Heavy Tetraquarks

Rubén Oncala Mesa.
*Facultat de Física, Universitat de Barcelona, Diagonal 645, 08028 Barcelona, Spain.**

Advisor: Dr. Joan Soto Riera.
(Dated: January 18, 2015)

This final degree project aims at studying the stability of a heavy tetraquark in nature using QCD formalism and some known approximations.We have chosen $bb\bar{c}\bar{c}$ configuration. The procedure has been: firstly we have calculated the colour factor involved in the interaction of each quark inside a tetraquark system; secondly, using the colour factor, we have obtained the potential between the different quarks to thirdly, be able to determine the energy of this system using the Rayleigh-Ritz variational approximation. We finally have compared the thetraquark ground state energy with the energy of two mesons $b\bar{c}$ and we have observed that the first is smaller. Therefore we have concluded that our tetraquark is stable.

I. INTRODUCTION

A tetraquark is a subatomic particle formed by two quarks and two antiquarks. It is classified as an exotic meson. Here we are going to study a heavy tetraquark, hence our system is composed by two bottom quarks and two charm antiquarks, so we have $bb\bar{c}\bar{c}$ configuration.

If we want to understand the interactions in a tetraquark it is necessary to analyse the interaction between two quarks, which is mediated by a gluon. Gluons are bosons responsible for the interaction between two quarks, exchanging its colour or anticolour charge (i.e mediation of the strong force), they also carry a colour charge which leads to the possibility of self interaction. There are three different colour charges: red(R), green(G) and blue(B).

SU(3) is the gauge group of QCD and the standard generators in physics are $T^a = \frac{1}{2}\lambda_a$, where λ_a are the eight Gel-Mann matrices. There are 8 different gluon types, 6 with colour/anticolour and 2 without colour, which form a SU(3) octet. Exact SU(3) symmetry of colour gives: $3 \times 3^* = 8 + 1$.

II. THE EXISTENCE OF TETRAQUARKS

The wave function of a hadron is:

$$\psi = \psi_{space} \times \psi_{spin} \times \psi_{flavour} \times \psi_{colour} \tag{II.1}$$

Quarks are fermions so the total wave function must be antisimetric since Pauli exclusion principle must be respected.

It is interesting to study spin and parity of a tetraquark. Each quark has 1/2 intrinsic spin, therefore tetraquark has a total spin S_t result to coupling all individual spins:

$$\begin{aligned} &S_t = 1/2 \times 1/2 \times 1/2 \times 1/2 = \\ &(0+1) \times (0+1) = (0) + (1) + (1) + (0+1+2) \end{aligned} \tag{II.2}$$

We are going to see later that, in our tetraquark, the quarks b are linked forming a diquark bb. In the same way, the antiquarks $\bar{c}$ are forming an anti-diquark $\bar{c}\bar{c}$. So, the space, spin and flavour wave function in eq.(II.1) of the diquark and anti-diquark must be symmetric so the colour component must be antisymmetric.

Therefore, the total spin of our tetraquark eq.(II.2) will be from 1x1=0+1+2 and the colours of the diquark and antidiquark must be x_1x_2 for bb and $\bar{x_1}\bar{x_2}$ for $\bar{c}\bar{c}$, where x_i is the colour R, G, B or anticolour $\bar{R}, \bar{G}, \bar{B}$

If we want to calculate the parity of a tetraquark, we must bear in mind that $P_q = 1$ and $P_{\bar{q}} = -1$. Moreover we use the fact that the two quarks and the two antiquarks are linked. So the parity is:

$$\begin{aligned} &P = (P_{b1}P_{b2}(-1)^{L_{12}})(P_{\bar{c3}}P_{\bar{c4}}(-1)^{L_{34}})(-1)^{L_{1234}} = \\ &= (-1)^{L_{12}}(-1)^{L_{34}}(-1)^{L_{1234}} = 1 \end{aligned} \tag{II.3}$$

Where $L_{12} = L_{34} = L_{1234} = 0$ for the ground state.

An important consideration for the consistency of the gauge theory is that the colour of the physical particle is null. In other words, the physical particle must be a singlet colour state.

In the following we work out the colour charge of mesons, baryons and tetraquarks in order to find singlet colour states. Because we are working in a SU(3) space we always use dimension N=3 referring to the three possible colours in quarks(3) or anticolors in antiquarks(3*). The result of the coupling gives the possible states of the particle.

Meson:

$$3 \times 3^* = 8 + 1 \tag{II.4}$$

Baryon:

$$\begin{aligned} &3 \times 3 \times 3 = (6 + 3^*) \times 3 = \\ &= 6 \times 3 + 3^* \times 3 = 10 + 8 + 8 + 1 \end{aligned} \tag{II.5}$$

The $bb\bar{c}\bar{c}$ tetraquark:

$$\begin{aligned} &3^*_{\bar{c}} \times 3^*_{\bar{c}} \times 3_b \times 3_b = (6^*_{\bar{c}\bar{c}} + 3_{\bar{c}\bar{c}}) \times (6_{bb} + 3^*_{bb}) = \\ &= 6^*_{\bar{c}\bar{c}} \times 6_{bb} + 6^*_{\bar{c}\bar{c}} \times 3^*_{bb} + 3^*_{\bar{c}\bar{c}} \times 6_{bb} + 3_{\bar{c}\bar{c}} \times 3^*_{bb} = \\ &= [(35+1) + (10^* + 8^*) + (10+8) + (1+8)]_{\bar{c}\bar{c}bb} \end{aligned} \tag{II.6}$$

This representation is extensively illustrated in[1]. We can note in eq.(II.6) we have not done an arbitrary coupling, since we are assuming that $m_b > m_c$ and therefore that bb form a compact core. For this reason we coupling bb and $\bar{c}\bar{c}$ first. We are going to extend this idea further.

*Electronic address: rmesaonc13@alumnes.ub.edu

We can see in eq.(II.6) that the components that give the singlet state are $3 \times 3^*$ and $6 \times 6^*$, which we denote as T and S respectively. Moreover this can also be denoted using δ nomenclature, as shown in Appendix (A.1 and A.2).

III. THE COLOUR FACTOR

The coupling strength in QCD is $\frac{1}{2}c_1c_2\alpha_s$, where $\alpha_s = \frac{g^2}{4\pi}$ and g is the strong coupling constant, depend on energy [2], and c_1, c_2 are colour coefficients. The term $|\frac{1}{2}c_1c_2|$ is called colour factor.

The free Lagrangian in QCD is:

$$L_{QCD} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - g\bar{\psi}\gamma^\mu T_a\psi G^a_\mu - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a \quad \text{(III.1)}$$

Where ψ is the wave function of the quark, γ^μ are the Dirac matrices and $G^a_{\mu\nu}$ is the field strength tensor [2].

The first term in the Lagrangian eq.(III.1) is the quark propagator, the second one is the quark-gluon vertex and the third one is the gluon propagator. We want to study the interaction between quarks, so we are interested just in second term.

We can interpret this term as a Feymann diagram [3].

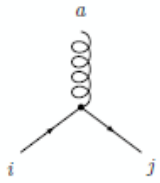

FIG. 1: Feymann diagram of quark-gluon vertex

Corresponding to

$$-igT^a_{ji} \quad \text{(III.2)}$$

The colour factor (c.f) of an interaction (Int) in a tetraquark system, also called projection operator, is the result of applying two vertex and two invariant deltas in a wave function eq.(III.4). This gives us the same wave function multiplied by a number, this number is the c.f of the interaction.

$$(Int) * \psi = c.f * \psi' \quad \text{(III.3)}$$

We have written ψ' on the final wave function because we want to distinguish the incoming and outgoing function wave. Even so there is no difference between them. The interaction between a quark and an antiquark in a tetraquark system is:

$$Int = (T^a)^\alpha_{\alpha'}(T^a)^{\gamma'}_\gamma \delta^\beta_{\beta'}\delta^{\delta'}_\delta \quad \text{(III.4)}$$

Note that we write in-quarks as a super-index and out-quarks as a sub-index. So the interaction $\alpha\gamma$applied over the T state give us a colour factor of $(1/3)_{T'}$ and $(1/2)_{S'}$ eq.(III.5).

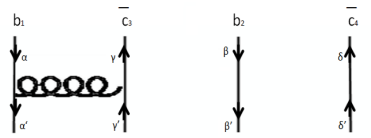

FIG. 2: Feymann diagram of quark-antiquark interaction in a tetraquark system

$$Int_{\alpha\gamma} * (\mathtt{T}) = (T^a)^\alpha_{\alpha'}(T^a)^{\gamma'}_\gamma \delta^\beta_{\beta'}\delta^{\delta'}_\delta \times (\frac{1}{2}(\delta^\gamma_\alpha\delta^\delta_\beta - \delta^\delta_\alpha\delta^\gamma_\beta)) =$$
$$= \frac{N^2-N-2}{4N}\mathtt{T'} + \frac{N-1}{4}\mathtt{S'} = \frac{1}{3}\mathtt{T'} + \frac{1}{2}\mathtt{S'} \quad \text{(III.5)}$$

Where we have used the delta's properties in [6] and appendix relations (A.1) and (A.3). We can do this for all possible combinations of interactions and we get:

$$\begin{aligned}
Int_{\alpha\beta} * (\mathtt{T}) &= -\frac{2}{3}\mathtt{T'} \\
Int_{\alpha\delta} * (\mathtt{T}) &= \frac{1}{3}\mathtt{T'} - \frac{1}{2}\mathtt{S'} \\
Int_{\gamma\beta} * (\mathtt{T}) &= \frac{1}{3}\mathtt{T'} - \frac{1}{2}\mathtt{S'} \\
Int_{\alpha\delta} * (\mathtt{T}) &= -\frac{2}{3}\mathtt{T'} \\
Int_{\beta\delta} * (\mathtt{T}) &= \frac{1}{3}\mathtt{T'} + \frac{1}{2}\mathtt{S'}
\end{aligned} \quad \text{(III.6)}$$

And the same for the S state:

$$\begin{aligned}
Int_{\alpha\gamma} * (\mathtt{S}) &= 1\mathtt{T'} + \frac{5}{6}\mathtt{S'} \\
Int_{\alpha\beta} * (\mathtt{S}) &= \frac{2}{3}\mathtt{S'} \\
Int_{\alpha\delta} * (\mathtt{S}) &= 1\mathtt{T'} + \frac{5}{6}\mathtt{S'} \\
Int_{\gamma\beta} * (\mathtt{S}) &= -1\mathtt{T'} + \frac{5}{6}\mathtt{S'} \\
Int_{\alpha\delta} * (\mathtt{S}) &= \frac{1}{3}\mathtt{S'} \\
Int_{\beta\delta} * (\mathtt{S}) &= 1\mathtt{T'} + \frac{5}{6}\mathtt{S'}
\end{aligned} \quad \text{(III.7)}$$

We can see that we obtain a direct term for which applying the interaction gives us the same out-state and a crossed term that gives us the other out-state.

IV. POTENTIAL IN TETRAQUARKS

Colour factor has a strong relation with the potential. However, we cannot directly take the potential between two quarks as $V = c.f\frac{\alpha_s}{r}$, we must take some previous considerations.

1. Sign:

   On one hand, we must consider if the potential is attractive or repulsive. Actually, colour interaction at

short distance is the same as electromagnetic interaction, provided we replace α by α_S and multiply by QCD colour factor. So for $q\bar{q}$ interaction we put a minus sign in the potential, contrarily we impose a plus sign in the potential if interaction is between two quarks/antiquarks. Note that even so total sign can change because c.f has an extra sign:

$$V_{q\bar{q}} \sim -c.f\frac{\alpha_S}{r} \qquad V_{qq} \sim +c.f\frac{\alpha_S}{r} \tag{IV.1}$$

2. Normalization:

On the other hand, we must consider the normalization factor. As we can see, in the previous expressions of the potential, we have not written an equal. The reason is that colour factor must be normalized. In a simple case where we don't have crossed terms, normalization factors are trivial. However we must consider these terms here.

$$[Int]\begin{pmatrix} N_T\mathtt{T} \\ N_S\mathtt{S} \end{pmatrix} = \begin{pmatrix} V_{TT} & V_{TS} \\ V_{ST} & V_{SS} \end{pmatrix}\begin{pmatrix} N_T\mathtt{T'} \\ N_S\mathtt{S'} \end{pmatrix} \tag{IV.2}$$

So we get:

$$[Int]N_T\mathtt{T} = N_TV_{TT}\mathtt{T'} + N_SV_{TS}\mathtt{S'}$$
$$[Int]N_S\mathtt{S} = N_TV_{ST}\mathtt{T'} + N_SV_{SS}\mathtt{S'}$$

We can obtain the normalization factor doing:

$$N_T^2(\mathtt{T}\times\mathtt{T}) = 1$$

Where $T \times T$ is, using appendix (A.1):

$$\frac{1}{2}(\delta^\gamma_\alpha\delta^\delta_\beta - \delta^\delta_\alpha\delta^\gamma_\beta)\frac{1}{2}(\delta^\gamma_\alpha\delta^\delta_\beta - \delta^\delta_\alpha\delta^\gamma_\beta) = \frac{1}{2}(N^2 - N)$$

So:

$$N_T = \frac{1}{\sqrt{\frac{1}{2}(N^2-N)}} = \frac{1}{\sqrt{3}} \tag{IV.3}$$

In the same way we can obtain:

$$N_S = \frac{1}{\sqrt{\frac{1}{2}(N^2+N)}} = \frac{1}{\sqrt{6}} \tag{IV.4}$$

At this point we are able to write the components of the potential:

For the T structure we have:

$$V_{TT} = \alpha_S[-\frac{2/3}{r_{12}} - \frac{1/3}{r_{13}} - \frac{1/3}{r_{14}} - \frac{1/3}{r_{23}} - \frac{1/3}{r_{24}} - \frac{2/3}{r_{34}}] \tag{IV.5}$$

$$V_{TS} = \alpha_S(\frac{N_S}{N_T})[\frac{0}{r_{12}} - \frac{1/2}{r_{13}} + \frac{1/2}{r_{14}} + \frac{1/2}{r_{23}} - \frac{1/2}{r_{24}} + \frac{0}{r_{34}}] \tag{IV.6}$$

On the other hand for the S structure the potential is:

$$V_{SS} = \alpha_S[\frac{1/3}{r_{12}} - \frac{5/6}{r_{13}} - \frac{5/6}{r_{14}} - \frac{5/6}{r_{23}} - \frac{5/6}{r_{24}} + \frac{1/3}{r_{34}}] \tag{IV.7}$$

$$V_{ST} = \alpha_S(\frac{N_T}{N_S})[\frac{0}{r_{12}} - \frac{1}{r_{13}} + \frac{1}{r_{14}} + \frac{1}{r_{23}} - \frac{1}{r_{24}} + \frac{0}{r_{34}}] \tag{IV.8}$$

Where r_{12} is $|\bar{r_1} - \bar{r_2}|$, index 1,2,3 and 4 are refereed to quarks $b_1, b_2, \bar{c}_3$ and $\bar{c}_4$

In the eq.(IV.5) potential component, for the direct term (TT), every single components of potentials between quarks are attractive. Moreover quarks with same flavour are more strongly linked. As a first approximation, we can assume that b quarks form a heavy core because $m_b > m_c$, and $\bar{c}$ quarks are orbiting around this. We could exploit that $\bar{c}\bar{c}$ are strongly linked and think in a two bodies system. However, we are going to use the similarities between this structure and He atom to make a better approximation.

On the other hand, we can assume, in a first approximation, that $r_{13} \approx r_{14}$ and $r_{23} \approx r_{24}$ if we think in the He structure. So, crossed term (TS), the potential eq.(IV.6), will be at first approximation cancelled because colour factors involves are equals with different sign. We can neglect this crossed term in front of the direct term.

The differences we have to consider between our state and He atom are: firstly, electrons are repulsive between them and here $\bar{c}$ quarks are attractive; and secondly, the core mass in our tetraquark is not "infinite", which is a very good approximation in He atom. However we are assuming for the calculations that $m_b >> m_c$ which is not exactly correct, look at this in the appendix (A.10). Therefore we have the ground state in the limit of the nucleus has an infinite mass.

In the eq.(IV.7) potential, the components of the direct term (SS) draw a configuration where we cannot cancel easily the crossed term (ST), the potential component eq.(IV.8). We have attractive and repulsive terms that locate the quarks in a similar configuration to the hydrogen molecule H_2 . So, we cannot solve the problem analytically and should use computational methods. Is for this reason we have focus on the first case.

V. A PROPOSAL TO APPROXIMATE THE ENERGY OF THE GROUND STATE

In this section we want to deeply analyse the T state energy. We have taken the potential eq.(IV.5) neglecting crossed terms and assuming: In one hand $r_{13} = r_{14} = r_1$ and $r_{23} = r_{24} = r_2$; and on the other hand $m_b >> m_c$. So we have a ground state potential like:

$$V_{TT} = \alpha_S[-\frac{2/3}{r_{12}} - \frac{1/3}{r_1}Z - \frac{1/3}{r_2}Z - \frac{2/3}{r_{34}}] \tag{V.1}$$

Where we have taken Z=2.

From here on we will use this potential and perturbation methods. Our objective is obtaining the ground state energy. We know that a good first approximation for the He atom is the Rayleigh-Ritz variational method [5]. So our proposal is applying this method to tetraquark system. This consist in exchanging the nucleus of the system by an effective nucleus

so that this new nucleus contains the interaction between the orbiting particles [5]. Lastly we will introduce the energy of our real nucleus to obtain the state energy of our tetraquark.

The expected value of the crust's Hamiltonian is:

$$E = \frac{<\phi|H|\phi>}{<\phi|\phi>=1} = \\ =<\phi|T_1+T_2-\frac{1/3}{r_1}Z\alpha_s-\frac{1/3}{r_2}Z\alpha_s-\frac{2/3}{r_{34}}\alpha_s|\phi> \tag{V.2}$$

Starting from the first term:

$$<\phi|T_1|\phi> =<\varphi_{1s}^{Z_{ef}}(1)\varphi_{1s}^{Z_{ef}}(2)|T_1|\varphi_{1s}^{Z_{ef}}(1)\varphi_{1s}^{Z_{ef}}(2)> \\ =<\varphi_{1s}^{Z_{eff}}(1)|T_1|\varphi_{1s}^{Z_{eff}}(1)>=\{A.4\}= \\ =-E_{n=1}=\{A.5\}=\frac{Z_{eff}^2}{2}\frac{1}{a_{bbc}}\frac{1}{3}\alpha_s$$

Where a_{bbc} is the Bohr radius of a system of two linked quarks bb in the centre and one antiquark $\bar{c}$ orbiting around, in a tetraquark environment. We can look at this in appendix (A.7). Note that we would have one different α_s for each interaction since α_s depend on energy[2]. We approach, consistently with the rest of approximations, $\alpha_s(bc)$ and $\alpha_s(bbc)$ as $\alpha_s(c)$ because μ_{bc} and μ_{bbc} is nearest to m_c than m_b. Look at this in[9].

On the other hand, we have obtained the colour factor involved in this expression $-1/3$ of the second and third term of eq.(V.1)

In the same way we can do:

$$<\phi|T_2|\phi>=\frac{Z_{eff}^2}{2}\frac{1}{a_{bbc}}\frac{1}{3}\alpha_s$$

A known property in atomic physic is:

$$<\phi|\frac{1}{r_1}|\phi>=<\varphi_{1s}^{Z_{eff}}(1)|\frac{1}{r_1}|\varphi_{1s}^{Z_{eff}}(1)>= \\ =\{A.6\}=Z_{eff}\frac{1}{a_{bbc}}$$

And hence:

$$<\phi|\frac{1}{r_2}|\phi>=Z_{eff}\frac{1}{a_{bbc}}$$

Lastly we can prove that:

$$<\phi|\frac{1}{r_{34}}|\phi>=\frac{5}{8}Z_{eff}\frac{1}{a_{bbc}}$$

This is a known expression in atomic physic. We can find this developed in [5].

If we joining everything in eq.(V.2), we obtain a first expression for the energy:

$$E=2(\frac{Z_{eff}^2}{2}\frac{\alpha_s}{a_{bbc}}\frac{1}{3})-2(Z_{eff}Z\frac{\alpha_s}{a_{bbc}}\frac{1}{3})-\frac{5}{8}Z_{eff}\frac{\alpha_s}{a_{bbc}}\frac{2}{3} \\ =(\frac{Z_{eff}^2}{3}-\frac{2}{3}ZZ_{eff}-\frac{5}{12}Z_{eff})\frac{\alpha_s}{a_{bbc}} \tag{V.3}$$

Minimizing the energy with respect Z_{eff}:

$$\frac{\partial E}{\partial Z_{eff}}=(\frac{2}{3}Z_{eff}-\frac{2}{3}Z-\frac{5}{12})\frac{\alpha_s}{a_{bbc}}=0$$

Obtaining an effective nucleus charge, using Z=2 we get:

$$Z_{eff}=Z+\frac{5}{8}=\frac{21}{8} \tag{V.4}$$

Now if we introduce this effective charge in the energy expression eq.(V.3) we obtain the energy of the crust:

$$E_{crust}=-2.2969\frac{\alpha_s}{a_{bbc}}=-1.3349\alpha_s^2m_c=-1978MeV \tag{V.5}$$

Where a_{bbc} is the Bohr radius of a system of two linked quarks bb in the centre and one antiquark $\bar{c}$ orbiting around, in a tetraquark environment.

Finally we must add the energy of the nucleus to obtain the energy of the tetraquark system. We calculate it as a diquark bb:

$$E_{nucleous}=\{A.4\}=\frac{1}{2}V=\{IV.1\}=\frac{1}{2}(c.f\frac{\alpha_s}{r})= \\ =\frac{1}{2}(-\frac{2/3}{a_{bb}}\alpha_s)=-0.1111\alpha_s^2m_b=-161MeV \tag{V.6}$$

Where a_{bb} is the Bohr radius of two bb quarks in a tetraquark system, look at this in appendix (A.7). In this last expression, we have used the colour factor $-2/3$ between bb, it was obtained in eq.(V.1). Lastly, we have used the appendix expression (A.7).

So, the final bond energy of the ground state is:

$$E_{bb\bar{c}\bar{c}}^{total}=E_{nucleous}+E_{crust}=-2139MeV \tag{V.7}$$

VI. STABILITY OF THE TETRAQUARK IN GROUND STATE

Once obtained the energy of our tetraquark, it is interesting to compare it with the energy of two mesons $b\bar{c}$.

$$E_{b\bar{c}}=\{A.4\}=\frac{1}{2}V=\frac{1}{2}(c.f\frac{\alpha_s}{r})=\frac{1}{2}(-\frac{4/3}{a_{bc}^m}\alpha_s)=-0.6869\alpha_s^2m_c \tag{VI.1}$$

Where now, the Bohr radius involved is between b and $\bar{c}$ in a meson system. We can look at this in appendix (A.8), moreover the colour factor involved is $-4/3$. We can find the colour factor of a meson in [1].

So the energy of two mesons is:

$$2E_{b\bar{c}}=-1.3737\alpha_s^2m_c=-2035MeV \tag{VI.2}$$

We can observe that the energy of a tetraquark is smaller than the two mesons, so our tetraquark in ground state is stable.

If we had obtained $E_{bb\bar{c}\bar{c}}^{total}>2E_{b\bar{c}}$, then tetraquark could decay into the two mentioned mesons, emitting two pions. In

this hypothetical case, it would be necessary for the energy difference to be grater than, or equal to, the energy of two pions. Contrary, the tetraquark in ground state would not decay in the two mentioned mesons and the tetraquark wolud be able to exist.

So, the real condition for its existent is that

$$\Delta E = |E^{total}_{bb\bar{c}\bar{c}} - 2E_{b\bar{c}}| = 2m_{\pi^0}$$

VII. CONCLUSION

We have done a first approximation to get the energy of the ground state for a physical tetraquark composed by $bb\bar{c}\bar{c}$. It is an analytical resolution where we have used the Rayleigh-Ritz approximation and the similarities between our particle and He atom. This work does not have as an objective obtaining a precise tetraquark energy, but rather having a first physical idea of this system. Finally we have compared our binding energy with the energy of two $b\bar{c}$ mesons and we have seen that first is grater than second. It mean that our tetraquark is a stable configuration and it dose not decay in two $b\bar{c}$ mesons.

We could improve the calculus in different ways, for example: using numerical methods to treat the four bodies system; no neglect the crossed terms; and do a better approximation for α_s involved.

VIII. ACKNOWLEDGEMENTS

I want to thank Dr. Joan Soto for his patience, his advice and mostly his help throughout these months. I want to thanks my family and friends for being such support in my life, specially mentioning Marc Borrell, Bernat Jiménez for listen to me and Núria Torres for everything said and helping me with my English.

IX. APPENDIX

$$A.1 \quad \mathtt{T} = 3 \times 3^* = 1/2(\delta^\gamma_\alpha\delta^\delta_\beta - \delta^\delta_\alpha\delta^\gamma_\beta)$$

$$A.2 \quad \mathtt{S} = 6 \times 6^* = 1/2(\delta^\gamma_\alpha\delta^\delta_\beta + \delta^\delta_\alpha\delta^\gamma_\beta)$$

$$A.3 \quad (T^a)^\alpha_{\alpha'}(T^a)^{\gamma'}_\gamma = [4] = \frac{1}{2}(\delta^{\gamma'}_{\alpha'}\delta^\alpha_\gamma - \frac{1}{N}\delta^\alpha_{\alpha'}\delta^\gamma_{\gamma'})$$

The Virial theorem tells us that:

$$A.4 \quad \begin{aligned} < T >= & -\frac{< V >}{2} \\ E_n = & < T > + < V >= - < T >= \frac{1}{2}V \end{aligned}$$

Energy of a hydrogen-like system in a tetraquark system:

$$A.5 \quad E_n = \frac{1}{2}V = -\frac{Z^2}{2n^2}c.f\frac{\alpha_s}{a_\mu}$$

$$A.6 \quad < \varphi^{Z_{eff}}_{1s}(1)|\frac{1}{r_1}|\varphi^{Z_{eff}}_{1s}(1) >=< \frac{1}{r} >_{100} = \{^{[5]}\} = \frac{Z_{eff}}{a_\mu n^2}$$

Bohr radius $a = \frac{1}{|c.f|\alpha_s\mu}$ in our tetraquark system are:

$$A.7 \quad \begin{aligned} a_{bb} &= \frac{1}{\frac{2}{3}\alpha_s\mu_{bb}} = 3\frac{1}{\alpha_s m_b} \\ a_{bc} &= \frac{1}{\frac{1}{3}\alpha_s\mu_{bc}} = \frac{66}{17}\frac{1}{\alpha_s m_c} \\ a_{bbc} &= \frac{1}{2\frac{1}{3}\alpha_s\mu_{bb,c}} = \frac{117}{68}\frac{1}{\alpha_s m_c} \end{aligned}$$

Bohr radius in a meson system used are:

$$A.8 \quad a^m_{bc} = \frac{1}{\frac{4}{3}\alpha_s\mu_{bc}} = \frac{33}{34}\frac{1}{\alpha_s m_c}$$

Where the reduced masses are:

$$A.9 \quad \mu_{bb} = \frac{1}{2}m_b; \quad \mu_{bc} = \frac{17}{22}m_c; \quad \mu_{cc} = \frac{1}{2}m_c; \quad \mu_{bbc} = \frac{34}{39}m_c$$

$$A.10 \quad m_b \cong 4250 MeV/c^2; \quad m_c \cong 1250 MeV/c^2; \quad m_b = 3.4 m_c$$

We can obtain a estimation of the $m_b\alpha_s^2$ using the difference between the mass of Upsilon (1s) and (2s), and $m_c\alpha_s^2$ using the difference between ψ' and J/ψ from [8]. We have gotten:

$$A.11 \quad m_b\alpha_s^2 = (1453 \pm 5)MeV \qquad m_c\alpha_s^2 = (1482 \pm 1)MeV$$

[1] D.Griffith: *Introduction to the elementary particles*, Pysics Textbooks, Wiley-Vch
[2] Richard D. Feld: *Applications of Perturvative QCD*, Addison-Wesley Publishing Company, (1995)
[3] R.K Ellise, W.J. Stirling and B.R. Webber: *QCD and Collider Phisics*, Cambridge University Press, (1996)
[4] V. Derya, I. Schienbein: *Color factors in QCD*, (2008)
[5] Ira N. Levine: *Química cuántica*, Prenctice Hall, AC Madrid, (1977)
[6] H. Georgi, Sheldon L. Glashow: *Lie algebras in particle physics: from isospin to unified theories*, Benjamin/Cummings, (1982)
[7] P. Pascual, R. Tarrach: *Renormalization for the practitioner* Berlin, Springer, (1984)
[8] K.A. Olive et al. : *Particle data group*, http://pdg.lbl.gov/, Regents of the University of California, (2014)
[9] P. Pascual, R. Tarrach : *QCD: Renormalization for the Practitioner*, Springer-Verlag, (1984)