X(D'))$ is non-trivial.

Moreover, the Cayley-Bacharach property can be generalized to construct higher rank vector bundles on smooth projective surfaces, by an extension of a line bundle and a direct sum of ideal sheaves.

Theorem 1.1.19. *Let $Z_1, \dots, Z_{r-1}$ be reduced 0-dimensional subschemes of a smooth projective surface X such that, for every $i \neq j$, $Z_i \cap Z_j = \emptyset$. Consider $D', D_1, D_2, \dots, D_{r-1} \in \mathrm{Pic}(X)$ and assume that E is given by an exact sequence of type*

$$e: \quad 0 \to \mathcal{O}_X(D') \to E \to \bigoplus_{i=1}^{r-1} I_{Z_i}(D_i) \to 0.$$

Then, E is a vector bundle if and only if, for every i, the 0-dimensional subschemes Z_i satisfies the Cayley-Bacharach property with respect to the linear system $|D_i - D' + K_X|$.

Proof. See [43, Proposition 2.2]. □

Remark 1.1.20. If, for every i, Z_i satisfies the Cayley-Bacharach property with respect to the linear system $|D_i - D' + K_X|$, then E is a vector bundle, which in particular implies that the extension $e = (e_1, \dots, e_{r-1})$ is non-trivial. Moreover, from the proof of [43, Proposition 2.2], one can assume that each $e_i \in \mathrm{Ext}^1(I_{Z_i}(D_i), \mathcal{O}_X(D))$ is such that $e_i \neq 0$.

We will end this section by recalling that, under certain conditions, the family that parametrizes rank 2 vector bundles on X given by a non-trivial extension is an algebraic variety.

In the case when X is a smooth projective surface, we have the following result.

Proposition 1.1.21. *Let $\mathcal{F}$ be the family of rank 2 vector bundles E on a smooth projective surface X, given by a non-trivial extension of type*

$$0 \to \mathcal{O}_X(D) \to E \to I_Z(c_1 - D) \to 0,$$

where $Z \subset X$ is a 0-dimensional subscheme of length $|Z| > 0$. If $\pm(-2D + c_1)$ and $-2D + c_1 + K_X$ are non-effective divisors, then $\mathcal{F}$ is an irreducible rational quasi-projective variety.

Proof. The idea is to construct a locally free sheaf $\mathcal{E}$ that is in bijection with $\mathcal{F}$. Let us consider the scheme of Hilbert $\mathcal{H}ilb^l(X)$ parametrizing subschemes of X of length l and consider $X \times \mathcal{H}ilb^l$ with the natural projections π and p_X onto $\mathcal{H}ilb^l(X)$ and X, respectively. Let us set $G_1 = p_X^*(D)$ and $G_2 = I_Z \otimes p_X^*(c_1 - D)$, and let $\mathrm{Ext}^1_\pi(G_2, \cdot)$ be the right derived functor of $\pi_* \mathcal{H}om(G_2, -)$. Now, if we apply [61, Corollary 11.44] to the functors π_* and $\mathcal{H}om(G_2, -)$, we get the long exact sequence

$$\begin{array}{lll} \to R^1\pi_*(\mathcal{H}om(G_2, G_1)) & \to R^1(\pi_*\mathcal{H}om(G_2, \cdot))(G_1) & \to \pi_*(R^1\mathcal{H}om(G_2, G_1)) \\ \to R^2\pi_*(\mathcal{H}om(G_2, G_1)) & \to R^2(\pi_*\mathcal{H}om(G_2, \cdot))(G_1) & \to \cdots . \end{array}$$

By the base change theorem, arguing as in [27, Lemma 3.2], we get that $\mathrm{Ext}^1_\pi(G_2, G_1)$ is in fact a locally free sheaf of rank $r = \mathrm{ext}^1(I_Z(c_1 - D), \mathcal{O}_X(D))$ and there is a natural bijective morphism

$$\phi : \mathbb{P}(\mathrm{Ext}^1_\pi(G_2, G_1)) \to \mathcal{F}.$$

Therefore, $\mathcal{F}$ is an irreducible rational quasi-projective variety. In particular, if $|Z| = 0$, $\mathcal{F}$ is in bijection with

$$\mathbb{P}(\mathrm{Ext}^1(\mathcal{O}_X(c_1 - D), \mathcal{O}_X(D))) \cong \mathbb{P}(\mathrm{H}^1(\mathcal{O}_X(2D - c_1))) \cong \mathbb{P}^d,$$

where $d = \mathrm{ext}^1(\mathcal{O}_X(c_1 - D), \mathcal{O}_X(D)) - 1 = \mathrm{h}^1\, \mathcal{O}_X(2D - c_1) - 1.$ □

For a smooth projective variety X of dimension $n \geq 2$, we have the following result.

Proposition 1.1.22. *Let $\mathcal{F}$ be the family of rank 2 vector bundles E on a smooth projective variety X, given by a non-trivial extension of type*

$$0 \to \mathcal{O}_X(D) \to E \to \mathcal{O}_X(c_1 - D) \to 0,$$

where $D, D' \in \mathrm{Pic}(X)$. Then, $\mathcal{F} \cong \mathbb{P}^d$ with $d = \mathrm{h}^1\, \mathcal{O}_X(2D - c_1) - 1$.

Proof. We have $\mathcal{F} \cong \mathbb{P}(\mathrm{Ext}^1(\mathcal{O}_X(c_1 - D), \mathcal{O}_X(D))) \cong \mathbb{P}^d$ with

$$d = \mathrm{ext}^1(\mathcal{O}_X(c_1 - D), \mathcal{O}_X(D)) - 1 = \mathrm{h}^1\, \mathcal{O}_X(2D - c_1) - 1,$$

which concludes the proof. □

1.2 Moduli spaces and Brill-Noether Theory

Classification problems are one of the main topics studied within Algebraic Geometry. Their goal is to classify the different classes of certain algebraic objects. To do so, the idea is to construct a space with a well-behaved algebraic structure that parametrizes these objects up to a certain equivalence relation.

In our case, we are interested in classifying the isomorphism classes of vector bundles on a smooth projective variety X with certain fixed invariants. Since the set of all vector bundles on X is unwieldy, it does not form an algebraic variety and, in order to endow it with an algebraic structure, we introduce a notion of stability.

In the literature, there are several notions of stability that allow one to equip the set of vector bundles with an algebraic structure. In this work, we will consider the slope stability, also known as Mumford-Takemoto stability. Mumford first introduced this notion in the 1960's for higher torsion-free sheaves on smooth curves (see [52]). Later, in the early 1970's, Takemoto generalized this definition to higher rank vector bundles on smooth projective varieties of arbitrary dimension (see [62] and [63]).

Definition 1.2.1. Let X be a smooth projective variety of dimension n, H an ample divisor on X and E a rank r vector bundle on X with fixed Chern classes $c_i(E)$ for $1 \le i \le \min\{r, n\}$. We say that E is stable with respect to H (or H-stable) if, for any subbundle F of E such that $0 < \operatorname{rank}(F) < \operatorname{rank}(E)$, we have

$$\mu_H(F) = \frac{c_1(F) \cdot H^{n-1}}{\operatorname{rank}(F)} < \frac{c_1(E) \cdot H^{n-1}}{\operatorname{rank}(E)} = \mu_H(E).$$

The following properties can be deduced from the definition.

Lemma 1.2.2. *Let X be a smooth projective variety of dimension n, H an ample divisor on X and E a rank r vector bundle on X. Then,*

i) E is H-stable if and only if there exists a line bundle $\mathcal{L}$ such that $E \otimes \mathcal{L}$ is H-stable, if and only if $E \otimes \mathcal{L}$ is H-stable for any line bundle $\mathcal{L}$.

ii) E is H-stable if and only if its dual E^* is H-stable.

iii) E is H-stable if and only if, for any vector subbundle F of E such that $0 < \operatorname{rank}(F) < \operatorname{rank}(E)$ and the quotient E/F is torsion-free, we have $\mu_H(F) < \mu_H(E)$.

iv) E is H-stable if and only if, for all torsion-free quotients Q of E such that $0 < \operatorname{rank}(Q) < \operatorname{rank}(E)$, we have $\mu_H(Q) > \mu_H(E)$.

v) If E is H-stable, then it is simple, that is, $\operatorname{Hom}(E, E) \cong K$.

Proof. See [25, Lemma 5, Chapter 4]. □

The space parametrizing H-stable rank r vector bundles E on X with fixed Chern classes $c_i := c_i(E)$, $i = 1, \dots, s := \min\{r, n\}$, is a moduli space. Its existence was established by Maruyama in the late 1970's (see [45] and [46]). More precisely,

Theorem 1.2.3. *There exists a scheme $M_{X,H}(r; c_1, c_2, \dots, c_s)$ that is a moduli space of H-stable vector bundles on X with Chern classes $c_1, \dots, c_s$, up to numerical equivalence.*

Proof. See [45, Theorem 5.6]. □

Notation 1.2.4. If there is no confusion, we will denote the moduli space $M_{X,H}(r; c_1, \ldots, c_s)$ by $M_H(r; c_1, \ldots, c_s)$, or simply by M_H.

The following result gives a description of the tangent space of the moduli space.

Theorem 1.2.5. *Let X be a smooth projective variety of dimension n and let $[E] \in M_{X,H}(r; c_1, \ldots, c_{\min\{r,n\}})$. Then, the Zarisky tangent space of the moduli space $M_{X,H}(r; c_1, \ldots, c_{\min\{r,n\}})$ at $[E]$ is canonically given by*

$$T_{[E]}M_{X,H}(r; c_1, \ldots, c_{\min\{r,n\}}) \cong \mathrm{Ext}^1(E, E).$$

If $\mathrm{Ext}^2(E, E) = 0$, then $M_{X,H}(r; c_1, \ldots, c_{\min\{r,n\}})$ is smooth at $[E]$. In general, we have the bound

$$\mathrm{ext}^1(E, E) \geq \dim(T_{[E]}M_{X,H}(r; c_1, \ldots, c_{\min\{r,n\}})) \geq \mathrm{ext}^1(E, E) - \mathrm{ext}^2(E, E).$$

Proof. See, for instance, [35, Corollary 4.5.2]). □

Moreover, in the particular case of vector bundles on a smooth projective surface X, we have a general result concerning the irreducibility, smoothness and dimension of the moduli space $M_H(r; c_1, c_2)$.

Theorem 1.2.6. *Let X be smooth projective surface, H an ample divisor on X and $c_1, c_2 \in \mathrm{H}^{2i}(X, \mathbb{Z})$ Chern classes. If $c_2 >> 0$, then the moduli space $M_H(r; c_1, c_2)$ is a generically smooth irreducible quasi-projective variety of dimension*

$$2rc_2 - (r-1)c_1^2 - (r^2-1)\chi(\mathcal{O}_X).$$

Proof. See [26] and [56]. □

Also, in the case in which X is a smooth projective surface, one can give sufficient conditions for the moduli space to be fine, that is, for it to admit a universal family: a bundle $\mathcal{U}$ over $X \times M_H(r; c_1, c_2)$ such that, for every $[E] \in M_H(r; c_1, c_2)$, its fiber $\mathcal{U}_{|X \times [E]}$ corresponds exactly to the bundle E.

Theorem 1.2.7. *Let X be a smooth projective surface, H be an ample divisor on X, $r \geq 1$ be an integer and c_1 and c_2 Chern classes. Let $t(c_1)$ be the greatest integer such that c_1 is divisible by $t(c_1)$ in $\mathrm{Pic}(X)$. Then, if*

$$\gcd(r, t(c_1), \chi(r; c_1, c_2)) = 1,$$

the moduli space $M_H(r; c_1, c_2)$ has a universal family.

Proof. See [22, Theoreme D]. □

In Chapter 2, we will use the following technical lemma.

Lemma 1.2.8. *Let X be a smooth projective surface and $E \in M_H(r; c_1, c_2)$ with $(c_1(E) - rK_X) \cdot H \geq 0$. Then,* $\mathrm{h}^2 E = 0$.

Proof. Let us assume that $\mathrm{h}^2 E > 0$. By Serre duality, this implies that there exists a non-zero section $s \in \mathrm{H}^0 E^*(K_X)$, which yields an inclusion $\mathcal{O}_X(-K_X) \hookrightarrow E^*$. Since E is H-stable, the same is true for E^*, and thus we have

$$\mu_H(\mathcal{O}_X(-K_X)) < \mu_H(E^*),$$

which is equivalent to

$$(-rK_X - c_1(E^*)) \cdot H = (-rK_X + c_1(E)) \cdot H < 0,$$

a contradiction. Hence, $\mathrm{h}^2 E = 0$. □

Notice that the definition of stability strongly depends on the fixed ample divisor H. Thus, a natural question that arises is how different are the moduli spaces $M_H(r; c_1, \ldots, c_s)$ and $M'_H(r; c_1, \ldots, c_s)$ when one considers two different ample divisors H and H'. The study of how moduli spaces vary when we change the ample divisor is related to the theory of walls and chambers. For instance, Qin introduced one of these theories for H-stable rank 2 bundles on projective varieties in [59]. We will discuss this topic in detail in Chapters 4 and 5.

Once the construction of an algebraic variety that parametrizes the set of H-stable bundles with fixed rank and Chern classes is given, our goal is to study its geometry. One way to face this is to describe some of its subvarieties. In particular, one can study the subvarieties called Brill-Noether loci.

Classical Brill-Noether theory deals with line bundles on smooth curves. It was introduced by Brill and Noether by the end of the 19th century in [11]. The main results in the classical case can be found, for instance, in [2, Chapter V]. If C is a smooth curve and $\mathrm{Pic}^d(C)$ is the variety parametrizing line bundles on C of degree d, the Brill-Noether locus is defined as

$$W_d^k(C) = \{L \in \mathrm{Pic}^d(C) \mid \mathrm{h}^0(C, L) \geq k+1\}.$$

Questions related to aspects such as the non-emptiness, connectedness, irreducibility, smoothness, or dimension of the Brill-Noether locus, among others, have been answered when the curve C is generic on the moduli space of curves of genus g.

Once these Brill-Noether loci were well understood, it was natural to look for generalizations. One way to generalize the Brill-Noether locus is to consider, instead of line bundles, higher rank bundles on smooth curves. During the last two decades, a great amount of work has been made around the stratification of the moduli space of stable rank r vector bundles of degree d on algebraic curves in terms of the Brill-Noether loci, giving rise to nice and interesting descriptions of these subvarieties (see, for instance [7], [9], [10], [12], [30], [36], [38], [68]). The other way of generalizing this theory is by considering line bundles on varieties of arbitrary dimension, which corresponds to the study of its Picard group. Eventually, one can go in both directions and consider the following sets.

Definition 1.2.9. Let X be a smooth projective variety of dimension n and H an ample divisor on X. The Brill-Noether locus $W_H^k := W_H^k(r; c_1, c_2, \dots, c_s)$, is the subvariety of the moduli space $M_H(r; c_1, c_2, \dots, c_s)$ which parametrizes vector bundles $E \in M_H$ having at least k independent sections, that is,

$$\operatorname{Supp}(W_H^k(r; c_1, c_2, \dots, c_s)) = \{E \in M_H(r; c_1, c_2, \dots, c_s) \mid \mathrm{h}^0(E) \geq k\}.$$

The existence of these subvarieties was first given in [18, Theorem 2.3]. The result states that, under certain cohomological conditions, the Brill-Noether locus has the structure of a determinantal variety. Let us recall the construction.

A determinantal variety is an algebraic variety (actually a scheme) defined by minors of a matrix representing a morphism of vector bundles. Let E and F be vector bundles on X of ranks r and s, respectively, and $\phi : E \to F$ a morphism. Then, if we choose a trivialization of E and F over an open set $U \subset X$, since E and F are locally free sheaves, the morphism ϕ is represented by an $s \times r$ matrix A of regular functions. The $(k+1) \times (k+1)$ minors of A define an open subset $U_k \subset U$. It is proven that U_k does not depend on the choice of trivialization and, thus, there exists an algebraic subvariety of X, called $X_k(\phi)$, such that $U_k = U \cap X_k(\phi)$ for every open set $U \subset X$ and which is supported on the set

$$\{p \in X \mid \operatorname{rank}(\phi_p) \leq k\}.$$

The variety $X_k(\phi)$, is the k-th determinantal variety, also known as the k-th degeneracy locus of ϕ.

Theorem 1.2.10. *Let X be a smooth projective variety of dimension n and consider the moduli space $M_H = M_{X,H}(r; c_1, c_2, \dots, c_s)$ of rank r, H-stable vector bundles E on X with Chern classes c_i, $1 \leq i \leq \min\{r, n\} := s$. Assume* *that, for any* $E \in M_H(r; c_1, c_2, \ldots, c_s)$, $\mathrm{h}^i\, E = 0$ *for* $i \geq 2$. *Then, for any* $k \geq 0$, *there exists a determinantal variety* $W_H^k = W_H^k(r; c_1, c_2, \ldots, c_s)$, *such that*

$$\mathrm{Supp}(W_H^k(r; c_1, c_2, \ldots, c_s)) = \{E \in M_H \mid \mathrm{h}^0\, E \geq k\}.$$

Moreover, each non-empty irreducible component of $W_H^k(r; c_1, c_2, \ldots, c_s)$ *has dimension at least*

$$\dim(M_H(r; c_1, c_2, \ldots, c_s)) - k(k - \chi(r; c_1, c_2, \ldots, c_s)),$$

and

$$W_H^{k+1}(r; c_1, c_2, \ldots, c_s) \subset \mathrm{Sing}(W_H^k(r; c_1, c_2, \ldots, c_s)),$$

whenever

$$W_H^k(r; c_1, c_2, \ldots, c_s) \neq M_H(r; c_1, c_2, \ldots, c_s).$$

Proof. Let us give a sketch of the proof. First of all, assume that M_H is a fine moduli space and, thus, there exists a universal family $p : \mathcal{U} \to X \times M_H$ such that, for any $t \in M_H$, $\mathcal{U}_{X \times \{t\}} = E_t \in M_H$. Let D be an effective divisor on X such that, for any $t \in M_H$, $\mathrm{h}^i\, E_t(D) = 0$ and set $\mathcal{D} = D \times M_H$. Then, using the base change theorem and applying the functor p_* to the exact sequence

$$0 \to \mathcal{U} \to \mathcal{U}(D) \to \mathcal{U}(D)/\mathcal{U} \to 0,$$

we get a morphism of bundles

$$\phi : p_*(\mathcal{U}(\mathcal{D})) \to p_*(\mathcal{U}(\mathcal{D})/\mathcal{U}).$$

This morphism defines a determinantal variety whose support is exactly the set

$$\{E_t \in M_H \mid \mathrm{h}^0\, E_t \geq k\}.$$

In the case that M_H is not a fine moduli space, the proof is given locally, by carrying out the construction just described. □

Remark 1.2.11. By construction, the Brill-Noether loci define a filtration

$$M_H \supseteq W_H^1 \supseteq W_H^2 \supseteq \cdots \supseteq W_H^k \supseteq W_H^{k+1} \supseteq \cdots$$

of the moduli space $M_H = M_H(r; c_1, c_2, \ldots, c_s)$.

Definition 1.2.12. The expected dimension of the Brill-Noether locus is defined as

$$\rho_H^k(r; c_1, c_2, \ldots, c_s) = \dim(M_H(r; c_1, c_2, \ldots, c_s)) - k[k - \chi(r; c_1, c_2, \ldots, c_s)].$$

If there is no confusion, we will simply denote it by ρ_H^k.

Remark 1.2.13. By the Riemann-Roch theorem and Theorem 1.2.6, if X is a smooth projective surface, we can explicitly compute the expected dimension. Its value is expressed as

$$2rc_2 - (r-1)c_1^2 - (r^2-1)\chi\mathcal{O}_X - k \cdot \left(k - [r(1+p_a(X)) - \frac{c_1 K_X}{2} + \frac{c_1^2}{2} - c_2]\right).$$

Recently, in [55], an alternative proof for endowing the Brill-Noether locus with a scheme structure was given. What is significant is that the results in [55] avoid the cohomological assuptions needed in [18] and, hence, one can always ensure the existence of these loci for varieties of arbitrary dimension.

The given statement is the following.

Theorem 1.2.14. *Let S and T be locally noetherian schemes and consider*

$$\mathcal{E}^{\cdot} : \mathcal{E}_0 \stackrel{\delta_0}{\to} \mathcal{E}_1 \stackrel{\delta_1}{\to} \mathcal{E}_2 \stackrel{\delta_2}{\to} \cdots$$

a complex of bundles on T. Denote by $r_0 := \operatorname{rank}(\mathcal{E}_0)$. Then, the ideals generated by the $(r_0-k)\times(r_0-k)$ minors of δ^0 give a determinantal scheme structure to the set

$$\{t \in T \mid \mathrm{h}^0(S_t, \mathcal{F}_t) \geq k+1\}.$$

Proof. Let us sketch the proof. In general, if $f : S \to T$ is a proper morphism of locally noetherian schemes and $\mathcal{F}$ is a coherent sheaf on S that is flat over T, it follows from semicontinuity (see [31, Chapter III, Section 12.8]) that the set

$$\{t \in T \mid \mathrm{h}^0(S_t, \mathcal{F}_t) \geq k+1\}$$

is closed in T. By [51], we may choose a bounded complex of locally free sheaves on T that is quasi-isomorphic to the pushforward of the Čech complex of $\mathcal{F}$ with respect to an affine covering of S. In addition, this complex would satisfy $\mathrm{H}^0(S_t, \mathcal{F}_t) = \ker(\delta_t^0)$ and therefore, the ideals $I_{r_0-k}(\delta^0)$ generated by the $(r_0-k)\times(r_0-k)$ minors of $\delta^0 : \mathcal{E}_0 \to \mathcal{E}_1$ define the set

$$\{t \in T \mid \mathrm{h}^0(S_t, \mathcal{F}_t) \geq k+1\}.$$

It is proved that the ideals $I_{r_0-k}(\delta^0)$ do not depend of the choice of complex and that they glue, which gives rise to the sheme structure of the set

$$\{t \in T \mid \mathrm{h}^0(S_t, \mathcal{F}_t) \geq k+1\}.$$

For the complete proof, see [55, Theorem 5.2]. □

Corollary 1.2.15. *Let X be a smooth projective variety and consider the moduli space $M_H = M_H(r; c_1, \dots, c_s)$. Then, the set*

$$\{E \in M_H \mid \mathrm{h}^0(X, E) \geq k\}$$

has a determinantal scheme structure.

Proof. It follows from Theorem 1.2.14. □

Once the existence of Brill-Noether locus is defined, it is natural to consider the following questions.

Question 1.2.16.

i) Can we determine the emptiness and non-emptiness of $W_H^k(r; c_1, \dots, c_s)$?

ii) Does $\rho_H^k(r; c_1, \dots, c_s) \geq 0$ imply that $W_H^k(r; c_1, \dots, c_s) \neq \emptyset$?
Does $\rho_H^k(r; c_1, \dots, c_s) < 0$ imply that $W_H^k(r; c_1, \dots, c_s) = \emptyset$?

iii) Is $W_H^k(r; c_1, \dots, c_s)$ smooth or irreducible?

iv) Can we compute the dimension of $W_H^k(r; c_1, \dots, c_s)$?
If so, is it the expected one? That is, is $W_H^k(r; c_1, \dots, c_s)$ of dimension $\rho_H^k(r; c_1, \dots, c_s)$?

If the underlying variety is a smooth curve C of genus g, there are several results that contribute to these questions. For instance, the Brill-Noether locus $W_d^k(C)$ is non-empty and connected whenever $\rho := g-(k+1)(g-d+r) \geq 0$. This was independenly proven by Kemp and Kleiman-Laksov. In the case when C is a generic curve, Griffiths and Harris proved that $\rho < 0$ implies the emptiness of the Brill-Noether locus and Gieseker proved that, if non-empty, $W_d^k(C)$ is smooth of dimension exactly ρ. In addition, if $\rho \geq 1$, Fulton and Lazarsfeld proved that it is irreducible. The compilation of these results is presented in [2, Chapter V]. The next-step case, involving higher rank vector bundles on curves, has been the subject of several contributions in the literature. For instance, we find the work of Teixidor, Newstead, Le Potier and Brambila, among others (see, for instance, [7], [9], [10], [12], [30], [36], [38] and [68]). If the underlying variety has dimension greater than or equal to two, few results are known. For example there are partial results if X is the projective plane $\mathbb{P}^2$, or a $K3$ surface, or an Enriques surface or a ruled surface (the reader may refer to [14], [19], [24], [39], [40], [54] or [44], and references quoted there). Some partial results when the underlying variety has dimension greater than or equal to 3 can be found in [18].

In the next chapters, we will extensively contribute to Question 1.2.16. More precisely, in Chapter 2, we will study the Brill-Noether Theory of stable vector bundles on ruled surfaces, corresponding to the results in [15], and in Chapter 5, we will focus on the case where X is a ruled 3-fold over $\mathbb{P}^2$, as in [17].

1.3 Generalities on ruled varieties

In this work, we will mainly focus on the study of vector bundles on ruled varieties. Hence, we find necessary to end this chapter by reviewing their construction and fixing some notation. We will pay special attention to ruled surfaces over a smooth curve and to ruled 3-fold over $\mathbb{P}^2$. Anyhow, we will start with the general construction.

Let $\mathcal{E}$ be a rank r vector bundle on a smooth projective variety Y. A ruled variety is a projective variety $X = \mathbb{P}(\mathcal{E})$ of dimension $\dim(X) = \dim(Y) + r - 1$, that comes with a natural projection $\pi : X = \mathbb{P}(\mathcal{E}) \to Y$ with the property that, for every point $y \in Y$, the fiber $F := \pi^{-1}(y)$ is isomorphic to $\mathbb{P}^{r-1}$ and it admits a section. Recall that there is a 1-1 correspondence between ruled varieties $\mathbb{P}(\mathcal{E})$ and equivalence classes of rank r vector bundles $\mathcal{E}$ on X under the equivalence $\mathcal{E} \sim \mathcal{E}'$ if and only if $\mathcal{E} \cong \mathcal{E}' \otimes \mathcal{L}$ for some line bundle $\mathcal{L}$ on Y.

We have

$$\operatorname{Pic}(X) \cong \mathbb{Z} \oplus \pi^* \operatorname{Pic}(Y), \tag{1.3}$$

where $\mathbb{Z}$ is generated by $\mathcal{O}_X(1)$, the tautological line bundle on X. We will denote by $S \in A^1(\mathbb{P}(\mathcal{E}))$ the class of the divisor corresponding to $\mathcal{O}_X(1)$. Then, π^* makes $A(\mathbb{P}(\mathcal{E}))$ into a free $A(Y)$-module generated by $1, S, S^2, \dots, S^{r-1}$ with the relation

$$\sum_{i=0}^{r} (-1)^i \pi^* c_i(\mathcal{E}) \cdot S^{r-i} = 0. \tag{1.4}$$

Thus, a divisor on X can be uniquely written as $D = aS + \pi^*\mathfrak{b}$, where $a \in \mathbb{Z}$ and $\mathfrak{b} \in \operatorname{Pic}(Y)$. According to this notation, the canonical divisor of X is $K_X = -rS + \pi^*(c_1(\mathcal{E}) + K_Y)$, where K_Y denotes the canonical divisor on Y.

In addition, we can relate the cohomology of line bundles on X with the cohomology of vector bundles on Y. More precisely, if $a \geq 0$, it follows from Lemma V-2-4, Exercises III.8.3 and III-8.4 of [31] that, for any $i \geq 0$,

$$\mathrm{H}^i(X, \mathcal{O}_X(aS + \pi^*\mathfrak{b})) = \mathrm{H}^i(Y, S^a(\mathcal{E}) \otimes \mathcal{O}_Y(\mathfrak{b})), \tag{1.5}$$

where $S^a(\mathcal{E})$ is the a-th symmetric power of $\mathcal{E}$. Moreover, for $-r < a < 0$ and $i \geq 0$,

$$\mathrm{H}^i(X, \mathcal{O}_X(aS + \pi^*\mathfrak{b})) = 0, \tag{1.6}$$

and the case $a \leq -r$ follows from Serre duality.

Now, let us introduce a characterization of ample divisors on ruled varieties.

Lemma 1.3.1. *Let $\mathcal{E}$ be a rank r vector bundle on a projective variety Y and $X = \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} Y$. Then, a divisor $D = \alpha S + \pi^*\mathfrak{b}$ on X is ample if and only if, for any irreducible curve $C \subset Y$,*

$$\alpha > 0 \quad \textit{and} \quad \alpha \cdot \gamma_{\min}(\mathcal{E}_{|C}) + \beta > 0,$$

where $\beta = \deg(\mathfrak{b})$ and

$$\gamma_{\min}(\mathcal{E}_{|C}) := \min\{\deg(Q) \mid \mathcal{E}_{|C} \to Q \to 0\}.$$

Proof. See [49, Theorem 3.1]. □

In this last part of the chapter, we will study in more detail the intersection product and the cohomology of line bundles on the ruled varieties we will work on in the next chapters: ruled surfaces over a non-singular curve C and ruled 3-folds over $\mathbb{P}^2$. Let us start with ruled surfaces over a smooth curve.

1.3.1 Ruled surfaces over a smooth curve

Let $Y = C$ be a non-singular curve of genus $g \geq 0$ and let $\mathcal{E}$ be a normalized rank 2 vector bundle on C, that is, a rank 2 vector bundle on C such that $\mathrm{h}^0\,\mathcal{E} \neq 0$ but $\mathrm{h}^0(\mathcal{E} \otimes \mathcal{L}) = 0$ for any line bundle $\mathcal{L}$ on C of negative degree. Let us consider the ruled surface $X = \mathbb{P}(\mathcal{E})$, which is a smooth projective surface endowed with a natural projection $\pi : X = \mathbb{P}(\mathcal{E}) \to C$ over C.

In this context, we will denote by f the class of the generic fiber of π and, if $\mathfrak{b}$ is a divisor on C, we will write $\mathfrak{b}f$ instead of $\pi^*\mathfrak{b}$. Thus, following (1.3), a divisor D on X can be uniquely written as $D = aS + \mathfrak{b}f$, being $\mathfrak{b} \in \mathrm{Pic}(C)$.

Notation 1.3.2. By abuse of notation, we will denote the tensor product $\mathcal{O}_X(aS + \mathfrak{b}f) \otimes \mathcal{O}_X(f)$ by $\mathcal{O}_X(aS + (\mathfrak{b}+1)f)$.

It is well known that $p_a(X) = -g$. Additionaly, since we are assuming that $\mathcal{E}$ is normalized, if $\mathfrak{e}$ is the divisor on C corresponding to the invertible sheaf $\wedge^2\mathcal{E}$, then $e := -\deg(\mathfrak{e})$ is an invariant of X. As a consequence, since $f \cong \mathbb{P}^1$ and according to (1.4), $\mathrm{Pic}(X) \cong \mathbb{Z} \oplus \pi^*\,\mathrm{Pic}(C)$ implies that $\mathrm{Num}(X) \cong \mathbb{Z} \oplus \mathbb{Z}$ is generated by S and f, with intersection product

$$S^2 = -e, \quad S \cdot f = 1 \quad \text{and} \quad f^2 = 0.$$

Hence, if $D = aS + \mathfrak{b}f$ is a divisor on X, its numerical class can be written as $D \equiv aS + bf \in \mathrm{Num}(X)$, where $b = \deg(\mathfrak{b})$. In addition, the canonical divisor is $K_X = -2S + (K_C + \mathfrak{e})f$, where K_C is the canonical divisor on C of degree $2g-2$, and its class is $K_X \equiv -2S + (2g-2-e)f$. To simplify, we will denote the canonical divisor by $K_X = -2S - \bar{\mathfrak{e}}f$ and its class by $K_X \equiv -2S - \bar{e}f$, where $\bar{\mathfrak{e}} \in \mathrm{Pic}(C)$ of degree $\bar{e} := e - 2(g-1)$.

In the particular case where $C = \mathbb{P}^1$, (i.e. X is a Hirzebruch surface), we get $\mathrm{Pic}(X) \cong \mathrm{Num}(X)$ and, therefore, we will not make any distinction between a divisor $D = aS + \mathfrak{b}f$ and its class $D \equiv aS + bf$.

We will study the case when $e \geq 0$ and thus, from now on, let us assume that we are under this condition.

Since $\mathcal{E}$ is a rank 2 normalized vector bundle on C, for any divisor $\mathfrak{d}$ on C, we have

$$\mathrm{h}^0(C, \mathcal{O}_C(\mathfrak{d})) \leq \mathrm{h}^0(C, (S^t\mathcal{E})(\mathfrak{d})) \leq \sum_{i=0}^{t} \mathrm{h}^0(C, \mathcal{O}_C(\mathfrak{d} + i\mathfrak{e})) \qquad (1.7)$$

(see, for instance, [1, Section 2]).

Using these inequalities, we can give a bound on the dimension of the space of global sections of a line bundle on X.

Lemma 1.3.3. *Let X be a ruled surface over a non-singular curve C of genus $g \geq 0$. Assume that $e \geq 0$. Let $D = aS + \mathfrak{b}f$ be a divisor on X such that $a \geq 0$ and $b := \deg(\mathfrak{b})$. Then,*

$$h^0\mathcal{O}_X(aS + \mathfrak{b}f) \leq \begin{cases} (a+1)(b+1) & \textit{for } 0 \leq b \leq g-1,\ g \geq 1 \\ (a+1)g & \textit{for } g \leq b \leq 2g-1,\ g \geq 1 \\ (a+1)(b+1-g) & \textit{for } b \geq 2g. \end{cases}$$

Proof. Let $D = aS + \mathfrak{b}f$ be a divisor on X with $a \geq 0$. First, let us consider the case $0 \leq b \leq g-1$. Since $a \geq 0$, by (1.5) and (1.7) we have

$$\mathrm{h}^0\, \mathcal{O}_X(aS + \mathfrak{b}f) = \mathrm{h}^0\, S^a\mathcal{E}(\mathfrak{b}) \leq \sum_{i=0}^{a} \mathrm{h}^0\, \mathcal{O}_C(\mathfrak{b} + i\mathfrak{e}).$$

By [31, Chapter IV, Exercise 1.5], for any effective divisor $\mathfrak{d}$ on C,

$$\mathrm{h}^0\, \mathcal{O}_C(\mathfrak{d}) \leq \deg(\mathfrak{d}) + 1. \qquad (1.8)$$

In particular, $\mathrm{h}^0\,\mathcal{O}_C(\mathfrak{b}+i\mathfrak{e})=0$ if $\mathfrak{b}+i\mathfrak{e}$ is non-effective or, otherwise, we have

$$\mathrm{h}^0\,\mathcal{O}_C(\mathfrak{b}+i\mathfrak{e})\leq b-ie+1\leq b+1.$$

Thus,

$$\mathrm{h}^0\,\mathcal{O}_X(aS+\mathfrak{b}f)\leq (a+1)(b+1).$$

Now, let us assume that $g\leq b\leq 2g-1$. Since $a\geq 0$, again by (1.5) and (1.7),

$$\mathrm{h}^0\,\mathcal{O}_X(aS+\mathfrak{b}f)=\mathrm{h}^0\,S^a\mathcal{E}(\mathfrak{b})\leq\sum_{i=0}^{a}\mathrm{h}^0\,\mathcal{O}_C(\mathfrak{b}+i\mathfrak{e})\leq(a+1)\,\mathrm{h}^0\,\mathcal{O}_C(\mathfrak{b})\leq(a+1)g,$$

where the last inequality follows from the fact that, since $b\leq 2g-1$, if $\mathfrak{d}$ is a divisor of degree $2g-1$, we have

$$\mathrm{h}^0\,\mathcal{O}_C(\mathfrak{b})\leq\mathrm{h}^0\,\mathcal{O}_C(\mathfrak{d})=g.$$

Finally, let us prove the case $b\geq 2g$. Since $a\geq 0$, by (1.5) and (1.7),

$$\begin{aligned}\mathrm{h}^0\,\mathcal{O}_X(aS+\mathfrak{b}f)\quad&=\mathrm{h}^0\,S^a\mathcal{E}(\mathfrak{b})\quad\leq\textstyle\sum_{i=0}^{a}\mathrm{h}^0\,\mathcal{O}_C(\mathfrak{b}+i\mathfrak{e})\quad\leq(a+1)\,\mathrm{h}^0\,\mathcal{O}_C(\mathfrak{b})\\&=(a+1)(b+1-g),\end{aligned}$$

where the last equality follows from the fact that, since $b\geq 2g>2g-1$, we have $\mathrm{h}^0\,\mathcal{O}_C(\mathfrak{b})=b+1-g$. $\square$

Recall that, by [31, Corollary V.2.18], if $e\geq 0$ and $D=aS+\mathfrak{b}f$ is a divisor, D is very ample if and only if D is ample if and only if $a>0$ and $\deg(\mathfrak{b})>ae$.

Now, let us give a description of effective divisors that we will use later on.

Lemma 1.3.4. *Let X be a ruled surface over a non-singular curve C of genus $g\geq 0$. Assume that $e\geq 0$. Let $D=aS+\mathfrak{b}f$ be a divisor on X with $b:=\deg(\mathfrak{b})$. If D is effective, then $a,b\geq 0$.*

Proof. Let us assume that D is effective. If $D=0$ there is nothing to prove. So let us assume that $D=aS+\mathfrak{b}f\neq 0$.

First of all, let us see that $a\geq 0$. Since D is an effective divisor, by Theorem 1.1.11, $D\cdot H>0$ for any ample divisor H on X. In particular, taking the ample divisor $H\equiv S+\beta f$ with $\beta>>0$, we have

$$0<D\cdot H=-ae+a\beta+b=a(\beta-e)+b,$$

which implies that $a\geq 0$.

Now, let us see that $b\geq 0$. Since D is effective and $a\geq 0$, it follows from (1.5) and (1.7) that

$$0<\mathrm{h}^0\,\mathcal{O}_X(D)=\mathrm{h}^0\,S^a\mathcal{E}(\mathfrak{b})\leq\sum_{i=0}^{a}\mathrm{h}^0\,\mathcal{O}_C(\mathfrak{b}+i\mathfrak{e}).$$

In particular, since $\deg(\mathfrak{e})=-e\leq 0$, $\mathrm{h}^0\,\mathcal{O}_C(\mathfrak{b})\neq 0$ and hence $b\geq 0$. $\square$

1.3.2 Ruled 3-folds over $\mathbb{P}^2$

In this subsection, we will give a detailed description of the varieties we will work on in Chapter 5. Let $\mathcal{E}$ be a globally generated rank 2 vector bundle over $\mathbb{P}^2$. Denote by $c_i' := c_i(\mathcal{E})$. Notice that, since $\mathcal{E}$ is generated by global sections, $c_1' \geq 0$ and ${c_1'}^2 \geq c_2' \geq 0$. Let us consider $\pi : X = \mathbb{P}(\mathcal{E}) \to \mathbb{P}^2$ the projectivized vector bundle associated to $\mathcal{E}$ with the natural projection π, that is, X is a ruled 3-fold over $\mathbb{P}^2$ which is a smooth projective variety of dimension 3. The tautological line bundle S on X is uniquely determined by the conditions $S_{|F} \cong \mathcal{O}_F(1)$ and $\pi_* S \cong \mathcal{E}$, being F a fiber of π.

Since, by (1.3), $\operatorname{Pic}(X) \cong \mathbb{Z}S \bigoplus \pi^* \operatorname{Pic}(\mathbb{P}^2)$, if we denote by $L := \pi^* h$, being h the hyperplane class of $\mathbb{P}^2$, any divisor D on X can be written as $D = aS + bL$ with $a, b \in \mathbb{Z}$. From this and by (1.4), $\operatorname{Num}(X) \cong \mathbb{Z} \oplus \mathbb{Z}$ is generated by S and L with the intersection product given by

$$L^3 = 0, \quad S \cdot L^2 = 1, \quad S^2 \cdot L = c_1'$$

and

$$0 = \sum_{i=0}^{2} (-1)^i c_i' S^{2-i} \cdot L^i.$$

In addition, from this last relation we deduce that $S^3 = {c_1'}^2 - c_2'$. Notice that a divisor $D = aS + bL$ is numerically equivalent to 0 if and only if $D = 0$. Hence, $\operatorname{Pic}(X) \cong \operatorname{Num}(X)$ and we will make no distinction between a divisor and its numerical class.

By the relation (1.6), we have

$$\mathrm{H}^i(X, \mathcal{O}_X(-S + bL)) = 0 \quad \text{for} \quad i \geq 0, \tag{1.9}$$

and the case $a \leq -2$ follows from Serre duality. Finally, we recall that the canonical divisor is given by $K_X = -2S + (\pi^* c_1' + \pi^* K_{\mathbb{P}^2}) = -2S + (c_1' - 3)L$.

Now, let us study how ample and effective divisors are on these ruled 3-folds.

Lemma 1.3.5. *An element $H = \alpha S + \beta L \in \operatorname{Pic}(X)$ with $\alpha \geq 1$ and $\beta \geq 1$, is an ample divisor on X.*

Proof. Since $\mathcal{E}$ is globally generated, $\alpha S + \pi^* D$ is an ample divisor on X if $\alpha \geq 1$ and D is an ample divisor on $\mathbb{P}^2$, this is, $D = \beta h$ with $\beta > 0$. Hence, $D = \alpha S + \beta L$ with $\alpha \geq 1$ and $\beta \geq 1$ is an ample divisor on X. $\square$

As a consequence of Lemma 1.3.1, for this particular ruled 3-fold, we have the following characterization.

Lemma 1.3.6. *Let $\mathcal{E}$ be a globally generated rank 2 vector bundle on $\mathbb{P}^2$ and $X = \mathbb{P}(\mathcal{E})$. Then, a divisor $H \equiv \alpha S + \beta L$ is ample if and only if, for any line $l \subset \mathbb{P}^2$,*

$$\alpha > 0 \quad \text{and} \quad \alpha \cdot \delta_{\min}(\mathcal{E}) + \beta > 0,$$

where

$$\delta_{\min}(\mathcal{E}) := \min_{l \subset \mathbb{P}^2}\{d \,|\, \mathcal{E}_{|l} \cong \mathcal{O}_{\mathbb{P}^2}(c_1' - d) \oplus \mathcal{O}_{\mathbb{P}^2}(d)\} \geq 0.$$

Proof. First of all notice that, for any irreducible curve $C \subset \mathbb{P}^2$ of degree d and for any vector bundle $\mathcal{F}$ on $\mathbb{P}^2$, $\deg(\mathcal{F}_{|C}) = d \deg(\mathcal{F})$. Thus, to apply Lemma 1.3.1, it is enough to consider lines l. In addition, since $\mathcal{E}$ is globally generated, $\mathcal{E}_{|l}$ is globally generated for every line $l \subset \mathbb{P}^2$, and thus for any quotient Q_l of $\mathcal{E}_{|l}$ we always have $\deg(Q_l) \geq 0$. The rest follows from Lemma 1.3.1. □

The effective divisors have the following description.

Lemma 1.3.7. *Let $X = \mathbb{P}(\mathcal{E})$ be a ruled 3-fold over $\mathbb{P}^2$, where $\mathcal{E}$ is a globally generated bundle on $\mathbb{P}^2$. Let $D = aS + bL$ be a divisor on X. If D is effective, then $D = 0$, $D = aS + bL$ with $a > 0$ or $D = bL$ with $b > 0$. In addition, if $\mathcal{E}$ is normalized and $a = 1$, then $b \geq 0$.*

Proof. Let us assume that D is effective. If $D = 0$ there is nothing to prove. So, let let us assume that $D = aS + bL \neq 0$. First of all, we will see that $a \geq 0$. Since D is an effective divisor, $D \cdot H > 0$ for any ample divisor H on X. In particular, taking the ample divisor $H = S + \beta L$ with $\beta >> 0$,

$$0 < D \cdot H = a\beta^2 + 2\beta(ac_1' + b) + a(c_1'^2 - c_2')$$

implies that we have two possibilities: either $a > 0$ or $a = 0$ and $b > 0$. Finally, if $\mathcal{E}$ is normalized,

$$\mathrm{h}^0\, \mathcal{O}_X(S + bL) = \mathrm{h}^0\, \mathcal{E}(b) = 0$$

whenever $b < 0$, and thus we get $b \geq 0$. □

We end this subsection with a technical result that we will use in Chapter 5 to study moduli spaces of rank 2 stable vector bundles on these ruled 3-folds.

Lemma 1.3.8. *Let $X = \mathbb{P}(\mathcal{E})$ be a ruled 3-fold over $\mathbb{P}^2$, where $\mathcal{E}$ is a globally generated bundle on $\mathbb{P}^2$.*

i) If $[Z] = SL$, then $\bigwedge^2 \mathcal{N}_{Z/X} \cong \mathcal{O}_X(S + L)_{|Z}$.

ii) If $\mathcal{G}$ is the scheme parametrizing codimension 2 complete intersections Z of X of class $[Z] = SL$, then

$$\dim(\mathcal{G}) = \mathrm{h}^0\,\mathcal{E} - \mathrm{h}^0\,\mathcal{E}(-1) + 1.$$

Proof. i) Since $[Z] = SL$ is a complete intersection, we have the following exact sequence

$$0 \to \mathcal{O}_X(-S-L) \to \mathcal{O}_X(-S) \oplus \mathcal{O}_X(-L) \to I_Z \to 0. \tag{1.10}$$

On the other hand, the dual of the normal bundle, $(\mathcal{N}_{Z/X})^*$, is isomorphic to $I_Z/(I_Z)^2 \cong I_Z \otimes \mathcal{O}_Z$. Twisting the exact sequence (1.10) by $\mathcal{O}_Z$, we get $I_Z \otimes \mathcal{O}_Z \cong \mathcal{O}_Z(-S) \oplus \mathcal{O}_Z(-L)$ and thus

$$\bigwedge^2 \mathcal{N}_{Z/X} \cong \mathcal{O}_X(S+L)_{|Z}.$$

ii) Since any Z such that $[Z] = SL$ is given by an exact sequence of type (1.10), we have

$$\begin{aligned}\dim(\mathcal{G}) \quad &= \dim \mathrm{Hom}(\mathcal{O}_X(-S-L), \mathcal{O}_X(-S) \oplus \mathcal{O}_X(-L))) + \dim I_g \\ &- \dim \mathrm{Aut}(\mathcal{O}_X(-S-L)) - \dim \mathrm{Aut}(\mathcal{O}_X(-S) \oplus \mathcal{O}_X(-L)),\end{aligned}$$

where $g \in \mathrm{Hom}(\mathcal{O}_X(-S-L), \mathcal{O}_X(-S) \oplus \mathcal{O}_X(-L))$ is a general element and I_g denotes its isotropy group under the action of

$$\mathrm{Aut}(\mathcal{O}_X(-S-L)) \times \mathrm{Aut}(\mathcal{O}_X(-S) \oplus \mathcal{O}_X(-L)).$$

First of all, notice that, since

$$\mathrm{Hom}(\mathcal{O}_X(-S-L), \mathcal{O}_X(-S) \oplus \mathcal{O}_X(-L)) \cong \mathrm{H}^0\,\mathcal{O}_X(L) \oplus \mathrm{H}^0\,\mathcal{O}_X(S),$$

we have

$$\begin{aligned}\dim(\mathrm{Hom}(\mathcal{O}_X(-S-L), \mathcal{O}_X(-S) \oplus \mathcal{O}_X(-L))) \quad &= \mathrm{h}^0\,\mathcal{O}_X(L) + \mathrm{h}^0\,\mathcal{O}_X(S) \\ &= \mathrm{h}^0\,\mathcal{O}_{\mathbb{P}^2}(1) + \mathrm{h}^0\,\mathcal{E} \\ &= 3 + \mathrm{h}^0\,\mathcal{E}.\end{aligned}$$

On the other hand, since

$$\mathrm{Aut}(\mathcal{O}_X(-S) \oplus \mathcal{O}_X(-L)) \cong \mathrm{H}^0(\mathcal{O}_X^{\oplus 2}) \oplus \mathrm{H}^0\,\mathcal{O}_X(S-L) \oplus \mathrm{H}^0\,\mathcal{O}_X(-S+L),$$

we get

$$\dim \mathrm{Aut}(\mathcal{O}_X(-S) \oplus \mathcal{O}_X(-L)) = 2\,\mathrm{h}^0\,\mathcal{O}_X + \mathrm{h}^0\,\mathcal{O}_X(S-L) = 2 + \mathrm{h}^0\,\mathcal{E}(-1).$$

Finally, $\dim(\mathrm{Aut}(\mathcal{O}_X(-S-L))) = 1$ and $\dim I_g = 1$. Putting altogether, we obtain

$$\dim(\mathcal{G}) = \mathrm{h}^0\,\mathcal{E} - \mathrm{h}^0\,\mathcal{E}(-1) + 1,$$

which finishes the proof. $\square$

Chapter 2

Brill-Noether loci of rank 2 stable bundles on ruled surfaces

The main goal of this chapter is to address several of the problems posed in Question 1.2.16 in the case of rank 2 stable vector bundles on ruled surfaces. The principal results of this chapter are contained in [20].

As we pointed out in Chapter 1, since Maruyama's foundational work on the construction of moduli spaces of stable vector bundles on projective surfaces, these moduli spaces have been the object of intense study from several points of view, including their geometry, topology, and deformation theory. Maruyama's results laid the groundwork for the systematic investigation of moduli spaces of vector bundles beyond the curve case, where the theory had already been well developed through the work of Mumford, Narasimhan–Seshadri, and others.

Over the years, many general structural properties of these moduli spaces have been established. For instance, as we have seen in Chapter 1, under suitable conditions on the Chern classes, moduli spaces of stable vector bundles are irreducible and generically smooth; in such cases, its dimension is given by a Riemann–Roch computation involving the discriminant of the bundle. The number of connected components, the behavior under wall-crossing with respect to the polarization, and the existence of universal families have also been extensively studied. Important contributions in this direction include the works of Gieseker, Li, Qin, and Yoshioka, among many others (see, for instance, [26], [41], [42] and [67]).

A striking aspect of this theory is that the geometry of the moduli space strongly depends on the position of the underlying surface within the Kodaira classification. For example, when X is a K3 surface or an abelian surface, the moduli space of stable sheaves often inherits a rich geometric symplectic structure and it is deformation equivalent to Hilbert schemes of points. This phenomenon illustrates a deep principle: in favorable situations, the geometry of the moduli space reflects and amplifies the intrinsic geometry of the surface itself. For example, moduli spaces of stable bundles on rational surfaces are conjectured to be rational varieties (see, for instance, [6], [8], [20], [21], [23], [47]).

One fruitful approach to understanding the geometry of moduli spaces of vector bundles is through the study of their distinguished subvarieties. These subvarieties often arise from imposing additional cohomological or geometric conditions on the bundles, such as the existence of sections or special behavior upon restriction to curves. From this perspective, moduli spaces are stratified by loci defined by jumping phenomena, degenerations, or extensions, and these strata encode refined geometric information.

This point of view finds its natural conceptual framework in the theory of Brill–Noether filtration. As we explained in Chapter 1, inspired by classical Brill–Noether theory for vector bundles on curves, one studies loci in the moduli space defined by conditions on the dimension of spaces of sections (see Definition 1.2.9). Although these loci are often highly singular and may have unexpected dimensions, they provide a powerful tool for probing the geometry of the ambient moduli space.

Once the general definition is given, it is natural to ask whether the results that hold in the classical case remain true in the generalized one. Consequently, the Brill-Noether theory of stable bundles on surfaces has been studied by several authors. For instance, in the $K3$ case, we have the work of Leyenson in [39] and [40]. For other surfaces such as $\mathbb{P}^2$, Enriques surfaces or ruled surfaces, we may look at [14], [19], [24], [54] or [44].

Now, let us outline the structure of the chapter. First of all, in Section 2.1, following the notation and results introduced in Chapter 1, we will fix our setting. After this, in Section 2.2, we will push forward the study of the non-emptiness of the Brill-Noether locus $W^k_H(2; c_1, c_2)$ for $c_1 \in \{S, S+f\}$ (see Theorems 2.2.2 and 2.2.8). After this, in Section 2.3, we will characterize the non-emptiness of the Brill-Noether locus $W^k_H(2; f, c_2)$ in Theorems 2.3.1 and 2.3.3. Finally, in Section 2.4, we will prove in Theorem 2.4.1 that the locus $W^1_H(2; f, c_2) \backslash W^2_H(2; f, c_2)$ is smooth, irreducible, and of the expected dimension.

2.1 First steps

The notation we will use is the one intruduced in Chapter 1, Section 1.3.1. The setting is the following. We will work on $\pi : X = \mathbb{P}(\mathcal{E}) \to C$ a ruled surface over a smooth curve C of genus $g \geq 0$, when $\mathcal{E}$ is a normalized rank 2 vector bundle on C. We will assume that the invariant e is such that $e \geq 0$.

Let H be an ample divisor on X and $c_i \in \mathrm{H}^{2i}(X, \mathbb{Z})$, for $i = 1, 2$, Chern classes. Our aim is to study the moduli space $M_H(2; c_1, c_2)$ of rank 2 H-stable vector bundles on X by means of its Brill-Noether locus $W^k_H(2; c_1, c_2)$. To do so, let us set which first Chern classes we are interested in.

Notice that, it is convenient to take $c_2 \gg 0$. In fact, if we are under this assumption, by Theorem 1.2.6, we know that the moduli space $M_H(r; c_1, c_2)$ is a generically smooth irreducible quasi-projective variety of dimension

$$2rc_2 - (r-1)c_1^2 - (r^2-1)(1-g).$$

Hence, from now on, we will consider $c_2 \gg 0$.

On the other hand, if $c_1 \cdot H \leq 0$, then $W^k_H(r; c_1, c_2) = \emptyset$ for any $k \geq 1$. In fact, if we have $E \in W^k_H(r; c_1, c_2)$ then $\mathrm{h}^0 E \neq 0$ and thus $\mathcal{O}_X \hookrightarrow E$. By stability of E and since $c_1 \cdot H \leq 0$, we have $0 = \mu_H(\mathcal{O}_X) < \mu_H(E) \leq 0$, which is a contradiction. Therefore, if $c_1 \cdot H < 0$ or $c_1 = 0$, we get $W^1_H(2; c_1, c_2) = \emptyset$. To avoid this situation, from now on we will assume that c_1 is effective since, by Theorem 1.1.11, this implies that $c_1 \cdot H > 0$ for any ample divisor H on X. Furthermore, since, by Proposition 1.2.2, E is H-stable if and only if $E \otimes \mathcal{L}$ is H-stable for any line bundle $\mathcal{L}$, it is natural to consider the normalized first Chern class. Hence, for all these reasons, we will focus our attention on the cases $c_1 \in \{S, S+f, f\}$.

Moreover, if we take the first Chern class as $c_1 \in \{S, S+f, f\}$, it is not very restrictive to require that $(c_1 - 2K_X) \cdot H \geq 0$. Thus, by Lemma 1.2.8, Theorem 1.2.10 guarantees the existence of the Brill-Noether locus $W^k_H(2; c_1, c_2)$. In any case, we can directly get its existence following the results in [55] (see Theorem 1.2.14 and Corollary 1.2.15 in Chapter 1, Section 1.2).

Once the setting if fixed, we are ready to study the non-emptiness of the Brill-Noether loci $W^k_H(2; c_1, c_2)$, for $c_1 \in \{S, S+f, f\}$ and $c_2 \gg 0$ an integer. We will obtain non-emptiness results by explicitly constructing families of H-stable rank 2 vector bundles for certain fixed ample divisors H.

2.2 Non-emptiness of $\mathbf{W_H^k(2; S + \mathfrak{m}f, c_2)}$

We will start with the case $c_1 \in \{S, S + f\}$. Thus, we fix $c_1 = S + \mathfrak{m}f$ with $\mathfrak{m}$ a divisor on C of degree $\deg(\mathfrak{m}) = m \in \{0, 1\}$.

Before stating the first result, let us prove the following lemma, which is a generalization of [60, Lemma 1.10].

Lemma 2.2.1. *Let $H \equiv \alpha S + \beta f$ be an ample divisor on X and c_2 an integer. Let E be a rank 2 H-stable vector bundle with Chern classes $c_1(E) = S + \mathfrak{d}f$ and $c_2(E) = c_2$. Then, $\beta < \alpha(2c_2 + e - d)$.*

Proof. First of all, since every fiber f is isomorphic to $\mathbb{P}^1$, there exists an integer a' such that $E_f \cong \mathcal{O}_{\mathbb{P}^1}(a') \oplus \mathcal{O}_{\mathbb{P}^1}(1 - a')$. We may assume that $a' \geq 1$. Then, the injection $\mathcal{O}_{\mathbb{P}^1}(a') \hookrightarrow E_f$ induces an injection $\mathcal{O}_X(aS + \mathfrak{b}f) \hookrightarrow E$ for some $a \geq 1$ and $\mathfrak{b} \in \mathrm{Pic}(C)$ of degree b, such that $s \in E(-aS - \mathfrak{b}f)$ has a scheme of zeros Z of codimension 2. Therefore, E sits in a short exact sequence of type

$$0 \to \mathcal{O}_X(aS + \mathfrak{b}f) \to E \to I_Z((1 - a)S + (\mathfrak{d} - \mathfrak{b})f) \to 0. \tag{2.1}$$

On one hand, since E is H-stable,

$$(aS + \mathfrak{b}f) \cdot (\alpha S + \beta f) < \frac{1}{2} c_1(E) \cdot (\alpha S + \beta f),$$

which is equivalent to

$$-b > \frac{1}{2\alpha}[(2a - 1)(\beta - \alpha e) - \alpha d]. \tag{2.2}$$

On the other hand, by the exact sequence (2.1), we have

$$\begin{aligned} c_2(E) = c_2 \quad &= |Z| + (aS + bf) \cdot [(1 - a)S + (d - b)f] \\ &= a[(a - 1)e + d] + (-b)(2a - 1) + |Z|. \end{aligned}$$

Using the inequality (2.2), we obtain

$$c_2 > a[(a - 1)e + d] + \frac{(2a - 1)}{2\alpha}[(2a - 1)(\beta - \alpha e) - \alpha d]$$

and thus

$$(2a - 1)[(2a - 1)(\beta - \alpha e) + \alpha d] < \alpha[2c_2 - 2a[(a - 1)e + d]].$$

Finally, since $a \geq 1$, we get

$$\beta < \alpha[2c_2 - 2a[(a - 1)e + d] + e + d] \leq \alpha(2c_2 + e - d),$$

which finishes the proof. $\square$

Theorem 2.2.2. *Let X be a ruled surface over a non-singular curve C of genus $g \geq 0$, $m \in \{0, 1\}$, $c_2 >> 0$ an integer and $H \equiv \alpha S + \beta f$ an ample divisor on X with*

$$\alpha(e + m) < \beta.$$

Then, for any k in the range

$$\max\{1, g\} \leq k < \frac{1}{2\alpha}[\beta - \alpha(e - m + 2g - 2)],$$

the Brill-Noether locus $W^k_H(2; S + \mathfrak{m}f, c_2) \neq \emptyset$.

Proof. First of all notice that, since $c_2 \gg 0$, the moduli space $M_H(2; c_1, c_2)$ is non-empty and thus, by Lemma 2.2.1, we can assume that $\beta < \alpha(2c_2 + e - m)$. Let us consider the family $\mathcal{G}$ of rank two vector bundles E on X given by a non-trivial extension of type

$$0 \to \mathcal{O}_X(\mathfrak{b}f) \to E \to I_Z(S + (\mathfrak{m} - \mathfrak{b})f) \to 0, \tag{2.3}$$

where Z is a generic 0-dimensional subscheme of length $|Z| = c_2 - b$ with

$$b = \deg(\mathfrak{b}) = k - 1 + g$$

and

$$\max\{1, g\} \leq k < \frac{1}{2\alpha}[\beta - \alpha(e - m + 2g - 2)].$$

Claim: $\mathcal{G}$ is non-empty.
Proof of the Claim: Notice that, since the divisor $S + (\mathfrak{m} - 2\mathfrak{b})f + K_X$ is non-effective, Z satisfies the Cayley-Bacharach property with respect to the linear system $|S + (\mathfrak{m} - 2\mathfrak{b})f + K_X|$. Therefore, by Theorem 1.1.17, any E given by an extension of type (2.3) is a rank 2 vector bundle and, by Remark 1.1.18, the extension $e \in \mathrm{Ext}^1(I_Z(S + (\mathfrak{m} - \mathfrak{b})f), \mathcal{O}_X(\mathfrak{b}f))$ is non-trivial. Hence, $\mathcal{G}$ is non-empty.

In addition, by (2.3), we get $c_1(E) = S + \mathfrak{m}f$, $c_2(E) = c_2 >> 0$ and

$$\mathrm{h}^0\, E \geq \mathrm{h}^0\, \mathcal{O}_X(\mathfrak{b}f) = \mathrm{h}^0\, \mathcal{O}_C(\mathfrak{b}).$$

Moreover, since $k \geq \max\{1, g\}$, we have $\mathrm{h}^1\, \mathcal{O}_C(\mathfrak{b}) = 0$ and thus

$$\mathrm{h}^0\, \mathcal{O}_C(\mathfrak{b}) = \chi\mathcal{O}_C(\mathfrak{b}) = b + 1 - g = k.$$

Now, let us prove that any rank two vector bundle E in $\mathcal{G}$ is stable with respect to the polarization $H \equiv \alpha S + \beta f$ with $\alpha(e + m) < \beta < \alpha(2c_2 + e - m)$.

To this end, since E is a rank two vector bundle on X, we have to check that, for any line subbundle $\mathcal{O}_X(G) \hookrightarrow E$, we have

$$\mu_H(\mathcal{O}_X(G)) = G \cdot H < \mu_H(E) = \frac{c_1(E) \cdot H}{2}.$$

By construction, E is given by a non-trivial extension of type (2.3). Therefore, we have two possibilities: $\mathcal{O}_X(G) \hookrightarrow \mathcal{O}_X(\mathfrak{b}f)$ or $\mathcal{O}_X(G) \hookrightarrow I_Z(S + (\mathfrak{m} - \mathfrak{b})f)$.

Case 1: Assume that $\mathcal{O}_X(G) \hookrightarrow \mathcal{O}_X(\mathfrak{b}f)$. In this case, since

$$k < \frac{1}{2\alpha}[\beta - \alpha(e - m + 2g - 2)],$$

we get

$$\begin{aligned} \mu_H(\mathcal{O}_X(G)) &\leq (\mathfrak{b}f) \cdot H = \alpha b = \alpha k + \alpha(-1+g) \\ &< \tfrac{\alpha}{2\alpha}[\beta - \alpha(e - m + 2g - 2)] + \alpha(-1+g) \\ &= \tfrac{1}{2}(\beta - \alpha e + \alpha m) = \mu_H(E). \end{aligned}$$

Case 2: Assume that $\mathcal{O}_X(G) \hookrightarrow I_Z(S + (\mathfrak{m} - \mathfrak{b})f)$, where $G = sS + \mathfrak{t}f$ with $t = \deg(\mathfrak{t})$. In this case,

$$S + (\mathfrak{m} - \mathfrak{b})f - G = (1-s)S + (\mathfrak{m} - \mathfrak{b} - \mathfrak{t})f$$

is an effective divisor, which by Lemma 1.3.4 implies that $s \leq 1$ and $t \leq m - b$. Assume that $s \leq 0$. Since $t \leq m - b$, we have

$$\mu_H(\mathcal{O}_X(G)) = s(\beta - \alpha e) + t\alpha \leq \alpha(m - b) < \frac{1}{2}(\beta - \alpha e + \alpha m) = \mu_H(E),$$

where the last inequality follows from the fact that $b \geq 0$ and $\beta > \alpha(e + m)$. Finally, assume that $s = 1$. In this case, $G = S + \mathfrak{t}f$ and, since

$$\mathcal{O}_X(G) \hookrightarrow I_Z(S + (\mathfrak{m} - \mathfrak{b})f),$$

we have $\mathrm{h}^0\, I_Z(S + (\mathfrak{m} - \mathfrak{b})f - G) = \mathrm{h}^0\, I_Z((\mathfrak{m} - \mathfrak{b} - \mathfrak{t})f) \neq 0$. Since Z is a generic 0-dimensional subscheme, we must have

$$|Z| < \mathrm{h}^0\, \mathcal{O}_X((\mathfrak{m} - \mathfrak{b} - \mathfrak{t})f) = \mathrm{h}^0\, \mathcal{O}_C(\mathfrak{m} - \mathfrak{b} - \mathfrak{t}) \leq m - b - t + 1,$$

where the last inequality is given by (1.8). Thus, since $|Z| = c_2 - b$,

$$t \leq m - c_2. \tag{2.4}$$

Now, let us check that $\mu_H(\mathcal{O}_X(S + \mathfrak{t}f)) < \mu_H(E)$. In fact, notice that

$$\mu_H(\mathcal{O}_X(S + \mathfrak{t}f)) < \mu_H(E)$$

if and only if

$$-\alpha e + \beta + t\alpha < \frac{1}{2}(\beta - \alpha e + \alpha m),$$

which is equivalent to

$$-\alpha e + \beta - \alpha m < -2t\alpha.$$

This inequality holds since, by the upper bound of β, and according to (2.4),

$$-2t\alpha \geq -2\alpha(m - c_2) > -\alpha e + \beta - \alpha m.$$

Therefore $\mu_H(\mathcal{O}_X(S + \mathfrak{t}f)) < \mu_H(E)$ and putting altogether we get that E is H-stable.

We have proved that any E in $\mathcal{G}$ is a rank two H-stable vector bundle with $\mathrm{h}^0 E \geq k$, which implies that $\mathcal{G} \hookrightarrow W_H^k(2; S + \mathfrak{m}f, c_2)$. Hence, the Brill-Noether locus $W_H^k(2; S + \mathfrak{m}f, c_2)$ is non-empty. $\square$

Remark 2.2.3. Notice that the condition $\beta > \alpha(e+m)$ in Theorem 2.2.2 is not restrictive. In fact, it is verified for almost all ample divisors, since $m \in \{0, 1\}$ and $\beta > \alpha e$ for any ample divisor $H \equiv \alpha S + \beta f$.

Corollary 2.2.4. *Let X be a ruled surface over a non-singular curve C of genus $g \geq 0$, $m \in \{0, 1\}$, $c_2 >> 0$ an integer and $H \equiv \alpha S + \beta f$ an ample divisor on X with*

$$\alpha(e + m) < \beta.$$

Then, the Brill-Noether locus $W_H^k(2; S + \mathfrak{m}f, c_2)$ has an irreducible component of dimension at least $3c_2 - b + 2g - 1$.

Proof. Notice that we can compute the dimension of the family $\mathcal{F}$ in the proof of Theorem 2.2.2, which we have seen that is in $W_H^k(2; S + \mathfrak{m}f, c_2)$. Thus, the dimension of the family $\mathcal{F}$ is a lower bound of the dimension of the irreducible component containing it. In fact, by (1.2), we have

$$\mathrm{h}^0 I_Z(S + (\mathfrak{m} - 2\mathfrak{b})f + K_X) = 0 = \mathrm{h}^2 I_Z(S + (\mathfrak{m} - 2\mathfrak{b})f + K_X).$$

Therefore, by Serre duality and the Riemann-Roch theorem,

$$\begin{aligned} \operatorname{ext}^1(I_Z(S + (\mathfrak{m} - \mathfrak{b})f), \mathcal{O}_X(\mathfrak{b}f)) &= \mathrm{h}^1 I_Z(S + (\mathfrak{m} - 2\mathfrak{b})f + K_X) \\ &= |Z| - \chi\mathcal{O}_X(-S + (2\mathfrak{b} - \mathfrak{m})f) \\ &= |Z| = c_2 - b > 0, \end{aligned}$$

and hence $\dim(\mathcal{F}) = 3c_2 - b + 2g - 1$. $\square$

Remark 2.2.5. The bound given in Corollary 2.2.4 is stronger than the expected dimension

$$\rho_H^k(2; S+\mathfrak{m}f, c_2) = (4-k)c_2 + 3k - 3 + 3(1-k)g + (1-k)e + 2(k-1)m - k^2$$

for any $k \geq 2$.

Corollary 2.2.6. *Let X be a ruled surface over a non-singular curve C of genus $g \geq 0$, $m \in \{0,1\}$, $c_2 >> 0$ an integer and $H \equiv \alpha S + \beta f$ an ample divisor on X with*

$$\alpha(e+m) < \beta.$$

Then, for any

$$1 \leq k < \frac{1}{2\alpha}[\beta - \alpha(e-m+2g-2)],$$

$W_H^k(2; S+\mathfrak{m}f, c_2) \neq \emptyset$.

Proof. If $\max\{1,g\} \leq k < \frac{1}{2}\alpha[\beta - \alpha(e+m+2g-2)]$, the result follows from Theorem 2.2.2. Finally, since the Brill-Noether loci form a filtration (see Remark 1.2.11), we have

$$W_H^1 \supseteq W_H^2 \supseteq \cdots \supseteq W_H^q \supseteq \cdots$$

for $q = \max\{1,g\}$. Thus, the fact that $W_H^q \neq \emptyset$, implies that $W_H^l \neq \emptyset$ for any $1 \leq l \leq q$. □

Remark 2.2.7. Notice that, by Theorem 1.2.6 and Definition 1.2.12, the expected dimension of the Brill-Noether locus $W_H^k(2; S+\mathfrak{m}f, c_2)$ is

$$\rho_H^k(2; S+\mathfrak{m}f, c_2) = (4-k)c_2 + 3k - 3 + 3(1-k)g + (1-k)e + 2(k-1)m - k^2.$$

Therefore, for $k \geq 4$ we have $\rho_H^k < 0$ but, for

$$1 \leq g \leq k < \frac{1}{2\alpha}[\beta - \alpha(e-m+2g-2)],$$

we have seen that $W_H^k \neq \emptyset$. Hence, we have an example of a non-empty Brill-Noether locus with negative expected dimension and it is clear that, in general, $\rho_H^k < 0$ does not imply the emptiness of W_H^k, which provides a contribution to Question 1.2.16 *ii*).

So far, we have determined upper bounds of k, depending on an ample divisor H, that guarantee the non-emptiness of $W^k_H(2; S + \mathfrak{m}f, c_2)$. Our next goal is to see that the above bound is sharp, in the sense that, if

$$k \geq \frac{1}{2\alpha}[\beta - \alpha(e - m + 2g - 2)],$$

then $W^k_H(2; S + \mathfrak{m}f, c_2) = \emptyset$.

Theorem 2.2.8. *Let X be a ruled surface over a non-singular curve C of genus $g \geq 0$, $m \in \{0, 1\}$, $c_2 >> 0$ an integer and $H \equiv \alpha S + \beta f$ be an ample divisor on X such that*

$$\max\{\alpha(8 + e - m + 2g), \alpha(6g - 4 + e - m)\} < \beta.$$

Then, $W^k_H(2; S + \mathfrak{m}f, c_2) = \emptyset$ for any $k \geq \frac{1}{2\alpha}[\beta - \alpha(e - m + 2g - 2)]$.

Proof. Let us assume that $W^k_H(2; S + \mathfrak{m}f, c_2) \neq \emptyset$. Since $\mathrm{h}^0 E \neq 0$, we can take a non-zero section s of E. We denote by Y its scheme of zeros and by $D = aS + \mathfrak{b}f$ the maximal effective divisor contained in Y. Then, s can be regarded as a section of $E(-D)$ and its scheme of zeros has codimension greater than or equal to two. Thus, we have a short exact sequence

$$0 \to \mathcal{O}_X(D) \to E \to I_Z(S + \mathfrak{m}f - D) \to 0, \tag{2.5}$$

where Z is a codimension 2 locally complete intersection 0-cycle of length

$$|Z| = c_2 - D \cdot (S + \mathfrak{m}f - D).$$

Since D is effective, by Lemma 1.3.4, $a \geq 0$ and $b := \deg(\mathfrak{b}) \geq 0$. If $D = 0$, considering the exact sequence (2.5) and taking cohomology, we have

$$k \leq \mathrm{h}^0 E \leq \mathrm{h}^0 \mathcal{O}_X + \mathrm{h}^0 I_Z(S + \mathfrak{m}f) = 1 + \mathrm{h}^0 I_Z(S + \mathfrak{m}f).$$

By the short exact sequence (1.1) and Lemma 1.3.3, we get

$$\mathrm{h}^0 I_Z(S + \mathfrak{m}f) \leq \mathrm{h}^0 \mathcal{O}_X(S + \mathfrak{m}f) \leq 2(m + 1) - e \leq 4.$$

This implies that $k \leq 5$, but, since $\beta > \alpha(8 + e - m + 2g)$, this contradicts the fact that

$$k \geq \frac{1}{2\alpha}[\beta - \alpha(e - m + 2g - 2)] > 5.$$

Hence $D \neq 0$. Now, let us see that, if $D \neq 0$, then $D = \mathfrak{b}f$ with $b > 0$. Since E is a rank two H-stable vector bundle, $\mu_H(\mathcal{O}_X(D)) < \mu_H(E)$ and this is equivalent to

$$\alpha(2b - m) < (1 - 2a)(\beta - \alpha e). \tag{2.6}$$

Let us see that $a = 0$. If $a > 0$, since $\beta - \alpha e > \alpha(8 + 2g - m)$ and considering the inequality in (2.6), we get $\alpha(2b+m) < -\alpha(8+2g-m)$ which implies $b < 0$, a contradiction. Thus, $a = 0$ and hence $D = \mathfrak{b}f$.

In addition, since D is a non-zero effective divisor, we get $b > 0$. Therefore, by (2.5), E sits in the exact sequence

$$0 \to \mathcal{O}_X(\mathfrak{b}f) \to E \to I_Z(S + (\mathfrak{m} - \mathfrak{b})f) \to 0 \tag{2.7}$$

with $b \geq 1$ and $|Z| = c_2 - b$.

If we consider the exact sequence (2.7) and we take cohomology, we get

$$k \leq \mathrm{h}^0 E \leq \mathrm{h}^0 \mathcal{O}_X(\mathfrak{b}f) + \mathrm{h}^0 I_Z(S + (\mathfrak{m} - \mathfrak{b})f).$$

Notice that, since $m \in \{0, 1\}$ and $b \geq 1$, $m - b \leq 0$.

Let us first assume that $m - b = 0$. In this case, since $b \geq 1$, $m = b = 1$ and we get

$$k \leq \mathrm{h}^0 \mathcal{O}_X(f) + \mathrm{h}^0 I_Z(S).$$

Thus, since

$$\mathrm{h}^0 I_Z(S) \leq \mathrm{h}^0 \mathcal{O}_X(S) \leq \mathrm{h}^0 \mathcal{O}_C + \mathrm{h}^0 \mathcal{O}_C(-e) \leq 2,$$

and $\mathrm{h}^0 \mathcal{O}_X(f) = \mathrm{h}^0 \mathcal{O}_C(1) \leq 2$, we conclude that $k \leq 4$, which is a contradiction.

Now, let us analize the case $m - b < 0$. This assumption implies that

$$\mathrm{h}^0 I_Z(S + (\mathfrak{m} - \mathfrak{b})f) = 0$$

and hence $k \leq \mathrm{h}^0 E = \mathrm{h}^0 \mathcal{O}_X(\mathfrak{b}f)$. Let us see that this is a contradiction. First, let us assume that $b \geq 2g - 1$. In this case, $k \leq \mathrm{h}^0 \mathcal{O}_C(\mathfrak{b}) = b + 1 - g$. On the other hand, since $\mathcal{O}_X(\mathfrak{b}f) \hookrightarrow E$, by stability $b < \frac{1}{2\alpha}[\beta - \alpha(e - m)]$. Putting altogether, we get

$$k \leq b + 1 - g < \frac{1}{2\alpha}[\beta - \alpha(e - m + 2g - 2)],$$

which is a contradiction.

Finally, let us assume $b < 2g - 1$. This implies that

$$k \leq \mathrm{h}^0 \mathcal{O}_C(\mathfrak{b}) \leq b + 1 < 2g,$$

which is a contradiction. In fact, since $\beta > \alpha(6g - 4 + e - m)$, we get

$$k \geq \frac{1}{2\alpha}[\beta - \alpha(e - m + 2g - 2)] > 2g.$$

Hence, $W^k_H(2; S + \mathfrak{m}f, c_2) = \emptyset$ for any $k \geq \frac{1}{2\alpha}[\beta - \alpha(e - m + 2g - 2)]$. $\square$

Remark 2.2.9. Notice that, for $g \geq 3$, we have

$$\max\{\alpha(8+e-m+2g), \alpha(6g-4+e-m)\} = \alpha(6g-4+e-m).$$

Thus, for these values of g, in Theorem 2.2.8 we only need to assume that

$$\beta > \alpha(6g-4+e-m).$$

Analogously, for $g \leq 2$, it is sufficient to assume

$$\beta > \alpha(8+e-m+2g).$$

Furthermore, if we consider the particular case $g=1$, the result is also true if we expand the family of ample line bundles on X to the one of $H \equiv \alpha S + \beta f$ with

$$\alpha(6+e-m) < \beta.$$

As a consequence of the above results, we get the following equivalence.

Corollary 2.2.10. *Let X be a ruled surface over a non-singular curve C of genus $g \geq 0$, $m \in \{0,1\}$, $c_2 >> 0$ an integer and $H \equiv \alpha S + \beta f$ an ample divisor on X such that*

$$\max\{\alpha(8+e-m+2g), \alpha(6g-4+e-m)\} < \beta.$$

Then, $W^k_H(2; S+\mathfrak{m}f, c_2) \neq \emptyset$ if and only if

$$1 \leq k < \frac{1}{2\alpha}[\beta - \alpha(e-m+2g-2)].$$

Proof. It follows from Corollary 2.2.6 and Theorem 2.2.8. □

2.3 Non-emptiness of $W^k_H(2; f, c_2)$

Now, let us turn our attention to the case $c_1 = f$ and see under which conditions $W^k_H(2; f, c_2)$ is non-empty. The first result is about non-emptiness.

Theorem 2.3.1. *Let X be a ruled surface over a non-singular curve C of genus $g \geq 0$, $c_2 >> 0$ an integer and $H \equiv S + \beta f$ an ample divisor on X. Then $W^1_H(2; f, c_2) \neq \emptyset$.*

Proof. Let us consider the family $\mathcal{G}$ of rank two vector bundles E on X given by a non-trivial extension

$$0 \to \mathcal{O}_X \to E \to I_Z(f) \to 0, \tag{2.8}$$

where Z is a 0-dimensional subscheme of length $|Z| = c_2$.

Claim: $\mathcal{G}$ is non-empty.
Proof of the Claim: Notice that, since $K_K + f$ is non-effective, Z satisfies the Cayley-Bacharach property with respect to the linear system $|K_X + f|$ and thus, by Theorem 1.1.17, E is a rank 2 vector bundle and we may assume that the extension $e \in \mathrm{Ext}^1(I_Z(f), \mathcal{O}_X)$ is non-trivial. Hence, $\mathcal{G}$ is non-empty.

In addition, by the exact sequence (2.8), $c_1(E) = f$, $c_2(E) = |Z| = c_2 \gg 0$ and $\mathrm{h}^0 E \geq \mathrm{h}^0 \mathcal{O}_X = 1$.

Now, let us prove that any rank two vector bundle E in $\mathcal{G}$ is stable with respect to $H \equiv S + \beta f$. To this end, since E is a rank two vector bundle on X, we have to check that for any line subbundle $\mathcal{O}_X(G) \hookrightarrow E$ we have

$$\mu_H(\mathcal{O}_X(G)) = G \cdot H < \mu_H(E) = \frac{c_1(E) \cdot H}{2}.$$

By construction, E is given by a non-trivial extension of type (2.8). Therefore we have two possibilities: $\mathcal{O}_X(G) \hookrightarrow \mathcal{O}_X$ or $\mathcal{O}_X(G) \hookrightarrow I_Z(f)$.

Case 1 Assume that $\mathcal{O}_X(G) \hookrightarrow \mathcal{O}_X$. Then,

$$\mu_H(\mathcal{O}_X(G)) \leq 0 < \frac{1}{2} = \frac{c_1(E) \cdot H}{2} = \mu_H(E).$$

Case 2 Assume that $\mathcal{O}_X(G) \hookrightarrow I_Z(f)$, where $G = sS + \mathfrak{t}f$ with $t = \deg(\mathfrak{t})$. In this case

$$f - G = f - sS - \mathfrak{t}f$$

is an effective divisor and, thus, by Lemma 1.3.4, $s \leq 0$ and $t \leq 1$. Notice that since $\mathrm{h}^0 I_Z = 0$, $f \neq G$. Since $f - G$ is effective, we have $(f - G) \cdot H > 0$, which is equivalent to $t \leq s(e - \beta)$, and hence

$$\mu_H(\mathcal{O}_X(G)) = s(\beta - e) + t \leq 0 < \mu_H(E).$$

Thus, putting altogether, we have already seen that E is H-stable.

We have proved that any E in $\mathcal{G}$ is a rank two H-stable vector bundle with $\mathrm{h}^0 E \geq 1$, which implies that $\mathcal{G} \hookrightarrow W^1_H(2; f, c_2)$. Thus, $W^1_H(2; f, c_2) \neq \emptyset$. $\square$

Now, let us see what happens for values of $k \geq 2$. In [14, Proposition 7.7], it is proven that $W^2_H(2; f, c_2) \neq \emptyset$. We will study the case $k \geq 3$. To this end, we will use the following technical Lemma.

Lemma 2.3.2. *Let X be a ruled surface over a non-singular curve C of genus $g \geq 0$ and let E be a rank 2 vector bundle with $c_1(E) = f$. If E is stable with respect to the ample divisor $H \equiv S + \beta f$, then $\mathrm{h}^0 E \leq \mathrm{h}^0 E|_f$.*

Proof. First of all, let us see that $\mathrm{h}^0 E(-f) = 0$. If $\mathrm{h}^0 E(-f) \neq 0$, then $\mathcal{O}_X(f) \hookrightarrow E$. Since E is H-stable, we get

$$\mu_H(\mathcal{O}_X(f)) = 1 < \frac{1}{2} = \mu_H(E),$$

which is a contradiction. Hence $\mathrm{h}^0 E(-f) = 0$.

If we twist the short exact sequence

$$0 \to \mathcal{O}_X(-f) \to \mathcal{O}_X \to \mathcal{O}_f \to 0$$

by E and we take cohomology, since $\mathrm{h}^0 E(-f) = 0$, we get $\mathrm{h}^0 E \leq \mathrm{h}^0 E|_f$. □

Now, we are ready to study the Brill-Noether locus $W^k_H(2; f, c_2)$ for values of $k \geq 3$.

Theorem 2.3.3. *Let X be a ruled surface over a non-singular curve C of genus $g \geq 0$, $c_2 >> 0$ an integer and $H \equiv S + \beta f$ an ample divisor on X. Then, $W^k_H(2; f, c_2) = \emptyset$ for any $k \geq 3$.*

Proof. Let us assume that $W^k_H(2; f, c_2) \neq \emptyset$ and consider a vector bundle E in $W^k_H(2; f, c_2)$. Since $\mathrm{h}^0 E \geq k > 0$, we can take a non-zero section s of E. We denote by Y its scheme of zeros and by $D = aS + \mathfrak{b}f$ the maximal effective divisor contained in Y. Then s can be regarded as a section of $E(-D)$ and its scheme of zeros has codimension greater than or equal to two. Thus, we have a short exact sequence

$$0 \to \mathcal{O}_X(D) \to E \to I_Z(f - D) \to 0 \tag{2.9}$$

where Z is a locally complete intersection 0-cycle of length $|Z| = c_2 - D(f - D)$.

Since D is effective, by Lemma 1.3.4, $a \geq 0$ and $b := \deg(\mathfrak{b}) \geq 0$. Let us see that $D = 0$. Assume that $D \neq 0$. Since D is effective, $D \cdot H > 0$ for any ample divisor H on X. On the other hand, since E is stable with respect to $H \equiv S + \beta f$, we have $\mu_H(\mathcal{O}_X(D)) < \mu_H(E)$. Putting altogether, we get

$$0 < \mu_H(\mathcal{O}_X(D)) = D \cdot H < \frac{1}{2} = \mu_H(E),$$

which is a contradiction. Hence, $D = 0$.

Therefore, by (2.9), any vector bundle $E \in W_H^k(2; f, c_2)$ sits in a short exact sequence

$$0 \to \mathcal{O}_X \to E \to I_Z(f) \to 0. \tag{2.10}$$

If $\mathrm{h}^0\, I_Z(f) = 0$, then $\mathrm{h}^0\, E = \mathrm{h}^0\, \mathcal{O}_X = 1$. On the other hand, if $\mathrm{h}^0\, I_Z(f) \neq 0$ this would mean that the points of Z are contained in a line.

Let l be a line such that $Z \cap l = \emptyset$. Restricting the exact sequence (2.10) to l we get the short exact sequence

$$0 \to \mathcal{O}_l \to E_{|l} \to \mathcal{O}_l \to 0,$$

and taking cohomology we obtain $\mathrm{h}^0\, E|_l \leq 2$.

By Lemma 2.3.2, this implies that $\mathrm{h}^0\, E \leq 2$ and thus $W_H^k(2; f, c_2) = \emptyset$ for any $k \geq 3$. $\square$

Putting altogether, we get the following characterization.

Corollary 2.3.4. *Let X be a ruled surface over a non-singular curve C of genus $g \geq 0$, $c_2 >> 0$ an integer and $H \equiv S + \beta f$ an ample divisor on X. Then, $W_H^k(2; f, c_2) \neq \emptyset$ if and only if $1 \leq k \leq 2$.*

Proof. It follows from Theorem 2.3.1, [14, Proposition 7.7] and Theorem 2.3.3. $\square$

2.4 The locus $W_H^1(2; f, c_2) \backslash W_H^2(2; f, c_2)$

To end this chapter, we will analyze the irreducibility and smoothness of the locus $W_H^1(2; f, c_2) \backslash W_H^2(2; f, c_2)$.

Theorem 2.4.1. *Let X be a ruled surface over a non-singular curve C of genus $g \geq 0$, $c_2 >> 0$ an integer and $H \equiv S + \beta f$ an ample divisor on X. Then $W_H^1(2; f, c_2) \backslash W_H^2(2; f, c_2)$ is smooth and irreducible of the expected dimension, namely,*

$$\rho_H^1 = 3c_2 + g - 1.$$

Proof. Let us take a vector bundle $E \in W_H^1(2; f, c_2) \backslash W_H^2(2; f, c_2)$. Since, by definition, $\mathrm{h}^0\, E = 1$, we can take a non-zero section s of E. We denote by Y its scheme of zeros and by $D = aS + \mathfrak{b}f$ the maximal effective divisor contained in Y. Then, s can be regarded as a section of $E(-D)$ and its scheme of zeros has codimension greater than or equal to two. Thus, we have a short exact sequence

$$0 \to \mathcal{O}_X(D) \to E \to I_Z(f - D) \to 0, \tag{2.11}$$

where Z is a locally complete intersection 0-cycle of length $|Z| = c_2 - D(f - D)$.

Following the same arguments as in the proof of Theorem 2.3.3 we see that $D = 0$. Hence, any vector bundle E in $W^1_H(2; f, c_2)\backslash W^2_H(2; f, c_2)$ sits in a short exact sequence of the following type

$$0 \to \mathcal{O}_X \to E \to I_Z(f) \to 0. \tag{2.12}$$

Now, let us check that any vector bundle E in $W^1_H(2; f, c_2)\backslash W^2_H(2; f, c_2)$ is a smooth point of $M_H(2; f, c_2)$. To this end, it is enough to prove $\mathrm{h}^2(E^* \otimes E) = 0$.

First of all, let us see that $\mathrm{h}^2(E^*) = 0$. If we tensor the short exact sequence (2.12) by $\mathcal{O}_X(K_X)$ and we take cohomology, we get the long exact sequence

$$0 \to \mathrm{H}^0\, \mathcal{O}_X(K_X) \to \mathrm{H}^0\, E(K_X) \to \mathrm{H}^0\, I_Z(f + K_X) \to \cdots .$$

From the fact that $\mathrm{H}^0\, \mathcal{O}_X(K_X) = \mathrm{H}^0\, I_Z(f + K_X) = 0$, we deduce that

$$0 = \mathrm{h}^0\, E(K_X) = \mathrm{h}^2(E^*),$$

where the last equality follows from Serre duality. Now, we will see that

$$\mathrm{h}^2(E^* \otimes \mathcal{O}_X(f) \otimes I_Z) = 0.$$

Tensoring the short exact sequence (2.12) by $\mathcal{O}_X(K_X - f)$ and taking cohomology, we get the long exact sequence

$$0 \to \mathrm{H}^0\, \mathcal{O}_X(K_X - f) \to \mathrm{H}^0\, E(K_X - f) \to \mathrm{H}^0\, I_Z(K_X) \to \cdots .$$

Since $\mathrm{H}^0\, \mathcal{O}_X(K_X - f) = \mathrm{H}^0\, I_Z(K_X) = 0$, we get $\mathrm{H}^0\, E(K_X - f) = 0$ and, by Serre duality, $\mathrm{H}^2\, E^*(f) = 0$. Hence,

$$\mathrm{h}^2(E^* \otimes \mathcal{O}_X(f) \otimes I_Z) = \mathrm{h}^2(E^* \otimes \mathcal{O}_X(f)) = 0.$$

Finally, if we tensor the short exact sequence (2.12) by E^* and we take cohomology, we get

$$\cdots \to \mathrm{H}^2(E^*) \to \mathrm{H}^2(E^* \otimes E) \to \mathrm{H}^2(E^* \otimes \mathcal{O}_X(f) \otimes I_Z) \to 0$$

and, since $\mathrm{H}^2(E^*) = \mathrm{H}^2(E^* \otimes \mathcal{O}_X(f) \otimes I_Z) = 0$, we deduce that $\mathrm{H}^2(E^* \otimes E) = 0$. Hence, E is a smooth point of $M_H(2; f, c_2)$.

To end the proof, let us see that the map

$$\mu_E : \mathrm{H}^0(E) \otimes \mathrm{H}^1(E^* \otimes \mathcal{O}_X(K_X)) \to \mathrm{H}^1(E \otimes E^* \otimes \mathcal{O}_X(K_X))$$

is injective. Since $E \in W^1_H(2; f, c_2)\backslash W^2_H(2; f, c_2)$, $\mathrm{H}^0(E) \cong K$ and hence to prove that μ_E is injective is equivalent to prove that the map

$$\tilde{\mu}_E : \mathrm{H}^1(E^* \otimes \mathcal{O}_X(K_X)) \to \mathrm{H}^1(E \otimes E^* \otimes \mathcal{O}_X(K_X))$$

is injective.

If we tensor the exact sequence (2.12) by $\mathcal{O}_X(-f)$ and we take cohomology, we get the long exact sequence

$$\cdots \to \mathrm{H}^2\,\mathcal{O}_X(-f) \to \mathrm{H}^2\,E(-f) \to \mathrm{H}^2\,I_Z \to 0.$$

Since, by Serre duality,

$$\mathrm{h}^2\,\mathcal{O}_X(-f) = \mathrm{h}^0\,\mathcal{O}_X(f+K_X) = 0$$

and $\mathrm{h}^2\,I_Z = \mathrm{h}^2\,\mathcal{O}_X = 0$, we deduce that $\mathrm{h}^2\,E(-f) = 0$. This implies, again by Serre duality, that $\mathrm{h}^0\,E^*(K_X+f) = 0$. Moreover, if $\mathrm{h}^0\,E^*(K_X+f) = 0$, we get $\mathrm{h}^0(E^* \otimes I_Z(K_X+f)) = 0$. Therefore, using that $\mathrm{h}^0(E^* \otimes I_Z(K_X+f)) = 0$, if we tensor the exact sequence (2.12) by $E^* \otimes \mathcal{O}_X(K_X)$ and we take cohomology, we get

$$0 \to \mathrm{H}^1(E^* \otimes \mathcal{O}_X(K_X)) \xrightarrow{\tilde{\mu}_E} \mathrm{H}^1(E \otimes E^* \otimes \mathcal{O}_X(K_X)) \to \cdots,$$

which implies that $\tilde{\mu}_E$ is injective. Hence, μ_E is injective.

We have proved that, for any vector bundle $E \in W^1_H(2;f,c_2)\backslash W^2_H(2;f,c_2)$, E is a smooth point of $M_H(2;f,c_2)$ and μ_E is injective. Therefore, it follows from [24, Corollary 2.6], that $W^1_H(2;f,c_2)\backslash W^2_H(2;f,c_2)$ is smooth and of the expected dimension at E, for any $E \in W^1_H(2;f,c_2)\backslash W^2_H(2;f,c_2)$. Hence, the component $W^1_H(2;f,c_2)\backslash W^2_H(2;f,c_2)$ is smooth and of the expected dimension.

It only remais to prove the irreducibility of $W^1_H(2;f,c_2)\backslash W^2_H(2;f,c_2)$. We have seen that any $E \in W^1_H(2;f,c_2)\backslash W^2_H(2;f,c_2)$ sits in an extension of type (2.12). Let us denote by $\mathcal{S}$ the family of rank two vector bundles E on X given by a non-trivial extension of type (2.12). Notice that, by Serre duality,

$$\mathrm{ext}^1(I_Z(f), \mathcal{O}_X) = \mathrm{h}^1\,I_Z(f+K_X),$$

and, since

$$\begin{aligned} \mathrm{h}^0\,I_Z(f+K_X) &\leq \mathrm{h}^0\,\mathcal{O}_X(f+K_X) = 0 \\ \mathrm{h}^2\,I_Z(f+K_X) &= \mathrm{h}^0\,\mathcal{O}_X(-f) = 0, \end{aligned}$$

we get

$$\mathrm{h}^1\,I_Z(f+K_X) = -\chi I_Z(f+K_X) = |Z| - \chi\mathcal{O}_X(-f) = |Z| + g = c_2 + g.$$

Hence, $\mathrm{ext}^1(I_Z(f), \mathcal{O}_X) = c_2 + g$. Therefore, $\mathcal{S}$ is a (c_2+g-1)-projective bundle over de Hilbert scheme $\mathcal{H}ilb^{c_2}(X)$, which parametrizes 0-dimensional subschemes Z of length $|Z| = c_2$ and thus, $\mathcal{S}$ is an irreducible family of bundles of dimension

$$3c_2 + g - 1 = \rho^1_H(2;f,c_2).$$

Furthermore, there is a modular map π from an open dense subset of $\mathcal{S}$ to $W^1_H(2; f, c_2)\backslash W^2_H(2; f, c_2)$. Denote by W the irreducible component of the locus $W^1_H(2; f, c_2)\backslash W^2_H(2; f, c_2)$ which contains $\mathrm{im}(\pi)$. By the above arguments, the general point of $\mathrm{im}(\pi)$ is a smooth point of W. Hence, W has the expected dimension $3c_2 + g - 1$.

Putting altogether, $\dim W = \dim \mathcal{S}$, which implies that π must be generically finite onto an open subset of W, and thus $\mathcal{S}$ finitely dominates W.

Finally, since we have seen that any isomorphism class $[E]$ in the locus $W^1_H(2; f, c_2)\backslash W^2_H(2; f, c_2)$ must arise as a non-trivial extension of type (2.12) for some $Z \in \mathcal{H}ilb^{c_2}(X)$, $\mathcal{S}$ finitely dominates the whole $W^1_H(2; f, c_2)\backslash W^2_H(2; f, c_2)$, and then $W = W^1_H(2; f, c_2)\backslash W^2_H(2; f, c_2)$. Therefore, $W^1_H(2; f, c_2)\backslash W^2_H(2; f, c_2)$ is irreducible. $\square$

Chapter 3

Prioritary vector bundles on ruled surfaces

The main goal of this chapter is to establish the existence of indecomposable higher-rank prioritary vector bundles admitting a large number of global sections (most of the results are contained in [16]).

As explained in the preceding chapters, the whole category of vector bundles on a smooth projective variety is usually unwieldy and, thus, one has to impose some restrictions on the families of vector bundles under consideration. One of the approaches that has been recently used, apart from imposing stability conditions, deals with the cohomological properties of vector bundles. That is, to require certain cohomological conditions on the bundles in order to restrict the class of bundles under study.

Nowadays, there is a substantial amount of work on vector bundles without intermediate cohomology (ACM bundles) and, in particular, many results focus on those with the maximal number of global sections, namely Ulrich bundles. Another class of vector bundles sharing a strong cohomological property is the class of prioritary bundles (see Definition 3.1.3). This chapter is devoted to the study of prioritary vector bundles. These objects naturally connect classification problems governed by cohomological constraints with those arising from stability conditions since, as we are going to explain, under suitable hypotheses, stable vector bundles satisfy the prioritary condition.

Prioritary sheaves were introduced in the early nineties by Hirschowitz and Laszlo in [33], to study vector bundles on $\mathbb{P}^2$. Later on, in [66], the notion of prioritary was generalized for bundles on rules surfaces by Walter. Apart from the fact that prioritary bundles have a nice cohomological description, they also play an important role in the study of the moduli space of H-stable vector bundles (Definition 1.2.1 considered in Chapters 1 and 2). In fact, under certain conditions on the polarization H, H-stable vector bundles on ruled surfaces are prioritary, and the moduli space of H-stable bundles is an open substack inside the moduli stack of prioritary bundles. Hence, some questions concerning stable bundles can be deduced from the study of the corresponding prioritary bundles. Since stability is one of the classical notions used to restrict the class of bundles to consider, this fact has led to numerous contributions in the literature. For example, prioritary bundles have been used to describe the Picard group of the moduli space of sheaves on quadric surfaces (see [58]) and the weak Brill-Noether property (see [14]). Hence, in some sense, prioritary bundles link both approaches.

As we have mentioned in Chapters 1 and 2, apart from the existence of bundles satisfying certain convenient properties, it is also worthwhile to study their space of global sections. In particular, it is interesting to construct indecomposable bundles with a relatively large number of independent global sections. In general, it is not difficult to obtain good families of rank two prioritary bundles on algebraic surfaces, but, the difficulties appear when we try to deal with vector bundles of rank r bigger than two, that is, when the rank is bigger than the dimension of the base space. We will be focused on the existence of rank $r \geq 3$ prioritary bundles on ruled surfaces and we will also be interested in their module of global sections.

The structure of the chapter is as follows. First, in Section 3.1, we will fix some notation, we will recall basic facts concerning prioritary bundles on ruled surfaces and we will provide some first examples of lower rank prioritary bundles. The goal of Section 3.2 is to recursively construct prioritary bundles of arbitrary rank. In Theorem 3.2.1, we will prove a recursive construction that we will use in this and the next sections to construct simple prioritary bundles. Next, we will use it to prove Theorem 3.2.2, that allows us to recursively construct prioritary bundles of arbitrary rank. After this, we will apply Theorem 3.2.2 to explicitly construct prioritary bundles of arbitrary rank (see Theorems 3.2.3 and 3.2.4). In Section 3.3, we will construct rank 3 simple prioritary vector bundles with many sections (see Theorems 3.3.2, 3.3.5 and 3.3.7). We will use different constructions according to their normalized first Chern class. In Section 3.4, in Theorems 3.4.5 and 3.4.7, we will construct rank 4 prioritary vector bundles with many sections, using recursively the results given in the previous section. Finally, in Section 3.5 (see Theorems 3.5.2, 3.5.3, 3.5.4 and 3.5.5), we will prove the existence of simple prioritary bundles of arbitrarily large rank. To do so, we will construct general families of arbitrary rank prioritary bundles and we will give effective lower bounds for the dimension of their space of global sections.

3.1 Preliminaries on prioritary bundles

We will use the notation introduced in Section 1.3.1 of Chapter 1. We will work with prioritary bundles on a ruled surface $\pi : X = \mathbb{P}(\mathcal{E}) \to C$ over a non-singular curve C of genus $g \geq 0$ and $\mathcal{E}$ a rank 2 normalized bundle on C. Recall that the degree of the divisor corresponding to $\wedge^2\mathcal{E}$ is an invariant of X, which we denote by e and, according to this, we denote the canonical divisor by $K_X = -2S - \bar{\mathfrak{e}}f$, where $\bar{\mathfrak{e}} \in \mathrm{Pic}(C)$ has degree $\bar{e} = e - 2(g-1)$.

Let us start by recalling the definition of prioritary sheaves on $\mathbb{P}^2$, which motivates the definition in the context of ruled surfaces. As mentioned, it was introduced in the early nineties by Hirschowitz and Laszlo as a generalization of semistable sheaves.

Definition 3.1.1. A sheaf $\mathcal{F}$ on $\mathbb{P}^2$ is prioritary if it is torsion-free and

$$\mathrm{Ext}^2(\mathcal{F}, \mathcal{F}(-1)) = 0.$$

We have the following examples of prioritary sheaves on $\mathbb{P}^2$.

Example 3.1.2.

i) For an integer d, the sum of line bundles $E = \bigoplus_{i=1}^{s} \mathcal{O}_{\mathbb{P}^2}(d)$ is a prioritary vector bundle.

ii) If $Z \subset \mathbb{P}^2$ is a 0-dimensional scheme and d an integer, then the ideal sheaf $I_Z(d)$ is a prioritary sheaf.

iii) A vector bundle E given by a non-trivial extension of type

$$0 \to \mathcal{O}_{\mathbb{P}^2} \to E \to I_Z(1) \to 0,$$

where Z is a 0-dimensional subscheme of $\mathbb{P}^2$, is a prioritary vector bundle.

Later on, Walter introduced in [66] the notion of prioritary sheaf in a much general context, which is as follows.

Definition 3.1.3. Let X be a ruled surface over a non-singular curve C of genus $g \geq 0$. A coherent sheaf E on X is prioritary if it is torsion-free and $\mathrm{Ext}^2(E, E(-f)) = 0$.

Example 3.1.4.

i) For a divisor D on X, the sum of line bundles $E = \bigoplus_{i=1}^{s} \mathcal{O}_X(D)$ is a prioritary bundle.

ii) If we consider the rank 2 vector bundle $E = \mathcal{O}_X(2S) \oplus \mathcal{O}_X((2g-3-e)f)$, we get $\mathrm{Ext}^2(E, E(-f)) \cong \mathrm{H}^0\, \mathcal{O}_X(f)^* \neq 0$ and thus E is not prioritary.

Ideal sheaves of 0-dimensional schemes are also examples of prioritary sheaves on a ruled surface X. In fact, we prove the following result.

Lemma 3.1.5. *Let X be a ruled surface over a non-singular curve C of genus $g \geq 0$ and let Z be a 0-dimensional scheme on X of length $|Z| \geq 0$ and D a divisor in* $\mathrm{Pic}(X)$. *Then, $I_Z(D)$ is simple and prioritary. In particular, line bundles $\mathcal{O}_X(D)$ are simple and prioritary.*

Proof. Let us consider the exact sequence

$$0 \to I_Z(D) \to \mathcal{O}_X(D) \to \mathcal{O}_Z(D) \to 0. \tag{3.1}$$

First of all, let us see that $I_Z(D)$ is simple. Applying the covariant functor $\mathrm{Hom}(I_Z(D), -)$ to the exact sequence (3.1), we obtain the long exact sequence

$$0 \to \mathrm{Hom}(I_Z(D), I_Z(D)) \to \mathrm{Hom}(I_Z(D), \mathcal{O}_X(D)) \to \mathrm{Hom}(I_Z(D), \mathcal{O}_Z(D)) \to . \tag{3.2}$$

Notice that

$$\mathrm{Hom}(I_Z(D), \mathcal{O}_X(D)) \cong \mathrm{H}^2\, I_Z(K_X)^* \cong K.$$

Hence, by (3.2), $\mathrm{Hom}(I_Z(D), I_Z(D)) \cong K$ and thus $I_Z(D)$ is simple.

Now, let us see that $I_Z(D)$ is prioritary. Applying the contravariant functor $\mathrm{Hom}(-, I_Z(D-f))$ to the exact sequence (3.1), we get the long exact sequence

$$\begin{aligned} \cdots \quad &\to \mathrm{Ext}^2(\mathcal{O}_Z(D), I_Z(D-f)) \to \mathrm{Ext}^2(\mathcal{O}_X(D), I_Z(D-f)) \to \\ &\to \mathrm{Ext}^2(I_Z(D), I_Z(D-f)) \to 0. \end{aligned}$$

Since

$$\mathrm{Ext}^2(\mathcal{O}_X(D), I_Z(D-f)) \cong \mathrm{H}^2\, I_Z(-f) \cong \mathrm{H}^0\, \mathcal{O}_X(K_X+f)^* = 0,$$

we get $\mathrm{Ext}^2(I_Z(D), I_Z(D-f)) = 0$ and hence $I_Z(D)$ is also prioritary. □

In the case of ruled surfaces, the set of rank r prioritary sheaves on X with fixed Chern classes is parametrized by an open substack of the stack of coherent sheaves on X, denoted by $\mathrm{Prior}_X(r; c_1, c_2)$. Moreover in, [66, Proposition 2], Walter proved the following theorem.

Theorem 3.1.6. *Let X be a ruled surface, $r \geq 2$ an integer and Chern classes c_1 and c_2. Then, the stack* $\mathrm{Prior}_X(r; c_1, c_2)$ *of rank r prioritary sheaves on X with Chern classes c_1 and c_2 is smooth and irreducible.*

As we noted in the introduction of this chapter, since the work of Walter, prioritary sheaves have become a very important class of sheaves, in particular for their relation with the stable ones. For bundles on $\mathbb{P}^2$ it follows from Serre duality that any semistable sheaf is prioritary. Walter observed that the same is true on ruled surfaces and, in addition, he proved the following result.

Theorem 3.1.7. *Let X be a ruled surface, $r \geq 2$ an integer and H a polarization on H. If $(K_X + f) \cdot H < 0$, then any H-semistable rank r vector bundle is prioritary. Moreover, there exists an open immersion from the moduli space $M_H(r; c_1, c_2)$ parametrizing rank r H-stable vector bundles E on X with Chern classes c_i to the stack of prioritary sheaves* $\mathrm{Prior}_X(r; c_1, c_2)$ *and, in particular, both spaces have the same expected dimension.*

Proof. Notice that, if E is H-semistable, then any non-zero torsion-free quotient Q of E has a slope satisfying $\mu_H(Q) \geq \mu_H(E)$, while any non-zero subsheaf F of E has a slope such that $\mu_H(F) \leq \mu_H(E)$. Now, let us assume that E is not prioritary. Then, there exists a non-zero element

$$\phi \in \mathrm{Hom}(E, E(K_X + f)) \cong \mathrm{Ext}^2(E, E(-f))^* \neq 0,$$

which image satisfies

$$\mu_H(E) \leq \mu_H(\mathrm{im}(\phi)) \leq \mu_H(E(K_X + f)) = \mu_H(E) + (K_X + f) \cdot H,$$

which contradicts the fact that $(K_X + f) \cdot H < 0$. Hence, E is a prioritary sheaf. See [66, Proof of Theorem 1] for the rest of the proof. □

Since, for any ample divisor H on X such that $(K_X + f) \cdot H < 0$, any H-stable vector bundle is prioritary, we have several examples of rank two simple prioritary vector bundles with sections. For instance, as a consequence of the results given in Chapter 2, we have the following examples.

Example 3.1.8.

i) We have seen in the proof of Theorem 2.2.2 that the rank 2 vector bundle E given by the following non-trivial extension

$$0 \to \mathcal{O}_X(\mathfrak{b}f) \to E \to I_Z(S - \mathfrak{b}f) \to 0,$$

where Z is a 0-dimensional subscheme of length $|Z| \gg 0$ and $\mathfrak{b} \in \mathrm{Pic}(C)$ is a divisor of degree $b > 0$, is H-stable for $H \equiv \alpha S + \beta f$ with $\beta > \alpha e$. Thus, since $(K_X + f) \cdot H < 0$, E is a rank 2 prioritary bundle.

ii) In the proof of Theorem 2.3.1, we have seen that a rank 2 vector bundle E given by the following non-trivial extension

$$0 \to \mathcal{O}_X \to E \to I_Z(f) \to 0, \tag{3.3}$$

where Z a 0-dimensional scheme of length $|Z| \gg 0$, is stable with respect to the ample divisor with class $H \equiv S + \beta f$. In addition, $(K_X + f) \cdot H < 0$. Therefore, by Theorem 3.1.7, E is a rank 2 prioritary vector bundle.

Recently, prioritary sheaves have been used to get nice contributions in different problems. Nevertheless, many questions concerning them remain still open. As we have already mentioned, in this chapter we will focus on the existence of prioritary locally free sheaves, as well as on finding lower bounds for its number of sections and ensuring their indecomposability.

3.2 Prioritary bundles of arbitrary rank

In general, it is a difficult problem to construct H-stable bundles of rank $r \geq 3$ on surfaces. So, the argument used to obtain the above examples is not fruitful when facing the higher rank case. Hence, we need to look for other strategies in order to construct rank $r \geq 3$ prioritary bundles. In this section, we develop a recursive method to construct vector bundles of arbitrary rank with sections. In Section 3.5, we will see another construction, which is more direct but tedious.

Theorem 3.2.1. *Let X be a ruled surface over a non-singular curve C of genus $g \geq 0$ and let $r \geq 3$ be an integer. Let $\mathcal{F}_r$ be the family of rank r vector bundles E_r on X given by a non-trivial extension of type*

$$0 \to E_{r-1} \to E_r \to \mathcal{O}_X(L) \to 0 \tag{3.4}$$

where $L \in \operatorname{Pic}(X)$ and E_{r-1} is a rank $r-1$ vector bundle on X such that $\mathrm{h}^1 E_{r-1}(-L) \neq 0$.

i) If E_{r-1} is a simple vector bundle such that $\mathrm{h}^0 E_{r-1}(-L) = 0$ and such that $\mathrm{h}^2 E_{r-1}(K_X - L) = 0$, then E_r is a simple vector bundle.

ii) If E_{r-1} is a prioritary vector bundle such that $\mathrm{h}^0 E_{r-1}(K_X - L + f) = 0$ and $\mathrm{h}^2 E_{r-1}(-f - L) = 0$, then E_r is a prioritary vector bundle with $\mathrm{h}^0 E_r(K_X - L + f) = 0$ and $\mathrm{h}^2 E_r(-f - L) = 0$. Moreover, if $g \geq 1$, we also have $\mathrm{h}^1 E_r(-L) \neq 0$.

Proof. Notice that the condition $\mathrm{h}^1 E_{r-1}(-L) \neq 0$ is necessary in order to have non-trivial extensions of type (3.4).

i) Assume that E_{r-1} is a simple vector bundle with $\mathrm{h}^0 E_{r-1}(-L) = 0$ and $\mathrm{h}^2 E_{r-1}(K_X - L) = 0$. Applying the functor $\mathrm{Hom}(-, E_r)$ to the exact sequence (3.4), we obtain

$$\begin{aligned} 0 \ &\to \mathrm{Hom}(\mathcal{O}_X(L), E_r) \to \mathrm{Hom}(E_r, E_r) \to \mathrm{Hom}(E_{r-1}, E_r) \\ &\to \mathrm{Ext}^1(\mathcal{O}_X(L), E_r) \to \cdots . \end{aligned}$$

Let us first compute $\mathrm{Hom}(\mathcal{O}_X(L), E_r) \cong \mathrm{H}^0 E_r(-L)$. If we twist the exact sequence (3.4) by $\mathcal{O}_X(-L)$ and we take cohomology, we obtain

$$0 \to \mathrm{H}^0 E_{r-1}(-L) \to \mathrm{H}^0 E_r(-L) \to \mathrm{H}^0 \mathcal{O}_X \xrightarrow{\alpha} \mathrm{H}^1 E_{r-1}(-L) \to \cdots .$$

Since $\mathrm{H}^0 E_{r-1}(-L) = 0$, we have $\mathrm{H}^0 E_r(-L) \cong \ker(\alpha)$. Notice that the map α can be indentified with the map

$$K \to \mathrm{Ext}^1(\mathcal{O}_X(L), E_{r-1}),$$

which sends $1 \in K$ to the non-trivial extension (3.4). Thus, α is an injection and $\mathrm{H}^0 E_r(-L) \cong \ker(\alpha) = 0$.

Now, let us compute $\mathrm{Hom}(E_{r-1}, E_r)$. Applying the functor $\mathrm{Hom}(E_{r-1}, -)$ to the exact sequence (3.4), we obtain

$$0 \to \mathrm{Hom}(E_{r-1}, E_{r-1}) \to \mathrm{Hom}(E_{r-1}, E_r) \to \mathrm{Hom}(E_{r-1}, \mathcal{O}_X(L)) \to \cdots \quad (3.5)$$

Since

$$\mathrm{Hom}(E_{r-1}, \mathcal{O}_X(L)) \cong \mathrm{Ext}^2(\mathcal{O}_X(L), E_{r-1}(K_X))^* \cong \mathrm{H}^2 E_{r-1}(K_X - L)^* = 0$$

and E_{r-1} is simple, we get $\mathrm{Hom}(E_{r-1}, E_r) \cong \mathrm{Hom}(E_{r-1}, E_{r-1}) \cong K$. Putting altogether, by (3.5), we get $1 \leq \dim \mathrm{Hom}(E_r, E_r) \leq 1$ and hence E_r is simple.

ii) Let us assume that E_{r-1} is a prioritary vector bundle that satisfies $\mathrm{h}^0 E_{r-1}(K_X - L + f) = 0$ and $\mathrm{h}^2 E_{r-1}(-f - L) = 0$.

Applying the functor $\mathrm{Hom}(-, E_r(-f))$ to (3.4), we get

$$\cdots \to \mathrm{Ext}^2(\mathcal{O}_X(L), E_r(-f)) \to \mathrm{Ext}^2(E_r, E_r(-f)) \to \mathrm{Ext}^2(E_{r-1}, E_r(-f)) \to 0. \quad (3.6)$$

Let us first compute $\mathrm{Ext}^2(\mathcal{O}_X(L), E_r(-f)) \cong \mathrm{H}^2 E_r(-f-L)$. If we consider the extension (3.4) and we take cohomology, we obtain

$$\cdots \to \mathrm{H}^2 E_{r-1}(-f - L) \to \mathrm{H}^2 E_r(-f - L) \to \mathrm{H}^2 \mathcal{O}_X(-f) \to 0.$$

Since $\mathrm{H}^2 E_{r-1}(-f - L) = 0$ and $\mathrm{H}^2 \mathcal{O}_X(-f) \cong \mathrm{H}^0 \mathcal{O}_X(K_X + f)^* = 0$, we conclude that $\mathrm{H}^2 E_r(-f - L) = 0$.

Now, let us compute $\mathrm{Ext}^2(E_{r-1}, E_r(-f))$. Applying the covariant functor $\mathrm{Hom}(E_{r-1}, -)$ to the exact sequence (3.4) twisted by $\mathcal{O}_X(-f)$, we get

$$\cdots \mathrm{Ext}^2(E_{r-1}, E_{r-1}(-f)) \to \mathrm{Ext}^2(E_{r-1}, E_r(-f)) \to \mathrm{Ext}^2(E_{r-1}, \mathcal{O}_X(L-f)) \to 0.$$

Since E_{r-1} is prioritary and by Serre duality,

$$\mathrm{Ext}^2(E_{r-1}, \mathcal{O}_X(L-f)) \cong \mathrm{H}^0\, E_{r-1}(K_X - L + f)^* = 0,$$

we get $\mathrm{Ext}^2(E_{r-1}, E_r(-f)) = 0$. Putting altogether, by (3.6), we deduce that E_r is prioritary.

Now, let us prove that $\mathrm{H}^0\, E_r(K_X + f - L) = 0$. If we twist the exact sequence (3.4) by $\mathcal{O}_X(K_X + f - L)$ and we take cohomology, we get

$$0 \to \mathrm{H}^0\, E_{r-1}(K_X + f - L) \to \mathrm{H}^0\, E_r(K_X + f - L) \to \mathrm{H}^0\, \mathcal{O}_X(K_X + f) \to \cdots .$$

Since $\mathrm{h}^0\, E_{r-1}(K_X + f - L) = 0$ and $K_X + f$ is non-effective, we obtain

$$\mathrm{H}^0\, E_r(K_X + f - L) = 0.$$

Moreover, again considering the exact sequence (3.4) twisted by $\mathcal{O}_X(-f - L)$ and taking cohomology, we get

$$\cdots \to \mathrm{H}^2\, E_{r-1}(-f - L) \to \mathrm{H}^2\, E_r(-f - L) \to \mathrm{H}^2\, \mathcal{O}_X(-f) \to 0,$$

and, since $\mathrm{h}^2\, E_{r-1}(-f - L) = 0$ and $\mathrm{H}^2\, \mathcal{O}_X(-f) \cong \mathrm{H}^0\, \mathcal{O}_X(K_X + f)^* = 0$, we obtain $\mathrm{H}^2\, E_2(-f - L) = 0$.

Finally, let us prove that, if $g \geq 1$, then $\mathrm{H}^1\, E_r(-L) \neq 0$. If we consider the exact sequence (3.4) twisted by $\mathcal{O}_X(-L)$ and we take cohomology, we get

$$\cdots \to \mathrm{H}^1\, E_{r-1}(-L) \to \mathrm{H}^1\, E_r(-L) \to \mathrm{H}^1\, \mathcal{O}_X \to \mathrm{H}^2\, E_{r-1}(-L) \to \cdots .$$

On the other hand, by assumption,

$$0 = \mathrm{H}^2\, E_{r-1}(-f - L) \cong \mathrm{H}^0\, E^*_{r-1}(L + f + K_X)^*.$$

Therefore, we get

$$\mathrm{H}^2\, E_{r-1}(-L) \cong \mathrm{H}^0\, E^*_{r-1}(L + K_X)^* = 0,$$

which implies that the morphism $\mathrm{H}^1\, E_r(-L) \twoheadrightarrow \mathrm{H}^1\, \mathcal{O}_X$ is surjective. Hence

$$\mathrm{h}^1\, E_r(-L) \geq \mathrm{h}^1\, \mathcal{O}_X = g > 0,$$

which finishes the proof. $\square$

As a consequence of Theorem 3.2.1, we have a powerful tool to recursively construct rank $r \geq 3$ prioritary bundles.

Theorem 3.2.2. *Let X be a ruled surface over a non-singular curve C of genus $g \geq 1$, $r \geq 3$ an integer and let $L \in \mathrm{Pic}(X)$. Assume that E_2 is a rank 2 prioritary bundle such that $\mathrm{h}^1 E_2(-L) \neq 0$, $\mathrm{h}^0 E_2(K_X + f - L) = 0$ and $\mathrm{h}^2 E_2(-f - L) = 0$. Then, there exists a rank r prioritary vector bundle E_r on X with $c_1(E_r) = c_1(E_2) + (r-2)L$ and*

$$c_2(E_r) = c_2(E_2) + (r-2)c_1(E_2) \cdot L + \binom{r-2}{2} L^2$$

such that

i) $\mathrm{h}^0 E_r \geq \mathrm{h}^0 E_2$,

ii) $\mathrm{h}^1 E_r(-L) \neq 0$,

iii) $\mathrm{h}^0 E_r(K_X + f - L) = 0$ *and*

iv) $\mathrm{h}^2 E_r(-f - L) = 0$.

Proof. Let us prove the statement by induction on $r \geq 3$. First of all, let us assume that $r = 3$. Let E_3 be a vector bundle given by a non-trivial extension of type

$$0 \to E_2 \to E_3 \to \mathcal{O}_X(L) \to 0.$$

Notice that, since E_2 is under the assumptions of Theorem 3.2.1 *ii*), E_3 is a prioritary bundle satisfying $\mathrm{h}^0 E_3(K_X + f - L) = 0$, $\mathrm{h}^2 E_3(-f - L) = 0$ and $\mathrm{h}^1 E_3(-L) \neq 0$. On the other hand, by construction, we get $\mathrm{H}^0 E_2 \subseteq \mathrm{H}^0 E_3$ and $c_1(E_3) = c_1(E_2) + L$, $c_2(E_3) = c_2(E_2) + c_1(E_2) \cdot L$.

Now, let us assume that there exists a rank $r-1$ prioritary bundle E_{r-1} on X satisfying the conditions *i*)-*iv*) and such that $c_1(E_{r-1}) = c_1(E_2) + (r-3)L$ and

$$c_2(E_{r-1}) = c_2(E_2) + (r-3)c_1(E_2) \cdot L + \binom{r-3}{2} L^2.$$

Let us consider a rank r vector bundle E_r given by a non-trivial extension of type

$$0 \to E_{r-1} \to E_r \to \mathcal{O}_X(L) \to 0.$$

Notice that, by induction hypothesis, E_{r-1} satisfies the assumptions of Theorem 3.2.1 *ii*) and, hence, E_r is a prioritary bundle such that $\mathrm{h}^0 E_r(K_X + f - L) = 0$, $\mathrm{h}^2 E_r(-f - L) = 0$ and $\mathrm{h}^1 E_r(-L) \neq 0$. Finally, again by construction and the hypothesis on E_{r-1}, we get

$$\mathrm{h}^0 E_r \geq \mathrm{h}^0 E_{r-1} \geq \mathrm{h}^0 E_2$$

and

$$\begin{aligned} c_1(E_r) &= c_1(E_{r-1}) + L = (c_1(E_2) + (r-3)L) + L = c_1(E_2) + (r-2)L, \\ c_2(E_r) &= c_2(E_{r-1}) + c_1(E_{r-1}) \cdot L \\ &= c_2(E_2) + (r-3)c_1(E_2) \cdot L + \tbinom{r-3}{2} L^2 + (c_1(E_2) + (r-3)L) \cdot L \\ &= c_2(E_2) + (r-2)c_1(E_2) \cdot L + \tbinom{r-2}{2} L^2, \end{aligned}$$

which finishes the proof. □

We conclude this section by proving the existence of prioritary vector bundles of rank r with a large number of sections, using the recursive construction established in Theorem 3.2.2. Notice that E is a prioritary vector bundle if and only if, for any divisor D on X, $E(D)$ is prioritary. Therefore, we will normalize the first Chern class of the rank r prioritary bundle we construct, that is, we will consider $c_1 = lS + \mathfrak{m}f$ with $0 \leq l, \deg(\mathfrak{m}) \leq r-1$. We will divide the discussion into two separate results, depending on the choice of the normalized first Chern class.

Theorem 3.2.3. *Let X be a ruled surface over a non-singular curve C of genus $g \geq 1$. Let $c_1 = lS + \mathfrak{m}f \in \operatorname{Pic}(X)$ with $0 \leq l \leq r-3$ and $\mathfrak{m} \in \operatorname{Pic}(C)$ of degree $0 \leq m := \deg(\mathfrak{m}) \leq r-1$. Fix $r \geq 3$, $c_2 \gg 0$ an integer and b an integer such that*

$$0 < b < \frac{1}{2(r-1-l)}[c_2 + f(l,m,r)] + \frac{\bar{e} + m - (r-1)}{2},$$

where

$$f(l,m,r) = (g-1)[l-(r-2)] + \frac{e}{2}l(l-1) - ml.$$

Then, there exists a rank r prioritary bundle E_r on X with $c_1(E_r) = c_1$, $c_2(E_r) = c_2$ and $\mathrm{h}^0 E_r \geq k := b + 1 - g$.

Proof. Since we are looking for a lower bound of the number of global sections of a bundle and $c_2 \gg 0$, we may assume that $b > \max\{0, 2g-1\}$. Let us consider the rank 2 vector bundle E_2 given by a non-trivial extension of type

$$0 \to \mathcal{O}_X(\mathfrak{b}f) \to E_2 \to I_Z(c_1 - (r-2)L - \mathfrak{b}f) \to 0, \tag{3.7}$$

where $\mathfrak{b} \in \operatorname{Pic}(C)$ is of degree b, $L = S + \mathfrak{t}f$ with

$$t := \deg(\mathfrak{t}) = \lceil \frac{1}{(r-2)(r-1-l)}(-c_2 - f(l,m,r)) \rceil + 1 < 0$$

and Z is a 0-dimensional subscheme of length

$$\begin{aligned} |Z| &= c_2 - (r-2)c_1 \cdot L + \tbinom{r-1}{2} L^2 - (\mathfrak{b}f) \cdot (c_1 - (r-2)L - \mathfrak{b}f) \\ &= c_2 - [l-(r-2)]b + \tfrac{e(r-2)}{2}[2l - (r-1)] + t(r-2)(r-1-l) - (r-2)m. \end{aligned}$$

Observe that, since

$$(r-2)(r-1-l)t \geq -\left(c_2 - [l-(r-2)]b + \frac{e(r-2)}{2}[2l-(r-1)] - (r-2)m\right),$$

we get $|Z| \geq 0$. Denote by $\overline{D} = c_1 - (r-2)L - \mathfrak{b}f$.

To prove the result, it is enough to check that E_2 is under the assumptions of Theorem 3.2.2 for $L = S + \mathfrak{t}f$. First of all, since $0 \leq l \leq r-3$, we get $\mathrm{h}^0\, \mathcal{O}_X(\overline{D} - \mathfrak{b}f + K_X) = 0$ and thus Z satisfies the Cayley-Bacharach property with respect to the linear system $|\overline{D} - \mathfrak{b}f + K_X|$ which, by Theorem 1.1.17 implies that E_2 is a rank 2 vector bundle and, in addition, by Remark 1.1.18, that the extension $e \in \mathrm{Ext}^1(I_Z(\overline{D}), \mathcal{O}_X(\mathfrak{b}f))$ is non-trivial.

Claim 1: E_2 is prioritary.
Proof of Claim 1: If we apply the covariant functor $\mathrm{Hom}(E_2, -)$ to the exact sequence (3.7) twisted by $\mathcal{O}_X(-f)$ we get

$$\cdots \to \mathrm{Ext}^2(E_2, \mathcal{O}_X((\mathfrak{b}-1)f)) \to \mathrm{Ext}^2(E_2, E_2(-f)) \to \mathrm{Ext}^2(E_2, I_Z(\overline{D}-f)) \to 0.$$

First of all, observe that

$$\mathrm{Ext}^2(E_2, \mathcal{O}_X((\mathfrak{b}-1)f)) \cong \mathrm{H}^0\, E_2(K_X - (\mathfrak{b}-1)f)^*$$

and, by (3.7), we have the long exact sequence

$$0 \to \mathrm{H}^0\, \mathcal{O}_X(K_X+f) \to \mathrm{H}^0\, E_2(K_X-(\mathfrak{b}-1)f) \to \mathrm{H}^0\, I_Z(\overline{D}-(\mathfrak{b}-1)f+K_X) \to \cdots.$$

Thus, since $K_X + f$ is a non-effective divisor and, by the bound $l \leq r-3$, the divisor $\overline{D}-(\mathfrak{b}-1)f+K_X$ is also non-effective, we get $\mathrm{H}^0\, E_2(K_X-(\mathfrak{b}-1)f) = 0$.

Now, let us check that $\mathrm{Ext}^2(E_2, I_Z(\overline{D}-f)) = 0$. Applying the contravariant functor $\mathrm{Hom}(-, I_Z(\overline{D}-f))$ to the exact sequence (3.7), we get

$$\mathrm{Ext}^2(I_Z(\overline{D}), I_Z(\overline{D}-f)) \to \mathrm{Ext}^2(E_2, I_Z(\overline{D}-f)) \to \mathrm{Ext}^2(\mathcal{O}_X(\mathfrak{b}f), I_Z(\overline{D}-f)) \to 0.$$

Notice that, by Lemma 3.1.5, $\mathrm{Ext}^2(I_Z(\overline{D}), I_Z(\overline{D}-f)) = 0$ and, since

$$(r-2)t < -2b - 1 + \overline{e} + m,$$

we get

$$\begin{aligned} \mathrm{Ext}^2(\mathcal{O}_X(\mathfrak{b}f), I_Z(\overline{D}-f)) &\cong \mathrm{H}^2\, I_Z(\overline{D}-(\mathfrak{b}+1)f) \\ &\cong \mathrm{H}^0\, \mathcal{O}_X((\mathfrak{b}+1)f - \overline{D} + K_X)^* = 0. \end{aligned}$$

Putting altogether, E_2 is prioritary.

Claim 2: $\mathrm{h}^1 E_2(-L) \neq 0$.
Proof of Claim 2: If we twist the exact sequence (3.7) by $\mathcal{O}_X(-L)$ and we take cohomology we get

$$\cdots \to \mathrm{H}^1 \mathcal{O}_X(\mathfrak{b}f - L) \to \mathrm{H}^1 E_2(-L) \to \mathrm{H}^1 I_Z(\overline{D} - L) \to \mathrm{H}^2 \mathcal{O}_X(\mathfrak{b}f - L) \to \cdots .$$

Notice that, since

$$\mathrm{H}^2 \mathcal{O}_X(\mathfrak{b}f - L) \cong \mathrm{H}^0 \mathcal{O}_X(L - \mathfrak{b}f + K_X)^* = 0,$$

we get a surjective morphism $\mathrm{H}^1 E_2(-L) \twoheadrightarrow \mathrm{H}^1 I_Z(\overline{D} - L)$, which implies that

$$\mathrm{h}^1 E_2(-L) \geq \mathrm{h}^1 I_Z(\overline{D} - L).$$

Therefore,

$$\mathrm{h}^1 E_2(-L) \geq -\chi I_Z(\overline{D} - L) = |Z| - \chi\mathcal{O}_X(\overline{D} - L),$$

where

$$\begin{aligned} |Z| - \chi\mathcal{O}_X(\overline{D} - L) &= c_2 + f(l, m, r) + tl \\ &\geq c_2 + f(l, m, r) + (r-1)t > 0. \end{aligned}$$

Hence,

$$\mathrm{h}^1 E_2(-L) \geq \mathrm{h}^1 I_Z(\overline{D} - L) > 0.$$

Claim 3: $\mathrm{H}^0 E_2(K_X + f - L) = 0$.
Proof of Claim 3: If we twist the exact sequence (3.7) by $\mathcal{O}_X(K_X + f - L)$ and we take cohomology, we obtain

$$0 \to \mathrm{H}^0 \mathcal{O}_X(K_X - L + (\mathfrak{b}+1)f) \to \mathrm{H}^0 E_2(K_X + f - L) \to \mathrm{H}^0 I_Z(\overline{D} - L + K_X + f)$$

Using the fact that both $K_X - L + (\mathfrak{b}+1)f$ and $\overline{D} - L + K_X + f$ are non-effective divisors, we get Claim 3.

Claim 4: $\mathrm{H}^2 E_2(-f - L) = 0$.
Proof of Claim 4: If we twist the exact sequence (3.7) by $\mathcal{O}_X(-f - L)$ and we take cohomology we get

$$\cdots \to \mathrm{H}^2 \mathcal{O}_X(-L + (\mathfrak{b}-1)f) \to \mathrm{H}^2 E_2(-f - L) \to \mathrm{H}^2 I_Z(\overline{D} - L - (\mathfrak{b}+1)f) \to 0.$$

Since $K_X + L - (\mathfrak{b} - 1)f$ is non-effective, we get

$$\mathrm{H}^2 \mathcal{O}_X(-L + (\mathfrak{b} - 1)f) \cong \mathrm{H}^0 \mathcal{O}_X(K_X + L - (\mathfrak{b} - 1)f)^* = 0$$

and, since

$$(r-1)t < -b + m + \overline{e} - 1,$$

we also have

$$\begin{aligned}\mathrm{H}^2\, I_Z(\overline{D}-L-(\mathfrak{b}+1)f) &\cong \mathrm{H}^2\, \mathcal{O}_X(\overline{D}-L-(\mathfrak{b}+1)f)\\ &\cong \mathrm{H}^0\, \mathcal{O}_X(K_X-\overline{D}+L+(\mathfrak{b}+1)f)^* = 0.\end{aligned}$$

Hence, we conclude that $\mathrm{H}^2\, E_2(-f-L)=0$.

Finally, notice that, by construction,

$$\mathrm{h}^0\, E_2 \geq \mathrm{h}^0\, \mathcal{O}_X(\mathfrak{b}f) = \mathrm{h}^0\, \mathcal{O}_C(\mathfrak{b}) = b+1-g = k.$$

Putting altogether, by Theorem 3.2.2, there exists a rank r prioritary vector bundle E_r with

$$\begin{aligned} c_1(E_r) &= c_1(E_2)+(r-2)L = c_1,\\ c_2(E_r) &= c_2(E_2)+(r-2)c_1(E_2)\cdot L+\tbinom{r-2}{2}L^2\\ &= |Z|+(\mathfrak{b}f)\cdot[c_1-(r-2)L]+(r-2)[c_1-(r-2)L]\cdot L+\tbinom{r-1}{2}L^2\\ &= |Z|+(\mathfrak{b}f)\cdot[c_1-(r-2)L]+(r-2)c_1\cdot L-\tbinom{r-1}{2}L^2\\ &= c_2 \gg 0,\end{aligned}$$

and such that $\mathrm{h}^0\, E_r \geq \mathrm{h}^0\, E_2 \geq b+1-g$. □

Theorem 3.2.4. *Let X be a ruled surface over a non-singular curve C of genus $g \geq 1$. Let $c_1 = lS+\mathfrak{m}f \in \mathrm{Pic}(X)$ with $0 \leq m := \deg(\mathfrak{m}) \leq r-1$ and $c_2 >> 0$.*

i) Assume that $l=r-1$ and let us fix b an integer such that

$$0 < b < c_2 + \frac{e}{2}(r-1)(r-2)-(r-2)m.$$

Then, there exists a rank r prioritary vector bundle E_r with $c_1(E)=c_1$ and $c_2(E_r)=c_2$, such that $\mathrm{h}^0\, E_r \geq b+1-g$.

ii) Assume that $l=r-2$ and fix $b \geq 1$ an integer. Then, there exists a rank r vector bundle E with $c_1(E_r)=c_1$ and $c_2(E_r)=c_2$ such that $\mathrm{h}^0\, E_r \geq b+1-g$.

Proof. i) The proof is analogous to the one of Theorem 3.2.3. Since we are looking for a lower bound of the number of global sections of a bundle and $c_2 \gg 0$, we may assume that $b > \max\{0, 2g-1\}$. Let us consider the rank 2 vector bundle E_2 given by a non-trivial extension of type

$$0 \to \mathcal{O}_X(\mathfrak{b}f) \to E_2 \to I_Z(S+(\mathfrak{m}-(r-2)\mathfrak{t}-\mathfrak{b})f)) \to 0, \tag{3.8}$$

where $\mathfrak{b} \in \mathrm{Pic}(C)$ is of degree b, $L = S+\mathfrak{t}f$ with

$$t := \deg(\mathfrak{t}) = \lceil \frac{1}{(r-1)}(-c_2-g(l,m,r)) \rceil + 1 < 0,$$

where

$$g(l,m,r) = (g-1) + \frac{e}{2}(r-1)(r-2) - (r-1)m$$

and Z is a 0-dimensional subscheme of length

$$\begin{aligned} |Z| &= c_2 - (r-2)c_1 \cdot L + \tbinom{r-1}{2}L^2 - (\mathfrak{b}f)\cdot(c_1 - (r-2)L - \mathfrak{b}f) \\ &= c_2 - b + \tfrac{e}{2}(r-1)(r-2) - (r-2)m. \end{aligned}$$

Notice that, by the upper bound of b, we have $|Z| > 0$. Let us denote by $\overline{D} = S + (\mathfrak{m} - (r-2)\mathfrak{t} - \mathfrak{b})f$.

Now, let us check that E_2 is under the assumptions of Theorem 3.2.2 for $L = S + \mathfrak{t}f$. First of all, since the divisor $\overline{D} - \mathfrak{b}f + K_X$ is non-effective, Z satisfies the Cayley-Bacharach property with respect to the linear system $|\overline{D} - \mathfrak{b}f + K_X|$. Thus, by Theorem 1.1.17, E_2 is a rank 2 vector bundle and, in addition, by Remark 1.1.18, the extension

$$e \in \operatorname{Ext}^1(I_Z(\overline{D}), \mathcal{O}_X(\mathfrak{b}f))$$

is non-trivial.

Claim 1: E_2 is prioritary.
Proof of Claim 1: If we apply the covariant functor $\operatorname{Hom}(E_2, -)$ to the exact sequence (3.8) twisted by $\mathcal{O}_X(-f)$ we get

$$\begin{aligned} \cdots \to \operatorname{Ext}^2(E_2, \mathcal{O}_X((\mathfrak{b}-1)f)) &\to \operatorname{Ext}^2(E_2, E_2(-f)) \\ &\to \operatorname{Ext}^2(E_2, I_Z(\overline{D}-f) \to 0. \end{aligned}$$

Notice that

$$\operatorname{Ext}^2(E_2, \mathcal{O}_X((\mathfrak{b}-1)f)) \cong \mathrm{H}^0\, E_2(K_X - (\mathfrak{b}-1)f)^* = 0.$$

In fact, by (3.8), we have the long exact sequence

$$0 \to \mathrm{H}^0\, \mathcal{O}_X(K_X+f) \to \mathrm{H}^0\, E_2(K_X-(\mathfrak{b}-1)f) \to \mathrm{H}^0\, I_Z(\overline{D}-(\mathfrak{b}-1)f+K_X) \to \cdots,$$

with both $K_X + f$ and $\overline{D} - (\mathfrak{b}-1)f + K_X$ non-effective divisors. Thus, $\mathrm{H}^0\, E_2(K_X - (\mathfrak{b}-1)f) = 0$. Next, let us check that $\operatorname{Ext}^2(E_2, I_Z(\overline{D}-f)) = 0$. If we apply the contravariant functor $\operatorname{Hom}(-, I_Z(\overline{D}-f))$ to the exact sequence (3.8), we get

$$\begin{aligned} \cdots \quad &\to \operatorname{Ext}^2(I_Z(\overline{D}), I_Z(\overline{D}-f)) \to \operatorname{Ext}^2(E_2, I_Z(\overline{D}-f)) \\ &\to \operatorname{Ext}^2(\mathcal{O}_X(\mathfrak{b}f), I_Z(\overline{D}-f)) \to 0. \end{aligned}$$

Notice that, by Lemma 3.1.5, $\operatorname{Ext}^2(I_Z(\overline{D}), I_Z(\overline{D}-f)) = 0$, and, by Serre duality,

$$\begin{aligned} \operatorname{Ext}^2(\mathcal{O}_X(\mathfrak{b}f), I_Z(\overline{D}-f)) &\cong \mathrm{H}^2\, I_Z(\overline{D}-(\mathfrak{b}+1)f) \\ &\cong \mathrm{H}^0\, \mathcal{O}_X((\mathfrak{b}+1)f - \overline{D} + K_X)^* = 0. \end{aligned}$$

Putting altogether, E_2 is prioritary.

Claim 2: $\mathrm{h}^1\, E_2(-L) \neq 0$.
Proof of Claim 2: If we twist the exact sequence (3.8) by $\mathcal{O}_X(-L)$ and we take cohomology we get

$$\cdots \to \mathrm{H}^1\, \mathcal{O}_X(\mathfrak{b}f - L) \to \mathrm{H}^1\, E_2(-L) \to \mathrm{H}^1\, I_Z(\overline{D} - L) \to \mathrm{H}^2\, \mathcal{O}_X(\mathfrak{b}f - L) \to \cdots .$$

Notice that, since

$$\mathrm{H}^2\, \mathcal{O}_X(\mathfrak{b}f - L) \cong \mathrm{H}^0\, \mathcal{O}_X(L - \mathfrak{b}f + K_X)^* = 0,$$

we get a surjective morphism $\mathrm{H}^1\, E_2(-L) \twoheadrightarrow \mathrm{H}^1\, I_Z(\overline{D} - L)$, which implies that

$$\mathrm{h}^1\, E_2(-L) \geq \mathrm{h}^1\, I_Z(\overline{D} - L).$$

Therefore,

$$\mathrm{h}^1\, E_2(-L) \geq -\chi I_Z(\overline{D} - L) = |Z| - \chi\mathcal{O}_X(\overline{D} - L),$$

where

$$|Z| - \chi\mathcal{O}_X(\overline{D} - L) = c_2 + g(l, m, r) + (r-1)t > 0.$$

Hence,

$$\mathrm{h}^1\, E_2(-L) \geq \mathrm{h}^1\, I_Z(\overline{D} - L) > 0.$$

Claim 3: $\mathrm{H}^0\, E_2(K_X + f - L) = 0$.
Proof of Claim 3: If we twist the exact sequence (3.8) by $\mathcal{O}_X(K_X + f - L)$ and we take cohomology, we obtain

$$0 \to \mathrm{H}^0\, \mathcal{O}_X(K_X - L + (\mathfrak{b}+1)f) \to \mathrm{H}^0\, E_2(K_X + f - L) \to \mathrm{H}^0\, I_Z(\overline{D} + K_X + f - L) \to .$$

Using the fact that both $K_X - L + (\mathfrak{b}+1)f$ and $\overline{D} + K_X + f - L$ are non-effective divisors, we get Claim 3.

Claim 4: $\mathrm{H}^2\, E_2(-f - L) = 0$.
Proof of Claim 4: If we twist the exact sequence (3.8) by $\mathcal{O}_X(-f - L)$ and we take cohomology we get

$$\cdots \to \mathrm{H}^2\, \mathcal{O}_X(-L + (\mathfrak{b}-1)f) \to \mathrm{H}^2\, E_2(-f - L) \to \mathrm{H}^2\, I_Z(\overline{D} - f - L) \to 0.$$

Since both $K_X + f + L - \mathfrak{b}f$ and $K_X - \overline{D} + f + L$ are non-effective divisors, we get

$$\mathrm{H}^2\, \mathcal{O}_X(-L + (\mathfrak{b}-1)f) \cong \mathrm{H}^0\, \mathcal{O}_X(K_X + L - (\mathfrak{b}-1)f)^* = 0$$

and
$$\begin{aligned} \mathrm{H}^2\, I_Z(\overline{D}-f-L) &\cong \mathrm{H}^2\, \mathcal{O}_X(\overline{D}-f-L) \\ &\cong \mathrm{H}^0\, \mathcal{O}_X(K_X-\overline{D}+f+L))^* = 0 \end{aligned}$$
and hence $\mathrm{H}^2\, E_2(-f-L) = 0$.

Finally, notice that, by construction,
$$\mathrm{h}^0\, E_2 \geq \mathrm{h}^0\, \mathcal{O}_X(\mathfrak{b}f) = \mathrm{h}^0\, \mathcal{O}_C(\mathfrak{b}) = b+1-g.$$
Putting altogether, by Theorem 3.2.2, there exists a rank r prioritary vector bundle E_r with
$$\begin{aligned} c_1(E_r) &= c_1(E_2)+(r-2)L = c_1, \\ c_2(E_r) &= c_2(E_2)+(r-2)c_1(E_2)\cdot L + \tbinom{r-2}{2}L^2 \\ &= |Z| + (\mathfrak{b}f)\cdot[c_1-(r-2)L] + (r-2)[c_1-(r-2)L]\cdot L + \tbinom{r-1}{2}L^2 \\ &= |Z| + (\mathfrak{b}f)\cdot[c_1-(r-2)L] + (r-2)c_1\cdot L - \tbinom{r-1}{2}L^2 \\ &= c_2 \gg 0, \end{aligned}$$
and such that $\mathrm{h}^0\, E_r \geq \mathrm{h}^0\, E_2 \geq b+1-g$.

ii) Since we are looking for a lower bound of the number of global sections of a bundle and b is not upper bounded, we may assume that $b > \max\{0, 2g-1\}$. Let us consider the rank 2 vector bundle E_2 given by a non-trivial extension of type
$$0 \to \mathcal{O}_X(\mathfrak{b}f) \to E_2 \to I_Z((\mathfrak{m}-\mathfrak{b})f) \to 0, \tag{3.9}$$
where $\mathfrak{b} \in \mathrm{Pic}(C)$ is of degree b and Z is a 0-dimensional subscheme of length
$$\begin{aligned} |Z| &= c_2-(r-2)c_1\cdot S + \tbinom{r-1}{2}S^2 - (\mathfrak{b}f)\cdot(c_1-(r-2)S-\mathfrak{b}f) \\ &= c_2 + \tfrac{e(r-2)}{2}(r-3)-(r-2)m > 0, \end{aligned}$$
where the last inequality follows from the fact that $c_2 \gg 0$. Let us check that the family (3.9) is under the assumptions of Theorem 3.2.2. First of all, since $(\mathfrak{m}-2\mathfrak{b})f + K_X$ is non-effective, Z satisfies the Cayley-Bacharach property with respect to the linear system $|(\mathfrak{m}-2\mathfrak{b})f+K_X|$, which by Theorem 1.1.17 and Remark 1.1.18 implies that E_2 is a rank 2 vector bundle and that we can assume that the extension $e \in \mathrm{Ext}^1(I_Z((\mathfrak{m}-\mathfrak{b})f), \mathcal{O}_X(\mathfrak{b}f))$ is non-trivial.

Claim 1: E_2 is prioritary.
Proof of Claim 1: If we apply the covariant functor $\mathrm{Hom}(E_2,-)$ to the exact sequence (3.9) twisted by $\mathcal{O}_X(-f)$ we get
$$\mathrm{Ext}^2(E_2, \mathcal{O}_X((\mathfrak{b}-1)f)) \to \mathrm{Ext}^2(E_2, E_2(-f)) \to \mathrm{Ext}^2(E_2, I_Z((\mathfrak{m}-\mathfrak{b}-1)f) \to 0.$$

First of all, notice that
$$\mathrm{Ext}^2(E_2, \mathcal{O}_X((\mathfrak{b}-1)f)) \cong \mathrm{H}^0\, E_2(K_X-(\mathfrak{b}-1)f)^* = 0.$$

In fact, by (3.9), we have the long exact sequence

$$0 \to \mathrm{H}^0\, \mathcal{O}_X(K_X+f) \to \mathrm{H}^0\, E_2(K_X-(\mathfrak{b}-1)f) \to \mathrm{H}^0\, I_Z((\mathfrak{m}-2\mathfrak{b}+1)f+K_X) \to \cdots,$$

with both $K_X + f$ and $(\mathfrak{m} - 2\mathfrak{b} + 1)f + K_X$ non-effective divisors which implies that $\mathrm{H}^0\, E_2(K_X - (\mathfrak{b} - 1)f) = 0$.

In addition, $\mathrm{Ext}^2(E_2, I_Z((\mathfrak{m} - \mathfrak{b} - 1)f)) = 0$. If we apply the contravariant functor $\mathrm{Hom}(-, I_Z(\mathfrak{m} - \mathfrak{b} - 1)f)$ to the exact sequence (3.9), we get

$$\mathrm{Ext}^2(I_Z((\mathfrak{m} - \mathfrak{b})f), I_Z((\mathfrak{m} - \mathfrak{b})f - f)) \to \mathrm{Ext}^2(E_2, I_Z((\mathfrak{m} - \mathfrak{b})f - f))$$
$$\to \mathrm{Ext}^2(\mathcal{O}_X(\mathfrak{b}f), I_Z((\mathfrak{m} - \mathfrak{b})f - f)) \to 0,$$

where, by Lemma 3.1.5, $\mathrm{Ext}^2(I_Z((\mathfrak{m}-\mathfrak{b})f), I_Z((\mathfrak{m}-\mathfrak{b})f-f)) = 0$ and, by Serre duality and by (1.1),

$$\begin{aligned}\mathrm{Ext}^2(\mathcal{O}_X(\mathfrak{b}f), I_Z((\mathfrak{m} - \mathfrak{b})f - f)) &\cong \mathrm{H}^2\, I_Z((\mathfrak{m} - 2\mathfrak{b} - 1)f) \\ &\cong \mathrm{H}^0\, \mathcal{O}_X(-(\mathfrak{m} - 2\mathfrak{b} - 1)f + K_X)^* = 0.\end{aligned}$$

Putting together, E_2 is prioritary.

Claim 2: $\mathrm{h}^1\, E_2(-S) \neq 0$.
Proof of Claim 2: Notice that, $\mathrm{h}^1\, E_2(-S) \geq -\chi E_2(-S)$ and, since

$$\begin{aligned} c_1(E_2(-S)) &= c_1(E_2) - 2S = -2S + \mathfrak{m}f \\ c_2(E_2(-S)) &= c_2(E_2) + c_1(E_2)\cdot(-S) + (-S)^2 = |Z| - m - e, \end{aligned}$$

by the Riemann-Roch Theorem, $-\chi E_2(-S) = |Z| > 0$, and thus we get $\mathrm{h}^1\, E_2(-S) \neq 0$.

Claim 3: $\mathrm{H}^0\, E_2(K_X + f - S) = 0$.
Proof of Claim 3: If we twist the exact sequence (3.9) by $\mathcal{O}_X(K_X + f - S)$ and we take cohomology, we obtain

$$\mathrm{H}^0\, \mathcal{O}_X(K_X+f-S+\mathfrak{b}f) \to \mathrm{H}^0\, E_2(K_X+f-S) \to \mathrm{H}^0\, I_Z(-S+(\mathfrak{m}-\mathfrak{b}+1)f+K_X).$$

Using the fact that both $K_X + f - L + \mathfrak{b}f$ and $-S + (\mathfrak{m} - \mathfrak{b} + 1)f + K_X$ are non-effective divisors, we get Claim 3.

Claim 4: $\mathrm{H}^2\, E_2(-f - S) = 0$.
Proof of Claim 4: If we twist the exact sequence (3.9) by $\mathcal{O}_X(-f - S)$ and we take cohomology we get

$$\cdots \to \mathrm{H}^2\, \mathcal{O}_X(-f-S+\mathfrak{b}f) \to \mathrm{H}^2\, E_2(-f-S) \to \mathrm{H}^2\, I_Z(-S+(\mathfrak{m}-\mathfrak{b}-1)f) \to 0.$$

Since both $K_X+f+S-\mathfrak{b}f$ and $K_X+S+(\mathfrak{b}+1-\mathfrak{m})f$ are non-effective divisors, we get

$$\mathrm{H}^2\, \mathcal{O}_X(-f - S + \mathfrak{b}f) \cong \mathrm{H}^0\, \mathcal{O}_X(K_X + f + S - \mathfrak{b}f)^* = 0$$

and

$$\begin{aligned}\mathrm{H}^2\, I_Z(-S+(\mathfrak{m}-\mathfrak{b}-1)f) &\cong \mathrm{H}^2\,\mathcal{O}_X(-S+(\mathfrak{m}-\mathfrak{b}-1)f)\\ &\cong \mathrm{H}^0\,\mathcal{O}_X(K_X+S+(\mathfrak{b}+1-\mathfrak{m})f)^* = 0.\end{aligned}$$

Hence, we conclude that $\mathrm{H}^2\, E_2(-f-S)=0$.

Finally, notice that, by construction, since $b>\max\{0,2g-1\}$,

$$\mathrm{h}^0\, E_2 \geq \mathrm{h}^0\,\mathcal{O}_X(\mathfrak{b}f) = \mathrm{h}^0\,\mathcal{O}_C(\mathfrak{b}) = b+1-g.$$

Putting altogether, by Theorem 3.2.2, there exists a rank r prioritary vector bundle E_r with

$$\begin{aligned}c_1(E_r) &= c_1(E_2)+(r-2)S = c_1,\\ c_2(E_r) &= c_2(E_2)+(r-2)c_1(E_2)\cdot S+\tbinom{r-1}{2}S^2\\ &= |Z|+(r-2)m-\tbinom{r-1}{2}e\\ &= c_2 \gg 0.\end{aligned}$$

and such that $\mathrm{h}^0\, E_r \geq \mathrm{h}^0\, E_2 \geq b+1-g$.

□

3.3 Rank 3 simple prioritary bundles

In Section 3.1, we saw examples of rank 1 and 2 simple prioritary vector bundles with sections and, in section 3.2, we constructed prioritary bundles of higher rank with sections, but which may not be simple. The main goal of this section is to prove the existence of simple prioritary rank 3 vector bundles on X and to find lower bounds for the dimension of their space of global sections. We want to emphasize that their simplicity ensures that these prioritary bundles are indecomposable.

Notice that E is a simple prioritary vector bundle if and only if, for any divisor D on X, $E(D)$ is simple and prioritary. Hence, in this section, we partially normalize the first Chern class of our rank 3 bundle E so that it is of the form $c_1(E)=sS+\mathfrak{t}f$ with $s=0,1,2$.

Let us start with the case $c_1=2S+\mathfrak{t}f$. We will apply Theorem 3.2.1 to construct rank 3 simple prioritary vector bundles with this first Chern class that have certain number of independent sections.

First of all, let us prove the following technical lemma.

Lemma 3.3.1. *Let X be a ruled surface over a non-singular curve C of genus $g \geq 0$. Let us consider $\mathcal{S}_2$ the family of rank 2 vector bundles on X given by a non-trivial extension of type*

$$0 \to \mathcal{O}_X(\mathfrak{b}f) \to E_2 \to I_Z(S + (\mathfrak{m} - \mathfrak{b})f) \to 0, \tag{3.10}$$

where $\mathfrak{m}$ and $\mathfrak{b}$ are divisors on C of degrees $m \in \{0, 1\}$ and $b \geq 1$, respectively, and Z is a generic 0-dimensional subscheme of length $|Z| \gg 0$. Let L be a divisor on X. Then, the following holds.

i) The family $\mathcal{S}_2$ is non-empty.

ii) The vector bundle E_2 is simple and prioritary.

iii) If $-L+\mathfrak{b}f$ is a non-effective divisor and $|Z| > \mathrm{h}^0\, \mathcal{O}_X(-L+S+(\mathfrak{m}-\mathfrak{b})f)$, then $\mathrm{h}^0\, E_2(-L) = 0$.

iv) If both $L - \mathfrak{b}f$ and $L - S - (\mathfrak{m} - \mathfrak{b})f$ are non-effective divisors, then $\mathrm{h}^2\, E_2(K_X - L) = 0$.

Proof. *i*) By Theorem 1.1.17 and Remark 1.1.18, it is sufficient to see that Z satisfies the Cayley-Bacharach property with respect to the linear system $|S+(\mathfrak{m}-2\mathfrak{b})f+K_X|$, which is straightfoward since the divisor $S+(\mathfrak{m}-2\mathfrak{b})f+K_X$ is non-effective.
ii) We have seen in Example 3.1.8 that there exists an ample divisor H with $(K_X + f) \cdot H < 0$ and such that $\mathcal{S}_2$ is a family of H-stable bundles. Then, by Theorem 3.1.7, any $E \in \mathcal{S}_2$ is a prioritary bundle and, by Lemma 1.2.2 *v*), any $E \in \mathcal{S}_2$ is simple.

Using the exact sequence (3.10) and Proposition 1.1.13, we directly prove *iii*). The proof of *iv*) follows using the exact sequence (3.10) and Serre duality. □

Now, let us apply Lemma 3.3.1 to construct rank 3 simple prioritary bundles with a certain number of sections.

Theorem 3.3.2. *Let X be a ruled surface over a non-singular curve C of genus $g \geq 0$. Let $c_2 \gg 0$ be an integer and $c_1 = 2S + \mathfrak{t}f \in \mathrm{Pic}(X)$, with $\mathfrak{t} \in \mathrm{Pic}(C)$ of degree $t = \deg(\mathfrak{t}) \leq \min\{-2, -2g + 1\}$. Then, there exists a rank 3 simple prioritary vector bundle E on X with $c_1(E) = c_1$ and $c_2(E) = c_2$ such that $\mathrm{h}^0\, E \geq k := -t - g$.*

Proof. Let us consider $\mathcal{G}_2$ the family of rank 2 vector bundles given by a non-trivial extension of type

$$0 \to \mathcal{O}_X(\mathfrak{b}f) \to E_2 \to I_Z(S - \mathfrak{b}f) \to 0, \tag{3.11}$$

where $\mathfrak{b}$ is a divisor on C of degree $b = \deg(\mathfrak{b}) = -t-1$ and Z is a generic 0-dimensional subscheme of length $|Z| = c_2 + e + 1 \gg 0$.

In addition, let us consider the family $\mathcal{G}_3$ of rank 3 vector bundles E_3 with $c_1(E_3) = c_1$ and $c_2(E_3) = c_2$ given by a non-trivial extension of type

$$0 \to E_2 \to E_3 \to \mathcal{O}_X(D) \to 0, \tag{3.12}$$

where $D = S + \mathfrak{t}f$ and $E_2 \in \mathcal{G}_2$.

Notice that, by construction,

$$\mathrm{h}^0\, E_3 \geq \mathrm{h}^0\, E_2 \geq \mathrm{h}^0\, \mathcal{O}_X(\mathfrak{b}f) = b + 1 - g = k = -t - g.$$

Claim 1: $\mathcal{G}_3 \neq \emptyset$.
Proof of Claim 1: Let us first prove that the dimension of the space of extensions of type (3.12) is positive. Since

$$\mathrm{ext}^1(\mathcal{O}_X(D), E_2) = \mathrm{h}^1\, E_2(-D) \geq -\chi E_2(-D),$$

it is enough to prove that $\chi E_2(-D) < 0$.

Since

$$\begin{aligned} c_1(E_2(-D)) &= c_1(E_2) - 2D = -S - 2\mathfrak{t}f \\ c_2(E_2(-D)) &= c_2(E_2) - c_1(E_2)\cdot D + D^2 = |Z| + b + t = |Z| - 1, \end{aligned}$$

by the Riemann-Roch theorem we get

$$\chi E_2(-D) = 2 - g - |Z| < 0,$$

where the last inequality follows from the fact that $|Z| \gg 0$. Hence,

$$\mathrm{ext}^1(\mathcal{O}_X(D), E_2) = \mathrm{h}^1\, E_2(-D) > 0.$$

On the other hand, E_2 is a rank 2 vector bundle with Chern classes $c_1(E_2) = S$ and $c_2(E_2) = |Z| + b$. Therefore, by construction, E_3 is a rank 3 vector bundle with $c_1(E_3) = S + D = 2S + \mathfrak{t}f$ and

$$c_2(E_3) = c_2(E_2) + D \cdot c_1(E_2) = |Z| + b - e + t = |Z| - 1 - e = c_2.$$

Hence, $\mathcal{G}_3 \neq \emptyset$.

Claim 2: E_3 is simple and prioritary.
Proof of Claim 2: Notice that, it is enough to see that E_2 is under assumptions of Theorem 3.2.1.

Let us first prove that E_3 is prioritary. By Lemma 3.3.1, E_2 is a prioritary vector bundle and thus, by Theorem 3.2.1 *ii*), it is sufficient to check that $\mathrm{h}^0\, E_2(K_X + f - D) = 0$ and $\mathrm{h}^2\, E_2(-f - D) = 0$. Since

$$\begin{aligned} K_X + f + \mathfrak{b}f - D &\equiv -3S + (-t + b + 1 - \overline{e})f \\ K_X + f - \mathfrak{b}f - D + S &\equiv -2S + (-t + b + 1 - \overline{e})f \end{aligned}$$

are a non-effective divisors, by Lemma 3.3.1, $\mathrm{H}^0\, E_2(K_X + f - D) = 0$. In addition, since

$$\begin{aligned} K_X + f + D - \mathfrak{b}f &\equiv -S + (2t - \overline{e} - 2)f \\ K_X + f + D - S + \mathfrak{b}f &\equiv -2S + (2t - \overline{e} - 2)f \end{aligned}$$

are non-effective divisors, by Lemma 3.3.1, $\mathrm{H}^2\, E_2(-f - D) = 0$. Hence, by Theorem 3.2.1 *ii)*, E_3 is prioritary.

Now, let us prove that E_3 is simple. Since, by Lemma 3.3.1, E_2 is a simple vector bundle, it only remains to check that we have the conditions $\mathrm{h}^0\, E_2(-D) = 0$ and $\mathrm{h}^2\, E_2(K_X - D) = 0$. Since $-D + \mathfrak{b}f = -S + (\mathfrak{b} - \mathfrak{t})f$ is a non-effective divisor and Z is a generic 0-dimensional subscheme such that

$$\mathrm{h}^0\, \mathcal{O}_X(-D + S - \mathfrak{b}f) = \mathrm{h}^0\, \mathcal{O}_C(1) < |Z|,$$

it follows, from Lemma 3.3.1, that $\mathrm{H}^0\, E_2(-D) = 0$. Moreover, since the divisors $D - \mathfrak{b}f = S + (2\mathfrak{t} + 1)f$ and $D - S + \mathfrak{b}f = -f$ are non-effective, by Lemma 3.3.1 we get $\mathrm{h}^2\, E_2(K_X - D) = 0$. Hence, by Theorem 3.2.1 *i)*, E_3 is simple.

Putting altogether, E_3 is a rank 3 simple prioritary vector bundle with $c_1(E) = c_1$ and $c_2(E) = c_2$, such that $\mathrm{h}^0\, E \geq k := -t - g$. □

To contruct rank 3 simple prioritary vector bundles for the remaining cases of c_1 that we want to study, we will use a different method.

Let us start by proving some preparatory lemmas.

Lemma 3.3.3. *Let X be a ruled surface over a non-singular curve C of genus $g \geq 0$ and $m = 0, 1$. Let E_2 be a rank 2 vector bundle on X given by a non-trivial extension of type*

$$0 \to \mathcal{O}_X(-S + (\mathfrak{b} + 1)f) \to E_2 \to I_Z((m+1)S + (\mathfrak{d} - 2\mathfrak{b} - 1)f) \to 0 \quad (3.13)$$

where $\mathfrak{d}, \mathfrak{b} \in \mathrm{Pic}(C)$ are divisors of degrees $d = \deg(\mathfrak{d}) \geq 0$ and

$$b = \deg(\mathfrak{b}) > \max\{\frac{d-1}{3}, \frac{d + 2(g-1) - e}{3}\},$$

respectively, and Z is a 0-dimensional subscheme of length $|Z| \geq 0$. Then,

i) *E_2 is a simple vector bundle,*

ii) $\mathrm{H}^2\, E_2(K_X - \mathfrak{b}f) = 0$,

iii) $\mathrm{H}^0\, E_2(K_X - (\mathfrak{b} - 1)f) = 0$, *iv)* $\mathrm{H}^0\, E_2(K_X + S - \mathfrak{b} f) = 0$,

v) $\mathrm{H}^0\, E_2(-\mathfrak{b} f) = 0$.

Proof. *i*) First of all, notice that, since $3b > d + 2(g-2) - e$, the divisor

$$(m+1)S + (\mathfrak{d} - 2\mathfrak{b} - 1)f + S - (\mathfrak{b}+1)f + K_X = mS + (\mathfrak{d} - 3\mathfrak{b} - \overline{\mathfrak{e}} - 2)f$$

is non-effective and thus Z satisfies the Cayley-Bacharach property with respect to the linear system $|mS + (\mathfrak{d} - 3\mathfrak{b} - \overline{\mathfrak{e}} - 2)f|$ which, by Theorem 1.1.17, implies that E_2 is a vector bundle and, by Remark 1.1.18, that the extension

$$e \in \mathrm{Ext}^1(I_Z((m+1)S + (\mathfrak{d} - 2\mathfrak{b} - 1)f), \mathcal{O}_X(-S + (\mathfrak{b}+1)f))$$

is non-trivial. To see that E_2 is simple, we apply the functor $\mathrm{Hom}(-, E_2)$ to the exact sequence (3.13) and we get

$$\begin{aligned} 0 &\to \mathrm{Hom}(I_Z((m+1)S + (\mathfrak{d} - 2\mathfrak{b} - 1)f), E_2) \to \mathrm{Hom}(E_2, E_2) \to \\ &\to \mathrm{Hom}(\mathcal{O}_X(-S + (\mathfrak{b}+1)f, E_2) \to \cdots . \end{aligned} \tag{3.14}$$

Let us first prove that $\mathrm{Hom}(\mathcal{O}_X(-S + (\mathfrak{b}+1)f), E_2) \cong K$. Notice that

$$\mathrm{Hom}(\mathcal{O}_X(-S + (\mathfrak{b}+1)f, E_2) \cong \mathrm{H}^0\, E_2(S - (\mathfrak{b}+1)f).$$

If we twist the exact sequence (3.13) by $\mathcal{O}_X(S - (\mathfrak{b}+1)f)$ and we take cohomology, we get

$$0 \to \mathrm{H}^0\, \mathcal{O}_X \to \mathrm{H}^0\, E_2(S - (\mathfrak{b}+1)f) \to \mathrm{H}^0\, I_Z((m+2)S + (\mathfrak{d} - 3\mathfrak{b} - 2)f) \to \cdots .$$

Since $3b > d - 1$, the divisor $(m+2)S + (\mathfrak{d} - 3\mathfrak{b} - 2)f$ is non-effective and thus $\mathrm{H}^0\, I_Z((m+2)S + (\mathfrak{d} - 3\mathfrak{b} - 2)f) = 0$, which implies that $\mathrm{H}^0\, E_2(S - (\mathfrak{b}+1)f) \cong K$.

Now, let us see that $\mathrm{Hom}(I_Z((m+1)S + (\mathfrak{d} - 2\mathfrak{b} - 1)f), E_2) = 0$. Denote by $\overline{D}$ the divisor $(m+1)S + (\mathfrak{d} - 2\mathfrak{b} - 1)f$. Applying the functor $\mathrm{Hom}(I_Z(\overline{D}), -)$ to the exact sequence (3.13) we get

$$\begin{aligned} 0 &\to \mathrm{Hom}(I_Z(\overline{D}), \mathcal{O}_X(-S + (\mathfrak{b}+1)f)) \to \mathrm{Hom}(I_Z(\overline{D}), E_2) \to \\ &\to \mathrm{Hom}(I_Z(\overline{D}), I_Z(\overline{D})) \xrightarrow{\alpha} \mathrm{Ext}^1(I_Z(\overline{D}), \mathcal{O}_X(-S + (\mathfrak{b}+1)f)) \to \cdots . \end{aligned} \tag{3.15}$$

Since $-(m+2)S + (3\mathfrak{b} - \mathfrak{d} + 2)f$ is non-effective,

$$\begin{aligned} \mathrm{Hom}(I_Z(\overline{D}), \mathcal{O}_X(-S + (\mathfrak{b}+1)f)) &\cong \mathrm{Ext}^2(\mathcal{O}_X(-S + (\mathfrak{b}+1)f), I_Z(\overline{D} + K_X))^* \\ &\cong \mathrm{H}^2\, I_Z((m+2)S - (3\mathfrak{b} - \mathfrak{d} + 2)f + K_X)^* \\ &\cong \mathrm{H}^0\, \mathcal{O}_X(-(m+2)S + (3\mathfrak{b} - \mathfrak{d} + 2)f) \\ &= 0. \end{aligned}$$

On the other hand, by Lemma 3.1.5, $\mathrm{Hom}(I_Z(\overline{D}), I_Z(\overline{D})) \cong K$. Hence, by (3.15),

$$\mathrm{Hom}(I_Z(\overline{D}), E_2) \cong \ker(\alpha),$$

where

$$\alpha : K \to \mathrm{Ext}^1(I_Z(\overline{D}), \mathcal{O}_X(-S + (\mathfrak{b}+1)f))$$

is the map that sends 1 to the non-trivial extension

$$e \in \mathrm{Ext}^1(I_Z(\overline{D}), \mathcal{O}_X(-S + (\mathfrak{b}+1)f)).$$

Thus, α is injective and

$$\mathrm{Hom}(I_Z(\overline{D}), E_2) \cong \ker(\alpha) = 0.$$

Finally, since it is always true that $1 \leq \dim \mathrm{Hom}(E_2, E_2)$ and, by (3.14), $\dim \mathrm{Hom}(E_2, E_2) \leq 1$, we get that E_2 is a simple vector bundle.

ii) Twisting the exact sequence (3.13) by $\mathcal{O}_X(K_X - \mathfrak{b}f)$ and taking cohomology we get

$$\mathrm{H}^2\,\mathcal{O}_X(K_X - S + f) \to \mathrm{H}^2\,E_2(K_X - \mathfrak{b}f) \to \mathrm{H}^2\,I_Z(\overline{D} - \mathfrak{b}f + K_X) \to 0$$

Since

$$\mathrm{H}^2\,\mathcal{O}_X(K_X - S + f) \cong \mathrm{H}^0\,\mathcal{O}_X(S - f)^* = 0$$

and

$$\mathrm{H}^2\,I_Z(\overline{D} - \mathfrak{b}f + K_X) \cong \mathrm{H}^0\,\mathcal{O}_X(-\overline{D} + \mathfrak{b}f)^* = 0,$$

we deduce that $\mathrm{H}^2\,E_2(K_X - \mathfrak{b}f) = 0$.

iii) Twisting the exact sequence (3.13) by $\mathcal{O}_X(K_X - (\mathfrak{b}-1)f)$ and taking cohomology, we get

$$\mathrm{H}^0\,\mathcal{O}_X(-3S-(\overline{\mathfrak{e}}-2)f) \to \mathrm{H}^0\,E_2(K_X-(\mathfrak{b}-1)f) \to \mathrm{H}^0\,I_Z((m-1)S+(\mathfrak{d}-3\mathfrak{b}-\overline{\mathfrak{e}})f).$$

Notice that, since $3b > d+2(g-1)-e$, $-3S-(\overline{\mathfrak{e}}-2)f$ and $(m-1)S+(\mathfrak{d}-3\mathfrak{b}-\overline{\mathfrak{e}})f$ are non-effective divisors. Hence, $\mathrm{H}^0\,E_2(K_X - (\mathfrak{b}-1)f) = 0$.

iv) Twisting the exact sequence (3.13) by $\mathcal{O}_X(K_X + S - \mathfrak{b}f)$ and taking cohomology we get

$$0 \to \mathrm{H}^0\,\mathcal{O}_X(K_X+f) \to \mathrm{H}^0\,E_2(K_X+S-\mathfrak{b}f) \to \mathrm{H}^0\,\mathcal{O}_X(mS+(\mathfrak{d}-3\mathfrak{b}-1-\overline{\mathfrak{e}})f) \to .$$

Since $K_X + f$ is non-effective and, by the bound $3b > d + 2g - e - 3$, the divisor $mS + (\mathfrak{d} - 3\mathfrak{b} - 1 - \overline{\mathfrak{e}})f$ is also non-effective, we get

$$\mathrm{H}^0\,\mathcal{O}_X(K_X + f) = \mathrm{H}^0\,\mathcal{O}_X(mS + (\mathfrak{d} - 3\mathfrak{b} - 1 - \overline{\mathfrak{e}})f) = 0$$

and, hence, $\mathrm{H}^0\,E_2(K_X + S - \mathfrak{b}f) = 0$.
Finally, *v*) can be proved using similar arguments. □

The following lemma provides a way to construct rank 3 simple prioritary vector bundles, obtained by applying Lemma 3.3.3.

Lemma 3.3.4. *Let X be a ruled surface over a non-singular curve C of genus $g \geq 0$. Let E_3 be a rank 3 vector bundle on X given by a non-trivial extension*

$$e : 0 \to \mathcal{O}_X(\mathfrak{b}f) \to E_3 \to E_2 \to 0 \tag{3.16}$$

where E_2 and $\mathfrak{b}$ are given as in Lemma 3.3.3 and satisfy $\mathrm{H}^1\, E_2(K_X - \mathfrak{b}f) \neq 0$. Then, E_3 is a simple prioritary vector bundle.

Proof. Notice that, the assumption $\mathrm{H}^1\, E_2(K_X - \mathfrak{b}f) \neq 0$ guarantees that the extension (3.16) is non-trivial.

First, let us prove that E_3 is simple, that is, $\mathrm{Hom}(E_3, E_3) \cong K$. If we apply the contravariant functor $\mathrm{Hom}(-, E_3)$ to the exact sequence (3.16), we get the long exact sequence

$$0 \to \mathrm{Hom}(E_2, E_3) \to \mathrm{Hom}(E_3, E_3) \to \mathrm{Hom}(\mathcal{O}_X(\mathfrak{b}f), E_3) \to \cdots \tag{3.17}$$

and

$$\mathrm{Hom}(\mathcal{O}_X(\mathfrak{b}f), E_3) \cong \mathrm{H}^0\, E_3(-\mathfrak{b}f) \cong K.$$

In fact, if we twist the exact sequence (3.16) by $\mathcal{O}_X(-\mathfrak{b}f)$ and we take cohomology, we get the exact sequence

$$0 \to \mathrm{H}^0\, \mathcal{O}_X \to \mathrm{H}^0\, E_3(-\mathfrak{b}f) \to \mathrm{H}^0\, E_2(-\mathfrak{b}f) \to \cdots$$

and, since by Lemma 3.3.3, $\mathrm{H}^0\, E_2(-\mathfrak{b}f) = 0$, we obtain

$$\mathrm{H}^0\, E_3(-\mathfrak{b}f) \cong \mathrm{H}^0\, \mathcal{O}_X \cong K.$$

Now, let us see that $\mathrm{Hom}(E_2, E_3) = 0$. In fact, since by Lemma 3.3.3

$$\mathrm{Hom}(E_2, \mathcal{O}_X(\mathfrak{b}f)) \cong \mathrm{H}^2\, E_2(K_X - \mathfrak{b}f)^* = 0$$

and E_2 is simple, if we apply the covariant functor $\mathrm{Hom}(E_2, -)$ to the exact sequence (3.16), we get

$$0 \to \mathrm{Hom}(E_2, E_3) \to K \xrightarrow{\beta} \mathrm{Ext}^1(E_2, \mathcal{O}_X(\mathfrak{b}f)) \to \cdots .$$

Therefore, $\mathrm{Hom}(E_2, E_3) \cong \ker(\beta)$ and, since β is the map that sends 1 to a non-trivial exact sequence $e \in \mathrm{Ext}^1(E_2, \mathcal{O}_X(\mathfrak{b}f))$, it is injective. Hence, $\mathrm{Hom}(E_2, E_3) = 0$ and, putting altogether, by the long exact sequence (3.17) we get $\mathrm{Hom}(E_3, E_3) \cong K$.

Now, let us see that E_3 is a prioritary bundle, that is, $\operatorname{Ext}^2(E_3, E_3(-f)) = 0$. Applying the covariant functor $\operatorname{Hom}(E_3, -)$ to the exact sequence (3.16) twisted by $\mathcal{O}_X(-f)$, we get

$$\cdots \to \operatorname{Ext}^2(E_3, \mathcal{O}_X((\mathfrak{b}-1)f)) \to \operatorname{Ext}^2(E_3, E_3(-f)) \to \operatorname{Ext}^2(E_3, E_2(-f)) \to 0. \tag{3.18}$$

First of all, let us see that

$$\operatorname{Ext}^2(E_3, \mathcal{O}_X((\mathfrak{b}-1)f)) \cong \mathrm{H}^0\, E_3(K_X - (\mathfrak{b}-1)f)^* = 0.$$

By the exact sequence (3.16), we get

$$0 \to \mathrm{H}^0\, \mathcal{O}_X(K_X + f) \to \mathrm{H}^0\, E_3(K_X - (\mathfrak{b}-1)f) \to \mathrm{H}^0\, E_2(K_X - (\mathfrak{b}-1)f) \to 0.$$

Therefore, since, by Lemma 3.3.3, $\mathrm{H}^0\, E_2(K_X - (\mathfrak{b}-1)f) = 0$, and $K_X + S$ is non-effective, we get $\mathrm{H}^0\, E_3(K_X - (\mathfrak{b}-1)f) = 0$. Now, let us check that $\operatorname{Ext}^2(E_3, E_2(-f)) = 0$. Applying the covariant functor $\operatorname{Hom}(E_3, -)$ to the exact sequence (3.13) twisted by $\mathcal{O}_X(-f)$, we get

$$\begin{aligned} \cdots \quad &\to \operatorname{Ext}^2(E_3, \mathcal{O}_X(-S + \mathfrak{b}f)) \to \operatorname{Ext}^2(E_3, E_2(-f)) \\ &\to \operatorname{Ext}^2(E_3, I_Z((m+1)S + (\mathfrak{d} - 2\mathfrak{b} - 2)f)) \to 0. \end{aligned} \tag{3.19}$$

First, let us see that

$$\operatorname{Ext}^2(E_3, \mathcal{O}_X(-S + \mathfrak{b}f)) \cong \mathrm{H}^0\, E_3(K_X + S - \mathfrak{b}f)^* = 0.$$

Notice that, by the exact sequence (3.16), we get

$$0 \to \mathrm{H}^0\, \mathcal{O}_X(K_X + S) \to \mathrm{H}^0\, E_3(K_X + S - \mathfrak{b}f) \to \mathrm{H}^0\, E_2(K_X + S - \mathfrak{b}f) \to 0.$$

Furthermore, since, by Lemma 3.3.3, $\mathrm{H}^0\, E_2(K_X + S - \mathfrak{b}f) = 0$, and $K_X + S$ is non-effective, we deduce that $\mathrm{H}^0\, E_3(K_X + S - \mathfrak{b}f) = 0$. In addition, we get

$$\operatorname{Ext}^2(E_3, I_Z((m+1)S + (\mathfrak{d} - 2\mathfrak{b} - 2)f)) = 0.$$

In fact, if we apply the functor $\operatorname{Hom}(-, I_Z((m+1)S + (\mathfrak{d} - 2\mathfrak{b} - 2)f))$ to the exact sequence (3.16), we get

$$\begin{aligned} &\to \operatorname{Ext}^2(E_2, I_Z((m+1)S+(\mathfrak{d}-2\mathfrak{b}-2)f)) \to \operatorname{Ext}^2(E_3, \mathcal{O}_X((m+1)S+(\mathfrak{d}-2\mathfrak{b}-2)f)) \\ &\qquad \to \operatorname{Ext}^2(\mathcal{O}_X(\mathfrak{b}f), I_Z((m+1)S + (\mathfrak{d} - 2\mathfrak{b} - 2)f)) \to 0. \end{aligned}$$

Since $\overline{D} := K_X - (m+1)S - (\mathfrak{d} - 3\mathfrak{b} - 2)f$ is not effective, by the exact sequence (1.2) and Serre duality,

$$\begin{aligned} \operatorname{Ext}^2(\mathcal{O}_X(\mathfrak{b}f), I_Z(K_X - \overline{D} + \mathfrak{b}f)) \cong \mathrm{H}^2\, I_Z(K_X - \overline{D}) &\cong \mathrm{H}^2\, \mathcal{O}_X(K_X - \overline{D}) \\ &\cong \mathrm{H}^0\, \mathcal{O}_X(\overline{D})^* = 0. \end{aligned}$$

Finally, applying the functor $\mathrm{Hom}(-, I_Z((m+1)S+(\mathfrak{d}-2\mathfrak{b}-2)f))$ to the exact sequence (3.13) and using the fact that I_Z is prioritary (see Lemma 3.1.5), we get

$$\mathrm{Ext}^2(E_2, I_Z((m+1)S+(\mathfrak{d}-2\mathfrak{b}-2)f)) = 0.$$

Putting altogether, $\mathrm{Ext}^2(E_3, E_2(-f)) = 0$ and hence, by the exact sequence (3.19), we get $\mathrm{Ext}^2(E_3, E_3(-f)) = 0$. □

Now, we are ready to apply Lemma 3.3.4 to get examples of rank 3 simple prioritary bundles.

First, let us consider the case $c_1 = S + \mathfrak{d}f$.

Theorem 3.3.5. *Let X be a ruled surface over a non-singular curve C of genus $g \geq 0$. Let us consider $c_2 >> 0$ an integer and $c_1 = S + \mathfrak{d}f$ with $\mathfrak{d} \in Pic(C)$ of degree $d = \deg(\mathfrak{d}) \in \{0,1,2\}$. Let $0 \leq l < 5$, be an integer equivalent to $c_2 - 2e - 3 + d$ module 5. Then, there exists a rank 3 simple prioritary vector bundle E with $c_1(E) = S + \mathfrak{d}f$ and $c_2(E) = c_2$, such that*

$$\mathrm{h}^0 E \geq k := \frac{1}{5}(c_2 - 2e - 3 + d - l) + 1 - g.$$

Proof. Let us consider the family $\mathcal{F}_2$ of rank two vector bundles E_2 on X given by a non-trivial extension of type

$$0 \to \mathcal{O}_X(-S + (\mathfrak{b}+1)f) \to E_2 \to I_Z(2S + (\mathfrak{d} - 2\mathfrak{b} - 1)f) \to 0, \tag{3.20}$$

where Z is a 0-dimensional subscheme of length $|Z| = l$ and $\mathfrak{b} \in \mathrm{Pic}(C)$ has degree $b = \frac{1}{5}(c_2 - 2e - 3 + d - l)$.

Since $c_2 \gg 0$, we may assume, without loss of generality, that

$$b > \max\{0, 2(g-1), \frac{d-1}{3}, \frac{d+2(g-1)-e}{3}\}.$$

Claim 1: $\mathcal{F}_2 \neq \emptyset$.
Proof of Claim 1: Notice that, since the divisor $3S + (\mathfrak{d} - 3\mathfrak{b} - 2)f + K_X$ is non-effective by the lower bound $3b > d-1$, Z satisfyies the Cayley-Bacharach property with respect to the linear system $|3S + (\mathfrak{d} - 3\mathfrak{b} - 2)f + K_X|$. Hence, by Theorem 1.1.17, E_2 is a rank 2 vector bundle and, in addition, considering Remark 1.1.18, the extension $e \in \mathrm{Ext}^1(I_Z(2S+(\mathfrak{d}-2\mathfrak{b}-1)f), \mathcal{O}_X(-S+(\mathfrak{b}+1)f))$ is non-trivial. Putting altogether, the family $\mathcal{F}_2$ is non-empty.

In addition, by (3.20), $c_1(E_2) = S + (\mathfrak{d} - \mathfrak{b})f$ and

$$c_2(E_2) = |Z| + (2e + 4b + 3 - d) = l + 2e + 4b + 3 - d.$$

Now, let us consider $\mathcal{F}_3$ the family of rank 3 vector bundles E_3 on X with $c_1(E_3) = S + \mathfrak{d}f$ and $c_2(E_3) = c_2$ given by a non-trivial extension of type

$$0 \to \mathcal{O}_X(\mathfrak{b}f) \to E_3 \to E_2 \to 0, \tag{3.21}$$

where $E_2 \in \mathcal{F}_2$.

Claim 2: $\mathcal{F}_3 \neq \emptyset$.
Proof of Claim 2: By construction, E_3 is a rank 3 vector bundle on X with $c_1(E_3) = S + \mathfrak{d}f$ and $c_2(E_3) = c_2$. Notice that

$$\begin{aligned} s := \mathrm{ext}^1(E_2, \mathcal{O}_X(\mathfrak{b}f)) \quad &= \mathrm{ext}^1(\mathcal{O}_X(\mathfrak{b}f), E_2(K_X)) \quad = \mathrm{h}^1\, E_2(K_X - \mathfrak{b}f) \\ &\qquad\qquad\qquad\qquad\qquad\qquad\quad \geq -\chi E_2(K_X - \mathfrak{b}f) \end{aligned}$$

and, by Serre duality, $-\chi E_2(K_X - \mathfrak{b}f) = -\chi E_2^*(\mathfrak{b}f)$. Since

$$\begin{aligned} c_1(E_2^*(\mathfrak{b}f)) = &\quad -c_1(E_2) + 2\mathfrak{b}f = -S + (3\mathfrak{b} - \mathfrak{d})f \\ c_2(E_2^*(\mathfrak{b}f)) = &\quad c_2(E_2) + (\mathfrak{b}f)c_1(E_2^*) + (\mathfrak{b}f)^2 = 2e - d + 3b + 3 + l, \end{aligned}$$

by the Riemann-Roch theorem we have

$$\begin{aligned} \chi E_2^*(\mathfrak{b}f) \quad &= 2(1-g) - \tfrac{1}{2}c_1(E_2^*(\mathfrak{b}f)) \cdot K_X + \tfrac{1}{2}c_1(E_2^*(\mathfrak{b}f))^2 - c_2(E_2^*(\mathfrak{b}f)) \\ &= -l - 2e + d - 3b - 2 - g. \end{aligned}$$

Hence, $s \geq l + 2e - d + 3b + 2 + g > 0$ and the family $\mathcal{F}_3$ is non-empty.

Since we have seen that $\mathrm{h}^1\, E_2(K_X - \mathfrak{b}f) > 0$, it follows from Lemma 3.3.4 that E_3 is a simple prioritary vector bundle. Moreover, by (3.21) and the fact that $b > \max\{0, 2(g-1)\}$, we have

$$\mathrm{h}^0\, E_3 \geq \mathrm{h}^0\, \mathcal{O}_X(\mathfrak{b}f) = \mathrm{h}^0\, \mathcal{O}_C(\mathfrak{b}) = b + 1 - g = \frac{1}{5}(c_2 - 2e - 3 + d) + 1 - g,$$

which completes the proof. $\square$

Remark 3.3.6. In Theorem 3.3.5, we are roughly assuming that if we take c_2 sufficiently positive then, in particular, we get

$$b = \frac{1}{5}(c_2 - 2e - 3 + d - l) > \max\{0, 2(g-1), \frac{d-1}{3}, \frac{d + 2(g-1) - e}{3}\}.$$

The explicit bound to take would be

$$c_2 > \max\{2e+3+l-d, 10g+3e+l-d-7, \frac{1}{3}(2d+8+3l+6e), \frac{1}{3}(2d+10g+e+3l-1)\}.$$

Finally, we will deal with the case $c_1 = \mathfrak{d}f$.

Theorem 3.3.7. *Let X be a ruled surface over a non-singular curve C of genus $g \geq 0$. Let $c_2 >> 0$ be an integer and $\mathfrak{d} \in \operatorname{Pic}(C)$ such that $d = \deg(\mathfrak{d}) \geq 0$. Then, there exists a rank 3 simple prioritary vector bundle with $c_1(E) = \mathfrak{d}f$ and $c_2(E) = c_2$ such that*

$$\mathrm{h}^0 E \geq \frac{1}{3}(c_2 + d + 1 - 3g - e - l) + 1 - g,$$

where l is an integer $1 \leq l \leq 3$ equivalent to $c_2 + d + 1 - 3g - e$ modulo 3.

Proof. The proof follows step by step as in the proof of Theorem 3.3.5. In this case, we consider the family $\mathcal{G}_2$ of rank two vector bundles E_2 on X given by a non-trivial extension of type

$$0 \to \mathcal{O}_X(-S + (\mathfrak{b}+1)f) \to E_2 \to I_Z(S + (\mathfrak{d} - 2\mathfrak{b} - 1)f) \to 0 \tag{3.22}$$

where $b = \deg(\mathfrak{b}) = \frac{1}{3}(c_2 + d - 2 - e - l)$ and Z is a 0-dimensional subscheme of length $|Z| = l$ and the family $\mathcal{G}_3$ of rank 3 vector bundles E_3 on X given by a non-trivial extension of type

$$0 \to \mathcal{O}_X(\mathfrak{b}f) \to E_3 \to E_2 \to 0, \tag{3.23}$$

where $E_2 \in \mathcal{G}_2$. It can be seen that the families $\mathcal{G}_2$ and $\mathcal{G}_3$ are non-empty and, applying Lemma 3.3.4, we can prove that E_3 is a simple prioritary vector bundle. Finally, it follows from (3.22) and (3.23) that $c_1(E_3) = \mathfrak{d}f$, $c_2(E_3) = c_2$ and

$$\mathrm{h}^0 E_3 \geq \mathrm{h}^0 E_2 \geq \frac{1}{3}(c_2 + d + 1 - 3g - e - l) + 1 - g,$$

which finishes the proof. □

Remark 3.3.8. In Theorem 3.3.7, we are roughly assuming that if we take c_2 sufficiently positive then, in particular, we get

$$b = \frac{1}{3}(c_2 + d - 2 - e - l) > \max\{0, 2(g-1), \frac{d-1}{3}, \frac{d + 2(g-1) - e}{3}\}.$$

The explicit bound to take would be

$$c_2 > \max\{l + e + 2 - d, 6g - 4 - d + e + l, e + l + 1, 2g + l\}.$$

3.4 Rank 4 simple prioritary bundles

The purpose of this section is to establish the existence of simple prioritary vector bundles of rank 4 and to obtain a lower bound for the dimension of their space of global sections. To this end, we make use of the families constructed in Section 3 and we apply Theorem 3.2.1.

First of all, to simplify the proof of some results in this section, we summarize some cohomological conditions in the following two lemmas.

Lemma 3.4.1. *Let X be a ruled surface over a smooth curve C of genus $g \geq 0$, $c_2 \gg 0$ an integer and $D \in \operatorname{Pic}(X)$ a divisor such that $\pm D$ and $K_X + f \pm 2D$ are non-effective divisors. Let us consider $\mathcal{S}_3$ the family of rank 3 vector bundles given by a non-trivial extension of type*

$$0 \to E_2 \to E_3 \to \mathcal{O}_X(D) \to 0 \tag{3.24}$$

where E_2 is a rank 2 vector bundle on X such that $\mathrm{H}^1 E_2(-D) \neq 0$. The following holds:

i) If $\mathrm{h}^0 E_2(D) = 0$, then $\mathrm{h}^0 E_3(D) = 0$.

ii) If $\mathrm{h}^0 E_2(K_X + f + D) = 0$, then $\mathrm{h}^0 E_3(K_X + f + D) = 0$.

iii) If $\mathrm{h}^2 E_2(K_X + D) = 0$, then $\mathrm{h}^2 E_3(K_X + D) = 0$.

iv) If $\mathrm{h}^2 E_2(-f + D) = 0$, then $\mathrm{h}^2 E_3(-f + D) = 0$.

Proof. The hypothesis $\mathrm{H}^1 E_2(-D) \neq 0$ is needed to guarantee the existence of non-trivial extensions of type (3.24).

i) Notice that, if we twist the exact sequence (3.24) by $\mathcal{O}_X(D)$ and we take cohomology, we get

$$0 \to \mathrm{H}^0 E_2(D) \to \mathrm{H}^0 E_3(D) \to \mathrm{H}^0 \mathcal{O}_X(2D) \to \cdots$$

and therefore, since $\mathrm{h}^0 E_2(D) = 0$ and D is non-effective, we get $\mathrm{h}^0 E_3(D) = 0$. We prove *ii*) analogously, using the fact that $K_X + f + 2D$ is a non-effective divisor.

iii) If we twist the exact sequence (3.24) by $\mathcal{O}_X(K_X + D)$ and we take cohomology we get

$$\cdots \to \mathrm{H}^2 E_2(K_X + D) \to \mathrm{H}^2 E_3(K_X + D) \to \mathrm{H}^2 \mathcal{O}_X(K_X + 2D) \to 0.$$

By Serre duality, $\mathrm{H}^2 \mathcal{O}_X(K_X + 2D) = \mathrm{H}^0 \mathcal{O}_X(-2D)^*$. Therefore, since we are assuming that the divisor $-D$ is non-effective and that $\mathrm{h}^2 E_2(K_X + D) = 0$, from the above exact sequence we deduce that $\mathrm{H}^2 E_3(K_X + D) = 0$. We prove *iv*) analogously, using the fact that $K_X + f - 2D$ is non-effective. $\square$

Lemma 3.4.2. *Let X be a ruled surface over a non-singular curve C of genus $g \geq 0$ and let $L \in \mathrm{Pic}(X)$. Let us consider the family $\mathcal{H}_2$ of rank 2 vector bundle on X given by a non-trivial extension of type*

$$0 \to \mathcal{O}_X \to E_2 \to I_Z(f) \to 0, \tag{3.25}$$

where Z is a generic 0-dimensional scheme of length $|Z| \gg 0$. The following holds.

i) If $-L$ is a non-effective divisor and $|Z| > \mathrm{h}^0\,\mathcal{O}_X(-L+f)$, then we get $\mathrm{h}^0\,E_2(-L) = 0$.

ii) If L is a non-effective divisor, then $\mathrm{h}^2\,E_2(K_X - L) = 0$.

Proof. *i*) Twisting the exact sequence (3.25) by $\mathcal{O}_X(-L)$ and taking cohomology, we get the long exact sequence

$$0 \to \mathrm{H}^0\,\mathcal{O}_X(-L) \to \mathrm{H}^0\,E_2(-L) \to \mathrm{H}^0\,I_Z(f-L) \to \cdots.$$

Since $-L$ is non-effective, $\mathrm{H}^0\,\mathcal{O}_X(-L) = 0$. Since Z is a generic 0-dimensional subscheme of length $|Z| > \mathrm{h}^0\,\mathcal{O}_X(f-L)$, it follows from Proposition 1.1.13 that $\mathrm{H}^0\,I_Z(f-L) = 0$. Hence $\mathrm{H}^0\,E_2(-L) = 0$.
ii) If we twist the exact sequence (3.25) by $\mathcal{O}_X(K_X - L)$ and we take cohomology, we get the long exact sequence

$$\cdots \to \mathrm{H}^2\,\mathcal{O}_X(K_X - L) \to \mathrm{H}^2\,E_2(K_X - L) \to \mathrm{H}^2\,I_Z(f + K_X - L) \to 0.$$

Notice that, since L is non-effective, by duality we have $\mathrm{H}^2\,\mathcal{O}_X(K_X - L) = 0$. In addition, since $L - f$ is a non-effective divisor, by duality and (1.2), we deduce that

$$\mathrm{H}^2\,I_Z(f + K_X - L) = \mathrm{H}^2\,\mathcal{O}_X(f + K_X - L) = \mathrm{H}^0\,\mathcal{O}_X(L-f)^* = 0.$$

Hence, $\mathrm{H}^2\,E_2(K_X - L) = 0$. □

Remark 3.4.3. Later on, when applying Lemma 3.4.2 to construct rank 4 simple prioritary bundles from rank 2 vector bundles given by an extension of type (3.25), we will need the assumption that $|Z| \gg 0$. Nevertheless, the proof of Lemma 3.4.2 holds for Z of length $|Z| \geq 1$.

Lemma 3.4.4. *Let X be a ruled surface over a non-singular curve C of genus $g \geq 0$. Let $\mathcal{H}_2$ be the family of rank 2 vector bundles given in Lemma 3.4.2 by a non-trivial extension*

$$0 \to \mathcal{O}_X \to E_2 \to I_Z(f) \to 0,$$

where Z is a generic 0-dimensional scheme of length $|Z| \gg 0$. Let D be a divisor on X such that *i)* $-D$ *is non-effective and* $|Z| > \mathrm{h}^0\, \mathcal{O}_X(-D+f)$,

ii) D *is a non-effective divisor.*

Let $\mathcal{F}_3$ *be the family of rank 3 vector bundles given by non-trivial extensions of type*

$$0 \to E_2 \to E_3 \to \mathcal{O}_X(D) \to 0$$

where $E_2 \in \mathcal{H}_2$. *Then,* E_3 *is a rank* 3 *simple prioritary vector bundle.*

Proof. First of all, since by construction E_3 is a rank 3 vector bundle and $\mathrm{ext}^1(\mathcal{O}_X(D), E_2) = \mathrm{h}^1\, E_2(-D) > 0$, we can assume that $\mathcal{F}_3 \neq \emptyset$. Now, let us prove that E_3 is a simple prioritary bundle. Notice that, by the assumptions *i*) and *ii*), if we apply Lemma 3.4.2 with $L = D$, we get $\mathrm{h}^0\, E_2(-D) = 0$ and $\mathrm{h}^2\, E_2(K_X - D) = 0$. Furthermore, since $\pm D$ and $K_X \pm f$ are non-effective divisors, the divisors $K_X \pm (D-f)$ are also non-effective. Therefore, applying Lemma 3.4.2 for the cases that $L = -(K_X - D + f)$ and $L = K_X + D + f$, we get $\mathrm{h}^0\, E_2(K_X - D + f) = 0$ and $\mathrm{h}^2\, E_2(-D-f) = 0$. Finally, according to Example 3.1.8, E_2 is a simple prioritary vector bundle. Putting altogether, E_2 is under assumptions of Theorem 3.2.1 and, hence, E_3 is a simple prioritary vector bundle. $\square$

First of all, let us focus our attention on rank 4 simple prioritary vector bundles E on X with $c_1(E) = f$. In this case we use two different constructions according to the sign of the invariant of X, $\overline{e} := e - 2(g-1)$.

Theorem 3.4.5. *Let* X *be a ruled surface over a non-singular curve* C *of genus* $g \geq 0$ *such that* $\overline{e} := e - 2(g-1) \geq 0$ *and let* $c_2 \gg 0$ *be an integer. Then, there exists a rank 4 simple prioritary vector bundle* E *with* $c_1(E) = f$ *and* $c_2(E) = c_2$ *such that* $\mathrm{h}^0\, E \geq 1$.

Proof. First we consider the family $\mathcal{F}_2$ of rank 2 vector bundles given by a non-trivial extension of type

$$0 \to \mathcal{O}_X \to E_2 \to I_Z(f) \to 0, \tag{3.26}$$

where Z is a generic 0-dimensional subscheme of length $|Z| = c_2 - 2 - e \gg 0$. Let us consider $D = S - f \in \mathrm{Pic}(X)$ and the family $\mathcal{F}_3$ of rank 3 vector bundles on X given by a non-trivial extension of type

$$0 \to E_2 \to E_3 \to \mathcal{O}_X(D) \to 0,$$

where $E_2 \in \mathcal{F}_2$. Using the exact sequence (3.26) and (1.6), we see that

$$\mathrm{h}^1\, E_2(-D) = |Z| > 0.$$

Finally, consider the family $\mathcal{F}_4$ of rank 4 vector bundles on X given by a non-trivial extension of type

$$0 \to E_3 \to E_4 \to \mathcal{O}_X(-D) \to 0, \tag{3.27}$$

where $E_3 \in \mathcal{F}_3$.

By construction, E_4 is a rank 4 vector bundle with

$$\begin{aligned} c_1(E_4) &= c_1(E_3) - D = c_1(E_2) = f \\ c_2(E_4) &= c_2(E_3) + c_1(E_3)\cdot(-D) = c_2(E_2) - D^2 = |Z| + e + 2 = c_2. \end{aligned}$$

On the other hand,

$$\operatorname{ext}^1(\mathcal{O}_X(-D), E_3) = \mathrm{h}^1\, E_3(D) \geq -\chi E_3(D) > 0.$$

In fact, since

$$\begin{aligned} c_1(E_3(D)) &= c_1(E_3) + 3D = f + 4D = 4S - 3f \\ c_2(E_3(D)) &= c_2(E_3) + 2c_1(E_3)\cdot D + 3D^2 = |Z| + 3f\cdot D + 5D^2 \\ &= c_2 - 6e - 9, \end{aligned}$$

by the Riemann-Roch theorem, we get

$$\begin{aligned} -\chi(E_3(D)) = &\ -3(1-g) + \tfrac{1}{2}c_1(E_3(D))\cdot K_X - \tfrac{1}{2}c_1(E_3(D))^2 + c_2(E_3(D)) \\ = &\ c_2 + 2e + 7g - 1 > 0. \end{aligned}$$

Hence, $\mathcal{F}_4 \neq \emptyset$.

Now, let us see that E_4 is simple and prioritary. Notice that the divisors $\pm D$, K_X+f-D and K_X+2f+D are non-effective, and $|Z| > 2 \geq \mathrm{h}^0\,\mathcal{O}_X(D+f)$. Therefore, by Lemma 3.4.2,

$$\begin{array}{lcl} \mathrm{h}^0\, E_2(D) = 0, & & \mathrm{h}^0\, E_2(K_X+f+D) = 0, \\ \mathrm{h}^2\, E_2(K_X+D) = 0 & \text{and} & \mathrm{h}^2\, E_2(-f+D) = 0. \end{array} \tag{3.28}$$

Furthermore, since $\overline{e} \geq 0$, $\pm D$ and $K_X + f \pm 2D$ are non-effective divisors. Thus, by Lemma 3.4.1, the conditions (3.28) imply that

$$\begin{array}{lcl} \mathrm{h}^0\, E_3(D) = 0, & & \mathrm{h}^0\, E_3(K_X+f+D) = 0, \\ \mathrm{h}^2\, E_3(K_X+D) = 0 & \text{and} & \mathrm{h}^2\, E_3(-f+D) = 0. \end{array} \tag{3.29}$$

Moreover, since D is under the assumptions of Lemma 3.4.4, E_3 is a simple prioritary rank 3 vector bundle. Putting altogether, E_3 is under the conditions of Theorem 3.2.1 and, hence, E_4 is a rank 4 simple prioritary vector bundle. Finally, by construction, $\mathrm{h}^0\, E_4 \geq \mathrm{h}^0\, E_3 \geq \mathrm{h}^0\, E_2 \geq 1$. $\square$

Now, let us consider the case $\overline{e} := e - 2(g-1) < 0$.

Theorem 3.4.6. *Let X be a ruled surface over a non-singular curve C of genus $g \geq 0$ such that $\overline{e} := e - 2(g-1) < 0$ and $c_2 \gg 0$ an integer. Then, there exists a rank 4 simple prioritary vector bundle E_4 with $c_1(E_4) = f$ and $c_2(E_4) = c_2$ such that $\mathrm{h}^0 E_4 \geq 1$.*

Proof. The proof follows step by step as in the proof of Theorem 3.4.5. In this case, we consider the family $\mathcal{F}_2$ of rank 2 vector bundles E_2 given by a non-trivial extension of type

$$0 \to \mathcal{O}_X \to E_2 \to I_Z(f) \to 0,$$

where Z is a generic 0-dimensional subscheme of length $|Z| = c_2+e-4g+2 \gg 0$. Now, we consider the family $\mathcal{F}_3$ of rank 3 vector bundles on X given by a non-trivial extension of type

$$0 \to E_2 \to E_3 \to \mathcal{O}_X(D) \to 0,$$

where $E_2 \in \mathcal{F}_2$ and $D = S + (\overline{e} - 1)f$. Finally, we consider the family $\mathcal{F}_4$ of rank 4 vector bundles on X given by a non-trivial extension of type

$$0 \to E_3 \to E_4 \to \mathcal{O}_X(-D) \to 0,$$

where $E_3 \in \mathcal{F}_3$ and $D = S + (\overline{e} - 1)f$. Applying Lemmas 3.4.1 and 3.4.4 we conclude. $\square$

We end the section by studying the case $c_1 = S + \mathfrak{m}f$.

Theorem 3.4.7. *Let X be a ruled surface over a non-singular curve C of genus $g \geq 0$ and $c_1 = S + \mathfrak{m}f \in \mathrm{Pic}(X)$ with $m = \deg(\mathfrak{m}) \in \{0, 1\}$. Let $c_2 \gg 0$ be an integer and $3 \leq l \leq 5$ be the numerical class of $c_2 - 2 - e$ module 3. Then, there exists a rank 4 simple prioritary vector bundle E with $c_1(E) = c_1$ and $c_2(E) = c_2$ such that $\mathrm{h}^0 E \geq k := \frac{1}{3}(c_2 - 2 - e - l) + 1 - g$.*

Proof. The proof is analogous to the one of Theorem 3.4.5. Let us consider the family $\mathcal{G}_2$ of rank 2 vector bundles on X given by a non-trivial extension of type

$$0 \to \mathcal{O}_X(\mathfrak{b}f) \to E_2 \to I_Z(S + (\mathfrak{m} - \mathfrak{b})f) \to 0,$$

where $\mathfrak{b} \in \mathrm{Pic}(X)$ is such that $b = \deg(\mathfrak{b}) = \frac{1}{3}(c_2 - 2 - e - l)$ and Z is a generic 0-dimensional subscheme of length $|Z| = l$. Notice that, since $c_2 \gg 0$, we can assume $b > \max\{0, 2(g-1)\}$. Let us fix $D = S - (\mathfrak{b} + 1)f$.

Now, consider the family $\mathcal{G}_3$ of rank 3 vector bundles on X given by a non-trivial extension of type

$$0 \to E_2 \to E_3 \to \mathcal{O}_X(D) \to 0,$$

where $E_2 \in \mathcal{G}_2$. Finally, consider the family $\mathcal{G}_4$ of rank 4 vector bundles on X given by a non-trivial extension of type

$$0 \to E_3 \to E_4 \to \mathcal{O}_X(-D) \to 0$$

where $E_3 \in \mathcal{G}_3$. Notice that, by Serre the Riemann-Roch theorem,

$$\operatorname{ext}^1(\mathcal{O}_X(-D), E_3) \geq -\chi E_3(D) > 0$$

and hence $\mathcal{G}_4 \neq \emptyset$. Moreover, by construction E_4 is a rank 4 vector bundle with $c_1(E_4) = S + \mathfrak{m}f$ and $c_2(E_4) = c_2$.

Claim : E_4 is simple and prioritary.
Proof of the Claim: First of all, notice that, for $\alpha \in \{0, 1\}$, the divisors

$$\begin{array}{ll} (-1)^\alpha[D + \mathfrak{b}f], & (-1)^\alpha[D + S + (\mathfrak{m} - \mathfrak{b})f], \\ K_X + f + D + (-1)^\alpha[S + (\mathfrak{m} - \mathfrak{b})f)], & K_X + f + D + (-1)^\alpha \alpha(\mathfrak{b}f), \end{array}$$

are non-effective. Therefore, by Lemma 3.3.1,

$$\begin{array}{lcl} \mathrm{h}^0\, E_2(D) = 0, & & \mathrm{h}^0\, E_2(K_X + f + D) = 0, \\ \mathrm{h}^2\, E_2(K_X + D) = 0, & \text{and} & \mathrm{h}^2\, E_2(-f - D) = 0. \end{array} \tag{3.30}$$

Moreover, since $\pm D$ and $K_X + f \pm 2D$ are non-effective divisors, the conditions (3.30) imply, by Lemma 3.4.1, that

$$\begin{array}{lcl} \mathrm{h}^0\, E_3(D) = 0, & & \mathrm{h}^0\, E_3(K_X + f + D) = 0, \\ \mathrm{h}^2\, E_3(K_X + D) = 0, & \text{and} & \mathrm{h}^2\, E_3(-f - D) = 0. \end{array} \tag{3.31}$$

Now, arguing as in the proof of Theorem 3.3.2, checking that E_2 is under assumptions of Theorem 3.2.1, we prove that E_3 is a simple prioritary rank 3 vector bundle. Putting altogether, E_3 verifies the conditions of Theorem 3.2.1 and, hence, E_4 is a simple prioritary vector bundle. Finally, by construction and since $b > \max\{0, 2(g-1)\}$,

$$\mathrm{h}^0\, E_4 \geq \mathrm{h}^0\, E_3 \geq \mathrm{h}^0\, E_2 \geq \mathrm{h}^0\, \mathcal{O}_X(\mathfrak{b}f) = \mathrm{h}^0\, \mathcal{O}_C(\mathfrak{b}) = b + 1 - g,$$

which finishes the proof. $\square$

3.5 Higher rank simple prioritary bundles

In this section, we will construct simple prioritary bundles of rank $r \geq 4$ with sections. In Section 3.2, we have seen the existence of rank $r \geq 4$ prioritary bundles with a large number of sections. Since we cannot guarantee that they are indecomposable, they could split as a direct sum of prioritary bundles of lower rank. In general, it is a difficult problem to construct indecomposable rank

r vector bundles on a smooth projective variety X when r is big compared with $\dim(X)$. We will achieve our goal using the generalization of Cayley-Bacharach property given in Theorem 1.1.19 and, in addition, we will see that the bundles that we construct are simple (and in particular indecomposable) and prioritary with a certain number of sections. To this end, we start with a technical lemma that we will use later on.

Lemma 3.5.1. *Let X be a ruled surface over a non-singular curve C of genus $g \geq 0$. Let E be a rank r vector bundle on X given by a non-trivial extension of type*

$$e: \quad 0 \to \mathcal{O}_X(\mathfrak{b}f) \to E \to \bigoplus_{i=1}^{r-1} I_{Z_i}(D_i) \to 0 \tag{3.32}$$

where, for every i, Z_i is a reduced 0-dimensional subscheme such that $Z_i \cap Z_j = \emptyset$ for $i \neq j$ and

$$e = (e_1, \ldots, e_{r-1}) \in \operatorname{Ext}^1(\bigoplus_{i=1}^{r-1} I_{Z_i}(D_i), \mathcal{O}_X(\mathfrak{b}f))$$

is a non-trivial extension such that, for every i, $e_i \in \operatorname{Ext}^1(I_{Z_i}(D_i), \mathcal{O}_X(\mathfrak{b}f))$ is non-trivial. Let us assume that

i) for every $i \neq j$, $D_i - D_j$ is non-effective,

ii) for every i, $\mathfrak{b}f - D_i$ is non-effective,

iii) for every $i \neq j$, $D_i - D_j + K_X + f$ is non-effective,

iv) for every i, $\mathfrak{b}f - D_i + K_X + f$ is non-effective,

v) for every i, $\mathrm{h}^0\, I_{Z_i}(D_i - \mathfrak{b}f) = 0$,

vi) for every i, $\mathrm{h}^0\, I_{Z_i}(K_X + f + D_i - \mathfrak{b}f) = 0$,

Then, E is a rank r simple prioritary vector bundle.

Proof. First of all, let us see that E is simple. Applying the functor $\operatorname{Hom}(-, E)$ to the exact sequence (3.32), we get

$$0 \to \operatorname{Hom}(\bigoplus_{i=1}^{r-1} I_{Z_i}(D_i), E) \to \operatorname{Hom}(E, E) \to \operatorname{Hom}(\mathcal{O}_X(\mathfrak{b}f), E) \to \cdots . \tag{3.33}$$

Therefore, since we always have $\dim \operatorname{Hom}(E, E) \geq 1$, by (3.33), it is enough to prove that $\operatorname{Hom}(\bigoplus_{i=1}^{r-1} I_{Z_i}(D_i), E) = 0$ and $\operatorname{Hom}(\mathcal{O}_X(\mathfrak{b}f), E) \cong K$.

Claim 1: $\operatorname{Hom}(\bigoplus_{i=1}^{r-1} I_{Z_i}(D_i), E) = 0$.
Proof of Claim 1: Since

$$\operatorname{Hom}(\bigoplus_{i=1}^{r-1} I_{Z_i}(D_i), E) \cong \bigoplus_{i=1}^{r-1} \operatorname{Hom}(I_{Z_i}(D_i), E),$$

it is enough to prove that, for any i, $\operatorname{Hom}(I_{Z_i}(D_i), E) = 0$.

Let us fix $i_0 \in I := \{1, \dots, r-1\}$. Applying the functor $\operatorname{Hom}(I_{Z_{i_0}}(D_{i_0}), -)$ to the exact sequence (3.32) we get the long exact sequence

$$\begin{aligned} 0 \quad &\to \operatorname{Hom}(I_{Z_{i_0}}(D_{i_0}), \mathcal{O}_X(\mathfrak{b}f)) \to \operatorname{Hom}(I_{Z_{i_0}}(D_{i_0}), E) \\ &\to \operatorname{Hom}(I_{Z_{i_0}}(D_{i_0}), \textstyle\bigoplus_{i=1}^{r-1} I_{Z_i}(D_i)) \to \operatorname{Ext}^1(I_{Z_{i_0}}(D_{i_0}), \mathcal{O}_X(\mathfrak{b}f)) \to \cdots . \end{aligned} \tag{3.34}$$

By assumption ii), we have

$$\begin{aligned} \operatorname{Hom}(I_{Z_{i_0}}(D_{i_0}), \mathcal{O}_X(\mathfrak{b}f)) &\cong \operatorname{Ext}^2(\mathcal{O}_X(\mathfrak{b}f), I_{Z_{i_0}}(D_{i_0} + K_X))^* \\ &\cong \mathrm{H}^2\, I_{Z_{i_0}}(D_{i_0} - \mathfrak{b}f + K_X)^* \\ &\cong \mathrm{H}^2\, \mathcal{O}_X(D_{i_0} - \mathfrak{b}f + K_X)^* \\ &\cong \mathrm{H}^0\, \mathcal{O}_X(\mathfrak{b}f - D_{i_0}) = 0. \end{aligned}$$

On the other hand,

$$\operatorname{Hom}(I_{Z_{i_0}}(D_{i_0}), \bigoplus_{i=1}^{r-1} I_{Z_i}(D_i)) \cong \bigoplus_{i=1}^{r-1} \operatorname{Hom}(I_{Z_{i_0}}(D_{i_0}), I_{Z_i}(D_i)).$$

If $i = i_0$, by Lemma 3.1.5, we get

$$\operatorname{Hom}(I_{Z_{i_0}}(D_{i_0}), I_{Z_i}(D_i)) \cong K.$$

Assume $i \neq i_0$. Applying the covariant functor $\operatorname{Hom}(I_{Z_{i_0}}, -)$ to the exact sequence

$$0 \to I_{Z_i} \to \mathcal{O}_X \to \mathcal{O}_{Z_i} \to 0$$

twisted by $\mathcal{O}_X(D_i - D_{i_0})$, we get

$$\hom(I_{Z_{i_0}}, I_{Z_i}(D_i - D_{i_0})) \leq \hom(I_{Z_{i_0}}, \mathcal{O}_X(D_i - D_{i_0})).$$

In addition, by Serre duality,

$$\begin{aligned} \operatorname{Hom}(I_{Z_{i_0}}, \mathcal{O}_X(D_i - D_{i_0})) &= \operatorname{Ext}^2(\mathcal{O}_X(D_i - D_{i_0}), I_{Z_{i_0}}(K_X))^* \\ &= \mathrm{H}^2\, I_{Z_{i_0}}(K_X + D_{i_0} - D_i)^* \\ &= \mathrm{H}^0\, \mathcal{O}_X(D_i - D_{i_0}), \end{aligned}$$

and $\mathrm{h}^0\, \mathcal{O}_X(D_i - D_{i_0}) = 0$ by assumption i). Putting altogether, we obtain

$$\operatorname{Hom}(I_{Z_{i_0}}, I_{Z_{i_0}}(D_i - D_{i_0})) = 0$$

whenever $i \neq i_0$ and hence

$$\bigoplus_{i=1}^{r-1} \mathrm{Hom}(I_{Z_{i_0}}(D_{i_0}), I_{Z_i}(D_i)) \cong K \oplus \left(\bigoplus_{i \neq i_0} \mathrm{Hom}(I_{Z_{i_0}}(D_{i_0}), I_{Z_i}(D_i)) \right) \cong K.$$

Moreover,

$$\mathrm{Hom}(I_{Z_{i_0}}(D_{i_0}), \mathcal{O}_X(\mathfrak{b}f)) = \mathrm{H}^2\, I_{Z_{i_0}}(D_{i_0} + K_X - \mathfrak{b}f)^* = \mathrm{H}^0\, \mathcal{O}_X(\mathfrak{b}f - D_{i_0}) = 0,$$

where the last equality follows from the assumption ii). Hence, the exact sequence (3.34) turns to be

$$0 \to \mathrm{Hom}(I_{Z_{i_0}}(D_{i_0}), E) \to K \stackrel{\gamma_{i_0}}{\to} \mathrm{Ext}^1(I_{Z_{i_0}}(D_{i_0}), \mathcal{O}_X(\mathfrak{b}f)) \to \cdots,$$

which means that

$$\mathrm{Hom}(I_{Z_{i_0}}(D_{i_0}), E) \cong \ker(\gamma_{i_0}).$$

By assumption, the non-trivial extension

$$e = (e_1, \ldots, e_{r-1}) \in \mathrm{Ext}^1(\bigoplus_{i=1}^{r-1} I_{Z_i}(D_i), \mathcal{O}_X(\mathfrak{b}f))$$

is defined by non-trivial extensions $e_i \in \mathrm{Ext}^1(I_{Z_i}(D_i), \mathcal{O}_X(\mathfrak{b}f))$ and thus γ_{i_0} is the map that sends $1 \in K$ to the non-trivial extension

$$e_{i_0} \in \mathrm{Ext}^1(I_{Z_{i_0}}(D_{i_0}), \mathcal{O}_X(\mathfrak{b}f)).$$

Therefore, γ_{i_0} is injective and hence

$$\mathrm{Hom}(I_{Z_{i_0}}(D_{i_0}), E) \cong \ker(\gamma_{i_0}) = 0.$$

Claim 2: $\mathrm{H}^0\, E(-\mathfrak{b}f) \cong K$.
Proof of Claim 2: Consider the exact sequence (3.32) twisted by $\mathcal{O}_X(-\mathfrak{b}f)$ and taking cohomology we get

$$0 \to \mathrm{H}^0\, \mathcal{O}_X \to \mathrm{H}^0\, E(-\mathfrak{b}f) \to \bigoplus_{i=1}^{r-1} \mathrm{H}^0\, I_{Z_i}(D_i - \mathfrak{b}f) \to \cdots.$$

Thus, by v),

$$\mathrm{H}^0\, E(-\mathfrak{b}f) \cong \mathrm{H}^0\, \mathcal{O}_X \cong K.$$

Therefore, by (3.33), we get $\hom(E,E) \leq 1$, and hence $\hom(E,E) = 1$, which means that E is simple.

Now, let us see that E is prioritary. Applying the functor $\mathrm{Hom}(E,-)$ to the exact sequence (3.32) twisted by $\mathcal{O}_X(-f)$, we obtain the exact sequence

$$\cdots \to \mathrm{Ext}^2(E,\mathcal{O}_X((\mathfrak{b}-1)f)) \to \mathrm{Ext}^2(E,E(-f)) \to \mathrm{Ext}^2(E,\bigoplus_{i=1}^{r-1} I_{Z_i}(D_i-f)) \to 0. \tag{3.35}$$

Claim 3: $\mathrm{Ext}^2(E,\mathcal{O}_X((\mathfrak{b}-1)f))=0$.
Proof of Claim 3: Notice that, by duality,

$$\mathrm{Ext}^2(E,\mathcal{O}_X((\mathfrak{b}-1)f)) = \mathrm{H}^0\, E(K_X-(\mathfrak{b}-1)f)^*.$$

Twisting (3.32) by $\mathcal{O}_X(K_X-(\mathfrak{b}-1)f)$ and taking cohomology we get

$$0 \to \mathrm{H}^0\,\mathcal{O}_X(K_X+f) \to \mathrm{H}^0\, E(K_X-(\mathfrak{b}-1)f) \to \bigoplus_{i=1}^{r-1} \mathrm{H}^0\, I_{Z_i}(D_i+K_X-(\mathfrak{b}-1)f) \to .$$

By $vi)$, $\mathrm{H}^0\, I_{Z_i}(D_i+K_X-(\mathfrak{b}-1)f)=0$ for any i and, since K_X+f is non-effective, we get

$$\mathrm{H}^0\, E(K_X-(\mathfrak{b}-1)f)=0.$$

Claim 4: $\mathrm{Ext}^2(E,\bigoplus_{i=1}^{r-1} I_{Z_i}(D_i-f))=0$.
Proof of Claim 4: Notice that

$$\mathrm{Ext}^2(E,\bigoplus_{i=1}^{r-1} I_{Z_i}(D_i-f)) \cong \bigoplus_{i=1}^{r-1} \mathrm{Ext}^2(E,I_{Z_i}(D_i-f)).$$

Then, it is sufficient to prove that, for a fixed $i_0\in I:=\{1,\dots,r-1\}$, we have

$$\mathrm{Ext}^2(E,I_{Z_{i_0}}(D_{i_0}-f))=0.$$

Applying the contravariant functor $\mathrm{Hom}(-,I_{Z_{i_0}}(D_{i_0}-f))$ to the exact sequence (3.32), we get

$$\begin{aligned}\cdots \to &\mathrm{Ext}^2(\bigoplus_{i=1}^{r-1} I_{Z_i}(D_i),\, I_{Z_{i_0}}(D_{i_0}-f)) \to \mathrm{Ext}^2(E,\, I_{Z_{i_0}}(D_{i_0}-f)) \\ &\to \mathrm{Ext}^2(\mathcal{O}_X(\mathfrak{b}f),\, I_{Z_{i_0}}(D_{i_0}-f)) \to 0.\end{aligned} \tag{3.36}$$

Let us first check that

$$\mathrm{Ext}^2(\bigoplus_{i=1}^{r-1} I_{Z_i}(D_i),I_{Z_{i_0}}(D_{i_0}-f)) \cong \bigoplus_{i=1}^{r-1} \mathrm{Ext}^2(I_{Z_i}(D_i),I_{Z_{i_0}}(D_{i_0}-f))=0.$$

If $i = i_0$, by Lemma 3.1.5, we get

$$\mathrm{Ext}^2(I_{Z_i}(D_i), I_{Z_{i_0}}(D_{i_0} - f)) = 0.$$

Let us assume $i \neq i_0$.

Since for any i_0 we have the exact sequence

$$\cdots \to \mathrm{Ext}^2(\mathcal{O}_X(D_i), I_{Z_{i_0}}(D_{i_0} - f)) \to \mathrm{Ext}^2(I_{Z_i}(D_i), I_{Z_{i_0}}(D_{i_0} - f)) \to 0$$

and, by *iii*),

$$\begin{aligned} \mathrm{Ext}^2(\mathcal{O}_X(D_i), I_{Z_{i_0}}(D_{i_0} - f)) &\cong \mathrm{H}^2\, \mathcal{O}_X(D_{i_0} - D_i - f) \\ &\cong \mathrm{H}^0\, \mathcal{O}_X(D_i - D_{i_0} + K_X + f)^* = 0, \end{aligned}$$

we get

$$\mathrm{Ext}^2(I_{Z_i}(D_i), I_{Z_{i_0}}(D_{i_0} - f)) = 0.$$

Putting altogether,

$$\mathrm{Ext}^2(\bigoplus_{i=1}^{r-1} I_{Z_i}(D_i), I_{Z_{i_0}}(D_{i_0} - f)) = 0.$$

Finally, by *iv*),

$$\mathrm{Ext}^2(\mathcal{O}_X(\mathfrak{b}f), I_{Z_{i_0}}(D_{i_0} - f)) \cong \mathrm{H}^0\, \mathcal{O}_X(\mathfrak{b}f + f - D_{i_0} + K_X)^* = 0.$$

Hence, by (3.36), we get Claim 4.

Finally, putting altogether, by (3.35) we conclude that E is a prioritary vector bundle. $\square$

Now, let us apply Lemma 3.5.1 in order to construct higher rank simple prioritary bundles with at least certain number of sections.

We will study separately the results according to the value of $\overline{e} := e - 2(g-1)$.

Let us start with the case $\overline{e} := e - 2(g-1) > 0$.

Theorem 3.5.2. *Let X be a ruled surface over a non-singular curve C of genus $g \geq 0$ such that $\overline{e} := e - 2(g-1) > 0$. Fix $c_2 \gg 0$ and $r \geq 4$ integers, and $c_1 = sS + \mathfrak{t}f \in \mathrm{Pic}(X)$ with $0 \leq s \leq r-1$ and $t := \deg(\mathfrak{t}) \geq \max\{0, 2g - r\}$. Then, there exists a rank r simple prioritary vector bundle with $c_1(E) = c_1$ and $c_2(E) = c_2$ such that $\mathrm{h}^0\, E \geq k := r - 1 + t - g$.*

Proof. Let us fix $\mathfrak{b} \in \mathrm{Pic}(X)$ a divisor of degree $b = r - 2 + t \geq 0$ and consider $r-1$ divisors $D_i \in \mathrm{Pic}(X)$ such that

$$\begin{array}{lll} D_1 & \equiv -(\sum_{j=2}^{r-1} j)S + (r-1+t)f, & \\ D_i & \equiv iS + (r-i)f, & \text{for} \quad 2 \leq i \leq r-2, \\ D_{r-1} & \equiv (r-1+s)S - (b + \sum_{j=2}^{r-1} j)f. & \end{array}$$

Let Z_i be a reduced generic 0-dimensional subscheme of length $|Z_i| = 1$ for $1 \leq i \neq 2 \leq r-2$, $|Z_2| = 4$ and

$$|Z_{r-1}| \quad = c_2 - (r+2) - \textstyle\sum_{i<j} D_i \cdot D_j - (\mathfrak{b}f)(\sum_{i=1}^{r-1} D_i),$$

such that $Z_i \cap Z_j = \emptyset$ for any $i \neq j$. Notice that, since c_2 is independent of b and the divisors D_i, we may assume that $|Z_{r-1}| > 0$.

Keeping the above notation, let us consider the family $\mathcal{F}$ of rank r vector bundles E on X with $c_1(E) = c_1$ and $c_2(E) = c_2$ given by a non-trivial extension of type

$$0 \to \mathcal{O}_X(\mathfrak{b}f) \to E \to \bigoplus_{i=1}^{r-1} I_{Z_i}(D_i) \to 0. \tag{3.37}$$

Claim : $\mathcal{F} \neq \emptyset$.
Proof of the Claim : Notice that, since $b \geq 1$ and $\overline{e} \geq 1$, the divisors

$$\begin{array}{lll} D_1 - \mathfrak{b}f + K_X & \equiv -(2 + \sum_{j=2}^{r-1} j)S + (r-1+t-b-\overline{e})f & \\ D_i - \mathfrak{b}f + K_X & \equiv (i-2)S + (r-i-b-\overline{e})f & 2 \leq i \leq r-2 \\ D_{r-1} - \mathfrak{b}f + K_X & \equiv (r-3+s)S - (2b - \overline{e} + \sum_{j=2}^{r-1} j)f & \end{array}$$

are non-effective and thus, for any i, Z_i satisfies the Cayley-Bacharach property with respect to the linear system $|D_i - \mathfrak{b}f + K_X|$. Hence, by Theorem 1.1.19, E is a rank r vector bundle and, by Remark 1.1.20, we can assume that

$$e = (e_1, \ldots, e_{r-1}) \in \mathrm{Ext}^1(\bigoplus_{i=1}^{r-1} I_{Z_i}(D_i), \mathcal{O}_X(\mathfrak{b}f))$$

is a non-trivial extension with $e_i \neq 0$ for $1 \leq i \leq r-1$. Moreover, by (3.37), $c_1(E) = sS + \mathfrak{t}f$ and

$$c_2(E) = r - 3 + |Z_2| + |Z_{r-1}| + \sum_{i<j} D_i \cdot D_j + (\mathfrak{b}f)(\sum_{i=1}^{r-1} D_i) = c_2.$$

Putting altogether, the family $\mathcal{F}$ is non-empty.

Notice that, for any i, the divisors D_i verify all the assumptions of Lemma 3.5.1. In fact, since $s,t \geq 0$, $r \geq 4$ and $b = r-2+t$, for any $2 \leq i \neq j \leq r-2$

$$\begin{array}{ll}
D_1 - D_i & \equiv -(i + \sum_{j=2}^{r-1} j)S + (i-1+t)f \\
D_i - D_j & \equiv (i-j)S + (j-i)f \\
D_i - D_1 & \equiv (i + \sum_{j=2}^{r-1} j)S - (t+i-1)f \\
D_i - D_{r-1} & \equiv [i-(r-1+s)]S + (r-i+b+\sum_{j=2}^{r-1} j) \\
D_{r-1} - D_i & \equiv (r-1+s-i)S - (b+r-i+\sum_{j=2}^{r-1} j)f \\
D_{r-1} - D_1 & \equiv (r-1+s+\sum_{j=2}^{r-1} j)S - (b+r-1+t+\sum_{j=2}^{r-1} j)f \\
D_1 - D_{r-1} & \equiv -(r-1+s+\sum_{j=2}^{r-1} j)S + (r-1+t+b+\sum_{j=2}^{r-1} j)f
\end{array}$$

are non-effective divisors and thus $i)$ holds. Again, by construction,

$$\begin{array}{lll}
\mathfrak{b}f - D_1 & \equiv (\sum_{j=2}^{r-1} j)S - f & \\
\mathfrak{b}f - D_i & \equiv -iS + (b-r+i)f & 2 \leq i \leq r-2 \\
\mathfrak{b}f - D_{r-1} & \equiv -(r-1+s)S + (2b + \sum_{j=2}^{r-1} j)f &
\end{array}$$

are non-effective divisors and we get $ii)$. Now, let us check that all D_i's satisfy $iii)$. On one hand, since $D_1 - D_i$, $D_1 - D_{r-1}$ and $K_X = -2S - \overline{e}f$ are non-effective divisors, it is straightforward to check that $D_1 - D_i + K_X + f$ and $D_1 - D_{r-1} + K_X + f$ are also non-effective divisors. On the other hand, since $\overline{e} > 0$, $t \geq 0$ and $b = r-2+t$, for any $2 \leq i \neq j \leq r-2$,

$$\begin{array}{ll}
D_1 - D_i + K_X + f & \equiv -(i+2+\sum_{j=1}^{r-1} j)S + (t-i-\overline{e})f \\
D_i - D_j + K_X + f & \equiv (i-j-2)S + (j-i-\overline{e}+1)f \\
D_i - D_1 + K_X + f & \equiv (i-2+\sum_{j=2}^{r-1} j)S - (\overline{e}+i+t-2)f \\
D_{r-1} - D_i + K_X + f & \equiv (r-3+s-i)S - (b+\overline{e}+\sum_{j=2}^{r-1} j + r-i-1)f \\
D_i - D_{r-1} + K_X + f & \equiv -(r+1+s-i)S + (r-i+b-\overline{e}+1+\sum_{j=1}^{r-1} j)f \\
D_{r-1} - D_1 + K_X + f & \equiv (r-3+s+\sum_{j=2}^{r-1} j)S - (2b+\overline{e}+\sum_{j=2}^{r-1} j)f \\
D_1 - D_{r-1} + K_X + f & \equiv -(r+1+s+\sum_{j=2}^{r-1} j)S + (2b+2-\overline{e}+\sum_{j=2}^{r-1} j)
\end{array}$$

are also non-effective divisors. Moreover, notice that, since $K_X = -2S - \overline{e}f$ with $\overline{e} > 0$, is straightfoward to check that $\mathfrak{b}f - D_i + K_X + f$ is non-effective for any $1 \leq i \leq r-1$.

Finally, let us check that, for any i, the D_i's satisfy conditions $v)$ and $vi)$. Notice that, if $D_i - \mathfrak{b}f$ and $K_X + f + D_i - \mathfrak{b}f$ are non-effective divisors, we directly get $\mathrm{h}^0 I_{Z_i}(D_i - \mathfrak{b}f) = 0$ and $\mathrm{h}^0 I_{Z_i}(K_X + f + D_i - \mathfrak{b}f) = 0$. The divisor $D_i - \mathfrak{b}f$ is always non-effective unless we have $i = 2$ and $t = 0$. The divisor $K_X + f + D_i - \mathfrak{b}f$ is non-effective apart from the case $i = 2$, $t = 0$ and $\overline{e} = 1$. Anyhow, for these cases, the conditions $\mathrm{h}^0 I_{Z_2}(D_2 - \mathfrak{b}f) = 0$ and $\mathrm{h}^0 I_{Z_2}(K_X + f + D_2 - \mathfrak{b}f) = 0$ follow from the fact that Z_2 is generic and

$$\begin{array}{llll}
\mathrm{h}^0 \mathcal{O}_X(D_2 - \mathfrak{b}f) & = \mathrm{h}^0 \mathcal{O}_X(2S) & \leq 3 & < |Z_2| \\
\mathrm{h}^0 \mathcal{O}_X(D_2 - \mathfrak{b}f + K_X + f) & = \mathrm{h}^0 \mathcal{O}_X & = 1 & < |Z_2|
\end{array}$$

(see Proposition 1.1.13). In addition, by Remark 1.1.20, the non-trivial extension

$$e = (e_1, \dots, e_{r-1}) \in \operatorname{Ext}^1(\bigoplus_{i=1}^{r-2} I_{Z_i}(D_i), \mathcal{O}_X(\mathfrak{b}f))$$

is such that, for any i, $e_i \in \operatorname{Ext}^1(I_{Z_i}(D_i), \mathcal{O}_X(\mathfrak{b}f))$ is a non-trivial extension.

Putting altogether, we are under assumptions of Lemma 3.5.1 and, hence, E is a rank r simple prioritary vector bundle. Moreover, since

$$b = r - 2 + t \geq \max\{2(g-1), r-2\},$$

by the long exact sequence of cohomology associated to (3.37) we get

$$\mathrm{h}^0 E \geq \mathrm{h}^0 \mathcal{O}_X(\mathfrak{b}f) = b + 1 - g = r - 1 + t - g,$$

which finishes the proof. □

The case $\overline{e} := e - 2(g-1) = 0$ can be proved analogously. We get the following result.

Theorem 3.5.3. *Let X be a ruled surface over a non-singular curve C of genus $g \geq 0$ such that $\overline{e} := e - 2(g-1) = 0$. Fix $c_2 \gg 0$ and $r \geq 4$ integers, and $c_1 = sS + \mathfrak{t}f \in \operatorname{Pic}(X)$ with $0 \leq s \leq r-1$ and $t := \deg(\mathfrak{t}) \geq \max\{0, 2g-r\}$. Then, there exists a rank r simple prioritary vector bundle with $c_1(E) = c_1$ and $c_2(E) = c_2$ such that $\mathrm{h}^0 E \geq k := r - 2 + t - g$.*

Proof. The proof is analogous to the one of Theorem 3.5.2. The difference is that, according to the definition of the divisor D_1 given there, if $b = r - 2 + t$ and $\overline{e} = 0$, the divisor $\mathfrak{b}f - D_1 + K_X + f$ would be effective. Thus, in this case we change the value of b and we fix $b = r - 3 + t$. The divisors D_i are as in the proof of Theorem 3.5.2, as well as the 0-dimensional subschemes Z_i.

We consider the family $\mathcal{G}$ of rank r vector bundles E on X given by a non-trivial extension of type

$$0 \to \mathcal{O}_X(\mathfrak{b}f) \to E \to \bigoplus_{i=1}^{r-1} I_{Z_i}(D_i) \to 0. \tag{3.38}$$

Since any Z_i satisfies the Cayley-Bacharach property with respect to the linear system $|D_i - \mathfrak{b}f + K_X|$, by Theorem 1.1.19 and (3.38), $E \in \mathcal{G}$ is a rank r vector bundle with $c_1(E) = c_1$ and $c_2(E) = c_2$ and, by Remark 1.1.20, we can assume that $e = (e_1, \dots, e_{r-1}) \in \operatorname{Ext}^1(\bigoplus_{i=1}^{r-1} I_{Z_i}(D_i), \mathcal{O}_X(\mathfrak{b}f))$ is a non-trivial extension with $e_i \neq 0$ for $1 \leq i \leq r-1$. In addition, all the divisors D_i are under the assumptions of Lemma 3.5.1. Hence any $E \in \mathcal{G}$ is a simple prioritary vector bundle. Finally, by construction,

$$\mathrm{h}^0 E \geq \mathrm{h}^0 \mathcal{O}_X(\mathfrak{b}f) = b + 1 - g = r - 2 - g + t,$$

which finishes the proof. $\square$

If $\overline{e} = -1$, keeping the notation of Theorem 3.5.2, for $i \geq j+2$, the divisor $D_i - D_j + K_X + f = 0$ is clearly effective. Hence, in this case, we need an alternative proof.

Theorem 3.5.4. *Let X be a ruled surface over a non-singular curve C of genus $g \geq 0$ such that $\overline{e} := e - 2(g-1) = -1$. Fix $c_2 \gg 0$ and $r \geq 4$ integers, and $c_1 = sS + \mathfrak{t}f \in \operatorname{Pic}(X)$ with $0 \leq s \leq r-1$ and $t := \deg(\mathfrak{t}) \geq \max\{0, 2g-r\}$. Then, there exists a rank r simple prioritary vector bundle with $c_1(E) = c_1$ and $c_2(E) = c_2$ such that $\mathrm{h}^0 E \geq k := 2(r-1) + t - g$.*

Proof. We also follow the same steps as the proof of Theorem 3.5.2 but considering $b = 2r - 3 + t$, the divisors D_i as

$$\begin{array}{lll} D_1 & \equiv -(\sum_{j=2}^{r-1} j)S - (\sum_{j=2}^{r-1}(r-3j) - t)f & \\ D_i & \equiv iS + (r-3i)f & 2 \leq i \leq r-2 \\ D_{r-1} & \equiv (r-1+s)S - (b+2r-3)f, & \end{array}$$

and Z_i a reduced 0-dimensional subscheme of length $|Z_i| = 1$ for $1 \leq i \leq r-2$ and

$$|Z_{r-1}| = c_2 - (r-2) - \sum_{i<j} D_i \cdot D_j - (\mathfrak{b}f) \cdot (\sum_{i=1}^{r-1} D_i),$$

such that $Z_i \cap Z_j = \emptyset$ for any $i \neq j$. Next, we consider the family of rank r vector bundles E given by a non-trivial extension of type

$$0 \to \mathcal{O}_X(\mathfrak{b}f) \to E \to \bigoplus_{i=1}^{r-1} I_{Z_i}(D_i) \to 0,$$

where $\mathfrak{b} \in \operatorname{Pic}(C)$ is a divisor of degree $b = 2r - 3 + t$. Since, for every i, $D_i - \mathfrak{b}f + K_X$ is non-effective, Z_i satisfies the Cayley-Bacharach property with respect to the linear system $D_i - \mathfrak{b}f + K_X$ and thus, by Theorem 1.1.19, E is a rank r vector bundle. In addition, by Remark 1.1.20, the extension

$$e = (e_1, \ldots, e_{r-1}) \in \operatorname{Ext}^1(\bigoplus_{i=1}^{r-1} I_{Z_i}(D_i), \mathcal{O}_X(\mathfrak{b}f))$$

is non-trivial and has all its components $e_i \neq 0$. Moreover, the divisors D_i satisfy the conditions of Lemma 3.5.1 and, hence, putting altogether, E is a rank r simple prioritary vector bundle with $c_1(E) = c_1$ and $c_2(E) = c_2$. Finally,

$$\mathrm{h}^0 E \geq b + 1 - g = 2(r-1) + t - g,$$

which finishes the proof. □

To conclude, let us consider the case $\overline{e} < -1$, for which we provide an alternative proof.

Theorem 3.5.5. *Let X be a ruled surface over a non-singular curve C of genus $g \geq 0$ such that $\overline{e} := e - 2(g-1) < -1$. Fix $r \geq 4$ and $c_2 \gg 0$ integers and $c_1 = sS + \mathfrak{t}f \in \operatorname{Pic}(X)$ with $0 \leq s \leq r-1$ and*

$$t := \deg(\mathfrak{t}) > \max\{3, 4g-1\}.$$

Then, there exists a rank r simple prioritary vector bundle E with $c_1(E) = c_1$ and $c_2(E) = c_2$, such that

$$\mathrm{h}^0 E \geq \frac{2t - (r-1)(r-2) + 4 - l}{4}\overline{e} - 1,$$

where l is an integer, $0 \leq l \leq 3$ such that

$$l \equiv 2t - (r-1)(r-2) \mod (4).$$

Proof. The steps of the proof are the same as the ones in the proof of Theorem 3.5.2, but we take different divisors D_i and subschemes Z_i. Let us consider $\mathfrak{b} \in \operatorname{Pic}(X)$ of degree

$$b = \frac{2t - (r-1)(r-2) + 4 - l}{4}\overline{e} - 1.$$

According to this, let us define the divisors

$$\begin{array}{lll} D_i & \equiv iS + i\overline{e}f, & 1 \leq i \leq r-2 \\ D_{r-1} & \equiv c_1 - bf - \sum_{i=1}^{r-2} D_i & \end{array}$$

and Z_i a reduced 0-dimensional subscheme of length $|Z_i| = 1$, for $1 \leq i \leq r-2$, and

$$|Z_{r-1}| \;\; = c_2 - (r-2) - \textstyle\sum_{i<j} D_i \cdot D_j - (\mathfrak{b}f)(\sum_{i=1}^{r-1} D_i),$$

such that $Z_i \cap Z_j = \emptyset$ for $i \neq j$. Since c_2 is independent of b and the divisors D_i, we can assume $|Z_{r-1}| > 0$.

Keeping the above notation, we consider the family of rank r vector bundles E on X given by a non-trivial extension of type

$$0 \to \mathcal{O}_X(\mathfrak{b}f) \to E \to \bigoplus_{i=1}^{r-1} I_{Z_i}(D_i) \to 0, \tag{3.39}$$

where $\mathfrak{b} \in \operatorname{Pic}(C)$ is a divisor of degree b. Since, for every i, $D_i - \mathfrak{b}f + K_X$ is non-effective, Z_i satisfies the Cayley-Bacharach property for every i and thus, by Theorem 1.1.19, E is a rank r vector bundle. In addition, by Remark 1.1.20, the extension

$$e = (e_1, \dots, e_{r-1}) \in \operatorname{Ext}^1(\bigoplus_{i=1}^{r-1} I_{Z_i}(D_i), \mathcal{O}_X(\mathfrak{b}f))$$

is non-trivial and has all its components $e_i \neq 0$. Moreover, the divisors D_i satisfy the conditions of Lemma 3.5.1 and hence, putting altogether, E is a rank r simple prioritary vector bundle with $c_1(E) = c_1$ and $c_2(E) = c_2$. Finally, since $\overline{e} < -1$ and $t > \max\{3, 4g-1\} + 1$, we get $b > \max\{0, 2g-1\}$ and thus, by (3.39),

$$\mathrm{h}^0 E \geq \mathrm{h}^0 \mathcal{O}_X(\mathfrak{b}f) = b + 1 - g = \frac{2t - (r-1)(r-2) + 4 - l}{4}\overline{e} - g,$$

which finishes the proof. $\square$

Remark 3.5.6. Notice that for the case $c_1 = S + \mathfrak{m}f$ and $r = 4$, Theorem 3.4.7 gives us a stronger result than the results obtained in this section for $r = 4$.

Chapter 4

Wall-and-chamber Theory in higher-dimensional varieties

The geometry of the moduli space $M_H = M_H(r; c_1, \ldots, c_s)$ of H-stable (see Definition 1.2.1) rank r vector bundles with Chern classes c_i, $1 \leq i \leq \min\{r, n\}$ strongly depends on the geometry of the underlying variety X, but also on the choice of the ample divisor H. A fundamental and highly interesting problem is to understand how two moduli spaces M_H and $M_{H'}$ are related when one varies the polarization, that is, when one considers two different ample divisors H and H'.

There exists a general theory of walls and chambers asserting that the moduli space only depends on the chamber containing the polarization. In general, when two polarizations lie in different chambers, the corresponding moduli spaces are not isomorphic. This wall–and–chamber theory has made it possible to describe the changes between two different moduli spaces. Moreover, it has allowed to describe moduli spaces of stable rank 2 vector bundles on smooth projective surfaces and to uncover rich geometric structures on them. Unfortunately, in its current form, this theory has been less fruitful when the underlying variety has dimension greater than or equal to 3.

In this chapter, inspired by the theory of walls and chambers introduced by Qin (see [59]), we develop a slightly different theory that allows us to describe moduli spaces of stable rank 2 vector bundles on varieties of dimension $n \geq 3$.

Now, let us outline the structure of this chapter. First of all, in Section 4.1, we will introduce the modification of the theory of walls and chambers of Qin and we will prove that two polarizations in the same chamber are equivalent (see Proposition 4.1.9). In Section 4.2, in Theorem 4.2.4, we will use walls and chambers to construct the family $E_\xi(c_1, c_2)$ of rank 2 stable vector bundles. To end, in Section 4.3, we will give a general result describing the decomposition of the moduli space in terms of the families $E_\xi(c_1, c_2)$ (see Theorem 4.3.2). The results of this chapter are contained in [17].

4.1 Walls and chambers

Inspired by the theory of walls and chambers introduced by Qin (see [59]), in this section we will introduce a slightly modified definition of walls and chambers and we will prove that the same main properties still hold in this new setting. The advantage of this approach is that it provides a more straightforward description of moduli spaces of stable rank 2 bundles on varieties of dimension $n \geq 3$.

Throughout this chapter, let X be a smooth projective variety of dimension n. Recall that a polarization is an element $H \in \mathrm{Num}(X)$ that is ample and the Kähler cone $\mathcal{C}_X$ of X is an open cone in $\mathrm{Num}(X) \otimes \mathbb{R}$ spanned by polarizations.

Definition 4.1.1. Let X be a smooth projective variety of dimension n. Let $\xi \in \mathrm{Num}(X) \otimes \mathbb{R}$ and $c_i \in \mathrm{H}^{2i}(X, \mathbb{Z})$, for $i = 1, 2$. We define

$$W^\xi := \{x \in \mathrm{Num}(X) \otimes \mathbb{R} \,|\, \xi \cdot x^{n-1} = 0\} \cap \mathcal{C}_X.$$

We define $\mathcal{W}(c_1, c_2)$ to be the set of elements W^ξ where ξ is the numerical equivalence class of a divisor D on X such that $\mathcal{O}_X(D + c_1)$ is divisible by 2 in $\mathrm{Pic}(X)$ and there exists a codimension 2 locally complete intersection Z on X such that

$$[Z] = c_2 + \frac{D^2 - {c_1}^2}{4}.$$

A wall of type (c_1, c_2) is an element of $\mathcal{W}(c_1, c_2)$. A chamber $\mathcal{C}$ is a connected component of $\mathcal{C}_X \backslash \mathcal{W}(c_1, c_2)$ and a $\mathbb{Z}$-chamber is the intersection $\mathcal{C} \cap \mathrm{Num}(X)$. We define a face as $\mathcal{F} := \mathrm{Closure}(\mathcal{C}) \cap W^\xi$.

Definition 4.1.2. Let W^ξ be a wall of type (c_1, c_2) defined by $\xi \in \mathrm{Num}(X) \otimes \mathbb{R}$. We say that the wall W^ξ separates two polarizations H_1 and H_2 if

$$\xi \cdot {H_1}^{n-1} < 0 < \xi \cdot {H_2}^{n-1}.$$

We say that the wall W^ξ separates two chambers $\mathcal{C}_1$ and $\mathcal{C}_2$ if W^ξ separates two polarizations $H_1 \in \mathcal{C}_1$ and $H_2 \in \mathcal{C}_2$.

Remark 4.1.3. If $\xi \in \mathrm{Num}(X) \otimes \mathbb{R}$ is numerically equivalent to a divisor D and defines a non-empty wall W^ξ of type (c_1, c_2) that separates H_1 and H_2, then, by [59, Theorem 1.2.5], there exist integers i, j, $0 \leq i < j \leq n-1$ such that, if

$$T := {H_1}^{n-1-j} \cdot {H_2}^{i} \cdot (H_1 + H_2)^{j-i-1},$$

then

$$({c_1}^2 - 4c_2) \cdot T \leq D^2 \cdot T < 0.$$

In particular, if ξ defines a non-empty wall W^ξ of type (c_1, c_2) that separates two polarizations, then there exists only a finite number of $\eta \in \mathrm{Num}(X) \otimes \mathbb{R}$ that defines the wall W^ξ.

Remark 4.1.4. By abuse of notation, we will write ξ instead of the divisor $D \in \mathrm{Pic}(X)$ such that $\xi \equiv D$.

Lemma 4.1.5. *Let us consider $\xi \in \mathrm{Num}(X) \otimes \mathbb{R}$ defining a non-empty wall W^ξ of type (c_1, c_2). Then, there exists $H \in \mathrm{Num}(X) \otimes \mathbb{R}$ such that $\xi \cdot H^{n-1} > 0$.*

Proof. Notice that, if W^ξ is a non-empty wall, then it separates two chambers. Therefore, in particular, there exists an element $H \in \mathrm{Num}(X) \otimes \mathbb{R}$ such that $\xi \cdot H^{n-1} > 0$, which corresponds to an element in one of the adjacent chamber to W^ξ. $\square$

The main reason to introduce the theory of walls and chambers is to study moduli spaces of H-stable rank 2 vector bundles (according to Definition 1.2.1). In more precise terms, it aims to describe how these moduli spaces vary when we change the ample divisor H.

As in [59], we will our definition of walls and chambers to the equivalence of polarizations. First of all, let us introduce some notation.

Notation 4.1.6. Let H_1 and H_2 be two polarizations and let E be a rank 2 vector bundle. We say that $H_1 \overset{s}{\geq} H_2$ if and only if E is H_2-stable whenever it is H_1-stable. We say that $H_1 \overset{s}{=} H_2$ if and only if $H_1 \overset{s}{\geq} H_2$ and $H_2 \overset{s}{\geq} H_1$, and we set

$$\Delta_H := \{H' |\, H' \overset{s}{\geq} H\} \quad \text{and} \quad \mathcal{X}_H := \{H' |\, H' \overset{s}{=} H\}.$$

Lemma 4.1.7. *Let X be a smooth projective variety of dimension n and let $\xi \in \mathrm{Num}(X) \otimes \mathbb{R}$ define a wall W^ξ of type (c_1, c_2). Assume that there exist two polarizations H_1 and H_2 on X such that $\xi \cdot {H_1}^{n-1} < 0 < \xi \cdot {H_2}^{n-1}$. Then, there exists an element $a_0 \in \mathbb{R}^+$ such that $\xi \cdot (H_1 + a_0 H_2)^{n-1} = 0$. In other words, the wall W^ξ is non-empty.*

Proof. Let us consider the following polynomial in a of degree $n-1$

$$P(a) := \xi \cdot (H_1 + aH_2)^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} ({H_1}^{n-1-k} \cdot {H_2}^k \cdot \xi) a^k$$

with coefficients

$$h_k := \binom{n-1}{k} {H_1}^{n-1-k} \cdot {H_2}^k \cdot \xi \in \mathbb{Z}.$$

In particular,

$$h_0 = \xi \cdot {H_1}^{n-1} < 0 \quad \text{and} \quad h_{n-1} = \xi \cdot {H_2}^{n-1} > 0,$$

and thus $h_0 \cdot h_{n-1} < 0$, which implies that there exists a root $a_0 \in \mathbb{R}^+$ of $P(a)$. Hence, $H = H_1 + a_0 H_1$ is ample and $\xi \cdot H^{n-1} = 0$. □

The following result establishes a relation between chambers and equivalence classes of polarizations.

Lemma 4.1.8. *Let H_1 and H_2 be two polarizations and E a rank 2 vector bundle with $c_i := c_i(E)$ for $i = 1, 2$. Assume that E is H_2-unstable and H_1-stable. Then, there exists $\xi \in \mathrm{Num}(X) \otimes \mathbb{R}$ defining a non-empty wall of type (c_1, c_2) such that $\xi \cdot {H_1}^{n-1} < 0 \leq \xi \cdot {H_2}^{n-1}$ and thus separating H_1 and H_2, whenever H_2 is not in W^ξ.*

Proof. Notice that, since E is H_2-unstable, there exists a vector subbundle $\mathcal{O}_X(G) \hookrightarrow E$ of E such that the quotient $E/\mathcal{O}_X(G)$ is torsion-free and satisfies $G \cdot {H_2}^{n-1} \geq \frac{c_1 \cdot {H_2}^{n-1}}{2}$.

Let us fix $\xi \equiv 2G - c_1$. First of all, let us see that ξ defines a wall of type (c_1, c_2). By definition, $\mathcal{O}_X(\xi + c_1)$ is divisible by 2 in $\mathrm{Pic}(X)$. Moreover, since $E/\mathcal{O}_X(G)$ is torsion-free, we have the short exact sequence

$$0 \to \mathcal{O}_X(G) \to E \to I_Z(c_1 - G) \to 0,$$

where Z is a codimension 2 locally complete intersection with $[Z] = \frac{\xi^2 - {c_1}^2}{4} + c_2$. Therefore, by definition, ξ defines a wall W^ξ of type (c_1, c_2).

Let us see that W^ξ is non-empty. Notice that $G \cdot {H_2}^{n-1} \geq \frac{c_1 \cdot {H_2}^{n-1}}{2}$ is equivalent to $\xi \cdot {H_2}^{n-1} \geq 0$. On the other hand, since E is H_1-stable, we have $\xi \cdot {H_1}^{n-1} < 0$. If $\xi \cdot {H_2}^{n-1} = 0$, then $H_2 \in W^\xi$ and we are done. Assume that $\xi \cdot {H_2}^{n-1} > 0$. Then, since $\xi \cdot {H_1}^{n-1} < 0 < \xi \cdot {H_2}^{n-1}$, by Lemma 4.1.7 there exists $a_0 \in \mathbb{R}^+$ such that $\xi \cdot (H_1 + a_0 H_2)^{n-1} = 0$. Thus, there exists $H = H_1 + a_0 H_2 \in (\mathrm{Num}(X) \otimes \mathbb{R}) \cap \mathcal{C}_X$ such that $\xi \cdot H^{n-1} = 0$ and hence $H \in W^\xi$. □

Proposition 4.1.9. *Let $\mathcal{C}$ be a chamber and $H_1, H_2 \in \mathcal{C}$ be two polarizations. Then $H_1 \stackrel{s}{=} H_2 \stackrel{s}{=} H_1 + H_2$.*

Proof. Let us assume that $H_1 \stackrel{s}{\neq} H_2$. Then, by definition, there exists a rank 2 vector bundle E on X with $c_i(E) = c_i$ such that E is H_1-stable but H_2-unstable. By Lemma 4.1.8, there exists a non-empty wall separating H_1 and H_2 which contradicts the fact that $H_1, H_2 \in \mathcal{C}$.

Finally, since by construction a chamber is convex and closed under the action of $\mathbb{R}^+$, $H_1 \stackrel{s}{=} H_2$ implies that $H_i \stackrel{s}{=} H_1 + H_2$ for $i = 1, 2$. $\square$

Remark 4.1.10. As a consequence of Proposition 4.1.9, Theorem 2.2.2 and Theorem 2.3.1 of Chapter 2 can be stated in more generality in the following sense. Any $E \in W^k_H(2; c_1, c_2)$ is also an element of $W^k_{\tilde{H}}(2; c_1, c_2)$ for any $\tilde{H}$ lying in the same chamber than H. Hence, for these $\tilde{H}$, $W^k_H(2; c_1, c_2) \neq \emptyset$ implies that $W^k_{\tilde{H}}(2; c_1, c_2) \neq \emptyset$.

As an inmediate consequence of Proposition 4.1.9, we get the following.

Corollary 4.1.11. *Every $\mathbb{Z}$-chamber is contained in an equivalence class and every equivalence class is a union of $\mathbb{Z}$-chambers and possible polarizations lying in walls.*

According to Proposition 4.1.9, we introduce the following definition.

Definition 4.1.12. We say that a vector bundle E is $\mathcal{C}$-stable if E is H-stable for some $H \in \mathcal{C}$. Analogously, a vector bundle E is $\mathcal{F}$-stable if E is H-stable for some $H \in \mathcal{F}$.

Notation 4.1.13. Given a chamber $\mathcal{C}$, let us denote by $M_{\mathcal{C}} := M_{\mathcal{C}}(2; c_1, c_2)$ the moduli space rank 2 vector bundles E on X with Chern classes $c_1(E) = c_1$ and $c_2(E) = c_2$ that are H-stable for some polarization $H \in \mathcal{C}$. Equivalently, given a face $\mathcal{F}$, let us denote by $M_{\mathcal{F}}(2; c_1, c_2)$ the moduli space of rank 2 vector bundles E on X with $c_1(E) = c_1$ and $c_2(E) = c_2$, that are H-stable for some polarization $H \in \mathcal{F}$.

The next result states not only that two polarizations in the same chamber are equivalent, but also that, under certain conditions, two chambers separated by a certain wall belong to different equivalence classes.

Proposition 4.1.14. *Let $\mathcal{C}$ and $\mathcal{C}'$ be two chambers having a unique common face that is part of a wall W. Let us assume that we have* $\mathrm{Num}(X) \cap \mathcal{C} \neq \emptyset$ *and* $\mathrm{Num}(X) \cap \mathcal{C}' \neq \emptyset$*. Then,* $\mathrm{Num}(X) \cap \mathcal{C} \stackrel{s}{=} \mathrm{Num}(X) \cap \mathcal{C}'$ *if and only if W is not of the form W^{ξ} where $\xi \equiv 2G - c_1$ for some subbundle $\mathcal{O}_X(G) \hookrightarrow E$ of a rank 2 vector bundle E which is $\mathcal{C}$-stable or $\mathcal{C}'$-stable.*

Proof. If $\mathrm{Num}(X) \cap \mathcal{C} \overset{s}{\neq} \mathrm{Num}(X) \cap \mathcal{C}'$, then there exists a rank two vector bundle E on X with $c_1(E) = c_1$ and polarizations $H_1 \in \mathcal{C} \cap \mathrm{Num}(X)$ and $H_2 \in \mathcal{C}' \cap \mathrm{Num}(X)$ such that E is H_2-unstable and H_1-stable. By Lemma 4.1.8, there is $\xi \in \mathrm{Num}(X)$ and a non-empty wall W^ξ which separates $\mathcal{C}$ and $\mathcal{C}'$. Hence, $W = W^\xi$, a contradiction.

Now, let us prove the converse. Assume that W is of the form W^ξ where $\xi \equiv 2G - c_1$ with $\mathcal{O}_X(G) \hookrightarrow E$, being E a rank 2 vector bundle $\mathcal{C}$-stable or $\mathcal{C}'$-stable. Take $H_1 \in \mathcal{C}$ and $H_2 \in \mathcal{C}'$. Since we are assuming that $H_1 \overset{s}{=} H_2$, E is H_1-stable and H_2-stable and, hence, $\xi \cdot {H_1}^{n-1} < 0$ and $\xi \cdot {H_2}^{n-1} < 0$. Thus, W^ξ cannot separate the chambers $\mathcal{C}$ and $\mathcal{C}'$, which again is a contradiction. $\square$

We will end this section with a result that contributes to drawn the picture between the notion of chambers and equivalence classes of polarizations.

Proposition 4.1.15. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two chambers having a unique common face which is part of the wall W. Let $\mathcal{F} = W \cap \mathrm{Closure}(\mathcal{C}_i) \neq \emptyset$, $i = 1, 2$, be the common face and $H \in \mathrm{Num}(X) \cap \mathcal{F}$. Then,*

i) $\mathrm{Num}(X) \cap \mathcal{F}$ *is in one equivalence class.*

ii) $\Delta_H \supseteq (\mathrm{Num}(X) \cap \mathcal{C}_1) \cap (\mathrm{Num}(X) \cap \mathcal{C}_2)$.

iii) $\mathrm{Num}(X) \cap \mathcal{C}_1 \overset{s}{=} \mathrm{Num}(X) \cap \mathcal{C}_2$ *if and only if*

$$\mathcal{X}_H \supseteq (\mathrm{Num}(X) \cap \mathcal{C}_1) \cup (\mathrm{Num}(X) \cap \mathcal{C}_2).$$

Proof. *i*) Assume that $\mathrm{Num}(X) \cap \mathcal{F}$ is not in an equivalence class. Then, there exists $H_1, H_2 \in \mathrm{Num}(X) \cap \mathcal{F}$ and a rank 2 vector bundle E such that E is H_1-stable but H_2-unstable. By Lemma 4.1.8, there exists $\xi \in \mathrm{Num}(X)$ defining a non-empty wall W^ξ separating H_1 and H_2 and hence $W^\xi \cap \mathcal{F} \neq \emptyset$, which is a contradiction.

ii) It is enough to prove that $\Delta_H \supseteq \mathrm{Num}(X) \cap \mathcal{C}_1$. Let $H_1 \in \mathrm{Num}(X) \cap \mathcal{C}_1$ and assume that $H_1 \notin \Delta_H$. Then there exists a rank 2 vector bundle E which is H-stable but H_1-unstable. By Lemma 4.1.8, there exists a non-empty wall W^ξ separating H_1 and H. Moreover, since $\xi \cdot H^{n-1} < 0$, we get $H \notin W^\xi$. But the only wall separating H and H_1 is W, which contains H, and hence we get a contradiction.

iii) By definition, if $\mathcal{X}_H \supseteq (\mathrm{Num}(X) \cap \mathcal{C}_1) \cup (\mathrm{Num}(X) \cap \mathcal{C}_2)$, then we have $(\mathrm{Num}(X) \cap \mathcal{C}_1) \overset{s}{=} (\mathrm{Num}(X) \cap \mathcal{C}_2)$. Now, let us prove the converse. Assume that $(\mathrm{Num}(X) \cap \mathcal{C}_1) \overset{s}{=} (\mathrm{Num}(X) \cap \mathcal{C}_2)$ and consider a rank 2 vector bundle E which is $\mathcal{C}_i$-stable for $i = 1$ or $i = 2$. If E is not H-stable, then by Lemma 4.1.8 there exists a wall W^ξ with $\xi = 2G - c_1$ for some $\mathcal{O}_X(G) \hookrightarrow E$, that separates H and H_i. But since H belongs to $W \cap \text{Closure}(\mathcal{C})$ and there is a unique common face in the wall W, $W = W^{\xi}$. But if $W = W^{\xi}$, then ξ separates $H_1 \in \mathcal{C}_1$ and $H_2 \in \mathcal{C}_2$ and $\xi \cdot H_1^{n-1} > 0 > \xi \cdot H_2^{n-1}$ contradicts the fact that $(\text{Num}(X) \cap \mathcal{C}_1) \stackrel{s}{=} (\text{Num}(X) \cap \mathcal{C}_2)$. Thus, E must be H-stable and therefore $H \stackrel{s}{=} \text{Num}(X) \cap \mathcal{C}_i$ and $\mathcal{X}_H \supseteq (\text{Num}(X) \cap \mathcal{C}_1) \cup (\text{Num}(X) \cap \mathcal{C}_2)$. □

Remark 4.1.16. From the above proposition, E is $(\text{Num}(X) \cap \mathcal{F})$-stable if and only if E is $(\text{Num}(X) \cap \mathcal{C}_i)$-stable for $i = 1, 2$.

To end this section, let us give an example of some structure of walls and chambers of type (c_1, c_2) in the case when X is a ruled surface. We follow the notation introduced in Section 1.2.1 of Chapter 1 and used in Chapter 2.

Example 4.1.17. Let us consider $c_1 = f$ and $c_2 \gg 0$ an integer. Notice that the class $\xi_1 \equiv -2S+f$ defines a wall W^{ξ_1} of type (c_1, c_2). In fact, $\xi_1 + c_1 \equiv -2S+2f$ is clearly divisible by 2 in $\text{Pic}(X)$ and there exists a 0-dimensional subscheme Z such that $|Z| = \frac{\xi_1^2 - c_1^2}{4} + c_2 = c_2 - e - 1 \gg 0$. Moreover, $H_1 \equiv 2S + (2e+1)f$ is an ample divisor such that $\xi_1 \cdot H_1 = 0$ and thus $H_1 \in W^{\xi_1}$, which implies that W^{ξ_1} is non-empty. Using the same arguments, both $\xi_0 \equiv -4S+f$ and $\xi_2 \equiv -2S+3f$ define two walls of type (c_1, c_2) denoted by W^{ξ_0} and W^{ξ_2}, respectively, which are non-empty since $H_0 \equiv 4S+(4e+1)f \in W^{\xi_0}$ and $H_2 \equiv 2S+(2e+3)f \in W^{\xi_2}$.

One can go further and determine the adjacent chambers to the wall W^{ξ_1}, that is, the chambers separated by the wall W^{ξ_1}. It can be proven that the next walls to W^{ξ_1} are W^{ξ_0} and W^{ξ_2}. In fact, if we assume that there exists $\eta_{p,q} \equiv -2pS + (2q+1)f$, with $p, q \geq 1$, defining a wall of type (c_1, c_2) such that, for $\tilde{H} \in W^{\eta_{p,q}}$, we have

$$\xi_2 \cdot \tilde{H} > 0 \quad \text{and} \quad \xi_1 \cdot \tilde{H} < 0$$

or

$$\xi_0 \cdot \tilde{H} < 0 \quad \text{and} \quad \xi_1 \cdot \tilde{H} > 0,$$

we get a contradiction. Hence, the adjacent chambers to W^{ξ_1} are the connected components $\mathcal{C}$ and $\mathcal{C}'$ in $\mathcal{C}_X \backslash (W^{\xi_0} \sqcup W^{\xi_1} \sqcup W^{\xi_2})$ satisfying

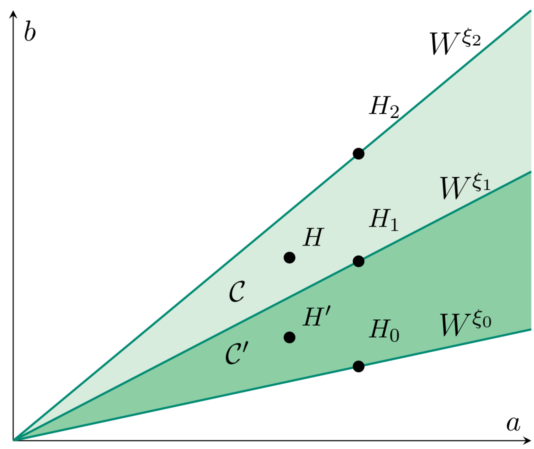
$$\text{Closure}(\mathcal{C}) \cap W^{\xi_1} \neq \emptyset, \quad \text{Closure}(\mathcal{C}) \cap W^{\xi_2} \neq \emptyset$$

and

$$\text{Closure}(\mathcal{C}') \cap W^{\xi_0} \neq \emptyset, \quad \text{Closure}(\mathcal{C}') \cap W^{\xi_1} \neq \emptyset.$$

For this reason, a polarization H is in the chamber $\mathcal{C}$ (respectively, a polarization H' is in $\mathcal{C}'$) if $\xi_1 \cdot H < 0 < \xi_2 \cdot H$ (respectively, if $\xi_0 \cdot H' < 0 < \xi_1 \cdot H'$). For instance, $H \equiv S + (e+1)f \in \mathcal{C}$ and $H' \equiv 3S + (3e+1)f \in \mathcal{C}'$.

Finally, thanks to the geometric interpretation of walls and chambers given by Qin in [60] for the case of ruled surfaces, we are able to identify a class $\xi \equiv aS + bf \in \mathrm{Num}(X) \otimes \mathbb{R}$ with a point $(a, b) \in \mathbb{R} \times \mathbb{R}$, the Kähler cone $\mathcal{C}_X$ with $\mathbb{R}^+ \times \mathbb{R}^+$, a wall of type (c_1, c_2) with a ray in $\mathbb{R}^+ \times \mathbb{R}^+$, and a chamber with a connected region in $\mathbb{R}^+ \times \mathbb{R}^+$ between two consecutive walls. Hence, the geometric picture behind the situation described above is the following:



4.2 The family $\mathbf{E}_\xi(\mathbf{c_1}, \mathbf{c_2})$

Now, we want to construct families $E_\xi(c_1, c_2)$ of rank 2 stable vector bundles on X as in [59], but in the context of X being a projective variety of dimension greater than or equal to two, not necessarily a surface. In this case, the situation is trickier and we will use the well-known Hartshorne-Serre correspondence (see Section 1.1, Theorem 1.1.14).

We can construct rank 2 vector bundles E on X by considering extensions of two line bundles, that is, extensions of type

$$0 \to \mathcal{O}_X(D) \to E \to \mathcal{O}_X(D') \to 0,$$

where D and D' are divisors. Moreover, by Theorem 1.1.14, we can also construct rank 2 vector bundles E on X from non-trivial extensions of type

$$0 \to \mathcal{O}_X \to E \to I_Z(D) \to 0, \tag{4.1}$$

if Z is a codimension 2 locally complete intersection and D is a divisor satisfying $\mathrm{H}^2(X, \mathcal{O}_X(-D)) = 0$ and $\mathrm{H}^0(Z, \bigwedge^2 \mathcal{N}_{Z/X} \otimes \mathcal{O}_X(-D)_{|Z}) \neq 0$, where $\mathcal{N}_{Z/X}$ is the normal bundle of Z in X.

This fact motivates the following definition.

Definition 4.2.1. Let us consider $\xi \in \mathrm{Num}(X) \otimes \mathbb{R}$ defining a non-empty wall W^ξ of type (c_1, c_2) and Z a codimension 2 locally complete intersection such that $[Z] = \frac{\xi^2 - {c_1}^2}{4} + c_2$. We say that ξ satisfies the C-condition if

$$\mathrm{H}^2(X, \mathcal{O}_X(\xi)) = 0 \quad \text{and} \quad \mathrm{H}^0(Z, \bigwedge^2 \mathcal{N}_{Z/X} \otimes \mathcal{O}_X(\xi)_{|Z}) \neq 0.$$

The constructions discussed above motivate the definition of families of rank 2 vector bundles associated to classes ξ satisfying the C-condition or such that $\xi^2 \equiv c_1^2 - c_2$.

Definition 4.2.2. *Let X be a smooth projective variety of dimension n and $\xi \in \mathrm{Num}(X) \otimes \mathbb{R}$ defining a non-empty wall W^ξ of type (c_1, c_2) satisfying the C-condition or such that $\xi^2 \equiv c_1^2 - 4c_2$. We define $E_\xi := E_\xi(c_1, c_2)$ as the family of rank two vector bundles E on X given by a non-trivial extension of type*

$$0 \to \mathcal{O}_X(G) \to E \to I_Z(c_1 - G) \to 0, \tag{4.2}$$

where G is a divisor on X such that $\xi \equiv 2G - c_1$ and Z is a codimension 2 locally complete intersection such that $[Z] = \frac{\xi^2 - c_1^2}{4} + c_2$.

Remark 4.2.3. It easily follows from the construction and Theorem 1.1.14, that any $E \in E_\xi(c_1, c_2)$ is a rank 2 vector bundle with Chern classes $c_1(E) = c_1$ and $c_2(E) = c_2$. Also, by Proposition 1.1.21, if X is a smooth projective surface and $-\xi + K_X$ is non-effective, $E_\xi(c_1, c_2)$ is an irreducible rational quasi-projective variety. In addition, by Proposition 1.1.22, if $\xi \equiv c_1^2 - 4c_2$, $E_\xi(c_1, c_2)$ is a projective space of dimension $d = \mathrm{h}^1\, \mathcal{O}_X(\xi) - 1$.

Moreover, the families $E_\xi(c_1, c_2)$ allow us to construct stable vector bundles. In fact, according to the proof of [59, Theorem 1.2.3], we get the following result.

Theorem 4.2.4. *Let X be a smooth projective variety of dimension n and $\xi \in \mathrm{Num}(X) \otimes \mathbb{R}$ defining a non-empty wall W^ξ of type (c_1, c_2) satisfying the C-condition or such that $\xi^2 \equiv c_1^2 - 4c_2$. Let us take $E \in E_\xi(c_1, c_2)$. The following holds:*

i) If H_1 is a polarization such that $\xi \cdot {H_1}^{n-1} > 0$, then E is H_1-unstable.

ii) If $\mathcal{C}$ is a chamber such that $W^\xi \cap \mathrm{Closure}(\mathcal{C}) \neq \emptyset$ and $\xi \cdot H^{n-1} < 0$ for $H \in \mathcal{C}$, then E is H-stable for any polarization $H \in \mathcal{C}$.

iii) If $\mathcal{C}$ is a chamber such that $W^\xi \cap \mathrm{Closure}(\mathcal{C}) \neq \emptyset$ and $\xi \cdot H^{n-1} \leq 0$ for $H \in \mathcal{C}$, then E is H-semistable for any polarization $H \in \mathcal{C}$ and E is strictly H-semistable for $H \in W^\xi \cap \mathrm{Num}(X)$.

Proof. It follows from Definition 4.2.2 that any $E \in E_\xi(c_1, c_2)$ is given by a non-trivial extension of type

$$0 \to \mathcal{O}_X(G) \to E \to I_Z(c_1 - G) \to 0, \tag{4.3}$$

where G is a divisor on X such that $\xi \equiv 2G - c_1$ and Z is a locally complete intersection codimension 2 cycle such that $[Z] = c_2 + \frac{\xi^2 - {c_1}^2}{4}$.

i) Notice that $\xi \cdot {H_1}^{n-1} > 0$ is equivalent to $G \cdot {H_1}^{n-1} > \frac{c_1 \cdot {H_1}^{n-1}}{2}$. Hence, the line subbundle $\mathcal{O}_X(G) \hookrightarrow E$ desestabilizes E with respect to H_1.

ii) Let us consider $\mathcal{O}_X(D) \hookrightarrow E$ a subbundle of E such that the quotient $E/\mathcal{O}_X(D)$ is torsion-free. Since E is given by (4.3), we have two possibilities: either $\mathcal{O}_X(D) \hookrightarrow \mathcal{O}_X(G)$ or $\mathcal{O}_X(D) \hookrightarrow I_Z(c_1 - G)$.

Assume that $\mathcal{O}_X(D) \hookrightarrow \mathcal{O}_X(G)$. In this case, $G - D$ is effective, which implies that $(G - D) \cdot H^{n-1} \geq 0$. Hence,

$$D \cdot H^{n-1} \leq G \cdot H^{n-1} < \frac{c_1 \cdot H^{n-1}}{2},$$

where the last inequality follows from the fact that $\xi \cdot H^{n-1} < 0$. Now, let us assume that $\mathcal{O}_X(D) \hookrightarrow I_Z(c_1 - G)$. Notice that $c_1 - G - D$ is an effective divisor. If $c_1 - G - D = 0$, we have $\mathrm{h}^0\, I_Z > 0$, which implies that $[Z] = 0$ and thus $\mathcal{O}_X(D)$ would be both a subbundle and a quotient bundle of E, which is a contradiction. Hence $c_1 - G - D$ is strictly effective and thus $(c_1 - G - D) \cdot {H'}^{n-1} > 0$ for any divisor $H' \in (\mathrm{Num}(X) \otimes \mathbb{R}) \cap \mathcal{C}_X$. In particular, $(c_1 - G - D) \cdot {H_0'}^{n-1} > 0$ for $H_0' \in W^\xi \cap \mathrm{Closure}(\mathcal{C})$. Moreover, by definition, $\xi \cdot (H_0')^{n-1} = 0$. Putting altogether,

$$(2D - c_1) \cdot (H_0')^{n-1} < 0. \tag{4.4}$$

We want to prove that $D \cdot H^{n-1} < \frac{c_1 \cdot H^{n-1}}{2}$. Assume that $D \cdot H^{n-1} \geq \frac{c_1 \cdot H^{n-1}}{2}$, which is equivalent to

$$(2D - c_1) \cdot H^{n-1} \geq 0. \tag{4.5}$$

Let us define $\xi' \equiv 2D - c_1$. Notice that ξ' defines a wall $W^{\xi'}$ of type (c_1, c_2). In fact, by definition $\mathcal{O}_X(\xi' + c_1)$ is divisible by 2 in $\mathrm{Pic}(X)$. Since $\mathcal{O}_X(D) \hookrightarrow E$ and $E/\mathcal{O}_X(D)$ is torsion-free, E sits on the short exact sequence

$$0 \to \mathcal{O}_X(D) \to E \to I_{Z'}(c_1 - D) \to 0,$$

where Z' is a codimension 2 locally complete intersection with

$$[Z'] = c_2 + \frac{(\xi')^2 - {c_1}^2}{4}.$$

Therefore, ξ' defines a wall $W^{\xi'}$ of type (c_1, c_2). Finally, let us see that the wall $W^{\xi'}$ is non-empty. By (4.4) and (4.5), we have

$$\xi' \cdot (H_0')^{n-1} < 0 \leq \xi' \cdot H^{n-1}.$$

If $\xi' \cdot H^{n-1} = 0$, by definition, $H \in W^{\xi'}$ and thus $W^{\xi'}$ is non-empty. Otherwise, we have $\xi' \cdot (H_0')^{n-1} < 0 < \xi' \cdot H^{n-1}$ which, by Lemma 4.1.7, implies that $\xi' \cdot (H_0' + a_0 L)^{n-1} = 0$ for some $a_0 \in \mathbb{R}^+$, and thus $H_0' + a_0 H \in W^{\xi'}$. Hence, there exists a non-empty wall $W^{\xi'}$ of type (c_1, c_2) which separates H_0' and H and, clearly satisfying $H_0' \notin W^{\xi'}$. But, by construction, for any $H \in \mathcal{C}$ and $H_0' \in W^{\xi} \cap \mathrm{Closure}(\mathcal{C})$, any wall separating H and H_0' must contain H_0'. Therefore we get a contradiction and hence $D \cdot H^{n-1} < \frac{c_1 \cdot H^{n-1}}{2}$.

Putting altogether, we have seen that, for any $\mathcal{O}_X(D) \hookrightarrow E$, we have $\mu_H(\mathcal{O}_X(D)) < \mu_H(E)$, which proves the H-stability of E.

iii) Using the same arguments as in *ii*), we prove that E is H-semistable for $H \in \mathcal{C}$. On the other hand, if $H_0 \in W^{\xi}$, then $\xi \cdot {H_0}^{n-1} = 0$, which is equivalent to $G \cdot {H_0}^{n-1} = \frac{c_1 \cdot {H_0}^{n-1}}{2}$ and hence E is strictly H_0-semistable. □

According to Notation 4.1.13 and as a consequence of Theorem 4.2.4, we get the following result.

Corollary 4.2.5. *Let $\xi \in \mathrm{Num}(X) \otimes \mathbb{R}$ defining a non-empty wall W^{ξ} of type (c_1, c_2) satisfying the C-condition or such that $\xi^2 \equiv c_1^2 - c_2$. Assume that $W^{\xi} \cap \mathrm{Closure}(\mathcal{C}) \neq \emptyset$ and $\xi \cdot H^{n-1} < 0$ for $H \in \mathcal{C}$. Then, there exists an embedding $E_{\xi}(c_1, c_2) \hookrightarrow M_{\mathcal{C}}(2; c_1, c_2)$.*

To prove Corollary 4.2.5 we need the following technical lemma.

Lemma 4.2.6. *Let $\xi \in \mathrm{Num}(X) \otimes \mathbb{R}$ defining a non-empty wall W^{ξ} of type (c_1, c_2) satisfying the C-condition or such that $\xi^2 \equiv c_1^2 - c_2$. Let $E \in E_{\xi}(c_1, c_2)$ and $\mathcal{O}_X(G) \hookrightarrow E$ coming from the construction of E in Definition 4.2.2. Assume that there exists $H' \in \mathrm{Num}(X)$ such that $\xi \cdot (H')^{n-1} < 0$. Then, $\mathrm{Hom}(\mathcal{O}_X(G), E) \cong K$. Moreover, for any $\mathcal{O}_X(D) \hookrightarrow E$ with quotient $E/\mathcal{O}_X(D)$ torsion-free and $\xi \equiv 2D - c_1$, we have*

$$\mathcal{O}_X(D) \cong \mathcal{O}_X(G).$$

Proof. First of all, let us prove that $\mathrm{Hom}(\mathcal{O}_X(G), E) \cong \mathrm{H}^0 E(-G) \cong K$. Twisting the exact sequence defining $E \in E_{\xi}(c_1, c_2)$ by $\mathcal{O}_X(-G)$ and taking cohomology, we get the long exact sequence

$$0 \to \mathrm{H}^0 \mathcal{O}_X \to \mathrm{H}^0 E(-G) \to \mathrm{H}^0 I_Z(c_1 - 2G) \to \cdots .$$

Claim: $\mathrm{H}^0\, I_Z(c_1 - 2G) = 0$.
Proof of the Claim: Since $c_1 - 2G \equiv -\xi$ and, by Lemma 4.1.5, there exists a polarization H such that $\xi \cdot H^{n-1} > 0$, it follows that $c_1 - 2G$ is non-effective and thus $\mathrm{H}^0\, I_Z(c_1 - 2G) = 0$. Hence $\mathrm{H}^0\, E(-G) \cong \mathrm{H}^0\, \mathcal{O}_X \cong K$.

Now, let us prove the second statement. Assume that $\xi \equiv 2D - c_1$. Notice that $\xi \equiv 2D - c_1$ is equivalent to $c_1 \equiv 2D - \xi \equiv 2(D-G) + c_1$ and $0 \equiv 2(D-G)$, which implies $D \equiv G$. Hence, $c_1 - G - D \equiv c_1 - 2G \equiv -\xi$. On the other hand, there exists a polarization H such that $\xi \cdot H^{n-1} > 0$ and thus, since $-\xi \equiv c_1 - G - D \equiv c_1 - 2G$, both $c_1 - G - D$ and $c_1 - 2G$ are non-effective divisors. In particular,

$$\mathrm{Hom}(\mathcal{O}_X(G), E/\mathcal{O}_X(D)) = \mathrm{Hom}(\mathcal{O}_X, I_Z(c_1 - D - G)) = 0$$

and

$$\mathrm{Hom}(\mathcal{O}_X(D), E/\mathcal{O}_X(G)) = \mathrm{Hom}(\mathcal{O}_X, I_Z(c_1 - G - D)) = 0.$$

Hence, the conclusion follows from the standart fact that asserts that two invertible subsheaves of a sheaf with torsion-free quotients coincide if the map of one to the quotient by the other is zero. $\square$

Now, we are ready to prove Corollary 4.2.5.

Proof. By Theorem 4.2.4, there exists a map $\phi : E_\xi(c_1, c_2) \to M_{\mathcal{C}}(2; c_1, c_2)$. Assume that it is not an embedding and $E \in E_\xi(c_1, c_2)$ is given by two different extensions which, by Lemma 4.2.6, we can assume that are of type

$$e_1 : 0 \to \mathcal{O}_X(G) \overset{\alpha_1}{\to} E \overset{\alpha_2}{\to} I_{Z_1}(c_1 - G) \to 0$$

and

$$e_2 : 0 \to \mathcal{O}_X(G) \overset{\beta_1}{\to} E \overset{\beta_2}{\to} I_{Z_2}(c_1 - G) \to 0.$$

Since

$$\mathrm{Hom}(\mathcal{O}_X(G), I_{Z_1}(c_1 - G)) \cong \mathrm{H}^0\, I_{Z_1}(c_1 - 2G) = 0$$

and

$$\mathrm{Hom}(\mathcal{O}_X(G), I_{Z_2}(c_1 - G)) \cong \mathrm{H}^0\, I_{Z_2}(c_1 - G) = 0,$$

we get $\beta_2 \circ \alpha_1 = 0 = \alpha_2 \circ \beta_1$. Therefore, there exists $\gamma \in Aut(\mathcal{O}_X(G)) \cong K$ such that $\alpha_1 = \gamma \circ \beta_1$. Hence, ϕ is an injection. $\square$

Therefore, we conclude that $\mathcal{C}$-stable vector bundles can be obtained from the family $E_\xi(c_1, c_2)$ and in this way, we generalize the construction given in [59] to the case of higher-dimensional varieties.

4.3 Decomposition of moduli spaces

In the case of surfaces, if $\mathcal{C}$ and $\mathcal{C}'$ are two chambers with a unique common wall W^ξ of type (c_1, c_2), one can say more and describe the moduli space $M_{\mathcal{C}}(2; c_1, c_2)$ in terms of the moduli $M_{\mathcal{C}'}(2; c_1, c_2)$ and the families $E_\xi(c_1, c_2)$. That is, when the underlying variety is a surface we have the following set theoretical decomposition

$$M_{\mathcal{C}}(2; c_1, c_2) \sqcup (\bigsqcup_{\xi} E_{-\xi}(2; c_1, c_2)) = M_{\mathcal{C}'}(2; c_1, c_2) \sqcup (\bigsqcup_{\xi} E_{\xi}(2; c_1, c_2)),$$

where ξ runs over all numerical classes defining the wall W^ξ and such that $\xi \cdot H^{n-1} < 0$ for $H \in \mathcal{C}$ (see [59, Proposition 1.3.1]).

While dealing with higher-dimensional varieties, having this description is trickier, since it is no longer true that any $E \in M_{\mathcal{C}}(c_1, c_2) \backslash M_{\mathcal{C}'}(2; c_1, c_2)$ is an element of $E_\xi(c_1, c_2)$. In fact, if $E \in M_{\mathcal{C}}(2; c_1, c_2) \backslash M_{\mathcal{C}'}(2; c_1, c_2)$, then it is given by a non-trivial extension of type

$$0 \to \mathcal{O}_X(G) \to E \to I_Z(c_1 - G) \to 0$$

but, for $\xi \equiv 2G - c_1$, it may fail that $\mathrm{H}^2(X, \mathcal{O}_X(\xi)) = 0$ or $\xi^2 \equiv c_1^2 - 4c_2$.

Fortunately, if the classes defining the wall separating two chambers $\mathcal{C}$ and $\mathcal{C}'$ satisfy certain conditions, one can relate the corresponding moduli spaces $M_{\mathcal{C}}(2; c_1, c_2)$ and $M_{\mathcal{C}'}(2; c_1, c_2)$. The idea is that their differences are described by the families $E_\xi(c_1, c_2)$ and next theorem proves this precisely.

To state the result that describes different decompositions of moduli spaces, let us fix some notation.

Notation 4.3.1. For $\eta \in \mathrm{Num}(X) \otimes \mathbb{R}$ defining a non-empty wall W^η of type (c_1, c_2), we denote by Z_η the codimension 2 locally complete intersection such that $[Z]_\eta = \frac{\eta^2 - c_1^2}{4} + c_2$. Notice that $[Z_\eta] = [Z_{-\eta}]$ and thus we will make no distinction.

The following general result gives the decomposition of some moduli spaces.

Theorem 4.3.2. *Let X be a smooth projective variety of dimension n and let $c_i \in \mathrm{H}^{2i}(X, \mathbb{Z})$, $i = 1, 2$, be Chern classes. Let W be a non-empty wall of type (c_1, c_2) and let $\mathcal{C}$ and $\mathcal{C}'$ be two chambers with polarizations lying in them and only separated by the wall W, that is, such that*

$$\mathcal{C} \cap \mathrm{Num}(X) \neq \emptyset, \quad \mathcal{C}' \cap \mathrm{Num}(X) \neq \emptyset$$

and

$$\text{Closure}(\mathcal{C}) \cap W \neq \emptyset, \quad \text{Closure}(\mathcal{C}') \cap W \neq \emptyset.$$

Set

$$T := \{\eta \in \text{Num}(X) \otimes \mathbb{R} \quad \textit{defining} \quad W \,|\, \eta \cdot H^{n-1} < 0 \quad \textit{for} \quad H \in \mathcal{C}\}.$$

Then, we have the following set-theoretical relations.

i) If $\mathrm{H}^0(\bigwedge^2 \mathcal{N}_{Z_\eta/X} \otimes \mathcal{O}_X(\eta)_{|Z_\eta}) = 0$ *or* $\mathrm{H}^{n-1} I_{Z_\eta}(-\eta + K_X) = 0$ *for any* $\eta \in T$ *and* $\mathrm{H}^0(\bigwedge^2 \mathcal{N}_{Z_\eta/X} \otimes \mathcal{O}_X(-\eta)_{|Z_\eta}) = 0$ *or* $\mathrm{H}^{n-1} I_{Z_\eta}(\eta + K_X) = 0$ *for any* $\eta \in T$*, then*

$$M_{\mathcal{C}}(2; c_1, c_2) = M_{\mathcal{C}'}(2; c_1, c_2).$$

ii) If $E_\eta(c_1, c_2) \neq \emptyset$ *for every* $\eta \in T$ *and* $\mathrm{H}^0(\bigwedge^2 \mathcal{N}_{Z_\eta/X} \otimes \mathcal{O}_X(-\eta)_{|Z_\eta}) = 0$ *or* $\mathrm{H}^{n-1} I_{Z_\eta}(\eta + K_X) = 0$ *for any* $\eta \in T$*, then*

$$M_{\mathcal{C}}(2; c_1, c_2) = M_{\mathcal{C}'}(2; c_1, c_2) \sqcup (\bigsqcup_{\eta \in T} E_\eta(c_1, c_2)).$$

iii) If $\mathrm{H}^0(\bigwedge^2 \mathcal{N}_{Z_\eta/X} \otimes \mathcal{O}_X(\eta)_{|Z_\eta}) = 0$ *or* $\mathrm{H}^{n-1} I_{Z_\eta}(\eta + K_X) = 0$ *for any* $\eta \in T$ *and* $E_{-\eta}(c_1, c_2) \neq \emptyset$ *for any* $\eta \in T$*, then*

$$M_{\mathcal{C}'}(2; c_1, c_2) = M_{\mathcal{C}}(2; c_1, c_2) \sqcup (\bigsqcup_{\eta \in T} E_{-\eta}(c_1, c_2)).$$

iv) If $E_\eta(c_1, c_2) \neq \emptyset$ *and* $E_{-\eta}(c_1, c_2) \neq \emptyset$ *for every* $\eta \in T$*, then*

$$M_{\mathcal{C}'}(2; c_1, c_2) \sqcup (\bigsqcup_{\eta \in T} E_\eta(c_1, c_2)) = M_{\mathcal{C}}(2; c_1, c_2) \sqcup (\bigsqcup_{\eta \in T} E_{-\eta}(c_1, c_2)).$$

Proof. *i*) First, let us prove the equality assuming that both $M_{\mathcal{C}}(2; c_1, c_2)$ and $M_{\mathcal{C}'}(2; c_1, c_2)$ are non-empty. Let us first prove that $M_{\mathcal{C}} \subseteq M_{\mathcal{C}'}$. If there exists a bundle E that is $\mathcal{C}$-stable but $\mathcal{C}'$-unstable then, by Lemma 4.1.8, there exists a wall W^η of type (c_1, c_2) defined by $\eta \equiv 2G - c_1$ for some $\mathcal{O}_X(G) \hookrightarrow E$ such that the quotient $E/\mathcal{O}_X(G)$ is torsion-free and satisfying $\eta \cdot H^{n-1} < 0 \leq \eta \cdot (H')^{n-1}$ for $H \in \mathcal{C}$ and $H' \in \mathcal{C}'$. In addition, since W is the only wall separating $\mathcal{C}$ and $\mathcal{C}'$, we get $W^\eta = W$, and hence $\eta \in T$. Putting altogether, E sits in the exact sequence

$$0 \to \mathcal{O}_X(G) \to E \to I_{Z_\eta}(c_1 - G) \to 0$$

with $2G - c_1 \equiv \eta \in T$. Since E is a vector bundle this implies

$$\mathrm{H}^0(Z, \bigwedge^2 \mathcal{N}_{Z_\eta/X} \otimes \mathcal{O}_X(\eta)_{|Z_\eta}) \neq 0$$

and, by the fact that E is a stable vector bundle, it also implies that

$$\mathrm{ext}^1(I_{Z_\eta}(c_1 - G), \mathcal{O}_X(G)) = \mathrm{h}^{n-1}\, I_{Z_\eta}(-\eta + K_X) > 0,$$

a contradiction. Hence $M_{\mathcal{C}} \subseteq M_{\mathcal{C}'}$. The inclusion $M_{\mathcal{C}'} \subseteq M_{\mathcal{C}}$ follows in the same way interchanging the roles of η and $-\eta$ and $\mathcal{C}$ and $\mathcal{C}'$. Hence $M_{\mathcal{C}} = M_{\mathcal{C}'}$.

Now, let us prove that, if some of the moduli spaces is empty, then the other one is also empty. For instance, let us prove that, if $M_{\mathcal{C}}(2; c_1, c_2)$ is empty, then we get the emptiness of $M_{\mathcal{C}'}(2; c_1, c_2)$. If there exists $E \in M_{\mathcal{C}'}(2; c_1, c_2)$ then, since $M_{\mathcal{C}}(2; c_1, c_2) = \emptyset$, it is $\mathcal{C}$-unstable. Following the same arguments as before, we get that E sits in an exact sequence of type

$$0 \to \mathcal{O}_X(G) \to E \to I_Z(c_1 - G) \to 0,$$

where $2G - c_1 \equiv \eta \in T$ and thus

$$\mathrm{H}^0(Z_\eta, \bigwedge^2 \mathcal{N}_{Z_\eta/X} \otimes \mathcal{O}_X(\eta)) \neq 0 \quad \text{and} \quad \mathrm{h}^{n-1}\, I_Z(-\eta + K_X) \neq 0,$$

a contradiction. The other case is analogous.

ii) Notice that, if $E_\eta(c_1, c_2) \neq \emptyset$ for every $\eta \in T$, we can assume that every $\eta \in T$ satisfies the C-condition or $\eta^2 \equiv c_1^2 - c_2$. Moreover, by Corollary 4.2.5, we can assume that $M_{\mathcal{C}}(2; c_1, c_2) \neq \emptyset$.

First, let us prove the equality in the case that $M_{\mathcal{C}'}(2; c_1, c_2)$ is non-empty. Let us first prove that, for any $\eta \in T$, $M_{\mathcal{C}'} \cap E_\eta = \emptyset$. If $E \in E_\eta$ then, by definition, E sits in a non-trivial extension of type

$$0 \to \mathcal{O}_X(G) \to E \to I_{Z_\eta}(c_1 - G) \to 0,$$

for some $\mathcal{O}_X(G) \hookrightarrow E$ such that $2G - c_1 \equiv \eta$. The fact that $\eta \cdot (H')^{n-1} > 0$ for $H' \in \mathcal{C}'$, which is equivalent to $G \cdot (H')^{n-1} > \frac{c_1 \cdot (H')^{n-1}}{2}$, implies that E is not H'-stable. Now, let us prove the inclusion $M_{\mathcal{C}} \subseteq M_{\mathcal{C}'} \sqcup (\bigsqcup_{\eta \in T} E_\eta)$. Following the same steps as in the proof of *i*), if $E \in M_{\mathcal{C}} \backslash M_{\mathcal{C}'}$, we get that E is given by a non-trivial extension of type

$$0 \to \mathcal{O}_X(G) \to E \to I_{Z_\eta}(c_1 - G) \to 0,$$

where $2G - c_1 \equiv \eta \in T$. Therefore, since in this case we can assume that η satisfies the C-condition or verifies $\eta^2 \equiv c_1^2 - 4c_2$, by definition we conclude that $E \in E_\eta$ and hence $M_{\mathcal{C}} \backslash M_{\mathcal{C}'} \subseteq \bigsqcup_{\eta \in T} E_\eta$, which is equivalent to have the inclusion $M_{\mathcal{C}} \subseteq M_{\mathcal{C}'} \sqcup (\bigsqcup_{\eta \in T} E_\eta)$. Now, let us prove the inclusion $M_{\mathcal{C}'} \sqcup (\bigsqcup_{\eta \in T} E_\eta) \subseteq M_{\mathcal{C}}$. By Theorem 4.2.4, for any $\eta \in T$ we get $E_\eta \subseteq M_{\mathcal{C}}$, and thus $\bigsqcup_{\eta \in T} E_\eta \subseteq M_{\mathcal{C}}$. To end, let us prove that $M_{\mathcal{C}'} \subseteq M_{\mathcal{C}}$. Following the same arguments as in the proof of *i*), we get $\mathrm{H}^0(\bigwedge^2 \mathcal{N}_{Z'_\eta} \otimes \mathcal{O}_X(\eta')_{|Z_{\eta'}}) \neq 0$ and

$$\mathrm{ext}^1(I_{Z'_\eta}(c_1 - G), \mathcal{O}_X(G)) = \mathrm{h}^{n-1}\, I_{Z_{\eta'}}(-\eta + K_X) > 0,$$

a contradiction. Putting altogether,

$$M_{\mathcal{C}}(2;c_1,c_2)=M_{\mathcal{C}'}(2;c_1,c_2)\sqcup(\bigsqcup_{\eta\in T}E_\eta(c_1,c_2)).$$

Finally, let us prove that, if $M_{\mathcal{C}'}(2;c_1,c_2)=\emptyset$, we get

$$M_{\mathcal{C}}(2;c_1,c_2)=\bigsqcup_{\eta\in T}E_\eta(c_1,c_2).$$

In fact, if $E\in M_{\mathcal{C}}(2;c_1,c_2)$, since $M_{\mathcal{C}'}(2;c_1,c_2)=\emptyset$, E is $\mathcal{C}$-stable but $\mathcal{C}'$-unstable. Therefore, following the arguments as before we get that $E\in E_\eta(c_1,c_2)$ for some $\eta\in T$ and thus $M_{\mathcal{C}}(2;c_1,c_2)\subseteq\bigsqcup_{\eta\in T}E_\eta(c_1,c_2)$. The other-side inclusion follows from Corollary 4.2.5.

iii) The proof is analogous to the one of *ii*) but changing the roles of η and $-\eta$ and the ones of $\mathcal{C}$ and $\mathcal{C}'$.

iv) We have already seen that $M_{\mathcal{C}}\cap E_\eta=\emptyset$ and $M_{\mathcal{C}'}\cap E_{-\eta}=\emptyset$ for any $\eta\in T$. In addition, we have argued that we can assume that, for any $\eta\in T$, η and $-\eta$ satisfy the C-condition or verify $\eta^2\equiv c_1^2-4c_2$, and also that both $M_{\mathcal{C}}(2;c_1,c_2)$ and $M_{\mathcal{C}'}(2;c_1,c_2)$ are non-empty.

Let us first prove that $M_{\mathcal{C}}\sqcup(\bigsqcup_{\eta\in T'}E_{-\eta})\subseteq M_{\mathcal{C}'}\sqcup(\bigsqcup_{\eta\in T}E_\eta)$. On one hand, by Theorem 4.2.4, $E_{-\eta}\subseteq M_{\mathcal{C}'}$ for any $\eta\in T$ and thus $\bigsqcup_{\eta\in T}E_{-\eta}\subseteq M_{\mathcal{C}'}$. On the other hand, by the proof of *ii*), if $E_\eta\neq\emptyset$ for any $\eta\in T$ we get $M_{\mathcal{C}}\subseteq M_{\mathcal{C}'}\sqcup(\bigsqcup_{\eta\in T}E_\eta)$. Therefore $M_{\mathcal{C}}\sqcup(\bigsqcup_{\eta\in T}E_{-\eta})\subseteq M_{\mathcal{C}}\sqcup(\bigsqcup_{\eta\in T}E_\eta)$. The other inclusion follows exactly in the same way but considering $-\eta$ and $\mathcal{C}'$. Hence

$$M_{\mathcal{C}'}\sqcup(\bigsqcup_{\eta\in T}E_\eta)=M_{\mathcal{C}}\sqcup(\bigsqcup_{\eta\in T}E_{-\eta}),$$

which finishes the proof. □

In the next chapter, we will deal with stability conditions which we can completely describe in terms of walls and chambers. In fact, we will obtain a detailed description of the walls and chambers and this will allow us to apply Theorem 4.3.2 to describe the corresponding moduli spaces.

Chapter 5

Moduli spaces of stable bundles on ruled 3-folds

The main goal of this chapter is to study moduli spaces of rank 2 stable bundles on the ruled 3-folds over $\mathbb{P}^2$ introduced in Chapter 1, by applying the wall-and-chamber theory developed in Chapter 4. The results in this chapter are contained in [17].

Let X be a smooth projective variety of dimension n and let $M_H(r; c_1, \ldots, c_s)$ denote the moduli space of H-stable vector bundles E on X of rank r with prescribed Chern classes $c_i(E) = c_i$, $1 \leq i \leq s := \min\{r, n\}$. As discussed in the preceding chapters, the investigation of these moduli spaces constitutes a central classification problem in Algebraic Geometry. Since their introduction in the seminal work of Maruyama (see [45] and [46]), such moduli spaces have been an object of extensive study from a wide range of perspectives.

If the underlying variety is a surface, there are general results that determine the irreducibility, smoothness, number of connected components, among other properties, of these moduli spaces. Moreover, there are rich results describing their main geometrical properties according to the position of the surface in the Kodaira classification. In contrast, when $\dim X \geq 3$, the study of moduli spaces of stable vector bundles becomes substantially more intricate and, comparatively, little is known. In fact, moduli spaces of stable bundles over varieties of higher dimension are often non-reduced and non-smooth (see, for instance, [4], [5] and [48]).

However, recent developments have highlighted the rich and subtle geometry exhibited by moduli spaces of stable bundles on higher-dimensional varieties. In fact, these moduli spaces of stable bundles encapsulate deep geometric information about the underlying variety. Understanding their structure is therefore crucial, not only for the intrinsic study of vector bundles, but also for advancing broader programs.

To face these challenges, the theory of walls and chambers have shown to be a powerful tool. In particular, studying the wall-crossing of the moduli space could help us to understand birational geometry, deformation theory, and how different moduli spaces are connected. Also, the understanding of the wall-crossing phenomenon allows the description of moduli spaces of H-stable bundles for smart choices of the ample divisor H.

As mentioned in Chapter 4, the transition from surfaces to 3-folds introduces a host of new geometric and analytical challenges, but also opens the door to deeper insights and applications. In addition, the study of moduli spaces of stable bundles on 3-folds is valuable since it advances our understanding of higher-dimensional Algebraic Geometry. In the literature, there are several works concerning moduli spaces of rank two stable bundles on $\mathbb{P}^3$ (see, for instance, [13], [32], [64]), but there are scattered results when the underlying variety has reacher geometry in terms, for instance, of its ample cone $\mathcal{C}_X$. In this chapter, our contribution focuses on the case in which X is a ruled 3-fold over $\mathbb{P}^2$, constructed from a rank 2 globally generated vector bundle on $\mathbb{P}^2$ (as described in Section 1.3.2, Chapter 1). We will apply Theorem 4.3.2 to give a decomposition of moduli spaces $M_H(2; c_1, c_2)$ of rank 2 stable bundles on ruled 3-folds in terms of the families $E_\xi(c_1, c_2)$ introduced in Chapter 4, Section 4.1. As a byproduct, we will be able to prove the existence of non-empty, smooth and irreducible moduli spaces and we will obtain results concerning the non-emptiness of the corresponding Brill-Noether loci. As we have explained in Chapter 1, Brill-Noether theory was introduced in the 19th century in the context of line bundles on curves and, later on, it was generalized to rank r stable vector bundles on projective varieties. In the general case, the Brill-Noether locus $W_H^k(r; c_1, \dots, c_s)$ is defined as the subvariety of the moduli space $M_H(r; c_1, \dots, c_s)$ that parametrizes H-stable rank r vector bundles on X with at least k independent sections, and the work of [55] makes it possible to prove their existence in the 3-fold case. In Chapter 2, we have mentioned several works in the case when X is a surface, as well as introduced our own results, but little is known for the higher-dimensional case. Appart from our work in the context of ruled 3-folds over $\mathbb{P}^2$, some results can be found in [18] and [54].

The chapter is organized as follows. In Section 5.1, we will give a geometric interpretation and a description of the walls and chambers of type (c_1, c_2) on these ruled 3-folds. After this, in Section 5.2, we will determine the walls and chambers structure for the cases of $c_1 \in \{S, S+L, L, 0\}$ and, then, we will study the moduli space corresponding to a polarization lying in a certain chamber. We proceed as follows. We first fix an element defining a wall of type (c_1, c_2) and we study the wall it defines and how the chambers adjacent to this wall are . After this, we apply Theorem 4.3.2 to get results concerning the decomposition and the emptiness and non-emptiness of the moduli spaces. Moreover, in some cases, this decomposition allows us to describe rational components of the moduli space (see Theorems 5.2.7, 5.2.8, 5.2.9 and 5.2.10; Theorems 5.2.17, 5.2.18 and 5.2.21; Theorems 5.2.26 and 5.2.28; Theorems 5.2.34 and 5.2.38). At the end of this chapter, in Section 5.3, we will apply the results in Section 5.2 to study the non-emptiness of the Brill-Noether locus in this context (see Theorems 5.3.2, 5.3.3, 5.3.4, 5.3.5, 5.3.6, 5.3.7, 5.3.8 and 5.3.9).

5.1 Walls and chambers on ruled 3-folds

As we mentioned in the introduction of this chapter, our goal is to describe some moduli spaces of rank 2 stable vector bundles on these ruled 3-folds and, as a main tool, we will apply the theory of walls and chambers developed in Chapter 4. To this end, in this section, we will give a detailed description of the walls and chambers of type (c_1, c_2) that we have for certain values of c_1 and c_2.

Within this chapter, our ambient variety will be a ruled 3-fold over $\mathbb{P}^2$ $\pi : X = \mathbb{P}(\mathcal{E}) \to \mathbb{P}^2$, where $\mathcal{E}$ is a globally generated vector bundle on $\mathbb{P}^2$ and we will keep the notation introduced in Section 1.3.2. Moreover, to provide a geometric interpretation of walls and chambers, we will assume that there exists a line $l \subset \mathbb{P}^2$ such that

$$\mathcal{E}_{|l} \cong \mathcal{O}_l(c_1') \oplus \mathcal{O}_l. \tag{5.1}$$

According to Lemma 1.3.6, this implies that a divisor $H \equiv \alpha S + \beta L$ is ample if and only if $\alpha, \beta > 0$. This characterization enables us to give a detailed geometric description of some walls in the Kähler cone $\mathcal{C}_X$ and allows us to draw the picture that will help us to visualize the main results of this chapter.

To start with, since $\mathrm{Pic}(X) \cong \mathrm{Num}(X) \cong \mathbb{Z} \oplus \mathbb{Z}$ (see Section 1.3.2), any ξ defining a wall of type (c_1, c_2) has integer coefficients and, in general, one can identify a divisor $D = aS + bL$ with a point $(a, b) \in \mathbb{Z} \times \mathbb{Z}$. Moreover, since we have also seen that $H \equiv \alpha S + \beta L$ is ample if and only if $\alpha, \beta > 0$, the Kähler cone $\mathcal{C}_X$ can be identified with $\mathbb{R}^+ \times \mathbb{R}^+$.

For some values of c_1 and c_2, the walls of type (c_1, c_2) are rays on $\mathbb{R}^+ \times \mathbb{R}^+$ and thus the chambers are connected regions separated by these rays. We will focus our attention on the walls that we will use in next sections. To this end, let us assume that $\xi = aS + (2b + c)L$ with $a \in \{-1, -2\}$, $b > c'_1 + 1$ an integer and $c \in \{0, 1\}$.

Notice that, $H = \alpha S + \beta L \in W^{\xi}$ if and only if $\xi \cdot H^2 = 0$, which is equivalent to

$$\alpha^2[a(c_1'^2 - c'_2) + (2b + c)c'_1] + \alpha\beta[2ac'_1 + 2(2b + c)] + a\beta^2 = 0.$$

Let us define

$$F_{\xi}(\alpha, \beta) := \alpha^2[a(c_1'^2 - c'_2) + (2b + c)c'_1] + \alpha\beta[2ac'_1 + 2(2b + c)] + a\beta^2.$$

Therefore, if we consider the identification that we have explained above, we can describe the walls as

$$W^{\xi} = \{(\alpha, \beta) \in \mathbb{R} \times \mathbb{R} \mid F_{\xi}(\alpha, \beta) = 0\} \cap (\mathbb{R}^+ \times \mathbb{R}^+)$$

and a chamber $\mathcal{C}$ is a connected region contained in

$$\{F_{\xi}(\alpha, \beta) < 0\} \cap (\mathbb{R}^+ \times \mathbb{R}^+)$$

if $\xi \cdot H^2 < 0$ for $H \in \mathcal{C}$, or in the region

$$\{F_{\xi}(\alpha, \beta) > 0\} \cap (\mathbb{R}^+ \times \mathbb{R}^+)$$

if $\xi \cdot H^2 > 0$ for $H \in \mathcal{C}$. Now, let us see that the region

$$\{(\alpha, \beta) \in \mathbb{R} \times \mathbb{R} \mid F_{\xi}(\alpha, \beta) = 0\}$$

corresponds to two different lines passing through the origin $(0, 0)$ and thus dividing the plane into four regions. Let us write

$$F_{\xi}(\alpha, \beta) = A\beta^2 + B\alpha\beta + C\alpha^2,$$

where A, B and C are the following integers:

$$A = a, \quad B = 2ac'_1 + 4b + 2c \quad \text{and} \quad C = a(c_1'^2 - c'_2) + 2bc'_1 + cc'_1.$$

Since $a \in \{-1, -2\}$, $b > c'_1 + 1$ and $c \in \{0, 1\}$, we get

$$A < 0, \quad B > 0 \quad \text{and} \quad C > 2(c'_1 + 1)c'_1 - 2(c_1'^2 - c'_2) = 2(c'_1 + c'_2) \geq 0 \quad (5.2)$$

and, therefore, we conclude that $-4AC$ and $B^2 - 4AC$ are positive integers. Thus, the equation $\{F_{\xi}(\alpha, \beta) = 0\}$ has as solutions the lines $\beta = \alpha(\frac{-B+\sqrt{B^2-4AC}}{2A})$ and $\beta = \alpha(\frac{-B-\sqrt{B^2-4AC}}{2A})$, which clearly pass through the origin $(0, 0)$ and have slopes of opposite sign. In fact, by (5.2), $-B+\sqrt{B^2-4AC}>0$ and $-B-\sqrt{B^2-4AC}<0$. Hence

$$\{(\alpha,\beta)\in\mathbb{R}\times\mathbb{R}\mid F_\xi(\alpha,\beta)=0\}$$

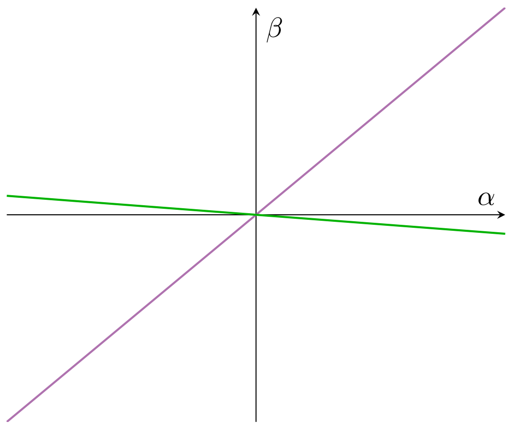
are two lines lying in opposite quadrants of $\mathbb{R}^2$, which implies that W^ξ is a ray in $\mathbb{R}^+\times\mathbb{R}^+$.

The following examples illustrate the situation.

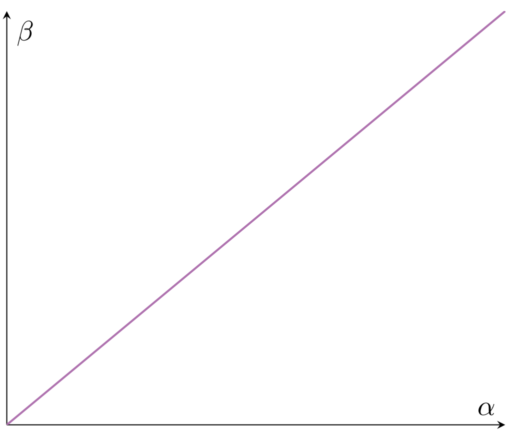
Example 5.1.1. Take $c_1'=c_2'=1$ and consider $\xi=-S+6L$, that is, $a=-1$, $b=3$ and $c=0$. If we plot the region concerning the points $(\alpha,\beta)\in\mathbb{R}^2$ such that

$$F_\xi(\alpha,\beta)=6\alpha^2+10\alpha\beta-\beta^2=0,$$

we get the lines $\beta=(\frac{-5+\sqrt{31}}{6})\alpha$ and $\beta=(\frac{-5-\sqrt{31}}{6})\alpha$:



Thus, W^ξ is the ray:



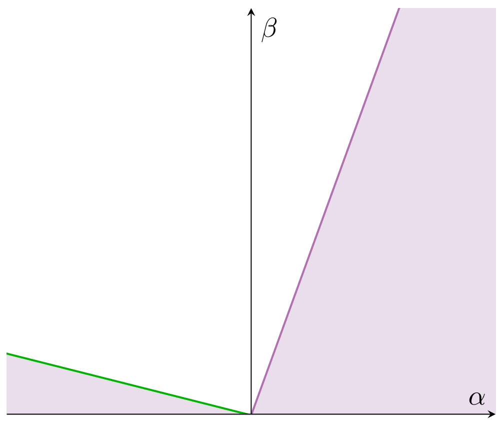
Moreover, following the geometry behind the walls, a chamber $\mathcal{C}$ such that $\mathcal{C} \cap \mathrm{Closure}(W^{\xi}) \neq \emptyset$ is a connected region in $\mathbb{R}^+ \times \mathbb{R}^+$, whose boundary (in the euclidean topology of $\mathbb{R}^2$) contains the ray W^{ξ}. In fact, if $\mathcal{C}$ is the chamber such that $\mathcal{C} \cap \mathrm{Closure}(W^{\xi}) \neq \emptyset$ and $\xi \cdot H^2 > 0$ for $H \in \mathcal{C}$, it is contained in the region

$$\{(\alpha, \beta) \mid F_{\xi}(\alpha, \beta) > 0\} \cap (\mathbb{R}^+ \times \mathbb{R}^+).$$

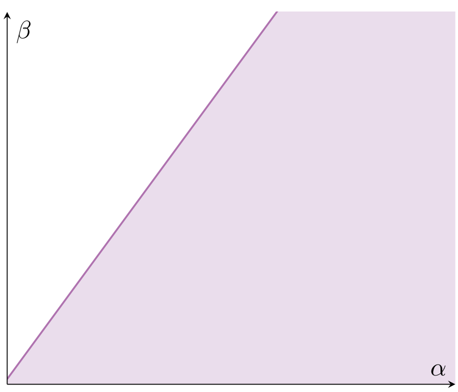
If we plot the region

$$\{(\alpha, \beta) \mid F(\alpha, \beta) > 0\},$$

we get:

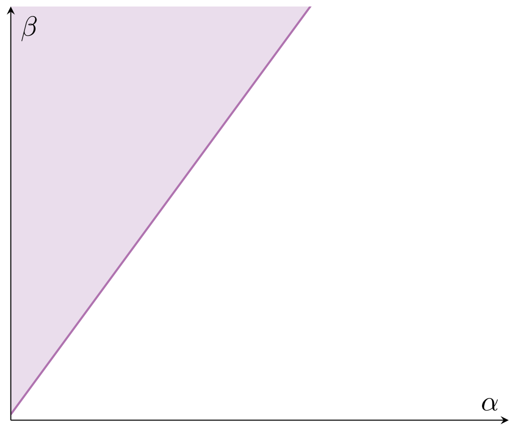


Hence, a chamber of that kind has its boundary containing W^{ξ} and it is a connected component of the region under this ray:

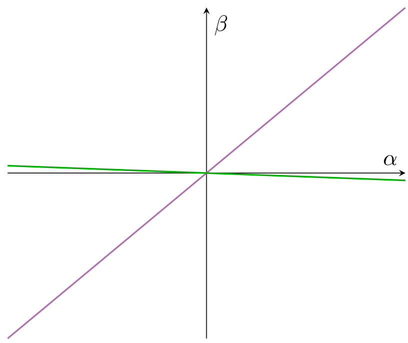


Analogously, a chamber $\mathcal{C}$ such that $\mathcal{C} \cap \mathrm{Closure}(W^{\xi}) \neq \emptyset$ and $\xi \cdot H^2 < 0$ for $H \in \mathcal{C}$, has its boundary containing the ray W^{ξ} and it is a connected component of the region above the ray W^{ξ}.

This region corresponds to the picture:



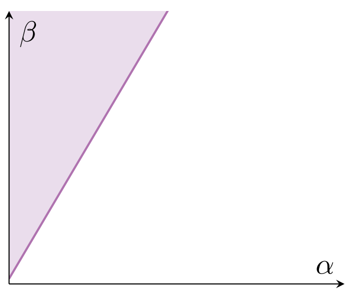
Example 5.1.2. Take $\xi = -S + 7L$, that is $a = 1$, $b = 3$ and $c = 1$, and for instance fix $c_1' = c_2' = 1$. The following picture

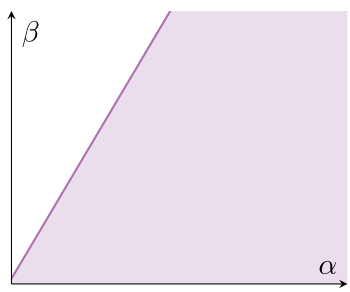


corresponds to

$$W^\xi = \{(\alpha, \beta) \in \mathbb{R} \times \mathbb{R} \mid 7\alpha^2 + 12\alpha\beta - \beta^2 = 0\} \cap (\mathbb{R}^+ \times \mathbb{R}^+).$$

Therefore, a chamber $\mathcal{C}$ such that $\xi \cdot H^2 < 0$ for $H \in \mathcal{C}$ and a chamber $\mathcal{C}$ such that $\xi \cdot H^2 > 0$ for $H \in \mathcal{C}$, are connected components of the following regions, respectively:





Once the geometric description has been established, we return to our original assumptions and henceforth assume only that $\mathcal{E}$ is globally generated.

The preceding geometric picture, which was also noticed by Qin in [60] in the context of ruled surfaces, suggests the following definition.

Definition 5.1.3. Assume that $\xi = aS + bH \in \mathrm{Num}(X)$ with $a < 0$ is a class defining a wall W^ξ and $H \in (\mathrm{Num}(X) \otimes \mathbb{R}) \cap \mathcal{C}_X$. We say that H is above (under) the wall W^ξ if $\xi \cdot H^2 < 0$ ($\xi \cdot H^2 > 0$). We say that a wall W is above (under) the wall W^ξ if any $H \in W$ is above (under) the wall W^ξ.

The following technical lemma will be instrumental in establishing the results of the next section.

Lemma 5.1.4. *If ξ and η are two elements in $\mathrm{Num}(X) \otimes \mathbb{R}$ defining the same non-empty wall of type (c_1, c_2), then $\eta = t\xi$ for some non-zero $t \in \mathbb{Z}$.*

Proof. Let $\xi = aS + bL \in \mathrm{Num}(X) \otimes \mathbb{R}$ defining the non-empty wall W^ξ and consider $\eta \in \mathrm{Num}(X) \otimes \mathbb{R}$ defining a non-empty wall W^η such that $W^\eta = W^\xi$. Notice that we can always find $D = cS + dL \in \mathrm{Num}(X) \otimes \mathbb{R}$ such that $\eta = \xi + D$. Let us see that $D = l\xi$ for some $l \in \mathbb{Z}$. Consider $H = \alpha S + \beta L \in W^\xi$. Then,

$$\begin{aligned} \xi \cdot H^2 &= (aS + bL) \cdot (\alpha S + \beta L)^2 \\ &= a(\alpha^2 S^3 + 2\alpha\beta S^2 \cdot L + \beta^2 S \cdot L^2) + b(\alpha^2 S^2 \cdot L + 2\alpha\beta S \cdot L^2) \\ &= a[\alpha^2(c_1'^2 - c_2') + 2\alpha\beta c_1' + \beta^2] + b[\alpha^2 c_1' + 2\alpha\beta] = 0. \end{aligned} \tag{5.3}$$

Since $\alpha, \beta > 0$ and $\mathcal{E}$ is globally generated, we get

$$\alpha^2 S^3 + 2\alpha\beta S^2 \cdot L + \beta^2 S \cdot L^2 = \alpha^2[c_1'^2 - c_2'] + 2\alpha\beta c_1' + \beta^2 > 0$$

and thus (5.3) is equivalent to

$$\frac{\alpha(\alpha c_1' + 2\beta)}{\alpha^2[(c_1')^2 - c_2'] + 2\alpha\beta c_1' + \beta^2} = -\frac{a}{b}. \tag{5.4}$$

On the other hand, since $W^\xi = W^\eta$, we have

$$0 = \eta \cdot H^2 = (\xi + D) \cdot H^2 = \xi \cdot H^2 + D \cdot H^2 = D \cdot H^2$$

and $D \cdot H^2 = 0$ is equivalent to

$$c = -d\frac{\alpha(\alpha c_1' + 2\beta)}{\alpha^2[(c_1')^2 - c_2'] + 2\alpha\beta c_1' + \beta^2}.$$

Hence, by (5.4) we get $\frac{a}{b} = \frac{c}{d}$ and

$$D = cS + dL = \frac{da}{b}S + dL = \frac{d}{b}(aS + bL) = \frac{d}{b}\xi.$$

Notice that, since $D \in \mathrm{Pic}(X)$, $\frac{d}{b} \in \mathbb{Z}$ and this proves the Claim. Finally, taking $t = 1 + \frac{d}{b}$ we get $\eta = t\xi$. $\square$

5.2 Moduli spaces of rank 2 stable bundles

In this section, we will apply Theorem 4.3.2 to describe different moduli spaces and, in particular, to prove whether they are non-empty. Moreover, in some cases, we will prove that they are smooth and irreducible, we will compute their dimension and we will show that they are rational varieties. In those cases, we will see that properties of the underlying variety, for instance being rational, are translated into the moduli space.

Remark 5.2.1. Since a rank 2 vector bundle E on X is H-stable if and only if $E \otimes \mathcal{O}_X(D)$ is H-stable for any line bundle $\mathcal{O}_X(D)$, we will normalize the first Chern class and we will assume that $c_1 = c_1(E) \in \{S, S+L, L, 0\}$.

Let us fix the following notation.

Notation 5.2.2. Let $c_1 = lS + mL$ and $b \in \mathbb{Z}$. We will denote by ξ_b the class $\xi_b := (l-2)S + (2b+m)L$, denote by W^{ξ_b} the wall defined by ξ_b and, by $\mathcal{C}_b$, the chamber such that $W^{\xi_b} \cap \mathrm{Closure}(\mathcal{C}_b) \neq \emptyset$ and $\xi_b \cdot H^2 < 0$ for $H \in \mathcal{C}_b$ (that is, adjacent to the wall W^{ξ_b} and above it) and by $\mathcal{C}'_b$ the chamber such that $W^{\xi_b} \cap \mathrm{Closure}(\mathcal{C}'_b) \neq \emptyset$ and $\xi_b \cdot H^2 > 0$ for $H \in \mathcal{C}'_b$ (that is, adjacent to the wall W^{ξ_b} and under it). When there is no confusion, we will simply denote by Z the codimension 2 locally complete intersection such that $[Z] = \frac{\xi_b^2 - c_1^2}{4} + c_2$.

In order to describe the moduli spaces $M_{\mathcal{C}_b}(2; c_1, c_2)$ and $M_{\mathcal{C}'_b}(2; c_1, c_2)$ for certain c_1 and c_2, we first give a description of some walls of type (c_1, c_2) and the chambers separated by them.

We will discuss the results separately according to the value of c_1. Let us start with the case $c_1 = S$.

5.2.1 Case $\mathbf{c_1 = S}$

According to Notation 5.2.2, $\xi_b = -S + 2bL$ and our first goal is to prove that ξ_b defines a non-empty wall of type (c_1, c_2) for $c_2 = (b+m)SL - b^2L^2$ with $m \in \{0, 1\}$ and, then, to describe its adjacent chambers.

Lemma 5.2.3. *Let $X = \mathbb{P}(\mathcal{E}) \to \mathbb{P}^2$ be a ruled 3-fold over $\mathbb{P}^2$. Let us fix $c_1 = S$, $c_2 = (b+m)SL - b^2L^2$ for $m \in \{0, 1\}$ and $\xi_b = -S + 2bH$ in* $\mathrm{Num}(X)$ *with $c'_1 + 1 < b \in \mathbb{Z}$. Then,*

i) ξ_b defines a non-empty wall W^{ξ_b} of type (c_1, c_2). Moreover, if $m = 1$, then ξ_b satisfies the C-condition.

ii) The wall W^{ξ_b} is only defined by ξ_b and $-\xi_b$. In addition, if $m=1$, then we get $\mathrm{H}^0(\bigwedge^2\mathcal{N}_{Z/X}\otimes\mathcal{O}_X(-\xi_b)_{|Z})=0$, where $[Z]=\frac{\xi_b^2-c_1^2}{4}+c_2=SL$.

Proof. $i)$ First, let us prove that ξ_b defines a wall W^{ξ_b} of type (c_1,c_2). In fact, $\xi_b+S=2bL$ is clearly divisible by 2 in $\mathrm{Pic}(X)$ and

$$[Z]=\frac{{\xi_b}^2-S^2}{4}+(b+m)SL-b^2L^2=\begin{cases}0 & \text{if} \quad m=0,\\ SL & \text{if} \quad m=1,\end{cases}$$

is a complete intersection in X. Moreover, this wall is non-empty. In fact, if $H=\alpha S+\beta L$, then $\xi_b\cdot H^2=0$ if and only if

$$\begin{aligned}\beta &=\alpha\cdot\left(\tfrac{2(2b-c_1')\pm\sqrt{2^2(2b-c_1')^2-2^2({c_1'}^2-c_2'-2bc_1')}}{2}\right)\\ &=\alpha\cdot\left(2b-c_1'\pm\sqrt{2b(2b-c_1')+c_2'}\right).\end{aligned}$$

In particular, since $2b>c_1'$,

$$\beta=\alpha\cdot\left(2b-c_1'+\sqrt{2b(2b-c_1')+c_2'}\right)\in\mathbb{R}^+,$$

and thus, if $\alpha\geq 1$, $H=\alpha S+\beta L\in(\mathrm{Num}(X)\otimes\mathbb{R})\cap\mathcal{C}_X$, which implies that $H\in W^{\xi_b}$.

Now, let us see that ξ_b satisfies the C-condition if $c_2=(b+1)SL-b^2L^2$. First of all since, by Lemma 1.3.8, we have $\bigwedge^2\mathcal{N}_{Z/X}\cong\mathcal{O}_X(S+L)_{|Z}$, we get

$$\mathrm{H}^0(\bigwedge^2\mathcal{N}_{Z/X}\otimes\mathcal{O}_X(\xi_b)_{|Z})\cong\mathrm{H}^0\,\mathcal{O}_X((2b+1)L)_{|Z}\neq 0.$$

On the other hand, it follows from (1.5) that $\mathrm{H}^2\,\mathcal{O}_X(\xi_b)=\mathrm{H}^2\,\mathcal{O}_X(-S+2bL)=0$ and hence ξ_b satisfies the C-condition.

$ii)$ By Lemma 5.1.4, the wall W^{ξ_b} is defined by $t\xi_b=-tS+2btL\in\mathrm{Num}(X)$ with $t\neq 0$. If $\eta=t\xi_b$ defines a non-empty wall of type $(S,(b+m)SL-b^2L^2)$, then

$$[Z]=\frac{t^2(\xi_b)^2-S^2}{4}+(b+m)SL-b^2L^2$$

is a codimension 2 locally complete intersection. Therefore, $[Z]\cdot H\geq 0$ for any $H\in\mathcal{C}_X$. Let us consider $H_0=S+\beta L$ with $\beta\gg 0$. Thus, $[Z]\cdot H_0\geq 0$, which implies

$$\frac{t^2-1}{4}c_1'+(b+m-t^2b)=(t^2-1)[\frac{c_1'}{4}-b]+m\geq 0.$$

Hence, since $\frac{c_1'}{4}-b\leq -1$ and $0\neq t$, we must have $|t|=1$.

Moreover, since $-\xi_b + S = 2(S - bL)$, $(-\xi_b)^2 = \xi_b^2$ and $-\xi_b \cdot H^2 = 0$, we can conclude from $i)$ that $-\xi_b$ defines a non-empty wall of type (c_1, c_2).

Finally, since $b > c_1' + 1$, by Lemma 1.3.8, we get

$$\mathrm{H}^0(\bigwedge^2 \mathcal{N}_{Z/X} \otimes \mathcal{O}_X(-\xi_b)_{|Z}) \cong \mathrm{H}^0\, \mathcal{O}_X(S - (2b-1)L)_{|Z} = 0,$$

which finishes the proof. □

Let us study the chambers separated by the wall W^{ξ_b}.

Proposition 5.2.4. *Let $c_1' + 1 < b \in \mathbb{Z}$ and consider the Chern classes $c_1 = S$ and $c_2 = (b+m)SL - b^2L^2$ for $m \in \{0,1\}$. Then, the following holds.*

i) There is no wall of type (c_1, c_2) above W^{ξ_b} and under the non-empty wall $W^{\xi_{b+1}}$ of type (c_1, c_2) defined by ξ_{b+1}.

ii) There is no wall of type (c_1, c_2) under W^{ξ_b} and above the set

$$W^{\xi_{b-1}} = \{H \in (\mathrm{Num}(X) \otimes \mathbb{R}) \cap \mathcal{C}_X \mid \xi_{b-1} \cdot H^2 = 0\}.$$

Proof. $i)$ First of all, since $\xi_{b+1} + S = 2(b+1)L$ and

$$[Z] = \frac{{\xi_{b+1}}^2 - S^2}{4} + (b+1)SL - b^2L^2 = 2(b+m)L^2$$

is a complete intersection on X, ξ_{b+1} defines a wall $W^{\xi_{b+1}}$ of type (c_1, c_2). In addition, $W^{\xi_{b+1}}$ is non-empty since

$$H = S + \left(2(b+1) - c_1' + \sqrt{2(b+1) \cdot [2(b+1) - c_1'] + c_2'}\right) L$$

is such that $H \in (\mathrm{Num}(X) \otimes \mathbb{R}) \cap \mathcal{C}_X$ and $\xi_{b+1} \cdot H^2 = 0$. Moreover, if $H \in W^{\xi_{b+1}}$, we have

$$\xi_b \cdot H^2 = (\xi_{b+1} - 2L) \cdot H^2 = -2L \cdot H^2 < 0,$$

which implies that H is above the wall W^{ξ_b}. Thus, the wall $W^{\xi_{b+1}}$ is above the wall W^{ξ_b}.

Now, let us prove that there is no wall of type (c_1, c_2) above W^{ξ_b} and under $W^{\xi_{b+1}}$. To this end, we will study the possible cases of walls W^η of type (c_1, c_2). Notice that we can avoid the case when η or $-\eta$ is effective, since it would not define a wall of type (c_1, c_2).

Case 1: Let us first study the case $\eta_p = -S + 2pL$ with $p > b+1$. If $H \in W^{\eta_p}$, then

$$\xi_{b+1} \cdot H^2 = (\eta_p + 2(b+1-p)L) \cdot H^2 = 2(b+1-p)L \cdot H^2 < 0,$$

where the inequality follows from the fact that H is effective and $b+1-p<0$. Therefore the wall W^{η_p} lies above the wall $W^{\xi_{b+1}}$.

Case 2: Now, let us study the case $\eta_p = -S+2pL$ with $0<p<b$. If $H \in W^{\eta_p}$, then

$$\xi_b \cdot H^2 = (\eta_p + 2(b-p)L)\cdot H^2 = 2(b-p)L \cdot H^2 > 0,$$

where the inequality follows from the fact that L is effective and $p<b$. Therefore the wall W^{η_p} is under the wall W^{ξ_b}.

Case 3: Now, let us study the family $\eta_p = -(2p+1)S + 2bL$ with $p \geq 1$. If $H \in W^{\eta_p}$, then

$$\xi_b \cdot H^2 = (\eta_p + 2pS)\cdot H^2 = 2pS \cdot H^2 > 0,$$

where the inequality follows from the fact that S is effective and $p \geq 1$. Therefore the wall W^{η_p} is under the wall W^{ξ_b}.

Case 4: Now, let us study the family $\eta_{(a,p)} = -(2a+1)S + 2pL$ with $a \geq 1$ and $0<p<b$. Then, by Cases 2 and 3, the wall $\eta_{(a,p)}$ is under the wall W^{ξ_b}.

Case 5: Now, let us study the family of classes $\eta_{(a,p)} = -(2a+1)S+2pL$ with $a \geq 1$ and $p>b$. Let H be such that $\eta_{(a,p)} \cdot H^2 = 0$ and let assume that H is above W^{ξ_b} and under $W^{\xi_{b+1}}$, that is, such that

$$\xi_b \cdot H^2 < 0 < \xi_{b+1} \cdot H^2.$$

On one hand, since $\xi_b \cdot H^2 < 0 = \eta_{(a,p)} \cdot H^2$, we get

$$S \cdot H^2 < \frac{p-b}{a} L \cdot H^2. \tag{5.5}$$

On the other hand, since $\xi_{b+1} \cdot H^2 > 0 = \eta_{(a,p)} \cdot H^2$ and $L \cdot H^2 > 0$, we get

$$S \cdot H^2 > \frac{p-b-1}{a} L \cdot H^2. \tag{5.6}$$

Finally, since $\eta_{(a,p)} \cdot H^2 = 0$ is equivalent to $S \cdot H^2 = \frac{2p}{2a+1} L \cdot H^2$, by (5.5) and (5.6) we get

$$\frac{p-b-1}{a} < \frac{2p}{2a+1} < \frac{p-b}{a},$$

which is equivalent to

$$(2a+1)b < p < (2a+1)(b+1).$$

Therefore, we only have to study the family of $\eta_{(a,p)} = -(2a+1)S + 2pL$ with $a \geq 1$ and p such that $(2a+1)b < p < (2a+1)(b+1)$.

Let us prove that $\eta_{(a,p)}$ does not define a wall of type (c_1, c_2). In fact, if $\eta_{(a,p)}$ defines a wall of type (c_1, c_2), then

$$[Z] = \frac{\eta^2_{(a,p)} - S^2}{4} + (b+m)SL - b^2L^2$$

is a locally complete intersection and thus $[Z] \cdot H \geq 0$ for any $H \in \mathcal{C}_X$. In particular, if $H_0 = S + \beta L$ with $\beta \gg 0$, we have that $[Z] \cdot H_0 \geq 0$, which implies that

$$a(a+1)c_1' + b + m - (2a+1)p \geq 0.$$

Since, $p > (2a+1)b$, we get

$$a(a+1)c_1' + b + 1 - (2a+1)^2 b = a(a+1)(c_1' - 4b) + 1 \geq 0,$$

which is a contradiction since $a \geq 1$ and $c_1' - 4b \leq -1$.

ii) The proof follows step by step as the proof of *i*). □

Now, let us prove that we can always find polarizations in the chambers with common wall W^{ξ_b}. By Proposition 5.2.4, in order to get polarizations in $\mathcal{C}_b$ and $\mathcal{C}_b'$, it is enough to get polarizations above W^{ξ_b} and under $W^{\xi_{b+1}}$, and above $W^{\xi_{b-1}}$ and under W^{ξ_b}, respectively.

Proposition 5.2.5. *Let $X = \mathbb{P}(\mathcal{E}) \to \mathbb{P}^2$ be a ruled 3-fold over $\mathbb{P}^2$ and set $c_i' = c_i(\mathcal{E})$. Let us fix an integer $b > c_1' + 1$ and Chern classes $c_1 = S$ and $c_2 = (b+1)SL - b^2L^2$. Then,* $\mathrm{Num}(X) \cap \mathcal{C}_b \neq \emptyset$ *and* $\mathrm{Num}(X) \cap \mathcal{C}_b' \neq \emptyset$.

Proof. Let us first prove that $\mathrm{Num}(X) \cap \mathcal{C}_b' \neq \emptyset$. To this end, by Proposition 5.2.4, it is sufficient to find a polarization H verifying $\xi_{b-1} \cdot H^2 < 0 < \xi_b \cdot H^2$. Consider $H = S + \beta L \in \mathrm{Num}(X) \cap \mathcal{C}_X$ and define

$$\begin{aligned} f(\beta) :=\ & \xi_b \cdot H^2 && = -\beta^2 + (2b - c_1')\beta + 2bc_1' - [(c_1')^2 - c_2'], \\ g(\beta) :=\ & \xi_{b-1} \cdot H^2 && = -\beta^2 + [2(b-1) - c_1']\beta + 2(b-1)c_1' - [(c_1')^2 - c_2']. \end{aligned}$$

Notice that $g(\beta) = f(\beta) - 2\beta - 2c_1'$. If we denote by $A_1 := 2b - c_1'$ and $A_0 := 2bc_1' - [(c_1')^2 - c_2']$, then $f(\beta) = -\beta^2 + A_1\beta + A_0$ and $g(\beta) = f(\beta) - 2\beta - 2c_1'$ are two parabolas in the variable $\beta \in \mathbb{R}^+$. Let us prove that there exists an integer $\beta_0 > 0$ such that $f(\beta_0) > 0 > g(\beta_0)$. Notice that g has the vertex at the point

$$P = (\frac{A_1 - 2}{2}, \frac{(A_1 - 2)^2}{4} + A_0 - 2c_1').$$

Since $b > c_1' + 1$, P belongs to $\mathbb{R}^+ \times \mathbb{R}^+$, which implies that there exists a positive y_0 such that $g(y_0) = 0$. Then, since g is continuous an decreases in $[y_0, +\infty)$, we have $g(y_0 + 1) < 0$.

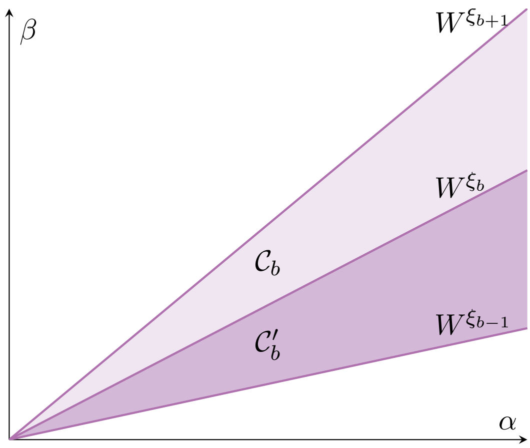
Now, let us check that $f(y_0+1) > 0$. Notice that

$$f(y_0+1) = g(y_0+1) + 2(y_0+1) + 2c_1' = g(y_0) - 1 + A_1 + 2c_1' = 2b + c_1' - 1 > 0.$$

Since f is continuous and $f(\beta) > g(\beta)$ for any β, we have that $f(\beta) > 0$ for $\beta \in [y_0, y_0+1]$. Putting altogether $f(\beta) > 0 > g(\beta)$ for any $\beta \in (y_0, y_0+1]$. If y_0 is an integer, take $\beta_0 = y_0 + 1$; otherwise, take $\beta_0 = \lceil y_0 \rceil \in (y_0, y_0+1)$. Hence we have proved that $H = S + \beta_0 L$ is a polarization in $\mathcal{C}_b'$.

Following the same argument, but in this case considering $f(\beta) := \xi_{b+1} \cdot H^2$ and $g(\beta) := \xi_b \cdot H^2$ for $L = S + \beta L$, we prove that there exists a polarization in $\mathcal{C}_b$. □

Remark 5.2.6. If the ruled 3-fold is constructed under the assumption (5.1), the geometrical description introduced in Section 5.1, gives the result stated in Proposition 5.2.5, and the situation described in Propositions 5.2.4 and 5.2.5 can be reflected in the following picture:



We are ready to study the moduli spaces $M_{\mathcal{C}_b}(2; c_1, c_2)$ and $M_{\mathcal{C}_b'}(2; c_1, c_2)$. Let us start by studying the decomposition of $M_{\mathcal{C}_b}(2; S, (b+1)SL - b^2L^2)$ in terms of $M_{\mathcal{C}_b'}(2; S, (b+1)SL - b^2L^2)$ and the family $E_{\xi_b}(S, (b+1)SL - b^2L^2)$, where $\mathcal{C}_b'$ is the chamber sharing the wall W^{ξ_b} with $\mathcal{C}_b$.

Theorem 5.2.7. *Let $X = \mathbb{P}(\mathcal{E}) \to \mathbb{P}^2$ be a ruled 3-fold over $\mathbb{P}^2$ with $c_i' = c_i(\mathcal{E})$. Let $c_1' + 1 < b \in \mathbb{Z}$ and Chern classes $c_1 = S$ and $c_2 = (b+1)SL - b^2L^2$. Then,*

$$M_{\mathcal{C}_b}(2; c_1, c_2) = M_{\mathcal{C}_b'}(2; c_1, c_2) \sqcup E_{\xi_b}(c_1, c_2).$$

In particular, $M_{\mathcal{C}_b}(2; c_1, c_2)$ is non-empty and contains a component of dimension $2b + \mathrm{h}^0\,\mathcal{E} - \mathrm{h}^0\,\mathcal{E}(-1) - 1$.

Proof. According to the above notations, by Lemma 5.2.3 *i*), $\xi_b = -S + 2bL$ defines the non-empty wall W^{ξ_b} of type (c_1, c_2) and satisfies the C-condition. Therefore, $E_{\xi_b}(c_1, c_2)$ is the family of rank 2 vector bundles E given by a non-trivial extension of type

$$0 \to \mathcal{O}_X(bL) \to E \to I_Z(S - bL) \to 0, \tag{5.7}$$

where $[Z] = \frac{\xi_b^2 - c_1^2}{4} + c_2 = SL$.

Claim: $E_{\xi_b}(c_1, c_2)$ is non-empty of dimension $2b + \mathrm{h}^0\,\mathcal{E} - \mathrm{h}^0\,\mathcal{E}(-1) - 1$.
Proof: Notice that

$$\begin{aligned} \mathrm{ext}^1(I_Z(S-bL), \mathcal{O}_X(bL)) &= \mathrm{ext}^2(\mathcal{O}_X(bL), I_Z(S-bL+K_X)) \\ &= \mathrm{h}^2\, I_Z(S-2bL+K_X). \end{aligned}$$

If we consider the exact sequence

$$0 \to \mathcal{O}_X(-S-L) \to \mathcal{O}_X(-S) \oplus \mathcal{O}_X(-L) \to I_Z \to 0 \tag{5.8}$$

twisted by $\mathcal{O}_X(S - 2bL + K_X)$ and we take cohomology, we get

$$\begin{aligned} \cdots \to\ & \mathrm{H}^2\,\mathcal{O}_X(-2bL+K_X) \oplus \mathrm{H}^2\,\mathcal{O}_X(S-(2b+1)L+K_X) \to \\ & \mathrm{H}^2\, I_Z(S-2bL+K_X) \to \mathrm{H}^3\,\mathcal{O}_X(-(2b+1)L+K_X) \to \\ & \mathrm{H}^3\,\mathcal{O}_X(-2bL+K_X) \oplus \mathrm{H}^3\,\mathcal{O}_X(S-(2b+1)L+K_X) \to \\ & \mathrm{H}^3\, I_Z(S-2bL+K_X) \to 0. \end{aligned} \tag{5.9}$$

Notice that we have

$$\begin{aligned} &\mathrm{h}^2\,\mathcal{O}_X(-2bL+K_X) = \mathrm{h}^1\,\mathcal{O}_X(2bL) = \mathrm{h}^1\,\mathcal{O}_{\mathbb{P}^2}(2b) = 0, \\ &\mathrm{h}^3\,\mathcal{O}_X(-(2b+1)L+K_X) = \mathrm{h}^0\,\mathcal{O}_X((2b+1)L) = \mathrm{h}^0\,\mathcal{O}_{\mathbb{P}^2}(2b+1) = \tbinom{2b+2}{2}, \\ &\mathrm{h}^3\,\mathcal{O}_X(-2bL+K_X) = \mathrm{h}^0\,\mathcal{O}_X(2bL) = \mathrm{h}^0\,\mathcal{O}_{\mathbb{P}^2}(2b) = \tbinom{2b+1}{2}, \\ &\mathrm{h}^2\,\mathcal{O}_X(S-(2b+1)L+K_X) = 0, \\ &\mathrm{h}^3\,\mathcal{O}_X(S-(2b+1)L+K_X) = 0, \\ &\mathrm{h}^3\, I_Z(S-2bL+K_X) = \mathrm{h}^3\,\mathcal{O}_X(S-2bL+K_X) = 0. \end{aligned}$$

Therefore, by (5.9), we obtain

$$\begin{aligned} 0\ &\to \mathrm{H}^2\, I_Z(S-2bL+K_X) \to \mathrm{H}^3\,\mathcal{O}_X(-(2b+1)L+K_X) \\ &\to \mathrm{H}^3\,\mathcal{O}_X(-2bL+K_X) \to 0 \end{aligned}$$

and thus

$$\begin{aligned} \mathrm{h}^2\, I_Z(S-2bL+K_X) &= \mathrm{h}^3\,\mathcal{O}_X(-(2b+1)L+K_X) - \mathrm{h}^3\,\mathcal{O}_X(-2bL+K_X) \\ &= \tbinom{2b+2}{2} - \tbinom{2b+1}{2} \\ &= 2b+1 > 0. \end{aligned}$$

Putting altogether, $E_{\xi_b}(c_1, c_2)$ is non-empty. Moreover, since any $E \in E_{\xi_b}(c_1, c_2)$ is given by a non-trivial extension of type (5.7), we get

$$\dim E_{\xi_b}(c_1, c_2) = \operatorname{ext}^1(I_Z(S - bL), \mathcal{O}_X(bL)) + \dim(\mathcal{G}) - 1,$$

where $\mathcal{G}$ is the scheme that parametrizes the codimension 2 complete intersections Z such that $[Z] = SL$. Hence,

$$\begin{aligned} \dim E_{\xi_b} &= \operatorname{ext}^1(I_Z(S - bL), \mathcal{O}_X(bL)) + \dim \mathcal{G} - 1 \\ &= 2b + 1 + \mathrm{h}^0\, \mathcal{E} - \mathrm{h}^0\, \mathcal{E}(-1) + 1 - 1 \\ &= 2b + \mathrm{h}^0\, \mathcal{E} - \mathrm{h}^0\, \mathcal{E}(-1) + 1, \end{aligned}$$

where the last equality follows from Lemma 1.3.8 *ii*).

Since, by Lemma 5.2.3 *ii*), the wall W^{ξ_b} is only defined by the classes ξ_b and $-\xi_b$ but $\mathrm{H}^0(\bigwedge^2 \mathcal{N}_{Z/X} \otimes \mathcal{O}_X(-\xi_b)) = 0$, by Theorem 4.3.2, $E_{\xi_b}(c_1, c_2) \neq \emptyset$ implies that

$$M_{\mathcal{C}_b}(2; c_1, c_2) = M_{\mathcal{C}'_b}(2; c_1, c_2) \sqcup E_{\xi_b}(c_1, c_2).$$

Thus, $E_{\xi_b}(c_1, c_2)$ is a component of the moduli space $M_{\mathcal{C}_b}(2; c_1, c_2)$ of dimension $2b + \mathrm{h}^0\, \mathcal{E} - \mathrm{h}^0\, \mathcal{E}(-1) - 1$, which in particular implies that the moduli space $M_{\mathcal{C}_b}(2; c_1, c_2)$ is non-empty. □

Now, we prove that upon crossing the wall $W^{\xi_{b+1}}$ into the adjacent chamber above, the moduli space becomes empty, that is, if we consider H in any chamber such that $\xi_{b+1} \cdot H^2 < 0$, then the corresponding moduli space is empty.

Theorem 5.2.8. *Let $X = \mathbb{P}(\mathcal{E}) \to \mathbb{P}^2$ be a ruled 3-fold over $\mathbb{P}^2$ with $c'_i = c_i(\mathcal{E})$. Let $c'_1 + 1 < b \in \mathbb{Z}$ and Chern classes $c_1 = S$ and $c_2 = (b+1)SL - b^2L^2$. If H is a polarization lying in $W^{\xi_{b+1}}$ or above this wall, that is, $\xi_{b+1} \cdot H^2 \leq 0$, then $M_H(2; c_1, c_2) = \emptyset$.*

Proof. First of all, notice that, for H lying in $W^{\xi_{b+1}}$ or above this wall, we have $\xi_{b+1} \cdot H^2 \leq 0$, which is equivalent to

$$2(b+1)L \cdot H^2 \leq S \cdot H^2. \tag{5.10}$$

Let us assume that $M_H(2; c_1, c_2) \neq \emptyset$ and take $E \in M_H(2; c_1, c_2)$. If $F \cong \mathbb{P}^1$ is a general fiber of $\pi : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^2$, we get $E_{|F} \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1-a)$, for some integer $a > 0$. Thus we have the inclusion $\mathcal{O}_{\mathbb{P}^1}(a) \hookrightarrow E_{|F}$, which yields $\mathcal{O}_X(aS + tL) \hookrightarrow E$ for some $t \in \mathbb{Z}$. Hence, for some integers $l \geq 1$ and m, there exists a non-zero section $s \in \mathrm{H}^0\, E(-lS - mL)$, whose scheme of zeros has codimension 2. Since E is H-stable, the inclusion $\mathcal{O}_X(lS + mL) \hookrightarrow E$ implies that $(lS + mL) \cdot H^2 < \frac{S \cdot H^2}{2}$, which is equivalent to

$$2mL \cdot H^2 < -(2l-1)S \cdot H^2. \tag{5.11}$$

By (5.10),

$$-S \cdot H^2 \leq -2(b+1)L \cdot H^2$$

and, since $l \geq 1$ and $S \cdot H^2 > 0$,

$$-(2l-1)S \cdot H^2 \leq -2(2l-1)(b+1)L \cdot H^2,$$

which together with (5.11) implies

$$2mL \cdot H^2 < -(2l-1)S \cdot H^2 \leq -2(2l-1)(b+1)L \cdot H^2.$$

Hence, since $L \cdot H^2 > 0$, we get

$$m < -(2l-1)(b+1).$$

Moreover, since $s \in \mathrm{H}^0\, E(-lS - mL)$ has a scheme of zeros of codimension 2,

$$c_2(E(-lS - mL)) = (b+1+(2l-1)m)SL + (m^2 - b^2)L^2 + l(l-1)S^2$$

is a codimension 2 locally complete intersection. In particular, if we consider $H_k = 2S + kL$ with $k \gg 0$, we have $c_2(E(-lS - mL)) \cdot H_k \geq 0$, which implies

$$b + 1 + (2l-1)m + c_1' l(l-1) \geq 0. \tag{5.12}$$

Finally, since $m < -(2l-1)(b+1)$, by (5.12) we get $l(l-1)(c_1' - 4b - 4) > 0$, a contradiction. Hence, $M_H(2; c_1, c_2) = \emptyset$. $\square$

We can also study the moduli space when considering $c_2 = bSL - b^2L^2$. Notice that, also in this case, by Proposition 5.2.4, there is no wall of type (c_1, c_2) above $W^{\xi_{b-1}}$ and under W^{ξ_b}, nor under $W^{\xi_{b+1}}$ and above W^{ξ_b}. Therefore, the structure of the chambers just above and under W^{ξ_b} is the same as in the other case of c_2. Hence, we will also denote the corresponding chambers by $\mathcal{C}_b$ and $\mathcal{C}_b'$.

Theorem 5.2.9. *Let $X = \mathbb{P}(\mathcal{E}) \to \mathbb{P}^2$ be a ruled 3-fold over $\mathbb{P}^2$ with $c_i' = c_i(\mathcal{E})$. Let $c_1' + 1 < b \in \mathbb{Z}$ and Chern classes $c_1 = S$ and $c_2 = bSL - b^2L^2$. If H is a polarization lying in the wall W^{ξ_b} of type (c_1, c_2) or above it, then $M_H(2; c_1, c_2) = \emptyset$. In particular, $M_{\mathcal{C}_b}(2; c_1, c_2) = \emptyset$.*

Proof. The proof is analogous to the one of Theorem 5.2.8. By assumption, $\xi_b \cdot H^2 \leq 0$, which is equivalent to

$$2bL \cdot H^2 \leq S \cdot H^2. \tag{5.13}$$

Let us assume that $M_H(2;c_1,c_2)\neq\emptyset$ and consider $E\in M_H(2;c_1,c_2)$. Since $c_1(E)=S$, for $l\geq 1$ and m some integers, there exists a non-zero section $s\in \mathrm{H}^0\,E(-lS-mL)$, whose scheme of zeros has codimension 2. Since E is H-stable, the inclusion $\mathcal{O}_X(lS+mL)\hookrightarrow E$ implies that $(lS+mL)\cdot H^2<\frac{S\cdot H^2}{2}$, which is equivalent to

$$2mL\cdot H^2<-(2l-1)S\cdot H^2.$$

Therefore we get $m<-(2l-1)b$. Moreover, since $s\in \mathrm{H}^0\,E(-lS-mL)$ has a scheme of zeros of codimension 2,

$$c_2(E(-lS-mL))=(b+(2l-1)m)SL+(m^2-b^2)L^2+l(l-1)S^2$$

is a codimension 2 locally complete intersection. In particular, if we consider $H_k=2S+kL$ with $k\gg 0$, we have $c_2(E(-lS-mL))\cdot H_k\geq 0$, which implies

$$b+(2l-1)m+c_1'l(l-1)\geq 0. \tag{5.14}$$

Finally, since $m<-(2l-1)b$, by (5.14) we get $l(l-1)(c_1'-4b)>0$, a contradiction. Hence $M_H(2;c_1,c_2)=\emptyset$. In particular, since any $H\in\mathcal{C}_b$ verifies $\xi_b\cdot H^2<0$, we get $M_{\mathcal{C}_b}(2;c_1,c_2)=\emptyset$. □

Theorem 5.2.10. *Let $X=\mathbb{P}(\mathcal{E})\to\mathbb{P}^2$ be a ruled 3-fold over $\mathbb{P}^2$ with $c_i'=c_i(\mathcal{E})$. Let $c_1'+1<b\in\mathbb{Z}$ and Chern classes $c_1=S$ and $c_2=bSL-b^2L^2$.*

i) If $\mathrm{h}^1\,\mathcal{E}(-2b)=0$, then $M_{\mathcal{C}_b'}(2;c_1,c_2)=\emptyset$.

ii) If $\mathrm{h}^1\,\mathcal{E}(-2b)\neq 0$, then $M_{\mathcal{C}_b'}(2;c_1,c_2)$ is a smooth irreducible projective variety of dimension $\mathrm{h}^1\,\mathcal{E}(-2b)-1$.

Proof. According to Lemma 5.2.3, the non-empty wall W^{ξ_b} of type (c_1,c_2) is only defined by ξ_b and $-\xi_b$. In addition,

$$[Z]=\frac{\xi_b-c_1^2}{4}+c_2=0$$

and we have

$$\begin{aligned}\mathrm{h}^1\,\mathcal{O}_X(\xi_b)&=\mathrm{h}^1\,\mathcal{O}_X(-S+2bL)=0,\\ \mathrm{h}^1\,\mathcal{O}_X(-\xi_b)&=\mathrm{h}^1\,\mathcal{O}_X(S-2bL)=\mathrm{h}^1\,\mathcal{E}(-2b).\end{aligned}$$

Therefore, by Theorem 4.3.2 *i*), if $\mathrm{h}^1\,\mathcal{E}(-2b)=0$ then

$$M_{\mathcal{C}_b'}(2;c_1,c_2)=M_{\mathcal{C}_b}(2;c_1,c_2).$$

Moreover, since $M_{\mathcal{C}_b}(2;c_1,c_2)=\emptyset$ by Theorem 5.2.9, we get $M_{\mathcal{C}'_b}(2;c_1,c_2)=\emptyset$.

Now, let us prove the case $\mathrm{h}^1\,\mathcal{E}(-2b)\neq 0$. Since $E_{-\xi_b}(c_1,c_2)$ is the family of rank 2 vector bundles given by a non-trivial extension of type

$$0\to\mathcal{O}_X(S-bL)\to E\to\mathcal{O}_X(bL)\to 0, \tag{5.15}$$

where

$$\mathrm{ext}^1(\mathcal{O}_X(bL),\mathcal{O}_X(S-bH))=\mathrm{h}^1\,\mathcal{O}_X(S-2bL)=\mathrm{h}^1\,\mathcal{E}(-2b)>0,$$

we conclude that it is non-empty. Therefore, it follows from Theorem 4.3.2 *iii)* that

$$M_{\mathcal{C}'_b}(2;c_1,c_2)=M_{\mathcal{C}_b}(2;c_1,c_2)\sqcup E_{-\xi_b}(c_1,c_2).$$

Finally, since by Theorem 5.2.9, $M_{\mathcal{C}_b}(2;c_1,c_2)=\emptyset$, we conclude that

$$M_{\mathcal{C}'_b}(2;c_1,c_2)=E_{-\xi_b}(c_1,c_2).$$

Therefore, since $\xi_b^2=c_1^2-4c_2$, by Proposition 1.1.22,

$$M_{\mathcal{C}'_b}(2;c_1,c_2)\cong\mathbb{P}(\mathrm{Ext}^1(\mathcal{O}_X(bL),\mathcal{O}_X(S-2bL)))\cong\mathbb{P}^d$$

for $d=\mathrm{h}^1\,\mathcal{E}(-2b)-1$. Hence, $M_{\mathcal{C}'_b}(2;c_1,c_2)$ is a smooth irreducible projective variety of dimension $\mathrm{h}^1\,\mathcal{E}(-2b)-1$. $\square$

Keeping the above notations, we can state the following result.

Corollary 5.2.11. *Let $\pi:X=\mathbb{P}(\mathcal{E})\to\mathbb{P}^2$ be a ruled 3-fold, where $\mathcal{E}$ is a rank 2 vector bundle on $\mathbb{P}^2$ such that $\mathrm{h}^1\,\mathcal{E}(-2b-1)\neq 0$. Then, the moduli space $M_{\mathcal{C}'_b}(2;c_1,c_2)$ is a rational variety with Picard group $\mathrm{Pic}(M_{\mathcal{C}'_b}(2;c_1,c_2))\cong\mathbb{Z}$.*

Proof. It follows from the fact that, by Theorem 5.2.10, if $\mathrm{h}^1\,\mathcal{E}(-2b)\neq 0$ then the moduli space $M_{\mathcal{C}'_b}(2;c_1,c_2)$ is isomorphic to a projective space. $\square$

Example 5.2.12.

i) The bundle $\mathcal{E}=\mathcal{O}_{\mathbb{P}^2}(d)\oplus\mathcal{O}_{\mathbb{P}^2}(d')$ with $d,d'\geq 0$ such that $d+d'=c'_1\geq 0$, is globally generated and $\mathrm{h}^1\,\mathcal{E}(-2b)=0$.

ii) Let us construct a globally generated rank 2 vector bundle $\mathcal{E}$ on $\mathbb{P}^2$ such that $\mathrm{h}^1\,\mathcal{E}(-2b)\neq 0$. Take $c'_1>0$ and consider the vector bundle $\mathcal{E}$ on $\mathbb{P}^2=\mathbb{P}(K[x,y,z])$ as the cokernel of the map $\varphi=(xq_1,xq_2,xq_3)$, where the q_i's are homogeneous forms of degree $c_1' - 1$. That is, $\mathcal{E}$ sits in the following exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-c_1') \xrightarrow{(xq_1,xq_2,xq_3)} \mathcal{O}_{\mathbb{P}^2}^{\oplus 3} \to \mathcal{E} \to 0.$$

It is clear that $\mathcal{E}$ is globally generated and, if we twist the exact sequence by $\mathcal{O}_{\mathbb{P}^2}(-2b)$ and we take cohomology, we get

$$0 \to \mathrm{H}^1\,\mathcal{E}(-2b) \to \mathrm{H}^2\,\mathcal{O}_{\mathbb{P}^2}(-c_1'-2b) \to \mathrm{H}^2\,\mathcal{O}_{\mathbb{P}^2}(-2b)^{\oplus 3} \to \mathrm{H}^2\,\mathcal{E}(-2b) \to 0,$$

and thus $\mathrm{H}^1\,\mathcal{E}(-2b) \neq 0$ if and only if the map

$$F : \mathrm{H}^2\,\mathcal{O}_{\mathbb{P}^2}(-2-2b) \to \mathrm{H}^2\,\mathcal{O}_{\mathbb{P}^2}(-2b)^{\oplus 3}$$

is not injective. Notice that, F is injective if and only if the dual map

$$\begin{array}{llll} F^*: & \mathrm{H}^0\,\mathcal{O}_{\mathbb{P}^2}(2b-3)^{\oplus 3} & \to & \mathrm{H}^0\,\mathcal{O}_{\mathbb{P}^2}(2b+c_1'-3) \\ & (f_1,f_2,f_3) & \mapsto & x(q_1f_1+q_2f_2+q_3f_2) \end{array}$$

is an epimorphism, which is not true since not every homogeneous polynomial of degree $2b+c_1'-3 \geq 1$ is of the form $x(q_1f_1+q_2f_2+q_3f_3)$ (for instance the monomial $y^{2b+c_1'-3}$). Hence, $\mathrm{H}^1\,\mathcal{E}(-2b) \neq 0$.

Now, let us study the case $c_1 = S + L$.

5.2.2 Case $\mathbf{c_1 = S + L}$

According to Notation 5.2.2, $\xi_b = -S + (2b+1)L$ and, as we have done for $c_1 = S$, we will start by proving that ξ_b defines a non-empty wall W^{ξ_b} of type (c_1, c_2). In this case, we fix $c_2 = (b+m)SL - b(b+1)L^2$ with $m \in \{1, 2\}$. After this, we will describe the chambers just above and under W^{ξ_b}.

Lemma 5.2.13. *Let $X = \mathbb{P}(\mathcal{E}) \to \mathbb{P}^2$ be a ruled 3-fold over $\mathbb{P}^2$, where $\mathcal{E}$ is a globally generated bundle on $\mathbb{P}^2$. Let us fix b an integer such that $b > c_1' + 1$. Consider $\xi_b = -S + (2b+1)L$ and fix the Chern classes $c_1 = S + L$ and $c_2 = (b+m)SL - b(b+1)L^2$ with $m = 1, 2$. Then,*

i) ξ_b defines a non-empty wall W^{ξ_b} of type (c_1, c_2). Moreover, for $m = 2$, ξ_b satisfies the C-condition.

ii) The wall W^{ξ_b} is only defined by the classes ξ_b and $-\xi_b$. Moreover, if $m = 2$, $\mathrm{H}^0(\bigwedge^2 \mathcal{N}_{Z/X} \otimes \mathcal{O}_X(-\xi_b)_{|Z}) = 0$.

Proof. The proof is analogous to the one of Lemma 5.2.4.

i) First, let us prove that ξ_b defines a wall W^{ξ_b} of type (c_1, c_2). In fact, $\xi_b + S + L = 2(b+1)L$ is clearly divisible by 2 in $\mathrm{Pic}(X)$ and

$$[Z] = \frac{{\xi_b}^2 - (S+L)^2}{4} + (b+1)SL - b(b+1)L^2 = \begin{cases} 0 & \text{if} \quad m=1, \\ SL & \text{if} \quad m=2, \end{cases}$$

is a complete intersection. Moreover, this wall is non-empty. Indeed, if we fix $H = \alpha S + \beta L$, then $\xi_b \cdot H^2 = 0$ if and only if

$$\begin{aligned} \beta &= \alpha \cdot \left(\tfrac{2(2b+1-c_1') \pm \sqrt{2^2(2b+1-c_1')^2 - 4({c_1'}^2 - c_2' - 2(b+1)c_1')}}{2} \right) \\ &= \alpha \cdot \left(2b+1-c_1' \pm \sqrt{(2b+1)(2b+1-c_1') + c_2'} \right). \end{aligned}$$

In particular, since $2b > c_1'$,

$$\beta = \alpha \cdot \left(2b - c_1' + \sqrt{2b(2b-c_1') + c_2'} \right) \in \mathbb{R}^+,$$

and thus, if $\alpha \geq 1$, $H = \alpha S + \beta L \in (\mathrm{Num}(X) \otimes \mathbb{R}) \cap \mathcal{C}_X$, which implies $H \in W^{\xi_b}$.

Finally, let us check that, for $m = 2$, ξ_b satisfies the C-condition. Notice that $\mathrm{H}^2\,\mathcal{O}_X(-S + (2b+1)L) = 0$. In addition, since $[Z] = SL$, by Lemma 1.3.8 we have

$$\mathrm{H}^0(\bigwedge^2 \mathcal{N}_{Z/X} \otimes \mathcal{O}_X(\xi_b)_{|Z}) \cong H^0 \mathcal{O}_X((2b+1)L)_{|Z} \neq 0,$$

and hence we conclude that ξ_b satisfies the C-condition.

ii) By Lemma 5.1.4, the wall W^{ξ_b} can only be defined by a class of the form $t\xi_b = -tS + (2b+1)tL \in \mathrm{Num}(X)$ with $t \neq 0$. If $\eta = t\xi_b$ defines a non-empty wall of type (c_1, c_2), then

$$[Z] = \frac{t^2(\xi_b)^2 - (S+L)^2}{4} + c_2$$

is a codimension 2 locally complete intersection. Then, in particular, $[Z] \cdot H_0 \geq 0$ for $H_0 = 2S + \beta H$ with $\beta \gg 0$, which implies

$$(t^2 - 1)(c_1' - b - m + 1) - t^2(3b + 2 - m + 1) \geq 0.$$

Thus, since $b > c_1' + 1$ and $t \neq 0$, we must have $|t| = 1$. Finally, if $m = 2$, by Lemma 1.3.8, we get

$$\mathrm{H}^0(\bigwedge^2 \mathcal{N}_{Z/X} \otimes \mathcal{O}_X(-\xi_b)_{|Z}) \cong \mathrm{H}^0\,\mathcal{O}_X(S - (2b-1)L)_{|Z} = 0,$$

where the last equality follows from the bound $b > c_1' + 1$. $\square$

Now, let us study the chambers separated by the wall W^{ξ_b}.

Proposition 5.2.14. *Let $X = \mathbb{P}(\mathcal{E}) \to \mathbb{P}^2$ be a ruled 3-fold over $\mathbb{P}^2$ with $c_i' = c_i(\mathcal{E})$. Let $c_1' + 1 < b \in \mathbb{Z}$ and fix the Chern classes $c_1 = S + L$ and $c_2 = (b+m)SL - b(b+1)L^2$ for $m = 1, 2$. Let us consider $\xi_b = -S + (2b+1)L$.*

i) There is no wall of type (c_1, c_2) above W^{ξ_b} and under the set

$$W^{\xi_{b+1}} = \{H \in (\mathrm{Num}(X) \otimes \mathbb{R}) \cap \mathcal{C}_X \mid \xi_{b+1} \cdot H^2 = 0\}.$$

ii) There is no wall of type (c_1, c_2) under W^{ξ_b} and above the set

$$W^{\xi_{b-1}} = \{H \in (\mathrm{Num}(X) \otimes \mathbb{R}) \cap \mathcal{C}_X \mid \xi_{b-1} \cdot H^2 = 0\}.$$

Proof. $i)$ Notice that, if $H \in W^{\xi_{b+1}}$, we have

$$\xi_b \cdot H^2 = (\xi_{b+1} - 2L) \cdot H^2 = -2L \cdot H^2 < 0,$$

which implies that $W^{\xi_{b+1}}$ is above the wall W^{ξ_b}.

Now, let us prove that there is no wall of type (c_1, c_2) above W^{ξ_b} and under $W^{\xi_{b+1}}$. To this end, we will study the possible walls W^{η} of type (c_1, c_2) that may exist. Observe that we can avoid the case when η or $-\eta$ is effective, since it would not define a wall of type (c_1, c_2).

Case 1: First, let us study the case $\eta_p = -S + (2p+1)L$ with $p > b+1$. If $H \in W^{\eta_p}$, then

$$\xi_{b+1} \cdot H^2 = [\eta_p + 2(b+1-p)L] \cdot H^2 = 2(b+1-p)L \cdot H^2 < 0,$$

where the inequality follows from the fact that L is effective and $b+1-p < 0$. Therefore, W^{η_p} lies above the wall $W^{\xi_{b+1}}$.

Case 2: Now, let us study the case $\eta_p = -S + (2p+1)L$ with $0 < p < b$. If $H \in W^{\eta_p}$, then

$$\xi_b \cdot H^2 = (\eta_p + 2(b-p)L) \cdot H^2 = 2(b-p)L \cdot H^2 > 0,$$

where the inequality follows from the fact that L is effective and $p < b$. Therefore, the wall W^{η} is under the wall W^{ξ_b}.

Case 3: Now, let us study the family $\eta_p = -(2p+1)S + (2b+1)L$ with $p \geq 1$. If $L \in W^{\eta_p}$, then

$$\xi_b \cdot H^2 = (\eta_p + 2pS) \cdot H^2 = 2pS \cdot H^2 > 0,$$

where the inequality follows from the fact that S is effective and $p \geq 1$. Therefore the wall W^{η_p} is under the wall W^{ξ_b}.

Case 4: Now, let us study the family $\eta_{(a,p)} = -(2a+1)S + (2p+1)L$ with $a \geq 1$ and $0 < p < b$. Then, by Cases 2 and 3, the wall $\eta_{(a,p)}$ is under the wall W^{ξ_b}.

Case 5: Now, let us study the family of classes $\eta_{(a,p)} = -(2a+1)S+(2p+1)L$ with $a \geq 1$ and $p > b$. Let H be such that $\eta_{(a,p)} \cdot H^2 = 0$ and assume that H is above W^{ξ_b} and under $W^{\xi_{b+1}}$, that is,

$$\xi_b \cdot H^2 < 0 < \xi_{b+1} \cdot H^2.$$

On one hand, since $\xi_b \cdot H^2 < 0 = \eta_{(a,p)} \cdot H^2$, we get

$$S \cdot H^2 < \frac{p-b}{a} L \cdot H^2. \tag{5.16}$$

On the other hand, since $\xi_{b+1} \cdot H^2 > 0 = \eta_{(a,p)} \cdot H^2$, we get

$$S \cdot H^2 > \frac{p-b-1}{a} L \cdot H^2. \tag{5.17}$$

Finally, since $\eta_{(a,p)} \cdot H^2 = 0$ is equivalent to $S \cdot H^2 = \frac{2p+1}{2a+1} L \cdot H^2$, by (5.16) and (5.17) we get

$$\frac{p-b-1}{a} < \frac{2p+1}{2a+1} < \frac{p-b}{a},$$

which is equivalent to

$$a(2b+1) + b < p < a(2b+3) + b + 1.$$

Therefore, we only have to study the family of $\eta_{(a,p)} = -(2a+1)S + 2pL$ with $a \geq 1$ and $a(2b+1) + b < p < a(2b+3) + b + 1$.

Let us prove that $\eta_{(a,p)}$ does not define a wall of type (c_1, c_2). In fact, if $\eta_{(a,p)}$ defines a wall of type (c_1, c_2), then

$$[Z] = \frac{\eta_{(a,p)}^2 - (S+L)^2}{4} + c_2$$

is a locally complete intersection and thus $[Z] \cdot H \geq 0$ for any $H \in \mathcal{C}_X$. In particular, if $H_0 = S + \beta L$ with $\beta \gg 0$, we have that $[Z] \cdot H_0 \geq 0$, which implies that

$$a(a+1)c_1' + b + 1 - (2a+1)p \geq 0.$$

Since, $p > a(2b+1)+b$, we get

$$(a+1)[a(c_1'-4b)-1]-a(2a+1)+m \geq 0,$$

which is a contradiction since $a \geq 1$ and $c_1'-4b<0$.

ii) The proof follows step by step the proof of *i*). □

Remark 5.2.15. Notice that, by Proposition 5.2.14, the structure of the adjacent chambers to W^{ξ_b} is the same for both values of c_2 and, thus, we will denote them by $\mathcal{C}_b$ and $\mathcal{C}_b'$, independently of the c_2 we are fixing.

Now, let us prove that we can always find polarizations in the chambers with common wall W^{ξ_b}. By Proposition 5.2.14, this holds if we get polarizations above $W^{\xi_{b-1}}$ and under W^{ξ_b}, and above W^{ξ_b} and under $W^{\xi_{b+1}}$.

Proposition 5.2.16. *Let $X = \mathbb{P}(\mathcal{E}) \rightarrow \mathbb{P}^2$ be a ruled 3-fold over $\mathbb{P}^2$ and denote $c_i' = c_i(\mathcal{E})$. Let us fix an integer $b > c_1'+1$. Consider $\xi_b = -S+(2b+1)L$ and Chern classes $c_1 = S+L$ and $c_2 = (b+m)SL - b(b+1)L^2$ with $m = 1, 2$. Then,* $\mathrm{Num}(X) \cap \mathcal{C}_b \neq \emptyset$ *and* $\mathrm{Num}(X) \cap \mathcal{C}_b' \neq \emptyset$.

Proof. Let us first prove that $\mathrm{Num}(X) \cap \mathcal{C}_b' \neq \emptyset$. To this end, by Proposition 5.2.14, it is enough to find a polarization H verifying $\xi_{b-1} \cdot H^2 < 0 < \xi_b \cdot H^2$. Consider $H = S + \beta L \in \mathrm{Num}(X) \cap \mathcal{C}_X$ and define

$$\begin{aligned} f(\beta) :=& \quad \xi_b \cdot H^2 &&= -\beta^2 + (4b+2-2c_1')\beta + (2b+1)c_1' - [(c_1')^2 - c_2'] \\ g(\beta) :=& \quad \xi_{b-1} \cdot H^2 &&= -\beta^2 + 2[(2b-1)-c_1']\beta + (2b-1)c_1' - [(c_1')^2 - c_2']. \end{aligned}$$

Notice that $g(\beta) = f(\beta) - 4\beta - 2c_1'$. If we denote by $A_1 := 4b+2-2c_1'$ and by $A_0 := (2b+1)c_1' - [(c_1')^2 - c_2']$, then $f(\beta) = -\beta^2 + A_1\beta + A_0$ and $g(\beta) = f(\beta) - 4\beta - 2c_1'$ are two parabolas in the variable $\beta \in \mathbb{R}^+$. Let us prove that there exists an integer $\beta_0 > 0$ such that $f(\beta_0) > 0 > g(\beta_0)$. Notice that g has the vertex at the point

$$P = (\frac{A_1-4}{2}, \frac{(A_1-4)^2}{4} + A_0 - 2c_1').$$

Since $b > c_1'+1$, P belongs to $\mathbb{R}^+ \times \mathbb{R}^+$, which implies that there exists a positive y_0 such that $g(y_0) = 0$. Then, since g is continuous an decreases in $[y_0, +\infty)$, we have $g(y_0+1) < 0$. Now, let us check that $f(y_0+1) > 0$. Notice that

$$f(y_0+1) = g(y_0+1) + 4(y_0+1) + 2c_1' = g(y_0) + 2y_0 + 3 + A_1 = 2y_0 + 5 + 4b - 2c_1' > 0.$$

Since f is continuos and $f(\beta) > g(\beta)$ for any β, we have that $f(\beta) > 0$ for $\beta \in [y_0, y_0+1]$. Putting altogether $f(\beta) > 0 > g(\beta)$ for any $\beta \in (y_0, y_0+1]$. If y_0 is an integer, take $\beta_0 = y_0 + 1$; otherwise, take $\beta_0 = \lceil y_0 \rceil \in (y_0, y_0+1)$. Hence we have proved that $H = S + \beta_0 L$ is a polarization in $\mathcal{C}'_b$.

Following the same argument, considering $h(\beta) := \xi_{b+1} \cdot H^2$ for $H = S + \beta L$, we prove that there exists a polarization in $\mathcal{C}_b$. In fact,

$$h(\beta) = f(\beta) + 4\beta + 2c'_1$$

and, since $f(\beta)$ has a vertex on

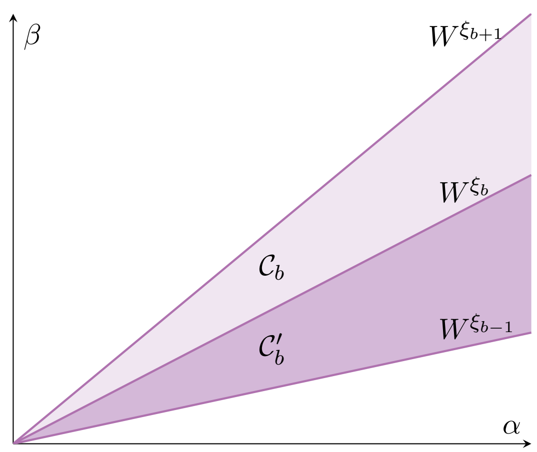
$$\left(\frac{A_1}{2}, \frac{(A_1)^2}{4} + A_0\right) \in \mathbb{R}^+ \times \mathbb{R}^+,$$

there exists $y_1 \in \mathbb{R}^+$ such that $f(y_1) = 0$ and therefore

$$f(\beta_1 + 1) < 0 \quad \text{and} \quad h(\beta_1+1) = f(\beta_1+1) + 4(\beta_1+1) + 2c'_1 > 0.$$

Hence, there exists an integer $\beta_1 \in (y_1, y_1+1]$ satisfying $h(\beta_1) > 0 > f(\beta_1)$. $\square$

As in the previous case, if the ruled 3-fold has been constructed under the assumpton (5.1), the geometrical description given in Section 5.1 gives the result stated in Proposition 5.2.16 and the situation described in Proposition 5.2.14 corresponds to the following picture:



Now, we are ready to describe the moduli space $M_{\mathcal{C}_b}(2; c_1, c_2)$ for the case $c_2 = (b+1)SL - b(b+1)L^2$ with $b > c'_1 + 1$ an integer.

Theorem 5.2.17. *Let $X = \mathbb{P}(\mathcal{E}) \to \mathbb{P}^2$ be a ruled 3-fold. Let us fix b an integer such that $b > c_1' + 1$. Let us consider $\xi_b = -S + (2b+1)L$ and Chern classes $c_1 = S + L$ and $c_2 = (b+1)SL - b(b+1)L^2$. If H is a polarization such that $\xi_b \cdot H^2 \leq 0$, then $M_H(2; c_1, c_2) = \emptyset$. In particular, $M_{\mathcal{C}_b}(2; c_1, c_2) = \emptyset$.*

Proof. Again, the proof is analogous to the one of Theorem 5.2.8. First of all, notice that $\xi_b \cdot H^2 \leq 0$ is equivalent to

$$(2b+1)L \cdot H^2 \leq S \cdot H^2. \tag{5.18}$$

Let us assume that $M_H(2; c_1, c_2) \neq \emptyset$ and take $E \in M_H(2; c_1, c_2)$. If $F \cong \mathbb{P}^1$ is a general fiber of $\pi : \mathbb{P}(\mathcal{E}) \to \mathbb{P}^2$, we get $E_{|F} \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(1-a)$, with $a > 0$ some integer. Thus $\mathcal{O}_{\mathbb{P}^1}(a) \hookrightarrow E_{|F}$, which implies $\mathcal{O}_X(aS + tL) \hookrightarrow E$ for some $t \in \mathbb{Z}$. Hence, for some integers $l \geq 1$ and m, we can find a section $s \in \mathrm{H}^0\, E(-lS - mL)$, whose scheme of zeros has codimension 2. Since E is H-stable, $\mathcal{O}_X(lS + mL) \hookrightarrow E$ implies $(lS + mL) \cdot H^2 < \frac{(S+L)\cdot H^2}{2}$, which is equivalent to

$$(2m-1)L \cdot H^2 < -(2l-1)S \cdot H^2. \tag{5.19}$$

By (5.18),

$$-S \cdot H^2 \leq -(2b+1)L \cdot H^2$$

and, since $l \geq 1$ and $S \cdot H^2 > 0$,

$$-(2l-1)S \cdot H^2 \leq -(2l-1)(2b+1)L \cdot H^2,$$

which together with (5.19) implies

$$2mL \cdot H^2 \leq -(2l-1)S \cdot H^2 \leq -(2l-1)(2b+1)L \cdot H^2.$$

Hence, since $L \cdot H^2 > 0$, we get

$$2m \leq -(2l-1)(2b+1).$$

Moreover, since $s \in \mathrm{H}^0\, E(-lS - mL)$ has a scheme of zeros of codimension 2,

$$c_2(E(-lS-mL)) = (b+1+(2l-1)m-l)SL + [m(m-1)-b(b+1)]L^2 + l(l-1)S^2$$

is a codimension 2 locally complete intersection. In particular, if $H_k = 2S + kL$ with $k \gg 0$, we have $c_2(E(-lS - mL)) \cdot H_k \geq 0$, which implies

$$b + 1 + (2l-1)m + l[c_1'(l-1) - 1] \geq 0. \tag{5.20}$$

Finally, since $2m \leq -(2l-1)(2b+1)$, by (5.20) we get

$$2l[(l-1)[c_1' - 4l(b-1) - 1] + 1 \geq 0,$$

a contradiction. Hence, $M_H(2; c_1, c_2) = \emptyset$. $\square$

We have studied the moduli space $M_H(2;c_1,c_2)$, when H lies in the wall W^{ξ_b} or in the adjacent chamber above it. Now, let us study the moduli space $M_{\mathcal{C}'_b}(2;c_1,c_2)$, that is, the one corresponding to polarizations lying in the adjacent chamber to W^{ξ_b} and under this wall.

Theorem 5.2.18. *Let $X=\mathbb{P}(\mathcal{E})\to\mathbb{P}^2$ be a ruled 3-fold. Let us fix an integer $b>c'_1+1$. Consider $\xi_b=-S+(2b+1)L$ and Chern classes $c_1=S+L$ and $c_2=(b+1)SL-b(b+1)L^2$. The following holds.*

i) If $\mathrm{h}^1\,\mathcal{E}(-2b-1)=0$, then $M_{\mathcal{C}'_b}(2;c_1,c_2)=\emptyset$.

ii) If $\mathrm{h}^1\,\mathcal{E}(-2b-1)\neq 0$, then $M_{\mathcal{C}'_b}(2;c_1,c_2)$ is a smooth irreducible projective variety of dimension $\mathrm{h}^1\,\mathcal{E}(-2b-1)-1$.

Proof. The proof is analogous to the one of Theorem 5.2.10. First, by Lemma 5.2.13 *i*), the wall W^{ξ_b} is non-empty and it is only defined by ξ_b and $-\xi_b$. In addition, in this case, $[Z]=\frac{\xi_b^2-c_1^2}{4}+c_2=0$ and
$\mathrm{h}^1\,\mathcal{O}_X(\xi_b)=0$,
$\mathrm{h}^1\,\mathcal{O}_X(-\xi_b)=\mathrm{h}^1\,\mathcal{O}_X(S-(2b+1)L)=\mathrm{h}^1\,\mathcal{E}(-2b-1)$.
Therefore, by Theorems 4.3.2 *i*) and 5.2.17, if $\mathrm{h}^1\,\mathcal{E}(-2b-1)=0$ we get

$$M_{\mathcal{C}'_b}(2;c_1,c_2)=M_{\mathcal{C}_b}(2;c_1,c_2)=\emptyset.$$

Otherwise, if $\mathrm{h}^1\,\mathcal{E}(-2b)\neq 0$, $E_{-\xi_b}(c_1,c_2)$ is non-empty by definition, which by Theorem 4.3.2 *iii*) implies that

$$M_{\mathcal{C}'_b}(2;c_1,c_2)=M_{\mathcal{C}_b}(2;c_1,c_2)\sqcup E_{-\xi_b}(c_1,c_2).$$

Finally, by Theorem 5.2.17 and Proposition 1.1.22, we get

$$M_{\mathcal{C}'_b}(2;c_1,c_2)=E_{-\xi_b}(c_1,c_2)\cong\mathbb{P}^d,$$

for $d=\mathrm{h}^1\,\mathcal{E}(-2b-1)-1$. □

As a consequence, we get the following Corollary.

Corollary 5.2.19. *Let $\pi:X=\mathbb{P}(\mathcal{E})\to\mathbb{P}^2$ be a ruled 3-fold, where $\mathcal{E}$ is a rank 2 vector bundle on $\mathbb{P}^2$ such that $\mathrm{h}^1\,\mathcal{E}(-2b-1)\neq 0$. Then, the moduli space $M_{\mathcal{C}'_b}(2;c_1,c_2)$ is a rational variety with Picard group $\mathrm{Pic}(M_{\mathcal{C}'_b}(2;c_1,c_2))\cong\mathbb{Z}$.*

Proof. It follows from Theorem 5.2.18. □

Remark 5.2.20. In Example 5.2.12 *i*), we have seen a vector bundle $\mathcal{E}$ such that $\mathrm{h}^1\,\mathcal{E}(-2b-1)=0$. On the other hand, in Example 5.2.12 *ii*) we have constructed a vector bundle $\mathcal{E}$ with $\mathrm{h}^1\,\mathcal{E}(-2b-1)\neq 0$.

Now, let us study the case $c_2 = (b+2)SL - b(b+1)L^2$.

Theorem 5.2.21. *Let $X = \mathbb{P}(\mathcal{E}) \to \mathbb{P}^2$ be a ruled 3-fold. Let us fix an integer $b > c_1' + 1$. Consider $\xi_b = -S + (2b+1)L$ and Chern classes $c_1 = S + L$ and $c_2 = (b+2)SL - b(b+1)L^2$. Then,*

$$M_{\mathcal{C}_b}(2; c_1, c_2) = M_{\mathcal{C}_b'}(2; c_1, c_2) \sqcup E_{\xi_b}(c_1, c_2).$$

In particular, $M_{\mathcal{C}_b}(2; c_1, c_2)$ is non-empty and has a component of dimension $2b + 1 + \mathrm{h}^0\,\mathcal{E} - \mathrm{h}^0\,\mathcal{E}(-1)$.

Proof. The proof is analogous to the one of Theorem 5.2.7. On one hand, by Lemma 5.2.13, $\xi_b = -S + (2b+1)L$ defines a non-empty wall of type W^{ξ_b} and satisfies the C-condition. Therefore, we define $E_{\xi_b}(c_1, c_2)$ as the family of rank 2 vector bundles given by a non-trivial extension of type

$$0 \to \mathcal{O}_X(bL) \to E \to I_Z(S - bL) \to 0, \tag{5.21}$$

where $[Z] = \frac{\xi_b^2 - c_1^2}{4} + c_2 = SL$.

Claim: $E_{\xi_b}(c_1, c_2)$ is non-empty of dimension $2b + 1 + \mathrm{h}^0\,\mathcal{E} - \mathrm{h}^0\,\mathcal{E}(-1)$.
Proof: Notice that

$$\begin{aligned} \mathrm{ext}^1(I_Z(S - bH), \mathcal{O}_X((b+1)L)) &= \mathrm{ext}^2(\mathcal{O}_X((b+1)L), I_Z(S - bL + K_X)) \\ &= \mathrm{h}^2\, I_Z(S - (2b+1)L + K_X). \end{aligned}$$

If we consider the resolution of the ideal sheaf I_Z given in (1.10) twisted by $\mathcal{O}_X(S - (2b+1)L + K_X)$, taking cohomology, we get

$$\begin{aligned} \cdots \quad &\to \mathrm{H}^2\,\mathcal{O}_X(-(2b+1)L + K_X) \oplus \mathrm{H}^2\,\mathcal{O}_X(S - 2(b+1)L + K_X) \\ &\to \mathrm{H}^2\, I_Z(S - (2b+1)L + K_X) \to \mathrm{H}^3\,\mathcal{O}_X(-2(b+1)L + K_X) \\ &\to \mathrm{H}^3\,\mathcal{O}_X(-(2b+1)L + K_X) \oplus \mathrm{H}^3\,\mathcal{O}_X(S - 2(b+1)L + K_X) \\ &\to \mathrm{H}^3\, I_Z(S - (2b+1)L + K_X) \to 0. \end{aligned}$$

Since

$$\begin{aligned} \mathrm{H}^2\,\mathcal{O}_X(-(2b+1)L + K_X) &= \mathrm{H}^2\,\mathcal{O}_X(S - 2(b+1)L + K_X) \\ &= \mathrm{H}^3\,\mathcal{O}_X(S - 2(b+1)L + K_X) = 0, \end{aligned}$$

we get

$$\mathrm{h}^2\, I_Z(S - (2b+1)L + K_X) = \mathrm{h}^3\,\mathcal{O}_X(-2(b+1)L + K_X) - \mathrm{h}^3\,\mathcal{O}_X(-(2b+1)L + K_X),$$

and

$$\begin{aligned} \mathrm{h}^3\,\mathcal{O}_X(-2(b+1)L + K_X) - \mathrm{h}^3\,\mathcal{O}_X(-(2b+1)L + K_X) = {} & \mathrm{h}^0\,\mathcal{O}_{\mathbb{P}^2}(2(b+1)) \\ & - \mathrm{h}^0\,\mathcal{O}_{\mathbb{P}^2}(2b+1) \\ = {} & 2b + 3 > 0. \end{aligned}$$

Hence $\mathrm{h}^2\, I_Z(S-(2b+1)L+K_X) = 2b+3$ and, putting altogether, $E_{\xi_b} \neq \emptyset$. Furthermore, by (5.21), we have

$$\begin{aligned} \dim(E_{\xi_b}) &= \mathrm{ext}^1(I_Z(S-bL), \mathcal{O}_X((b+1)L)) + \dim(\mathcal{G}) - 1 \\ &= 2b+3+\mathrm{h}^0(\mathcal{E}) - \mathrm{h}^0\,\mathcal{E}(-1), \end{aligned}$$

where the last equality follows from Lemma 1.3.8 *ii*).

Since, according to Lemma 5.2.13 *ii*), the wall W^{ξ_b} is only defined by the classes ξ_b and $-\xi_b$ and $\mathrm{H}^0(\bigwedge^2 \mathcal{N}_{Z/X} \otimes \mathcal{O}_X(-\xi_b)) = 0$, by Theorem 4.3.2 *ii*), $E_{\xi_b}(c_1, c_2) \neq \emptyset$ implies

$$M_{\mathcal{C}_b}(2; c_1, c_2) = M_{\mathcal{C}'_b}(2; c_1, c_2) \sqcup E_{\xi_b}(c_1, c_2).$$

In particular, the family $E_{\xi_b}(c_1, c_2)$ is a component of $M_{\mathcal{C}_b}(2; c_1, c_2)$ of dimension $2b+3+\mathrm{h}^0(\mathcal{E}) - \mathrm{h}^0\,\mathcal{E}(-1)$ and $M_{\mathcal{C}_b}(2; c_1, c_2)$ is non-empty. □

Now, let us move to the case $c_1 = L$.

5.2.3 Case $\mathbf{c_1 = L}$

As we have done in the previous cases, we will first study the wall and chamber structure defined by $\xi_b = -2S + (2b+1)L$. We fix the Chern classes $c_1 = L$ and $c_2 = -S^2 + (2b+m)SL - b(b+1)L^2$ with $m = 1, 2$.

Lemma 5.2.22. *Let $X = \mathbb{P}(\mathcal{E}) \to \mathbb{P}^2$ be a ruled 3-fold over $\mathbb{P}^2$. Let us fix an integer $b > 2c'_1 + 1$. Consider $\xi_b = -2S + (2b+1)L$ and Chern classes $c_1 = L$ and $c_2 = -S^2 + (2b+m)SL - b(b+1)L^2$ with $m = 1, 2$.*

i) ξ_b defines a non-empty wall W^{ξ_b} of type (c_1, c_2). Moreover, if $m = 2$, ξ_b satisfies the C-condition.

ii) The wall W^{ξ_b} is only defined by the classes ξ_b and $-\xi_b$. Moreover, if $m = 2$, we get $\mathrm{H}^0(\bigwedge^2 \mathcal{N}_{Z/X} \otimes \mathcal{O}_X(-\xi_b)_{|Z}) = 0$.

Proof. We follow the same steps as in the proof of Lemma 5.2.3.

i) First, let us prove that ξ_b defines a wall W^{ξ_b} of type (c_1, c_2). In fact, $\xi_b + L = -2S + 2(b+1)L$ is clearly divisible by 2 in $\mathrm{Pic}(X)$ and

$$[Z] = \frac{{\xi_b}^2 - L^2}{4} - S^2 + (2b+1)SL - 4b(b+1)L^2 = \begin{cases} 0 & \text{if} \quad m = 1, \\ SL & \text{if} \quad m = 2 \end{cases}$$

is a complete intersection. Moreover, this wall is non-empty since $H = \alpha S + \beta L$, with

$$\beta \;= \tfrac{\alpha}{2} \cdot \left(2b+1+c'_1 \pm \sqrt{(2b+1)(2b+1-2c'_1)+4c'_2}\right) \in \mathbb{R}^+,$$

satisfies $\xi_b \cdot H^2 = 0$, and thus $H \in W^{\xi_b}$. In addition, let us prove that, for $m = 2$, ξ_b satisfies the C-condition. First of all,

$$\begin{aligned} \mathrm{H}^2\,\mathcal{O}_X(\xi_b) &= \mathrm{H}^2\,\mathcal{O}_X(-2S + (2b+1)L) &&\cong \mathrm{H}^1\,\mathcal{O}_X(-(2b+4-c_1')L)^* \\ & &&\cong \mathrm{H}^1\,\mathcal{O}_{\mathbb{P}^2}(-2b-4+c_1')^* = 0. \end{aligned}$$

On the other hand, since $[Z] = SL$ and $b > 2c_1' + 1$, if we apply Lemma 1.3.8, we get

$$\mathrm{H}^0(\bigwedge^2 \mathcal{N}_{Z/X} \otimes \mathcal{O}_X(\xi_b)_{|Z}) \cong \mathrm{H}^0\,\mathcal{O}_X(-S + 2(b+1)L)_{|Z} \neq 0,$$

and thus, for $m = 2$, ξ_b satisfies the C-condition.

ii) By Lemma 5.1.4, the wall W^{ξ_b} can only be defined by a class of the form $t\xi_b = -2tS + (2b+1)tL \in \mathrm{Num}(X)$ with $t \neq 0$. If $\eta = t\xi_b$ defines a non-empty wall of type (c_1, c_2), then

$$[Z] = \frac{t^2(\xi_b)^2 - L^2}{4} + c_2$$

is a codimension 2 locally complete intersection. Then, in particular, $[Z]\cdot H_0 \geq 0$ for $H_0 = S + \beta H \in \mathcal{C}_X$ with $\beta \gg 0$, which implies

$$(t^2 - 1)(c_1' - 2b - 1) + m - 1 \geq 0.$$

Thus, since $b > 2c_1' + 1$ and $t \neq 0$, we must have $|t| = 1$. Finally, since $b > 2c_1' + 1$, we get $3c_1' - 2b < 0$ and hence

$$\mathrm{H}^0(\bigwedge^2 \mathcal{N}_{Z/X} \otimes \mathcal{O}_X(-\xi_b)_{|Z}) \cong \mathrm{H}^0\,\mathcal{O}_X(3S - 2bL)_{|Z} = 0,$$

which completes the proof. $\square$

Now, let us study the adjacent chambers of the wall W^{ξ_b}.

Proposition 5.2.23. *Let us fix an integer $b > 2c_1'+1$ and $\xi_b = -2S+(2b+1)L$. Let us consider the Chern classes $c_1 = L$ and $c_2 = -S^2+(2b+m)SL-b(b+1)L^2$ for $m = 1, 2$. Then,*

i) There is no wall of type (c_1, c_2) above W^{ξ_b} and under the set

$$W^{\xi_{b+1}} = \{H \in (\mathrm{Num}(X) \otimes \mathbb{R}) \cap \mathcal{C}_X \mid \xi_{b+1} \cdot H^2 = 0\}.$$

ii) There is no wall of type (c_1, c_2) under W^{ξ_b} and above the set

$$W^{\xi_{b-1}} = \{H \in (\mathrm{Num}(X) \otimes \mathbb{R}) \cap \mathcal{C}_X \mid \xi_{b-1} \cdot H^2 = 0\}.$$

Proof. The proof is analogous to the one of Proposition 5.2.4.

i) Again, if $H \in W^{\xi_{b+1}}$, we have $\xi_b \cdot H^2 = (\xi_{b+1} - 2L) \cdot H^2 = -2L \cdot H^2 < 0$, which implies that $W^{\xi_{b+1}}$ is above the wall W^{ξ_b}.

Let us prove that there is no wall of type (c_1, c_2) above W^{ξ_b} and under $W^{\xi_{b+1}}$. To this end, we will study the possible cases of walls W^η of type (c_1, c_2). Notice that we can avoid the case when η or $-\eta$ is effective, since it would not define a wall of type (c_1, c_2).

Case 1: Let us first study the case $\eta_p = -2S + (2p+1)L$ with $p > b+1$. If $H \in W^{\eta_p}$, then

$$\xi_{b+1} \cdot H^2 = (\eta_p + 2(b+1-p)L) \cdot H^2 = 2(b+1-p)L \cdot H^2 < 0,$$

where the inequality follows from the fact that H is effective and $b+1-p < 0$. Therefore, W^{η_p} lies above the wall $W^{\xi_{b+1}}$.

Case 2: Now, let us study the case $\eta_p = -2S + (2p+1)L$ with $0 < p < b$. If $H \in W^{\eta_p}$, then

$$\xi_b \cdot H^2 = (\eta_p + 2(b-p)L) \cdot H^2 = 2(b-p)L \cdot H^2 > 0,$$

where the inequality follows from the fact that L is effective and $p < b$. Therefore the wall W^η is under the wall W^{ξ_b}.

Case 3: Now, let us study the family $\eta_p = -2pS + (2b+1)L$ with $p \geq 1$. If $H \in W^{\eta_p}$, then

$$\xi_b \cdot H^2 = (\eta_p + 2pS) \cdot H^2 = 2pS \cdot H^2 > 0,$$

where the inequality follows from the fact that S is effective and $p \geq 1$. Therefore the wall W^{η_p} is under the wall W^{ξ_b}.

Case 4: Now, let us study the family $\eta_{(a,p)} = -2aS + (2p+1)L$ with $a \geq 1$ and $0 < p < b$. Then, by Cases 2 and 3, the wall $\eta_{(a,p)}$ is under the wall W^{ξ_b}.

Case 5: Now, let us study the family of classes $\eta_{(a,p)} = -2aS + (2p+1)L$ with $a \geq 1$ and $p > b$. Let H be such that $\eta_{(a,p)} \cdot H^2 = 0$ and assume that H is above W^{ξ_b} and under $W^{\xi_{b+1}}$, that is,

$$\xi_b \cdot H^2 < 0 < \xi_{b+1} \cdot H^2.$$

On one hand, since $\xi_b \cdot H^2 < 0 = \eta_{(a,p)} \cdot H^2$, we get

$$S \cdot H^2 < \frac{p-b+1}{a-1} L \cdot H^2. \tag{5.22}$$

On the other hand, since $\xi_{b+1} \cdot H^2 > 0 = \eta_{(a,p)} \cdot H^2$, we get

$$S \cdot H^2 > \frac{p-b}{a-1} L \cdot H^2. \tag{5.23}$$

Finally, since $\eta_{(a,p)} \cdot H^2 = 0$ is equivalent to $S \cdot H^2 = \frac{2p+1}{2a} L \cdot H^2$, by (5.22) and (5.23) we get

$$\frac{p-b}{a-1} < \frac{2p+1}{2a} < \frac{p-b+1}{a-1},$$

which in particular implies

$$p > \frac{a(2b-1)-1}{2}.$$

In addition, if $\eta_{(a,p)}$ defines a wall of type (c_1, c_2), then

$$[Z] = \frac{\eta_{(a,p)}^2 - L^2}{4} + c_2$$

is a locally complete intersection and thus, in particular, $[Z] \cdot H_0 \geq 0$ for any ample divisor $H_0 = S + \beta H$ with $\beta \gg 0$, which implies

$$a(ac_1' - 1) - 2ap - c_1' + 2b + m \geq 0.$$

Finally, since $p > \frac{a(2b-1)-1}{2}$ with $a \geq 1$ and $c_1' - 2b + 1 \leq -1$, we get

$$(a^2 - 1)(c_1' - 2b + 1) + m + 1 \geq 0,$$

a contradiction.

ii) The proof follows step by step the proof of *i*). $\square$

Remark 5.2.24. By Proposition 5.2.23, it is clear that the structure of the adjacent chambers to W^{ξ_b} is the same for both values of c_2 and, thus, as in the previous cases, we will denote them by $\mathcal{C}_b$ and $\mathcal{C}_b'$ in both cases.

Now, let us prove the existence of polarizations in the chambers with common wall W^{ξ_b}, which by Proposition 5.2.23 holds if we find polarizations above $W^{\xi_{b-1}}$ and under W^{ξ_b}, and above W^{ξ_b} and under $W^{\xi_{b+1}}$.

Proposition 5.2.25. *Let $X = \mathbb{P}(\mathcal{E}) \to \mathbb{P}^2$ be a ruled 3-fold over $\mathbb{P}^2$ with $c_i' = c_i(\mathcal{E})$. Let $b > 2c_1' + 1$ be an integer and fix the Chern classes $c_1 = L$ and $c_2 = -S^2 + (2b+m)SL - b(b+1)L^2$ with $m = 1, 2$. Then,* $\mathrm{Num}(X) \cap \mathcal{C}_b \neq \emptyset$ *and* $\mathrm{Num}(X) \cap \mathcal{C}_b' \neq \emptyset$.

Proof. The proof follows the same argument as the one of Proposition 5.2.16.

First of all, let us prove that $\mathrm{Num}(X) \cap \mathcal{C}'_b \neq \emptyset$. Notice that, according to Proposition 5.2.23, it is enough to determine a polarization H verifying $\xi_{b-1} \cdot H^2 < 0 < \xi_b \cdot H^2$. Consider $L = S + \beta H \in \mathrm{Num}(X) \cap \mathcal{C}_X$ and define

$$\begin{aligned} f(\beta) := \quad & \xi_b \cdot H^2 && = -2\beta^2 + 2(2b+1-c'_1)\beta + (2b+1)c'_1 - 2[(c'_1)^2 - c'_2] \\ g(\beta) := \quad & \xi_{b-1} \cdot H^2 && = -2\beta^2 + 2[2b-1-c'_1]\beta + (2b-1)c'_1 - 2[(c'_1)^2 - c'_2]. \end{aligned}$$

Notice that $g(\beta) = f(\beta) - 4\beta - 2c'_1$. If we denote by $A_1 := 2(2b+1-c'_1)$ and $A_0 := (2b+1)c'_1 - 2[(c'_1)^2 - c'_2]$, then $f(\beta) = -2\beta^2 + A_1\beta + A_0$ and $g(\beta) = f(\beta) - 4\beta - 2c'_1$ are two parabolas in the variable $\beta \in \mathbb{R}^+$. Let us prove that there exists an integer $\beta_0 > 0$ such that $f(\beta_0) > 0 > g(\beta_0)$. Notice that g has the vertex at the point

$$P = (\frac{A_1 - 4}{4}, \frac{(A_1-4)^2}{8} + A_0 - 2c'_1).$$

Since $b > c'_1 + 1$, P belongs to $\mathbb{R}^+ \times \mathbb{R}^+$, which implies that there exists a positive y_0 such that $g(y_0) = 0$. Then, since g is continuous an decreases in $[y_0, +\infty)$, we have $g(y_0+1) < 0$. Now, let us check that $f(y_0+1) > 0$. Notice that

$$f(y_0+1) = g(y_0+1) + 4(y_0+1) + 2c'_1 = g(y_0) + 4b = 4b > 0.$$

Since f is continuous and, for any β, $f(\beta) > g(\beta)$, we have $f(\beta) > 0$ for $\beta \in [y_0, y_0+1]$. Putting altogether $f(\beta) > 0 > g(\beta)$ for any $\beta \in (y_0, y_0+1]$. If y_0 is an integer, take $\beta_0 = y_0 + 1$; otherwise, take $\beta_0 = \lceil y_0 \rceil \in (y_0, y_0+1)$. Hence we have proved that $H = S + \beta_0 L$ is a polarization in $\mathcal{C}'_b$.

Following the same argument, considering $h(\beta) := \xi_{b+1} \cdot H^2$ for $H = S + \beta L$, we prove that there exists a polarization in $\mathcal{C}_b$. In fact,

$$h(\beta) = f(\beta) + 4\beta + 2c'_1$$

and, since $f(\beta)$ has a vertex on

$$(\frac{A_1}{4}, \frac{(A_1)^2}{8} + A_0) \in \mathbb{R}^+ \times \mathbb{R}^+,$$

there exists $y_1 \in \mathbb{R}^+$ such that $f(y_1) = 0$ and therefore

$$\begin{aligned} f(\beta_1 + 1) \quad & < 0 \\ h(\beta_1 + 1) \quad & = f(\beta_1+1) + 4(\beta_1+1) + 2c'_1 > 0. \end{aligned}$$

Hence, there exists an integer $\beta_1 \in (y_1, y_1+1]$ such that $h(\beta_1) > 0 > f(\beta_1)$. □

As in the previous cases, under the condition (5.1), Proposition 5.2.25 can also be deduced from the geometric interpretation given in Section 5.1, ending at the same picture.

Now, let us study the moduli spaces $M_{\mathcal{C}_b}(2; c_1, c_2)$ and $M_{\mathcal{C}'_b}(2; c_1, c_2)$. First, we fix $c_2 = -S^2 + (2b+1)SL - b(b+1)L^2$ for some integer $b > 2c'_1 + 1$.

Theorem 5.2.26. *Let $X = \mathbb{P}(\mathcal{E}) \to \mathbb{P}^2$ be a ruled 3-fold. Let us fix an integer $b > 2c'_1 + 1$. Consider $\xi_b = -2S + (2b+1)L$ and Chern classes $c_1 = L$ and $c_2 = -S^2 + (2b+1)SL - b(b+1)L^2$. Then,*

i) If $\mathrm{h}^1(S^2\mathcal{E})(-2b-1) \neq 0$, then

$$M_{\mathcal{C}'_b}(2; c_1, c_2) \sqcup E_{\xi_b}(c_1, c_2) = M_{\mathcal{C}_b}(2; c_1, c_2) \sqcup E_{-\xi_b}(c_1, c_2)$$

and $M_{\mathcal{C}'_b}(2; c_1, c_2)$ is non-empty and has a smooth irreducible component of dimension $\mathrm{h}^1(S^2\mathcal{E})(-2b-1) - 1$.

ii) If $\mathrm{h}^1(S^2\mathcal{E})(-2b-1) = 0$, then

$$M_{\mathcal{C}_b}(2; c_1, c_2) = M_{\mathcal{C}'_b}(2; c_1, c_2) \sqcup E_{\xi_b}(c_1, c_2).$$

In particular, $M_{\mathcal{C}_b}(2; c_1, c_2)$ is non-empty and has a smooth irreducible component of dimension $\binom{2b+3-c'_1}{2} - 1$.

Proof. One one hand, by Lemma 5.2.22, $\xi_b = -2S + (2b+1)L$ defines a non-empty wall W^{ξ_b} of type (c_1, c_2) and, in addition, $\xi_b^2 = c_1^2 - c_2$. Therefore, $E_{\xi_b}(c_1, c_2)$ is the family of rank 2 vector bundles given by a non-trivial extension of type

$$0 \to \mathcal{O}_X(-S + (b+1)L) \to E \to \mathcal{O}_X(S - bL) \to 0.$$

Notice that

$$\begin{aligned} \mathrm{ext}^1(\mathcal{O}_X(S - bL), \mathcal{O}_X(-S + (b+1)L)) &= \mathrm{h}^1\, \mathcal{O}_X(-2S + (2b+1)L) \\ &= \mathrm{h}^2\, \mathcal{O}_{\mathbb{P}^2}(-(2b+4-c'_1)) \\ &= \mathrm{h}^0\, \mathcal{O}_{\mathbb{P}^2}(2b+1-c'_1) \\ &= \tbinom{2b+3-c'_1}{2} > 0. \end{aligned}$$

Hence, since $\xi_b^2 = c_1^2 - 4c_2$, by Proposition 1.1.22,

$$E_{\xi_b}(c_1, c_2) \cong \mathbb{P}^d \quad \text{with} \quad d = \binom{2b+3-c'_1}{2} - 1.$$

On the other hand, $E_{-\xi_b}(c_1, c_2)$ is the family of rank 2 vector bundles given by a non-trivial extension of type

$$0 \to \mathcal{O}_X(S - bL) \to E \to \mathcal{O}_X(-S + (b+1)L) \to 0.$$

Notice that

$$\begin{aligned}\mathrm{ext}^1(\mathcal{O}_X(-S+(b+1)L),\mathcal{O}_X(S-bL)) &= \mathrm{h}^1\,\mathcal{O}_X(2S-(2b+1)L)\\ &= \mathrm{h}^1(S^2\mathcal{E})(-2b-1).\end{aligned}$$

If $\mathrm{h}^1(S^2\mathcal{E})(-2b-1)\neq 0$, again by Proposition 1.1.22, since $(-\xi_b)^2=c_1^2-4c_2$,

$$E_{-\xi_b}(c_1,c_2)\cong\mathbb{P}^d \quad \text{with} \quad d=\mathrm{h}^1(S^2\mathcal{E})(-2b-1)-1.$$

Hence, since the wall W^{ξ_b} is only defined by ξ_b and $-\xi_b$ (see Lemma 5.2.22 *ii*)) and both E_{ξ_b} and $E_{-\xi_b}$ are non-empty, by Theorem 4.3.2 *iv*) we get

$$M_{\mathcal{C}'_b}(2;c_1,c_2)\sqcup E_{\xi_b}(c_1,c_2)=M_{\mathcal{C}_b}(2;c_1,c_2)\sqcup E_{-\xi_b}(c_1,c_2).$$

In particular, this implies that $E_{-\xi_b}$ is a smooth irreducible component of $M_{\mathcal{C}'_b}$ of dimension $\mathrm{h}^1(S^2\mathcal{E})(-2b-1)-1$ and $M_{\mathcal{C}'_b}$ is non-empty.

Otherwise, if $\mathrm{h}^1(S^2\mathcal{E})(-2b-1)=0$, since the wall W^{ξ_b} is only defined by ξ_b and $-\xi_b$ and E_{ξ_b} is non-empty, by Theorem 4.3.2 *iii*), we get

$$M_{\mathcal{C}_b}(2;c_1,c_2)=M_{\mathcal{C}'_b}(2;c_1,c_2)\sqcup E_{\xi_b}(c_1,c_2).$$

Finally, in both cases, E_{ξ_b} is a smooth irreducible component of the moduli space $M_{\mathcal{C}_b}$ of dimension $\binom{2b+3-c'_1}{2}-1$ and $M_{\mathcal{C}_b}$ is non-empty. □

Example 5.2.27.

i) To obtain an example of a rank 2 vector bundle $\mathcal{E}$ satisfying the condition $\mathrm{H}^1(S^2\mathcal{E})(-2b-1)=0$, consider the bundle $\mathcal{E}=\mathcal{O}_{\mathbb{P}^2}(d)\oplus\mathcal{O}_{\mathbb{P}^2}(d')$ given in Example 5.2.12. Taking the second symmetric power we have

$$S^2\mathcal{E}\cong\mathcal{O}_{\mathbb{P}^2}(2d)\oplus\mathcal{O}_{\mathbb{P}^2}(2d')\oplus\mathcal{O}_{\mathbb{P}^2}(d+d').$$

Therefore, $\mathrm{H}^1(S^2\mathcal{E})(-2b-1)=0$.

ii) To have an example of a vector bundle $\mathcal{E}$ with $\mathrm{H}^1(S^2\mathcal{E})(-2b-1)\neq 0$, consider the rank two vector bundle $\mathcal{E}$ given in Example 5.2.12 *ii*) which is given by the exact sequence

$$0\to\mathcal{O}_{\mathbb{P}^2}(-c'_1)\to\mathcal{O}_{\mathbb{P}^2}^{\oplus 3}\to\mathcal{E}\to 0.$$

Taking the wedge product of this sequence, since $\bigwedge^2\mathcal{O}_{\mathbb{P}^2}(-c'_1)=0$, we get the short exact sequence

$$0\to\mathcal{O}_{\mathbb{P}^2}^{\oplus 3}\to\mathcal{E}^{\oplus 3}\to S^2\mathcal{E}\to 0. \tag{5.24}$$

Twisting the exact sequence (5.24) by $\mathcal{O}_{\mathbb{P}^2}(-2b-1)$ and taking cohomology, we get

$$0 \to \mathrm{H}^1\, \mathcal{O}_{\mathbb{P}^2}(-2b-1)^{\oplus 3} \to \mathrm{H}^1\, \mathcal{E}(-2b-1)^{\oplus 3} \to \mathrm{H}^1(S^2\mathcal{E})(-2b-1) \to \cdots .$$

Since $\mathrm{H}^1\, \mathcal{O}_{\mathbb{P}^2}(-2b-1)^{\oplus 3} = 0$, from this exact sequence and the fact that $\mathrm{H}^1\, \mathcal{E}(-2b-1)^{\oplus 3} \neq 0$, we deduce that $\mathrm{H}^1(S^2\mathcal{E})(-2b-1) \neq 0$.

Now, let us study the case $c_2 = -S^2 + 2(b+1)SL - b(b+1)L^2$, where $b > 2c_1' + 1$ is an integer.

Theorem 5.2.28. *Let $X = \mathbb{P}(\mathcal{E}) \to \mathbb{P}^2$ be a ruled 3-fold. Let us fix an integer $b > 2c_1' + 1$. Consider $\xi_b = -2S + (2b+1)L$ and Chern classes $c_1 = L$ and $c_2 = -S^2 + 2(b+1)SL - b(b+1)L^2$. Then,*

$$M_{\mathcal{C}_b}(2; c_1, c_2) = M_{\mathcal{C}_b'}(2; c_1, c_2) \sqcup E_{\xi_b}(c_1, c_2).$$

In particular, $M_{\mathcal{C}_b}(2; c_1, c_2)$ is non-empty and has a component of dimension

$$\binom{2b+3-c_1'}{2} + \mathrm{h}^0\, \mathcal{E} - \mathrm{h}^0\, \mathcal{E}(-1) - 2.$$

Proof. The proof is analogous to the one of Theorem 5.2.7. Notice that, by Lemma 5.2.22 *ii)*, $\xi_b = -2S + (2b+1)L$ defines a non-empty wall W^{ξ_b} of type (c_1, c_2) and satisfies the C-condition. Therefore, E_{ξ_b} is the family of rank 2 vector bundles given by a non-trivial extension of type

$$0 \to \mathcal{O}_X(-S + (b+1)L) \to E \to I_Z(S - bL) \to 0, \tag{5.25}$$

where $[Z] = \frac{\xi_b^2 - c_1^2}{4} + c_2 = SL$.

Claim: E_{ξ_b} is non-empty of dimension $\binom{2b+4-c_1'}{2} + \mathrm{h}^0\, \mathcal{E} - \mathrm{h}^0\, \mathcal{E}(-1) - 2$.
Proof: Notice that

$$\mathrm{ext}^1(I_Z(S - bL), \mathcal{O}_X(-S + (b+1)L)) = \mathrm{h}^2\, I_Z(-(2b+4-c_1')L).$$

Denote by $p = 2b + 4 - c_1'$. From the resolution of I_Z (see (1.10)), we obtain the long exact sequence

$$\begin{aligned} \cdots\ &\to \mathrm{H}^2\, \mathcal{O}_X(-S-(p+1)L) \to \mathrm{H}^2\, \mathcal{O}_X(-S-pL) \oplus \mathrm{H}^2\, \mathcal{O}_X(-(p+1)L) \\ &\to \mathrm{H}^2\, I_Z(-pL) \to \mathrm{H}^3\, \mathcal{O}_X(-S-(p+1)L) \to \cdots . \end{aligned}$$

Since

$$\mathrm{h}^2\, \mathcal{O}_X(-S-(p+1)L) = \mathrm{h}^2\, \mathcal{O}_X(-S-pL) = \mathrm{h}^3\, \mathcal{O}_X(-S-(p+1)L) = 0,$$

we get

$$\begin{aligned}\mathrm{h}^2\, I_Z(-pL) \quad &= \mathrm{h}^2\, \mathcal{O}_X(-(p+1)L) \quad = \mathrm{h}^0\, \mathcal{O}_{\mathbb{P}^2}(2b+2-c_1')\\ &\qquad\qquad\qquad\qquad\quad = \tbinom{2b+4-c_1'}{2} > 0.\end{aligned}$$

Putting altogether, E_{ξ_b} is non-empty and, by (5.25) and Lemma 1.3.8 ii), it is of dimension

$$\begin{aligned}\dim(E_{\xi_b}) \quad &= \mathrm{ext}^1(I_Z(S-bL), \mathcal{O}_X(-S+(b+1)L)) + \dim(\mathcal{G}) - 1\\ &= \tbinom{2b+4-c_1'}{2} + \mathrm{h}^0\, \mathcal{E} - \mathrm{h}^0\, \mathcal{E}(-1).\end{aligned}$$

Therefore, since by Lemma 5.2.22 *i)* the wall W^{ξ_b} is only defined by $\pm\xi_b$ and we have $\mathrm{H}^0(\mathcal{N}_{Z/X} \otimes \mathcal{O}_X(\xi_b)_{|Z}) = 0$, it follows from Theorem 4.3.2 *ii)* that

$$M_{\mathcal{C}_b}(2; c_1, c_2) = M_{\mathcal{C}_b'}(2; c_1, c_2) \sqcup E_{\xi_b}(c_1, c_2).$$

In particular, E_{ξ_b} is a component of the moduli space $M_{\mathcal{C}_b}$ of dimension

$$\binom{2b+4-c_1'}{2} + \mathrm{h}^0\, \mathcal{E} - \mathrm{h}^0\, \mathcal{E}(-1)$$

and $M_{\mathcal{C}_b}$ is non-empty. □

Finally, let us study the case $c_1 = 0$.

5.2.4 Case $\mathbf{c_1 = 0}$

As before, we first describe the wall defined by $\xi_b = -2S - 2bL$ and its adjacent chambers.

Lemma 5.2.29. *Let $X = \mathbb{P}(\mathcal{E}) \to \mathbb{P}^2$ be a ruled 3-fold over $\mathbb{P}^2$. Let us fix an integer $b > 2c_1' + 1$. Consider $\xi_b = -2S + 2bL$ and Chern classes $c_1 = 0$ and $c_2 = -S^2 + (2b+m)SL - b^2L^2$ with $m = 0, 1$. Then,*

i) ξ_b defines a non-empty wall W^{ξ_b} of type (c_1, c_2). Moreover, if $m = 1$, then ξ_b satisfies the C-condition.

ii) The wall W^{ξ_b} is only defined by ξ_b and $-\xi_b$. Furthermore, if $m = 1$, then $\mathrm{H}^0(\mathcal{N}_{Z|X} \otimes \mathcal{O}_X(-\xi_b)_{|Z}) = 0$.

Proof. i) Let us first prove that ξ_b defines a wall W^{ξ_b} of type (c_1, c_2). In fact, $\xi_b = -2S + 2bL$ is clearly divisible by 2 in $\mathrm{Pic}(X)$ and

$$[Z] = \frac{{\xi_b}^2}{4} - S^2 + 2bSL - b^2L^2 = \begin{cases} 0 & \text{if} \quad m = 0\\ SL & \text{if} \quad m = 1,\end{cases}$$

is a codimension 2 complete intersection. Moreover, this wall is non-empty since $H = \alpha S + \beta L$, with

$$\beta \quad = \alpha \cdot \left(b - c_1' \pm \sqrt{b(b - c_1') + c_2'} \right) \in \mathbb{R}^+,$$

satisfies $\xi_b \cdot H^2 = 0$.

Moreover, if $m = 1$, ξ_b satisfies the C-condition. In fact,

$$\mathrm{H}^2\, \mathcal{O}_X(-2S + 2bL) = \mathrm{H}^1\, \mathcal{O}_{\mathbb{P}^2}(c_1' - 3 - 2b)^* = 0$$

and, in addition, if Z is such that $[Z] = \frac{\xi_b{}^2}{4} - S^2 + (2b+1)SL - b^2L^2 = SL$, by Lemma 1.3.8 and the fact that $b > 2c_1' + 1$, we get

$$\mathrm{H}^0(\bigwedge^2 \mathcal{N}_{Z/X} \otimes \mathcal{O}_X(\xi_b)_{|Z}) \cong H^0\mathcal{O}_X(-S + (2b+1)L)_{|Z} \neq 0.$$

ii) By Lemma 5.1.4, the wall W^{ξ_b} is defined by $t\xi_b = -tS + 2btL \in \mathrm{Num}(X)$ with $t \neq 0$. If $\eta = t\xi_b$ defines a non-empty wall of type (c_1, c_2), then

$$[Z] = \frac{t^2(\xi_b)^2}{4} + c_2$$

is a codimension 2 locally complete intersection and thus, $[Z] \cdot H_0 \geq 0$ for $H_0 = S + \beta H$ with $\beta \gg 0$, which implies

$$(t^2 - 1)(c_1' - 2b) + m \geq 0.$$

Hence, since $b > 2c_1' + 1$ and $t \neq 0$, we must have $|t| = 1$. Finally, if $m = 1$, since $b > c_1' + 1$, we get

$$\mathrm{H}^0(\bigwedge^2 \mathcal{N}_{Z/X} \otimes \mathcal{O}_X(-\xi_b)_{|Z}) \cong \mathrm{H}^0\, \mathcal{O}_X(3S - (2b-1)L)_{|Z} = 0,$$

which completes the proof. □

Now, let us describe the chambers separated by the wall W^{ξ_b} defined by $\xi_b = -2S + 2bL$.

Proposition 5.2.30. *Let us fix an integer $b > 2c_1' + 1$ and Chern classes $c_1 = 0$ and $c_2 = -S^2 + (2b+m)SL - b^2L^2$, for $m = 0, 1$. Let us consider $\xi_b = -2S + (2b+1)L$. Then, we get the following.*

i) There is no wall of type (c_1, c_2) above W^{ξ_b} and under

$$W^{\xi_{b+1}} = \{H \in (\mathrm{Num}(X) \otimes \mathbb{R}) \cap \mathcal{C}_X |\, \xi_{b+1} \cdot H^2 = 0\}.$$

ii) There is no wall of type (c_1, c_2) under W^{ξ_b} and above

$$W^{\xi_{b-1}} = \{H \in (\mathrm{Num}(X) \otimes \mathbb{R}) \cap \mathcal{C}_X | \, \xi_{b-1} \cdot H^2 = 0\}.$$

Proof. The proof is analogous to the one of Proposition 5.2.14 and, since i) and ii) can be proved similarly, we will prove i).

Notice that, if $H \in W^{\xi_{b+1}}$, we have $\xi_b \cdot L^2 = (\xi_{b+1} - 2L) \cdot H^2 = 2L \cdot H^2 > 0$, which implies that $W^{\xi_{b+1}}$ is above the wall W^{ξ_b}.

Let us prove that there is no wall of type (c_1, c_2) above W^{ξ_b} and under $W^{\xi_{b+1}}$. To this end, we will study the possible walls W^{η} of type (c_1, c_2) we may have. Notice that we can avoid the case when η or $-\eta$ is effective, since it would not define a wall of type (c_1, c_2).

Case 1: Let us first study the case $\eta_p = -2S + 2pL$ with $p > b+1$. If $H \in W^{\eta_p}$, then

$$\xi_{b+1} \cdot H^2 = (\eta_p + 2(b+1-p)L) \cdot H^2 = 2(b+1-p)L \cdot H^2 < 0,$$

where the inequality follows from the fact that H is effective and $b+1-p < 0$. Therefore, W^{η_p} lies above the wall $W^{\xi_{b+1}}$.

Case 2: Now, let us study the case $\eta_p = -2S + 2pL$ with $0 < p < b$. If $H \in W^{\eta_p}$, then

$$\xi_b \cdot H^2 = (\eta_p + 2(b-p)L) \cdot H^2 = 2(b-p)L \cdot H^2 > 0,$$

where the inequality follows from the fact that L is effective and $p < b$. Therefore the wall W^{η} is under the wall W^{ξ_b}.

Case 3: Now, let us study the family $\eta_p = -2pS + 2bL$ with $p \geq 1$. If $H \in W^{\eta_p}$, then

$$\xi_b \cdot H^2 = (\eta_p + 2pS) \cdot H^2 = 2pS \cdot H^2 > 0,$$

where the inequality follows from the fact that S is effective and $p \geq 1$. Therefore the wall W^{η_p} is under the wall W^{ξ_b}.

Case 4: Now, let us study the family $\eta_{(a,p)} = -2aS + 2pL$ with $a \geq 1$ and $0 < p < b$. Then, by Cases 2 and 3, the wall $\eta_{(a,p)}$ is under the wall W^{ξ_b}.

Case 5: Now, let us study the family of classes $\eta_{(a,p)} = -2aS + 2pL$ with $a \geq 1$ and $p > b$. Let H be such that $\eta_{(a,p)} \cdot H^2 = 0$ and assume that H is above W^{ξ_b} and under $W^{\xi_{b+1}}$, that is,

$$\xi_b \cdot H^2 < 0 < \xi_{b+1} \cdot H^2.$$

On one hand, since $\xi_b \cdot H^2 < 0 = \eta_{(a,p)} \cdot H^2$, we get

$$S \cdot H^2 < \frac{p-b}{a-1} L \cdot H^2. \tag{5.26}$$

On the other hand, since $\xi_{b+1} \cdot H^2 > 0 = \eta_{(a,p)} \cdot H^2$, we get

$$S \cdot H^2 > \frac{p-b-1}{a-1} L \cdot H^2. \tag{5.27}$$

Finally, since $\eta_{(a,p)} \cdot H^2 = 0$ is equivalent to $S \cdot H^2 = \frac{2p+1}{2a} L \cdot H^2$, by (5.26) and (5.27) we get

$$\frac{p-b-1}{a-1} < \frac{p}{a} < \frac{p-b}{a-1},$$

which in particular implies

$$p > ab.$$

In addition, if $\eta_{(a,p)}$ defines a wall of type (c_1, c_2), then

$$[Z] = \frac{\eta_{(a,p)}^2}{4} + c_2$$

is a locally complete intersection. Thus, in particular, $[Z] \cdot H_0 \geq 0$ for the ample divisor $H_0 = S + \beta L$ with $\beta \gg 0$ and, since $p > ab$, this implies

$$(a^2 - 1)(c_1' - 2b) + m \geq 0,$$

a contradiction. □

Remark 5.2.31. From now on, as in the previous cases, we will use the same notation for the adjacent chambers, regardless of the value of c_2.

Now, let us prove that we can always find polarizations in the chambers with common wall W^{ξ_b} for $\xi_b = -2S + 2bL$. By Proposition 5.2.30, it is sufficiently to get polarizations above $W^{\xi_{b-1}}$ and under W^{ξ_b}, and above W^{ξ_b} and under $W^{\xi_{b+1}}$.

Proposition 5.2.32. *Let $X = \mathbb{P}(\mathcal{E}) \to \mathbb{P}^2$ be a ruled 3-fold over $\mathbb{P}^2$ with $c_i' = c_i(\mathcal{E})$. Let us fix an integer $b > 2c_1' + 1$. Consider $\xi_b = -2S + 2bL$ and Chern classes $c_1 = 0$ and $c_2 = -S^2 + (2b+m)SL - b^2L^2$ with $m = 0, 1$. Then,* $\mathrm{Num}(X) \cap \mathcal{C}_b \neq \emptyset$ *and* $\mathrm{Num}(X) \cap \mathcal{C}_b' \neq \emptyset$.

Proof. The proof follows the same steps as the one of Proposition 5.2.5. By Proposition 5.2.30, to prove $\mathrm{Num}(X)\cap\mathcal{C}'_b \neq \emptyset$, it is enough to find a polarization H verifying $\xi_{b-1}\cdot H^2 < 0 < \xi_b \cdot H^2$. Consider $H = S+\beta L \in \mathrm{Num}(X)\cap\mathcal{C}_X$ and define

$$\begin{aligned} f(\beta) :=\ & \xi_b \cdot H^2 && = -2\beta^2 + 4(b-c'_1)\beta + 2[bc'_1 - (c'_1)^2 + c'_2] \\ g(\beta) :=\ & \xi_{b-1}\cdot H^2 && = -2\beta^2 + 4(b-1-c'_1)\beta + 2[(b-1)c'_1 - (c'_1)^2 + c'_2]. \end{aligned}$$

Notice that $g(\beta) = f(\beta) - 4\beta - 2c'_1$. If we denote by $A_1 := 4(b-c'_1)$ and $A_0 := 2[bc'_1-(c'_1)^2+c'_2]$, then $f(\beta) = -2\beta^2+A_1\beta+A_0$ and $g(\beta) = f(\beta)-4\beta-2c'_1$ are two parabolas in the variable $\beta \in \mathbb{R}^+$. Let us prove that there exists an integer $\beta_0 > 0$ such that $f(\beta_0) > 0 > g(\beta_0)$. Notice that g has the vertex at the point

$$P = (\frac{A_1-4}{4}, \frac{(A_1-4)^2}{8} + A_0 - 2c'_1).$$

Since $b > c'_1 + 1$, P belongs to $\mathbb{R}^+ \times \mathbb{R}^+$, which implies that there exists a positive y_0 such that $g(y_0) = 0$. Then, since g is continuous an decreases in $[y_0, +\infty)$, we have $g(y_0+1) < 0$. Now, let us check that $f(y_0+1) > 0$. Notice that

$$f(y_0+1) = g(y_0+1)+4(y_0+1)+2c'_1 = g(y_0)+4b-2-2c'_1 = 2(2b-1-c'_1) > 0.$$

Since f is continuos and, for any β, $f(\beta) > g(\beta)$, we have $f(\beta) > 0$ for any $\beta \in [y_0, y_0+1]$. Putting altogether, $f(\beta) > 0 > g(\beta)$ for any $\beta \in (y_0, y_0+1]$. If y_0 is an integer, take $\beta_0 = y_0+1$; otherwise, take $\beta_0 = \lceil y_0 \rceil \in (y_0, y_0+1)$. Hence, we have proved that $H = S + \beta_0 L$ is a polarization in $\mathcal{C}'_b$.

Following the same argument, considering $h(\beta) := \xi_{b+1}\cdot H^2$ for $H = S+\beta L$, we prove that there exists a polarization in $\mathcal{C}_b$. □

Remark 5.2.33. If the ruled 3-fold is constructed under the assumption (5.1), the geometrical interpretation given in Section 5.1 also proves Proposition 5.2.32.

Now, we are ready to study the moduli spaces $M_H(2;0,c_2)$, for a polarization H in some adjacent chamber to W^{ξ_b}.

First, we take $c_2 = -S^2 + 2bSL - b^2L^2$ for some integer $b > 2c'_1 + 1$.

Theorem 5.2.34. *Let $X = \mathbb{P}(\mathcal{E}) \to \mathbb{P}^2$ be a ruled 3-fold. Let us fix an integer $b > 2c'_1 + 1$. Consider $\xi_b = -2S + 2bL$ and the Chern classes $c_1 = 0$ and $c_2 = -S^2 + 2bSL - b^2L^2$.*

i) If $\mathrm{h}^1(S^2\mathcal{E})(-2b) \neq 0$, *then*

$$M_{\mathcal{C}_b}(2; c_1, c_2) \sqcup E_{-\xi_b}(c_1, c_2) = M_{\mathcal{C}'_b}(2; c_1, c_2) \sqcup E_{\xi_b}(c_1, c_2).$$

ii) If $\mathrm{h}^1(S^2\mathcal{E})(-2b) = 0$, *then*

$$M_{\mathcal{C}_b}(2; c_1, c_2) = M_{\mathcal{C}'_b}(2; c_1, c_2) \sqcup E_{\xi_b}(c_1, c_2).$$

Proof. First of all, since $[Z] = \frac{\xi_b^2}{4} + c_2 = 0$, $E_{\xi_b}(c_1, c_2)$ is the family of rank 2 vector bundles given by a non-trivial extension of type

$$0 \to \mathcal{O}_X(-S + bL) \to E \to \mathcal{O}_X(S - bL) \to 0,$$

which is non-empty since

$$\begin{aligned} \mathrm{ext}^1(\mathcal{O}_X(S - bL), \mathcal{O}_X(-S + bL)) &= \mathrm{h}^2\,\mathcal{O}_X(2S - 2bL + K_X) \\ &= \mathrm{h}^0\,\mathcal{O}_{\mathbb{P}^2}(2b - c'_1) \\ &= \binom{2b+2-c'_1}{2} \neq 0. \end{aligned}$$

On the other hand, also due to the fact that $[Z] = 0$, $E_{-\xi_b}(c_1, c_2)$ is the family of rank 2 vector bundles given by a non-trivial extension of type

$$0 \to \mathcal{O}_X(S - bL) \to E \to \mathcal{O}_X(-S + bL) \to 0,$$

with

$$\mathrm{ext}^1(\mathcal{O}_X(-S + bL), \mathcal{O}_X(S - bL)) = \mathrm{h}^1(S^2\mathcal{E})(-2b).$$

Therefore, if $\mathrm{h}^1(S^2\mathcal{E})(-2b) \neq 0$, both E_{ξ_b} and $E_{-\xi_b}$ are non-empty and thus, by Theorem 4.3.2 *iv*), we get

$$M_{\mathcal{C}_b}(2; c_1, c_2) \sqcup E_{-\xi_b}(c_1, c_2) = M_{\mathcal{C}'_b}(2; c_1, c_2) \sqcup E_{\xi_b}(c_1, c_2).$$

Otherwise, if $\mathrm{h}^1(S^2\mathcal{E})(-2b) = 0$, by Theorem 4.3.2 *iii*), we get

$$M_{\mathcal{C}_b}(2; c_1, c_2) = M_{\mathcal{C}'_b}(2; c_1, c_2) \sqcup E_{\xi_b}(c_1, c_2),$$

which finishes the proof. □

Remark 5.2.35. Arguing as in Example 5.2.27, we can see that the bundles given in Example 5.2.12 also satisfy the conditions in Theorem 5.2.34.

Keeping the above notation, as a consequence of Theorem 5.2.34, we get the following results.

Corollary 5.2.36. *Let us fix an integer $b > 2c_1' + 1$ and consider the Chern classes $c_1 = 0$, $c_2 = -S^2 + 2bSL - b^2L^2$. Then, $M_{\mathcal{C}_b}(2; c_1, c_2)$ is non-empty and has a smooth irreducible rational component of dimension $\binom{2b+2-c_1'}{2} - 1$.*

Proof. By the proof of Theorem 5.2.34, $\emptyset \neq E_{\xi_b} \subseteq M_{\mathcal{C}_b}$ and, since any $E \in E_{\xi_b}$ is given by a non-trivial extension of type

$$0 \to \mathcal{O}_X(-S + bL) \to E \to \mathcal{O}_X(S - bL) \to 0,$$

by Proposition 1.1.22 we have

$$E_{\xi_b} \cong \mathbb{P}(\mathrm{Ext}^1(\mathcal{O}_X(S - bL), \mathcal{O}_X(-S + bL))) \cong \mathbb{P}^d,$$

with

$$d = \mathrm{ext}^1(\mathcal{O}_X(S - bL), \mathcal{O}_X(-S + (b+1)L)) - 1 = \binom{2b + 2 - c_1'}{2} - 1,$$

and we conclude. $\square$

Corollary 5.2.37. *Let $\pi : X = \mathbb{P}(\mathcal{E}) \to \mathbb{P}^2$ be a ruled 3-fold such that $\mathrm{h}^1(S^2\mathcal{E})(-2b) \neq 0$. Fix $b > \max\{\frac{\overline{e}}{2}, 2c_1' + 1\}$ an integer. Then, the moduli space $M_{\mathcal{C}_b'}(2; c_1, c_2)$ is non-empty and has a smooth irreducible rational component of dimension $\mathrm{h}^1(S^2\mathcal{E})(-2b) - 1$.*

Proof. By the proof of Proposition 5.2.34 *i)*, if $\mathrm{h}^1((S^2\mathcal{E})(-2b)) \neq 0$ then $\emptyset \neq E_{-\xi_b} \subseteq M_{\mathcal{C}_b}$ and, since any $E \in E_{-\xi_b}$ is given by a non-trivial extension of type

$$0 \to \mathcal{O}_X(S - bL) \to E \to \mathcal{O}_X(-S + bL) \to 0,$$

by Proposition 1.1.22 we get

$$E_{\xi_b} \cong \mathbb{P}(\mathrm{Ext}^1(\mathcal{O}_X(-S + bL), \mathcal{O}_X(S - bL))) \cong \mathbb{P}^d,$$

with

$$d = \mathrm{ext}^1(\mathcal{O}_X(-S + bL), \mathcal{O}_X(S - bL)) - 1 = \mathrm{h}^1(S^2\mathcal{E})(-2b) - 1,$$

and we conclude. $\square$

Finally, we will focus on the case $c_2 = -S^2 + (2b+1)SL - b^2L^2$.

Theorem 5.2.38. *Let $X = \mathbb{P}(\mathcal{E}) \to \mathbb{P}^2$ be a ruled 3-fold. Let us fix an integer $b > 2c_1' + 1$. Consider $\xi_b = -2S + 2bL$ and the Chern classes $c_1 = 0$ and $c_2 = -S^2 + (2b+1)SL - b^2L^2$. Then,*

$$M_{\mathcal{C}_b}(2; c_1, c_2) = M_{\mathcal{C}_b'}(2; c_1, c_2) \sqcup E_{\xi_b}(c_1, c_2).$$

In particular, $M_{\mathcal{C}_b}(2; c_1, c_2)$ is non-empty and has a component of dimension $\binom{2b+3-c_1'}{2} + \mathrm{h}^0(\mathcal{E}) - \mathrm{h}^0(\mathcal{E}(-1)) - 2$.

Proof. The proof is analogous to the one of Theorem 5.2.21.

i) Since Lemma 5.2.29 *i*) proves that ξ_b defines a non-empty wall W^{ξ_b} of type (c_1, c_2) and satisfies the C-condition, E_{ξ_b} is the family of rank 2 vector bundles given by a non-trivial extension of type

$$0 \to \mathcal{O}_X(-S + bL) \to E \to I_Z(S - bL) \to 0, \tag{5.28}$$

where $[Z] = SH$. Therefore, it is non-empty if

$$\mathrm{ext}^1(I_Z(S - bL), \mathcal{O}_X(-S + bL)) = \mathrm{h}^2\, I_Z(-(2b + 3 - c_1')L) \neq 0.$$

If we consider the exact sequence

$$0 \to \mathcal{O}_X(-S - L) \to \mathcal{O}_X(-S) \oplus \mathcal{O}_X(-L) \to I_Z \to 0$$

twisted by $\mathcal{O}_X(-pL) := \mathcal{O}_X(-(2b + 4 - c_1')L)$ and we take cohomology, we get

$$\begin{aligned} \cdots \quad &\to \mathrm{H}^2\, \mathcal{O}_X(-S - (p+1)L) \to \mathrm{H}^2\, \mathcal{O}_X(-S - pL) \oplus \mathrm{H}^2\, \mathcal{O}_X(-(p+1)L) \to \\ &\to \mathrm{H}^2\, I_Z(-pL) \to \mathrm{H}^3\, \mathcal{O}_X(-S - (p+1)L) \to \cdots \end{aligned}$$

and, since

$$\mathrm{h}^2\, \mathcal{O}_X(-S - (p+1)L) = \mathrm{h}^2\, \mathcal{O}_X(-S - pL) = \mathrm{h}^3\, \mathcal{O}_X(-S - (p+1)L) = 0,$$

we conclude

$$\begin{aligned} \mathrm{h}^2\, I_Z(-pL) &= \mathrm{h}^2\, I_Z\big(-(2b + 4 - c_1')L\big) \\ &= \mathrm{h}^2\, \mathcal{O}_X\big(-(2b + 5 - c_1')L\big) \\ &= \mathrm{h}^0\, \mathcal{O}_{\mathbb{P}^2}(2b + 2 - c_1') \\ &= \binom{2b + 4 - c_1'}{2} \neq 0. \end{aligned} \tag{5.29}$$

Therefore, E_{ξ_b} is non-empty and, by (5.28) and Lemma 1.3.8 *ii*), E_{ξ_b} is of dimension

$$\begin{aligned} \dim(E_{\xi_b}) \quad &= \mathrm{ext}^1(I_Z(S - bL), \mathcal{O}_X(-S + (b+1)L)) + \dim(\mathcal{G}) - 1 \\ &= \tbinom{2b+4-c_1'}{2} + \mathrm{h}^0\, \mathcal{E} - \mathrm{h}^0\, \mathcal{E}(-1). \end{aligned}$$

Hence, since $E_{\xi_b} \neq \emptyset$ and, by Lemma 5.2.29 *ii*), $\mathrm{H}^0(\bigwedge^2 \mathcal{N}_{Z/X} \otimes \mathcal{O}_X(\xi_b)) = 0$, applying Theorem 4.3.2 *iii*), we get

$$M_{\mathcal{C}_b}(2; c_1, c_2) = M_{\mathcal{C}'_b}(2; c_1, c_2) \sqcup E_{\xi_b}(c_1, c_2).$$

Finally, this implies that E_{ξ_b} is a component of the moduli space $M_{\mathcal{C}_b}(2; c_1, c_2)$ of dimension $\binom{2b+4-c'_1}{2} + \mathrm{h}^0\,\mathcal{E} - \mathrm{h}^0\,\mathcal{E}(-1)$ and $M_{\mathcal{C}_b}(2; c_1, c_2)$ is non-empty. □

5.3 Application to Brill-Noether Theory

In this section, we will apply some of the results of Section 5.2 to study the non-emptiness of the subvarieties of the moduli space called Brill-Noether locus.

As we have introduced in Section 1.2, and precisely stated in Definition 1.2.9, the Brill-Noether locus $W^k_H(2; c_1, c_2)$ is the subvariety of the moduli space $M_H(2; c_1, c_2)$, whose support is the set of H-stable vector bundles E in $M_H(2; c_1, c_2)$ such that $\mathrm{h}^0\,E \geq k$. As we have seen in Section 1.2, we can guarantee the existence of these loci for moduli spaces of stable bundles on arbitrary dimensional varieties, and thus, for ruled 3-folds.

Following Notation 4.1.13, we fix the following notation.

Notation 5.3.1. If $\mathcal{C}$ is a chamber, we denote by $W^k_{\mathcal{C}}(2; c_1, c_2)$ the Brill-Noether locus $W^k_H(2; c_1, c_2)$ for some $H \in \mathcal{C}$. If there is no confusion, we will simply denote by $W^k_{\mathcal{C}}$ the Brill-Noether locus $W^k_{\mathcal{C}}(2; c_1, c_2)$.

As a consequence of Theorems 5.2.7 and 5.2.10, we will study whether or not the Brill-Noether locus $W^k_H(2; S, c_2)$ is non-empty. We will start with the case $c_2 = (b+1)SL - b^2L^2$.

Theorem 5.3.2. *Let $X = \mathbb{P}(\mathcal{E}) \to \mathbb{P}^2$ be a ruled 3-fold over $\mathbb{P}^2$ with $c'_i = c_i(\mathcal{E})$. Let $c'_1 + 1 < b \in \mathbb{Z}$ and Chern classes $c_1 = S$ and $c_2 = (b+1)SL - b^2L^2$. Then, $W^k_{\mathcal{C}_b}(2; c_1, c_2) \neq \emptyset$ for k in range $1 \leq k \leq \binom{b+2}{2}$.*

Proof. By the proof of Theorem 5.2.7, the family $E_{\xi_b}(c_1, c_2)$ is a non-empty irreducible component of the moduli space $M_{\mathcal{C}_b}(2; c_1, c_2)$ and, since any $E \in E_\xi$ is given by a non-trivial extension of type

$$0 \to \mathcal{O}_X(bL) \to E \to I_Z(S - bL) \to 0,$$

where $[Z] = SL$, considering the associated long exact sequence of cohomology, in particular we get $\mathrm{H}^0\,\mathcal{O}_X(bL) \subseteq \mathrm{H}^0\,E$, which implies

$$\mathrm{h}^0\,E \geq \mathrm{h}^0\,\mathcal{O}_X(bL) = \mathrm{h}^0\,\mathcal{O}_{\mathbb{P}^2}(b) = \binom{b+2}{2}.$$

Therefore, by definition, $E \in W^d_{\mathcal{C}_b}$ for $d = \binom{b+2}{2}$ and, hence, for $1 \leq k \leq \binom{b+2}{2}$, $W^k_{\mathcal{C}_b} \neq \emptyset$. $\square$

Now, let us consider the case $c_2 = bSL - b^2L^2$.

Notice that, since Theorem 5.2.9 proves the emptiness of the moduli space $M_{\mathcal{C}_b}(2; c_1, c_2)$, it makes no sense to study the corresponding Brill-Noether locus $W^k_{\mathcal{C}_b}(2; c_1, c_2)$. In the same way, there is no point in studying the Brill-Noether locus $W^k_{\mathcal{C}'_b}(2; c_1, c_2)$ if $\mathrm{h}^1\,\mathcal{E}(-2b) = 0$, since in this case, by Theorem 5.2.10 *i*), we also get the emptiness of the moduli space $M_{\mathcal{C}'_b}(2; c_1, c_2)$. Hence, let us focus on discussing the non-emptiness of the Brill-Noether locus $W^k_{\mathcal{C}'_b}(2; c_1, c_2)$ whenever $\mathrm{h}^1\,\mathcal{E}(-2b) \neq 0$.

Theorem 5.3.3. *Let $X = \mathbb{P}(\mathcal{E}) \to \mathbb{P}^2$ be a ruled 3-fold over $\mathbb{P}^2$ with $c'_i = c_i(\mathcal{E})$. Let $c'_1 + 1 < b \in \mathbb{Z}$ and Chern classes $c_1 = S$ and $c_2 = bSL - b^2L^2$. Assume that $\mathrm{h}^1\,\mathcal{E}(-2b) \neq 0$. Then, $W^k_{\mathcal{C}'_b}(2; c_1, c_2) = \emptyset$ for any $k > \mathrm{h}^0\,\mathcal{E}(-b) + \binom{b+2}{2}$. In addition, if $\mathrm{h}^0\,\mathcal{E}(-b) \neq 0$, then*

$$M_{\mathcal{C}_b}(2; c_1, c_2) = W^1_{\mathcal{C}'_b}(2; c_1, c_2) = W^2_{\mathcal{C}'_b}(2; c_1, c_2) = \cdots = W^k_{\mathcal{C}'_b}(2; c_1, c_2) \neq \emptyset$$

for $k = \mathrm{h}^0\,\mathcal{E}(-b)$.

Proof. If $\mathrm{h}^1\,\mathcal{E}(-2b) \neq 0$, by Theorem 5.2.10, $M_{\mathcal{C}'_b}(2; c_1, c_2) = E_{-\xi_b}(c_1, c_2)$ and thus, any $E \in M_{\mathcal{C}'_b}(2; c_1, c_2)$ is given by a non-trivial extensions of type

$$0 \to \mathcal{O}_X(S - bL) \to E \to \mathcal{O}_X(bL) \to 0.$$

Therefore, for any $E \in M_{\mathcal{C}'_b}(2; c_1, c_2)$, we have

$$\mathrm{h}^0\,\mathcal{E}(-b) \leq \mathrm{h}^0\,E \leq \mathrm{h}^0\,\mathcal{E}(-b) + \mathrm{h}^0\,\mathcal{O}_X(bL) = \mathrm{h}^0\,\mathcal{E}(-b) + \binom{b+2}{2},$$

which finishes the proof. $\square$

Now, let us study the case $c_1 = S + L$.

It follows from Theorem 5.2.17 that $M_{\mathcal{C}_b}(2; c_1, c_2) = \emptyset$ and, in addition, Theorem 5.2.18 *i*) states that $M_{\mathcal{C}'_b}(2; c_1, c_2) = \emptyset$ if $\mathrm{h}^1\,\mathcal{E}(-2b-1) = 0$. Therefore, it is clear that for theses cases the corresponding Brill-Noether loci are empty. However, if $\mathrm{h}^1\,\mathcal{E}(-2b-1) \neq 0$, then by Theorem 5.2.18 we can study the non-emptiness of the Brill-Noether locus $W^k_{\mathcal{C}'_b}(2; c_1, c_2)$.

Theorem 5.3.4. *Let $X = \mathbb{P}(\mathcal{E}) \to \mathbb{P}^2$ be a ruled 3-fold. Let us fix an integer $b > c_1'+1$ and Chern classes $c_1 = S+L$ and $c_2 = (b+1)SL-b(b+1)L^2$. Assume that $\mathrm{h}^1\,\mathcal{E}(-2b-1) \neq 0$. Then, $W^k_{\mathcal{C}_b'}(2;c_1,c_2) = \emptyset$ for any $k > \mathrm{h}^0\,\mathcal{E}(-2b) + \binom{b+3}{2}$. Moreover, if $\mathrm{h}^0\,\mathcal{E}(-2b) \neq 0$, then*

$$M_{\mathcal{C}_b'}(2;c_1,c_2) = W^1_{\mathcal{C}_b'}(2;c_1,c_2) = W^2_{\mathcal{C}_b'}(2;c_1,c_2) = \cdots = W^k_{\mathcal{C}_b'}(2;c_1,c_2) \neq \emptyset$$

for $k = \mathrm{h}^0\,\mathcal{E}(-b)$.

Proof. Notice that, under these assumptions, by Theorem 5.2.18 *ii*) we get $M_{\mathcal{C}_b'}(2;c_1,c_2) = E_{-\xi_b}(c_1,c_2)$, and thus any $E \in M_{\mathcal{C}_b'}(2;c_1,c_2)$ is given by a non-trivial extension of type

$$0 \to \mathcal{O}_X(S-bL) \to E \to \mathcal{O}_X((b+1)L) \to 0.$$

Hence, for any $E \in M_{\mathcal{C}_b'}(2;c_1,c_2)$, the dimension of $\mathrm{H}^0\,E$ is such that

$$\mathrm{h}^0\,\mathcal{E}(-b) \leq \mathrm{h}^0\,E \leq \mathrm{h}^0\,\mathcal{E}(-b) + \binom{b+3}{2},$$

and we conclude. □

Now we will apply Theorem 5.2.21 to study the case in which the second Chern class is $c_2 = (b+2)SL - b(b+1)L^2$.

Theorem 5.3.5. *Let $X = \mathbb{P}(\mathcal{E}) \to \mathbb{P}^2$ be a ruled 3-fold. Let us fix an integer $b > c_1'+1$ and Chern classes $c_1 = S+L$ and $c_2 = (b+2)SL-b(b+1)L^2$. Then, $W^k_{\mathcal{C}_b}(2;c_1,c_2)$ is non-empty for $1 \leq k \leq \binom{b+2}{2}$.*

Proof. By Theorem 5.2.21, for $\xi_b = -S+(2b+1)L$, the family $E_{\xi_b}(c_1,c_2)$ defines a non-empty irreducible component of $M_{\mathcal{C}_b}(2;c_1,c_2)$. Since any $E \in E_{\xi_b}$ sits in a non-trivial extension of type

$$0 \to \mathcal{O}_X(bL) \to E \to I_Z(S-bL) \to 0,$$

with $[Z] = SL$, for any $E \in E_{\xi_b}$,

$$\mathrm{h}^0\,E \geq \mathrm{h}^0\,\mathcal{O}_X(bL) = \binom{b+2}{2}.$$

Hence, $E_{\xi_b}(c_1,c_2) \hookrightarrow W^{\binom{b+2}{2}}_{\mathcal{C}_b}(2;c_1,c_2)$, and thus

$$W^k_{\mathcal{C}_b}(2;c_1,c_2) \neq \emptyset \quad \text{for} \quad 1 \leq k \leq \binom{b+2}{2},$$

which finishes the proof. □

Now, let us move to the case $c_1 = L$. For this first Chern class, the results arise as a consequence of Theorems 5.2.26 and 5.2.28. First, let us consider the case $c_2 = -S^2 + (2b+1)SL - b(b+1)L^2$.

Theorem 5.3.6. *Let $X = \mathbb{P}(\mathcal{E}) \to \mathbb{P}^2$ be a ruled 3-fold. Let us consider an integer $b > 2c_1' + 1$. Assume that $\mathrm{h}^0\,\mathcal{E}(-b) \neq 0$. Fix the Chern classes $c_1 = L$ and $c_2 = -S^2 + (2b+1)SL - b(b+1)L^2$. Then, $W^k_{\mathcal{C}_b}(2; c_1, c_2) \neq \emptyset$ for $1 \leq k \leq \mathrm{h}^0\,\mathcal{E}(-b)$. Moreover, if $\mathrm{h}^1(S^2\mathcal{E})(-2b-1) \neq 0$, $W^k_{\mathcal{C}'_b}(2; c_1, c_2) \neq \emptyset$ for $1 \leq k \leq \mathrm{h}^0\,\mathcal{E}(-b)$.*

Proof. By the proof of Theorem 5.2.26, for $\xi_b = -2S + (2b+1)L$, the family $E_{\xi_b}(c_1, c_2)$ of rank 2 vector bundles given by a non-trivial extension of type

$$0 \to \mathcal{O}_X(-S + (b+1)L) \to E \to \mathcal{O}_X(S - bL) \to 0,$$

is a non-empty irreducible component of $M_{\mathcal{C}_b}(2; c_1, c_2)$. In addition, any $E \in E_{\xi_b}$ satisfies $\mathrm{h}^0\,E = \mathrm{h}^0\,\mathcal{E}(-b)$ and thus $W^d_{\mathcal{C}'_b}(2; c_1, c_2) \neq \emptyset$ for $d = \mathrm{h}^0\,\mathcal{E}(-b)$.

Furthermore, again by Theorem 5.2.26, if $\mathrm{h}^1(S^2\mathcal{E})(-2b-1) \neq 0$, then we have $\emptyset \neq E_{-\xi_b}(c_1, c_2) \subseteq M_{\mathcal{C}'_b}(2; c_1, c_2)$. By construction, $E_{-\xi_b}(c_1, c_2)$ is the family of rank 2 vector bundles given by the exact sequence

$$0 \to \mathcal{O}_X(S - bL) \to E \to \mathcal{O}_X(-S + (b+1)L) \to 0.$$

Hence, for any $E \in E_{-\xi_b}$, $\mathrm{h}^0\,E = \mathrm{h}^0\,\mathcal{E}(-b)$. Therefore,

$$\emptyset \neq W^d_{\mathcal{C}'_b}(2; c_1, c_2) \subseteq \cdots \subseteq W^1_{\mathcal{C}'_b}(2; c_1, c_2) \subseteq M_{\mathcal{C}'_b}(2; c_1, c_2)$$

for $d = \mathrm{h}^0\,\mathcal{E}(-b)$. $\square$

Now, let us study the case $c_2 = -S^2 + 2(b+1)SL - b(b+1)L^2$.

Theorem 5.3.7. *Let $X = \mathbb{P}(\mathcal{E}) \to \mathbb{P}^2$ be a ruled 3-fold. Let us fix an integer $b > 2c_1' + 1$ and Chern classes $c_1 = L$ and $c_2 = -S^2 + 2(b+1)SL - b(b+1)L^2$. Assume that $\mathrm{h}^0\,\mathcal{E}(-b-1) \neq 0$. Then, $W^k_{\mathcal{C}_b}(2; c_1, c_2) \neq \emptyset$ for $1 \leq k \leq \mathrm{h}^0\,\mathcal{E}(-b-1)$.*

Proof. Notice that, by Theorem 5.2.28, the family E_{ξ_b} of rank 2 vector bundles given by a non-trivial extension of type

$$0 \to \mathcal{O}_X(-S + (b+1)L) \to E \to I_Z(S - bL) \to 0,$$

where $[Z] = \frac{\xi_b^2 - c_1^2}{4} + c_2 = SL$, defines a non-empty irreducible component of $M_{\mathcal{C}_b}(2; c_1, c_2)$. Moreover, by (1.9), for any $E \in E_{\xi_b}$, $\mathrm{h}^0\,E = \mathrm{h}^0\,I_Z(S - bL)$. Using the resolution of I_Z in (1.10), we obtain

$$\mathrm{H}^0\,I_Z(S - bL) \cong \mathrm{H}^0\,\mathcal{O}_X(S - (b+1)L) \cong \mathrm{H}^0\,\mathcal{E}(-b-1) \neq 0$$

and hence, $\emptyset \neq E_{\xi_b} \subseteq W^d_{\mathcal{C}_b}$ for $d = \mathrm{h}^0\,\mathcal{E}(-b-1)$. $\square$

Finally, let us study the case $c_1 = 0$. First of all, we will assume that $c_2 = -S^2 + 2bSL - b^2L^2$.

Theorem 5.3.8. *Let $X = \mathbb{P}(\mathcal{E}) \to \mathbb{P}^2$ be a ruled 3-fold. Let us fix an integer $b > 2c_1' + 1$ and Chern classes $c_1 = 0$ and $c_2 = -S^2 + 2bSL - b^2L^2$. Assume that $\mathrm{h}^0\,\mathcal{E}(-b) \neq 0$. Then, $W^k_{\mathcal{C}_b}(2; c_1, c_2)$ is non-empty for $1 \leq k \leq \mathrm{h}^0\,\mathcal{E}(-b)$. Moreover, if $\mathrm{h}^1(S^2\mathcal{E})(-2b) \neq 0$, $W^k_{\mathcal{C}_b'}(2; c_1, c_2) \neq \emptyset$ for $1 \leq k \leq \mathrm{h}^0\,\mathcal{E}(-b)$.*

Proof. Let us consider $\xi_b = -2S+2bL$. The result follows from Theorem 5.2.34. In fact, on one hand, since $E_{\xi_b}(c_1, c_2) \hookrightarrow M_{\mathcal{C}_b}(2; c_1, c_2)$ is a non-empty family of rank 2 vector bundles given by a non-trivial extension of type

$$0 \to \mathcal{O}_X(-S + bL) \to E \to \mathcal{O}_X(S - bL) \to 0,$$

and any $E \in E_{\xi_b}$ is a $\mathcal{C}_b$-stable bundle such that $\mathrm{h}^0\,E = \mathrm{h}^0\,\mathcal{E}(-b)$, any $E \in E_{\xi_b}$ is inside the Brill-Noether locus $W^d_{\mathcal{C}_b}$ for $d = \mathrm{h}^0\,\mathcal{E}(-b)$. On the other hand, $E_{-\xi_b}(c_1, c_2) \subseteq M_{\mathcal{C}_b'}$ is non-empty and, since any $E \in E_{-\xi_b}$ sits in a non-trivial extension of type

$$0 \to \mathcal{O}_X(S - bL) \to E \to \mathcal{O}_X(-S + bL) \to 0,$$

$\mathrm{h}^0\,E = \mathrm{h}^0\,\mathcal{E}(-b)$ and hence $E \in W^d_{\mathcal{C}_b'}(2; c_1, c_2)$ for $d = \mathrm{h}^0\,E = \mathrm{h}^0\,\mathcal{E}(-b)$, which implies that $W^k_{\mathcal{C}_b'}(2; c_1, c_2)$ is non-empty for $1 \leq k \leq \mathrm{h}^0\,\mathcal{E}(-b)$ and thus $W^k_{\mathcal{C}_b'}(2; c_1, c_2)$ is non-empty for $1 \leq k \leq \mathrm{h}^0\,\mathcal{E}(-b)$. $\square$

Finally, let us consider $c_2 = -S^2 + (2b+1)SL - b^2L^2$.

Theorem 5.3.9. *Let $X = \mathbb{P}(\mathcal{E}) \to \mathbb{P}^2$ be a ruled 3-fold. Let us fix an integer $b > 2c_1'+1$ and Chern classes $c_1 = 0$ and $c_2 = -S^2+(2b+1)SL-b^2L^2$. Assume that $\mathrm{h}^0\,\mathcal{E}(-b-1) \neq 0$. Then, for $1 \leq k \leq \mathrm{h}^0\,\mathcal{E}(-b-1)$, the Brill-Noether locus $W^k_{\mathcal{C}_b}(2; c_1, c_2)$ is non-empty.*

Proof. Let us consider $\xi_b = -2S + 2bL$. Since any E_{ξ_b} sits in a non-trivial extension of type

$$0 \to \mathcal{O}_X(-S + bL) \to E \to I_Z(S - bL) \to 0,$$

where $[Z] = SH$, we get

$$\mathrm{h}^0\,E = \mathrm{h}^0\,I_Z(S - bL) = \mathrm{h}^0\,\mathcal{O}_X(S - (b+1)L) = \mathrm{h}^0\,\mathcal{E}(-b-1).$$

Therefore, since by Theorem 5.2.38 we have $\emptyset \neq E_{\xi_b} \subseteq M_{\mathcal{C}_b}$, it follows that $\emptyset \neq E_{\xi_b} \subseteq W^d_{\mathcal{C}_b}$ for $d = \mathrm{h}^0\,\mathcal{E}(-b-1)$. Hence, if $\mathrm{h}^0\,\mathcal{E}(-b-1) \neq 0$, then the Brill–Noether locus $W^k_{\mathcal{C}_b}(2; c_1, c_2)$ is non-empty for $1 \leq k \leq \mathrm{h}^0\,\mathcal{E}(-b-1)$. $\square$

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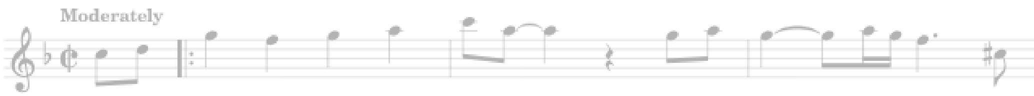
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