Non-minimal coupling and the gyromagnetic factor of spin-1 and higher-spin fields

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Abstract: This thesis investigates non-minimal electromagnetic couplings for spin $s = 1$ and higher-spin fields, focusing on the universality of the gyromagnetic factor (g). From the non-relativistic limit of the non-minimal Proca equation up to $\mathcal{O}(1/m^2)$, where m is the particle's mass, we derive an effective Hamiltonian, identifying the g-factor (M1), Darwin, spin-orbit, and electric quadrupole (E2) corrections. We show that a generic spin-1 field has $g = 1 + a/q$, where q is its electric charge and a is the anomalous moment coupling constant, whereas $g = 2$ arises from the non-Abelian structure of electroweak $W^{\pm}$ bosons, countering Belinfante's conjecture ($g = 1/s$). New physics or internal substructures are parameterized via a dimension-4, E1-generating operator, and dimension-6 operators that modify the electric corrections and generate their magnetic duals, including M2. Finally, dimensional analysis reveals that an arbitrary, anomalous, renormalizable coupling is allowed for elementary bosons in Minkowski space, rendering g arbitrary rather than fixed at 2.
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I. INTRODUCTION

In classical electromagnetism, a massive, charged, point-like particle following a trajectory $\vec{r}(t)$ with velocity $\vec{v} = \dot{\vec{r}}(t)$ has an associated current density of $\vec{J}(\vec{x}, t) = q\vec{v}\,\delta(\vec{x} - \vec{r}(t))$. Its magnetic dipole moment is thusly given, in natural units, by

$$\vec{\mu} = \frac{1}{2}\int \mathrm{d}^3\vec{x}\left(\vec{x} \times \vec{J}\right) = \frac{q}{2m}\,\vec{r} \times m\vec{v} = \frac{q}{2m}\vec{L}\,.$$

In the quantum mechanical framework, extrapolating this definition to the particle's intrinsic angular momentum, its spin, would require adding a dimensionless proportionality constant, as $[\vec{S}] = [\vec{L}]$ and the ratio $q/2m$ already capture the magnetic moment's dimensions:

$$\vec{\mu} \equiv g\,\frac{q}{2m}\vec{S}\,. \tag{1}$$

This is the renowned gyromagnetic factor or g-factor of a spin-s particle. While classical physics implies $g = 1$, the earliest record of a computation of this constant dates back to 1927, with Dirac's celebrated result that, at tree level, $g = 2$ for the electron ($s = 1/2$). In the later years, subsequent computations for higher spins led Belinfante (1953) to conjecture that $g = 1/s$ for any elementary particle of spin s, a premise seemingly validated by applying minimal coupling substitution to arbitrary-spin equations [1]. However, this paradigm shifted drastically in 1970 through the work of Weinberg [2]. Utilizing the Drell-Hearn sum rule for high-energy photon scattering amplitudes, it was shown that, for charged particles unaffected by strong interactions—such as the $W^{\pm}$ bosons—, the requirement that scattering amplitudes remain well-behaved at infinity enforces $\mu \simeq s(q/m) = 2(q/2m)\langle S_z\rangle|_{s_z=s}$. This condition dictates a universal value of $g \simeq 2$, regardless of the particle's spin. Indeed, as early as in 1940, Corben and Schwinger already contemplated the possibility to modify the Proca's theory g-factor to 2. This universality was later and more recently advocated for and consolidated theoretically through non-minimal electromagnetic couplings and string theory arguments [3, 4]. Experimentally, the most precise measurement of the $W^{\pm}$ boson g-factor to date published by the DELPHI collaboration [5] gives $g_{W^{\pm}} = 2.22^{+0.20}_{-0.19}$, in contraposition to Belinfante's conjecture, and aligning seamlessly with Electroweak theory (EWT) and Weinberg's predictions. Although $g = 2$ seems theoretically inherent to arbitrary-spin elementary particles, the absence of massive, charged $s > 1$ elementary particles in the SM and in nature leaves this universality unverified.

In this TFG we provide a thorough study of the effects of coupling spin-1—and higher—particles to an electromagnetic (EM) field in a non-minimal manner. The work is structured as follows: in Section II A, we perform the non-relativistic (NR) expansion up to $\mathcal{O}(1/m^2)$ of the Proca equation (PE) coupled non-minimally to an EM field in order to discuss the value of g and, in this spirit, comment on its associated fine structure; in Section II B, we see what non-minimal coupling terms are permitted in the Lagrangian that may characterize new physics or physics related to composite spin-1 particles and, lastly, in Section III, we analyze the plausibility of modifying g for higher-spin particles, tying it to its implications for Belinfante's conjecture and Weinberg's theoretical expectations. We will hereafter make use of natural units, $\hbar = 1 = c$, and we will adopt the $(+,-,-,-)$ metric signature as well.

II. SPIN-1

The Proca Lagrangian describing a free, massive, charged, elementary, spin-1 vector field B^{μ} reads

$$\mathcal{L} = -\frac{1}{2}\hat{B}^{\star}_{\mu\nu}\hat{B}^{\mu\nu} + m^2 B^{\star}_{\mu}B^{\mu},$$

with $\hat{B}_{\mu\nu} = \partial_{\mu}B_{\nu} - \partial_{\nu}B_{\mu}$ the antisymmetric field strength. The B^{μ} object transforms according to the $(\frac{1}{2}, \frac{1}{2})$ representation of the homogeneous Lorentz group, and it can be decomposed into a spin-0, scalar part, B^0, and a spin-1, vector part, B^i, which we are interested in: $\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$. The EM field A^μ is real, massless and has unit spin, so its free Lagrangian is $\mathcal{L}_\gamma = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$, where the additional $1/2$ accounts for the factor of 2 we get when differentiating with respect to a real field instead of a complex one, which would appear only once in the terms of $\mathcal{L}$. To minimally couple an EM field to a Proca field, we only need to add the Lagrangians, $\mathcal{L} = \mathcal{L}_B + \mathcal{L}_\gamma$, and prescribe $\partial_\mu \mapsto D_\mu = \partial_\mu + iqA_\mu$ exclusively to $\hat{B}_{\mu\nu}$, $\hat{B}_{\mu\nu} \to B_{\mu\nu}$, where D_μ is called the covariant derivative and q is the particle's charge:

$$\mathcal{L} = -\frac{1}{2} B^\star_{\mu\nu} B^{\mu\nu} + m^2 B^\star_\mu B^\mu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \,. \qquad (2)$$

Now, this Lagrangian and any other minimally coupled to the photon is unchanged under the $U(1)$ local gauge transformation, $B'^\mu(x) = e^{iq\theta(x)} B^\mu(x)$, $A'_\mu(x) = A_\mu(x) + \partial_\mu \theta(x)$, $\theta(x)$ being a real function evaluated at some space-time point x; this can easily be seen using the fact that the covariant derivative is "transparent" to the exponential factor: $D'^\mu B'^\nu = e^{iq\theta(x)} D^\mu B^\nu$ (see Appendix A). Indeed, that being the case, we have that $B^{\mu\nu}$ transforms like B^μ: $B'^{\mu\nu} = e^{iq\theta(x)} B^{\mu\nu}$. And, of course, we already know from electrodynamics that said gauge transformation leaves $F_{\mu\nu}(x)$ invariant. Remarkably, were the photon not massless, $\mathcal{L}$ would include a term proportional to $A_\mu A^\mu$, which would not be gauge invariant. Given this freedom, one can always add real—to ensure a real action—, gauge- and Lorentz- —as to not violate the CPT symmetry—invariant terms to the Lagrangian without affecting the underlying physics; therefore, following the introductory discussion, we add a non-minimal, anomalous, Corben-Schwinger term to (2):

$$\mathcal{L} = -\frac{1}{2} B^\star_{\mu\nu} B^{\mu\nu} + m^2 B^\star_\mu B^\mu + iaF_{\mu\nu} B^{\star\mu} B^\nu - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \,. \quad (3)$$

Effectively, $\mathcal{L}$ remains real as long as a is real too: $(iaF_{\mu\nu} B^{\star\mu} B^\nu)^\dagger = -ia^\star F_{\mu\nu} B^\mu B^{\star\nu} = ia^\star F_{\mu\nu} B^{\star\mu} B^\nu$. Also, given that, in natural units, $[\mathcal{L}] = \mathrm{M}^4$, $[B^\mu] = \mathrm{M}$, and $[F_{\mu\nu}] = \mathrm{M}^2$, it is readily verified that a is a dimensionless coupling ($[a] = 1$). Importantly, since this operator does not introduce field derivatives of the type DB, its contribution to the resulting equations of motion scales directly with the field strength tensor $F_{\mu\nu}$, which is associated to the EM fields $\vec{E}, \vec{B}$ and, thus, to the g-factor.

A. Gyromagnetic factor and fine structure of elementary spin-1 particles

We next consider the equations to be reduced in the NR limit. Varying the Lagrangian or making use of the Euler-Lagrange equations yields the following equations of motion for B^μ, $B^{\star\mu}$ and A^μ (see Appendix B):

$$D_\nu B^{\nu\mu} + m^2 B^\mu + iaF^\mu{}_\nu B^\nu = 0 \,; \qquad (4)$$

$$D^\star_\nu B^{\star\nu\mu} + m^2 B^{\star\mu} - iaF^\mu{}_\nu B^{\star\nu} = 0 \,; \qquad (5)$$

$$\partial_\nu F^{\nu\mu} = J^\mu . \qquad (6)$$

We will not work with (5). Note how (6) is nothing more than Maxwell's equations, having identified the field current density as

$$J^\mu \equiv iq \left(B^{\nu\mu} B^\star_\nu - B^{\star\nu\mu} B_\nu\right) + ia\, \partial_\nu (B^{\star\nu} B^\mu - B^\nu B^{\star\mu}) \,.$$

The first current is merely the bare Noether current, whereas the second is linked to the current causing the anomalous magnetic moment. Owing to the fact that the first current has terms of the form $B^\star DB, (DB)^\star B$, and the second of the form $\partial(B^\star B)$, it should be possible (see property in Appendix C) to manipulate the first into something similar to the second, thus getting some current plus the second with $a \to q + a$, signifying a modification in, at least, the first multipole moment. As a consistency check, it can be proven that the divergence of (6) vanishes identically, and hence conveys no new information (see Appendix D). Furthermore, the covariant divergence of the first equation results in

$$D_\mu B^\mu = \frac{i(q-a)}{m^2} F_{\mu\nu} D^\mu B^\nu - \frac{ia}{m^2} (\partial_\mu F^{\mu\nu}) B_\nu , \qquad (7)$$

where we used that $[D_\mu, D_\nu] = iqF_{\mu\nu}$ (see Appendix E). Since we will be in the NR regime, in order to simplify calculations, we will be ignoring all the 2-body terms appearing in J^μ, that is, we will take $\partial_\nu F^{\nu\mu} = 0$. Developing $B^{\nu\mu}$ and using the aforementioned commutation relation in the PE, we are left with the following system of equations:

$$(D^2 + m^2) B^\mu - D^\mu D_\nu B^\nu + i(q+a) F^\mu{}_\nu B^\nu = 0 \,; \quad (8)$$

$$\partial_\nu F^{\nu\mu} = 0 \,; \qquad D_\mu B^\mu = \frac{i(q-a)}{m^2} F_{\mu\nu} D^\mu B^\nu \,. \quad (9, 10)$$

Despite seeming intuitive, we will not be substituting (10) in (8), simply, because, in doing so, we will see that would get us a non-Hermitian Hamiltonian. Note that imposing Hermiticity will be crucial in obtaining the definitive, consistent Hamiltonian. Before proceeding, let us define an inner product and state the rules to check for Hermiticity. We choose a Lorentz-invariant, action-like inner product for Proca solutions,

$$(U, \mathcal{O}V) := \int_M \mathrm{d}^4x \sqrt{-g}\; U^\star_\mu \mathcal{O}^\mu{}_\nu V^\nu ,$$

where $g = \det \eta_{\mu\nu} = -1$ and M is Minkowski space. This inner product enables us to prove D_μ is anti-Hermitian, as required by $(p^\mu)^\dagger = p^\mu = iD^\mu$, making it a suitable candidate. Note, though, that this product is flawed, as it need not be positive-definite; these are usually called indefinite inner products. Knowing sesquilinear inner products verify $(U, \mathcal{O}V)^\star = (V, \mathcal{O}^\ddagger U)$, with $\ddagger$ the dagger adapted to tensor-differential operators that act on vector fields, then, one can show that $(\mathcal{O}^\ddagger)^\mu{}_\nu \equiv (\mathcal{O}_\nu{}^\mu)^\dagger$ (see Appendix F). This is analogous in the NR case, where we work with 3-vectors and $iD_0 \vec{v} = H\vec{v}$, $H = H^\ddagger$.

In the NR limit we will restrict the calculations to the single-particle subspace, similar to how it is done in 1st quantization. The assertion simply follows from the fact that the $2m$ energy gap needed to bridge the particle and antiparticle can not be covered in our regime inasmuch as $|\vec{p}| \ll m$. Under this assumption, the general solution may be of the sorts of a free Proca field solution,

$$B^\mu(x) \sim \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3\sqrt{2E}} \sum_{\lambda=1}^{3} e^{-ip\cdot x}\,\epsilon^\mu_\lambda(\vec{p})a_\lambda(\vec{p})\,;$$

$a_\lambda(\vec{p})$ is no longer an operator, but rather an arbitrary function. Since $e^{-ip\cdot x} = e^{-i(p^0 t - \vec{p}\cdot\vec{x})}$, with $p^0 = \sqrt{m^2+\vec{p}^2} \approx m + \vec{p}^2/2m$, we can factor out an e^{-imt} out of the integral, in turn suggesting that we employ the substitution $B^\mu(x) = e^{-imt}\,\psi^\mu(x)$ if we want to excite the NR structure of the PE. Applying the substitution and taking into account that $m\psi \gg |D_0\psi| \gg |D_i\psi|$, (8) and (10) become (see Appendix G)

$$(2m+iD_0)iD_0\psi^\mu = -\vec{D}^2\psi^\mu + i(q+a)F^\mu{}_\nu\psi^\nu - e^{imt}D^\mu D_\nu e^{-imt}\psi^\nu; \quad (11)$$

$$-im\psi^0 + D_\mu\psi^\mu = \frac{i(q-a)}{m^2}F_{\mu\nu}D^\mu\psi^\nu + \frac{q-a}{m}F^0{}_\nu\psi^\nu, \quad (12)$$

where, whenever it is not explicited, the covariant derivatives act on all that is on its right.

Let us just focus on (11). Note how it looks like the Schrödinger equation (SE) except for the operator $2m + iD_0$ on the LHS. It is thus desirable that we could invert it somehow. In fact, according to the previous hierarchy, one can formally expand its inverse operator as

$$\frac{1}{2m+iD_0} = \frac{1}{2m}\sum_{n=0}^{\infty}\left(-\frac{iD_0}{2m}\right)^n = \frac{1}{2m} - \frac{iD_0}{4m^2} + \cdots .$$

Nonetheless, to preserve Hermiticity, we can not simply multiply by its inverse from the left. For now, we will leave it be. The time component of (11) or (12) allows us to express ψ^0 in terms of ψ^i,

$$\psi^0 = \frac{D_j\psi^j}{im} + \frac{D_0D_j\psi^j}{m^2} - \frac{i(q+a)}{m^2}F^0{}_j\psi^j + \mathcal{O}(1/m^3), \quad (13)$$

having expanded $(m^2 - \vec{D}^2)^{-1}$ or $(D_0 - im)^{-1}$, respectively, and the spatial component of (11) will too read

$$(2m+iD_0)iD_0\psi^i = -\vec{D}^2\psi^i + i(q+a)F^i{}_0\psi^0 + i(q+a)F^i{}_j\psi^j - D^i(D_0 - im)\psi^0 + D^iD_j\psi^j. \quad (14)$$

Next, we can substitute (13) in (14) and iterate it recursively, yielding, up to at least $\mathcal{O}(1/m^2)$,

$$\left(1+\frac{iD_0}{2m}\right)iD_0\psi^i = -\frac{\vec{D}^2}{2m}\delta^i{}_j\psi^j + \frac{i(q+a)}{2m}F^i{}_j\psi^j + \frac{a(F^i{}_0D_j + D^iF^0{}_j)}{2m^2}\psi^j + \frac{iD_0D^iD_j + D^iD_jiD_0}{2m^2}\psi^j \quad (15)$$

(see Appendix H). We neglected terms containing explicit $1/m^3$ factors in the process. Now, to solve the Hermiticity issue, we can redefine the wave function as $\psi^\mu = \varphi^\mu/\sqrt{1+iD_0/2m}$. Indeed, (15) can be recasted into $iD_0\vec{\varphi} = MhM\vec{\varphi}$, and both $h^i{}_j$—acting on ψ^j in (15)—and $M = (1+iD_0/2m)^{-1/2}\,\mathbb{1}_3$ are Hermitian. To obtain the Hamiltonian up to second order, we will approximate $M \simeq (1 - iD_0/4m)\,\mathbb{1}_3$. Note $iD_0/4m$ will only apply to the $\mathcal{O}(1/m)$ terms; hence, the Hamiltonian will consist of the operators in (15) plus ones of the type $-(iD_0A + AiD_0)/4m^2$. Since temporal derivatives induce a $1/m$ order, we can neglect all $-AiD_0/4m^2$-like operators; moreover, the $-iD_0A/4m^2$ ones can be brought to the LHS, giving $iD_0(\mathbb{1}_3 + A/4m^2)\vec{\varphi}$. We can thus redefine a new wavefunction as $(\mathbb{1}_3 + A/4m^2)\vec{\varphi}$—note this redefinition would not affect operators on the RHS because it would introduce $\mathcal{O}(1/m^{\geq 3})$ terms. We can also recover the $\mathcal{O}(1/m^3)$ relativistic term from the novel one with $A = -\vec{D}^2/2m$, before redefining: ignoring the rest of $\mathcal{O}(1/m^3)$, by redefining $\varphi^\mu = (1+\vec{D}^2/8m^2)\tilde{\varphi}^\mu$ and using $iD_0\tilde{\varphi}^i = -\vec{D}^2/2m\,\tilde{\varphi}^i + \cdots$, there appears $-\vec{D}^4/8m^3\tilde{\varphi}^i$ in the new Hamiltonian. To understand the emergence of the magnetic moment, spin-orbit, and Darwin terms, we examine the EM tensor structures. Specifically, the former originates from the $F^i{}_j = -\epsilon^i{}_j{}^kB^k = \epsilon^{ijk}B^k$ term ($\epsilon^{123} = 1$), because, in the spherical basis, the spin-1 matrices ($SO(3)$ generators) are expressed as $(S^i)^{jk} = -i\epsilon^{ijk}$. Concerning the latter, they arise from $F^i{}_0D_j + D^iF^0{}_j = E^iD_j - D_iE^j$: of course, this expression consists of direct products of 3-vectors, that is, objects transforming like $(1,1)$, $1\otimes 1 = 2\oplus 1\oplus 0$, so, once decomposed, we expect there to be a scalar, spin-0 term (Darwin, trace), a vector, spin-1 term (spin-orbit, antisymmetric 3×3 matrix), and a new tensor, spin-2 term corresponding to an electric quadrupole (symmetric, traceless 3×3 matrix). In fact, this is already predicted by the Wigner-Eckart theorem (WET): for a spin-s particle, the allowed multipole operators are of rank $\leq 2s$. Bearing this in mind, after the successive redefinitions, $\psi^\mu \overset{\cdots}{\to} \phi^i$, the SE up to $\mathcal{O}(1/m^2)$—with Hermitian Hamiltonian—for a spin-1 elementary particle is

$$i\partial_0\vec{\phi} = qA_0(x)\vec{\phi} - \frac{\vec{D}^2}{2m}\vec{\phi} - \frac{\vec{D}^4}{8m^3}\vec{\phi} - \frac{q+a}{2m}(\vec{S}\cdot\vec{B})\vec{\phi} - \frac{a}{6m^2}(\vec{\nabla}\cdot\vec{E})\vec{\phi} + \frac{ia}{4m^2}[\vec{S}\cdot(\vec{E}\times\vec{D} - \vec{D}\times\vec{E})]\vec{\phi} + \frac{a}{6m^2}(Q^{\ell m}\partial_\ell E^m)\vec{\phi}. \quad (16)$$

(see Appendix I). Here, the quadrupole adimensional operator is $Q^{\ell m} = \frac{3}{2}[S^\ell S^m + S^m S^\ell + \frac{4}{3}\,\mathbb{1}_3\,\eta^{\ell m}]$; indeed, it is symmetric and traceless ($Q^{\ell\ell} = 0$, with $\vec{S}^2 = 2\,\mathbb{1}_3$). The $3/2$ is to impose the $1/6$ coefficient of the last term on the RHS, like in the classical multipole expansion of the electrostatic interaction energy,

$$E_{\text{int}} = qV(0) - \vec{d}\cdot\vec{E}(0) - \frac{1}{6}\mathcal{Q}^{ij}\partial_iE^j(0) + \cdots .$$

In addition, (16) passes the self-consistency check whereby inserting (13) and (15) into (12) yields an identity up to $\mathcal{O}(1/m^2)$, meaning that no new physical information is generated at this order (see Appendix J).

Comparing to (1), we can identify the g-factor as $g = 1+a/q$, where, in the $W^\pm$ case, a will be fixed by the EWT. Note that the Darwin, spin-orbit, and quadrupole terms are all proportional to $a = (g-1)q$; consequently, under minimal coupling ($a = 0$), the gyromagnetic factor reduces to $g = 1$, thereby verifying Belinfante's conjecture and causing these $\mathcal{O}(1/m^2)$ contributions to vanish. More surprisingly, even though WET allows it, there are no electric dipole nor magnetic quadrupole moments (E1, M2) in our Hamiltonian, the reason being they violate P and T, while our QED Lagrangian does conserve these symmetries. Effectively, they would be of the form $d(\vec{S}\cdot\vec{E})$, $\mathcal{Q}^{ij}\partial_i B^j$ and, according to

$$\vec{S} \overset{\mathsf{P},\mathsf{T}}{\to} \pm\vec{S},\ \vec{E} \overset{\mathsf{P},\mathsf{T}}{\to} \mp\vec{E},\ Q \overset{\mathsf{P},\mathsf{T}}{\to} Q,\ \vec{\nabla} \overset{\mathsf{P},\mathsf{T}}{\to} \mp\vec{\nabla},\ \vec{B} \overset{\mathsf{P},\mathsf{T}}{\to} \pm\vec{B},$$

they would always change sign under P or T. If we particularize this model to the $W^\pm$ bosons, we should get $g = 2$, namely, $a = q$—recall a was dimensionless ($1 = [\alpha] = [e]^2$). In fact, we can do so comparing (3) to the gauge-kinetic part from the EW, $SU(2)\times U(1)$ sector of the SM Lagrangian, $\mathcal{L}^{\rm EW}_{\rm gauge} = -\frac{1}{4}W^i_{\mu\nu}W^{i\,\mu\nu} - \frac{1}{4}\breve{B}_{\mu\nu}\breve{B}^{\mu\nu}$, $i = 1,2,3$, where the $SU(2)$ gauge-invariant field strength reads $W^i_{\mu\nu} = \partial_\mu W^i_\nu - \partial_\nu W^i_\mu - \mathfrak{g}\,\epsilon^{ijk}W^j_\mu W^k_\nu$, $\mathfrak{g}$ being the $SU(2)$ coupling constant, and the $U(1)$ one reads $\breve{B}_{\mu\nu} = \partial_\mu\breve{B}_\nu - \partial_\nu\breve{B}_\mu$, with $\mathfrak{g}'$ its coupling constant. We can relate the real $i = 1,2$ gauge fields to the complex, charged ones through $W^\pm_\mu = (W^1_\mu \mp iW^2_\mu)/\sqrt{2}$—note $(W^\pm_\mu)^\dagger = W^\mp_\mu$—, and $i = 3$, $\breve{B}_\mu$ to the neutral ones, A_μ, Z_μ, through $\breve{B}_\mu = A_\mu\cos\theta_W$, $W^3_\mu = A_\mu\sin\theta_W$, where, for our purpose, $Z_\mu = 0$, and θ_W is the Weinberg angle that relates $\mathfrak{g}$ and $\mathfrak{g}'$. Schematically, the field strengths and their respective Lagrangian densities decompose as

$$W^{1,2}_{\mu\nu} \sim \partial W + \mathfrak{g}WA,\ W^3_{\mu\nu} \sim \partial A + \mathfrak{g}WW,\ \breve{B}_{\mu\nu} \sim \partial A;$$
$$\mathcal{L}_{W^{1,2}} \sim (\partial W)^2 + \mathfrak{g}(\partial W)WA + \mathfrak{g}^2 WWAA;$$
$$\mathcal{L}_{W^3} \sim (\partial A)^2 + \mathfrak{g}(\partial A)WW + \mathfrak{g}^2 WWWW,\ \mathcal{L}_{\breve{B}} \sim (\partial A)^2;$$

whilst, in our Lagrangian,

$$B_{\mu\nu} \sim DB \sim \partial B + qBA;$$
$$\mathcal{L}^{\rm gauge}_{B,\gamma} \sim (\partial B)^2 + q(\partial B)BA + q^2 BBAA + a(\partial A)BB + (\partial A)^2.$$

Thus, expanding $\mathcal{L}^{\rm EW}_{\rm gauge}$ and $\mathcal{L}^{\rm gauge}_{B,\gamma}$ we should be able to match each and every term except for the W quartic self-coupling. Additionally, the Corben-Schwinger term consists of a $WW\gamma$ 3-vertex that emanates from $\mathcal{L}_{W^3}$. Ignoring the free-kinetic terms and using $e = \mathfrak{g}\sin\theta_W$, the 3- and 4-point gauge boson self-couplings are

$$\mathcal{L}_{\mathfrak{g}} = -ie\big[(\hat{W}^+_{\mu\nu}W^{-\mu} - \hat{W}^-_{\mu\nu}W^{+\mu})A^\nu + F_{\mu\nu}W^{+\mu}W^{-\nu}\big];$$
$$\mathcal{L}_{\mathfrak{g}^2} = e^2\big[(W^+\cdot A)(W^-\cdot A) - |W|^2A^2\big] - \frac{\mathfrak{g}^2}{2}\big[|W|^4 - W^{+2}W^{-2}\big];$$

meanwhile, ours look like

$$\mathcal{L}_{q,a} = -iq\big[\hat{B}_{\mu\nu}B^{\star\mu} - \hat{B}^\star_{\mu\nu}B^\mu\big]A^\nu - iaF_{\mu\nu}B^\mu B^{\star\nu};$$
$$\mathcal{L}_{q^2} = q^2\big[(B\cdot A)(B^\star\cdot A) - |B|^2A^2\big],$$

where $\hat{W}^\pm_{\mu\nu} \equiv \partial_\mu W^\pm_\nu - \partial_\nu W^\pm_\mu$. Setting $B = W^+, W^-$, we get $a = e, -e$, so, in either case, we have $a = q$, as expected. From this, we infer that $g = 2$ is simply a consequence of the non-Abelian nature of the EWT, which we did not take into account in the minimal coupling. With $a = q$, we can see that our quadrupole, $\mathcal{Q}^{zz} = -a/m^2\,\langle Q^{zz}\rangle_{|11\rangle}$, coincides with the one predicted by the SM, $\mathcal{Q}_{W^\pm} = -q/m^2$ [7] (details of this $a = q$ part can be found in Appendix K).

B. New physics and composite spin-1 particles

Thus far, we have only considered minimal Proca terms and a non-minimal, renormalizable one. Nevertheless, as established earlier, Hermitian, gauge- and Lorentz-invariant operators are also permitted in the Lagrangian. Accordingly, in the context of SM effective field theory (EFT), we are free to add mass dimension $(4+N)$, non-renormalizable terms with a dimensionless coupling constant, that are suppressed by the Nth power of a cut-off scale, Λ. In the event that B_μ describes an elementary particle, Λ represents the threshold of a new physics, ultraviolet theory with new degrees of freedom; in contrast, if B_μ describes a composite state—such as mesons or tetraquarks—, Λ corresponds to the confinement scale, where the effects of the internal substructure become relevant. Here, we will consider the remaining dimension-4, -5 and -6 operators, as higher-dimensional ones would contribute to the SE at orders beyond our accuracy. We will focus exclusively on operators containing two B fields, which is suitable in our NR expansion. To build these we will combine contractions of well-behaved dimension-0, -1, -2 and -3 objects, respectively: ϵ; $B, B^\star$; $F, DB, (DB)^\star$, and $\partial F, D^2B, (D^2B)^\star$. Note that, except for ϵ, these objects have the same number of indices than dimensions, so dimension-5 operators have to contract 5 indices, violating Lorentz invariance. Indeed, by this reasoning, odd-dimension operators do not exist in this framework. Consequently, we restrict our attention to dimension-4 and -6 operators, specifically, to $BB\gamma$ classes rather than BB, topological ones.

In the 4-dimensional case, the only independent structure is the E1-generating term, $i\tilde{a}B^{\star\mu}\tilde{F}_{\mu\nu}B^\nu$, where $\tilde{F}_{\mu\nu}$ is the dual of $F_{\mu\nu}$. As for the dimension-6 operators, we have to take in mind the following: firstly, that (8), (10), and (9) and its dual version (Bianchi identity) hold and, secondly, since operators of the Hamiltonian acquire an $1/2m$ in the NR limit, terms without D, D^2 will not compensate $1/2m\Lambda^2 \sim \mathcal{O}(1/m^3)$ with m or m^2 factors. Thus, possibilities would be $(DB)^\star F(DB)$, $B^\star\partial F(DB)$ and $B^\star FD^2B$; however, one of these may always be related to the other two through a total divergence, so we will just consider the latter two. This way, we have

$$\left\{\frac{b_n}{\Lambda^2}(\mathcal{O}_n + \text{h.c.})\right\}^2_{n=1},\quad \left\{\frac{ib_n}{\Lambda^2}(\mathcal{O}_n - \text{h.c.})\right\}^2_{n=1},$$

where $\mathcal{O}_1 = B^\star_\mu\partial^\mu F_{\alpha\beta}D^\alpha B^\beta$, $\mathcal{O}_2 = B^\star_\mu\partial_\alpha F^{\alpha\beta}D^\mu B_\beta$, and we also have their dual counterparts with $\tilde{b}_1, \tilde{b}_2$. The first four modify Darwin, spin-orbit, and E2 terms, whereas dual ones generate the same with $\vec{E} \to \vec{B}$, including M2.

At last, in the 2nd quantization scheme [7], $\partial_\mu \overset{\mathsf{P,T}}{\to} \pm\partial^\mu$, $U_\mu(x) \overset{\mathsf{P,T}}{\to} U^\mu(\pm t, \mp\vec{x})$, $i \overset{\mathsf{T}}{\to} -i$, $U_\mu(x) \overset{\mathsf{C}}{\to} -U_\mu^\dagger(x)$, for $U_\mu = B_\mu, A_\mu$, with $A_\mu(x) = A_\mu^\dagger(x)$, so we can summarize their behaviours as follows (see Appendix L):

TABLE I: Properties of the operators under the P, T and C transformations. The a, ib_1, ib_2 terms correspond to EM or strong interactions, $\tilde{a}, i\tilde{b}_1, i\tilde{b}_2$ to EW interactions, and the rest to physics beyond the SM. Either way, CPT is conserved.

	b_1, b_2	a, ib_1, ib_2	$\tilde{b}_1, \tilde{b}_2$	$\tilde{a}, i\tilde{b}_1, i\tilde{b}_2$
P	+	+	−	−
T	−	+	+	−
C	−	+	−	+

III. HIGHER SPIN AND $g = 2$

Renormalizable theories are characterized because their associated coupling constant have dimensions $\mathrm{M}^{\geq 0}$, albeit they are normally dimensionless (QED, QCD, ...). Considering relativistic wave equations (RWEs) that verify Belinfante's conjecture, we will see whether the non-minimal term $\sim a\Psi^\star F\Psi$ enables us to alter g for elementary particles with arbitrary spin, tying to Weinberg's claim. Elementary particles are described by renormalizable Lagrangians, valid at any energy scale, so the condition to be met is that the non-minimal term be renormalizable—otherwise we would be describing composite particles. We assume a D dimensional space-time.

First, note the Lagrangian must have dimensions of M^D, so $[F] = \mathrm{M}^{D/2}$. Thus, $\mathrm{M}^D = [a\Psi^\star F\Psi] = [a] \cdot [\Psi]^2 \cdot \mathrm{M}^{D/2}$, and, comparing exponents, we get $D = 2d_a + 4d_\Psi$. Therefore, for renormalizable theories where a is dimensionless, $D = 4d_\Psi$; for non-renormalizable ones, $D < 4d_\Psi$, and, for super-renormalizable ones, $D > 4d_\Psi$. Now, it is hardly unnoticeable that the typical RWEs include ∂^2 for bosons, whilst the fermion ones include a single ∂. This pattern, if true for higher-spin equations, means that $[\Psi_b^\star \partial^2 \Psi_b] = \mathrm{M}^D = [\Psi_f^\star \partial \Psi_f]$, that is, $d_{\Psi_b} = (D-2)/2$ and $d_{\Psi_f} = (D-1)/2$. Note d_Ψ does not depend on spin explicitly. For bosons and fermions, respectively, we then get $d_a = (4-D)/2$ and $d_a = (2-D)/2$. Hence, fixing Minkowski space ($D = 4$), we conclude that we can modify g for integer-spin elementary particles. Regarding half-integer ones, the non-minimal term is not renormalizable, concretely, it is of $\mathcal{O}(1/\Lambda)$. A typical example would be the anomalous Pauli term, $a'\,\bar{\psi}\sigma^{\mu\nu}\psi F_{\mu\nu}/m$, with $d_{a'} = 0$, which can reproduce the g-factor of protons and neutrons. Interestingly, for non-fixed D, we have that half-integer-spin elementary particles allow altering g in a renormalizable manner if $D = 2$. Particularly, the term would also be renormalizable for bosons, being of $\mathcal{O}(\Lambda)$.

IV. CONCLUSIONS

In this work, we systematically investigated the non-minimal EM coupling of spin-1 fields via the Corben-Schwinger term, deriving the effective SE up to $\mathcal{O}(1/m^2)$. The resulting Hamiltonian identifies $g = 1 + a/q$ for a generic spin-1 field, alongside Darwin, spin-orbit, and electric quadrupole terms, while all P- and T-violating multipoles are absent. Under minimal coupling ($a = 0$), we have that $g = 1$, matching Belinfante's conjecture, and that the $\mathcal{O}(1/m^2)$ corrections, which are proportional to $g - 1$, vanish. In additon, embedding the field into the EW Lagrangian fixes $a = q$ for the $W^\pm$ bosons, establishing $g = 2$ as a direct consequence of the non-Abelian $SU(2)$ gauge structure.

Within an EFT framework, we were able to exhaust all the independent operators permitted by our theory, including the remaining dimension-4 operator generating the missing E1 moment, as well as dimension-6 operators that modify the existing electric corrections and that generate their magnetic versions, including M2. We also classified the operators according to the nature of the interaction they describe.

At last, dimensional analysis in D space-time dimensions shows that the non-minimal Corben-Schwinger coupling is renormalizable only for integer-spin fields in $D = 4$, allowing for an arbitrary g-factor rather than the value of 2. Conversely, for half-integer spins, this term scales as $\mathcal{O}(1/\Lambda)$, strictly implying a composite nature or an ultraviolet cutoff, as exemplified by the anomalous magnetic moments of nucleons.

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Acoblament no-mínim i el factor giromagnètic de camps d'espín 1 i superior

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Resum: En aquesta tesi s'investiga l'acoblament no-mínim per a camps d'espín $s = 1$ i superior, focalitzant-la en la universalitat del factor giromagnètic (g). Fent el límit no-relativista de l'equació de Proca no-mínima fins a $\mathcal{O}(1/m^2)$, amb m la massa de la partícula, deduïm un Hamiltonià efectiu, podent-hi identificar les correccions degudes al factor g (M1) i als termes de Darwin, d'espín-òrbita, i del quadrupol elèctric (E2). Demostrem que és $g = 1 + a/q$ per a un camp d'espín 1 qualsevol, on q és la seva càrrega elèctrica i a és la constant d'acoblament que quantitza el moment magnètic anòmal, mentre que $g = 2$ prové de l'estructura no-Abeliana dels bosons electrofebles $W^{\pm}$, contradient la conjectura de Belinfante ($g = 1/s$). Així mateix, nova física o subestructures internes poden ser parametritzades amb un operador de dimensió 4 que genera el moment dipolar elèctric, E1, i també per operadors de dimensió 6, típics de teoria efectiva de camps, que poden modificar les correccions elèctriques i generar-ne de magnètiques, inclòs el moment quadrupolar magnètic, M2. Per últim, fent anàlisi dimensional es conclou que acoblaments arbitraris, renormalitzables, no-mínims, que modifiquin el moment magnètic, són permesos pels bosons elementals (partícules d'espín enter) en el cas de l'espai de Minkowski ($D = 4$ dimensions espaitemporals), fent que el seu valor de g no estigui necessàriament fixat a 2.
Paraules clau: Lagrangià, espín, factor g, Darwin, espín-òrbita, quadrupol
ODSs: El contingut d'aquest TFG, part d'un grau universitari de Física, es relaciona amb l'ODS 4, concretament, amb la fita 4.4, ja que contribueix a l'educació a nivell universitari.

Objectius de Desenvolupament Sostenible (ODSs o SDGs)

1. Fi de la es desigualtats		10. Reducció de les desigualtats	
2. Fam zero		11. Ciutats i comunitats sostenibles	
3. Salut i benestar		12. Consum i producció responsables	
4. Educació de qualitat	X	13. Acció climàtica	
5. Igualtat de gènere		14. Vida submarina	
6. Aigua neta i sanejament		15. Vida terrestre	
7. Energia neta i sostenible		16. Pau, justícia i institucions sòlides	
8. Treball digne i creixement econòmic		17. Aliança pels objectius	
9. Indústria, innovació, infraestructures			

Appendix A

Let Ψ, Φ be arbitrary, differentiable fields. The Leibniz' rule for D_μ is

$$\begin{aligned} D_\mu(\Psi\Phi) &= \partial_\mu(\Psi\Phi) + iqA_\mu\Psi\Phi \\ &= (\partial_\mu\Psi)\Phi + \Psi(\partial_\mu\Phi) + iqA_\mu\Psi\Phi \\ &= (D_\mu\Psi)\Phi + \Psi(\partial_\mu\Phi) \\ &= (\partial_\mu\Psi)\Phi + \Psi(D_\mu\Phi)\,. \end{aligned}$$

Thus,

$$\begin{aligned} &D'^\mu B'^\nu \\ &= (\partial^\mu + iqA'^\mu)\left[e^{iq\theta(x)}B^\nu\right] \\ &= \Big(\partial^\mu + iqA^\mu - iq\,\partial^\mu\theta(x)\Big)[e^{iq\theta(x)}B^\nu] \\ &= D^\mu\left[e^{iq\theta(x)}B^\nu\right] - iq\,\partial^\mu\theta(x)e^{iq\theta(x)}B^\nu \\ &= \left(\partial^\mu e^{iq\theta(x)}\right)B^\nu + e^{iq\theta(x)}D^\mu B^\nu - iq\,\partial^\mu\theta(x)e^{iq\theta(x)}B^\nu \\ &= e^{iq\theta(x)}D^\mu B^\nu, \end{aligned}$$

as was expected.

Appendix B

Let us vary the action and later impose that the functional derivative be zero. Firstly, vary $B^\star_\mu$:

$$\begin{aligned} \delta\mathcal{S}|_{A,B} &= \int \mathrm{d}^4x\,\delta\mathcal{L}|_{A,B} \\ &= \int \mathrm{d}^4x - \tfrac{1}{2}B^{\mu\nu}\delta B^\star_{\mu\nu} + m^2B^\mu\delta B^\star_\mu + iaF^\mu{}_\nu B^\nu\delta B^\star_\mu \\ &\overset{\clubsuit_1}{=} \int \mathrm{d}^4x\; B^{\mu\nu}D^\star_\nu\,\delta B^\star_\mu + m^2B^\mu\delta B^\star_\mu + iaF^\mu{}_\nu B^\nu\delta B^\star_\mu \\ &\overset{\clubsuit_2}{=} \int \mathrm{d}^4x\left[-D_\nu B^{\mu\nu} + m^2B^\mu + iaF^\mu{}_\nu B^\nu\right]\delta B^\star_\mu\,, \end{aligned}$$

In $\clubsuit_1$, we did

$$-\tfrac{1}{2}B^{\mu\nu}\delta B^\star_{\mu\nu} = \tfrac{1}{2}B^{\mu\nu}\delta B^\star_{\nu\mu} = B^{\mu\nu}\delta(D_\nu B_\mu)^\star = B^{\mu\nu}D^\star_\nu\delta B^\star_\mu;$$

and, in $\clubsuit_2$, we exploited the property (see Apendix C)

$$\partial_\mu(\Psi^\star\Phi) = (D_\mu\Psi)^\star\Phi + \Psi^\star D_\mu\Phi$$

for $\Psi \equiv D_\nu\,\delta B_\mu$ and $\Phi \equiv B^{\mu\nu}$, and used the fact that $\delta B^\star_\mu|_{\text{boundary}} = 0$. Setting

$$\frac{\delta\mathcal{L}}{\delta B^\star_\mu} = 0,$$

we get the desired equation,

$$D_\nu B^{\nu\mu} + m^2B^\mu + iaF^\mu{}_\nu B^\nu = 0\,.$$

Similarly, varying B_μ:

$$\begin{aligned} \delta\mathcal{S}|_{A,B^\star} &= \int \mathrm{d}^4x - \tfrac{1}{2}B^{\star\mu\nu}\delta B_{\mu\nu} + m^2B^{\star\mu}\delta B_\mu + iaF_\nu{}^\mu B^{\star\nu}\delta B_\mu \\ &= \int \mathrm{d}^4x\; B^{\star\mu\nu}D_\nu\,\delta B_\mu + m^2B^{\star\mu}\delta B_\mu + iaF_\nu{}^\mu B^{\star\nu}\delta B_\mu \\ &= \int \mathrm{d}^4x\left[-D^\star_\nu B^{\star\mu\nu} + m^2B^{\star\mu} + iaF_\nu{}^\mu B^{\star\nu}\right]\delta B_\mu\,, \end{aligned}$$

Hence,

$$D^\star_\nu B^{\star\nu\mu} + m^2B^{\star\mu} - iaF^\mu{}_\nu B^{\star\nu} = 0\,,$$

which is simply the complex conjugated version of the previous equation of motion. This result makes sense intuitively, for $\mathcal{L}$ is invariant under the application of complex conjugation. As for A_μ, it is not as direct:

$$\begin{aligned} \delta\mathcal{S}|_{B,B^\star} &= \int \mathrm{d}^4x\; - \tfrac{1}{2}(\delta B^\star_{\mu\nu})B^{\mu\nu} - \tfrac{1}{2}B^\star_{\mu\nu}\delta B^{\mu\nu} \\ &\qquad + ia\,(\delta F_{\mu\nu})B^{\star\mu}B^\nu - \tfrac{1}{4}(\delta F_{\mu\nu})F^{\mu\nu} - \tfrac{1}{4}F_{\mu\nu}\delta F^{\mu\nu} \\ &= \int \mathrm{d}^4x\;(B^{\star\nu\mu}\delta(D_\mu B_\nu) + \text{c.c.}) \\ &\qquad + ia(B^{\star\nu}B^\mu - B^{\star\mu}B^\nu)\,\partial_\nu\delta A_\mu + F^{\mu\nu}\partial_\nu\delta A_\mu \\ &= \int \mathrm{d}^4x\Big[iq\,(B^{\star\nu\mu}B_\nu - B^{\nu\mu}B^\star_\nu) \\ &\qquad - ia\,\partial_\nu(B^{\star\nu}B^\mu - B^{\star\mu}B^\nu) - \partial_\nu F^{\mu\nu}\Big]\delta A_\mu. \end{aligned}$$

Here,

$$\begin{aligned} \delta(D_\mu B_\nu) &= \delta(\partial_\mu B_\nu + iq\,A_\mu B_\nu) \\ &= \partial_\mu\delta B_\nu + iq\left((\delta A_\mu)B_\nu + A_\mu\delta B_\nu\right) = iqB_\nu\,\delta A_\mu, \end{aligned}$$

because $\delta B_\nu = 0$. At last, this boils down to

$$\partial_\nu F^{\nu\mu} = iq\,(B^{\nu\mu}B^\star_\nu - B^{\star\nu\mu}B_\nu) + ia\,\partial_\nu(B^{\star\nu}B^\mu - B^\nu B^{\star\mu})\,.$$

Appendix C

It is pretty simple to prove. Let Ψ, Φ be arbitrary, differentiable fields too. Then:

$$\begin{aligned} \partial_\mu(\Psi^\star\Phi) &= (\partial_\mu\Psi)^\star\Phi + \Psi^\star\partial_\mu\Phi \\ &= (\partial_\mu\Psi)^\star\Phi + \Psi^\star(iqA_\mu\Phi) + (-iqA_\mu\Psi^\star)\Phi + \Psi^\star\partial_\mu\Phi \\ &= [(\partial_\mu + iqA_\mu)\Psi]^\star\Phi + \Psi^\star(\partial_\mu + iqA_\mu)\Phi \\ &= (D_\mu\Psi)^\star\Phi + \Psi^\star D_\mu\Phi. \end{aligned}$$

Note that $\partial^\star_\mu = \partial_\mu$.

Appendix D

The divergence of equation (6) reads

$$\begin{aligned} \partial_\mu\partial_\nu F^{\nu\mu} &= iq\,\partial_\mu(B^{\nu\mu}B^\star_\nu - B^{\star\nu\mu}B_\nu) \\ &\qquad + ia\,\partial_\mu\partial_\nu(B^{\star\nu}B^\mu - B^\nu B^{\star\mu})\,. \end{aligned}$$

To begin with, the LHS and second term in RHS are immediately zero because $[\partial_\mu, \partial_\nu] = 0$:

$$\partial_\mu \partial_\nu F^{\nu\mu} = \partial_{[\mu}\partial_{\nu]} F^{\nu\mu} = \tfrac{1}{2}[\partial_\mu, \partial_\nu] F^{\nu\mu} = 0;$$
$$\partial_\mu \partial_\nu (B^{\star\nu} B^\mu - B^{\star\mu} B^\nu) = [\partial_\mu, \partial_\nu](B^{\star\nu} B^\mu) = 0.$$

So we simply need to prove that

$$0 = iq\, \partial_\mu (B^{\nu\mu} B^\star_\nu - B^{\star\nu\mu} B_\nu)$$

is satisfied identically. First, notice that it suffices computing the first term given that

$$\partial_\mu (B^{\nu\mu} B^\star_\nu - B^{\star\nu\mu} B_\nu) = \partial_\mu (B^{\nu\mu} B^\star_\nu) - \text{c.c.}\ .$$

Employing the property derived in Appendix C leads to

$$\partial_\mu (B^{\nu\mu} B^\star_\nu) = (D_\mu B^{\nu\mu}) B^\star_\nu + B^{\nu\mu} (D_\mu B_\nu)^\star,$$

so substituting in the Proca equation, (4), gives

$$\begin{aligned}
\partial_\mu (B^{\nu\mu} B^\star_\nu) &= (D_\mu B^{\nu\mu}) B^\star_\nu + B^{\nu\mu} (D_\mu B_\nu)^\star \\
&= (m^2 B^\nu + iaF^\nu{}_\lambda B^\lambda) B^\star_\nu - \tfrac{1}{2} B^{\mu\nu} B^\star_{\mu\nu} \\
&= -\tfrac{1}{2} B^{\mu\nu} B^\star_{\mu\nu} + m^2 B^\star_\nu B^\nu + iaF^\nu{}_\lambda B^\star_\nu B^\lambda \\
&= \mathcal{L} + \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu}.
\end{aligned}$$

Since the Lagrangian and EM field strength tensor are real, it follows that

$$\partial_\mu (B^{\nu\mu} B^\star_\nu - B^{\star\nu\mu} B_\nu) = \mathcal{L} + \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - (\mathcal{L} + \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu})^\star = 0,$$

thus finishing the proof.

Appendix E

First, let us derive the commutation relation for covariant derivatives; it will prove useful in the future. Since we are working in the 1st quantization scheme, $[A_\mu, A_\nu] = 0$, so:

$$\begin{aligned}
[D_\mu, D_\nu] &= [\partial_\mu + iqA_\mu, \partial_\nu + iqA_\nu] \\
&= [\partial_\mu, \partial_\nu] + iq[\partial_\mu, A_\nu] + iq[A_\mu, \partial_\nu] - q^2 [A_\mu, A_\nu] \\
&= iq([\partial_\mu, A_\nu] + [A_\mu, \partial_\nu]) \\
&= iq(\partial_\mu (A_\nu -) - A_\nu \partial_\mu - + A_\mu \partial_\nu - - \partial_\nu (A_\mu -)) \\
&= iq(\, (\partial_\mu A_\nu) - + A_\nu \partial_\mu - - A_\nu \partial_\mu - \\
&\qquad + A_\mu \partial_\nu - - (\partial_\nu A_\mu) - - A_\mu \partial_\nu -) \\
&= iq(\, (\partial_\mu A_\nu) - - (\partial_\nu A_\mu) -) = iqF_{\mu\nu} -\ .
\end{aligned}$$

Taking the divergence of (4) gives

$$D_\mu D_\nu B^{\nu\mu} + m^2 D_\mu B^\mu + iaD_\mu F^\mu{}_\nu B^\nu = 0\,.$$

From Appendix A property,

$$iaD_\mu F^\mu{}_\nu B^\nu = ia(\partial_\mu F^{\mu\nu}) B_\nu + iaF_{\mu\nu} D^\mu B^\nu;$$

moreover,

$$\begin{aligned}
D_\mu D_\nu B^{\nu\mu} &= D_{[\mu} D_{\nu]} B^{\nu\mu} = \tfrac{1}{2}[D_\mu, D_\nu] B^{\nu\mu} \\
&= iqF_{\mu\nu} \tfrac{1}{2} B^{\nu\mu} = iqF_{\mu\nu} D^{[\nu} B^{\mu]} \\
&= -iqF_{\mu\nu} D^{[\mu} B^{\nu]} = -iqF_{\mu\nu} D^\mu B^\nu.
\end{aligned}$$

Isolating $D_\mu B^\mu$, we finally have

$$D_\mu B^\mu = \frac{i(q-a)}{m^2} F_{\mu\nu} D^\mu B^\nu - \frac{ia}{m^2} (\partial_\mu F^{\mu\nu}) B_\nu.$$

Appendix F

To see why $(\mathcal{O}^\ddagger)^\mu{}_\nu \equiv (\mathcal{O}_\nu{}^\mu)^\dagger$, let us part from the inner product definition:

$$\begin{aligned}
(U, \mathcal{O}V)^\star &= \left[\int_M \mathrm{d}^4x\ U^\star_\mu \mathcal{O}^\mu{}_\nu V^\nu\right]^\star \\
&= \left[\int_M \mathrm{d}^4x\ \Big((\mathcal{O}^\mu{}_\nu)^\dagger U_\mu\Big)^\star V^\nu\right]^\star \\
&= \int_M \mathrm{d}^4x\ V^{\star\nu} (\mathcal{O}^\mu{}_\nu)^\dagger U_\mu \\
&= \int_M \mathrm{d}^4x\ V^\star_\nu (\mathcal{O}_\mu{}^\nu)^\dagger U^\mu \\
&= \int_M \mathrm{d}^4x\ V^\star_\mu (\mathcal{O}_\nu{}^\mu)^\dagger U^\nu \\
&\overset{!}{=} (V, \mathcal{O}^\ddagger U) = \int_M \mathrm{d}^4x\ V^\star_\mu (O^\ddagger)^\mu{}_\nu U^\nu.
\end{aligned}$$

Notice how we were forced to manipulate U and V's indices in order to be able to compare both inner products.

Now, let us see whether the controversial tensor-differential operators we have mentioned, $D^\mu D_\nu$ and $D^\mu iF_{\alpha\nu} D^\alpha$—the same operator, but when substituting (12)—are Hermitian. As for the rest, they are trivial to check for. It does not matter if we first apply the dagger and transpose later as $(\mathcal{O}_{\hat\nu}{}^{\hat\mu})^\dagger = \eta^{\hat\mu\nu}\eta_{\hat\nu\mu}(\mathcal{O}^\mu{}_\nu)^\dagger$. We start by naively applying the dagger:

$$D^\mu D_\nu \longrightarrow (-D_\nu)(-D^\mu)\,;$$
$$D^\mu iF_{\alpha\nu} D^\alpha \longrightarrow (-D^\alpha)(-iF_{\alpha\nu})(-D^\mu)\,.$$

Finally, we raise and lower free indices and interchange them:

$$D_\nu D^\mu \longrightarrow D^\mu D_\nu\,;$$
$$D^\alpha(-iF_{\alpha\nu}) D^\mu \longrightarrow D^\alpha iF^\mu{}_\alpha D_\nu \neq D^\mu iF_{\alpha\nu} D^\alpha\,.$$

As we commented, once we substitute $D_\nu B^\nu$, the PE looses its "Hermiticity". From the above result it is pretty easy to see that $D^\mu (iD_0)^n D_\nu$ is Hermitian too $\forall n$.

Appendix G

It is straightforward substituting

$$e^{imt}D^2e^{-imt}\psi^\mu = -m^2\psi^\mu - 2imD_0\psi^\mu + D_0^2\psi^\mu - \vec{D}^2\psi^\mu$$

in (8),

$$e^{imt}D^2e^{-imt}\psi^\mu + m^2\psi^\mu - e^{imt}D^\mu D_\nu e^{-imt}\psi^\nu + i(q+a)F^\mu{}_\nu\psi^\nu = 0,$$

and

$$e^{imt}D_\mu e^{-imt}\psi^\mu = -im\psi^0 + D_\mu\psi^\mu,$$
$$e^{imt}D^\mu e^{-imt}\psi^\nu = -im\eta^{0\mu}\psi^\nu + D^\mu\psi^\nu$$

in (10),

$$e^{imt}D_\mu e^{-imt}\psi^\mu = \frac{i(q-a)}{m^2}F_{\mu\nu}e^{imt}D^\mu e^{-imt}\psi^\nu\,.$$

Appendix H

Let us first demonstrate

$$e^{imt}D^\mu D_\nu e^{-imt}\psi^\nu = (D^\mu - im\eta^{\mu0})[(D_0-im)\psi^0 + D_j\psi^j].$$

To do so, notice

$$[e^{imt}, D^\mu] = e^{imt}D^\mu - D^\mu e^{imt} = -(\partial^\mu e^{imt}) = -im\eta^{\mu0}e^{imt},$$

where $\partial^\mu t = \partial^\mu x^0 = \eta^{\mu\lambda}\partial_\lambda x^0 = \eta^{\mu\lambda}\delta_\lambda{}^0 = \eta^{\mu0}$; then,

$$e^{imt}D^\mu D_\nu e^{-imt}\psi^\nu = (D^\mu - im\eta^{\mu0})e^{imt}D_\nu e^{-imt}\psi^\nu,$$

but, from the previous appendix, we had

$$e^{imt}D_\nu e^{-imt}\psi^\nu = -im\psi^0 + D_\mu\psi^\mu = (D_0-im)\psi^0 + D_j\psi^j.$$

Having proven the result, we can now use it. The time component of (11) is thus

$$(2m+iD_0)iD_0\psi^0 = -\vec{D}^2\psi^0 + i(q+a)F^0{}_j\psi^j - (D_0-im)^2\psi^0 - (D_0-im)D_j\psi^j.$$

Now, the whole LHS cancels with $-(D_0-im)^2\psi^0$, leaving $m^2\psi^0$, the only ψ^0 dependent term besides $-\vec{D}^2\psi^0$ and the one encoded in $D_0D_j\psi^j$:

$$0 = -\vec{D}^2\psi^0 + i(q+a)F^0{}_j\psi^j + m^2\psi^0 - (D_0-im)D_j\psi^j.$$

Isolating ψ^0 in terms of ψ^i, the spin-1 field, yields

$$\psi^0 = \frac{1}{1-\dfrac{\vec{D}^2}{m^2}}\left[-\frac{i(q+a)}{m^2}F^0{}_j\psi^j + \frac{(D_0-im)D_j}{m^2}\psi^j\right].$$

In a similar fashion to the main work, we can expand this inverse operator as $(1-\vec{D}^2/m^2)^{-1} \simeq 1 + \vec{D}^2/m^2$ given that $|m\psi| \gg |D_i\psi|$, so, up to at least $\mathcal{O}(1/m^3)$, it results in

$$\psi^0 \simeq \frac{D_j\psi^j}{im} + \frac{D_0D_j}{m^2}\psi^j - \frac{i(q+a)}{m^2}F^0{}_j\psi^j + \frac{\vec{D}^2D_j\psi^j}{im^3},$$

which is nothing else but (13). The last term appears here, albeit not in (13), because (12) includes $-im\psi^0$, and to prove (12) is an identity up to $\mathcal{O}(1/m^2)$ we need to include it. Also, it does not appear in (13) due to the fact that the SE is presented to second order and, once there are no more derivatives of e^{-imt} or explicit positive mass terms, the order of $\mathcal{O}(1/m^3)$ operators can not be reduced further as in the previous example.

Concerning the spatial components of (11), (14) follows immediately just by substituting $e^{imt}D^iD_\nu e^{-imt}\psi^\nu$.

To derive (15), firstly, we divide (14) by $2m$ and, afterwards, we substitute (13):

$$\left(1+\frac{iD_0}{2m}\right)iD_0\psi^i = -\frac{\vec{D}^2}{2m}\psi^i + \frac{i(q+a)}{2m}F^i{}_0\psi^0 + \frac{i(q+a)}{2m}F^i{}_j\psi^j - \frac{D^i(D_0-im)}{2m}\psi^0 - \frac{D^iD_j}{2m}\psi^j,$$

where

$$\frac{i(q+a)}{2m}F^i{}_0\psi^0 = \frac{i(q+a)}{2m}F^i{}_0\left(\frac{D_j\psi^j}{im}+\cdots\right) = \frac{q+a}{2m^2}F^i{}_0D_j\psi^j + \mathcal{O}(1/m^3)$$

and

$$-\frac{D^i(D_0-im)}{2m}\psi^0 = -\frac{D^iD_0}{2m}\left(\frac{D_j\psi^j}{im}+\cdots\right) + \frac{D^i}{2m}\left(D_j\psi^j + \frac{q+a}{m}F^0{}_j\psi^j + \frac{iD_0D_j\psi^j}{m}+\cdots\right)$$
$$= \frac{D^iiD_0D_j}{2m^2}\psi^j + \frac{D^iD_j}{2m}\psi^j + \frac{q+a}{2m^2}D^iF^0{}_j\psi^j + \frac{D^iiD_0D_j}{2m^2}\psi^j + \mathcal{O}(1/m^3).$$

This leaves us with

$$\left(1+\frac{iD_0}{2m}\right)iD_0\psi^i = -\frac{\vec{D}^2}{2m}\psi^i + \frac{i(q+a)}{2m}F^i{}_j\psi^j \frac{q+a}{2m^2}(F^i{}_0D_j + D^iF^0{}_j)\psi^j + \frac{D^iiD_0D_j}{m^2}\psi^j,$$

where

$$\frac{D^iiD_0D_j}{m^2}\psi^j = \frac{D^iiD_0D_j + D^iiD_0D_j}{2m^2}\psi^j = \frac{iD_0D^iD_j + D^iD_jiD_0}{2m^2}\psi^j + \frac{i[D^i,D_0]D_j + D^ii[D_0,D_j]}{2m^2}\psi^j = \frac{iD_0D^iD_j + D^iD_jiD_0}{2m^2}\psi^j - \frac{q}{2m^2}(F^i{}_0D_j + D^iF^0{}_j)\psi^j,$$

thus arriving at (15). The second term simply represents a generalized Thomas precession correction.

Appendix I

Let us start by doing the $\psi^\mu \to \varphi^\mu$ redefinition to (15):

$$\sqrt{1+\frac{iD_0}{2m}}iD_0\varphi^i = \left(-\frac{\vec{D}^2}{2m}\delta^i{}_j + \frac{i(q+a)}{2m}F^i{}_j\right)\frac{1}{\sqrt{1+\frac{iD_0}{2m}}}\varphi^j + \frac{a}{2m^2}(F^i{}_0 D_j + D^i F^0{}_j)\frac{1}{\sqrt{1+\frac{iD_0}{2m}}}\varphi^j + \frac{iD_0 D^i D_j + D^i D_j iD_0}{2m^2}\frac{1}{\sqrt{1+\frac{iD_0}{2m}}}\varphi^j\,.$$

Observe how $[iD_0, (1+iD_0/2m)^{-1/2}] = 0$. Upon inverting the LHS square root operator, we have

$$iD_0\varphi^i = \frac{\delta^i{}_k}{\sqrt{1+\frac{iD_0}{2m}}}h^k{}_\ell \frac{\delta^\ell{}_j}{\sqrt{1+\frac{iD_0}{2m}}}\varphi^j,$$

which clearly displays an Hermitian structure. Naming $h^{(N)}$ the operators divided by an m^N explicit factor in h and by expanding $(1+iD_0/2m)^{-1/2}$—given that $|m\varphi| \gg |D_0\varphi|$—, we now have

$$\begin{aligned} iD_0\vec{\varphi} &= \left(1-\frac{iD_0}{4m}+\cdots\right)(h^{(1)}+h^{(2)})\left(1-\frac{iD_0}{4m}+\cdots\right)\vec{\varphi} \\ &= \left(h - \frac{iD_0 h^{(1)} + h^{(1)} iD_0}{4m}\right)\vec{\varphi} + \text{explicit } 1/m^{\geq 3} \text{ terms.} \end{aligned}$$

To sum up, there will appear new, at least, second order terms in the Hamiltonian. They read

$$\begin{aligned} h'^{(2)} &\equiv -\frac{iD_0 (h^{(1)})^i{}_j + (h^{(1)})^i{}_j iD_0}{4m} \\ &= \frac{iD_0\vec{D}^2 + \vec{D}^2 iD_0}{8m^2}\delta^i{}_j - \frac{i(q+a)}{8m^2}(iD_0 F^i{}_j + F^i{}_j iD_0). \end{aligned}$$

By redefining $\varphi^\mu \to \tilde{\varphi}^\mu$, we now get

$$\left(iD_0 + \frac{iD_0\vec{D}^2}{8m^2}\right)\vec{\tilde{\varphi}} = \left(h + h'^{(2)} + \frac{h^{(1)}\vec{D}^2}{8m^2}\right)\vec{\tilde{\varphi}},$$

having truncated to third order, and with

$$\frac{(h^{(1)})^i{}_j\vec{D}^2}{8m^2} = -\frac{\vec{D}^4}{16m^3}\delta^i{}_j + \frac{i(q+a)}{16m^3}F^i{}_j\vec{D}^2.$$

Ignoring this second term, the Schrödinger equation looks like

$$\left(iD_0 + \frac{iD_0\vec{D}^2}{8m^2}\right)\vec{\tilde{\varphi}} = -\frac{\vec{D}^2}{2m}\vec{\tilde{\varphi}} - \frac{\vec{D}^4}{16m^3}\vec{\tilde{\varphi}} + \frac{iD_0\vec{D}^2 + \vec{D}^2 iD_0}{8m^2}\vec{\tilde{\varphi}} + \cdots,$$

that is,

$$iD_0\vec{\tilde{\varphi}} = -\frac{\vec{D}^2}{2m}\vec{\tilde{\varphi}} - \frac{\vec{D}^4}{16m^3}\vec{\tilde{\varphi}} + \frac{\vec{D}^2}{8m^2}iD_0\vec{\tilde{\varphi}} + \cdots.$$

Ultimately, we can substitute $iD_0\vec{\tilde{\varphi}}$ recursively, getting in turn

$$\begin{aligned} iD_0\vec{\tilde{\varphi}} &= -\frac{\vec{D}^2}{2m}\vec{\tilde{\varphi}} - \frac{\vec{D}^4}{16m^3}\vec{\tilde{\varphi}} + \frac{\vec{D}^2}{8m^2}\left(-\frac{\vec{D}^2}{2m}\vec{\tilde{\varphi}}+\cdots\right)+\cdots \\ &= -\frac{\vec{D}^2}{2m}\vec{\tilde{\varphi}} - \frac{\vec{D}^4}{8m^3}\vec{\tilde{\varphi}} + \cdots. \end{aligned}$$

This, of course, concurs with the classical expansion

$$T = \sqrt{\vec{p}^2 + m^2} - m = \frac{\vec{p}^2}{2m} - \frac{\vec{p}^4}{8m^3} + \frac{\vec{p}^6}{16m^5} + \cdots,$$

where $\vec{p} = \gamma m\vec{v} \sim -i\vec{D}$. As of now, the SE looks like

$$\begin{aligned} iD_0\tilde{\varphi}^i &= -\frac{\vec{D}^2}{2m}\tilde{\varphi}^i - \frac{\vec{D}^4}{8m^3}\tilde{\varphi}^i + \frac{i(q+a)}{2m}F^i{}_j\tilde{\varphi}^j \\ &+ \frac{a}{2m^2}(F^i{}_0 D_j + D^i F^0{}_j)\tilde{\varphi}^j + (iD_0\mathcal{A}^i{}_j + \mathcal{A}^i{}_j iD_0)\tilde{\varphi}^j, \end{aligned}$$

with

$$\begin{aligned} H^i{}_j &\equiv -\frac{\vec{D}^2}{2m}\delta^i{}_j - \frac{\vec{D}^4}{8m^3}\delta^i{}_j + \frac{i(q+a)}{2m}F^i{}_j + \frac{a}{2m^2}(F^i{}_0 D_j + D^i F^0{}_j); \\ \mathcal{A}^i{}_j &\equiv \frac{D^i D_j}{2m^2} - \frac{i(q+a)}{8m^2}F^i{}_j. \end{aligned}$$

It is worth pointing out that all terms, and even $\mathcal{A}$, are Hermitian. We can at least show it for the last two terms:

$$\begin{aligned} F^i{}_0 D_j + D^i F^0{}_j &\xrightarrow{\dagger} (-D_j)F^i{}_0 + F^0{}_j(-D^i) \\ &\xrightarrow{\mathsf{T}} -D^i F_{j0} - F^{0i}D_j = F^i{}_0 D_j + D^i F^0{}_j; \\ (iD_0\mathcal{A} + \mathcal{A}iD_0)^\ddagger &= \mathcal{A}^\ddagger (iD_0)^\ddagger + (iD_0)^\ddagger \mathcal{A}^\ddagger = \mathcal{A}iD_0 + iD_0\mathcal{A}. \end{aligned}$$

Again, by the same exact argument from the main text, $\mathcal{A}^i{}_j iD_0\tilde{\varphi}^j$ can be neglected as it is of $\mathcal{O}(1/m^3)$—it can be easily proven by iterating the SE equation—and $iD_0\mathcal{A}^i{}_j\tilde{\varphi}^j$ can be brought to the LHS; hence, the SE reduces to

$$\begin{aligned} iD_0(\delta^i{}_j - \mathcal{A}^i{}_j)\tilde{\varphi}^j &= -\frac{\vec{D}^2}{2m}\tilde{\varphi}^i - \frac{\vec{D}^4}{8m^3}\tilde{\varphi}^i \\ &+ \frac{i(q+a)}{2m}F^i{}_j\tilde{\varphi}^j + \frac{a}{2m^2}(F^i{}_0 D_j + D^i F^0{}_j)\tilde{\varphi}^j. \end{aligned}$$

Redefining $\phi^i = (\delta^i{}_j - \mathcal{A}^i{}_j)\tilde{\varphi}^j$, it yields

$$\begin{aligned} iD_0\phi^i &= -\frac{\vec{D}^2}{2m}\phi^i - \frac{\vec{D}^4}{8m^3}\phi^i \\ &+ \frac{i(q+a)}{2m}F^i{}_j\phi^j + \frac{a}{2m^2}(F^i{}_0 D_j + D^i F^0{}_j)\phi^j, \end{aligned}$$

where $\mathcal{A}$, being of $\mathcal{O}(1/m^2)$, does not introduce any new $\mathcal{O}(1/m^{\leq 2})$ terms on the RHS.

Lastly, we can simplify index notation into vector notation. For instance, the third operator may be identified as a magnetic moment,

$$\begin{aligned}\frac{i(q+a)}{2m}F^i{}_j\phi^j &= \frac{i(q+a)}{2m}\epsilon^{ijk}B^k\phi^j \\ &= -\frac{q+a}{2m}(-i\epsilon^{ijk})B^k\phi^j \\ &= -\frac{q+a}{2m}(S^kB^k)^{ij}\phi^j \\ &= -\frac{q+a}{2m}(\vec{S}\cdot\vec{B})\vec{\phi} \\ &\equiv \left[-\left(\frac{q+a}{2m}\vec{S}\right)\cdot\vec{B}\right]\vec{\phi} \sim -(\vec{\mu}\cdot\vec{B})\vec{\phi}.\end{aligned}$$

Regarding the last one, it would correspond to the Darwin, spin-orbit and quadrupole terms; let us decompose it into its spin-0, spin-1 and spin-2 parts, respectively:

$$\frac{a}{2m^2}(F^i{}_0D_j + D^iF^0{}_j) = \frac{a}{2m^2}(E^iD_j - D_iE^j),$$

where

$$\begin{aligned}E^iD_j &= \frac{1}{3}(\vec{E}\cdot\vec{D})\delta^i{}_j + \frac{E^iD_j - E^jD_i}{2} \\ &\quad + \left(\frac{E^iD_j + E^jD_i}{2} - \frac{1}{3}(\vec{E}\cdot\vec{D})\delta^i{}_j\right); \\ D_iE^j &= \frac{1}{3}(\vec{D}\cdot\vec{E})\delta^i{}_j + \frac{D_iE^j - D_jE^i}{2} \\ &\quad + \left(\frac{D_iE^j + D_jE^i}{2} - \frac{1}{3}(\vec{D}\cdot\vec{E})\delta^i{}_j\right).\end{aligned}$$

So it reads

$$\begin{aligned}&\frac{a}{2m^2}(E^iD_j - D_iE^j) \\ &= -\frac{a}{6m^2}(\vec{\nabla}\cdot\vec{E})\delta^i{}_j + \frac{a}{4m^2}\left[(E^iD_j - E^jD_i) - (D_iE^j - D_jE^i)\right] \\ &\quad - \frac{a}{2m^2}\left(\frac{\partial_iE^j + \partial_jE^i}{2} - \frac{1}{3}(\vec{\nabla}\cdot\vec{E})\delta^i{}_j\right),\end{aligned}$$

where

$$\begin{aligned}E^iD_j - E^jD_i &= \epsilon^{ijk}(\vec{E}\times\vec{D})^k = i[\vec{S}\cdot(\vec{E}\times\vec{D})]^{ij}; \\ D_iE^j - D_jE^i &= \epsilon^{ijk}(\vec{D}\times\vec{E})^k = i[\vec{S}\cdot(\vec{D}\times\vec{E})]^{ij}.\end{aligned}$$

Moreover,

$$\begin{aligned}&\frac{\partial_iE^j + \partial_jE^i}{2} - \frac{1}{3}(\vec{\nabla}\cdot\vec{E})\delta^i{}_j \\ &= \left[\frac{1}{2}(\delta_i{}^\ell\delta^j{}_m + \delta_j{}^\ell\delta^i{}_m) - \frac{1}{3}\delta^i{}_j\delta^\ell{}_m\right]\partial_\ell E^m \\ &= \left[\frac{1}{2}(\delta_i{}^\ell\delta^j{}_m - \delta_i{}^j\delta^\ell{}_m + \delta_j{}^\ell\delta^i{}_m - \delta_j{}^i\delta^\ell{}_m) + \frac{2}{3}\delta^i{}_j\delta^\ell{}_m\right]\partial_\ell E^m \\ &= \left[\frac{1}{2}(\epsilon^{mik}\epsilon^{\ell kj} + \epsilon^{\ell ik}\epsilon^{mkj}) + \frac{2}{3}\delta^i{}_j\delta^\ell{}_m\right]\partial_\ell E^m \\ &= \left[-\frac{1}{2}(S^mS^\ell + S^\ell S^m)^{ij} + \frac{2}{3}\delta^i{}_j\delta^\ell{}_m\right]\partial_\ell E^m \\ &= -\frac{1}{2}\left[(S^mS^\ell + S^\ell S^m)^{ij} - \frac{4}{3}\eta^{ij}\,\eta^{\ell m}\right]\partial_\ell E^m \\ &\equiv -\frac{1}{3}(Q^{\ell m})^{ij}\partial_\ell E^m.\end{aligned}$$

To be consistent with indices we did $\eta^{ij} = -\delta^i{}_j$. Also, since the delta's are symmetric, $\delta^i{}_j = \delta_j{}^i = \delta_i{}^j = \delta^j{}_i$. In conclusion, it all amounts to

$$\begin{aligned}&\frac{a}{2m^2}(E^iD_j - D_iE^j) \\ &= -\frac{a}{6m^2}(\vec{\nabla}\cdot\vec{E})\delta^i{}_j + \frac{a}{6m^2}(Q^{\ell m})^{ij}\partial_\ell E^m \\ &\qquad + \frac{ia}{4m^2}\left[\vec{S}\cdot(\vec{E}\times\vec{D} - \vec{D}\times\vec{E})\right]^{ij}.\end{aligned}$$

By bringing the qA_0 (E0 moment) in $iD_0 = i\partial_0 - qA_0(x)$ to the RHS, we finally arrive at (16).

Appendix J

To validate (16), we should see if substituting (16) and (13)—in terms of ϕ—in (12) yields an identity up to $\mathcal{O}(1/m^2)$. Since expressing (13) in function of ϕ is quite cumbersome, it should be equivalent to workout the case with ψ's, in other words, substituting (15) and (13) in (12). Here we will bypass Hermiticity, as we simply want to achieve an algebraic identity. Let us commence with the LHS of (13):

$$\begin{aligned}-im\psi^0 &= -D_i\psi^i + \frac{D_0D_j}{im}\psi^j - \frac{q+a}{m}F^0{}_j\psi^j - \frac{\vec{D}^2D_j}{m^2}\psi^j \\ &= -D_i\psi^i - \frac{D_j}{m}iD_0\psi^j - \frac{a}{m}F^0{}_j\psi^j - \frac{\vec{D}^2D_j}{m^2}\psi^j; \\ D_0\psi^0 &= -\frac{D_j}{m}iD_0\psi^j + \frac{q}{m}F^0{}_j\psi^j + \frac{D_0^2D_j}{m^2}\psi^j - \frac{i(q+a)}{m^2}D_0F^0{}_j\psi^j.\end{aligned}$$

Given that

$$\begin{aligned}
iD_0\psi^j &= -\frac{\vec{D}^2}{2m}\psi^j + \frac{i(q+a)}{2m}F^j{}_k\psi^k + \cdots;\\
\vec{D}^2 D_j &= D_j\vec{D}^2 + [\vec{D}^2, D_j]\\
&= D_j\vec{D}^2 + iq(D_iF_{ij} + F_{ij}D_i);\\
D_0^2 D_j &= D_0D_jD_0 + D_0[D_0, D_j];\\
D_0F^0{}_j &= \partial_0F^0{}_j + F^0{}_jD_0,
\end{aligned}$$

we now have, ignoring $\mathcal{O}(1/m^3)$ terms,

$$\begin{aligned}
-im\psi^0 = -D_i\psi^i - \frac{D_j\vec{D}^2}{2m^2}\psi^j - \frac{a}{m}F^0{}_j\psi^j&\\
- \frac{iq}{m^2}F_{ij}D_i\psi^j + \frac{i(a-q)}{2m^2}D_iF_{ij}\psi^j;&\\
D_0\psi^0 = \frac{D_j\vec{D}^2}{2m^2}\psi^j + \frac{i(q+a)}{2m^2}D_iF_{ij}\psi^j&\\
+\frac{q}{m}F^0{}_j\psi^j - \frac{ia}{m^2}(\partial_0F^0{}_j)\psi^j.&
\end{aligned}$$

Combining both results the LHS becomes

$$\begin{aligned}
-im\psi^0 + D_\mu\psi^\mu &= \frac{q-a}{m}F^0{}_j\psi^j - \frac{iq}{m^2}F_{ij}D_i\psi^j\\
&\quad + \frac{ia}{m^2}D_iF_{ij}\psi^j - \frac{ia}{m^2}(\partial_0F^0{}_j)\psi^j\\
&= \frac{q-a}{m}F^0{}_j\psi^j + \frac{i(q-a)}{m^2}F_{ij}D^i\psi^j\\
&\quad + \frac{ia}{m^2}(\partial_iF_{ij})\psi^j - \frac{ia}{m^2}(\partial_0F^0{}_j)\psi^j\\
&= \frac{q-a}{m}F^0{}_j\psi^j + \frac{i(q-a)}{m^2}F_{ij}D^i\psi^j\\
&\quad - \frac{ia}{m^2}(\partial_\mu F^\mu{}_j)\psi^j.
\end{aligned}$$

But $\partial_\mu F^{\mu\nu} = 0$, so

$$-im\psi^0 + D_\mu\psi^\mu = \frac{q-a}{m}F^0{}_j\psi^j + \frac{i(q-a)}{m^2}F_{ij}D^i\psi^j + \mathcal{O}(1/m^3).$$

Meanwhile, the RHS reads

$$\begin{aligned}
&\frac{i(q-a)}{m^2}F_{\mu\nu}D^\mu\psi^\nu + \frac{q-a}{m}F^0{}_j\psi^j\\
&= \frac{i(q-a)}{m^2}(F_{0\nu}D^0\psi^\nu + F_{i\nu}D^i\psi^\nu) + \frac{q-a}{m}F^0{}_j\psi^j\\
&= \frac{i(q-a)}{m^2}(F_{0i}D^0\psi^i + F_{i0}D^i\psi^0 + F_{ij}D^i\psi^j) + \frac{q-a}{m}F^0{}_j\psi^j\\
&= \frac{i(q-a)}{m^2}F_{ij}D^i\psi^j + \frac{q-a}{m}F^0{}_j\psi^j + \mathcal{O}(1/m^3),
\end{aligned}$$

where we neglected terms with $\psi^0 \sim \mathcal{O}(1/m)$, $D_0\psi^j \sim \mathcal{O}(1/m)$, and thus finishing the proof.

Appendix K

Consider the generators of a Lie group G, $\{T^a\}_{a=1}^n$, with $[T^a, T^b] = if^{abc}T^c$. Introducing a gauge field for each generator, A^a_μ, the covariant derivative should look like $D_\mu = \partial_\mu + igA^a_\mu T^a$, g the coupling constant. Now, we need that the covariant derivative of the matter field we couple the gauge field to, $D_\mu\psi$, behave well under transformations $\psi \to \Omega\psi$, with $\Omega(x) = e^{-i\alpha^a(x)T^a} \in G$:

$$D'_\mu\Omega\psi = \Omega D_\mu\psi \Rightarrow D'_\mu = \Omega D_\mu\Omega^{-1}.$$

Let us see what conditions it imposes on the transformation of A^a_μ. For infinitesimal transformations, we have

$$\begin{aligned}
D'_\mu &= \partial_\mu + igA'^a_\mu T^a\\
&\overset{!}{=} \Omega D_\mu\Omega^{-1}\\
&= (1 - i\alpha^aT^a)(\partial_\mu + igA^a_\mu T^a)(1 + i\alpha^aT^a)\\
&= \partial_\mu + igA^a_\mu T^a + i(\partial_\mu\alpha^a)T^a - gA^b_\mu\alpha^c f^{abc}T^a,
\end{aligned}$$

namely, the gauge field transforms as

$$A'^a_\mu = A^a_\mu + \frac{1}{g}(\partial_\mu\alpha^a) + if^{abc}A^b_\mu\alpha^c.$$

Summing up, under the gauge transformation $\psi \to \psi' = \Omega\psi$, $A_\mu \to A'_\mu$, the invariant field strength should be

$$\begin{aligned}
\mathcal{F}_{\mu\nu} &= D_\mu A_\nu - D_\nu A_\mu\\
&= (\partial_\mu\mathbb{1} + igA^a_\mu T^a)A^a_\nu T^a - (\partial_\nu\mathbb{1} + igA^a_\nu T^a)A^a_\mu T^a\\
&= (\partial_\mu A^c_\nu - \partial_\nu A^c_\mu - gf^{abc}A^a_\mu A^b_\nu)T^c.
\end{aligned}$$

Effectively, if $G = SU(2)$, with $f^{abc} = \epsilon^{abc}$, $g = \mathfrak{g}$, $A^a_\mu \equiv W^a_\mu$, we get $W^a_{\mu\nu}$. As for $G = U(1)$, since it is abelian, $f^{abc} = 0$, getting $\breve{B}_{\mu\nu}$ too.

Now we will obtain $\mathcal{L}_\mathfrak{g}, \mathcal{L}_{\mathfrak{g}^2}, \mathcal{L}_{q,a}$ and $\mathcal{L}_{q^2}$. We will be using

$$\begin{pmatrix} W^1_\mu\\ W^2_\mu\\ W^3_\mu\\ \breve{B}_\mu \end{pmatrix} = \left(\begin{array}{cc|cc} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & & \\ -\frac{1}{i\sqrt{2}} & \frac{1}{i\sqrt{2}} & & \\ \hline & & s & c\\ & & c & -s \end{array}\right) \begin{pmatrix} W^+_\mu\\ W^-_\mu\\ A_\mu\\ Z_\mu \end{pmatrix},$$

where $s \equiv \sin\theta_W$, $c \equiv \cos\theta_W$, and where we expressed $W^{1,2}$ in terms of $W^\pm$. We later set $Z_\mu = 0$. First, due to linearity,

$$\begin{aligned}
\partial_\mu W^1_\nu - \partial_\nu W^1_\mu &= \frac{\hat{W}^-_{\mu\nu} + \hat{W}^+_{\mu\nu}}{\sqrt{2}};\\
\partial_\mu W^2_\nu - \partial_\nu W^2_\mu &= \frac{\hat{W}^-_{\mu\nu} - \hat{W}^+_{\mu\nu}}{i\sqrt{2}};\\
\partial_\mu W^3_\nu - \partial_\nu W^3_\mu &= sF_{\mu\nu} + cZ_{\mu\nu};\\
\partial_\mu \breve{B}_\nu - \partial_\nu \breve{B}_\mu &= cF_{\mu\nu} - sZ_{\mu\nu}.
\end{aligned}$$

The gauge-kinetic terms then read

$$
\begin{aligned}
-\frac{1}{4}W^i_{\mu\nu}W^{i\,\mu\nu} &= -\frac{1}{4}(\hat{W}^i_{\mu\nu} - \mathfrak{g}\epsilon^{ijk}W^j_\mu W^k_\nu)^2 \\
&= -\frac{1}{4}\hat{W}^i_{\mu\nu}\hat{W}^{i\,\mu\nu} + \frac{\mathfrak{g}}{2}\hat{W}^i_{\mu\nu}\epsilon^{ijk}W^{j\,\mu}W^{k\,\nu}; \\
&\qquad - \frac{\mathfrak{g}^2}{4}\epsilon^{ijk}\epsilon^{i\ell m}W^j_\mu W^k_\nu W^{\ell\,\mu}W^{m\,\nu} \\
-\frac{1}{4}\breve{B}_{\mu\nu}\breve{B}^{\mu\nu} &= -\frac{1}{4}(cF_{\mu\nu} - sZ_{\mu\nu})^2,
\end{aligned}
$$

where

$$
\begin{aligned}
&-\frac{1}{4}\hat{W}^1_{\mu\nu}\hat{W}^{1\,\mu\nu} - \frac{1}{4}\hat{W}^2_{\mu\nu}\hat{W}^{2\,\mu\nu} \\
&= -\frac{1}{4}\begin{pmatrix}\hat{W}^1_{\mu\nu} & \hat{W}^2_{\mu\nu}\end{pmatrix}\begin{pmatrix}\hat{W}^{1\,\mu\nu} \\ \hat{W}^{2\,\mu\nu}\end{pmatrix} \\
&= -\frac{1}{4}\begin{pmatrix}\hat{W}^+_{\mu\nu} & \hat{W}^-_{\mu\nu}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{i\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{i\sqrt{2}}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{i\sqrt{2}} & \frac{1}{i\sqrt{2}}\end{pmatrix}\begin{pmatrix}\hat{W}^{+\,\mu\nu} \\ \hat{W}^{-\,\mu\nu}\end{pmatrix} \\
&= -\frac{1}{4}\begin{pmatrix}\hat{W}^+_{\mu\nu} & \hat{W}^-_{\mu\nu}\end{pmatrix}\begin{pmatrix}0 & 1 \\ 1 & 0\end{pmatrix}\begin{pmatrix}\hat{W}^{+\,\mu\nu} \\ \hat{W}^{-\,\mu\nu}\end{pmatrix} = -\frac{1}{2}\hat{W}^-_{\mu\nu}\hat{W}^{+\,\mu\nu}; \\
&-\frac{1}{4}\hat{W}^3_{\mu\nu}\hat{W}^{3\,\mu\nu} = -\frac{1}{4}(sF_{\mu\nu} + cZ_{\mu\nu})^2 \\
&+\frac{\mathfrak{g}}{2}\hat{W}^3_{\mu\nu}\epsilon^{3jk}W^{j\,\mu}W^{k\,\nu} = \mathfrak{g}\hat{W}^3_{\mu\nu}W^{1\,\mu}W^{2\,\nu} \\
&\qquad = \mathfrak{g}\hat{W}^3_{\mu\nu}\frac{W^{-\mu}+W^{+\mu}}{\sqrt{2}}\frac{W^{-\nu}-W^{+\nu}}{i\sqrt{2}} \\
&\qquad = -i\mathfrak{g}\hat{W}^3_{\mu\nu}W^{+\mu}W^{-\nu}; \\
&+\frac{\mathfrak{g}}{2}\hat{W}^1_{\mu\nu}\epsilon^{1jk}W^{j\,\mu}W^{k\,\nu} = \mathfrak{g}\hat{W}^1_{\mu\nu}W^{2\,\mu}W^{3\,\nu}; \\
&+\frac{\mathfrak{g}}{2}\hat{W}^2_{\mu\nu}\epsilon^{2jk}W^{j\,\mu}W^{k\,\nu} = -\mathfrak{g}\hat{W}^2_{\mu\nu}W^{1\,\mu}W^{3\,\nu}; \\
&\epsilon^{ijk}\epsilon^{i\ell m} = \delta^{j\ell}\delta^{km} - \delta^{jm}\delta^{k\ell}.
\end{aligned}
$$

Therefore, the free-field kinetic terms are

$$
\begin{aligned}
-\frac{1}{4}\hat{W}^i_{\mu\nu}\hat{W}^{i\,\mu\nu} - \frac{1}{4}\breve{B}_{\mu\nu}\breve{B}^{\mu\nu} &= -\frac{1}{2}\hat{W}^-_{\mu\nu}\hat{W}^{+\,\mu\nu} \\
&\quad -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu},
\end{aligned}
$$

the 3-vertex terms,

$$
\begin{aligned}
\frac{\mathfrak{g}}{2}\hat{W}^i_{\mu\nu}\epsilon^{ijk}W^{j\,\mu}W^{k\,\nu} &= -i\mathfrak{g}\hat{W}^3_{\mu\nu}W^{+\mu}W^{-\mu} \\
&\quad + \mathfrak{g}(\hat{W}^1_{\mu\nu}W^{2\,\mu} - \hat{W}^2_{\mu\nu}W^{1\,\mu})W^{3\,\nu},
\end{aligned}
$$

with

$$
\begin{aligned}
&\hat{W}^1_{\mu\nu}W^{2\,\mu} - \hat{W}^2_{\mu\nu}W^{1\,\mu} \\
&= \begin{pmatrix}\hat{W}^1_{\mu\nu} & \hat{W}^2_{\mu\nu}\end{pmatrix}\begin{pmatrix}0 & 1 \\ -1 & 0\end{pmatrix}\begin{pmatrix}\hat{W}^{1\,\mu} \\ \hat{W}^{2\,\mu}\end{pmatrix} \\
&= \begin{pmatrix}\hat{W}^+_{\mu\nu} & \hat{W}^-_{\mu\nu}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & -\frac{1}{i\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{i\sqrt{2}}\end{pmatrix}\begin{pmatrix}0 & 1 \\ -1 & 0\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{i\sqrt{2}} & \frac{1}{i\sqrt{2}}\end{pmatrix}\begin{pmatrix}\hat{W}^{+\mu} \\ \hat{W}^{-\mu}\end{pmatrix} \\
&= \begin{pmatrix}W^+_{\mu\nu} & W^-_{\mu\nu}\end{pmatrix}\begin{pmatrix}0 & -i \\ i & 0\end{pmatrix}\begin{pmatrix}W^{+\mu} \\ W^{-\mu}\end{pmatrix} \\
&= i(\hat{W}^-_{\mu\nu}W^{+\mu} - \hat{W}^+_{\mu\nu}W^{-\mu}),
\end{aligned}
$$

and the 4-vertex ones,

$$
\begin{aligned}
&-\frac{\mathfrak{g}^2}{4}\epsilon^{ijk}\epsilon^{i\ell m}W^j_\mu W^k_\nu W^{\ell\,\mu}W^{m\,\nu} \\
&= -\frac{\mathfrak{g}^2}{4}\Big[(W^j_\mu W^{j\,\mu})^2 - (W^j_\mu W^{j\,\nu})^2\Big],
\end{aligned}
$$

wherein, by a similar change of basis procedure,

$$
\begin{aligned}
W^j_\mu W^{j\,\nu} &= W^+_\mu W^{-\,\nu} + W^-_\mu W^{+\,\nu} + W^3_\mu W^{3\,\nu}; \\
W^j_\mu W^{j\,\mu} &= 2|W|^2 + (W^3)^2,
\end{aligned}
$$

and so,

$$
\begin{aligned}
&(W^j_\mu W^{j\,\nu})^2 = 2W^{+2}W^{-2} + 2|W|^4 \\
&\qquad\qquad + 4(W^+\cdot W^3)(W^-\cdot W^3) + (W^3)^4; \\
&(W^j_\mu W^{j\,\mu})^2 = 4|W|^4 + 4|W|^2(W^3)^2 + (W^3)^4; \\
&(W^j_\mu W^{j\,\mu})^2 - (W^j_\mu W^{j\,\nu})^2 = 2|W|^4 + 4|W|^2(W^3)^2 \\
&\qquad\qquad -2W^{+2}W^{-2} - 4(W^+\cdot W^3)(W^-\cdot W^3).
\end{aligned}
$$

Setting $W^3 = sA$ and using $e = \mathfrak{g}s$, we finally have

$$
\begin{aligned}
\mathcal{L}^{\mathrm{EW}}_{\mathrm{gauge}} &= -\frac{1}{2}\hat{W}^-_{\mu\nu}\hat{W}^{+\,\mu\nu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}Z_{\mu\nu}Z^{\mu\nu} \\
&\quad - ie\Big[F_{\mu\nu}W^{+\mu}W^{-\nu} + (\hat{W}^+_{\mu\nu}W^{-\mu} - \hat{W}^-_{\mu\nu}W^{+\mu})A^\nu\Big] \\
&-\frac{\mathfrak{g}^2}{2}\Big[|W|^4 - W^{+2}W^{-2}\Big] + e^2\Big[(W^+\cdot A)(W^-\cdot A) - |W|^2A^2\Big].
\end{aligned}
$$

As for the case of Proca,

$$
\mathcal{L}^{\mathrm{gauge}}_{\gamma,B} = -\frac{1}{2}B^\star_{\mu\nu}B^{\mu\nu} + iaF_{\mu\nu}B^{\star\mu}B^\nu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu},
$$

with

$$
\begin{aligned}
B^{\mu\nu} &= \hat{B}^{\mu\nu} + 2iqA^{[\mu}B^{\nu]}; \\
B^\star_{\mu\nu} &= \hat{B}^\star_{\mu\nu} - 2iqA_{[\mu}B^\star_{\nu]},
\end{aligned}
$$

and, hence,

$$
\begin{aligned}
-\frac{1}{2}B^\star_{\mu\nu}B^{\mu\nu} &= -\frac{1}{2}\hat{B}^\star_{\mu\nu}\hat{B}^{\mu\nu} - iq(\hat{B}^\star_{\mu\nu}A^{[\mu}B^{\nu]} - \hat{B}^{\mu\nu}A_{[\mu}B^\star_{\nu]}) \\
&\qquad - 2q^2 A_{[\mu}B^\star_{\nu]}A^{[\mu}B^{\nu]} \\
&= -\frac{1}{2}\hat{B}^\star_{\mu\nu}\hat{B}^{\mu\nu} - iq(\hat{B}_{\mu\nu}B^{\star\mu} - \hat{B}^\star_{\mu\nu}B^\mu)A^\nu \\
&\qquad + q^2\Big[(B\cdot A)(B^\star\cdot A) - |B|^2A^2\Big],
\end{aligned}
$$

thus ending the derivation.

Appendix L

Here, we demonstrate that the term with a and $\tilde{a}$, and the operators built from $\mathcal{O}_1$ and $\mathcal{O}_2$ exhaust all possibilities for dimension-4 and -6 operators in our case. Moreover, we will provide a demonstration to how these behave under the usual discrete transformations.

In effect, if we discard topological operators, the only allowed 4 dimensional structures including two B fields are $B^\star FB$ and $B^\star\tilde{F}B$. In the dimension-6 case, we had $B^\star FFB$, $B^\star F\tilde{F}B$, $B^\star(\partial^2F)B$, $B^\star(\partial^2\tilde{F})B$, $(DB)^\star F(DB)$, $(DB)^\star\tilde{F}(DB)$, $B^\star(\partial F)(DB)$, $B^\star(\partial\tilde{F})(DB)$, $B^\star F(D^2B)$, $B^\star\tilde{F}(D^2B)$, and their Hermitian conjugates. Note $\tilde{F}\tilde{F}$ can be expressed as FF, or that contractions including more than one ϵ may be expressed as combinations of Kronecker deltas, or that it can not be contracted to symmetric direct products. Also, F has to be contracted with the other objects, as it is traceless. Now, the first four structures become of $\mathcal{O}(1/m^3)$ in the NR limit, which is why we will ignore them; moreover, it was already mentioned that the $(DB)^\star F(DB)$'s can be expressed in terms of $B^\star F(D^2B)$ and $B^\star(\partial F)(DB)$ through a total divergence. Hence, we are pretty much left with the structures we included in the main work. The first kind of operators are of the form $B^\star_\mu F^{\alpha\beta}D_\rho D_\sigma B_\nu$, so there would be $\binom{4}{2}=6$ ways to contract the two indices from F with the remaining 4. These read

$$
\begin{aligned}
&B^\star_\mu F^{\alpha\beta}D_\alpha D_\beta B^\mu,\ \ B^\star_\mu F_{\alpha\beta}D^\alpha D^\mu B^\beta,\ \ B^\star_\mu F_{\alpha\beta}D^\mu D^\alpha B^\beta,\\
&B^\star_\mu F^{\mu\nu}D_\nu D_\alpha B^\alpha,\ \ B^\star_\mu F^{\mu\nu}D_\alpha D_\nu B^\alpha,\ \ B^\star_\mu F^{\mu\nu}D^2B_\nu.
\end{aligned}
$$

Since $D_{[\alpha}D_{\beta]}\propto F_{\alpha\beta}$, the first reduces to a $B^\star FFB$-type opertor; commuting $D^\alpha D^\mu$ in the second, you would get something like $B^\star FFB$ plus the third, which we will leave for now; by substituting (10) in the fourth, we get a higher order operator; in a similar fashion, by commuting $D_\alpha D_\nu$ in the fifth, we would get something like $B^\star FFB$ plus the fourth, and, finally, by substituting (8), with $a=0$, in the sixth, we will be obtaining familiar operators or higher order contributions. Similarly, the second kind of operators look like $B^\star_\mu\partial_\rho F^{\alpha\beta}D_\sigma B_\nu$, so there would be $\binom{3}{2}=3$ ways to contract the two indices from F with the remaining 4, except for the one from the ∂, because the divergence of F vanishes, (9). These are

$$
B^\star_\mu\partial^\mu F^{\alpha\beta}D_\alpha B_\beta,\ \ B^\star_\alpha\partial_\mu F^{\alpha\beta}D^\mu B_\beta,\ \ B^\star_\alpha\partial_\mu F^{\alpha\beta}D_\beta B^\mu.
$$

Given that divergence of $\tilde{F}$ is zero identically —it represents the homogeneous Maxwell equations—, which is nothing else but the Bianchi identity,

$$
\partial_\mu F_{\alpha\beta}+\partial_\beta F_{\mu\alpha}+\partial_\alpha F_{\beta\mu}=0,
$$

then, the third operator reduces to

$$
\begin{aligned}
B^\star_\alpha\partial_\mu F^{\alpha\beta}D_\beta B^\mu &= B^{\star\alpha}\partial_\mu F_{\alpha\beta}D^\beta B^\mu \\
&= -B^{\star\alpha}\partial_\beta F_{\mu\alpha}D^\beta B^\mu - B^{\star\alpha}\partial_\alpha F_{\beta\mu}D^\beta B^\mu \\
&= B^\star_\alpha\partial_\mu F^{\alpha\beta}D^\mu B_\beta - B^\star_\mu\partial^\mu F^{\alpha\beta}D_\alpha B_\beta,
\end{aligned}
$$

namely, a linear combination of the first two, $\mathcal{O}_1$ and $\mathcal{O}_2$. Regarding the other third operator, $B^\star_\mu F_{\alpha\beta}D^\mu D^\alpha B^\beta$, we now clearly see that, employing a total divergence, it can be related to $\mathcal{O}_1$ and $(D^\mu B_\mu)^\star F_{\alpha\beta}D^\alpha B^\beta$, which is negligible. Therefore, the only independent structures in this framework are these two. To construct real or Hermitian operators we either do $(\mathcal{O}+\mathcal{O}^\ddagger)$ or $i(\mathcal{O}-\mathcal{O}^\ddagger)$. Following the same procedure for $\tilde{F}$, we arrive at the same conclusion. It is worth noting that the Bianchi identity also holds for $\tilde{F}$ since we assumed (9) as to not introduce more than two Proca fields in the operators:

$$
\begin{aligned}
&\partial_\mu\tilde{F}_{\alpha\beta}+\partial_\beta\tilde{F}_{\mu\alpha}+\partial_\alpha\tilde{F}_{\beta\mu}=0\\
\iff\ &\partial_{[\mu}\tilde{F}_{\alpha\beta]}=0\iff\epsilon^{\delta\mu\alpha\beta}\partial_{[\mu}\tilde{F}_{\alpha\beta]}=0\\
\iff\ &\epsilon^{\delta\mu\alpha\beta}\partial_\mu\tilde{F}_{\alpha\beta}=-2\partial_\mu F^{\mu\delta}=0.
\end{aligned}
$$

Lastly, we will explicitly check how the definitive operators transform. Luckily enough, bosonic fields commute with each other, $[U^\mu(x),U_\nu(y)]=i\delta^3(\vec{x}-\vec{y})\delta^\mu{}_\nu$. We need only focus on how $\mathcal{O}_1$, $\mathcal{O}_2$, $\tilde{\mathcal{O}}_1$ and $\tilde{\mathcal{O}}_2$ transform under P, T and C. In fact, we will only check for $\mathcal{O}_1$ and $\tilde{\mathcal{O}}_1$, as the respective others exhibit the same structure. Let us present how all objects transform from the rules stated in the text—we will omit the coordinates. Under P:

$$
\begin{aligned}
&\partial_\mu\to\partial^\mu,\ A_\mu\to A^\mu,\ B_\mu\to B^\mu,\\
&D_\mu\to D^\mu,\ F_{\mu\nu}\to F^{\mu\nu},\ \tilde{F}_{\mu\nu}\to-\tilde{F}^{\mu\nu}.
\end{aligned}
$$

The $F_{\mu\nu}$ transformation is obvious since $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$. As for its dual, it is a bit technical. From its definition we would have $\tilde{F}_{\mu\nu}=\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}\to\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}$, but, in view that

$$
\epsilon_{\mu\nu\rho\sigma}=\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\rho\gamma}\eta_{\sigma\delta}\epsilon^{\alpha\beta\gamma\delta}=\det\eta_{ab}\,\epsilon^{\mu\nu\rho\sigma}=-\epsilon^{\mu\nu\rho\sigma},
$$

it simplifies to $-\tilde{F}^{\mu\nu}$. Alternatively, when ϵ is contracted with other tensors, the non-zero contributions are due to ϵ_{0123} and its permutations, so by raising all indices, three of them being spatial, you get an additional minus sign too. Under T, via similar arguments, we have

$$
\begin{aligned}
&\partial_\mu\to-\partial^\mu,\ A_\mu\to A^\mu,\ B_\mu\to B^\mu,\ i\to-i\\
&D_\mu\to-D^\mu,\ F_{\mu\nu}\to-F^{\mu\nu},\ \tilde{F}_{\mu\nu}\to\tilde{F}^{\mu\nu}.
\end{aligned}
$$

And, under C, it is

$$\begin{aligned}&\partial_\mu \to \partial_\mu, A_\mu \to -A_\mu, B_\mu \to -B^\dagger_\mu\\ &D_\mu \to D^\dagger_\mu,\, F_{\mu\nu} \to -F_{\mu\nu},\, \tilde{F}_{\mu\nu} \to -\tilde{F}^{\mu\nu}.\end{aligned}$$

Therefore—note for scalar, that is, index-free operators in 2nd quantization it is $\ddagger = \dagger$—,

$$\begin{aligned}\mathcal{O}_1 = B^\dagger_\mu \partial^\mu F_{\alpha\beta} D^\alpha B^\beta &\xrightarrow{\mathsf{P}} B^{\dagger\mu}\partial_\mu F^{\alpha\beta} D_\alpha B_\beta = \mathcal{O}_1\\ &\xrightarrow{\mathsf{T}} -B^{\dagger\mu}\partial_\mu F^{\alpha\beta} D_\alpha B_\beta = -\mathcal{O}_1\\ &\xrightarrow{\mathsf{C}} -B_\mu \partial^\mu F_{\alpha\beta}(D^\alpha B^\beta)^\dagger = -\mathcal{O}^\dagger_1;\\ \tilde{\mathcal{O}}_1 = B^\dagger_\mu \partial^\mu \tilde{F}_{\alpha\beta} D^\alpha B^\beta &\xrightarrow{\mathsf{P}} -B^{\dagger\mu}\partial_\mu \tilde{F}^{\alpha\beta} D_\alpha B_\beta = -\tilde{\mathcal{O}}_1\\ &\xrightarrow{\mathsf{T}} B^{\dagger\mu}\partial_\mu \tilde{F}^{\alpha\beta} D_\alpha B_\beta = \tilde{\mathcal{O}}_1\\ &\xrightarrow{\mathsf{C}} -B_\mu \partial^\mu F_{\alpha\beta}(D^\alpha B^\beta)^\dagger = -\mathcal{O}^\dagger_1.\end{aligned}$$

This way and having in mind that $i \xrightarrow{\mathsf{T}} -i$, we retrieve the results in Table I. We omitted the calculations for a and $\tilde{a}$ operators, but they are straightforward.