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OPTIMAL PORTFOLIOS AND PRICING MODELS

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Abstract

This final degree project aims to introduce the bases of portfolio theory in order to understand mathematical and economic foundations which are used in optimal portfolios models. So it will be seen the models of Markowitz, Sharpe, the Capital Asset Pricing Model and the Arbitrage Pricing Theory in a theoretical way and in a practical case, so all the models can be embraced.

Resum

Aquest treball de final de grau pretén introduir els fonaments de la cartera de valors a fi d'entendre el seu rerefons económic i matemàtic i veure com s'aplica dins els models d'optimització de valors. Aixídoncs, es mostraran els models de Markowitz, Sharpe, el Model de Valoració dels Actius Financers i la Teoria de l'Arbitratge de forma teórica i práctica, de forma que els models es puguin consolidar.

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Chapter 1

Introduction

Optimization of resources and finding the optimal investment are recurrent issues in economic history and in our day-to-day.

However, this optimisation cannot be realised without rigorous, clear and objective definitions or without basic hypotheses; we have to know what we are trying to optimise, what information we have and what properties have the optimal objects.

All the same, when we take an abstract theory to its application in a social science, some of the hypothesis can be broken or we have to estimate some parameters that the abstract model consider as known; so the models do not fit completely with its application, and we have to validate if, despite that, the model explains sufficiently the reality.

So, this work is useful as an introduction to portfolios and some basic models that have tried to optimise them, from an economic and mathematical way. We are going to define the models with accuracy, explaining why they would optimise the portfolio if the hypotheses meet, if they apply in reality and some of their principal critics.

Markowitz (1952) was the first to substantiate the bases of modern portfolio theory, considering it as an optimization problem of the trade-off between risk and return with some restrictions.

Through diversification theory we observe that we can reduce the risk without reducing the return by combining assets with some characteristics, so we can understand why we are trying to find an optimal portfolio instead of an optimal asset.

Altering Markowitz's model with other hypotheses, we introduce the Capital Asset Pricing Model, a model which uses simpler calculations and introduces concepts as risk premium, two-fund theorem or one-fund theorem, as well as the principal critics of the model.

Once the models are explained, we are going to propose a #C program which optimises a portfolio given its information and we are going to solve a case using the different models.

In the conclusion we are going to show the most relevant results and we are going to recommend other models and themes that can complement this work. We have to emphasise that this work serves as an introduction to portfolios and to understand the rigour of the explained models, however, they exist multitude of models, as the APT, that depart from other hypotheses, that can further the theme, but they need detailed explanations too.

Every correctly formulated model allows for explaining the reality if their hypotheses meet. The difficult thing is finding a simple and easy model which can explain reality in a global way. For this reason, the existence of other models, with other hypotheses, do not annul the models explained, but they complement them and they allow us to understand portfolios in other standpoints.

Chapter 2

Previous notions

In this chapter we introduce fundamental bases that allow us to understand the main points of this work. It is useful to know its notation and the principal concepts of the work, as long as it simplifies the main text, because we are going to use the prepositions and lemmas of Previous Notions to make demonstrations in the project.

2.1 Finance notions

In this section we are going to define the principal financial assets and the way they are valued.

A financial instrument that gives to its buyer the right to receive future incomes from the seller is known as financial asset. In this work we are going to consider financial assets and securities as synonyms.
From this set we can find fixed income assets and equity assets as the most important subsets, which we are going to explain below:

- Fixed income assets: Financial assets which all its maturity and flow charts (Timeline of the flows, not the amounts) are known at the beginning of the operation. Some important fixed income assets are treasury bills, bonds and debentures. The principal differences are shown in the table below:

Type	**Maturity**	**Incomes**
Bills	≤ 1 year	One income at maturity
Bonds	Between 1 and 5 years	More than an income
Debentures	>5 years	More than an income

  We notice that the interest that the buyer will receive can depend on a external value that can be variable as the EURIBOR of a determined maturity. However, in case the interest rate is fixed we know a minimal return at the beginning of the operation.

- Equity assets: Financial assets which maturity and flow charts are unknown. The most important equity assets are the stocks.

Once we know the definitions of financial assets, we have to know how is calculated their value, in order we can compare them and define a return between what we have payed and what we have charged.
In all cases the values V_t of financial assets in a time t are obtained as the discount of the future cash flows C_s, either known or expected, by a periodic interest rate k, which is the market discount rate. If we consider a set of times I where are expected some payments or incomes, then we have:

$$V_t = \sum_{s \in I} \frac{C_s}{(1+k)^s}, \tag{2.1}$$

where s $\geq$ t, is the time it lasts to arrive from t to the moment the cash flow will be payed.

From here we see that the value of a security depends of the cash flows, the time in which they are reached and the market discount rate, so a change in any of this values will affect to the value of the security.

- As long as time passes, some of the cash flows will be charged and the time to reach other cash flows will be reduced, so the value will change constantly.
- If the entity has economic problems and can not pay the expected cash flows, then the expected cash flows have to be reduced, and then the security value is reduced too. This is called credit risk.
- If the market discount rate increases, that would mean that the interest offered by the market increases, then the security would be comparably worse, so its value would be less. The risk that the market discount rate reduces the value of a security is called market risk.

So the value of a security changes constantly, however if we consider a fixed income asset with all its incomes periodic and with fixed rates, and without credit and market risks, we can calculate r the minimal expect return that the buyer would have if he keeps the asset until its maturity.
Let be C the capital payed to have the financial asset, I the periodic effective interest given to the buyer, the periodic incomes of the financial asset I $\cdot$ C, in periods 1,...,n and a extra income of C in period n. Then we define r, this minimal periodic expected return as the solution of:

$$C = \sum_{i=1}^{n} \frac{C \cdot I}{(1+r)^i} + \frac{C}{(1+r)^n}.$$

It is easy to see that in this case r=I, so it is a good expected return that has implicit the different value of money in time.
In the same way, we can define the expected return of a financial asset with V_0 as the capital payed for having the asset, J the set of times measured in relation of periods from 0 where a expected income C_t is given, then we can define a return r as the solution of:

$$V_0 = \sum_{i \in J} \frac{C_t}{(1+r)^t}.$$

We can consider the expected return of an asset as a percentage of interest payed additionally to the original price, in order to buy an asset; so we are going to define some properties about interests and their bases.

Let be C an original amount which defines the interests of a operation, C $\cdot$ I, where I is constant and represents a percentage of the amount.
If these interests are paid and they are not aggregated to the original amount, it is called a simple interest operation. On the other hand, if the interests are aggregated to the original amount, it is called a composed interest operation.

In order to we can compare effective interests of different periods, we can use the equalities we have using composed interest bases:
Let be I_m the effective interest of an asset which has periodic m incomes per year; let be I_k the effective interest of this asset if it has k periodic incomes per year; then:

$$(1 + I_m)^m = (1 + I_k)^k.$$

However, for simple operations or for most of operations which have a maturity t fewer than a year, simple interest bases are used. In this bases,the interests are not reinvested increasing the amount from which interests are calculated. Let be i the nominal interest of a operation, C the nominal paid at the beginning of the operation and C′ the final income you receive, then

$$C' = C + C \cdot i.$$

From here we obtain that the nominal interest can be calculated as:

$$i = \frac{C'}{C} - 1.$$

Nevertheless, i corresponds a benefit produced in a maturity of t days, so in order we can compare two nominal interests with different maturities, we can consider them as effective interests with period $\frac{t}{360}$. So we would have

$$i = i_{\frac{360}{t}}.$$

$$I_{\frac{360}{t}} = \frac{i_{\frac{360}{t}}}{\frac{360}{t}}$$

and we can compare them as effective interests[1].
We are going to compare them as I_1, so from the equations above we would have

$$I_1 = (1 + \frac{t \cdot (\frac{C'}{C} - 1)}{360})^{\frac{360}{t}} - 1.$$

A combination of some assets is defined as a portfolio. It will be the main theme of this project, its study and some models that try to find the optimal portfolio given some

[1] We have used the convention ACTUAL/360, because we are going to consider treasury bills as the risk-free assets to consider.

assumptions. Let be A a portfolio, T the total investment in the portfolio A and T_i the investment in asset i, then we define the weight of the asset i in the portfolio A as:

$$X_i = \frac{T_i}{T}.$$

In order we can negotiate with more assets without the restriction of a total investment amount T in the portfolio, we can use short sells, which is the sale of an asset which the seller does not own.
This process works as follows: the short seller borrows an asset which sells at the market and then he repurchases it in order to give it back to the lender, who lends the asset for a commission. The short seller has to return the asset, with independence of its value. The relation between the price at which he sold it and the price at which he repurchased it determines the profitability of the short sell.
One example could be as follows: The short seller borrows a stock from an entity at an initial price of x (which does not pay at the moment) and puts it at the market at the same price. How he has to return the asset, he repurchases it on time t for price y, and then returns the asset to the entity. Then, if y<x the short seller would have a benefit, so he would have bought it cheaper than he sold, and if y>x he would have a loss.
We are going to see that we are not going to consider short sells. However, we introduce them because it can be studied in other more specific projects, which do not have such an introductory objective as our project.

2.2 Diversification theory

Diversification theory shows how we can achieve a financial instrument with less risk by combining some financial assets so the reduction of the risk is bigger than the reduction of the return.
This theory considers the returns of financial assets as random variables, so first of all we remember some basic properties.

Remark 2.1. The expected return of a portfolio A consisting of the assets $A_1,...,A_n$ with weights $X_1,...,X_n$ is the weighted mean of the expected return of its assets, if the expected returns are finite.

Remark 2.2. Let be Σ the covariances matrix of the portfolio A consisting of the assets $A_1,...,A_n$, with finite first and second ordinay moments, and weights $X_1,...,X_n$; let be $\sigma_{i,j}$ the covariances between r_{A_i} and r_{A_j} for i,j $\in$ {1,...,n}x{1,...,n}, X the array with weights $X_1,...,X_n$; $r_{A_1},...,r_{A_n}$ the expected returns of the assets and r the return of the portfolio. Then

$$Var(r) = \sum_{i=1}^{n}\sum_{j=1}^{n} X_i \cdot X_j \cdot \sigma_{i,j} = \sum_{i=1}^{n} X_i^2 \cdot \sigma_{i,i} + 2 \cdot \sum_{i=1}^{n}\sum_{j>i} X_i \cdot X_j \cdot \sigma_{i,j} = X^T \cdot \Sigma \cdot X.$$

Remark 2.3. Let be P a portfolio of n assets, with positive weights $X_1,...,X_n$, which expected returns have finite variances and covariances $\sigma_{i,j}$, i,j $\in$ {1,...,n}x{1,...,n}. Let be

$\sqrt{\sigma_{i,i}}$=σ_i the standard deviation of the i^{th} asset, let be σ_P=$\sqrt{Var(r_P)}$ the standard deviation of the expected return of the portfolio P. Then

$$\sigma_P \leq \sum_{i=1}^{n} X_i \cdot \sigma_i \text{ and } \sigma_P = \sum_{i=1}^{n} X_i \cdot \sigma_i \Leftrightarrow \rho_{i,j} = 1, \forall i, j$$

where $\rho_{i,j}$=$\frac{\sigma_{i,j}}{\sigma_i \cdot \sigma_j}$.

Proof. We see when the standard deviation of the portfolio is the weighted mean of the standard deviations of the assets in the portfolio. How $\sigma_{i,i} \geq 0$ for i=1,...,n, we can consider $\sqrt{\sigma_{i,i}}$:=σ_i.
If the standard deviation of the portfolio is the weighted mean of the standard deviations of the assets in the portfolio, then:

$$\begin{aligned} Var(r_P) &= (\sum_{i=1}^{n} X_i \cdot \sigma_i)^2 \\ &= \sum_{i=1}^{n} X_i^2 \cdot \sigma_{i,i} + 2 \cdot \sum_{i=1}^{n} \sum_{j>i} X_i \cdot X_j \cdot \sigma_i \cdot \sigma_j \end{aligned}$$

$$Var(r_P) = \sum_{i=1}^{n} \sum_{j=1}^{n} X_i \cdot X_j \sigma_{i,j} = \sum_{i=1}^{n} X_i^2 \cdot \sigma_{i,i} + 2 \cdot \sum_{i=1}^{n} \sum_{j>i} X_i \cdot X_j \cdot \sigma_{i,j}.$$

Combining both equations, we obtian

$$\sum_{i=1}^{n} \sum_{j \neq i} X_i \cdot X_j \cdot (\sigma_{i_j} - \sigma_i \cdot \sigma_j) = 0.$$

We have that $X_i \cdot X_j \geq 0$ $\forall$ i,j because X_i,X_j>0, and ($\sigma_{i,j}$-$\sigma_i \cdot \sigma_j$) ≤ 0 due to Cauchy-Schwarz inequality.
So the equality holds only if all the addends are 0, so $\sigma_{i,j}$=$\sigma_i \cdot \sigma_j$ and $\rho_{i,j}$=$\frac{\sigma_{i,j}}{\sigma_i \cdot \sigma_j}$=1 for every i,j $\in$ {1,...,n}x{1,...,n}.
In case it exists one or more assets with $\rho_{i,j}$<1, then $\sigma_{i,j}$<$\sigma_i \cdot \sigma_j$ and

$$\begin{aligned} Var(r_P) &= \sum_{i=1}^{n} X_i^2 \cdot \sigma_{i,i} + 2 \cdot \sum_{i=1}^{n} \sum_{j>i} X_i \cdot X_j \cdot \sigma_{i,j} \\ &< \sum_{i=1}^{n} X_i^2 \cdot \sigma_{i,i} + 2 \cdot \sum_{i=1}^{n} \sum_{j>i} X_i \cdot X_j \cdot \sigma_i \cdot \sigma_j = (\sum_{i=1}^{n} X_i \cdot \sigma_i)^2. \end{aligned}$$

□

Remark 2.4. Let be X_1,...,X_n the weights of a portfolio P consisting of n assets. Let be r_P the expected return of the portfolio, $\sigma_{i,j}$ the covariance between the returns of assets i and j; i,j $\in$ {1,...,n}x{1,...,n}.Then the variance of the return of the portfolio can be reduced by increasing its size.

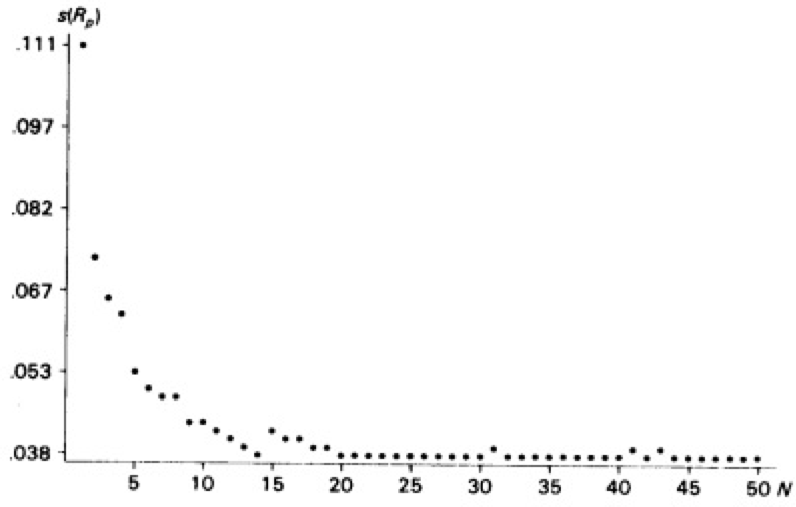

Figure 2.1: From Foundations of Finance of Eugene Fama

$$Var(r_P) = \sum_{i=1}^{n} X_i^2 \cdot \sigma_{i,i} + 2 \cdot \sum_{i=1}^{n} \sum_{j>i} X_i \cdot X_j \cdot \sigma_{i,j}.$$

We consider a portfolio with weights $X_i = \frac{1}{n}$, so in this case, we have:

$$\begin{aligned} Var(r_P) =& \frac{1}{n^2} \sum_{i=1}^{n} \sigma_{i,i} + \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j \neq i} \sigma_{i,j} \\ =& \frac{1}{n} \cdot \overline{\sigma_{i,i}} + \frac{n-1}{n} \cdot \overline{\sigma_{i,j}}, \end{aligned}$$

where

$$\begin{aligned} \overline{\sigma_{i,i}} =& \frac{\sum_{i=1}^{n} \sigma_{i,i}}{n}. \\ \overline{\sigma_{i,j}} =& \frac{\sum_{i=1}^{n} \sum_{j \neq i} \sigma_{i,j}}{n \cdot (n-1)}. \end{aligned}$$

Here we can see that the weight of the variances tends to 0 as long as the size increases, and the variance tends to the mean of the covariances. In the figure 2.1, it can be seen the result of a study by Eugene Fama where appears the deviation of different portfolios consisting of N different assets.It is easy to see that the deviation tends to be lower as long as N increases, and that for big N's the variation of deviations is fewer.

So we have seen that it exists a part of the variance of the return that can be reduced by investing in a greater number of assets. However, this increase in the number of assets will reduce the risk only if the correlation between all assets is not 1. The introduction of new assets affects also in the expected return of the portfolio, so we have to consider new assets with similar expected returns and variances as the assets which are already in our portfolio so we can construct a new portfolio with a similar expected value of the return and less variance.

As it is shown in the figure 2.1, an increase of the number of assets could not suppose a reduction of the risk, for this reason we have to know the relation between the assets, so we do not chose assets with correlations close to 1.
However, other studies of Fama show that the process of diversification hardly reduces risk if the market does not have stability. A intuitive way to see it is as follows, in times of recession, the expectations for most of the companies will be negative, so most of them will have a drop of their returns, so the correlation of most of them will approach to 1 and vice versa in times of continuous expansion.

Then, diversification is a good way to reduce risk, but we have to know the relation between assets and the market has to have some stability.

2.3 Utility theory

In case we have uncertainty, we can apply to utility theory in case we want to decide which is the better option considering our personal preferences. In this chapter we consider some of their principal options.

Definition 2.5. We denote a simple gamble which an individual has a probability α of receiving outcome x and a probability of 1-α of receiving outcome z as G(x,z;α)

Definition 2.6. (Von Neumann and Morgenstern's axioms) Let be S a set of events ,the sign $\succ$ an order relation, the sign $\sim$ a equivalence relation, (+,$\cdot$) internal operations in the set, p_1,...,p_n the probabilities of occurrence of events B_1,...,B_n, $\sum_{i=1}^{n} p_i$=1; then we can define a complex event B as the event consisting of the possibility of occurring B_i with probability p_i for i=1,...,n. Then the following properties are known as Von Neumann and Morgenstern's axioms:

1) Comparability (Completeness): x $\succ$ y or y $\succ$ x or x $\sim$ y $\forall$ x,y $\in$ S.

2) Transitivity (Consistency): If x $\succ$ y and y $\succ$ z $\Rightarrow$ x $\succ$ z. If x $\sim$ y and y $\sim$ z $\Rightarrow$ x $\sim$ z.

3) Continuity (Measurable) Let be x,y,z $\in$ S such that x $\succ$ y $\succ$ z. Then, it exists an unique $\alpha \in$ (0,1) such that y $\sim$ G(x,z,α); if x $\sim$ y $\succ$ z then y $\sim$ G(x,z,1) and if

$$x \succ y \sim z$$

then y $\sim$ G(x,z,0).

4) Independence: If x $\succ$ y, then $\forall$ p $\in$ [0,1], $\forall$ z $\in$ S, G(x,z,p) $\succ$ G(y,z,p).

Definition 2.7. *Given a set S, given a utility function u: S $\longrightarrow \mathbb{R}$. Let be G any gamble with events in S, then:*

- *A individual is risk adverse if u(E(G))>E(u(G)), so he prefers to do not make the gamble.*
- *A individual is risk neutral if u(E(G))=E(u(G)), so he has the same utility making the gamble that without making it.*
- *A individual is risk loving if E(u(G))>u(E(G)), so he prefers to make the gamble.*

Under conditions of uncertainty, the preferences of the individual impact on which is the best option for him. In order we can consider the preferences of an individual, we assume that Von Neumann and Morgenstern's axioms hold, so it exists an utility function that can measure its preferences.

Theorem 2.8. (Von Neumann and Morgenstern's utility theorem) *For any individual and set satisfying Von Neumann and Morgenstern's axioms, it exists a function u: $S \longrightarrow \mathbb{R}$ such that for any two gambles G_1, G_2; $G_1 \succ G_2 \Leftrightarrow u(E(G_1)) \succ u(E(G_2))$.*

Proof. We consider only the case of #{x $\in$ S}< ∞. Let be x_1,...,x_n the different possible outcomes.
By the comparability axiom, we can order the events so $x_1 \succ ... \succ x_n$. If all events were equivalents, the utility function would be constant, so we consider the case that at least two events are not equivalent.
By the continuity axiom, we can find $\alpha_i \in$ [0,1] such that $x_i \sim$ G(x_n,x_1,α_i).
We define the utilities of the events as u(x_i)=α_i.
If we consider a complex gamble G_1 consisting of the possibility of occurrence of x_1,...,x_n with probabilities p_1,...,p_n, then u(G_1)=$\sum_{i=1}^{n} p_i \cdot$ u(x_i)=$\sum_{i=1}^{n} p_i \cdot \alpha_i$, by the independence axiom.
By the continuity axiom again, we have that $x_i \sim$ G(x_n,x_1,α_i),
so $G_1 \sim$ G(x_1,x_n,$\sum_{i=1}^{n} p_i \cdot \alpha_i$)=G($x_1$,$x_n$,u($G_1$)).
So in gamble G_1 we win the best event with probability u(G_1) and the worst event with probability 1-u(G_1).
Then it is clear that if G_1,G_2 are two gambles, then u(G_1) $\succ$ u(G_2) $\Leftrightarrow G_1 \succ G_2$. □

2.4 Covariance matrix properties

Let be Σ the covariance matrix of a portfolio of n risky assets. Then it is obvious that it is a symmetric matrix, so Σ^T=Σ, but we can see more interesting properties about the covariance matrix.

Definition 2.9. *Let be A a matrix nxn which has all its elements in $\mathbb{R}$, we define it as a positive-definite matrix if, and only if, $\forall$ $x \in \mathbb{R}^n \setminus 0$, $x^T \cdot A \cdot x$ >0.*

Remark 2.10. If A is a nxn positive-definite matrix, then it has an inverse A^{-1}, and A^{-1} is a positive-definite and symmetric matrix.

Remark 2.11. Let be (Ω,P,$\mathcal{A}$) a probabiliy space, Σ the covariance matrix of a portfolio with n risky assets A_1,...,A_n; r_{A_1},...,r_{A_n} its expected returns, x^T=(x_1,...,x_n) $\in \mathbb{R}^n \setminus 0$, then Σ is a nxn positive-definite matrix.

Proof. If we consider the product $x^T \cdot \Sigma \cdot$ x:

$$x^T \cdot \Sigma \cdot x = Var(\sum_{i=1}^{n} x_i \cdot r_{A_i}) > 0.$$

□

Remark 2.12. (Factorization of Cholesky) Given a symmetric and definite-positive matrix A, it exists a triangular inferior matrix L so

$$A = L \cdot L^T$$

where L^T is the transposed matrix of L.

Through this remarks we can save the covariance matrix Σ like a triangular matrix, and calculate its inverse as a triangular matrix, having to save memory only for $\frac{n^2+n}{2}$ elements, instead of n^2 elements.

Chapter 3

Markowitz's model

Harry Markowitz in 1952 wrote Portfolio Selection, which would be the basis of the modern portfolio theory.Since most of portfolio theory departs from Markowitz's model or it uses some of their hypothesis, we have to know and explain this model with some accuracy in order to understand the assumptions of the model and its limitations.

3.1 Classical model

[1] Harry Markowitz considered the need of two stages in order to select a portfolio. The first stage is the observation and experience so we can have relevant beliefs about future securities; the second stage starts with the relevant beliefs and ends with the choice of portfolio.However, he only developed the second stage, considering the future beliefs as known.
The hypotheses of the model were:

- The investor considers expected return as a desirable thing, so he wants to maximise it.
- The investor considers variance as an undesirable thing, so he wants to minimise it.
- Diversification is both observed and sensible. A rule of behaviour which does not imply the superiority of diversification must be rejected.
- Expected returns are random variables.
- The investor and the assets accomplish Von Neumann and Morgenstern's axioms.
- To simplify, they are consider static probability beliefs and short sales are not allowed.

Implicitly there are considered this hypotheses so the model has sense:

[1]In this section we have obtained the information from Portfolio Selection of Harry Markowitz.

- The assets are perfectly divisible, so you can invest any amount you want in an asset.
- The market is complete, so there are no transaction costs all possible combinations of assets are well defined.
- It exists perfect information.

Through this hypothesis, he constructs a model to look for optimal portfolios for simple cases and then describes them in a geometrical way. After we explain his work, we are going to generalise it for greater cases.

Remark 3.1. The model which only considers that the investor should maximise discounted return must be rejected.

Proof. Suppose that there are N securities, let $r_{i,t}$ be the anticipated return at time t per monetary unit invested in security i; let $d_{i,t}$ be the rate of discount of security i to the present; let X_i be the relative amount invested in i. We exclude short sales, so $X_i \geq 0$ for i=1,...,N.
Then the expected return of the portfolio is:

$$R = \sum_{t=1}^{\infty} \sum_{i=1}^{N} d_{i,t} \cdot r_{i,t} \cdot X_i = \sum_{i=1}^{N} X_i \cdot (\sum_{t=1}^{\infty} d_{i,t} \cdot r_{i,t}).$$

If we name R_i=$\sum_{t=1}^{\infty} d_{i,t} \cdot r_{i,t}$, then we have that R=$\sum_{i=1}^{N} X_i \cdot R_i$; how $X_i \geq 0$ and $\sum_{i=1}^{N} X_i$=1, it is a weighted mean of R_i. In order to optimise R we would catch the asset having the maximum R_i or a combination of assets having that maximum. In no case is a diversified portfolio preferred to all non-diversified portfolios, so we have to reject this model. □

Then he considers a model which can fit with diversification theory. He a model which tries to maximise expected value and minimise variance.
First of all, he defines μ_i as the expected return of the i^{th} asset, and $\sigma_{i,j}$ the covariance between asset i and asset j for (i,j) ∈ {1,...,N}x{1,...,N}, with N the number of assets; and X_i as the weight in asset i.

So for fixed (μ_i, $\sigma_{i,j}$), the investor has different choices of portfolios, altering the weights invested in each asset. The set of all possible portfolios is defined as the attainable set.

In a similar way, we can define the frontier set as a set of portfolios in which all of them have minimal variances for its expected values. In the case that it is in the frontier set and it does not exist a portfolio with more expected value and the same variance, then we say that it is in the efficient set.

Given the preferences of the investor and given a efficient set, the investor can find the optimal portfolio if he wants to use (E-V) criteria and if we can find reasonable (μ_i, $\sigma_{i,j}$), so Markowitz shows the properties of this sets for N=3 and N=4.

In the case of N=3: Every point in the attainable set accomplishes:

1) E=$\sum_{i=1}^{3} X_i \cdot \mu_i$.
2) V=$\sum_{i=1}^{3} \sum_{j=1}^{3} X_i \cdot X_j \cdot \sigma_{i,j}$.
3) $\sum_{i=1}^{3} X_i$=1.
4) $X_i \geq 0$, $\forall$ i $\in$ {1,2,3}.

Where E is its expected value and V is its variance.
From point 3) we have

3') X_3=1-X_2-X_1.

So applying 3') into 1) and 2) we obtain (E,V) as functions of two parameters:

1') E=μ_3+$X_1 \cdot (\mu_1$-$\mu_3)$+$X_2 \cdot (\mu_2$-$\mu_3)$.
2') V=$X_1^2 \cdot (\sigma_{1,1}-2\sigma_{1,3}$+$\sigma_{3,3})$+$X_2^2 \cdot (\sigma_{2,2}-2\sigma_{2,3}$+$\sigma_{3,3})$+2$X_1 \cdot X_2 \cdot (\sigma_{1,2}$-$\sigma_{1,3}$-$\sigma_{2,3}$+$\sigma_{3,3})$+2$X_1 \cdot (\sigma_{1,3}$-$\sigma_{3,3})$ + 2$X_2 \cdot (\sigma_{2,3}$-$\sigma_{3,3})$+$\sigma_{3,3}$.

Through 1') and 2') is easy to see the form of the isomean curves and the isovariance curve, that is, the set of points with the same mean and the set of points with the same variance. Through 3') We can obtain the attainable set as a plot where the reference axis are X_1 and X_2. So we would have that the attainable set in this plot is a triangle which its vertex are (0,0),(0,1) and (1,0) where the first coordinate indicates X_1 and the second coordinate indicates X_2.
We can consider that it exists $\mu_i \neq \mu_j$ for i $\neq$ j. If they were all the same, all the possible combinations would have the same expected value and we would choose that with less variance. So we can consider, reordering the securities if necessary, that μ_2-$\mu_3 \neq 0$, and solving 1'):

$$X_2 = \frac{\mu_3 - E}{\mu_2 - \mu_3} + \frac{\mu_1 - \mu_3}{\mu_2 - \mu_3} \cdot X_1.$$

So the isomean curves are a system of parallel straight lines.
In the same way, it can be seen that the isovariance curves are a system of concentric ellipses where its center X is the point with minimum variance.

Let be $\widehat{E}$ the expected value in point X, and $\widehat{V}$ its variance. Given a expected value E' we consider X(E') the point of the isomean curve with less variance, so it is tangent to a isovariance curve. If we vary E', we obtain a straight line that we define as critical line, which increases its variance as long as it distances from X. All the points in the critical line that are in the attainable set are also in the efficient set. Then we can have the following cases:

- If X is in the attainable set, then X is in the efficient set, because it does not exist a point with less variance than $\widehat{V}$, and it is also in the critical line, because it minimises the variance for all points with expected value $\widehat{E}$.So the efficient set is formed by all the points in the critical line which are in the attainable set and that points of the boundary of the attainable set with less variance; as is shown in the figure 3.1, where X is the centre of the ellipses and l the critical line.

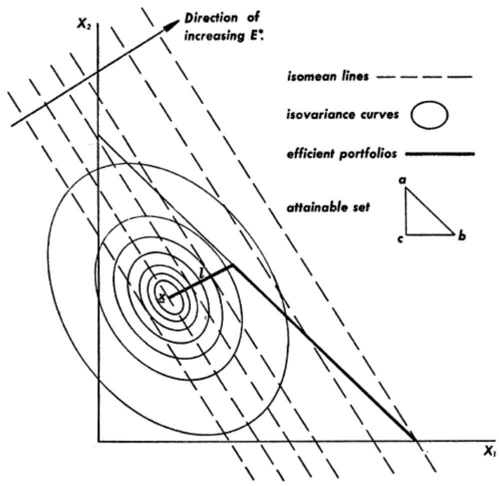

Figure 3.1: Efficient Set from Portfolio Selection of Harry Markowitz

- If X is not in the attainable set and the critical line cuts the attainable set, then the efficient set will include the point which minimises the total variance in the attainable set (if it is not in the boundaries of the attainable set); that points which are in the critical line and in the attainable set; and the boundary of the attainable set from the point which intersects with the critical line and has less variance. Or the boundaries of the attainable set with less variance and the critical line if the minimal variance point in the attainable set lies in one of the boundary lines. The figure 3.2 shows the second case.

- If X is not in the attainable set and the critical line does not cut the attainable set, then the efficient set would lie into one of the boundaries, so it would be a security that would not be in any efficient portfolio (in our case X_3).

In the case of N=4 we have:
As in the case N=3, we can reduce the problem one dimension using that

$$X_4 = 1 - X_3 - X_2 - X_1.$$

So the attainable set represented in the 3-dimensional space with axes X_1,X_2,X_3 would be the tetrahedron with vertexes (0,0,0),(1,0,0),(0,1,0),(0,0,1).
We can define $S_{a_1,\dots,a_k}$={x $\in \mathbb{R}^3$: X_i=0, i $\notin$ {a_1,...,a_k}}, $l_{a_1,\dots,a_k}$ as the critical line of the space $S_{a_1,\dots,a_k}$.
Then the efficient set would be constructed beginning at the point of minimum variance moving continuously through various critical lines until it intersects to a larger subspace (changing the critical line) or intersects to a boundary (and arrives to a fewer subspace); until it arrives at the point with maximum expected value.

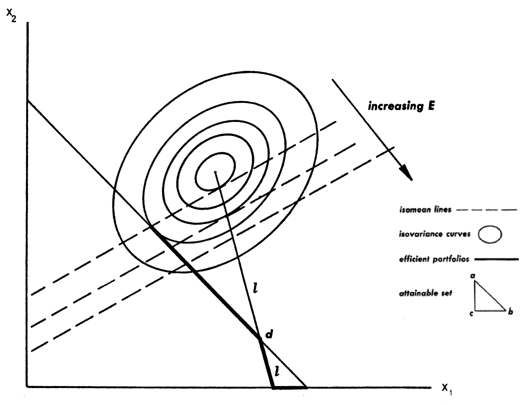

Figure 3.2: Efficient Set from Portfolio Selection of Harry Markowitz

3.2 Model for n risky assets

[2] Markowitz's model shows simple cases, however we can generalise his model for greater numbers. However, in order we can generalise, we have to add the following new condition to the conditions explained before:

- We consider that assets of the portfolio can have weights smaller than 0.

If we consider an asset with a negative weight, if it is already in our portfolio, this would mean that we would have to sell it in order to invest into other assets. If the asset is not in our portfolio, this would mean that we would have to do a short sell, so we expect a reduction of its future value, and with the future profit and the liability of the asset we can increase the investment into other assets.

However, we are going to consider only the case in which we dispose of all assets, so we are not considering short sells because of two reasons:

- The first reason to not introducing them is that in some countries exist some limitations in the amount in which you can make short sells, so it would be unrealistic in some cases.
- The second reason is because if we consider them, the return of an asset changes depending on if we use short sells or not. So if we make a short sell, we pay a commission and it reduces the return and we would have to consider different returns depending in the situation. That it would mean the calculation of different covariance matrix increasing considerably the difficulty of the optimisation problem.

Given a portfolio P consisting of n risky assets, we have to minimise the variance of P given a expected return μ_b, so we have to optimise V=X$' \cdot \Sigma \cdot$ X, where X is the arrow with

[2]In this section we have used some information of the study An Introduction to Portfolio Theory of Paul J.Atzberger.

the weights of the portfolio and Σ is its covariance matrix. We define μ as the array of the expected returns and e^T=(1,...,1).
With the restrictions:

$$\mu_b = \sum_{i=1}^{n} X_i \cdot \mu_i. \tag{3.1}$$

$$\sum_{i=1}^{n} X_i = 1. \tag{3.2}$$

From this equalities we can define two functions g and h, which represent the restrictions when their image is 0, as

$$g(X_1, ..., X_n) = X^T \cdot \mu - \mu_b.$$
$$h(X_1, ..., X_n) = X^T \cdot e - 1.$$

In order to simplify the optimisation we are going to consider the function

$$f(X_1, ..., X_n) = \frac{1}{2} \cdot X^T \cdot \Sigma \cdot X.$$

so as Σ is a symmetric matrix, we can obtain a simplier derivative and the optimisation does not change multiplying the function by an escalar distinct of 0. So we have to minimise the function f with the restrictions (3.1) and (3.2).
We notice that

- $f, g, h \in C^\infty$ $(\mathbb{R}^n)$.
- $\bigtriangledown$ g=μ.
- $\bigtriangledown$ h=e.
- $\bigtriangledown$ h and $\bigtriangledown$ g are linearly independent, so if not all the expected returns would be equal.

where $\bigtriangledown$ represents the gradient of the function.

Proposition 3.2. *(Lagrange multipliers) Let be $\Omega \subset \mathbb{R}^n$ an open set. Let be*

$$f, g_1, ..., g_k : \Omega \longrightarrow \mathbb{R}$$

functions of class C^1, M={$x \in \Omega$: $g_j(x)$=0,1 $\leq j \leq k$} and suppose that the following conditions meet:

1) $x_0 \in \Omega$.

2) $\bigtriangledown$ g_1 (x_0),$\bigtriangledown$ g_2 (x_0),...,$\bigtriangledown$ g_k (x_0) are linearly independent.

3) The restriction of f in M has a local extremum in x_0.

Then, $\exists$ μ_1,...,$\mu_k \in \mathbb{R}$ which accomplish

$$\bigtriangledown f(x_0) = \mu_1 \bigtriangledown g_1(x_0) + ... + \mu_k \bigtriangledown g_k(x_0).$$

We suppose that it exists a local extremum in M={x $\in \mathbb{R}^n$: g(x)=0,h(x)=0 }. Then the conditions to apply Lagrange multipliers hold and we can consider

$$F(X_1, ..., X_n, \lambda_1, \lambda_2) = \frac{1}{2} \cdot X' \cdot \Sigma \cdot X - \lambda_1 \cdot g(X_1, ..., X_n) - \lambda_2 \cdot h(X_1, ..., X_n).$$

We calculate the derivatives of F:

$$\frac{\partial F}{\partial \lambda_1} = g(X_1, ..., X_n).$$
$$\frac{\partial F}{\partial \lambda_2} = h(X_1, ..., X_n).$$
$$D_X F = \Sigma \cdot X - \lambda_1 \cdot \mu - \lambda_2 \cdot e.$$

And if we make the derivatives of F equal to 0 in order to find this extremum we have

$$X^T \cdot \mu = \mu_b. \tag{3.3}$$
$$X^T \cdot e = 1. \tag{3.4}$$
$$\Sigma \cdot X - \lambda_1 \cdot \mu - \lambda_2 \cdot e = 0. \tag{3.5}$$

We notice that D_X^2=Σ is a positive-definite matrix, then if it exists the solution to the Lagrange multipliers, then it will be a local minimum of f.
How Σ is a positive-definite matrix, it exists Σ^{-1}, so we can solve equation (3.5) in the following way

$$X = \lambda_1 \cdot \Sigma^{-1} \cdot \mu + \lambda_2 \cdot \Sigma^{-1} \cdot e. \tag{3.6}$$

Introducing equation (3.6) into equations (3.4) and (3.3), we have the following equalities

$$\begin{aligned}
\mu_b =& (\lambda_1 \cdot \Sigma^{-1} \cdot \mu + \lambda_2 \cdot \Sigma^{-1} \cdot e)^T \cdot \mu \\
=& (\lambda_1 \cdot \Sigma^{-1} \cdot \mu)^T \cdot \mu + (\lambda_2 \cdot \Sigma^{-1} \cdot e)^T \cdot \mu \\
=& \lambda_1 \cdot \mu^T \cdot \Sigma^{-1} \cdot \mu + \lambda_2 \cdot e^T \cdot \Sigma^{-1} \cdot \mu,
\end{aligned}$$

$$\begin{aligned}
1 =& (\lambda_1 \cdot \Sigma^{-1} \cdot \mu + \lambda_2 \cdot \Sigma^{-1} \cdot e)^T \cdot e \\
=& \lambda_1 \cdot \mu^T \cdot \Sigma^{-1} \cdot e + \lambda_2 \cdot e^T \cdot \Sigma^{-1} \cdot e.
\end{aligned}$$

We notice that $e^T \cdot \Sigma^{-1} \cdot \mu$=$\mu^T \cdot \Sigma^{-1} \cdot$ e, because Σ^{-1} is symmetric, so we can define

$$A = \mu^T \cdot \Sigma^{-1} \cdot \mu > 0,$$
$$C = e \cdot \Sigma^{-1} \cdot e > 0,$$

because Σ^{-1} is positive-defined;

$$B = \mu^T \cdot \Sigma^{-1} \cdot e.$$

So if we put the equalities in a matrix form we have:

$$\begin{pmatrix} A & B \\ B & C \end{pmatrix} \cdot \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \mu_b \\ 1 \end{pmatrix}$$

We have to determine that the determinant of the matrix $\Delta \neq 0$, so we can solve the system of equations. So we consider the vector v=B $\cdot$ μ -A $\cdot$ e $\neq 0$, because if v=0 then

$$\mu^T \cdot \Sigma^{-1} \cdot e \cdot \mu - \mu^T \cdot \Sigma^{-1} \cdot \mu \cdot e = 0.$$

So a $\cdot$ μ=b$\cdot$e, with a,b $\in \mathbb{R} \setminus 0$, so all expected values would be equal, but we have seen that all expected values can not be equal, because if they were, we would consider that asset with less variance and discard the others.
Using vector v we have

$$\begin{aligned} 0 < v^T \cdot \Sigma^{-1} v =& (B \cdot \mu - A \cdot e)^T \cdot \Sigma^{-1} \cdot (B \cdot \mu - A \cdot e) \\ =& B^2 \cdot \mu^T \cdot \Sigma^{-1} \cdot \mu - 2AB \cdot \mu^T \cdot \Sigma^{-1} \cdot e + A^2 \cdot e^T \cdot \Sigma^{-1} \cdot e. \\ =& B^2 A - 2AB^2 + A^2 C \\ =& - AB^2 + A^2 C \\ =& A(AC - B^2) = A\Delta. \end{aligned}$$

A>0 and A $\cdot$ Δ>0 $\Rightarrow$ Δ>0, so $\Delta \neq 0$ and we can solve the system

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} C & -B \\ -B & A \end{pmatrix} \cdot \begin{pmatrix} \mu_b \\ 1 \end{pmatrix}$$

So we have

$$\lambda_1 = \frac{1}{\Delta}(C \cdot \mu_b - B).$$

$$\lambda_2 = \frac{1}{\Delta}(A - B \cdot \mu_b).$$

And introducing these results into equation (3.6) we obtain the optimal weights for the portfolio for a expected return of μ_b

$$\begin{aligned} X =& \frac{1}{\Delta}(C \cdot \mu_b - B) \cdot \Sigma^{-1} \cdot \mu + \frac{1}{\Delta}(A - B \cdot \mu_b) \cdot \Sigma^{-1} \cdot e \\ =& \frac{1}{\Delta}(C \cdot \Sigma^{-1} \cdot \mu - B \cdot \Sigma^{-1} \cdot e) \cdot \mu_b + \frac{1}{\Delta}(A \cdot \Sigma^{-1} \cdot e - B \cdot \Sigma^{-1} \cdot \mu) \\ =& g \cdot \mu_b + h. \end{aligned} \tag{3.7}$$

So we deduce the two fund theorem. In the case that we have n risky assets, we can obtain efficient assets by the combination of only two portfolios, g and h.

Once we have calculated the optimal weights, we can calculate which are their variances

$$
\begin{aligned}
\sigma^2 =& Var(P) = X^T \cdot \Sigma \cdot X \\
=& X^T \cdot \Sigma \cdot (g \cdot \mu_b + h) \\
=& X^T \cdot \Sigma^{-1} \cdot (\frac{1}{\Delta}(C \cdot \Sigma^{-1} \cdot \mu - B \cdot \Sigma^{-1} \cdot e) \cdot \mu_b + \frac{1}{\Delta}(A \cdot \Sigma^{-1} \cdot e - B \cdot \Sigma^{-1} \cdot \mu) \\
=& X^T \cdot (\frac{1}{\Delta}(C \cdot \mu - B \cdot e) \cdot \mu_b + \frac{1}{\Delta}(A \cdot e - B \cdot \mu)) \\
=& \frac{1}{\Delta^2}[(C \cdot \Sigma^{-1} \cdot \mu - B \cdot \Sigma^{-1} \cdot e) \cdot \mu_b + (A \cdot \Sigma^{-1} \cdot e - B \cdot \Sigma^{-1} \cdot \mu)]^T \cdot (C \cdot \mu - B \cdot e) \cdot \mu_b + (A \cdot e - B \cdot \mu)] \\
=& \frac{1}{\Delta^2}(C \cdot \mu_b^2 \cdot (AC - B^2) + 2B \cdot \mu_b \cdot (B^2 - AC) + A(AC - B^2) \\
=& \frac{1}{\Delta}(C \cdot \mu_b^2 - 2B \cdot \mu_b + A) \\
=& \frac{C}{\Delta}(\mu_b - \frac{B + \sqrt{-\Delta}}{C})(\mu_b - \frac{B - \sqrt{-\Delta}}{C}) \\
=& \frac{C}{\Delta}(\mu_b - \frac{B}{C})^2 + \frac{1}{C}.
\end{aligned}
\tag{3.8}
$$

So we have that the minimum possible variance for the portfolio is $\frac{1}{C}$ and it is reached if μ_b=$\frac{B}{C}$, so a risk-adverse investor would choose μ_b=$\frac{\mu^T \cdot \Sigma^{-1} \cdot e}{e^T \cdot \Sigma^{-1} \cdot e}$ as the expected value to calculate the optimal weights for his portfolio.
From equation (3.8) have the relation (μ_b,σ) between the expected value and the standard deviation of optimal portfolios, so we can observe how the curve of optimal portfolios is in a plot where the axes are μ_b and σ:

$$
\begin{aligned}
\sigma^2 - \frac{C}{\Delta}(\mu_b - \frac{B}{C})^2 = \frac{1}{C} \\
\frac{\sigma^2}{\frac{1}{C}} - \frac{(\mu_b - \frac{B}{C})^2}{\frac{\Delta}{C^2}} = 1.
\end{aligned}
$$

We observe that the plot (μ_b,σ) is a hyperbola with center ($\frac{A}{C}$,0) and vertex ($\frac{A}{C}$,$\sqrt{\frac{1}{C}}$).
In the figure below we can observe an example of the attainable set and the frontier set of a portfolio. The drawn curve shows all the points which are efficient for a given expected value. However, not all the points are in the efficient set, so for all that points with a expected value below the expected value of the portfolio with minimal variance, it exists an asset with the same variance and greater expected value. So the efficient set is the curve defined since the point with minimal variance moving it through points with greater expected value. All the curve and the space that lies into its plot is the attainable set.

3.3 Model for n risky assets and a risk-free asset

As in the past section, we have to introduce some restrictions to generalise the model. We are going to add to the previous assumptions the following conditions:

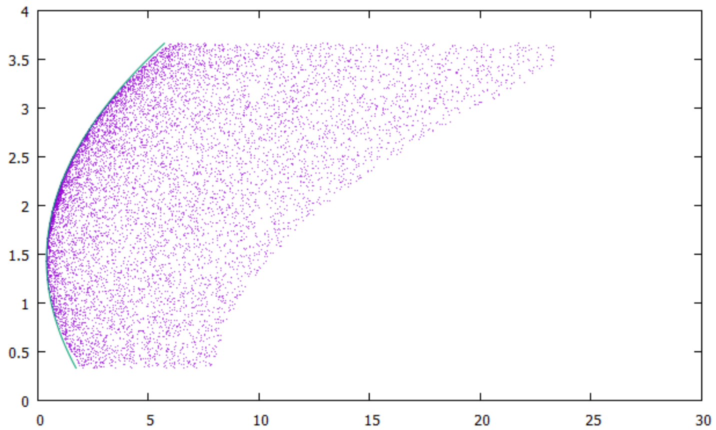

Figure 3.3: Image of attainable and frontier sets for a case of 3 assets.

- We consider that it exists a risk-free asset A_{rf}, so let be r_{rf} its expected return, then Var(r_{rf})=0 and its covariance with any asset is 0.
- We consider that we can lend or borrow money at r_{rf}, so the interest for lending is the same that the interest for borrowing.

From this assumptions we can reduce the restrictions in our problem to find the optimal weights X_1,...,X_n of the n risky assets of portfolio P, with expected returns μ_1,...,μ_n, and the weight X_{rf} consisting of the weight corresponding to the risk-free asset. We can remove condition (3.2), so:

- If $\sum_{i=1}^{n} X_i$<n we would lend the excess at risk-free expected rate.
- If $\sum_{i=1}^{n} X_i$>n we would borrow the difference at risk-free expected rate.

So the new expected return μ_b of the portfolio would be:

$$\mu_b = \sum_{i=1}^{n} X_i \cdot \mu_i + (1 - \sum_{i=1}^{n} X_i) \cdot r_{rf}.$$

So we could define a function which shows this restriction as

$$g(X) = X^T(\mu - e \cdot r_{rf}) + r_{rf} - \mu_b.$$

And the problem we would have to solve would be minimize f(X) subject to g(X)=0.
Applying Lagrange multipliers to this problem we would have the function

$$F(X, \lambda) = \frac{1}{2} X^T \cdot \Sigma \cdot X - \lambda \cdot g(X).$$

where $\lambda \in \mathbb{R}$.
We calculate the derivatives of F:

$$\begin{aligned} \frac{\partial F}{\partial \lambda} =& g(X). \\ D_X F =& \Sigma \cdot X - \lambda \cdot (\mu - r_{rf} \cdot e) \end{aligned}$$

If we make the derivatives of F equal to 0 in order to find the extremum, we would have

$$X^T(\mu - e \cdot r_{rf}) + r_{rf} - \mu_b = 0. \tag{3.9}$$

$$\Sigma \cdot X - \lambda \cdot (\mu - r_{rf} \cdot e) = 0. \tag{3.10}$$

We notice that D^2_X F = Σ, and Σ is a positive-definite matrix, so if it exists a solution to this problem, then it will be a minimum.
By (3.10) we have

$$X = \lambda \cdot \Sigma^{-1} \cdot (\mu - r_{rf} \cdot e). \tag{3.11}$$

So putting (3.11) into (3.9) we obtain:

$$\begin{aligned} \mu_b - r_{rf} &= (\lambda \cdot \Sigma^{-1} \cdot (\mu - r_{rf} \cdot e))^T \cdot (\mu - e \cdot r_{rf}) \\ &= \lambda \cdot (\mu - r_{rf} \cdot e)^T \cdot \Sigma^{-1} \cdot (\mu - e \cdot r_{rf}). \end{aligned} \tag{3.12}$$

We notice that μ-r_{rf} $\cdot$ e $\neq 0$, because if it was 0, then all the expected returns of the portfolio would be equal, and we have considered that at least there are two assets with different expected value, so (μ-r_{rf} $\cdot$ e) $\neq 0$ and how Σ^{-1} is a positive-definite matrix $(\mu - r_{rf} \cdot e)^T \cdot \Sigma^{-1} \cdot$ (μ-e $\cdot$ r_{rf})>0, so we can consider by (3.12)

$$\lambda = \frac{\mu_b - r_{rf}}{(\mu - r_{rf} \cdot e)^T \cdot \Sigma^{-1} \cdot (\mu - e \cdot r_{rf})}. \tag{3.13}$$

So we obtain that the solution to the problem is:

$$X = \frac{(\mu_b - r_{rf}) \cdot \Sigma^{-1} \cdot (\mu - r_{rf} \cdot e)}{(\mu - r_{rf} \cdot e)^T \cdot \Sigma^{-1} \cdot (\mu - e \cdot r_{rf})} \tag{3.14}$$

As in the case without a risk-free asset, we can consider the variance of this optimal portfolios, to simplify we can consider:

$$\begin{aligned} A &= \mu^T \cdot \Sigma^{-1} \cdot \mu. \\ B &= \mu^T \cdot \Sigma^{-1} \cdot e. \\ C &= e \cdot \Sigma^{-1} \cdot e. \\ D &= (\mu - r_{rf} \cdot e)^T \cdot \Sigma^{-1} \cdot (\mu - e \cdot r_{rf}) = (A - 2Br_{rf} + Cr_{rf}^2). \end{aligned} \tag{3.15}$$

So we have:

$$\begin{aligned} \sigma^2 &= X^T \Sigma X = \frac{X^T \cdot (\mu_b - r_{rf}) \cdot (\mu - r_{rf} \cdot e)}{D} \\ &= \frac{(\mu_b - r_{rf}) \cdot \Sigma^{-1} \cdot (\mu - r_{rf} \cdot e)^T}{D^2} \cdot (\mu_b - r_{rf}) \cdot (\mu - r_{rf} \cdot e) \\ &= (\frac{\mu_b - r_{rf}}{D})^2 \cdot [(\mu - r_{rf} \cdot e)^T \cdot \Sigma^{-1} \cdot (\mu - r_{rf} \cdot e)] \\ &= (\frac{\mu_b - r_{rf}}{D})^2 \cdot D = \frac{(\mu_b - r_{rf})^2}{D}. \end{aligned}$$

Then, we can see that in a (μ_b,σ) plane, we would have represented variance compared to expected value as two semi-lines with vertex $(r_{rf},0)$ and slopes $\sqrt{D}$ or $-\sqrt{D}$. So the efficient set would be the semi-line with positive slope.

An important remark is that we can find the relationship between the efficient set between a portfolio of n assets without a risk-free asset and the same portfolio with a risk free asset. We can consider that $r_{rf}<\frac{B}{C}$, which it supposes that we expect less return without risk than with risk.

Proposition 3.3. *Let be P a portfolio with risky assets $A_1,...,A_n$, let be μ the array with the expected return of the assets,Σ the covariance matrix of the return and r_{rf} the return of a risk-free asset. Let be A,B,C,D as defined in (3.15), Δ=AC-B^2, (σ,μ_b) the pair of standard deviation and expected return of the portfolios in the frontier set. Then, the semi-lines which define the frontier set of P with the risk-free asset are tangent to the hyperbola which defines the frontier set of P without the risk-free asset, in case that $r_{rf}<\frac{B}{C}$.*

Proof. From equation (3.8) we can express μ_b as a function of σ to the case of the frontier set without considering the risk-free asset, so we have

$$\mu_b = \frac{1}{C} + \epsilon \cdot \sqrt{(\sigma^2 - \frac{1}{C}) \cdot \frac{\Delta}{C})} \tag{3.16}$$

where $\epsilon \in \{1,-1\}$. In case it exists a point tangent to the hyperbola which corresponds to the semi-line which defines the frontier set with the risk-free asset, then they have to have the same slope, so we need to calculate the derivative of the function.

$$\frac{d\mu_b}{d\sigma}(\sigma) = \frac{\epsilon \cdot \sigma \cdot \frac{\Delta}{C}}{\sqrt{(\sigma^2 - \frac{1}{C}) \cdot \frac{\Delta}{C}}} \tag{3.17}$$

We notice that the derivative is well defined in all points but when σ^2=$\frac{1}{C}$, or what is the same, when μ_b=$\frac{B}{C}$, but the tangent line to the hyperbola through this point is parallel to the μ_b axis, so it would never cross the line and it can not be the semi-line which joins this point to $(0,rrf)$. So if we consider all the other points and find out if one of them can have slope $\tau \cdot \sqrt{D}$, where $\tau \in \{1,-1\}$, we would have

$$\begin{aligned}
\tau \cdot \sqrt{D} = \frac{\epsilon\sigma_M \cdot \frac{\Delta}{C}}{\sqrt{(\sigma_M^2 - \frac{1}{C}) \cdot \frac{\Delta}{C}}} &\Leftrightarrow \epsilon \cdot \sigma \cdot \frac{\Delta}{C} = \tau \cdot \sqrt{D} \cdot \sqrt{(\sigma_M^2 - \frac{1}{C}) \cdot \frac{\Delta}{C}} \\
&\Leftrightarrow \sigma_M^2 \cdot \frac{\Delta^2}{C^2} = D \cdot (\sigma_M^2 - \frac{1}{C}) \cdot \frac{\Delta}{C} \\
&\Leftrightarrow \sigma_M^2 = \frac{-D}{\Delta - DC}.
\end{aligned}$$

So we only have to proof that Δ-DC<0 if $r_{rf}<\frac{B}{C}$, so we can consider the points above.

$$\begin{aligned}
\Delta - DC =& AC - B^2 - (A - 2Br_{rf} + Cr_{rf}^2) \cdot C \\
=& - B^2 + 2BCr_{rf} - C^2 r_{rf}^2 \\
=& - C^2 \cdot (r_{rf} - \frac{B}{C})^2 < 0.
\end{aligned}$$

So these points are well defined. Finally, we check that

$$\sigma_{M,+} = \sqrt{\frac{-D}{\Delta - DC}}$$
$$\sigma_{M,-} = -\sqrt{\frac{-D}{\Delta - DC}}$$

give us the two possible slopes we wanted.

$$\begin{aligned}\frac{d\mu_b}{d\sigma}(\sigma_{M,+}) =& \frac{\sqrt{\frac{-D}{\Delta-DC}} \cdot \frac{\Delta}{C}}{\sqrt{\frac{-D}{\Delta-DC} - \frac{1}{C}} \cdot \sqrt{\frac{\Delta}{C}}} \\ =& \frac{\sqrt{\frac{-D\Delta}{C\cdot(\Delta-DC)}}}{\sqrt{\frac{-DC-\Delta+DC}{C\cdot(\Delta-DC)}}} \\ =& \sqrt{\frac{-D\cdot\Delta}{-\Delta}} = \sqrt{D}.\end{aligned}$$

And in a similar way

$$\frac{d\mu_b}{d\sigma}(\sigma_{M,-}) = -\sqrt{D}.$$

□

From here we demonstrate the one fund theorem. In the case we have n risky assets and a risk-free asset, we can obtain the efficient set by the combination of only one risky asset (that which makes the tangency with the frontier set) and the risk-free asset.

We have seen some important consequences, like the geometry of the efficient portfolios in the case of n risky assets and how it changes if we add a free-risk asset; the one fund theorem or the two funds theorem in case the conditions hold. Empirically we can consider that it exist an asset with risk almost 0, like we are going to see in the practical field. However, we can not consider that the lending rate is equal at the borrowing rate, because in most of the cases one is bigger to the other.

3.4 Limitations

We have shown that the model explained before and its extensions hold in some assumptions, however some of them do not fit if we apply them in reality or they require a lot of calculations.

The model considers variance as the risk which the investor wants to minimise, however Markowitz considers that there are some risks which could adjust better to what the investor really wants to minimise, as the semivariance. However, he considers variance because its simplicity.

In order to Von Neumann and Morgenstern's axioms hold, investors have to know the preference between all the possible combinations of assets, and this is so unrealistic. So we cannot maximise a utility function that is unknown.

It is considered that all assets are perfectly divisible, however, in most cases we can buy only multiple amounts of a given nominal. We can correct this using OTC markets and financial derivatives, but in this work we do not consider them, because they have more specific models which consider them as Black-Scholes model.

Transaction costs exist, so if we consider a low horizon of time the possible found optimisation can be neglected by the effects of transaction costs.
It exists a lot of information for every asset, however not all individuals have the same accessibility.

It does not exist a pure risk-free asset, however we can consider treasury bills as a realistic approximation of them.

There are some limitations on the amount of money you can borrow at a determinate rate, and there are discrepancies between lending rates and borrowing rates, so borrowing rates are higher than lending rates.

Finally, in order to find the covariance matrix of a portfolio of n assets it is required to make $\frac{n^2+n}{2}$ estimations and the n expected returns estimations. So it is necessary to find an appropriate approximation of the behaviour of the returns and have enough memory to save all this estimations.

Chapter 4

Other models derived from Markowitz model

4.1 Sharpe's model

As we have seen before, Markowitz's model has optimal solutions given some fixed parameters, however, the number of parameters needed to estimate in order to find the optimal solution increases in a quadratic way in function of the number of the assets in the portfolio; so for a portfolio of n risky assets are needed n expected returns, n variances and $\frac{n^2-n}{2}$ covariances, so there are needed $\frac{n^2+3n}{2}$ estimations. We have to consider that nowadays that is not such a great problem due to the improvement of computers, nevertheless, in early sixties that was a great problem, first of all because of the great number of calculations needed and in a second way, because of the memory needed to keep the information needed to operate.

In order to minimise the calculations needed to do, William F.Sharpe in 1963 published *A Simplified Model for Portfolio Analysis*, where appears the model that we are going to explain below.

His principal assumption is that it exists a known index I so all the expected returns of the assets depend on this index and other random factors. Let be P a portfolio with n risky assets with weights $X_1,...,X_n$, let be (r_j,σ_j^2) the expected return and variance of the j^{th} asset, $\alpha_i,\beta_j \in \mathbb{R}$, and ϵ_i random variables for j=1,...,n and i=1,...,n+1 with the following properties:

a) E(ϵ_i)=0 $\forall$ i=1,...,n+1. So the random factors do not have more probability to achieve positive or negative numbers.

a') We notice that from a) we have that Var(ϵ_i)=E(ϵ_i^2).

b) Homoscedasticity: Var(ϵ_i)=$\sigma_{\epsilon,i}^2$ does not depend of time and remains constant for each period $\forall$ i=1,...,n+1.

c) Cov(ϵ_i,ϵ_j)=0 $\forall$ i $\neq$ j. So the random factors of the assets are independent.

c′) We notice that combining a) and c) we obtain that E($\epsilon_i \cdot \epsilon_j$)=0 $\forall$ i $\neq$ j.

d) $\epsilon_i \sim$ N(0,$\sigma^2_{\epsilon,i}$) for i=1,...,n+1. So the random factors have normal distributions.

We define a random vector ϵ=(ϵ_1,...,ϵ_n) with this properties as white noise.
Then we can consider the following equalities:

$$r_i = \alpha_i + \beta_i \cdot I + \epsilon_i,$$

for i=1,...,n.

$$I = \alpha_{n+1} + \epsilon_{n+1}.$$

We consider r_P as the return of the portfolio P; E(I)=α_{n+1} and Var(I)=σ^2_I.

$$\begin{aligned} r_P &= \sum_{i=1}^{n} X_i \cdot r_i \\ &= \sum_{i=1}^{n} X_i \cdot (\alpha_i + \beta_i \cdot I + \epsilon_i) \\ &= \sum_{i=1}^{n} X_i \cdot (\alpha_i + \epsilon_i) + X_i \cdot \beta_i \cdot I \\ &= \sum_{i=1}^{n} X_i \cdot (\alpha_i + \epsilon_i) + X_i \cdot \beta_i \cdot (\alpha_{n+1} + \epsilon_{n+1}), \end{aligned}$$

So we can see that the expected return can be divided in investment in the assets and investment in the index I.
Defining

$$\sum_{i=1}^{n} X_i \cdot \beta_i = X_{n+1},$$

then we have that

$$r_P = \sum_{i=1}^{n+1} X_i \cdot (\alpha_i + \epsilon_i).$$

Then we can calculate the expected return and the variance of the returns of the portfolio:
Using assumptions a), c′), a′), b) and that the square of a sum can be expressed as it is shown below, we obtain

$$\begin{aligned} E(r_P) =& E[\sum_{i=1}^{n+1} X_i \cdot (\alpha_i + \epsilon_i)] \\ &= \sum_{i=1}^{n+1} (E(X_i \cdot \alpha_i) + X_i \cdot E(\epsilon_i)) \\ &= \sum_{i=1}^{n+1} X_i \cdot \alpha_i. \end{aligned}$$

$$\begin{aligned}
Var(r_P) =& E[(r_P - E(r_P))^2] \\
=& E(\sum_{i=1}^{n+1} X_i \cdot (\alpha_i + \epsilon_i) - \sum_{i=1}^{n+1} X_i \cdot \alpha_i)^2 \\
=& E((\sum_{i=1}^{n+1} X_i \cdot \epsilon_i)^2) \\
=& E[(\sum_{i=1}^{n+1} X_i^2 \cdot \epsilon_i^2) + (2 \cdot \sum_{i=1}^{n+1} \sum_{j>i} X_i \cdot X_j \cdot \epsilon_i \cdot \epsilon_j)] \\
=& (\sum_{i=1}^{n+1} X_i^2 \cdot E(\epsilon_i^2)) + (2 \cdot \sum_{i=1}^{n+1} \sum_{j>i} X_i \cdot X_j \cdot E(\epsilon_i \cdot \epsilon_j)) \\
=& \sum_{i=1}^{n+1} X_i^2 \cdot E(\epsilon_i^2), \\
=& \sum_{i=1}^{n+1} X_i^2 \cdot \sigma_{\epsilon,i}^2.
\end{aligned}$$

Let be X^T=(X_1,...,X_{n+1}), D a diagonal matrix with ($\sigma_{\epsilon,1}^2$,$\sigma_{\epsilon,2}^2$,$\sigma_{\epsilon,3}^2$,...,$\sigma_{\epsilon,n+1}^2$) as its diagonal components, then we obtain that

$$Var(r_P) = W^T \cdot D \cdot W.$$

And we can use Markowitz's model to this process in order to find a solution of the portfolio with minimal variance fixed a determined expected value.
With this process we have reduced the parameters to estimate in a portfolio of n assets into:

- n+1 parameters α_i.
- n+1 parameters $\sigma_{\epsilon,i}^2$.
- n parameters β_i.

So we reduce the problem of having to estimate $\frac{n^2+3n}{2}$ parameters into having to estimate 3n+2 parameters.
In the work of Sharpe is shown how important this reduction was: *The computing time required by the diagonal code is considerably smaller than that required by standard quadratic programming codes. The RAND QP code required 33 minutes to solve a 100-security example on an IBM 7090 computer; the same problem was solved in 30 seconds with the diagonal code. Moreover, the reduced storage requirements allow many more securities to be analyzed: with the IBM 709 or 7900 the RAND QP code can be used for no more than 249 securities, while the diagonal code can analyze up to 2000 securities.*
One of the problems is to consider an optimal index *I*. Most of the studies and William Sharpe himself had considered the Gross National Product, the level of the stock market as a whole or some important index of the area in which the portfolios are located as possible indexes. However, this index can vary in function of the area and the portfolios chosen, so it is so important to consider a good index.

However, once we have considered an index, we can consider which are the optimal estimators due to the assumptions done.

We consider a sample of m data $(r_{i,k}, I_k)$ of independent and identical distributed vairables for each asset's return r_i and I_k the value of the index, i=1,...,n; k=1,...,m. How return r_i is a linear function depending on I, we can consider the lineal regression that reduces the square of the differences between real value $r_{i,k}$ and the estimation $\widehat{r_{i,k}}$=α_i+$\beta_i \cdot I_k$.
So, we consider

$$F_i(\alpha_i, \beta_i) = \sum_{k=1}^{m} (R_{i,k} - \alpha_i - \beta_i \cdot I_k)^2,$$

and we calculate its partial derivatives and equal them to 0 so we can find its minimum

$$\begin{aligned}
\frac{\partial F_i}{\partial \alpha_i} &= \sum_{k=1}^{m} -2 \cdot (r_{i,k} - \alpha_i - \beta_i \cdot I_k) = 0. \\
&\Rightarrow \sum_{k=1}^{m} r_{i,k} = n \cdot \alpha + \beta \sum_{k=1}^{m} I_k \Rightarrow \bar{r_i} = \alpha_i + \beta_i \cdot \bar{I}. \qquad (4.1)\\
\frac{\partial F_i}{\partial \beta_i} &= \sum_{k=1}^{m} 2 \cdot (r_{i,k} - \alpha_i - \beta_I \cdot I_k) \cdot (-I_k)) = 0. \\
&\Rightarrow \sum_{k=1}^{m} (r_{i,k} - \alpha_i - \beta_i \cdot I_k) \cdot (I_k) = 0. \\
&\Rightarrow \sum_{k=1}^{m} r_{i,k} \cdot I_k - \alpha_i \cdot I_k^2 = \beta_i \cdot \sum_{k=1}^{m} I_k^2. \\
&\Rightarrow \beta_i = \frac{\sum_{k=1}^{m} r_{i,k} \cdot I_k - \alpha_i \cdot \sum_{k=1}^{m} I_k}{\sum_{k=1}^{m} I_k^2}. \qquad (4.2)
\end{aligned}$$

Combining (4.1) and (4.2), we have

$$\beta_i = \frac{\sum_{k=1}^{m} r_{i,k} \cdot I_k - m \cdot \bar{I} \cdot \bar{r_i}}{\sum_{k=1}^{m} I_k^2 - m \cdot \bar{I_k}^2} = \frac{Cov(r_i, I)}{Var(I)}.$$

We notice that we have used also that the mean is a non biased estimator of the expected value of a random variable.
We see that it is a minimum considering the second order derivatives of F

$$\begin{aligned}
\frac{\partial^2 F}{\partial \alpha_i^2} &= 2m. \\
\frac{\partial^2 F}{\partial \beta_i^2} &= 2 \sum_{k=1}^{m} I_k^2. \\
\frac{\partial^2 F}{\partial \beta_i \alpha_i} &= 2 \sum_{k=1}^{m} I_k.
\end{aligned}$$

So the Hessian matrix would be:

$$\begin{pmatrix} 2m & 2\sum_{k=1}^{m} I_k \\ 2\sum_{k=1}^{m} I_k & 2\sum_{k=1}^{m} I_k^2 \end{pmatrix}$$

How 2m>0, we only need to see that its determinant Δ is positive, so the matrix would be positive definite and the critical point would be a minimum.

$$\Delta = 4m \cdot \sum_{k=1}^{m} I_k^2 - 4 \cdot (\sum_{k=1}^{m} I_k)^2 = 4 \cdot [(\sum_{k=1}^{m} 1^2) \cdot (\sum_{k=1}^{m} I_k^2) - (\sum_{k=1}^{m} 1 \cdot I_k)^2].$$

By Cauchy-Schwartz's inequality:

$$(\sum_{k=1}^{m} 1^2) \cdot (\sum_{k=1}^{m} I_k^2) \geq (\sum_{k=1}^{m} 1 \cdot I_k)^2$$

and if the equity holds, then it would mean that I_k=I $\forall$ k, so I would be always constant, so it would not be a random variable, and it would not fit with the model. So Δ>0 and this estimators define a minimum.

In the case of the residuals, we know that they follow a normal distribution with expected value 0 and variance $\sigma^2_{\epsilon,i}$, so a non biased estimator of its variance is the sample Bessel's correction variance. So we can consider the estimators:

$$\widehat{\sigma}^2_{\epsilon,i} = \frac{\sum_{k=1}^{m}(r_{i,k} - \widehat{r_{i,k}})^2}{m-1}.$$

4.2 Capital Asset Pricing Model-CAPM

In this section we are going to explain the model [1] published in 1964 by William F. Sharpe, which tries to extend the models explained before to construct a market equilibrium theory of asset prices under conditions of risk. We notice that Jack Treynor, John Lintner and Jan Mossin made similar independent studies that lead to a similar model, however, we are going to explain the model proposed by William F.Sharpe.
First of all, we introduce the hypotheses he considered:

- Individual views the outcome of any instrument in probabilistic terms. However, he is willing to act on the basis of simply expected value and standard deviation.
- Individual assumes preference to instruments with higher expected value and fewer standard deviation, so utility indifference curves are upward-sloping. In his study, he considers that the curves can be represented by a quadratic function.
- It exists a common pure interest rate, without risk, with all investors able to borrow or lend funds on equal terms at this interest rate.
- It exists homogeneity of investor expectations, so they have the same information and they agree with the values of the expected values, standard deviations and correlations of all assets.
- All investors want to invest in a same horizon time.

Implicitly, he has considered these assumptions so the model can fit:

[1] *Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk*

- All instruments are perfectly divisible.
- The market is complete, so all possible combinations of assets are well defined and there are no transaction costs.
- The market is efficient, so for a given information we can not have two different prices for the same combination of assets, depending on how we have combined the assets; and the prices adjusts immediately to new information.
- It exists perfect competition, so the trades of a single individual do not affect to the price of assets.

He knows that they are highly restrictive and unrealistic assumptions, nevertheless they lead to an equilibrium which fits with the models explained before and it complements them[2].

Proposition 4.1. *Let be P the asset which gives the pure interest rate, let be A another asset, E_{r_P}, E_{r_A} the expected values of their returns and σ_{r_A} the standard deviation of the return of asset A, let be α the weight of asset P, (1-α) the weight of A, E_r the expected value and σ_r the standard deviation of the expected return of the investment, then all the possible combinations are contained in a straight line in the (σ_r,E_r) plot.*

Proof. We calculate the expected value and the standard deviation of the investments, and we notice that they are in the same straight line:

$$E_r = \alpha \cdot E_{r_P} + (1-\alpha) \cdot E_{r_A}.$$
$$\sigma_r = \sqrt{\alpha^2 \sigma_{r_P}^2 + 2 \cdot \alpha \cdot (1-\alpha) \cdot Cov(r_A, r_P) + (1-\alpha)^2 \cdot \sigma_{r_A}^2} = \alpha \cdot 0 + |1-\alpha| \cdot |\sigma_{r_A}|.$$

□

So for each asset or combination of assets we can consider a straight line which goes from P, the asset which gives the pure interest rate, to the asset or combination of assets, as it is shown in figure 4.1. We define the straight line which is tangent to the frontier set as the capital market line (CML). We notice that the CML is better than any of the other straight lines which go from P to another combination of assets of the frontier set. So for a given level of standard deviation has the greater level of expected value. In our figure, it is the line that passes through P and Φ and corresponds to the efficient set found in the Markowitz's model with n assets and a risk-free asset.

How all investors have the same expectations about the assets, the frontier set is the same for all of them and we can put their preferences in the same plot. We consider the case of three investors and we see that all of them would consider options in the capital market line, as it is shown in figure 4.2. We consider three types of utility functions A,B and C,

[2] Needless to say, these are highly restrictive and undoubtedly unrealistic assumptions. However, since the proper test of a theory is not the realism of its assumptions but the acceptability of its implications, and since these assumptions imply equilibrium conditions which form a major part of classical financial doctrine, it is far from clear that this formulation should be rejected.

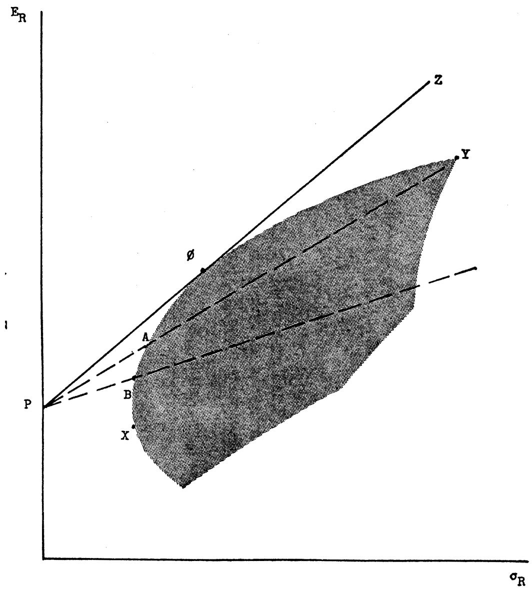

Figure 4.1: Capital market line from Sharpe's project.

one for each investor. A_i,B_i,C_i, i=1,...,4 are some levels of utility, we notice that the level of utility is better as long as the sub-index increases. The options which maximised its utility without a risk-free asset were G for the first investor, Φ for the second investor and F for the third investor. However, with the risk free asset, they can maximise their utility using options A*, Φ and C*, all of them in the capital market line. So the first investor would do a combination of investing in Φ and lending at risk-free to reach A*; the second investor would invest all his money in Φ and the third investor would invest in Φ and he would borrow money at risk-free to reach C*.

So we have seen that all investors would invest using Φ and the free-risk asset. So the attempts of all investors to purchase Φ and the lack of interest of other assets would make a revision of the price of the assets. The prices of the assets contained in Φ would rise and the prices of the assets which are not in Φ would fall. So the expected value of the assets would change, being lower if the price has been increased or higher if the price has been reduced[3]. This alteration will move all that securities which expected values have been reduced downwards; and all the securities which expected values have been increased upwards, so the frontier portfolios would change, inducing other points of tangency and other assets or combination of assets in this points. The process will continue, making the frontier set more linear in each step, until a set of prices makes that all assets are into a combination of assets that lie into the CML.

[3]The expected return depends of the initial amount paid, so if the price raises it would reduce the expected value and vice versa.

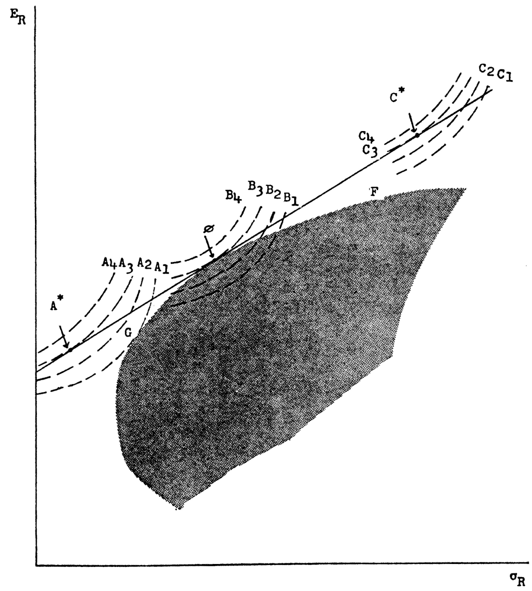

Figure 4.2: Figure from Sharpe's project.

So we obtain that in equilibrium it exists a simple linear relationship between the expected return and standard deviation of return for efficient combinations of risky assets.

Most of the individual assets will no lie into the CML, however we can expect a relationship between them and the efficient assets which include them.
We can consider the possible expected returns and standard deviations of combinations of an individual asset A and G a efficient security which includes A. The curve defined by all the possible combinations has to be continuous and tangent to the CML, if not it would exist some points above the CML, which it is impossible, because the CML is the efficient set.
Let be α the weight of A, (1-α) the weight of G; r_A,r_G their returns, σ_{r_A},σ_{r_G} their standard deviations and $r_{A,G}$ the correlation between them. If we denote (E_α,σ_α) the expected return and the standard deviation of the return of the investments, we have

$$\sigma_\alpha = \sqrt{\alpha^2\sigma_{r_A}^2 + 2\cdot\alpha\cdot(1-\alpha)\cdot\sigma_{r_A}\cdot\sigma_{r_G}\cdot r_{A,G} + (1-\alpha)^2\cdot\sigma_{r_G}^2}.$$

$$E_\alpha = \alpha\cdot E(r_A) + (1-\alpha)\cdot E(r_G).$$

$$\frac{d\sigma}{d\alpha}(0) = r_{A,G}\cdot\sigma_{r_A} - \sigma_{r_G}.$$

$$\frac{dE}{d\alpha}(0) = E(r_A) - E(r_G).$$

$$\frac{dE}{d\sigma}(0) = \frac{E(r_A)-E(r_G)}{r_{A,G}\cdot\sigma_{r_A} - \sigma_{r_G}} = \frac{E(r_G)-r_P}{\sigma_G}.$$

So we obtain that

$$E(r_A) = r_P + \beta_{A,G} \cdot (E(r_G) - r_P)$$

where $\beta_{A,G}=\frac{r_{A,G} \cdot \sigma_{r_A}}{\sigma_{r_G}}=\frac{Cov(r_A,r_G)}{\sigma^2_{r_G}}$, so it coincides with a regression line between the expected returns of A and G. Then, we can consider the systematic risk as the deviation explained by the regression line as the important risk to consider; meanwhile the unexpected risk becomes the risk unexplained that can be reduced by diversification.

Proposition 4.2. *The regression line obtained by the comparison of the expected return of an asset A and a efficient combination of assets G that contains A and passes through r_P the return of a risk-free asset P in the intercept of the line with the axis representing the standard deviation does not depend on the choice of G.*

Proof. We consider another efficient portfolio G′ containing A. We denote $r_{i,j}$ as the correlation coefficient between the return of portfolios i and j. How G and G′ are efficient, both are in the CML so they are perfectly correlated. Then

$$r_{A,G} = r_{A,G'}.$$
$$\frac{E(r_G) - r_P}{\sigma_G} = \frac{E(r_{G'}) - r_P}{\sigma_{G'}}.$$

So we obtain that

$$\begin{aligned} E(r_A) =& r_P + \frac{r_{A,G} \cdot \sigma_{r_A}}{\sigma_{r_G}} \cdot (E(r_G) - r_P) \\ =& \frac{r_{A,G'} \cdot \sigma_{r_A}}{\sigma_{r_{G'}}} \cdot (E(r_{G'}) - r_P). \end{aligned}$$

□

How the regression line does not depend on the efficient portfolio we have considered, we can consider a portfolio that include all individual assets, so we can use the same regression line for all of the assets.
How the market is complete and efficient and in the last CML we can find a efficient portfolio which includes an asset, for each individual asset; we can consider an efficient combination of asset M which includes all individual assets and define it as the market portfolio. In conditions of equilibrium we can consider that all investors have invested rationally, so according to the model explained before, the prices tend to equilibrium and we can consider the total participation of each single asset in the market as this market portfolio. So we obtain

$$E(r_A) = r_P + \beta_{A,M} \cdot (E(r_M) - r_P)$$

where $\beta_{A,M}=\frac{r_{A,M} \cdot \sigma_{r_A}}{\sigma_{r_M}}=\frac{Cov(r_A,r_M)}{\sigma^2_{r_M}}$. We understand (E($r_M$)-$r_P$) as the risk premium, which shows the additional expected return produced by investing with risk with the market expectations. Furthermore, $\beta_{A,M}$ shows the risk of the asset compared to the risk of the market, so

- If $\beta_{A,M}$<1, investing on the portfolio A takes less risk than investing on the market, so it is expected less expected return.
- If $\beta_{A,M}$=1, investing on the portfolio A takes the same risk than investing on the market, so it is expected the same expected return.
- If $\beta_{A,M}$>1, investing on the portfolio A takes more risk than investing on the market, so it is expected more expected return.

Finally we notice how we can obtain the expected return of a portfolio P.

Remark 4.3. Let be P a porfolio consisting of the individual assets A_1,...,A_n, with weights X_1,...,X_n, where M is a market portfolio, $\beta_{A_i,M}$=$\frac{Cov(r_{A_i},r_M}{\sigma^2_{r_M}}$ and $\sigma^2_{r_M}$ the variance of the return of the market portfolio M, then

$$\beta_{P,M} = \sum_{i=1}^{n} X_i \cdot \beta_{A_i,M}$$

and

$$E(r_P) = r_f + \beta_{P,M} \cdot (r_M - r_f),$$

where r_f is the risk-free return.
We would like to remark that this model shows an equilibrium which uses less estimations than the needed by the Markowitz model.
However, we have to add to all the other limitations of Markowitz model, the introduction of the same expectations for all individuals, as much from the same time horizon as from the same return expectations; a quadratic representation of the utility function, the normality of the errors, the expectation that it follows a line and a efficient market.

4.3 Arbitrage Pricing Theory

Stephen A. Ross developed in 1976 the arbitrage pricing theory, which shows[4] an alternative to the CAPM model explained before.
Let be P a initial portfolio consisting of the assets A_1...,A_n, we can define new portfolios which are obtained from P without additional investment or risk as arbitrage portfolios.For example, if we have a portfolio with assets A_1, A_2 and A_3, with weights X_1=0.25, X_2=0.25 and X_3=0.5, we can sell some amounts of asset A_1 and invest it into assets A_2 and A_3, so we could form a new portfolio, as for example, X_1=0.1, X_2=0.3, X_3=0.6. So in this case the portfolio which has no additional risk would be a portfolio with the following weights (X_1=-0.15,X_2=0.05,X_3=0.1). If the risk of this portfolio is 0, then we know it as an arbitrage portfolio.
Once we have defined what are the arbitrage portfolios, we can consider the assumptions[5] of the model:

[4] We have considered the publications *The Capital Asset Pricing model and the Arbitrage pricing theory* and the chapter 6 of *Finacial Theory and Corporate Policy* to explain this model.

[5] In some adaptations of the principal article are found more assumptions as agents which do not know exactly the returns of the assets, but they know that they are bounded and it also presupposes homogeneity of expectations

1) Asset returns are explained by a linear function of k factors, and all the investors have the same expectations in terms of expected returns and risk.

2) Investors can build a portfolio of assets where specific risk is eliminated through diversification.

3) No arbitrage opportunity exists among well-diversified portfolios.

4) We have a complete, perfectly competitive and frictionless capital market.

5) Assets are perfectly divisible.

So, if we define the additional investment in the i^th asset as w_i, w^T=(w_1,...,w_n), e^T=(1,...,1) $\in \mathbb{R}^n$ and F_1,...,F_k the factors which explain the asset returns r_i, i=1,...,n we have

$$\begin{aligned} w^T \cdot e =& 0, \\ r_i =& E(r_i) + b_{i,1} \cdot F_1 + ... + b_{i,k} \cdot F_k + \epsilon_i. \end{aligned}$$

where F_1,...,F_k are zero mean factors and ϵ_1,...,ϵ_n are white noises[6].

Let be r the vector with the returns of assets A_1,...,A_n; b_i^T=($b_{1,i}$,...,$b_{n,i}$), i=1,..,k; ϵ=(ϵ_1,...,ϵ_n), then the return of the arbitrage portfolio r_{AP} would be

$$r_{AP} = w^T \cdot r = w^T \cdot E(r) + w^T \cdot (b_1 \cdot F_1 + ... + b_k \cdot F_k) + w^T \cdot \epsilon.$$

If we have a large number of assets n,and we consider $w_i \simeq \frac{1}{n}$ $\forall$ i , then by the law of large numbers we have that

$$\lim_{n \to \infty} w^T \cdot \epsilon = 0.$$

So we have

$$r_{AP} = w^T \cdot r = w^T \cdot E(r) + w^T \cdot (b_1 \cdot F_1 + ... + b_k \cdot F_k).$$

If n>k, we can find w such that

$$w^T \cdot b_i = 0, \forall i \in \{1, ..., n\}$$

and $w_i \simeq \frac{1}{n}$.Then we obtain that the return of the arbitrage portfolio would be

$$r_{AP} = w^T \cdot E(r).$$

We notice that w^T and E(r) are fixed, so r_{AP} is a value. If it was different than 0, then we could obtain an infinite profitability without additional investment and without risk, so it is not factible. So we obtain

$$w^T \cdot E(r) = 0. \tag{4.3}$$

Remark 4.4. If the fact that a vector w is orthogonal to N-1 vectors implies that w has to be orthogonal to a N^{th} vector v, then v can be expressed as a linear combination of that N-1 vectors.

[6] In his work Stephen Ross does not require ϵ to be a joint normal variable, he only requires that it exists σ^2<∞ such that σ_i^2<σ^2, where σ_i is the standard deviation of ϵ_i. But we will assume it in order to simplify it.

So using the previous remark we obtain that

$$E(r_i) = \lambda_0 + \lambda_1 \cdot b_{i,1} + ... + \lambda_k \cdot b_{i,k}$$

where $\lambda_i \in \mathbb{R}\ \forall$ i $\in 1, ..., n$.
If it exists a risk-free asset with return r_f necessarily λ_0=r_f.
If we define δ_i as the expected return of a portfolio with unitary sensibility to factor i and 0 sensitivity to the other factors, then we can write the equality as

$$E(r_i) = r_f + (\delta_1 - r_f) \cdot b_{i,1}+, ..., +(\delta_k - r_f) \cdot b_{i,k}.$$

If we consider the existence of a unique factor, and the assumptions to consider a regression line hold, we find that

$$b_i = \frac{Cov(r_i, \hat{\delta_1})}{\sigma^2_{\hat{\delta}_1}}.$$

One of the problems of the model is that the factors are not defined, so it is needed a estimation of the number of factors that has to have the model and a estimation of which are this factors.
Stephen A. Ross in *Economic Forces and Stock Market* define some factors that can be considered:

- Inflation.
- Treasury-bill rate.
- Long-term government bonds.
- Industrial production.
- Low-grade bonds.
- Equally weighted equities.
- Value-weighted equities.
- Consumption.
- Oil prices.

So we obtain that the CAPM and the APT converge to the same estimations with the assumption of a unique factor and with the conditions which make a lineal regression a good estimation.

Chapter 5

Practical case

In this chapter we are going to see the application of some of the models explained before in order to evaluate a portfolio with real data. CAPM and APT models are not considered because they try to explain the equilibrium and we would have to make additional suppositions about this equilibrium, whereas Markowitz and Sharpe models are considered, so they only try to explain individual assets.

We have considered day-to-day and monthly data from the return of IBEX assets from January of 2008 until October of 2018, and we have divided it into different periods, always beginning in January and finishing in October, so we can find a relation between the results and the economic situation. We notice that we have considered the equivalent I_1 to daily and monthly returns, so we can compare them.
Investment in variable assets are usually set for long periods, however if we want to compare different situations of the market we have to use reduced periods of time[1]. On the other hand we need several data in order we can obtain solid information, so we are going to consider daily and monthly returns.
As we have considered principal companies working in Spain we will consider spanish treasury bills with similar maturity as free-risk assets[2].

We have selected a portfolio consisting of the twenty companies of IBEX-35 which have stayed in this index during the period of 2008-2018, so we can have the same companies during all the period studied.
First of all we have calculated the weighted means, covariance matrix and its inverse in order to have the necessary data to use the models. The program considered in the annexes works for low number of assets; however, as the expected covariances and variances are so close to 0, as long as we introduce new assets the determinant of the covariance matrix tends to 0 and it can not determine or write this determinant. This problem possibly could be corrected by considering a multiple of the matrix.
We have calculated the portfolio which gives the point of tangency between the frontier set

[1]We notice that the return of treasury bills have less variations as long as they have lower periods, so they approximate more to the theoretical free-risk return.

[2]We have chosen the last treasury bill with a maturity of three months of each period as the risk free asset.

where free-risk assets are not considered and the frontier set which considers the free-risk asset.
In the tables considered in the annexes we can find which it is the point of tangency for each method and for each period considered.
It is clear that this point has a great relation with which it is the free-risk asset considered, so here comes the first problem. We have considered the risk-free asset as the last period one, for each period. If it was a risk-free asset, it would remain constant and it would not be relevant which one we are considering. However, it changes and in the considered period has changed from 0,000273434 to -0,00650087, which is a sharp change. So we have that the empirical risk-free asset differs from the the theoretical definition.
In the tables mentioned above, we can see that the tangent portfolios of Sharpe's model are more diversified and they present less sharp movements, like very negative weights, compared with the tangent portfolio of Markowitz. The model of Markowitz uses the correlations between the assets so when it exists a negative correlation, it can be done a short-sell of the asset which is worse to increase the amount invested on the asset which expects to go upwards. For this reason, Markowitz model can be more sharp in this example than the model of Sharpe.

However, we also notice that in the periods with more instability, like in 2008-2010, both models present really sharp movements, that can not fit with reality. This sentence fits with one of the studies of Fama in which he shows that diversification does not apply in situations of regression. Moreover, as seen before, it exists some limitations about short sells, so they are not allowed or it exists an amount limit, so we could not make operations which imply great negative weights. We remember that we let negative weights, but we supposed that we had the necessary amount invested in that asset, so we could sell it without doing short sells.
For this reason, we have considered also the classical Markowitz model, where all weights must be positive, or an alternative Markowitz model[3] where it is allowed a negative weight, until -$\frac{1}{n}$=-5%.

Both models show the properties of the classical model of Markowitz, so the frontier set contains several boundary weights, where the classical boundary weight is 0 and in the alternative model it is -0.05. We notice that we have considered this alternative model so it is factible to have an amount near to $\frac{1}{n}$, so short sells would not be needed.
In figure 5.1 we can find the differences between a part of the frontier sets of Markowitz without restrictions, Markowitz with weight superior of -5% and the classical model of Markowitz.
From left to right, the frontier sets are from Markowitz model without restrictions, Markowitz model with restrictions of -5% and the classical model. It is obvious that the model without restrictions seems better to the others, so it present really low variances and if we consider variances of the order of the other models, it has a expected return of 3% while in the other models they barely achieve 0.5%. However, this efficiency is attached using short selling of the assets which show downwards trends. So if it exists some limitations

[3]We have used Solver complement from Excel in order to calculate this models.

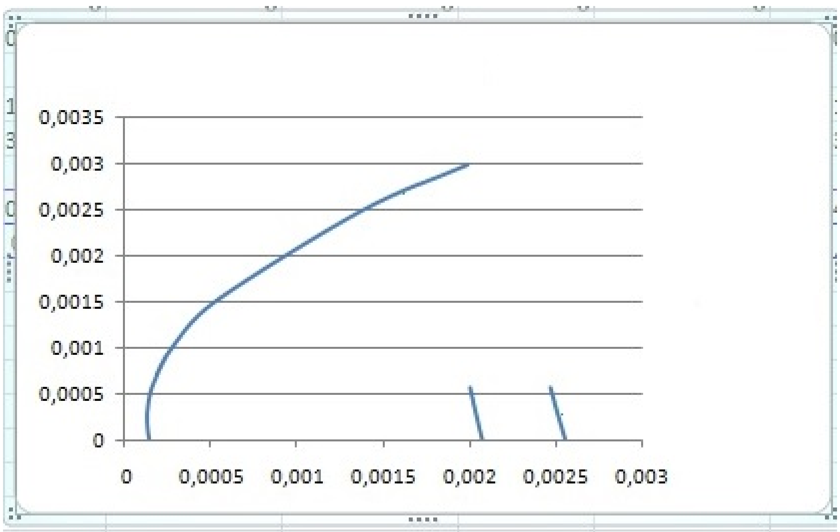

Figure 5.1: Comparison between frontier sets of Markowitz models.

in reality, this solutions cannot be applied and therefore do not fit with reality.

If we observe the tables attached in annexes, there are several differences in the estimations of Markowitz and Sharpe, moreover, in some cases one suppose a great investment in an asset and the other considers a negative weight for this asset. We have noticed also that some of the hypotheses which differ from Markowitz, as it is homocedasticity does not hold, so it can be seen a relation with the time when there is not a stability on the market, as it is shown in figure 5.2. We have used the weighted mean of the returns and the Bessel's correction of the variance as the estimators of the expected return of the assets and their variance, so they are nonbiased estimators. Some studies have considered that the expected returns follow a normal distribution, however, we have seen that the histogram of the returns is more leptokurtic and it is not symmetric, so there is more points in the positive region. We can see this situation in figure 5.3.

We can see that we can not estimate the returns as a normal.

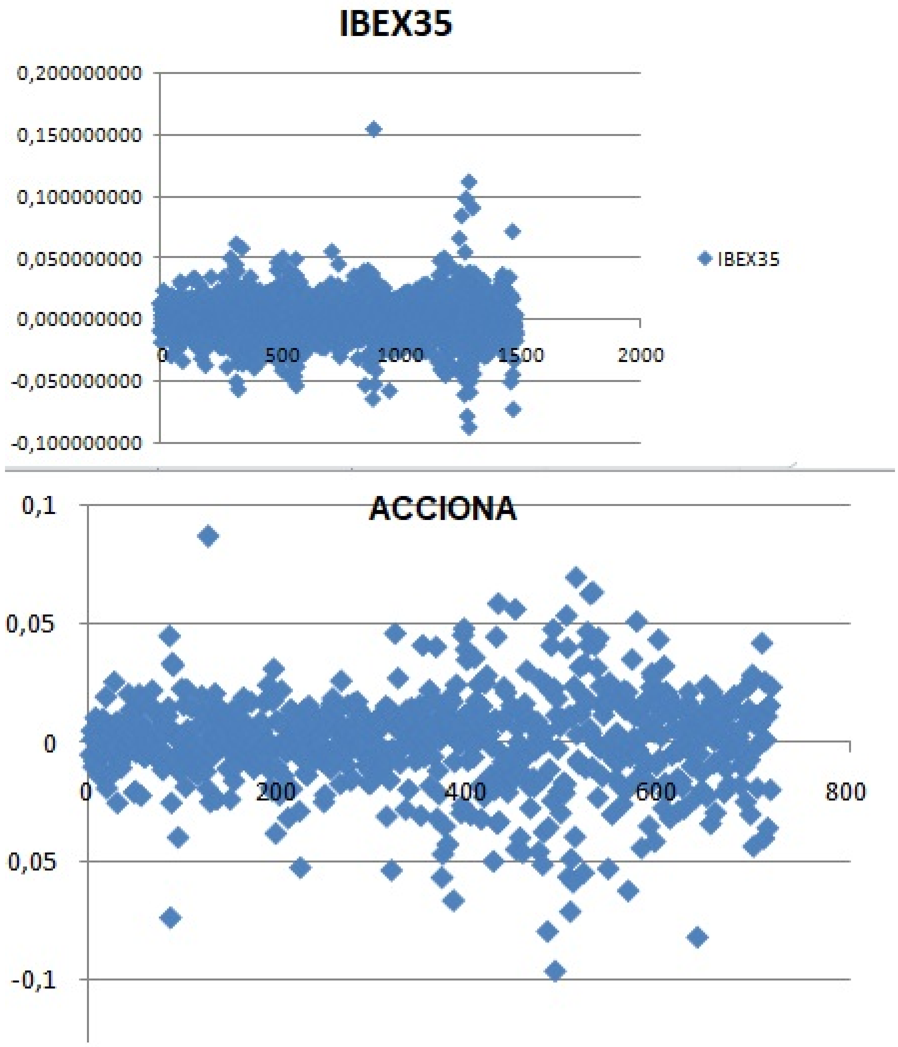

Figure 5.2: Dispersion of daily errors of IBEX35 and Acciona in the model of Sharpe

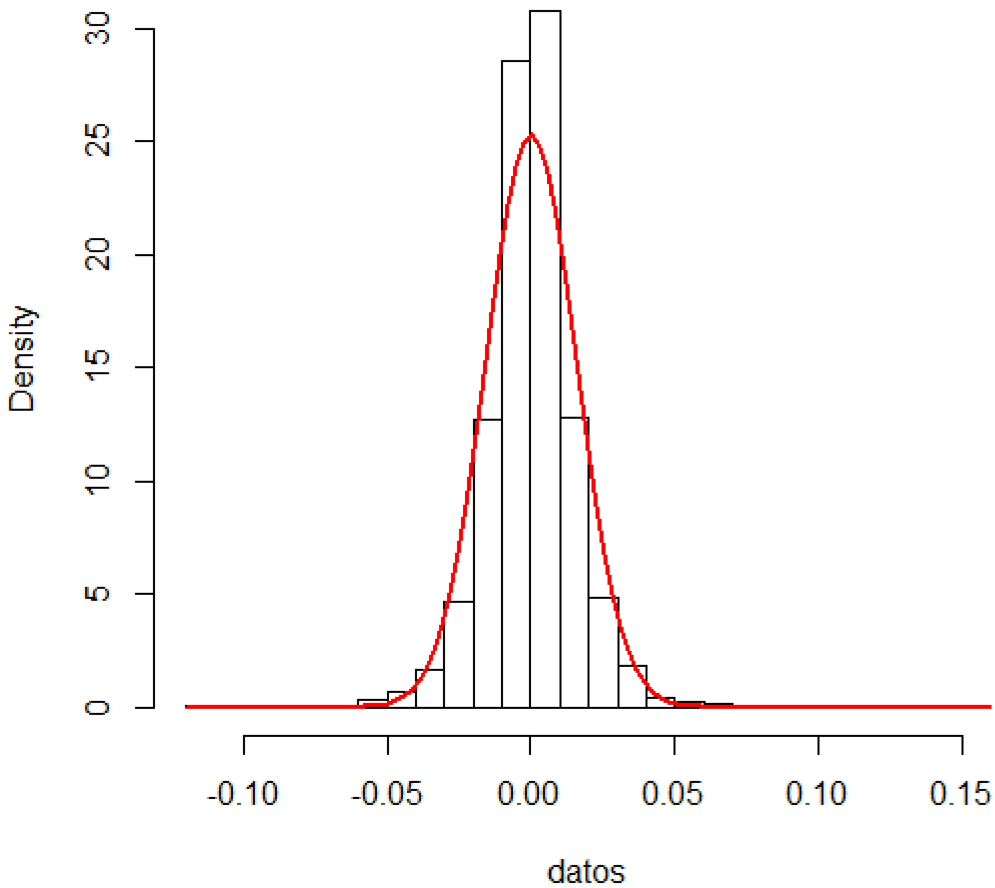

Figure 5.3: Histogram of expected returns of IBEX-35 against a normal distribution.

Chapter 6

Conclusions

Harry Markowitz developed a simple model which could explain the best way to invest given the preferences of the investor, the information of the market, the diversification theory as certain and some limitations. However, in the reality most people do not know their preferences, and they can change depending on the way an issue is informed; we do not have all the information of the market, and we have to do estimations in order to have basic information as expected return and risk; diversification can not hold in case the market is not stable and most of the assumptions that he makes, in order to make the problem simpler, do not apply in reality neither, assets are not perfectly divisible and it exists transaction costs.
It is a simple model which does not fit reality in a lot of issues, however it is a starting point where they can be defined new models with different assumptions and conclusions, for this reason it is important to understand this model.

It is needed to comment the differences existent between mathematics and finance. Mathematics part from assumptions to get a valid result, however, this theoretical assumptions are really restrictive or are not well defined in reality. Some of this cases are the risk-free asset, that it does not match its theoretical foundation or the risk itself. Every author has its consideration of risk, so we can find different finance models using variance, semivariance, value at risk, beta, and a greater list of typologies.
We can not process all the types of risk in a same way, for this reason we have considered risks that are related in this project. However, it is important to understand how other types of risk are defined, so considering other models that treat with other risk can be a interesting way to expand the notions of finance.

In the same way, it can be performed more deeper projects about APT model itself or extensions of the models studied in this project, investments considering dividends, foreign currency or transaction costs, which it would give models with less differences from reality. It can be considered KKT restrictions in order to study restrictions with inequalities; also we could consider more possible financial assets as the derivatives, and models as the Black-Scholes.

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Annexes

6.1 Code

In this section we include the codes in C which have been used in order to find efficient sets in Markowitz's Model:

6.1.1 Main code

The main code is code uses the functions showed before in order to find the frontier set of a portfolio given μ the vector with its expected returns and Σ the inferior triangular matrix of its covariance matrix.

```
/*Determina cartera eficient d'un portfolio i l'omple a la sortida*/
#include <stdio.h>
#include <stdlib.h>
#include <math.h>
#include "random_portfolios.h"
#include "prodVecVec.h"
#include "prodVecMatInf.h"
#include "cholesky.h"
#include "cholesky_inv.h"
#include "randEV_inf.h"
int main(void){
int i,j,n;
double* mu,*e,*aux,*aux2,*max,*min;
double a,b,c,d,det,res,rf=-0.2,muTan;
double **sigma;
FILE *entrada,*sortida1,*sortida2;
entrada=fopen("matrixD.txt","r");
fscanf(entrada,"%d",&n);
/*Guardem memoria als punters*/
mu=(double*)malloc(n*sizeof(double));
if(mu==NULL){
    printf("Error de memoria en el punter mu\n");
    exit(1);
}
```

```
aux=(double*)malloc(n*sizeof(double));
if(aux==NULL){
    printf("Error de memoria en el punter aux\n");
    exit(1);
}
aux2=(double*)malloc(n*sizeof(double));
if(aux2==NULL){
    printf("Error de memoria en el punter aux2\n");
    exit(1);
}
aux2[0]=1;
for(i=1;i<n;i++){
    aux2[i]=0;
}
e=(double*)malloc(n*sizeof(double));
if(e==NULL){
    printf("Error de memoria en el punter e\n");
    exit(1);
}
max=(double*)malloc(1*sizeof(double));
if(max==NULL){
    printf("Error en la memoria al guardar max\n");
    exit(1);
}
min=(double*)malloc(1*sizeof(double));
if(min==NULL){
    printf("Error en la memoria al guardar min\n");
    exit(1);
}
sigma=(double**)malloc(n*sizeof(double*));
for(i=0;i<n;i++){
        sigma[i]=(double*)malloc((i+1)*sizeof(double*));
        if(sigma[i]==NULL){
            printf("Error de memoria en el punter sigma[%d]\n",i);
            exit(1);
        }
}
printf("n=%d\n",n);
for(i=0;i<n;i++){
    fscanf(entrada,"%le",&mu[i]);
}
printf("mu= ");
for(i=0;i<n;i++){
    printf("%le ",mu[i]);
}
```

```
printf("\n");
for(i=0;i<n;i++){
    for(j=0;j<=i;j++){
        fscanf(entrada,"%le",&sigma[i][j]);
    }
}
fclose(entrada);
printf("Ha tancat la entrada\n");
printf("Matriu sigma=\n");
for(i=0;i<n;i++){
    for(j=0;j<=i;j++){
        printf("%le ",sigma[i][j]);
    }
    printf("\n");
}
printf("\n");

for(i=0;i<n;i++){
    aux[i]=0;
    e[i]=1;
}
/*Inicialitzem el maxim i el minim i els calculem per a les carteres aleatories*/
min[0]=mu[0];
max[0]=mu[0];
for(i=1;i<n;i++){
        if(mu[i]>max[0]){
            max[0]=mu[i];
        }
        if(mu[i]<min[0]){
            min[0]=mu[i];
        }
}
/*Calculem m carteres diferents i les posem al document aleat.txt*/
/*random_portfolios(n,100000);*/
printf("Random_portfolios calculated\n");
random_EV_inf(n,100000,mu,sigma,min,max);
printf("RandomEV calculated\n");
/*Calculem la variancia de la cartera eficient per cada rentabilitat des de min fins
/*Calculem la inversa*/
det=cholesky(n,sigma);
printf("Cholesky calculated, det=%le\n",det);
/*if(fabs(det)>1e-16){*/
    cholesky_inv(n,sigma);
    printf("Cholesky_inv calculated\n");
/*}
```

```
else{
    printf("La matriu no te inversa\n");
    exit(1);
}
*/
printf("Matriu inversa\n");
for(i=0;i<n;i++){
    for(j=0;j<=i;j++){
        printf("%le ",sigma[i][j]);
    }
    printf("\n");
}
printf("\n");
prodVecMatInf(n,mu,sigma,aux2);
a=prodVecVec(n,aux2,mu);
b=prodVecVec(n,aux2,e);
prodVecMatInf(n,e,sigma,aux);
c=prodVecVec(n,aux,e);
d=a-2*b*rf+c*rf*rf;
printf("a=%le , b=%le , c=%le, d=%le\n",a,b,c,d);
printf("min=%le max=%le\n",min[0],max[0]);
det=a*c-b*b; /*Redefinm el determinant*/
/*We calculate the tangent portfolio and its expected return*/
muTan=1./c+sqrt((-d/(det-d*c)-1./c)*det/c);
/*We notice that sigma^-1 * e and sigma^-1 * mu are already saved in aux and aux2*/
for(i=0;i<n;i++){
    aux[i]=aux[i]*(a-b*muTan)/det;
    aux2[i]=aux2[i]*(c*muTan-b)/det;
}
sortida1=fopen("fronteraC1.txt","w");
/*We save the tangent portfolio as the first line of "fronteraC1.txt"*/
fprintf(sortida1,"Tangent portfolio=( ");
for(i=0;i<n-1;i++){
    fprintf(sortida1,"%le, ",aux[i]+aux2[i]);
}
fprintf(sortida1,"%le). Expected return=%le \n",aux[n-1]+aux2[n-1],muTan);
/*We calculate the frontier points*/
aux2[0]=min[0];
while(min[0]<max[0]){
    res=sqrt((c/det)*(min[0]-b/c)*(min[0]-b/c)+1./c);
    fprintf(sortida1,"%le %le\n",res,min[0]);
    min[0]=min[0]+0.001;
}
fclose(sortida1);
sortida2=fopen("fronteraC2.txt","w");
```

```
min[0]=aux2[0];
while(min[0]<max[0]){
    res=sqrt(((min[0]-rf)*(min[0]-rf))/d);
    fprintf(sortida2,"%le %le\n",res,min[0]);
    min[0]=min[0]+0.001;
}
fclose(sortida2);
for(i=0;i<n;i++){
    free(sigma[i]);
}
free(e);
free(mu);
free(aux);
free(aux2);
free(sigma);
free(min);
free(max);
return 0;
}
```

6.1.2 Functions

1.Function randomportfolios(int n,int m) lets us calculate m random portfolios given n individual assets, it considers negative weights but just a few.

```
/*Donat un nombre d'actius n,un nombre de resultats m,calcula m portfolios aleatoris
#include <stdio.h>
#include <stdlib.h>
#include <math.h>
void random_portfolios(int,int);
void random_portfolios(int n,int m){
int i,j,k;
double aux=0;
double *w;
FILE *sortida;
sortida=fopen("aleatW.txt","w");
if(sortida==NULL){
    printf("It could not be possible to find sortida\n");
    exit(1);
}
w=(double*)malloc(n*sizeof(double));
if(w==NULL){
printf("w could not be saved\n");
exit(1);
}
```

```
srand(time(NULL));
for(i=0;i<m;i++){
    for(j=0;j<n-1;j++){
        w[j]=(double)rand()/(RAND_MAX*n);
        /*We consider that it is possible negative weights, but we consider them ir
        if(rand()%(2*n)==0){
            w[j]=-w[j];
        }
        aux+=w[j];
        while(aux>1){
        aux=aux-w[j];
        w[j]=(double)rand()/(RAND_MAX*n);
        if(rand()%(2*n)==0){
            w[j]=-w[j];
        }
        aux+=w[j];
        }
    }
    w[n-1]=1-aux;
        for(k=0;k<n;k++){
            fprintf(sortida,"%le  ",w[k]);
        }
        fprintf(sortida,"\n");
        aux=0;
}
fclose(sortida);
free(w);
return;
}
```

2.prodVecVec(int n, double *v, double *u) gives the escalar product between two vectors.

```
/*Calcula el producte esclar de dos vectors de
```

3.prodVecMatInf(int n, double *v, double **a, double *p) gives the product between vector v and matrix a, considering that matrix a is saved in a triangular way, the result is assigned to p.

```
/*Function that gives the product between a vector v and a matrix A, symmetric whic
#include <stdio.h>
#include <stdlib.h>
void prodVecMatInf(int,double *,double **,double *);
void prodVecMatInf(int n,double *v,double **a,double *prod){
    int i, j;
    double sum=0;
    for(i=0;i<n;i++){
        for(j=0;j<n;j++){
```

```
            if(j<=i){
                sum+=a[i][j]*v[j];
            }
            else{
                sum+=a[j][i]*v[j];
            }
        }
        prod[i]=sum;
        sum=0;
    }
    return;
}
```

4. cholesky(int n, double **a) calculates the Cholesky descomposition L*L^T=A and saves matrix L into a and gives the determinant of the matrix A.

```
/*Let be A a symmetric and positive-definite matrix saved in a inferior triangular w
#include <stdio.h>
#include <stdlib.h>
#include <math.h>
double cholesky(int,double **);
double cholesky(int n,double **a){
int i,j,k;
double sum=0,det=1;
for(i=0;i<n;i++){
    for(k=0;k<i;k++){
        sum+=a[i][k]*a[i][k];
    }
    a[i][i]=sqrt(a[i][i]-sum);
    sum=0;
    for(j=i+1;j<n;j++){
        for(k=0;k<i;k++){
            sum+=a[i][k]*a[j][k];
        }
        a[j][i]=(a[j][i]-sum)/a[i][i];
        sum=0;
    }
    det=det*a[i][i];
}
det=det*det;
return det;
}
```

5.*cholesky$_i$nv*(int n, double **l) returns the inverse matrix of a, once this has been transformed into L by cholesky.

```
/*Calculates the inverse of a symmetric positive-definite matrix A, which has been c
```

```
#include <stdio.h>
#include <stdlib.h>
#include <math.h>
#include "cholesky_solve.h"
void cholesky_inv(int,double **);
void cholesky_inv(int n,double **l){
int i,j;
double *b;
double **inv;
b=malloc(n*sizeof(double));
if(b==NULL){
    printf("Error trying to save b\n");
    exit(1);
}
inv=(double**)malloc(n*sizeof(double*));
if(inv==NULL){
    printf("Error trying to save inv\n");
    exit(1);
}
for(i=0;i<n;i++){
    inv[i]=(double*)malloc((i+1)*sizeof(double));
    if(inv[i]==NULL){
        printf("Error trying to save inv[%d]\n",i);
        exit(1);
    }
}
for(i=0;i<n;i++){
    b[i]=0;
}
for(i=0;i<n;i++){
    b[i]=1; /*We consider b as the i^th canonic vector in each step*/
    cholesky_solve(n,l,b);
    for(j=i;j<n;j++){
        inv[j][i]=b[j];
    }
    for(j=0;j<n;j++)
    b[j]=0;
}
for(i=0;i<n;i++){
    for(j=i;j<n;j++){
        l[j][i]=inv[j][i];
    }
}
free(b);
for(i=0;i<n;i++){
```

```
    free(inv[i]);
}
free(inv);
return;
}
```

6.*cholesky$_s$olve* gives the solution of a system of the type AX=b once A is transformed into L, so it solves L*L^T*x=b.

```
/*Solves the problem Ax=b using the decomposition A=L*L^T from cholesky, where A is
void cholesky_solve(int,double**,double*);
void cholesky_solve(int n,double **l,double *b){
    int i,j;
    double sum=0;
    /*We calculate the solution of Ly=b and save the solution into b*/
    for(i=0;i<n;i++){
        for(j=0;j<i;j++){
            sum+=l[i][j]*b[j];
        }
        b[i]=(b[i]-sum)/l[i][i];
        sum=0;
    }
    /*We calculate the solution of (L^T)x=y and save the solution into b*/
    for(i=n-1;i>=0;i--){
        for(j=n-1;j>i;j--){
            sum+=l[j][i]*b[j];
        }
        b[i]=(b[i]-sum)/l[i][i];
        sum=0;
    }
    return;
}
```

7.*randEV$_i$nf*(int n, int m, double mu*, double **sigma, double *mini, double *maxi) calculates the expected values and standard deviation of the random portfolios created above, mu is the vector with the expected values of individual assets and sigma is the covariance matrix saved as a triangular matrix. the minimal and maximal expected returns are calculated in order that the plots from data are adjusted.

```
/*Given random weights, it calculates their expected value and variance, from a cov
#include <stdio.h>
#include <stdlib.h>
#include <math.h>
void random_EV_inf(int,int,double*,double**,double*,double*);
void random_EV_inf(int n,int m, double *mu,double **sigma,double* mini, double *max
int i,j;
double res,aux;
```

```
double *w,*prod;
w=(double*)malloc(n*sizeof(double));
if(w==NULL){
printf("Error en la memoria al guardar w\n");
exit(1);
}
prod=(double*)malloc(n*sizeof(double));
if(prod==NULL){
printf("Error en la memoria al guardar prod\n");
exit(1);
}
FILE *entrada,*sortida;
entrada=fopen("aleatW.txt","r");
sortida=fopen("randEV_inf.txt","w");
for(i=0;i<m;i++){
     for(j=0;j<n;j++){
          fscanf(entrada,"%le",&w[j]);
     }
     res=prodVecVec(n,mu,w);
     prodVecMatInf(n,w,sigma,prod);
     aux=sqrt(prodVecVec(n,w,prod));
     /*Calculamos el mÃnimo y el mÃ¡ximo*/
     if(res>maxi[0]){
          maxi[0]=res;
     }
     if(res<mini[0]){
          mini[0]=res;
     }
     fprintf(sortida,"%le %le\n",aux,res);
}
fclose(entrada);
fclose(sortida);
free(w);
free(prod);
return;
}
```

6.2 Tables

Company	Total_D	Total_M	2013_2015_D	2013_2015_M	2013_2018_D	2013_2018_M	2008_2010_D	2008_2010_M	2008_2013_D	2008_2013_M
Acciona	0,0304675	0,028353	0,00039869	0,00575918	-0,02372314	0,00449022	-1,7275256	-0,56171055	-0,09218202	-0,07330054
ACS	0,0277701	0,043542	-0,00118393	0,01851013	-0,03170358	-0,00559504	1,37483654	0,13942053	0,00729459	0,01273317
Bankinter	0,00227902	0,02131449	-0,03082915	-0,00856954	-0,09331018	-0,02583032	-1,0346061	-0,30700918	0,01123337	9,8554E-05
BBVA	0,14370714	0,08824023	0,16498099	0,12088478	0,19092261	0,13447031	-0,13138307	-0,03644656	0,11569027	0,07424638
Caixabank	0,01352848	0,0380238	0,01512032	0,01844033	-0,03192218	0,00109881	0,38355122	-0,00318694	0,06110144	0,0833945
Enagaás	0,02036516	0,05966593	0,00393266	0,02228462	-0,00017722	0,03633695	-0,19587291	-0,00221041	0,08517285	0,10247023
Ferrovial	0,00488617	0,02826284	-0,04093111	0,00739699	0,04056405	0,04479999	0,57620336	0,13263764	0,15632383	0,12637808
Grifols	-0,01817133	0,01758426	0,00154304	0,00814118	0,01853752	-0,00803567	0,15249311	0,00638681	0,19225339	0,11197472
Iberdrola	0,07895389	0,07765144	0,05920566	0,05738139	0,02587265	0,05191016	-0,55474911	-0,31712895	-0,07987046	-0,09202729
Inditex	-0,01649141	0,05086299	0,00620852	0,03402628	0,05083823	0,06934493	2,63926092	0,71734546	0,36531796	0,38150423
Indra	0,03475726	0,04031956	0,02010567	0,02953961	0,01908748	0,03174158	-0,40849227	-0,63920074	-0,00573717	-0,03881195
Mapfre	-0,00383789	0,04416412	0,05444011	0,06069403	0,03127836	0,0450431	0,79497969	0,0759687	0,12560787	0,13715714
Mediaset	0,01797746	0,02164586	-0,03055457	-0,0092611	-0,00687508	0,02476057	0,02309957	0,03330402	0,03780434	0,04124182
Naturgy	0,06413969	0,04400422	0,03343637	0,03612756	0,04430325	0,01603059	-3,19614456	-0,90450192	-0,08545939	-0,04302866
REE	0,01956638	0,07121925	-0,03716833	-0,00599803	0,02237088	-0,00165524	0,03603543	-0,03235465	0,11258732	0,13901221
Repsol	0,02891423	0,05946158	0,18460184	0,14589032	0,03575838	0,06327379	1,24048154	2,92513059	0,14428436	0,08159022
Sabadell	0,08090579	0,02384304	0,0379137	0,03803263	0,05399371	0,06304111	-2,97461612	-0,05350023	-0,15498577	-0,02693728
Santander	0,12539416	0,07659579	0,30776524	0,20039338	0,20423273	0,15509158	1,5745392	0,59914179	0,05372611	0,0032986
Tećnicas Reunidas	0,0136459	0,02917523	0,04362484	0,08037443	0,06122567	0,07160936	1,35818971	0,10938667	0,09903516	0,05938213
Telefońica	0,28260795	0,08998864	0,16330744	0,07654348	0,31490122	0,1438181	1,23762981	-1,13497118	-0,13974795	-0,06890939
IBEX35	0,04863435	0,04608172	0,04408201	0,06340835	0,07382464	0,08425513	-0,16791036	0,25349909	-0,00945009	-0,01146687

Table 6.1: Tangent portfolios given by Sharpe Model.

Company	Total_D	Total_M	2013_2015_D	2013_2015_M	2013_2018_D	2013_2018_M	2008_2010_D	2008_2010_M	2008_2013_D	2008_2013_M
Acciona	-0,065693912	-0,177053124	-0,086327135	-0,137766277	-0,086969045	-0,087631491	-7,563067924	-0,566236763	-4,293093549	-0,536249984
ACS	0,028989703	0,072779691	0,001053108	0,065008915	-0,054253597	0,213999805	5,981508752	-0,553235814	-0,154751515	0,458318048
Bankinter	0,001941244	0,123928888	-0,022784863	0,135014078	-0,065779807	0,341708149	-2,604695841	0,194006773	0,136552864	-0,018040766
BBVA	-0,121270068	-0,23461162	-0,008952371	-0,037869723	0,066774185	-0,279456879	0,256403822	-0,577005081	6,060193276	0,626485366
Caixabank	0,049408541	0,223864369	0,095584409	-0,174498356	0,006293295	0,352744046	3,892161195	0,129267642	2,262599135	0,185943308
Enagaś	0,300356005	-0,046360878	0,176460147	0,14348054	0,265600988	-0,218872068	0,021236666	0,809807467	-0,331941476	-0,501967336
Ferrovial	0,024451327	0,09280234	0,242096141	0,421713009	0,065616816	0,44650522	1,149405906	0,059898218	2,845608554	0,363248062
Grifols	0,153864374	0,226746481	0,084959548	-0,065903094	0,090666062	0,122753712	-0,042355484	-0,020588637	3,634098714	0,297471743
Iberdrola	-0,040196749	-0,00496927	0,33086056	0,437153572	0,215301114	0,807990416	-1,338321712	0,309936571	-2,774660933	-0,688252504
Inditex	0,088428582	0,637878705	0,057105537	-0,04106743	0,059277993	-0,061807914	6,800100855	0,881797274	8,598276508	2,318286909
Indra	0,12084808	0,112826261	0,0141028	0,086905699	0,052159985	0,049309919	-0,891382612	-0,267350108	-1,445830767	-0,329173563
Mapfre	0,002694266	0,166274006	-0,085726367	0,106305521	0,020343009	-0,064036105	3,291721827	0,464780982	3,425607053	1,334941291
Mediaset	-0,002731496	-0,082627449	0,049204985	0,235479796	0,042096623	0,171207475	-1,374823071	0,053959721	-0,256262168	-0,162082196
Naturgy	0,03577931	-0,150130538	0,071713469	0,003863446	0,040766094	0,189091679	-8,970200846	-0,570379243	-4,323248356	-0,419555801
REE	0,118159244	0,49751655	0,088815721	-0,019786403	0,075800691	-0,030343874	1,922178934	-0,186186252	2,246380593	0,231636347
Repsol	0,014838992	0,243183679	-0,034937069	-0,305201268	0,010229657	0,236993782	1,423807763	1,819909639	2,3189845	0,709007405
Sabadell	0,130738329	-0,060117937	-0,053611477	0,187682724	0,038550224	-0,253049107	-8,898808754	0,483576324	-7,548572252	-0,48764945
Santander	-0,221203326	-0,234964684	-0,2362637	-0,118173334	-0,172084589	-0,2815307	1,22202437	-0,574559852	-3,860425957	-1,687738203
Tećnicas Reunidas	0,068839068	-0,001230302	0,175141421	0,368507239	0,163123033	-0,267627658	4,65479115	-0,210314855	1,385047282	0,255888232
Telefońica	0,311758488	-0,405735168	0,141505136	-0,290848656	0,166487269	-0,387948406	2,068315004	-0,681084005	-6,924561507	-0,950516909

Table 6.2: Tangent portfolios given by Markowitz Model.