# BEURLING-LANDAU'S DENSITY ON COMPACT MANIFOLDS

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ABSTRACT. Given a compact Riemannian manifold $M$, we consider the subspace of $L^2(M)$ generated by the eigenfunctions of the Laplacian of eigenvalue less than $L \geq 1$. This space behaves like a space of polynomials and we have an analogy with the Paley-Wiener spaces. We study the interpolating and Marcienkiewicz-Zygmund (M-Z) families and provide necessary conditions for sampling and interpolation in terms of the Beurling-Landau densities. As an application, we prove the equidistribuition of the Fekete arrays on some compact manifolds.

## INTRODUCTION

Let $(M, g)$ be a smooth, connected, compact Riemannian manifold without boundary, of dimension $m \geq 2$. Let $dV$ and $\Delta_M$ be the volume element and the Laplacian on $M$ associated to the metric $g$, respectively. The Laplacian is given in local coordinates by

$$\Delta_M f = \frac{1}{\sqrt{|g|}} \sum_{i,j} \frac{\partial}{\partial x_i} \left( \sqrt{|g|} g^{ij} \frac{\partial f}{\partial x_j} \right),$$

where $|g| = |\det(g_{ij})|$ and $(g^{ij})_{ij}$ is the inverse matrix of $(g_{ij})_{ij}$. Since $M$ is compact, $g_{ij}$ and all its derivatives are bounded and we assume that the metric $g$ is non-singular at each point of $M$.

By the compactness of $M$, the spectrum of the Laplacian is discrete and there is a sequence of eigenvalues

$$0 \leq \lambda_1^2 \leq \lambda_2^2 \leq \cdots \to \infty$$

and an orthonormal basis $\phi_i$ of smooth real eigenfunctions of the Laplacian i.e. $\Delta_M \phi_i = -\lambda_i^2 \phi_i$. Thus, $L^2(M)$ decomposes into an orthogonal direct sum of eigenfunctions of the Laplacian.

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We consider the following subspaces of $L^2(M)$.

$$E_L = \left\{ f \in L^2(M) \ : \ f = \sum_{i=1}^{k_L} \beta_i \phi_i, \ \Delta_M \phi_i = -\lambda_i^2 \phi_i, \ \lambda_{k_L} \leq L \right\},$$

where $L \geq 1$ and $k_L = \dim E_L$. $E_L$ consists of functions in $L^2(M)$ with a restriction on the support of its Fourier transform. It is, in a sense, the Paley-Wiener space on $M$ with bandwidth $L$. Such spaces have been studied by I. Z. Pesenson and his co-authors (see [GP11] for a detailed discussion).

The goal of this work is to extend the theory of Beurling-Landau on the discretization of functions in the Paley-Wiener space on $\mathbb{R}^n$ to functions in $M$. This should be possible because there is already a literature on the subject in the case $M = \mathbb{S}^m$ (see [Mar07] for more details). In the present work, we study the interpolating and Marcinkiewicz-Zygmund families for the spaces $E_L$. We prove some basic facts about them and give necessary conditions in terms of the Beurling-Landau's density. More precisely, our main result is:

**Theorem 1.** *Let $\mathcal{Z}$ be a triangular family in $M$. If $\mathcal{Z}$ is an $L^2$-*M-Z *family then there exists a uniformly separated $L^2$-*M-Z *family $\tilde{\mathcal{Z}} \subset \mathcal{Z}$ such that*

$$D^-(\tilde{\mathcal{Z}}) \geq 1.$$

*If $\mathcal{Z}$ is an $L^2$-interpolating family then it is uniformly separated and*

$$D^+(\mathcal{Z}) \leq 1,$$

*where $D^+$ and $D^-$ are the upper and lower Beurling-Landau's density (see Definition 4 for more details), respectively,*

We want to point out that there are no good sufficient conditions in terms of the density for the Paley-Wiener space. Thus, we only obtain necessary density conditions for a family to be M-Z or interpolating. Nevertheless, we prove some qualitative sufficient conditions for M-Z and interpolating families: a family that is sufficiently dense (sparse) is M-Z (interpolating) (see Theorem 7 and Proposition 2) but we do not know in a quantitative manner how dense or sparse a family should be.

In the last section, we study the Fekete families for the spaces $E_L$. Fekete points are the points that maximize a Vandermonde-type determinant that appears in the polynomial Lagrange interpolation formula. We show their connection with the interpolating and M-Z families and prove the asymptotic equidistribution of the Fekete points on some compact manifolds. Our main result in this direction is:

**Theorem 2.** *Let $M$ be an admissible manifold and $\mathcal{Z} = \{\mathcal{Z}(L)\}_{L \geq 1}$ be any array such that $\mathcal{Z}(L)$ is a set of Fekete points of degree $L$. Consider the* *measures* $\mu_L = \frac{1}{k_L}\sum_{j=1}^{k_L}\delta_{z_{Lj}}$. *Then* $\mu_L$ *converges in the weak-*$*$ *topology to the normalized volume measure on* $M$*.*

For the precise definition of an admissible manifold see Definition 7. The key idea in proving the above theorem is the necessary condition for the interpolating and Marcienkiewicz-Zygmund arrays in terms of the Beurling-Landau densities given by Theorem 1.

In what follows, when we write $A \lesssim B$, $A \gtrsim B$ or $A \simeq B$, we mean that there are constants depending only on the manifold such that $A \leq CB$, $A \geq CB$ or $C_1 B \leq A \leq C_2 B$, respectively. Also, the value of the constants appearing during a proof may change but they will be still denoted with the same letter. A geodesic ball in $M$ and an Euclidean ball in $\mathbb{R}^m$ are represented by $B(\xi, r)$ and $\mathbb{B}(z, r)$, respectively.

## 1. Kernels associated to $E_L$

Let

$$K_L(z,w) := \sum_{i=1}^{k_L} \phi_i(z)\phi_i(w) = \sum_{\lambda_i \leq L} \phi_i(z)\phi_i(w).$$

This function is the reproducing kernel of the space $E_L$, i.e.

$$\forall f \in E_L \quad f(z) = \langle f, K_L(z,\cdot)\rangle.$$

Note that $\dim(E_L) = k_L = \#\{\lambda_i \leq L\}$. The function $K_L$ is also called the spectral function associated to the Laplacian. Hörmander in [Hör68], proved the following estimates.

(1) $K_L(z,z) = \frac{\sigma_m}{(2\pi)^m} L^m + O(L^{m-1})$ (uniformly in $z \in M$), where $\sigma_m = \frac{2\pi^{m/2}}{m\Gamma(m/2)}$.

(2) $k_L = \frac{\mathrm{vol}(M)\sigma_m}{(2\pi)^m} L^m + O(L^{m-1})$.

In fact, in [Hör68], there are estimates for the spectral function associated to any elliptic operator of order $n \geq 1$ with constants depending only on the manifold.

So, for $L$ big enough we have $k_L \simeq L^m$ and

$$\|K_L(z,\cdot)\|_2^2 = K_L(z,z) \simeq L^m \simeq k_L$$

with constants independent of $L$ and $z$.

We will also use the Bochner-Riesz Kernel associated to the Laplacian that is defined as

$$S_L^N(z,w) := \sum_{i=1}^{k_L} \left(1 - \frac{\lambda_i}{L}\right)^N \phi_i(z)\phi_i(w).$$

Here $N \in \mathbb{N}$ is the order of the kernel. Using the definition, one has that for all $g \in L^2(M)$, the Bochner-Riesz transform of $g$ is

$$S_L^N(g)(z) = \int_M S_L^N(z,w)g(w)dV(w) = \sum_{i=1}^{k_L} \left(1 - \frac{\lambda_i}{L}\right)^N c_i\phi_i(z) \in E_L,$$

where $c_i = \langle g, \phi_i \rangle$. Observe that $\|S_L^N(g)\|_2 \le \|g\|_2$.

Note that $S_L^0(z,w) = K_L(z,w)$. The Bochner-Riesz Kernel satisfies the following estimate.

$$|S_L^N(z,w)| \le C_N L^m \left(1 + Ld(z,w)\right)^{-N-1}, \tag{1}$$

where $C_N$ is a constant depending on the manifold and the order $N$. This estimate has its origins in Hörmander's article [Hör69, Theorem 5.3]. Estimate (1) can be found also in [Sog87, Lemma 2.1].

Note that on the diagonal, $S_L^N(z,z) \simeq C_N L^m$. The upper bound is trivial by the definition and the lower bound follows from

$$S_L^N(z,z) \ge \sum_{\lambda_i \le L/2} \left(1 - \frac{\lambda_i}{L}\right)^N \phi_i(z)\phi_i(z) \ge 2^{-N} K_{L/2}(z,z) \simeq C_N L^m.$$

Similarly we observe that $\left\|S_L^N(\cdot,\xi)\right\|_2^2 \simeq C_N L^m$.

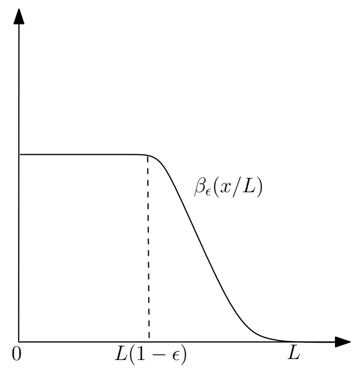
We can consider other Bochner-Riesz type kernels. From now on, we fix an $\epsilon > 0$ and $B_L^\epsilon$ will denote a transform from $L^2(M)$ to $E_L$ with kernel

$$B_L^\epsilon(z,w) = \sum_{i=1}^{k_L} \beta_\epsilon\left(\frac{\lambda_i}{L}\right)\phi_i(z)\phi_i(w), \tag{2}$$

i.e.

$$B_L^\epsilon(f)(z) = \int_M B_L^\epsilon(z,w)f(w)dV(w) = \sum_{i=1}^{k_L} \beta_\epsilon\left(\frac{\lambda_i}{L}\right)\langle f, \phi_i\rangle \phi_i(z),$$

where $\beta_\epsilon : [0,+\infty) \to [0,1]$ is a function of class $\mathcal{C}^\infty$ supported in $[0,1]$ such that $\beta_\epsilon(x) = 1$ for $x \in [0, 1-\epsilon]$ and $\beta_\epsilon(x) = 0$ if $x \notin [0,1)$.



Observe that for $\epsilon = 0$, the transform $B_L^0$ is just the orthogonal projection of the space $E_L$ i.e. the kernel $B_L^0(z,w) = K_L(z,w)$.

We recall now an estimate for the kernel $B_L^\epsilon(z,w)$ that is similar to the Bochner-Riesz kernel estimate (1).

**Lemma 1.** *Let $H : [0,+\infty) \to [0,1]$ be a function with continuous derivatives up to order $N > m$ with compact support in $[0,1]$. Then there exists a constant $C_N$ independent of $L$ such that*

$$\left|\sum_{i=1}^{k_L} H(\lambda_i/L)\phi_i(z)\phi_i(w)\right| \le C_N L^m \frac{1}{(1+Ld(z,w))^N}, \quad \forall z,w \in M. \tag{3}$$

For a proof see [FM10b, Theorem 2.1] and some ideas can be traced back from [Sog87].

## 2. Definitions and Notations

Given $L \ge 1$ and $m_L \in \mathbb{N}$, we consider a triangular family of points in $M$, $\mathcal{Z} = \{\mathcal{Z}(L)\}_L$, denoted as

$$\mathcal{Z}(L) = \{z_{Lj} \in M \ : \ 1 \le j \le m_L\}, L \ge 1,$$

and we assume that $m_L \to \infty$ as $L$ increases.

**Definition 1.** A triangular family of points $\mathcal{Z}$ in $M$ is **uniformly separated** if there exists a positive $\epsilon$ such that for all $L \ge 1$

$$d_M(z_{Lj}, z_{Lk}) \ge \frac{\epsilon}{L}, \quad j \ne k,$$

where $\epsilon$ is called the separation constant of $\mathcal{Z}$.

*Remark* 1. The natural separation is of order $1/L$ in view of Proposition 1 (see below) that shows that a necessary condition for interpolation is that the family should be uniformly separated with this order of separation. The key idea is Bernstein's inequality:

$$\|\nabla f_L\|_\infty \lesssim L\|f_L\|_\infty, \quad \forall f_L \in E_L. \tag{4}$$

This estimate has been proved recently in [FM10b, Theorem 2.2]. Thus, on balls of radius $1/L$, a bounded function of $E_L$ oscillates little.

**Definition 2.** Let $\mathcal{Z} = \{\mathcal{Z}(L)\}_{L\ge1}$ be a triangular family in $M$ with $m_L \ge k_L$ for all $L$. Then $\mathcal{Z}$ is a $L^2$-**Marcinkiewicz-Zygmund** (M-Z) family, if there exists a constant $C > 0$ such that for all $L \ge 1$ and $f_L \in E_L$

$$\frac{C^{-1}}{k_L}\sum_{j=1}^{m_L} |f_L(z_{Lj})|^2 \le \int_M |f_L|^2 dV \le \frac{C}{k_L}\sum_{j=1}^{m_L} |f_L(z_{Lj})|^2.$$

*Remark* 2. The condition of being M-Z can be expressed in terms of the reproducing kernel of $E_L$: a family $\mathcal{Z}$ is M-Z if and only if the normalized reproducing kernels form a frame with uniform bounds in $L$, i.e.

$$\sum_{j=1}^{m_L} |\langle f_L, \tilde{K}_L(z_{Lj}, \cdot)\rangle|^2 \simeq \|f_L\|_2^2,$$

with constants independent of $L$, where $\tilde{K}_L(z,w) = \frac{K_L(z,w)}{\|K_L(z,\cdot)\|_2}$.

**Definition 3.** Let $\mathcal{Z} = \{\mathcal{Z}(L)\}_{L\geq 1}$ be a triangular family in $M$ with $m_L \leq k_L$ for all $L$. Then $\mathcal{Z}$ is a $L^2$-**interpolating family** if for all family of values $c = \{c(L)\}_{L\geq 1}$, $c(L) = \{c_{Lj}\}_{1\leq j\leq m_L}$ such that

$$\sup_{L\geq 1} \frac{1}{k_L} \sum_{j=1}^{m_L} |c_{Lj}|^2 < \infty,$$

there exists a sequence of functions $f_L \in E_L$ with

$$\sup_{L\geq 1} \|f_L\|_2 < \infty$$

and $f_L(z_{Lj}) = c_{Lj}$ $(1 \leq j \leq m_L)$. That is, $f_L(\mathcal{Z}(L)) = c(L)$ for all $L \geq 1$.

*Remark* 3. Equivalently, a family is interpolating if the normalized reproducing kernels form a Riesz sequence, i.e.

$$\frac{1}{k_L} \sum_j |c_{Lj}|^2 \simeq \|\sum_j c_{Lj} \tilde{K}_L(z_{Lj}, \cdot)\|_2^2,$$

with constants independent of $L$, whenever $c = \{c_{Lj}\}_{j,L}$ is a family satisfying

$$\sup_L \frac{1}{k_L} \sum_{j=1}^{m_L} |c_{Lj}|^2 < \infty.$$

Intuitively, a M-Z family should be *dense* in order to recover the $L^2$-norm of functions of the space $E_L$ and an interpolating family should be *sparse*.

We recall a result that is a Plancherel-Pólya type inequality. For a proof, see [OCP11, Theorem 4.6].

**Theorem 3** (Plancherel-Pólya Theorem)**.** *Let $\mathcal{Z}$ be a triangular family of points in $M$, i.e. $\mathcal{Z} = \{z_{Lj}\}_{j\in\{1,\dots,m_L\},L\geq 1} \subset M$. Then $\mathcal{Z}$ is a finite union of uniformly separated families, if and only if there exists a constant $C > 0$ such that for all $L \geq 1$ and $f_L \in E_L$,*

$$\frac{1}{k_L} \sum_{j=1}^{m_L} |f_L(z_{Lj})|^2 \leq C \int_M |f_L(\xi)|^2 dV(\xi). \tag{5}$$

*Remark* 4. The above result is interesting because the inequality (5) means that the sequence of normalized reproducing kernels is a Bessel sequence for $E_L$, i.e.

$$\sum_{j=1}^{m_L} |\langle f, \tilde{K}_L(\cdot, z_{Lj})\rangle|^2 \lesssim \|f\|_2^2 \quad \forall f \in E_L,$$

where $\left\{\tilde{K}_L(\cdot, z_{Lj})\right\}_j$ are the normalized reproducing kernels. Note that $|\tilde{K}_L(\cdot, z_{Lj})|^2 \simeq |K_L(\cdot, z_{Lj})|^2 k_L^{-1}$. That's the reason why the quantity $k_L$ appears in the inequality (5) and in Definitions 2 and 3.

## 3. Interpolating and M-Z families

In this section we will present some qualitative results about the interpolating and M-Z families.

### 3.1. Interpolating families.

The following result shows that the interpolation can be done in a stable way.

**Lemma 2.** *Let $\mathcal{Z}$ be a triangular family in $M$. Assume $\mathcal{Z}$ is interpolating. Then the interpolation can be done by functions $f_L \in E_L$ such that*

$$\|f_L\|_2^2 \leq \frac{C}{k_L} \sum_{j=1}^{m_L} |f_L(z_{Lj})|^2,$$

*where $C$ is independent of $L$.*

The proof follows from the Closed Graph's Theorem (check the basic ideas in [You80, Proposition 2, Page 129]).

Now, we provide a necessary condition for an interpolating family.

**Proposition 1.** *Let $\mathcal{Z}$ be an $L^2$-interpolating triangular family in $M$. Then $\mathcal{Z}$ is uniformly separated.*

*Proof.* Fix $L_0 \geq 1$ and $1 \leq j_0 \leq m_{L_0}$. Using Lemma 2, there exist functions $f_{L_0} \in E_{L_0}$ such that $f_{L_0}(z_{L_0 j}) = \delta_{jj_0}$ and $\|f_{L_0}\|_2^2 \leq C/k_{L_0}$ ($C$ independent of $L$). Applying [OCP11, Proposition 3.4], we get the following.

$$\begin{aligned} 1 &= |f_{L_0}(z_{L_0 j_0}) - f_{L_0}(z_{L_0 j})| \leq \|\nabla f_{L_0}\|_\infty \, d_M(z_{L_0 j_0}, z_{L_0 j}) \\ &\lesssim \sqrt{k_{L_0}} L_0 \, \|f_{L_0}\|_2 \, d_M(z_{L_0 j_0}, z_{L_0 j}) \lesssim L_0 \sqrt{k_{L_0}} \frac{1}{\sqrt{k_{L_0}}} d_M(z_{L_0 j_0}, z_{L_0 j}) \\ &\simeq L_0 d_M(z_{L_0 j_0}, z_{L_0 j}). \end{aligned}$$

Thus,

$$d_M(z_{L_0 j_0}, z_{L_0 j}) \gtrsim \frac{1}{L_0}, \quad \forall L_0 \geq 1, \; j \neq j_0,$$

where the constant does not depend on $L_0$ and $j_0$. □

**Theorem 4.** *Let $\mathcal{Z}$ and $\mathcal{Z}'$ be two triangular families in $M$. Assume that $\mathcal{Z}$ is an $L^2$-interpolating family. Then there exists $\delta_0 > 0$ such that for all $0 < \delta \leq \delta_0$, $Z'$ is also $L^2$-interpolating provided*

$$d_M(z'_{Lj}, z_{Lj}) < \delta/L, \quad \forall j = 1, \cdots, m_L; L \geq 1.$$

The proof follows by a perturbation argument and some gradient estimates proved in [OCP11]. The next proposition gives us a sufficient condition for interpolation. It says, essentially, that a *sparse* family should be interpolating.

**Proposition 2.** *Let $\mathcal{Z} = \{\mathcal{Z}(L)\}_L = \{z_{Lj}\}_{L\geq 1, j=1,\dots,m_L} \subset M$ be a triangular family of points with $m_L \leq k_L$. Assume $\mathcal{Z}$ is separated enough, i.e. there exists $R > 0$ (big enough) such that*

$$d(z_{Lj}, z_{Lk}) \geq \frac{R}{L}, \quad \forall j \neq k, \quad \forall L.$$

*Then $\mathcal{Z}$ is an interpolating family.*

*Proof.* In the following, we consider the Banach spaces:

$$\mathcal{A} = \left\{ v = \{v_L\}_L = \{v_{Lj}\}_{j=1}^{m_L} : \quad \|v\|_{\mathcal{A}}^2 := \sup_L \frac{1}{k_L} \sum_{j=1}^{m_L} |v_{Lj}|^2 < \infty \right\}$$

and $E := \{f = (f_L)_L; \quad f_L \in E_L\}$ endowed with the norm

$$\|f\|_E^2 = \sup_L \|f_L\|_2^2.$$

Let $\mathcal{R} : E \to \mathcal{A}$ be the evaluating operator, i.e. if $v := \mathcal{R}(f)$ for some $f \in E$, then $v_{Lj} = f_L(z_{Lj})$. This operator is linear and continuous by the Plancherel-Pólya type inequality (Theorem 3). Now, consider the operator $\mathcal{S} : \mathcal{A} \to E$ defined as follows: if $v \in \mathcal{A}$, then $\mathcal{S}(v) =: f$ with

$$f_L(z) := \sum_{j=1}^{m_L} v_{Lj} \frac{S_L^N(z_{Lj}, z)}{S_L^N(z_{Lj}, z_{Lj})},$$

where $S_L^N(z, w)$ is the Bochner-Riesz Kernel of order $N$ associated to the Laplacian (see Section 1 for the definition). The order $N$ will be chosen later. Note that the functions $f_L$ belong to $E_L$ and

$$f_L(z_{Lk}) = v_{Lk} + \sum_{j \neq k} v_{Lj} \frac{S_L^N(z_{Lj}, z_{Lk})}{S_L^N(z_{Lj}, z_{Lj})}.$$

The operator $\mathcal{S}$ is well defined. Indeed, let $v \in \mathcal{A}$ and $f := \mathcal{S}(v)$. We need to prove that $f \in E$. Using Cauchy-Schwarz inequality, we obtain:

$$\|f_L\|_2 = \sup_{\|g\|_2=1} |\langle f_L, g \rangle| = \sup_{\|g\|_2=1} \left| \sum_{j=1}^{m_L} v_{Lj} \frac{\langle S_L^N(z_{Lj}, \cdot), g \rangle}{S_L^N(z_{Lj}, z_{Lj})} \right|$$
$$\lesssim \|v\|_{\mathcal{A}} \sup_{\|g\|_2=1} \|S_L^N g\|_2 \leq \|v\|_{\mathcal{A}},$$

where we have applied Theorem 3 to $S_L^N(g)$. Therefore, $\|f\|_E \lesssim \|v\|_{\mathcal{A}} < \infty$. This proves that $\mathcal{S}$ is well defined and continuous. Obviously this operator is linear.

If $\|\mathcal{R} \circ \mathcal{S} - Id\| < 1$, then $\mathcal{R}$ is invertible. Furthermore, $\mathcal{R}$ is exhaustive and as a consequence the family $\mathcal{Z}$ is interpolating. We only need to check that $\|\mathcal{R} \circ \mathcal{S} - Id\| < 1$. We claim that

$$\sum_{j \neq k} \left| \frac{S_L^N(z_{Lj}, z_{Lk})}{S_L^N(z_{Lj}, z_{Lj})} \right| << 1, \tag{6}$$

uniformly in $L$ for $R$ big enough, provided $N + 1 > m$. Thus,

$$\|\mathcal{R} \circ \mathcal{S} - Id\|^2 = \sup_{v \in \mathcal{A}; \|v\|_{\mathcal{A}} = 1} \|\mathcal{R}(\mathcal{S}(v)) - v\|_{\mathcal{A}}^2 = \sup_{v \in \mathcal{A}; \|v\|_{\mathcal{A}} = 1} \|w\|_{\mathcal{A}}^2,$$

where $w = \{w_{Lk}\}_{k;L}$ with

$$w_{Lk} = \sum_{j \neq k} v_{Lj} \frac{S_L^N(z_{Lj}, z_{Lk})}{S_L^N(z_{Lj}, z_{Lj})}.$$

Using the claim (6), we get a control of the $L^\infty$-norm of $w$:

$$\sup_L |w_{Lk}| \leq \sup_L \sum_{j \neq k} \left| \frac{S_L^N(z_{Lj}, z_{Lk})}{S_L^N(z_{Lj}, z_{Lj})} \right| << 1,$$

for all $v = \{v_{Lj}\}_{j;L}$ such that $\sup_L |v_{Lj}| = 1$. Moreover, using again (6), we have the same control of the $L^1$-norm of $w$:

$$\sup_L \frac{1}{k_L} \sum_{k=1}^{m_L} |w_{Lk}| \lesssim \sup_L \frac{1}{k_L} \sum_{j=1}^{m_L} |v_{Lj}| \sum_{k \neq j} \left| \frac{S_L^N(z_{Lk}, z_{Lj})}{S_L^N(z_{Lk}, z_{Lk})} \right| << 1,$$

for all $v = \{v_{Lj}\}_{j;L}$ such that $\sup_L \frac{1}{k_L} \sum_j |v_{Lj}| = 1$. Thus, interpolating between the $L^1$-norm and $L^\infty$-norm, we get the same result for the $L^2$-norm of $w$ and the proof is complete. Now we proceed in order to prove the claim (6). Let

$$g_k(z) := \frac{1}{(1 + L d_M(z, z_{Lk}))^{N+1}}$$

and $B_j := B(z_{Lj}, 1/L)$. It is easy to check that

$$\inf_{B_j} g_k(z) \geq \frac{1}{2^{N+1}} g_k(z_{Lj}).$$

Using the fact that $\mathcal{Z}$ is separated enough, we know that $B_j$ are pairwise disjoint and $\cup_{j\neq k} B_j \subset M \setminus B(z_{Lk},(R-1)/L)$. Therefore, applying (1),

$$\sum_{j\neq k}\left|\frac{S_L^N(z_{Lj},z_{Lk})}{S_L^N(z_{Lj},z_{Lj})}\right| \leq C_N \sum_{j\neq k} g_k(z_{Lj}) \leq C_N L^m \sum_{j\neq k}\int_{B_j} g_k(z)dV(z)$$
$$\leq C_N L^m \int_M \frac{1}{(Ld_M(z,z_{Lk}))^{N+1}}\chi_{M\setminus B(z_{Lk},(R-1)/L)}(z)dV.$$

Consider for any $t\geq 0$, the following set $A_t$.

$$A_t = \left\{ z\in M : \quad d_M(z,z_{Lk}) \geq \frac{R-1}{L}, \quad d_M(z,z_{Lk}) < \frac{t^{-1/(N+1)}}{L}\right\}.$$

Using the distribution function, one can compute that

$$\sum_{j\neq k}\left|\frac{S_L^N(z_{Lj},z_{Lk})}{S_L^N(z_{Lj},z_{Lj})}\right| \leq C_N L^m \int_0^{(R-1)^{-(N+1)}} \mathrm{vol}(A_t)dt$$
$$\lesssim C_N \frac{1}{(R-1)^{(N+1)-m}},$$

provided $N+1>m$. Taking $R$ big enough we get the desired claim. □

3.2. **Marcinkiewicz-Zygmund families.** In what follows, we present some qualitative results concerning the M-Z families. The proof of these results follows from standard techniques and the ideas in [Mar07, Theorem 4.7], replacing the corresponding gradient estimates obtained in [OCP11].

The following theorem allows us to assume, without loss of generality, that a M-Z familiy is uniformly separated.

**Theorem 5.** *Let $\mathcal{Z}\subset M$ be an $L^2$-M-Z family. Then there exists a uniformly separated family $\tilde{\mathcal{Z}}\subset\mathcal{Z}$ which is also an $L^2$-M-Z family.*

The next result shows us that a small perturbation of a M-Z family is still a M-Z family.

**Theorem 6.** *Let $\mathcal{Z}$ be an $L^2$-M-Z family. There exists $\epsilon_0>0$ such that if $\mathcal{Z}'$ is a uniformly separated family with*

$$d_M(z_{Lj},z'_{Lj}) < \frac{\epsilon}{L},$$

*for some $\epsilon\leq\epsilon_0$, then the family of points $\mathcal{Z}'$ is $L^2$-M-Z.*

Now we provide a sufficient condition for a family to be $L^2$-M-Z. Intuitively, a family should be *dense* in order to be M-Z.

**Theorem 7.** *There exists $\epsilon_0>0$ such that if $\mathcal{Z}$ is an $\epsilon$-dense family (not necessarily uniformly separated), i.e. for all $L\geq 1$*

$$\sup_{\xi\in M} d_M(\xi,\mathcal{Z}(L)) < \frac{\epsilon}{L}, \quad (\epsilon\leq\epsilon_0),$$

*then there exists a uniformly separated subfamily which is $\tilde{\epsilon}$-dense and is an $L^2$-M-Z family provided that $\tilde{\epsilon} \leq \epsilon_0$.*

*Remark* 5. Theorem 7 has been also proved by F. Fibir and H.N. Mhaskar using other techniques (see [FM10a, Theorem 5.1]).

## 4. Beurling-Landau density

In this section, we provide necessary conditions for a family to be interpolating or M-Z in terms of the following Beurling-Landau type densities.

**Definition 4.** Let $\mathcal{Z}$ be a triangular family of points in $M$. We define the upper and lower Beurling-Landau density, respectively, as

$$D^+(\mathcal{Z}) = \limsup_{R\to\infty}\left(\limsup_{L\to\infty}\left(\max_{\xi\in M}\left(\frac{\frac{1}{k_L}\#(\mathcal{Z}(L)\cap B(\xi,R/L))}{\frac{\mathrm{vol}(B(\xi,R/L))}{\mathrm{vol}(M)}}\right)\right)\right),$$

$$D^-(\mathcal{Z}) = \liminf_{R\to\infty}\left(\liminf_{L\to\infty}\left(\min_{\xi\in M}\left(\frac{\frac{1}{k_L}\#(\mathcal{Z}(L)\cap B(\xi,R/L))}{\frac{\mathrm{vol}(B(\xi,R/L))}{\mathrm{vol}(M)}}\right)\right)\right).$$

*Remark* 6. Let $\mu_L$ be the normalized counting measure, i.e.

$$\mu_L = \frac{1}{k_L}\sum_{j=1}^{m_L}\delta_{z_{Lj}}$$

and $\sigma$ the normalized volume measure, i.e. $d\sigma = dV/\mathrm{vol}(M)$. Then the densities defined above can be viewed as the asymptotic behaviour of the quantity

$$\frac{\mu_L(B(\xi,R/L))}{\sigma(B(\xi,R/L))}.$$

Our main result is:

**Theorem 8.** *Let $M$ be an arbitrary smooth compact Riemannian manifold without boundary of dimension $m \geq 2$ and $\mathcal{Z}$ a triangular family in $M$. If $\mathcal{Z}$ is an $L^2$-M-Z family then there exists a uniformly separated $L^2$-M-Z family $\tilde{\mathcal{Z}} \subset \mathcal{Z}$ such that*

$$D^-(\tilde{\mathcal{Z}}) \geq 1.$$

*If $\mathcal{Z}$ is an $L^2$-interpolating family then it is uniformly separated and*

$$D^+(\mathcal{Z}) \leq 1.$$

This result was proved in the particular case when $M = \mathbb{S}^m$ in [Mar07]. Following the ideas of [Mar07], we prove Theorem 8 in the general case of a manifold. In [Mar07], the key idea to prove this result was the comparision of the trace of the concentration operator and its square with an estimate of the eigenvalues of this operator. In general, the main difference from the case of the Sphere is that we lack of an explicit expression of the reproducing kernel. Thus, in the general setting, we need to work with a "modified" concentration operator. Before we proceed, we shall introduce the concept of the classical and modified concentration operator.

4.1. **Classical Concentration Operator.**

**Definition 5.** The *classical concentration operator* $\mathcal{K}_A^L$, over a set $A \subset M$, is defined for $f_L \in E_L$ as

$$\mathcal{K}_A^L f_L(z) = \int_A K_L(z,\xi) f_L(\xi) dV(\xi). \tag{7}$$

This operator is the composition of the restriction operator to $A$ with the orthogonal projection of $E_L$, i.e. $\mathcal{K}_A^L(f_L) = P_{E_L}(\chi_A f_L)$ for all $f_L \in E_L$. The operator $\mathcal{K}_A^L$ is self-adjoint. Indeed, if $f_L, g_L \in E_L$ then:

$$\begin{aligned}\langle \mathcal{K}_A^L f_L, g_L\rangle &= \langle P_{E_L}(\chi_A \cdot f_L), g_L\rangle = \langle \chi_A \cdot f_L, P_{E_L}(g_L)\rangle = \langle \chi_A \cdot f_L, g_L\rangle \\ &= \langle f_L, \chi_A \cdot g_L\rangle = \langle P_{E_L} f_L, \chi_A \cdot g_L\rangle = \langle f_L, \mathcal{K}_A^L(g)\rangle.\end{aligned}$$

Alternatively, we can view the action of the concentration operator as a matrix acting on a sequence $\beta = \{\beta_i\}_{i=1,\dots,k_L}$ that are the Fourier coefficients of a function $f_L \in E_L$ (with respect to the orthonormal basis $\{\phi_i\}$). If we denote by $D_L := (d_{ij})_{i,j=1}^{k_L}$, where

$$d_{ij} = \int_A \phi_i \phi_j,$$

then $\mathcal{K}_A^L(f_L) \equiv D_L(\beta)$.
Using the spectral theorem, we know that the eigenvalues of $\mathcal{K}_A^L$ are all real and $E_L$ has an orthonormal basis of eigenvectors of $\mathcal{K}_A^L$. The trace of $\mathcal{K}_A^L$ is

$$\operatorname{tr}(\mathcal{K}_A^L) = \sum_{i=1}^{k_L} d_{ii} = \int_A K_L(z,z) dV(z).$$

Similarly, we can compute the trace of $\mathcal{K}_A^L \circ \mathcal{K}_A^L$.

$$\operatorname{tr}(\mathcal{K}_A^L \circ \mathcal{K}_A^L) = \sum_{i,j=1}^{k_L} d_{ij} d_{ji} = \int_{A\times A} |K_L(z,w)|^2 dV(w) dV(z).$$

We will choose $A$ as $B(\xi, R/L)$ for some fixed point $\xi \in M$ (note that all the constants in the estimates will not depend on the fixed point $\xi \in M$). Taking into account that

$$\operatorname{vol}(B(\xi, R/L)) \simeq \frac{R^m}{L^m}$$

and using Hörmander's estimates for the reproducing kernel and $k_L$ (see Section 1), we get

$$\operatorname{tr}(\mathcal{K}_{B(\xi,R/L)}^L) = k_L \frac{\operatorname{vol}(B(\xi,R/L))}{\operatorname{vol}(M)} + \frac{o(L^m)}{L^m}. \tag{8}$$

4.2. **Modified Concentration Operator.** From now on, we fix an $\epsilon > 0$ and consider the transform $B_L^\epsilon$ defined in Section 1 associated with the kernel

$$B_L^\epsilon(z,w) = \sum_{i=1}^{k_L} \beta_\epsilon\left(\frac{\lambda_i}{L}\right)\phi_i(z)\phi_i(w),$$

i.e. for all $f \in L^2(M)$,

$$B_L^\epsilon(f)(z) = \int_M B_L^\epsilon(z,w)f(w)dV(w) = \sum_{i=1}^{k_L} \beta_\epsilon\left(\frac{\lambda_i}{L}\right)\langle f, \phi_i\rangle\phi_i(z).$$

**Definition 6.** The *modified concentration operator* $T_{L,A}^\epsilon$, over a set $A \subset M$, is defined for $f_L \in E_L$ as:

$$T_{L,A}^\epsilon f_L(z) = B_L^\epsilon(\chi_A \cdot B_L^\epsilon(f_L))(z) = \int_M B_L^\epsilon(z,w)\chi_A(w)B_L^\epsilon(f_L)(w)dV(w).$$

Observe that for $\epsilon = 0$, the modified concentration operator is just the classical concentration operator defined previously.

An advantage of $T_{L,A}^\epsilon$ in contrast of $\mathcal{K}_A^L$ is that we have a nice estimate of its kernel: using Lemma 1, we know that for any $N > m$, there exists a constant $C_N$ independent of $L$ such that

$$|B_L^\epsilon(z,w)| \le C_N L^m \frac{1}{(1+Ld(z,w))^N}, \quad \forall z,w \in M.$$

The operator $T_{L,A}^\epsilon$ is self-adjoint and by the spectral theorem its eigenvalues are all real and $E_L$ has an orthonormal basis of eigenvectors of $T_{L,A}^\epsilon$. In fact, the main reason to do the first smooth projection in $T_{L,A}^\epsilon$ is to ensure the self-adjointness of the operator (but the calculations work even if we consider only $B_L^\epsilon(\chi_A\cdot)$).

As before, we can compute the trace of $T_{L,A}^\epsilon$ and $T_{L,A}^\epsilon \circ T_{L,A}^\epsilon$ that will be used later on.

$$\operatorname{tr}(T_{L,A}^\epsilon) = \sum_{i=1}^{k_L} \beta_\epsilon^2\left(\frac{\lambda_i}{L}\right)\int_A \phi_i^2(z)dV(z) =: \int_A \tilde{B}_L^\epsilon(z,z)dV(z),$$

where $\tilde{B}_L^\epsilon(z,w)$ is a kernel defined as

$$\tilde{B}_L^\epsilon(z,w) = \sum_{i=1}^{k_L} \alpha\left(\frac{\lambda_i}{L}\right)\phi_i(z)\phi_i(w),$$

with $\alpha(x) := \beta_\epsilon^2(x)$. Note that the function $\alpha$ has the same properties as $\beta_\epsilon$ and therefore we know that $\tilde{B}_L^\epsilon(z,w)$ has the estimate (3).

Similarly we can compute the trace of $T^{\epsilon}_{L,A} \circ T^{\epsilon}_{L,A}$.

$$\operatorname{tr}(T^{\epsilon}_{L,A} \circ T^{\epsilon}_{L,A}) = \int_{A\times A} |\tilde{B}^{\epsilon}_L(z,w)|^2 dV(z)dV(w).$$

Since the modified concentration operator is a *small* perturbation of $\mathcal{K}^L_A$, one can estimate $\operatorname{tr}(T^{\epsilon}_{L,A})$ in terms of $\operatorname{tr}(\mathcal{K}^L_A)$. Indeed, using the definition of $\beta_\epsilon(x)$,

$$\operatorname{tr}(\mathcal{K}^{L(1-\epsilon)}_A) \leq \operatorname{tr}(T^{\epsilon}_{L,A}) \leq \operatorname{tr}(\mathcal{K}^L_A).$$

Applying this computation to $A = A_L := B(\xi, R/L)$ and using (8), we get the following.

$$\frac{\operatorname{tr}(T^{\epsilon}_{L,B(\xi,R/L)})}{k_L\frac{\operatorname{vol}(B(\xi,R/L))}{\operatorname{vol}(M)}} \geq \frac{k_{L(1-\epsilon)}\frac{\operatorname{vol}(B(\xi,R/L))}{\operatorname{vol}(M)}}{k_L\frac{\operatorname{vol}(B(\xi,R/L))}{\operatorname{vol}(M)}} + \frac{o(L^m(1-\epsilon)^m)}{L^m(1-\epsilon)^m}\frac{1}{k_L\frac{\operatorname{vol}(B(\xi,R/L))}{\operatorname{vol}(M)}}.$$

Since $\operatorname{vol}(B(\xi,R/L)) \simeq R^m/L^m$, the second term tends to 0 when $L \to \infty$. Thus, using the expression for $k_L$ (see Section 1), we get:

$$\liminf_{L\to\infty} \frac{\operatorname{tr}(T^{\epsilon}_{L,A_L})}{k_L\frac{\operatorname{vol}(B(\xi,R/L))}{\operatorname{vol}(M)}} \geq (1-\epsilon)^m, \quad \forall \epsilon > 0. \tag{9}$$

The upper bound for this quantity is trivial since $\operatorname{tr}(T^{\epsilon}_{L,A_L}) \leq \operatorname{tr}(\mathcal{K}^L_{A_L})$ and has been computed previously. Hence, using (8) we have

$$\limsup_{L\to\infty} \frac{\operatorname{tr}(T^{\epsilon}_{L,A_L})}{k_L\frac{\operatorname{vol}(B(\xi,R/L))}{\operatorname{vol}(M)}} \leq 1. \tag{10}$$

Similarly, if $\rho > 0$ is a fixed number, then

$$\limsup_{L\to\infty} \frac{\operatorname{tr}(T^{\epsilon}_{L(1+\rho),A_L})}{k_L\frac{\operatorname{vol}(B(\xi,R/L))}{\operatorname{vol}(M)}} \leq (1+\rho)^m. \tag{11}$$

4.3. **Proof of the main result.** In the spirit of the original work of Landau, the proof of Theorem 8 relies on a trace estimate of $T^{\epsilon}_{L,A}$ and two technical lemmas (Lemma 3 and 4 below) that estimate the number of big eigenvalues of the modified concentration operator. First we state these technical results and show the proof of the main result and in Sections 4.4 and 4.5 we present a proof of them.

The following result is an estimate of the difference of the trace of $T^{\epsilon}_{L,A_L}$ and $T^{\epsilon}_{L,A_L} \circ T^{\epsilon}_{L,A_L}$. It will show us, later on, that most of the eigenvalues are either close to 1 or to 0.

**Proposition 3.** *Let $A_L = B(\xi, R/L)$. Then*

$$\limsup_{L\to\infty} \left(\operatorname{tr}(T^{\epsilon}_{L,A_L}) - \operatorname{tr}(T^{\epsilon}_{L,A_L} \circ T^{\epsilon}_{L,A_L})\right) \leq C_1(1-(1-\epsilon)^m)R^m + C_2R^{m-1},$$

*where $C_1$ (independent of $\epsilon$) and $C_2$ are constants independent of $L$ and $R$. Similarly, if $\rho > 0$ then*

$$\limsup_{L\to\infty} \left(\operatorname{tr}(T^{\epsilon}_{L(1+\rho),A_L}) - \operatorname{tr}(T^{\epsilon}_{L(1+\rho),A_L} \circ T^{\epsilon}_{L(1+\rho),A_L})\right) \\ \leq C_1(1+\rho)^m(1-(1-\epsilon)^m)R^m + C_2R^{m-1},$$

*where $C_1$ (independent of $\epsilon$ and $\rho$) and $C_2$ are constants independent of $L$ and $R$.*

Given $L \geq 1$ and $R > 0$, let $A_L, A_L^+ = A_L^+(t)$ and $A_L^- = A_L^-(t)$ be the balls centered at a fixed point $\xi \in M$ and radius $R/L$, $(R+t)/L$ and $(R-t)/L$, respectively, where $t$ is a parameter such that $s << t << R << L$ and $s$ is the separation constant of the family $\mathcal{Z}$. The value of $t$ will be chosen later on. We denote the eigenvalues of the modified concentration operator $T^{\epsilon}_{L,A_L}$ as

$$1 > \lambda_1^L \geq \cdots \geq \lambda_{k_L}^L > 0.$$

**Lemma 3.** *Let $\mathcal{Z}$ be an $s$-uniformly separated $L^2$-M-Z family. Then there exist $t_0 = t_0(M,s) > 0$ and a constant $0 < \gamma < 1$ (independent of $\epsilon$, $R$ and $L$) such that for all $t \geq t_0$,*

$$\lambda^L_{N_L+1} \leq \gamma,$$

*where*

$$N_L := N_L(t) = \#(\mathcal{Z}(L) \cap A_L^+) = \#(\mathcal{Z}(L) \cap B(\xi, (R+t)/L)).$$

*Remark* 7. In the conditions of Lemma 3,

$$\#\left\{\lambda_j^L > \gamma\right\} \leq N_L = \#(\mathcal{Z}(L) \cap A_L^+) \leq \#(\mathcal{Z}(L) \cap A_L) + O(R^{m-1}), R \to \infty,$$

where the constant in $O(R^{m-1})$ does not depend on $L$.

*Proof of Remark 7.* The first inequality is trivial by Lemma 3 and the second inequality follows using the separation of the family $\mathcal{Z}$. Moreover, $N_L \lesssim R^m/s^m$. □

**Lemma 4.** *Let $\mathcal{Z}$ be an $L^2$-interpolating family with separation constant $s$ and $\rho > 0$. Then there exist $t_1 = t_1(M,s) > 0$ and a constant $0 < \delta < 1$ independent of $R$ and $L$ such that for all $t \geq t_1$,*

$$\lambda^{L(1+\rho)}_{n_L-1} \geq \delta := C\beta_\epsilon^2\left(\frac{1}{1+\rho}\right),$$

*where $\lambda_k^{L(1+\rho)}$ are the eigenvalues associated to $T^{\epsilon}_{L(1+\rho),A_L}$, $C$ is independent of $\rho$ and $\epsilon$ and*

$$n_L := n_L(t) = \#(\mathcal{Z}(L) \cap A_L^-) = \#(\mathcal{Z}(L) \cap B(\xi, (R-t)/L)).$$

*Remark* 8. In the conditions of Lemma 4 we have

$$\#(\mathcal{Z}(L) \cap A_L) - O(R^{m-1}) \leq n_L = \#(\mathcal{Z}(L) \cap A_L^-) \leq \#\left\{\lambda_j^{L(1+\rho)} \geq \delta\right\} + 1,$$

where the constant in $O(R^{m-1})$ does not depend on $L$.

*Proof of Remark 8.* The second inequality is trivial by Lemma 4 and the first inequality follows using the separation of $\mathcal{Z}$. □

In what follows, we pick the parameter $t$ in the range $\max(t_0, t_1) \leq t << R$, where $t_0$ and $t_1$ are the values given by Lemmas 3 and 4.

Now we have all the tools in order to prove the main result concerning the notion of densities.

*Proof of Theorem 8.* Assume $\mathcal{Z}$ is an $L^2$-M-Z family. Without loss of generality, we may assume that $\mathcal{Z}$ is uniformly separated (see Theorem 5). Consider the following measures:

$$d\mu_L = \sum_{j=1}^{k_L} \delta_{\lambda_j^L}.$$

Note that

$$\operatorname{tr}(T^{\epsilon}_{L,A_L}) = \int_0^1 x d\mu_L(x), \quad \operatorname{tr}(T^{\epsilon}_{L,A_L} \circ T^{\epsilon}_{L,A_L}) = \int_0^1 x^2 d\mu_L(x).$$

Let $\gamma$ be given by Lemma 3. We have

$$\begin{aligned}\#\left\{\lambda_j^L > \gamma\right\} &= \int_\gamma^1 d\mu_L(x) \geq \int_0^1 x d\mu_L(x) - \frac{1}{1-\gamma}\int_0^1 x(1-x) d\mu_L(x) \\ &= \operatorname{tr}(T^{\epsilon}_{L,A_L}) - \frac{1}{1-\gamma}(\operatorname{tr}(T^{\epsilon}_{L,A_L}) - \operatorname{tr}(T^{\epsilon}_{L,A_L} \circ T^{\epsilon}_{L,A_L})),\end{aligned}$$

Using the remark following Lemma 3 and (9), we have

$$\begin{aligned}&\liminf_{L\to\infty} \frac{\#(\mathcal{Z}(L)\cap A_L) + O(R^{m-1})}{k_L \frac{\operatorname{vol}(B(\xi,R/L))}{\operatorname{vol}(M)}} \\ &\geq \liminf_{L\to\infty}\left[\frac{\operatorname{tr}(T^{\epsilon}_{L,A_L})}{k_L \frac{\operatorname{vol}(B(\xi,R/L))}{\operatorname{vol}(M)}} - \frac{1}{1-\gamma}\frac{\operatorname{tr}(T^{\epsilon}_{L,A_L}) - \operatorname{tr}(T^{\epsilon}_{L,A_L}\circ T^{\epsilon}_{L,A_L})}{k_L \frac{\operatorname{vol}(B(\xi,R/L))}{\operatorname{vol}(M)}}\right] \\ &\geq (1-\epsilon)^m - \frac{1}{1-\gamma}\limsup_{L\to\infty}\frac{\operatorname{tr}(T^{\epsilon}_{L,A_L}) - \operatorname{tr}(T^{\epsilon}_{L,A_L}\circ T^{\epsilon}_{L,A_L})}{k_L \frac{\operatorname{vol}(B(\xi,R/L))}{\operatorname{vol}(M)}}\end{aligned}$$

Observe that

$$k_L \frac{\operatorname{vol}(B(\xi,R/L))}{\operatorname{vol}(M)} \simeq R^m. \tag{12}$$

Applying (12) and Proposition 3, we have

$$\liminf_{L\to\infty} \frac{\#(\mathcal{Z}(L)\cap A_L) + O(R^{m-1})}{k_L \frac{\mathrm{vol}(B(\xi,R/L))}{\mathrm{vol}(M)}}$$
$$\geq (1-\epsilon)^m - \frac{C}{1-\gamma} \frac{\limsup_{L\to\infty}(\mathrm{tr}(T^\epsilon_{L,A_L}) - \mathrm{tr}(T^\epsilon_{L,A_L}\circ T^\epsilon_{L,A_L}))}{R^m}$$
$$\geq (1-\epsilon)^m - \frac{C}{1-\gamma}(1-(1-\epsilon)^m) - \frac{1}{1-\gamma}\frac{O(R^{m-1})}{R^m}.$$

Taking inferior limits when $R\to\infty$ in the last estimate, we get that

$$D^-(\mathcal{Z}) \geq (1-\epsilon)^m - \frac{C}{1-\gamma}(1-(1-\epsilon)^m) \quad \forall \epsilon > 0,$$

where $C$ and $\gamma$ are independent of $\epsilon$. Therefore, letting $\epsilon \to 0$ we get the claimed result:

$$D^-(\mathcal{Z}) \geq 1.$$

Assume now that $\mathcal{Z}$ is an $L^2$-interpolating family, in particular it is uniformly separated by Proposition 1. Fix $\rho > 0$. Let $\delta > 0$ be the value given by Lemma 4.

$$\#\left\{\lambda_j^{L(1+\rho)} \geq \delta\right\} \leq \frac{-1}{\delta}\mathrm{tr}(T^\epsilon_{L(1+\rho),A_L}\circ T^\epsilon_{L(1+\rho),A_L}) + \frac{1+\delta}{\delta}\mathrm{tr}(T^\epsilon_{L(1+\rho),A_L})$$
$$= \mathrm{tr}(T^\epsilon_{L(1+\rho),A_L}) + \frac{1}{\delta}(\mathrm{tr}(T^\epsilon_{L(1+\rho),A_L}) - \mathrm{tr}(T^\epsilon_{L(1+\rho),A_L}\circ T^\epsilon_{L(1+\rho),A_L})).$$

Using the remark following Lemma 4, (12), (11) and Proposition 3 we have

$$\limsup_{L\to\infty} \frac{\#(\mathcal{Z}(L)\cap A_L) - O(R^{m-1})}{k_L \frac{\mathrm{vol}(B(\xi,R/L))}{\mathrm{vol}(M)}} \leq \limsup_{L\to\infty} \frac{\mathrm{tr}(T^\epsilon_{L(1+\rho),A_L})}{k_L \frac{\mathrm{vol}(B(\xi,R/L))}{\mathrm{vol}(M)}}$$
$$+ \frac{1}{\delta}\limsup_{L\to\infty} \frac{\mathrm{tr}(T^\epsilon_{L(1+\rho),A_L}) - \mathrm{tr}(T^\epsilon_{L(1+\rho),A_L}\circ T^\epsilon_{L(1+\rho),A_L})}{k_L \frac{\mathrm{vol}(B(\xi,R/L))}{\mathrm{vol}(M)}} + \frac{C_1}{R^m}$$
$$\leq (1+\rho)^m + \frac{C(1+\rho)^m}{\delta}(1-(1-\epsilon)^m) + \frac{1}{\delta}\frac{O(R^{m-1})}{R^m} + \frac{C_1}{R^m}.$$

Taking superior limits in $R\to\infty$ in the last estimate and using the expression for $\delta$, we get

$$D^+(\mathcal{Z}) \leq (1+\rho)^m + \frac{C(1+\rho)^m}{\beta_\epsilon^2\left(\frac{1}{1+\rho}\right)}(1-(1-\epsilon)^m), \quad \forall \epsilon, \rho > 0,$$

where $C$ is independent of $\epsilon > 0$ and $\rho$. Thus, taking limits in $\epsilon \to 0$ and then in $\rho \to 0$, we get the claimed result:

$$D^+(\mathcal{Z}) \leq 1.$$

□

4.4. **Trace estimate.** In this section, we prove Proposition 3. For this purpose, we need the following computation.

**Lemma 5.** *Let $H : [0,\infty) \to [0,1]$ be a function of class $\mathcal{C}^\infty$ with compact support in $[0,1]$. Let $B(\xi, R/L)$ be a ball in $M$. Then*

$$I := \int_{B(\xi,R/L)} \int_{M\setminus B(\xi,R/L)} \left| \sum_{i=1}^{k_L} H(\lambda_i/L)\phi_i(z)\phi_i(w) \right|^2 dV(w)dV(z)$$
$$\leq CR^{m-1},$$

*where $C$ is independent of $L$ and $R$.*

The proof follows by using Lemma 1 and working in local coordinates.

*Proof of Proposition 3.* Let $A = B(\xi, R/L)$. Recall the definition of the kernels $B_L^\epsilon(z,w)$ and $\tilde{B}_L^\epsilon(z,w)$:

$$B_L^\epsilon(z,w) = \sum_{i=1}^{k_L} \beta_\epsilon \left(\frac{\lambda_i}{L}\right) \phi_i(z)\phi_i(w),$$

$$\tilde{B}_L^\epsilon(z,w) = \sum_{i=1}^{k_L} \alpha \left(\frac{\lambda_i}{L}\right) \phi_i(z)\phi_i(w) := \sum_{i=1}^{k_L} \beta_\epsilon^2 \left(\frac{\lambda_i}{L}\right) \phi_i(z)\phi_i(w).$$

First, we will compute the trace of $T_{L,A}^\epsilon \circ T_{L,A}^\epsilon$.

$$\mathrm{tr}(T_{L,A}^\epsilon \circ T_{L,A}^\epsilon) = \int_{A\times A} |\tilde{B}_L^\epsilon(z,w)|^2 dV(w)dV(z)$$
$$= \int_A \sum_{i=1}^{k_L} \alpha^2 \left(\frac{\lambda_i}{L}\right) \phi_i^2(z)dV(z) - \int_A \int_{M\setminus A} |\tilde{B}_L^\epsilon(z,w)|^2 dV(w)dV(z).$$

Thus, we have

$$\mathrm{tr}(T_{L,A}^\epsilon) - \mathrm{tr}(T_{L,A}^\epsilon \circ T_{L,A}^\epsilon) = \int_A \sum_{i=1}^{k_L} \left[\alpha\left(\frac{\lambda_i}{L}\right) - \alpha^2\left(\frac{\lambda_i}{L}\right)\right] \phi_i^2(z)dV(z)$$
$$+ \int_A \int_{M\setminus A} |\tilde{B}_L^\epsilon(z,w)|^2 dV(w)dV(z) =: I_1 + I_2.$$

By Lemma 5, $I_2 = O(R^{m-1})$ with constants independent of $L$ (the constant may depend on $\epsilon$). Now we need to estimate $I_1$. Note that $\alpha(x) \equiv 1$ for $0 \leq 0 \leq 1-\epsilon$. Hence,

$$I_1 = \int_A \sum_{\lambda_i \in (L(1-\epsilon),L]} \left[\alpha\left(\frac{\lambda_i}{L}\right) - \alpha^2\left(\frac{\lambda_i}{L}\right)\right] \phi_i^2(z)dV(z)$$
$$\leq \int_A \sum_{\lambda_i \in (L(1-\epsilon),L]} \phi_i^2(z)dV(z) = \int_A (K_L(z,z) - K_{L(1-\epsilon)}(z,z))dV(z).$$

Using the expression of the reproducing kernel (see Section 1), we obtain: $K_L(z,z) - K_{L(1-\epsilon)}(z,z) = c_m L^m(1-(1-\epsilon)^m) + O(L^{m-1})(1-(1-\epsilon)^{m-1})$. Thus,

$$\begin{aligned} I_1 &\le c_m(1-(1-\epsilon)^m)L^m \mathrm{vol}(B(\xi, R/L)) + \frac{o(L^m)}{L^m}(1-(1-\epsilon)^{m-1}) \\ &\le C(1-(1-\epsilon)^m)R^m + \frac{o(L^m)}{L^m}(1-(1-\epsilon)^{m-1}), \end{aligned}$$

where $C$ is independent of $L$, $R$ and $\epsilon$. Therefore,

$$\lim_{L\to\infty} I_1 \le C(1-(1-\epsilon)^m)R^m.$$

If $\rho > 0$ then a similar computation, working with $L(1+\rho)$ instead of $L$, shows the second claim of Proposition 3. □

4.5. **Technical results.** In this section, we present a proof of Lemma 3 and 4. First, we shall prove a localization type property of the functions $f_L$ of the space $E_L$.

**Lemma 6.** *Let $\mathcal{Z}$ be a s-separated family. Given $f_L \in E_L$ and $\eta > 0$, there exists $t_0 = t_0(\eta)$ such that for all $t \ge t_0$,*

$$\frac{1}{k_L} \sum_{z_{Lj} \notin A_L^+(t)} |f_L(z_{Lj})|^2 \le C_1 \int_{M\setminus A_L} |f_L|^2 + C_2 \eta \int_{A_L} |f_L|^2,$$

*where $A_L^+ = A_L^+(t) = B(\xi, (R+t)/L)$, $C_1$ and $C_2$ are constants depending only on the manifold $M$ and the separation constant $s$ of $\mathcal{Z}$.*

*Proof.* Let $f_L \in E_L$. Consider the kernel

$$B_{2L}(z,w) := B_{2L}^{1/2}(z,w),$$

where $B_L^\epsilon(z,w)$ is defined in (2). Note that the transform $B_{2L}|_{E_L}$ is the identity transform, by construction. Thus,

$$f_L(z) = B_{2L}(f_L)(z) = \int_M B_{2L}(z,w) f_L(w) dV(w), \quad \forall z \in M. \tag{13}$$

By Lemma 1, for any $N > m$, there exists a constant $C_N$ such that

$$|B_{2L}(z,w)| \le C_N L^m \frac{1}{(1+2Ld(z,w))^N}. \tag{14}$$

We will choose $N$ later on.

In order to prove the claimed result, we will show that

(1) Given $\eta > 0$ there exists $t_0 = t_0(\eta)$ such that for all $t \ge t_0$,

$$\frac{1}{k_L} \sum_{z_{Lj} \notin A_L^+} |f_L(z_{Lj})| \le C_1 \int_{M\setminus A_L} |f_L| + C_2 \eta \int_{A_L} |f_L|, \tag{15}$$

where $C_i$ are uniform constants.

(2) Given $\eta > 0$ there exists $t_0 = t_0(\eta)$ such that for all $t \geq t_0$,

$$\max_{z_{Lj} \notin A_L^+} |f_L(z_{Lj})| \leq C_1 \|f_L\|_{L^\infty(M \setminus A_L)} + C_2 \eta \|f_L\|_{L^\infty(A_L)}, \tag{16}$$

where $C_i$ are uniform constants.

Hence, by interpolating between the $L^1$-norm and $L^\infty$-norm, we will have the claimed result for the $L^2$-norm. Let's prove first that this is true in the $L^\infty$-norm.

Observe that the set of points $z_{Lj} \notin A_L^+$ is contained in $M \setminus B(\xi, (R+t)/L)$. Thus,

$$\max_{z_{Lj} \notin A_L^+} |f_L(z_{Lj})| \leq \|f_L\|_{L^\infty(M \setminus B(\xi,(R+t)/L))} \leq \|f_L\|_{L^\infty(M \setminus B(\xi, R/L))}.$$

Hence, (16) is trivially true.

Now we just need to prove (15). Let

$$0 \leq h_j(w) := \frac{1}{(1 + 2Ld(z_{Lj}, w))^N} \leq 1.$$

Using (13) and (14), we obtain:

$$\begin{aligned} \frac{1}{k_L} \sum_{z_{Lj} \notin A_L^+} |f_L(z_{Lj})| &\leq C_N \left\{ \int_{M \setminus B(\xi, R/L)} + \int_{B(\xi, R/L)} \right\} |f_L(w)| \sum_{z_{Lj} \notin A_L^+} h_j(w) \\ &=: I_1 + I_2. \end{aligned}$$

Observe that for all $w \in M$,

$$h_j(w) \lesssim \frac{L^m}{s^m} \int_{B(z_{Lj}, s/L)} \frac{dV(z)}{(1 + 2Ld(z,w))^N}.$$

Note that $B(z_{Lj}, s/L)$ are pairwise disjoint and for $w \in B(\xi, R/L)$,

$$\bigcup_{z_{Lj} \notin A_L^+} B\left(z_{Lj}, \frac{s}{L}\right) \subset M \setminus B\left(\xi, \frac{R+t-s}{L}\right) \subset M \setminus B\left(w, \frac{t-s}{L}\right),$$

Therefore, if $w \in B(\xi, R/L)$,

$$\sum_{z_{Lj} \notin A_L^+} h_j(w) \lesssim \frac{L^m}{s^m} \int_{M \setminus B\left(w, \frac{t-s}{L}\right)} \frac{dV(z)}{(1 + 2Ld(z,w))^N} \lesssim \frac{C_N}{s^m (t-s)^{N-m}} \leq \eta$$

for all $t \geq t_0(\eta, N)$, provided $N > m$. This implies that

$$I_2 \leq C_2 \eta \int_{B(\xi, R/L)} |f_L|.$$

The only thing left is to bound the integral $I_1$. Given $w$, let

$$\#J := \# \{ j : B(w, 2s/L) \cap B(z_{Lj}, s/L) \neq \emptyset \}.$$

Then there exists a uniform constant $C(s)$ (depending only on $s$) such that $\#J \le C(s)$. Hence,

$$\sum_{z_{Lj}\notin A_L^+} h_j(w) = \sum_{\substack{z_{Lj}\notin A_L^+\\ j\in J}} h_j(w) + \sum_{\substack{z_{Lj}\notin A_L^+\\ j\notin J}} h_j(w) \le C(s) + \sum_{j\notin J} h_j(w).$$

Note that for any $w \in M$,

$$\cup_{j\notin J} B(z_{Lj}, s/L) \subset M \setminus B(w, s/L).$$

Hence,

$$\sum_{j\notin J} h_j(w) \lesssim \frac{L^m}{s^m} \int_{M\setminus B(w,s/L)} \frac{dV(z)}{(1+2Ld(z,w))^N} \lesssim C_{s,N},$$

provided $N > m$. So we have that

$$I_1 \le (C(s) + C_{s,N}) \int_{M\setminus B(\xi,R/L)} |f_L| dV$$

and the claim is proved. □

**Lemma 7.** *Let $\mathcal{Z}$ be a $s$-separated family. Given $f_L \in E_L$ and $\eta > 0$, there exists $t_1 = t_1(\eta)$ such that for all $t \ge t_1$*

$$\frac{1}{k_L} \sum_{z_{Lj}\in A_L^-(t)} |f_L(z_{Lj})|^2 \le C_1 \int_{A_L} |f_L|^2 + C_2\eta \int_{M\setminus A_L} |f_L|^2,$$

*where $A_L^- = A_L^-(t) = B(\xi, (R-t)/L)$, $C_1$ and $C_2$ are constants depending only on the manifold $M$ and the separation constant $s$ of $\mathcal{Z}$.*

The proof of Lemma 7 is similar to the proof of Lemma 6.

Now we prove Lemma 3.

*Proof of Lemma 3.* Given $F_L \in E_L$, assume that

$$F_L(z_{Lj}) = 0, \quad \forall z_{Lj} \in A_L^+ = B(\xi, (R+t)/L).$$

Then, using the fact that $\mathcal{Z}$ is $L^2$-M-Z and Lemma 6, we have

$$\begin{aligned} \|F_L\|_2^2 &\lesssim \frac{1}{k_L} \sum_{j=1}^{m_L} |F_L(z_{Lj})|^2 = \frac{1}{k_L} \sum_{z_{Lj}\notin A_L^+} |F_L(z_{Lj})|^2 \\ &\le C_1 \int_{M\setminus A_L} |F_L|^2 + C_2\eta \int_{A_L} |F_L|^2 \le C_1 \int_{M\setminus A_L} |F_L|^2 + C_2\eta \|F_L\|_2^2. \end{aligned}$$

Picking $\eta > 0$ small enough (note that it is independent of $\epsilon$, $L$ and $R$), we get a $t_0(\eta)$ given by Lemma 6 so that for all $t \ge t_0$,

$$\|F_L\|_2^2 \le C_3 \int_{M\setminus A_L} |F_L|^2 dV, \tag{17}$$

where $F_L \in E_L$ is any function vanishing at the points $z_{Lj}$ that are contained in $A_L^+$. Observe that $C_3 > 1$.

Now, we consider an orthonormal basis of eigenvectors $G_j^L$ corresponding to the eigenvalues $\lambda_j^L$ of the modified concentration operator. Let

$$f_L(z) = \sum_{j=1}^{N_L+1} c_j^L G_j^L \in E_L.$$

Note that $f_L \in E_L$ since $N_L \leq CR^m \leq k_L$ for $L$ big enough, in view of the separation of $\mathcal{Z}$. Consider now $F_L := B_L^\epsilon(f_L) \in E_L$. We will apply inequality (17) to $F_L$. We pick $c_j^L$ such that $F_L(z_{Lj}) = 0$ for all $z_{Lj} \in A_L^+$. Observe that

$$\sum_{j=1}^{N_L+1} \lambda_j^L |c_j^L|^2 = \langle T_{L,A_L}^\epsilon f_L, f_L \rangle = \int_{A_L} |B_L^\epsilon f_L(w)|^2 dV(w).$$

Now, using inequality (17),

$$\lambda_{N_L+1}^L \sum_{j=1}^{N_L+1} |c_j^L|^2 \leq \sum_{j=1}^{N_L+1} \lambda_j^L |c_j^L|^2 = \left\{ \int_M - \int_{M \setminus A_L} \right\} |B_L^\epsilon f_L(z)|^2 dV$$

$$\leq \left(1 - \frac{1}{C_3}\right) \|B_L^\epsilon(f_L)\|_2^2 \leq \left(1 - \frac{1}{C_3}\right) \|f_L\|_2^2 = \left(1 - \frac{1}{C_3}\right) \sum_{j=1}^{N_L+1} |c_j^L|^2.$$

where the constant $C_3$ comes from (17) (independent of $\epsilon$, $L$ and $R$). Hence,

$$\lambda_{N_L+1}^L \leq 1 - \frac{1}{C_3} =: \gamma < 1.$$

□

Now we are going to prove the technical lemma corresponding to the interpolating case.

*Proof of Lemma 4.* Let $\mathcal{I} = \{j; \quad z_{Lj} \in A_L^-\}$ and $\rho > 0$ fixed.
Recall that, by Lemma 2, if $\mathcal{Z}$ is an interpolating sequence, then for each sequence $\{c_{Lj}\}_{Lj}$ such that

$$\sup_L \frac{1}{k_L} \sum_{j=1}^{m_L} |c_{Lj}|^2 < \infty,$$

we can construct functions $f_L \in e(L)$ with $\sup_L \|f_L\|_2 < \infty$ and $f_L(z_{Lj}) = c_{Lj}$, where

$$e(L) := \left\{ f_L \in E_L; \quad \|f_L\|_2^2 \lesssim \frac{1}{k_L} \sum_{j=1}^{m_L} |f_L(z_{Lj})|^2 \right\}.$$

In fact, these functions $f_L$ are the solution of the interpolation problem with minimal norm.
Since we have an interpolating family, we can construct for each $z_{Lj} \in \mathcal{Z}(L)$ a function $f_j \in e(L)$ such that

$$f_j(z_{Lj'}) = \delta_{jj'}.$$

Clearly these functions $f_j$ are linearly independent. Since $B^{\epsilon}_{L(1+\rho)}|_{E_L}$ is bijective, for each $j$ there exists a function $h_j \in E_L$ such that

$$f_j = B^{\epsilon}_{L(1+\rho)} h_j.$$

Let

$$F := \operatorname{span}\left\{h_j; \quad z_{Lj} \in A_L^-\right\}.$$

Note that $F$ has dimension $n_L$. Let $f_L \in F$ an arbitrary function and $g_L := B^{\epsilon}_{L(1+\rho)} f_L$. Since $f_L \in F$, we know that

$$f_L = \sum_{j \in \mathcal{I}} c_j h_j.$$

Hence,

$$g_L = B^{\epsilon}_{L(1+\rho)} f_L = \sum_{j \in \mathcal{I}} c_j B^{\epsilon}_{L(1+\rho)} h_j = \sum_{j \in \mathcal{I}} c_j f_j \in e(L),$$

where we have used that each $f_j \in e(L)$ and so this $g_L$ is the function of minimal norm that solves the interpolation problem with data $c_j \delta_{jj'}$. Therefore,

$$\|g_L\|_2^2 \lesssim \frac{1}{k_L} \sum_{j=1}^{m_L} |g_L(z_{Lj})|^2,$$

where the constant do not depend on $\epsilon$ and $L$.

Note that, by construction, $f_j$ vanishes in the points $z_{Lj'}$ with $j \neq j'$. Therefore, for each $j \in \mathcal{I}$ fixed, we have that $f_j(z_{Lk}) = 0$ for all $k \notin \mathcal{I}$. Thus,

$$g_L(z_{Lk}) = \sum_{j \in \mathcal{I}} c_j f_j(z_{Lk}) = 0, \quad \forall k \notin \mathcal{I}.$$

This shows that $g_L = 0$ for $z_{Lk} \notin A_L^-$. Hence, applying Lemma 7 to $g_L = B^{\epsilon}_{L(1+\rho)} f_L$, we get

$$\begin{aligned}
\|B^{\epsilon}_{L(1+\rho)} f_L\|_2^2 &\lesssim \frac{1}{k_L} \sum_{j=1}^{m_L} |g_L(z_{Lj})|^2 = \frac{1}{k_L} \sum_{j \in \mathcal{I}} |g_L(z_{Lj})|^2 \\
&\leq C_1 \int_{A_L} |B^{\epsilon}_{L(1+\rho)} f_L|^2 dV + C_2 \eta \int_{M \setminus A_L} |B^{\epsilon}_{L(1+\rho)} f_L|^2 dV \\
&\leq C_1 \int_{A_L} |B^{\epsilon}_{L(1+\rho)} f_L|^2 dV + C_2 \eta \|B^{\epsilon}_{L(1+\rho)} f_L\|_2^2.
\end{aligned}$$

Picking $\eta$ small enough (note that it is independent of $\rho$, $\epsilon$, $L$ and $R$ because all the constants appearing in the above computation are independent of these parameters), we get from Lemma 7 a value $t_1 = t_1(\eta)$ such that for all $t \geq t_1$,

$$\|B^{\epsilon}_{L(1+\rho)} f_L\|_2^2 \leq C_1 \int_{A_L} |B^{\epsilon}_{L(1+\rho)} f_L|^2 dV. \tag{18}$$

Thus, using this last estimate (18), we get the following.

$$\beta_\epsilon^2\left(\frac{1}{1+\rho}\right)\|f_L\|_2^2 \leq \sum_{i=1}^{k_L} \beta_\epsilon^2\left(\frac{\lambda_i}{L(1+\rho)}\right)|\langle f_L, \phi_i\rangle|^2$$
$$= \sum_{i=1}^{k_{L(1+\rho)}} \beta_\epsilon^2\left(\frac{\lambda_i}{L(1+\rho)}\right)|\langle f_L, \phi_i\rangle|^2 = \|B^{\epsilon}_{L(1+\rho)} f_L\|_2^2$$
$$\leq C_1 \int_{A_L} |B^{\epsilon}_{L(1+\rho)} f_L|^2 dV = C_1 \langle T^{\epsilon}_{L(1+\rho),A_L} f_L, f_L\rangle.$$

We have proved that for all $f_L \in F$,

$$\frac{\langle T^{\epsilon}_{L(1+\rho),A_L} f_L, f_L\rangle}{\langle f_L, f_L\rangle} \geq \delta := C\beta_\epsilon^2\left(\frac{1}{1+\rho}\right), \tag{19}$$

where $C$ does not depend on $L$, $\rho$, $\epsilon$ and $f_L$. Now, applying Weyl-Courant Lemma (see [DS67, Part 2, p. 908]), we know

$$\lambda_{k-1}^{L(1+\rho)} \geq \inf_{g \in E_{L(1+\rho)} \cap E} \frac{\langle T^{\epsilon}_{L(1+\rho),A_L} g, g\rangle}{\langle g, g\rangle}$$

for each subspace $E \subset E_{L(1+\rho)}$ with $\dim(E) = k$. Take $E := F \subset E_L \subset E_{L(1+\rho)}$ defined previously. Note that $\dim(E) = \dim(F) = n_L$ and hence, using (19)

$$\lambda_{n_L-1}^{L(1+\rho)} \geq \inf_{f_L \in F} \frac{\langle T^{\epsilon}_{L(1+\rho),A_L} f_L, f_L\rangle}{\langle f_L, f_L\rangle} \geq \delta.$$

Note that $0 < \delta = C\beta_\epsilon^2(1/(1+\rho)) < 1$. □

## 5. Fekete families

Throughout this section only, we assume that $M$ is an admissible manifold (at the end of this section we provide some examples of such manifolds). The precise definition of admissibility is the following.

**Definition 7.** We say that a manifold is **admissible** if it satisfies the following *product property*: there exists a constant $C > 0$ such that for all $0 < \epsilon < 1$ and $L \geq 1$:

$$E_L \cdot E_{\epsilon L} \subset E_{L(1+C\epsilon)}. \tag{20}$$

Thus, we are assuming that we may multiply two functions of our spaces and still obtain a function which is in some bigger space $E_L$.

**Definition 8.** Let $\{\phi_1^L, \dots, \phi_{k_L}^L\}$ be any basis in $E_L$. The points $\mathcal{Z}(L) = \{z_{L1}, \dots, z_{Lk_L}\}$ maximizing the determinant

$$|\Delta(x_1, \dots, x_{k_L})| = |\det(\phi_i^L(x_j))_{i,j}|$$

are called a set of **Fekete points** of *degree* $L$ for $M$.

A natural problem is to find the limiting distribution of points as $L \to \infty$. In [MOC10], J. Marzo and J. Ortega-Cerdà proved that as $L \to \infty$, the number of Fekete points of degree $L$ for $\mathbb{S}^m$ in a spherical cap $B(z, R)$ gets closer to $k_L \tilde{\sigma}(B(z, R))$, where $\tilde{\sigma}$ is the normalized Lebesgue measure on $\mathbb{S}^m$. They emphasize the connection of the Fekete points with the M-Z and interpolating arrays. In [BB08], Berman and Boucksom have found the limiting distribution in the context of line bundles over complex manifolds. The proof is based on a careful study of the weighted transfinite diameter and its differentiability.
Following the approach in [MOC10], we study the distribution of a set of Fekete points associated to the spaces $E_L$ as $L \to \infty$. The main difficulty in relating the Fekete points with the M-Z and interpolating families is to construct a weighted interpolation formula for $E_L$ where the weight has a fast decay off the diagonal. That is the reason why, we restrict our attention to manifolds that satisfy the product property (20). Under this hypothesis, we are able to prove the equidistribution of the Fekete points.

5.1. **Relation with interpolating and M-Z arrays.** The following two results give the relation of the Fekete points with the interpolating and M-Z arrays. Intuitively, Fekete families are almost interpolating and M-Z.

**Theorem 9.** *Given $\epsilon > 0$, let $L_\epsilon = [(1+\epsilon)L]$ and*

$$\mathcal{Z}_\epsilon(L) = \mathcal{Z}(L_\epsilon) = [z_{L_\epsilon 1}, \dots, z_{L_\epsilon k_{L_\epsilon}}],$$

*where $\mathcal{Z}(L)$ is a set of Fekete points of degree $L$. Then $\mathcal{Z}_\epsilon = \{\mathcal{Z}_\epsilon(L)\}_L$ is a* M-Z *array.*

*Proof.* Assume that $\mathcal{Z}$ is a Fekete family. We will prove that they are uniformly separated. Consider the Lagrange *polynomial* defined as

$$l_{Li}(z) := \frac{\Delta(z_{L1}, \dots, z_{L(i-1)}, z, z_{L(i+1)}, \dots, z_{Lk_L})}{\Delta(z_{L1}, \dots, z_{Lk_L})}.$$

Note that

- $\|l_{Li}\|_\infty = 1$.
- $l_{Li}(z_{Lj}) = \delta_{ij}$.
- $l_{Li} \in E_L$.

Thus, using the Bernstein inequality for the space $E_L$ (see (4)), we have for all $j \neq i$,

$$\begin{aligned} 1 = |l_{Li}(z_{Li}) - l_{Li}(z_{Lj})| &\le \|\nabla l_{Li}\|_\infty d_M(z_{Li}, z_{Lj}) \\ &\lesssim L\|l_{Li}\|_\infty d_M(z_{Li}, z_{Lj}) = L d_M(z_{Li}, z_{Lj}). \end{aligned}$$

Therefore,

$$d_M(z_{Li}, z_{Lj}) \geq \frac{C}{L},$$

i.e. $\mathcal{Z}$ is uniformly separated. This implies that $\mathcal{Z}_\epsilon$ is also uniformly separated because

$$d_M(z_{L_\epsilon i}, z_{L_\epsilon j}) \geq \frac{C}{L_\epsilon} \overset{L_\epsilon \leq (1+\epsilon)L}{\geq} \frac{C/(1+\epsilon)}{L}.$$

Using Theorem 3 we get for any $f_L \in E_L$,

$$\frac{1}{k_L} \sum_{j=1}^{k_{L_\epsilon}} |f_L(z_{L_\epsilon j})|^2 \lesssim \int_M |f_L|^2 dV.$$

In order to prove that $\mathcal{Z}_\epsilon$ is M-Z, we only need to prove the converse inequality, i.e.

$$\frac{1}{k_L} \sum_{j=1}^{k_{L_\epsilon}} |f_L(z_{L_\epsilon j})|^2 \gtrsim \|f_L\|_2^2.$$

Consider the Lagrange interpolation operator defined in $\mathcal{C}(M)$ as

$$\Lambda_L(f)(z) := \sum_{j=1}^{k_L} f(z_{Lj}) l_{Lj}(z).$$

Note that

$$\|\Lambda_L(f)\|_\infty \leq k_L \|f\|_\infty.$$

This estimate isn't enough. In order to have better control on the norms, we will make use of a weighted interpolation formula. Fix a point $z \in M$ and let $p(z, \cdot)$ be a function in the space $E_{\frac{\epsilon}{C}L}$ such that $p(z,z) = 1$, where $C$ is the constant appearing in (20). Then given $f_L \in E_L$ one has

$$R(w) = f_L(w) p(z,w) \in E_{L_\epsilon}.$$

Note that $R(z) = f_L(z)p(z,z) = f_L(z)$. Thus, we have a weigthed representation formula

$$f_L(z) = \sum_{j=1}^{k_{L_\epsilon}} p(z, z_{L_\epsilon j}) f_L(z_{L_\epsilon j}) l_{L_\epsilon j}(z).$$

We define the operator $Q_L$ from $\mathbb{C}^{k_{L_\epsilon}} \to E_{L_{2\epsilon}}$ as

$$Q_L[v](z) = \sum_{j=1}^{k_{L_\epsilon}} v_j p(z, z_{L_\epsilon j}) l_{L_\epsilon j}(z), \quad \forall v \in \mathbb{C}^{k_{L_\epsilon}}.$$

We want to prove that

$$\int_M |Q_L[v](z)|^2 dV(z) \lesssim \frac{1}{k_L} \sum_{j=1}^{k_{L_\epsilon}} |v_j|^2, \tag{21}$$

with constant independent of $L$. Once we have proved this estimate, choosing $v_j = f_L(z_{L_\epsilon j})$ we will have

$$\begin{aligned} Q_L[(f_L(z_{L_\epsilon j}))_j](z) &= \sum_{j=1}^{k_{L_\epsilon}} f_L(z_{L_\epsilon j}) p(z, z_{L_\epsilon j}) l_{L_\epsilon j}(z) \\ &= \sum_{j=1}^{k_{L_\epsilon}} R(z_{L_\epsilon j}) l_{L_\epsilon j}(z) = R(z) = f_L(z). \end{aligned}$$

Hence, applying the claimed inequality (21) we will obtain

$$\|f_L\|_2^2 \lesssim \frac{1}{k_L} \sum_{j=1}^{k_{L_\epsilon}} |f_L(z_{L_\epsilon j})|^2,$$

and thus $\mathcal{Z}_\epsilon$ is M-Z.
In order to prove (21), we need to choose the weight $p$ with care. We shall construct $p \in E_{L\epsilon/C}$ with a fast decay off the diagonal.

Let $\delta > 0$ and consider the kernels $B_L(z, w) := B_L^\delta(z, w)$ defined in Section 1. Let

$$p(z, w) = \frac{B_{L\frac{\epsilon}{C}}(z, w)}{B_{L\frac{\epsilon}{C}}(z, z)} \in E_{L\frac{\epsilon}{C}}.$$

Observe that

- $p(z, z) = 1$.
- $$\begin{aligned} \int_M |p(z, w)| dV(w) &= \frac{1}{B_{L\frac{\epsilon}{C}}(z, z)} \|B_{L\frac{\epsilon}{C}}(z, \cdot)\|_1 \\ &\lesssim \frac{1}{k_L}, \end{aligned}$$

  where we have used $\|B_L(z, \cdot)\|_1 \lesssim 1$ (see [FM10b, Equation (2.11), Theorem 2.1] for a proof).

Now we are ready to prove (21). Note that

$$\begin{aligned} \int_M |Q_L[v](z)| dV(z) &\le \int_M \sum_{j=1}^{k_{L_\epsilon}} |v_j| |p(z, z_{L_\epsilon j})| |l_{L_\epsilon j}(z)| dV(z) \\ &\le \sum_{j=1}^{k_{L_\epsilon}} |v_{Lj}| \|p(\cdot, z_{L_\epsilon j})\|_1 \lesssim \frac{1}{k_L} \sum_{j=1}^{k_{L_\epsilon}} |v_{Lj}|. \end{aligned}$$

On the other hand,

$$|Q_L[v](z)| \le \sup_j |v_j| \sum_{j=1}^{k_{L_\epsilon}} |p(z, z_{L_\epsilon j})|.$$

Let $s$ be the separation constant of $Z_{L_\epsilon}$ and

$$h(z,w)=\frac{1}{(1+L_\epsilon d_M(z,w))^N}\le 1.$$

Note that,

$$\inf_{w\in B(z_{L_\epsilon j},s/L_\epsilon)} h(z,w)\ge C_s h(z,z_{L_\epsilon j}).$$

Therefore,

$$\begin{aligned}\sum_{j=1}^{k_{L_\epsilon}}|p(z,z_{L_\epsilon j})| &= \frac{1}{B_{L\frac{\epsilon}{C}}(z_{L_\epsilon j},z_{L_\epsilon j})}\sum_{j=1}^{k_{L_\epsilon}}|B_{L\frac{\epsilon}{C}}(z_{L_\epsilon j},z)|\\ &\lesssim \sum_{j=1}^{k_{L_\epsilon}}\frac{1}{(1+L\frac{\epsilon}{C}d_M(z,z_{L_\epsilon j}))^N}\\ &\lesssim \frac{L_\epsilon^m}{s^m}\int_{\cup_{j=1}^{k_{L_\epsilon}}B(z_{L_\epsilon j},s/L_\epsilon)} h(z,w)dV(w)\\ &= \frac{L_\epsilon^m}{s^m}\int_{\cup_{j=1}^{k_{L_\epsilon}}B(z_{L_\epsilon j},s/L_\epsilon)\cap B(z,2s/L_\epsilon)} h(z,w)dV(w)\\ &+ \frac{L_\epsilon^m}{s^m}\int_{\cup_{j=1}^{k_{L_\epsilon}}B(z_{L_\epsilon j},s/L_\epsilon)\cap B(z,2s/L_\epsilon)^c} h(z,w)dV(w)\\ &\le C_{s,\epsilon}+C_sL_\epsilon^m\int_{M\setminus B(z,2s/L_\epsilon)} h(z,w)dV(w)\lesssim 1,\end{aligned}$$

where we have used that

$$\int_{M\setminus B(z,r/L_\epsilon)} h(z,w)dV(w)\lesssim\frac{1}{L_\epsilon^m(1+r)^{N-m}}.$$

This computation follows by integrating $h(z,w)$ using the distribution function.
Hence, we have proved that

$$\|Q_L[v]\|_\infty\lesssim\sup_j|v_j|.$$

The claimed estimate (21) follows by the Riesz-Thorin interpolation theorem. □

The following result relates the Fekete points with the interpolating families.

**Theorem 10.** *Given $\epsilon>0$, let $L_{-\epsilon}=[(1-\epsilon)L]$ and let*

$$\mathcal{Z}_{-\epsilon}(L)=\mathcal{Z}(L_{-\epsilon})=\left\{z_{L_{-\epsilon}1},\dots,z_{L_{-\epsilon}k_{L_{-\epsilon}}}\right\},$$

*where $\mathcal{Z}(L)$ is a set of Fekete points of degree $L$. Then the array $\mathcal{Z}_{-\epsilon}=\{\mathcal{Z}_{-\epsilon}(L)\}_L$ is an interpolating family.*

*Proof.* Given any array of values $\{v_{L-\epsilon j}\}_{j=1}^{k_{L-\epsilon}}$, we consider

$$R_L[v](z) = \sum_{j=1}^{k_{L-\epsilon}} v_{L-\epsilon j} p(z, z_{L-\epsilon j}) l_{L-\epsilon j}(z) \in E_L,$$

where $p(\cdot, z) \in E_{L\epsilon/C}$ defined in the proof of the previous Theorem. Note that

$$\begin{aligned} R_L[v](z_{L-\epsilon k}) &= \sum_{j=1}^{k_{L-\epsilon}} v_{L-\epsilon j} p(z_{L-\epsilon k}, z_{L-\epsilon j}) l_{L-\epsilon j}(z_{L-\epsilon k}) \\ &= v_{L-\epsilon k} p(z_{L-\epsilon k}, z_{L-\epsilon k}) = v_{L-\epsilon k}. \end{aligned}$$

Also, as in the proof of the previous theorem we have

$$\sum_{j=1}^{k_{L-\epsilon}} |p(z, z_{L-\epsilon j})| \lesssim 1$$

and

$$\int_M |p(z, z_{L-\epsilon j})| dV(z) \lesssim \frac{1}{k_L}.$$

Thus, as before we have that

$$|R_L[v](z)| \leq \sup_j |v_{L-\epsilon j}| \sum_{j=1}^{k_{L-\epsilon}} |p(z, z_{L-\epsilon j})| \lesssim \sup_j |v_{L-\epsilon j}|.$$

Hence

$$\|R_L[v]\|_\infty \lesssim \sup_j |v_{L-\epsilon j}|.$$

Also,

$$\|R_L[v]\|_1 \leq \sum_{j=1}^{k_{L-\epsilon}} |v_{L-\epsilon j}| \|p(\cdot, z_{L-\epsilon j})\|_1 \lesssim \frac{1}{k_L} \sum_{j=1}^{k_{L-\epsilon}} |v_{L-\epsilon j}|.$$

By the Riesz-Thorin interpolation theorem we get

$$\|R_L[v]\|_2^2 \lesssim \frac{1}{k_L} \sum_{j=1}^{k_{L-\epsilon}} |v_{L-\epsilon j}|^2.$$

□

5.2. **Equidistribuition of the Fekete families.** Now we are ready to prove the equidistribuition of the Fekete points. Since the Fekete families are, essentially, interpolating and M-Z, we will make use of the density result, proved in the previous section, that gives a necessary condition for interpolation and M-Z. In what follows, $\sigma$ will denote the normalized volume measure, i.e. $d\sigma = dV/\mathrm{vol}(M)$. Our main result is:

**Theorem 11.** *Let $\mathcal{Z} = \{\mathcal{Z}(L)\}_{L\geq 1}$ be any array such that $\mathcal{Z}(L)$ is a set of Fekete points of degree $L$ and $\mu_L = \frac{1}{k_L}\sum_{j=1}^{k_L}\delta_{z_{Lj}}$. Then $\mu_L$ converges in the weak-$*$ topology to the normalized volume measure on $M$.*

*Proof.* We know that for any $\epsilon > 0$ the array $\mathcal{Z}_\epsilon = \{\mathcal{Z}_\epsilon(L)\}_{L\geq 1}$ is M-Z, so if we use the density results (see Theorem 8), we get for any $\epsilon > 0$, a large $R = R(\epsilon)$ and $L(R(\epsilon))$ such that for all $L \geq L(R(\epsilon))$ and $\xi \in M$,

$$\frac{\frac{1}{k_L}\#(\mathcal{Z}(L)\cap B(\xi, R/L))}{\sigma(B(\xi, R/L))} \geq (1-\epsilon). \tag{22}$$

Similarly, since $\mathcal{Z}_{-\epsilon}$ is interpolating (because $\mathcal{Z}$ is a family of Fekete) we know that there exist $R = R(\epsilon)$ and $L(R(\epsilon))$ such that for all $L \geq L(R(\epsilon))$ and $\xi \in M$,

$$\frac{\frac{1}{k_L}\#(\mathcal{Z}(L)\cap B(\xi, R/L))}{\sigma(B(\xi, R/L))} \leq (1+\epsilon). \tag{23}$$

Note that

$$\mu_L(B(\xi, R/L)) = \frac{1}{k_L}\#(\mathcal{Z}(L)\cap B(\xi, R/L)).$$

Thus, for any $\epsilon > 0$ there is a large $R$ such that for any $L$ big enough and $\xi \in M$,

$$(1-\epsilon)\sigma(B(\xi, r_L)) \leq \mu_L(B(\xi, r_L)) \leq (1+\epsilon)\sigma(B(\xi, r_L)), \tag{24}$$

where $r_L = R/L$. Hence, we have that

$$\lim_{L\to\infty}\frac{\mu_L(B(z, r_L))}{\sigma(B(z, r_L))} = 1, \quad r_L \to 0, \tag{25}$$

uniformly in $z \in M$. This is enough to prove the equidistribuition of the Fekete points. We proceed now with the details. Let $f \in \mathcal{C}(M)$. We will use the notation

$$\nu(f) := \int_M f(z)d\nu(z),$$

where $\nu$ is a measure and $f_r$ will denote the mean of $f$ over a ball $B(z, r)$ with respect to the volume measure, i.e.

$$f_r(z) = \frac{1}{\sigma(B(z, r))}\int_{B(z,r)} f(w)d\sigma(w).$$

We want to show that $\mu_L(f) \to \sigma(f)$, when $L \to \infty$, for all $f \in \mathcal{C}(M)$.

$$\begin{aligned}
|\mu_L(f) - \sigma(f)| &\leq |(\mu_L - \sigma)(f - f_{r_L})| + |(\mu_L - \sigma)(f_{r_L})| \\
&\leq (\mu_L(M) + \sigma(M))\|f - f_{r_L}\|_\infty + |(\mu_L - \sigma)(f_{r_L})| \\
&\leq 2\|f - f_{r_L}\|_\infty + |(\mu_L - \sigma)(f_{r_L})|.
\end{aligned}$$

We will estimate the second term using [Blü90, Lemma 2] that says

$$\sup_{z\in M}\left|\frac{\sigma(B(z, r))}{|\mathbb{B}(0, cr)|} - 1\right| = O(r^2), \tag{26}$$

uniformly in $z \in M$, where $c$ is a constant depending only on the manifold and $|\cdot|$ denotes the Euclidean volume. Similarly, one has

$$\sup_{z\in M}\left|\frac{|\mathbb{B}(0,cr)|}{\sigma(B(z,r))}-1\right| = O(r^2), \tag{27}$$

because, by the compactness of $M$,

$$C_1 \le \frac{\sigma(B(z,r))}{|\mathbb{B}(0,cr)|} \le C_2, \tag{28}$$

thus,

$$\left|\frac{|\mathbb{B}(0,cr)|}{\sigma(B(z,r))}-1\right| = \left|\frac{1-\frac{\sigma(B(z,r))}{|\mathbb{B}(0,cr)|}}{\frac{\sigma(B(z,r))}{|\mathbb{B}(0,cr)|}}\right| \le \frac{Cr^2}{C_1} = O(r^2).$$

Similarly,

$$\sup_{w,z\in M}\left|\frac{\sigma(B(w,r))}{\sigma(B(z,r))}-1\right| = O(r^2). \tag{29}$$

Using Fubini, we obtain:

$$|(\mu_L-\sigma)(f_{r_L})| \le \int_M |f(w)| \left|\int_{B(w,r_L)} \frac{d\mu_L(z)-d\sigma(z)}{\sigma(B(z,r_L))}\right| d\sigma(w)$$

Now we deal with the second integral.

$$\int_{B(w,r_L)} \frac{d\mu_L(z)-d\sigma(z)}{\sigma(B(z,r_L))} = \int_{B(w,r_L)} \frac{d\mu_L(z)-d\sigma(z)}{\sigma(B(w,r_L))}\frac{\sigma(B(w,r_L))}{\sigma(B(z,r_L))}$$
$$= \int_{B(w,r_L)} \frac{d\mu_L(z)-d\sigma(z)}{\sigma(B(w,r_L))} + \int_{B(w,r_L)} \frac{d\mu_L(z)-d\sigma(z)}{\sigma(B(w,r_L))}\left(\frac{\sigma(B(w,r_L))}{\sigma(B(z,r_L))}-1\right)$$

Thus,

$$\left|\int_{B(w,r_L)} \frac{d\mu_L(z)-d\sigma(z)}{\sigma(B(z,r_L))}\right| \le \frac{1}{\sigma(B(w,r_L))}\,|\mu_L(B(w,r_L))-\sigma(B(w,r_L))|$$
$$+ \int_{B(w,r_L)} \frac{1}{\sigma(B(w,r_L))}\left|\frac{\sigma(B(w,r_L))}{\sigma(B(z,r_L))}-1\right|(d\mu_L(z)+d\sigma(z)).$$

Hence, using (29),

$$|(\mu_L-\sigma)(f_{r_L})| \le \sup_{w\in M}\left|\frac{\mu_L(B(w,r_L))}{\sigma(B(w,r_L))}-1\right| \|f\|_1$$
$$+ \sup_{z,w\in M}\left|\frac{\sigma(B(w,r_L))}{\sigma(B(z,r_L))}-1\right| \int_M |f(w)| \left(\frac{\mu_L(B(w,r_L))}{\sigma(B(w,r_L))}+1\right) d\sigma(w)$$
$$\le \|f\|_1 \left(\sup_{w\in M}\left|\frac{\mu_L(B(w,r_L))}{\sigma(B(w,r_L))}-1\right| + Cr_L^2\left(\sup_{w\in M}\left|\frac{\mu_L(B(w,r_L))}{\sigma(B(w,r_L))}\right|+1\right)\right).$$

Briefly, we have obtained

$$|\mu_L(f) - \sigma(f)| \le 2\|f - f_{r_L}\|_\infty$$
$$+ \|f\|_1 \left( \sup_{w\in M} \left| \frac{\mu_L(B(w,r_L))}{\sigma(B(w,r_L))} - 1 \right| + Cr_L^2 \left( \sup_{w\in M} \left| \frac{\mu_L(B(w,r_L))}{\sigma(B(w,r_L))} \right| + 1 \right) \right)$$

Letting $L \to \infty$ and using (25), we obtain the desired result:

$$\mu_L(f) \to \sigma(f), \quad L \to \infty, \forall f \in \mathcal{C}(M).$$

□

5.3. **Examples of manifolds.** The basic examples are the compact two-point homogeneous spaces. These spaces, essentially are $\mathbb{S}^m$, the projective spaces over the field $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ and the Cayley Plane. In these spaces we can multiply two functions of the spaces $E_L$ and obtain another function of some bigger space $E_L$. Indeed, in the case of the Sphere, $E_L$ represents the spherical harmonics of degree less than $L$, usually denoted by $\Pi_L$. In such spaces, we know that

$$\Pi_{2L} = \text{span}\Pi_L\Pi_L,$$

(see [Mar08, Lemma 4.5]). Moreover, in $\mathbb{S}^m$,

$$\Pi_L \cdot \Pi_{\epsilon L} \subset \Pi_{L(1+\epsilon)}.$$

Thus, the product property holds trivially in $\mathbb{S}^m$.

**Projective Spaces.**

The case of the Projective spaces is similar to the Sphere. In [Sha01, Sections 3.2 and 3.3], there is a description and an orthogonal decomposition of the harmonic polynomials on the projective spaces.

Let $\mathbb{K}$ be the field of $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Consider the sphere $\mathbb{S}^{m-1} \subset \mathbb{K}^m \approx \mathbb{R}^{dm}$, where $d = \dim_{\mathbb{R}}\mathbb{K}$. We define the projective space $\mathbb{KP}^{m-1}$ over the field $\mathbb{K}$ (of dimension $m-1$) as the quotient

$$\mathbb{KP}^{m-1} = \mathbb{S}^{m-1}/\sim,$$

where $x \sim y$ if and only if $y = \gamma x$ with $\gamma \in \mathbb{K}$ and $|\gamma| = 1$. Consider the space of homogeneous polynomials of degree less than $L$ on the projective spaces:

$$\text{Pol}_L = \left\{ p(x)|_{\mathbb{S}^{m-1}};\ x \in \mathbb{R}^{dm}, \deg(p) \le L, p(\gamma x) = |\gamma|^L p(x), \forall \gamma \in \mathbb{K} \right\}.$$

It is immediate that $\text{Pol}_L$ verify the product property (20). We will show that the spaces $E_L$ associated to $\mathbb{KP}^{m-1}$ are identified with the spaces $\text{Pol}_L$. This proves that the projective spaces are admissible. It is observed in [Sha01, Section 3.2], that $\text{Pol}_L$ coincide with its subspace of harmonic polynomials of degree less than $L$:

$$\text{Pol}_L = \text{Harm}_L = \{p \in \text{Pol}_L;\ \Delta_{\mathbb{R}^{dm}} p \equiv 0\}$$

and an orthogonal decomposition holds:

$$\mathrm{Harm}_L = \mathrm{Harm}(0) \oplus \mathrm{Harm}(2) \oplus \ldots \oplus \mathrm{Harm}(2[L/2]),$$

where $\mathrm{Harm}(2k)$ is the subspace of $\mathrm{Pol}_L$ of harmonics of degree $2k$. We claim that the spaces $E_L$ associated to the projective spaces are identified with the spaces $\mathrm{Harm}_L$. Thus, we need to show that $\mathrm{Harm}(2k)$ are the eigenspaces of $\Delta_{\mathbb{KP}^{m-1}}$. For this purpose, it is sufficient to prove that its reproducing kernel, $f(x,y)$, is an eigenfunction because then for any $Y \in \mathrm{Harm}(2k)$,

$$\begin{aligned}\Delta_{\mathbb{KP}^{m-1}} Y(x) &= \Delta_{\mathbb{KP}^{m-1}} \langle Y, f(x,\cdot)\rangle = \langle Y, \Delta_{\mathbb{KP}^{m-1}} f(x,\cdot)\rangle \\ &= -\lambda^2 \langle Y, f(x,\cdot)\rangle = -\lambda^2 Y(x).\end{aligned}$$

Let $h_{2k}$ be the dimension of $\mathrm{Harm}(2k)$ and $(s_{ki})_{I=1}^{h_{2k}}$ be an orthonormal basis in $\mathrm{Harm}(2k)$. Its kernel, can be expressed as the function

$$f(x,y) = \sum_{i=1}^{h_{2k}} \overline{s_{ki}(x)} s_{ki}(y), \quad x, y \in \mathbb{S}^{m-1}.$$

It is proved, in [Sha01, Section 3.3], that $f(x,y)$ is a function of $|\langle x,y\rangle|^2$,

$$f(x,y) = q_k(|\langle x,y\rangle|^2),$$

where $q_k$ is a function $[0,1] \to \mathbb{C}$. Moreover, in [Sha01, Section 3.3], we can find an explicit form of this function:

$$\sum_{i=1}^{h_{2k}} \overline{s_{ki}(x)} s_{ki}(y) = b_k^d P_k^{(\alpha,\beta)}(2|\langle x,y\rangle|^2 - 1) = b_k^d P_k^{(\alpha,\beta)}(\cos(\sqrt{2}\rho(x,y))),$$

where $\rho$ is the geodesic distance, $b_k^d$ is a constant of normalization and

$$\alpha = \frac{dm-d-2}{2}, \quad \beta = \frac{d-2}{2}, \quad d = \dim_{\mathbb{R}} \mathbb{K}.$$

Note that, since the reproducing kernel $f(x,y)$ depends only on $|\langle x,y\rangle|^2$, we only need to take account of the radial part of the Laplacian, i.e.

$$\frac{1}{A(r)} \frac{\partial}{\partial r} \left( A(r) \frac{\partial}{\partial r} \right), \tag{30}$$

where $A(r) = c' \sin^{d(m-2)}(r/\sqrt{2}) \sin^{d-1}(\sqrt{2}r)$ (see [Rag71, p. 168]). Since we want to calculate the radial part of the Laplacian of functions of the form $f(\cos(\sqrt{2}r))$, we will make a change of variable $t = \cos(\sqrt{2}r)$ in (30). We proceed with the details taking into account these basic identities:

$$\sin(\theta/2) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}, \quad \sin(\arccos(x)) = \sqrt{1-x^2}.$$

$$\begin{aligned} A(r) &= c' \sin^{d(m-2)}(\sqrt{2}r/2) \sin^{d-1}(\sqrt{2}r) \\ &= c'(1-\cos(\sqrt{2}r))^{\frac{d(m-2)}{2}} \sin^{d-1}(\arccos(t)) \\ &= c'(1-t)^{\frac{d(m-2)}{2}}(1-t^2)^{\frac{d-1}{2}} = c'(1-t)^{\frac{d(m-1)-1}{2}}(1+t)^{\frac{d-1}{2}}. \end{aligned}$$

Now the radial part of the Laplacian can be written also in the variable $t$ and it turns out to be:

$$\begin{aligned} &\frac{1}{A(r)}\frac{\partial}{\partial r}\left(A(r)\frac{\partial}{\partial r}\right) \\ &= c'(1-t)^{-\frac{d(m-1)-2}{2}}(1+t)^{-\frac{d-2}{2}}\frac{\partial}{\partial t}\left((1-t)^{\frac{d(m-1)}{2}}(1+t)^{\frac{d}{2}}\frac{\partial}{\partial t}\right). \end{aligned}$$

Thus, defining

$$\alpha = \frac{d(m-1)-2}{2}, \quad \beta = \frac{d-2}{2},$$

we get that the radial part of the Laplacian is of the form

$$c'(1-t)^{-\alpha}(1+t)^{-\beta}\frac{\partial}{\partial t}\left((1-t)^{\alpha+1}(1+t)^{\beta+1}\frac{\partial}{\partial t}\right).$$

It is well known (see [Sze39]) that the precise eigenfunctions of this operator are the Jacobi polynomials $P^{(\alpha,\beta)}(t)$ with eigenvalues $-k(k+\alpha+\beta+1) = -k(k+dm/2-1)$.

Observe that since the polynomials are dense in $L^2(\mathbb{KP}^{m-1})$,

$$L^2(\mathbb{KP}^{m-1}) = \bigoplus_{l\geq 0} \text{Harm}(2l),$$

For further details check [Rag72, Page 87]. Therefore, we know that all the eigenvalues of $\Delta_{\mathbb{KP}^{m-1}}$ are of the form $-k(k+dm/2-1)$. A simple calculation shows that the spaces $E_L$ in the projective spaces are identified with the space of spherical harmonics (of the projective spaces) with degree less than $L$. More precisely,

$$E_L = \text{Harm}_{L^*} = \bigoplus_{l=0}^{[L^*/2]} \text{Harm}(2l) = \text{Pol}_{L^*},$$

where $L^* = \sqrt{(dm/2-1)^2+4L^2} - (dm/2-1) > 0$ for $L > 0$ (note that $\frac{L^*}{2L} \to 1$, as $L \to \infty$). Therefore, $E_L$ satisfies the product property (20) because the spaces $\text{Pol}_{L^*}$ verify it. As a consequence, the projective spaces $\mathbb{KP}^{m-1}$ are admissible.

**Other examples.**

Another example with a different nature is the Torus, represented as the unit rectangle $[0,1]\times[0,1]$ with the identification $(x,y)\sim(x,y+1)$ and

$(x, y) \sim (x + 1, y)$.
The eigenfunctions of the Laplacian are of the form $e^{2\pi i(mx+ny)}$ with $m, n \in \mathbb{N}$. Now we are ready to prove the product property. Let $f_1 \in E_L$, i.e. $f_1$ is a linear combination of eigenvectors of eigenvalues less than $L^2$, i.e. we are taking pairs $(n, m)$ such that

$$4\pi^2 \left(n^2 + m^2\right) \leq L^2,$$

and let $f_2$ be a linear combination of eigenvectors of eigenvalue less than $\epsilon^2 L^2$ $(0 < \epsilon < 1)$, i.e. we are taking pairs $(k, l)$ such that

$$4\pi^2 \left(k^2 + l^2\right) \leq \epsilon^2 L^2,$$

We can compute the product of $f_1$ and $f_2$:

$$f_1(x, y) f_2(x, y) = \sum_{n,m,k,l} c_{n,m} d_{k,l} e^{i2\pi((n+k)y + x(m+l))}.$$

Thus, we have eigenvalues

$$V^2 := 4\pi^2 \left((n + k)^2 + (m + l)^2\right).$$

We will estimate $V$ by computing $(a + b)^2$ and using the fact that

$$\begin{cases} n, m \leq \frac{L}{2\pi} \\ k, l \leq \frac{\epsilon L}{2\pi} \\ \sqrt{1 + x} \leq 1 + x/2, \quad \forall x \geq 0 \end{cases}.$$

Then we get that $V^2 \leq L^2(1+\epsilon^2+4\epsilon) \leq L^2(1+5\epsilon)$. Hence, $V \leq L\sqrt{1 + 5\epsilon} \leq L(1 + 5/2\epsilon)$. Therefore, a Torus is admissible.

Similar computations show that the Klein bottle is also admissible.

**Product of admissible manifolds.**

More examples can be constructed by taking products of manifolds that satisfy the product assumption because if $f_1$ and $f_2$ are functions defined on two manifolds $M$ and $N$, respectively, then

$$\Delta_{M \times N}(f_1 \cdot f_2) = f_2 \Delta_M f_1 + f_1 \Delta_N(f_2).$$

More precisely, let $M$ and $N$ be admissible manifolds, i.e.

$$E_L^M \cdot E_{\epsilon L}^M \subset E_{L(1+C_1\epsilon)}^M,$$

and

$$E_{mL}^N \cdot E_{\epsilon L}^N \subset E_{L(1+C_2\epsilon)}^N,$$

where

$$E_L^M = \langle \big\{\phi_i; \quad \Delta_M \phi_i = -\lambda_i^2 \phi_i, \lambda_i \leq L\big\}\rangle,$$

and

$$E_L^N = \langle \big\{\psi_i; \quad \Delta_N \psi_i = -\mu_i^2 \psi_i, \mu_i \leq L\big\}\rangle.$$

Thus, if we consider the product manifold $M \times N$, then

$$E_L^{M\times N} = \langle \{\phi_i \psi_j; \quad \lambda_i^2 + \mu_j^2 \le L^2\} \rangle,$$

It is a straightforward computation that $M \times N$ satisfies the condition of admissibility:

$$E_L^{M\times N} \cdot E_{\epsilon L}^{M\times N} \subset E_{L(1+C\epsilon)}^{M\times N}$$

with $C = 2\max(C_1, C_2)$.

*Remark* 9. Note that the example of the torus can be reduced to this later case because it is the product of two $\mathbb{S}^1$.

## References


[BB08] R. Berman and S. Boucksom, *Equidistribution of Fekete points on complex manifolds*, Preprint available at http://arxiv.org/abs/0807.0035, 2008.

[Blü90] M. Blümlinger, *Asymptotic distribution and weak convergence on compact Riemannian manifolds*, Monatsh. Math. **110** (1990), no. 3-4, 177–188. MR 1084310 (92h:58033)

[DS67] N. Dunford and J. T. Schwartz, *Linear Operators. parts 1 and 2*, Interscience, New York, 1967, Third and fourth printing.

[FM10a] F. Filbir and H. N. Mhaskar, *Marcinkiewicz-zygmund measures on manifolds*, Preprint available at http://arxiv.org/abs/1006.5123, 2010.

[FM10b] ______, *A quadrature formula for diffusion polynomials corresponding to a generalized heat kernel*, J. Fourier Anal. Appl. **16** (2010), no. 5, 629–657. MR 2673702

[GP11] D. Geller and I. Z. Pesenson, *Band-limited localized Parseval frames and Besov spaces on compact homogeneous manifolds*, J. Geom. Anal. **21** (2011), no. 2, 334–371. MR 2772076 (2012c:43013)

[Hör68] L. Hörmander, *The spectral function of an elliptic operator*, Acta Math. **121** (1968), 193–218. MR 0609014 (58 #29418)

[Hör69] ______, *On the Riesz means of spectral functions and eigenfunction expansions for elliptic differential operators*, Some Recent Advances in the Basic Sciences, Vol. 2 (Proc. Annual Sci. Conf., Belfer Grad. School Sci., Yeshiva Univ., New York, 1965–1966), Belfer Graduate School of Science, Yeshiva Univ., New York, 1969, pp. 155–202. MR 0257589 (41 #2239)

[Lan67a] H. J. Landau, *Necessary density conditions for sampling and interpolation of certain entire functions*, Acta Math. **117** (1967), 37–52. MR 0222554 (36 #5604)

[Lan67b] ______, *Sampling, data transmission and the nyquist rate*, Proc. IEEE **55** (1967), no. 10, 1701–1704.

[Mar07] J. Marzo, *Marcinkiewicz-Zygmund inequalities and interpolation by spherical harmonics*, J. Funct. Anal. **250** (2007), no. 2, 559–587. MR 2352491 (2008k:33040)

[Mar08] ______, *Sampling sequences in spaces of bandlimited functions in several variables*, Ph.D. thesis, Universitat de Barcelona, 2008.

[MOC10] J. Marzo and J. Ortega-Cerdà, *Equidistribution of Fekete points on the sphere*, Constr. Approx. **32** (2010), no. 3, 513–521. MR 2726443

[OCP11] J. Ortega-Cerdà and B. Pridhnani, *Carleson measures and logvinenko-sereda sets on compact manifolds*, Forum Mathematicum (2011), To appear.

[Rag71] D. L. Ragozin, *Constructive polynomial approximation on spheres and projective spaces.*, Trans. Amer. Math. Soc. **162** (1971), 157–170. MR 0288468 (44 #5666)

[Rag72] ______, *Uniform convergence of spherical harmonic expansions*, Math. Ann. **195** (1972), 87–94. MR 0294964 (45 #4032)

[Sha01] O. Shatalov, *Isometrics embeddings $\ell_2^m \to \ell_p^n$ and cubature formulas over classical fields*, Ph.D. thesis, 2001.

[Sog87] C. D. Sogge, *On the convergence of Riesz means on compact manifolds*, Ann. of Math. (2) **126** (1987), no. 2, 439–447. MR 908154 (89b:35126)

[Sze39] G. Szegö, *Orthogonal Polynomials*, American Mathematical Society, New York, 1939, American Mathematical Society Colloquium Publications, v. 23. MR 0000077 (1,14b)

[You80] R. M. Young, *An introduction to nonharmonic Fourier series*, Pure and Applied Mathematics, vol. 93, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1980. MR 591684 (81m:42027)

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