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Treball final de grau

THE ESCAPING SET

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Abstract

The aim of this project is to understand the behaviour of holomorphic functions of one complex variable under iteration, focusing on polynomials and transcendental entire functions. Our study centres on the points whose orbits tend to infinity, which form the escaping set, a fundamental object in complex dynamics. The escaping set provides insight into the global behaviour of iterates and their relationship with the Julia and Fatou sets, which are also important sets in complex dynamics. To achieve this, we begin by establishing a foundational background in dynamical systems. We then proceed with a separate study to characterize the escaping set for both polynomials and transcendental entire functions, using the previous dynamical results as tools to analyse their structure and properties. In both cases, the most remarkable result is that the escaping set is non-empty, proving the existence of points whose orbits eventually escape to infinity under iteration.

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Contents

Introduction

Throughout history, humanity has sought to understand the world and explain its phenomena. Mathematics, serving as a universal language, was developed to decipher the complexities of the world around us. Among its many branches, complex dynamics stands out as a particularly interesting field. This area of study investigates the the long-term behaviour of the iterates of a given holomorphic function f under different initial conditions.

The origins of this field trace back to the late 19th century. In the 1870s, German mathematician Ernst Schröder[1] and British mathematician Arthur Cayley[2] were the first to study the iteration of holomorphic functions within a dynamical framework through their investigation of complex extensions of Newton's[3] root-finding method in [24] and [8] respectively. Their pioneering work laid the foundation for the study of holomorphic function iteration, inspiring a new perspective on mathematical dynamical systems.

While early contributions by mathematicians such as Böttcher[4], Koenigs[5], and Siegel[6] primarily focused on the local behaviour near fixed points, the development of Montel's[7] theorem on normal families marked a turning point. This theorem enabled the French mathematicians Pierre Fatou[8] and Gaston Julia[9] to establish the foundations of the modern theory during the period between 1910 and 1920, motivated by the prestigious *Grand Prix des Sciences mathématiques de 1918*.

Using Montel's theorem, Fatou and Julia independently proved in [16] and [17] that the extended complex plane $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$, also known as the Riemann sphere, could be divided into two complementary subsets with radically different dynamical behaviours. These were the stable set, now referred to as the Fatou set, and the chaotic set, now known as the Julia set. Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a holomorphic function. Informally, given two starting points $z_0, w_0 \in \widehat{\mathbb{C}}$ sufficiently close to each other, these points belong to the *Fatou set of* f, or $F(f)$, if, the sequences or orbits $\{z_n = f^n(z_0)\}_{n\in\mathbb{N}}$ and $\{w_n = f^n(z_0)\}_{n\in\mathbb{N}}$ exhibit roughly the same behaviour as n tends to infinity. Otherwise, the points are said to belong to the *Julia set of* f, or $J(f)$.

Beyond the division of the Riemann sphere into the Fatou and Julia sets, another fundamental concept in complex dynamics is the *escaping set*, which consists of all points

[1] Friedrich Wilhelm Karl Ernst Schröder: 1841-1902
[2] Arthur Cayley: 1821-1895
[3] Sir Isaac Newton: 1643-1727
[4] Lucjan Emil Böttcher: 1872-1937
[5] Gabriel Xavier Paul Koenigs: 1858-1931
[6] Carl Ludwig Siegel: 1896-1981
[7] Paul Antoine Aristide Montel: 1876-1975
[8] Pierre Joseph Louis Fatou: 1878-1929
[9] Gaston Maurice Julia: 1893-1978

in the complex plane whose orbits under iteration tend toward infinity, that is

$$I(f) = \{z \in \mathbb{C} \mid f^n(z) \to \infty\}.$$

While the Fatou set focuses on stability and the Julia set on chaotic behaviour, the escaping set centres on the trajectory of points as they diverge to infinity, providing insight into the global dynamics of holomorphic functions.

The aim of this project is to analyse and understand the escaping set $I(f)$ of two historically significant classes of holomorphic functions: polynomials, defined on the extended complex plane, and transcendental entire functions, defined on the complex plane. In both cases, the point at infinity plays a major role in shaping the structure and behaviour of their respective escaping sets.

For polynomials, infinity acts as a superattracting fixed point in the extended complex plane $\widehat{\mathbb{C}}$. Consequently, for any polynomial $p : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree $d \geq 2$, there exists an open set of points forming the basin of attraction of infinity, defined as

$$A_p(\infty) = \{z \in \mathbb{C} \mid p^n(z) \to \infty, \;\; as \;\; n \to \infty\}.$$

We will show that in this case, the Fatou and Julia sets of p can be characterized in a very simple way. Indeed we shall see that, if we define the *filled Julia* set as

$$K(p) = \mathbb{C} \setminus A_p(\infty) = \{z \in \mathbb{C} \mid p^n(z) \not\to \infty\},$$

then the Julia set is the common boundary between these complementary sets, that is

$$J(p) = \partial K(p) = \partial A_p(\infty)$$

and therefore, the Fatou set is $F(p) = A_p(\infty) \cup int(K(p))$. In Figure 1, we can observe the filled Julia set of a polynomial (yellow) and its escaping orbits (purple).

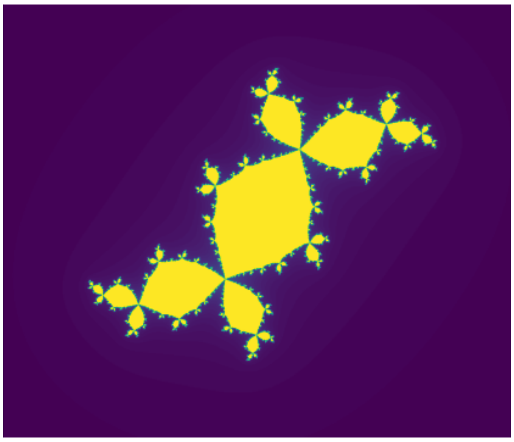


Figure 1: Julia set of the polynomial $p(z) = z^2 - 0.122 + 0.745i$. The yellow points represent the points whose orbits remain bounded under iteration of p. The boundary between the yellow region (bounded orbits) and the purple region (escaping orbits) is the Julia set of p.

This leads to an important first conclusion:

> For polynomials of degree $d \geq 2$, the escaping set coincides with the basin of attraction of infinity and is therefore an open set containing infinity in its interior.

One may then wonder about the connectivity properties of the basin of attraction of infinity, that is, the escaping set, which, as it turns out, depend only on the dynamical behaviour of some special points: the *critical points*, which are those $c \in \mathbb{C}$ such that $p'(c) = 0$. Our first important result is then an initial evidence of the deep connection between topology of the escaping set and dynamics. Recall that an open set is simply connected if its fundamental group is trivial or, informally, if it has no "holes".

Theorem A. *Given a polynomial $p : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with degree $d \geq 2$, $A_p(\infty)$ is always connected. Moreover,*

1. *$A_p(\infty)$ is simply connected if and only if all critical points of p belongs to $K(p)$; and*
2. *if all critical points of p belong to $A_p(\infty)$, then $K(p) = \widehat{\mathbb{C}} \setminus A_p(\infty)$ is totally disconnected.*

The proof of Theorem A will be based on the parametrization of $A_p(\infty)$ called the Böttcher coordinates (see Lemma 3.0.6) and the Green's function of $A_p(\infty)$ (see Definition 3.0.7), and can be found in chapter three.

In strong contrast to polynomials, transcendental entire functions exhibit more intricate and less predictable dynamics. This complexity arises from the nature of infinity as an essential singularity, as highlighted by the Great Picard[10] theorem (see Theorem 4.1.2). Building on this, Fatou made the first significant contributions to the iteration theory of transcendental entire functions in 1926. This field was further developed through the works of I.N. Baker[11] with important papers such as [1] and [2], and later formalized by A.E. Eremenko[12] in the late 1980s. Eremenko was the first to formally define the concept of the escaping set $I(f)$ in the context of transcendental entire functions, although, it had already appeared early, for instance, in McMullen's[13] work [20] on the area of the Julia sets of some transcendental entire functions.

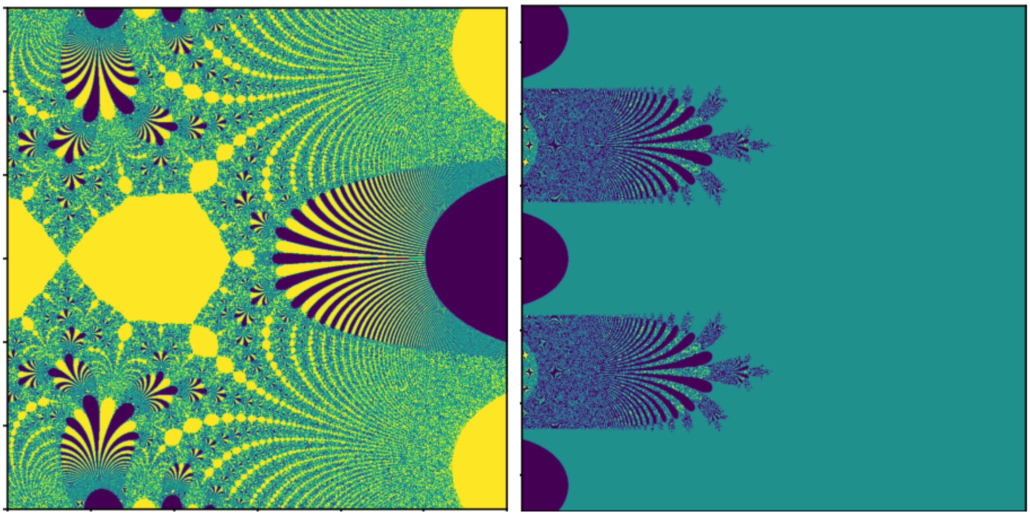
Observe that Picard's theorem tells us that any neighbourhood of infinity is mapped under f to the entire plane, except at most one point. So, a priori, it might seem plausible that orbits $\{f^n(z)\}_n$ could repeatedly "bounce back" as they get near infinity, leading to the possibility that no orbit actually escapes. Contrary to this intuition, Eremenko in [12] proved the following significant result:

Theorem B. *The escaping set $I(f)$ is non-empty for every transcendental entire function $f : \mathbb{C} \to \mathbb{C}$.*

In this project, we present a version of Eremenko's proof of Theorem B slightly different that the original one. The innovation in our approach lies in using Bergweiler[15], Rippon[16] and Stallard's[17] theorem to simplify the geometrical constructions of the proof. Eremenko also conjectured that every connected component of the escaping set of a transcendental entire function is unbounded. While much progress was made on this conjecture, it was recently disproven in [19].

[10] Charles Émile Picard: 1856-1941
[11] Irvine Noel Baker: 1932-2001
[12] Alexandre Emanuilovych Eremenko: 1954-/
[13] Curtis Tracy McMullen: 1958-/
[15] Walter Bergweiler: 1958-/
[16] Philip Jonathan Rippon: 1951-/
[17] Gwyneth Mary Stallard: 1967-/

Once the existence of the escaping set is established, it is natural to ask about its topological properties of f or about its dynamics. In an initial inspection, we find a much richer variety of possibilities than in the polynomial case. Indeed, there are examples of entire functions f for which $I(f)$ contains open sets, while others are Cantor sets of curves with no interior (see Figure 2). But even more remarkably, while in the polynomial case the escaping set was always part of the Fatou set, we see in the entire case that this is no longer necessarily true.



(a) $f(z) = 10ze^z$ (b) $f(z) = 1 + z + e^{-z}$

Figure 2: Julia sets of two transcendental entire functions: (a) corresponds to a function with bounded singular values, while (b) corresponds to one with unbounded singular values. The first picture is a Cantor set of curves with no interior, and the orbits of points in the yellow region converge to a fixed point. In the second picture, the escaping set contains open sets, as the orbits of points in the blue region escape to infinity. In both examples, the purple region is the Julia set, whose apparent interior is due to numerical errors.

Once again, we find that a set of special points, here called *singular values*, plays an important role in this matter. Singular values are points for which some branch of f^{-1} is not well defined in any arbitrary small neighbourhood (see Section 1.4 and Section 4.1). An important class of functions is the Eremenko-Lyubich class, which is defined as

$$B = \{f : \mathbb{C} \to \mathbb{C} \text{ entire} \mid Sing(f) \text{ is bounded}\},$$

where $Sing(f)$ denotes the sets of singular values. This particular class of functions, referred to as bounded-type functions or class B functions, often exhibits simpler dynamics compared to those with unbounded singular values. In collaboration with M. Lyubich[14], Eremenko showed in [13] that for the class of transcendental entire functions whose singular values are all bounded, their escaping set is entirely contained within the Julia set. Relaying on this result, we prove the following fundamental theorem:

Theorem C. *If $f \in B$, the escaping set has empty interior.*

[14]Mikhail Yurievich Lyubich: 1959-/

Under the framework outlined, the objective of this project is to analyse and compare the escaping set of polynomials and transcendental entire functions, using Eremnko's results for the latter. To facilitate this comparison, the structure of the project is organized as follows:

In chapter one, we introduce several preliminary concepts to aid understanding in the following chapters. This chapter is divided into five sections, each addressing a specific theme: relevant results in complex analysis, the extended complex plane, normal families, singular values and critical points, and covering maps.

Chapter two focuses on both the local and global theories of complex dynamics, which are necessary to analyse the dynamical behaviour of the escaping sets. In the local theory, we centre on periodic points and the linearization problem, while in the global theory, we define the Julia and Fatou sets and state several of their properties.

The final two chapters are dedicated to the study of the escaping set. Given the greater complexity of the escaping set for transcendental entire functions, due to the behaviour at infinity and their singular values, we first examine the case of polynomials.

In chapter three, we begin by proving that infinity is a superattracting fixed point. We then establish the relationship between its basin of attraction and the escaping and Julia sets. Using the results from the previous chapter, alongside the Green's function, we discuss the connectivity properties of the basin of attraction of infinity proving Theorem A.

Chapter four starts with an exposition of the analytical differences between transcendental entire functions and polynomials. We then prove the two key results we have mentioned before: first, that the escaping set of all transcendental functions is non-empty (Theorem B); and second, that the interior of the escaping set is empty if it is a bounded-type function (Theorem C).

Lastly, two appendices have been added: the first contains the Python code used to generate the Julia set images of polynomials presented in this work, while the second appendix contains the code for transcendental entire functions. The additional images in this project were created using Geogebra, Inkscape and Wolfram Mathematica.

Chapter 1

Preliminaries

The purpose of this chapter is to introduce a series of foundational results which, even thought we will not deepen them, will be needed when working in the upcoming chapters. This chapter is divided into five sections: the first focuses on complex analysis, the second is an overview of the extended complex plane, the third addresses normal families, the fourth introduces the concept of critical points and, more generally, singular values and the fifth briefly covers covering maps.

1.1 Complex analysis

In this section, we introduce fundamental definitions from complex analysis, along with theorems that we will not prove but will use in future sections. The contents of this section can be found in references [9], [10], [21], [23] and [26]. Since holomorphic functions are central to this project, we begin by defining them.

Definition 1.1.1 (Holomorphic function). *Let $\Omega \subset \mathbb{C}$ be an open set. Let f be a function such that $f : \Omega \to \mathbb{C}$. The function f is said to be* holomorphic *or* $\mathbb{C}$-differentiable *at $z_0 \in \Omega$ if the following limit exists*

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

This limit is denoted by $f'(z_0)$ and is given by:

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

We say that f is holomorphic in Ω, *and we denote it as $f \in H(\Omega)$, if f is holomorphic for every $z_0 \in \Omega$. If $\Omega = \mathbb{C}$, f is said to be an* entire function.

If $z_0 \in \Omega$ is a point where f is non-holomorphic, we say that z_0 is an *isolated singularity* if there exists $r > 0$ such that $f \in H(D(z_0, r) \setminus z_0)$. Moreover, isolated singularities can be classified into three categories:

1. **Pole:** z_0 is a *pole* if $\lim_{z \to z_0} |f(z)| = \infty$.
2. **Removable Singularity:** z_0 is a *removable singularity* if there exists $\varepsilon > 0$ and $g \in H(D(z_0, \varepsilon))$ such that $f(z) = g(z)\ \ \forall z \in D(z_0, \varepsilon) \setminus z_0$.

3. **Essential Singularity:** z_0 is an *essential singularity* if it is an isolated singularity of f, but is is neither a pole nor removable.

Definition 1.1.2 (Conformal maps). *A function f is called* conformal *or* univalent *if it is holomorphic and one-to-one.*

We conclude this section with a list of classical theorems that will be applied in the proofs presented in the upcoming chapters.

Theorem 1.1.3 (Weierstrass uniform convergence theorem). *If a sequence of holomorphic functions $f_n : \Omega \to \mathbb{C}$ converges uniformly on any compact subset to the limit function f, then f itself is holomorphic. Furthermore, the sequence of derivatives f_n' converges uniformly on any compact subset of Ω to the derivative f'.*

Theorem 1.1.4 (The Riemann mapping theorem). *If Ω is a non-empty simply connected open subset of $\mathbb{C}$ such that $\Omega \neq \mathbb{C}$, then there is a conformal function f from Ω onto $\mathbb{D}$, where $\mathbb{D}$ is the unit disc.*

Theorem 1.1.5 (Hurwitz's theorem). *Let Ω be an open subset of the complex pane and let $\{f_n\}$ be a sequence of holomorphic functions in Ω that converges to a holomorphic function f. If f is not identically 0 and there exists a closed ball $\overline{B}(a, r) \subset H(\Omega)$, where $r > 0$, such that $f(z) \neq 0$ for $|z - a| = r$, then there exists an integer N such that, for $n \geq N$, f and f_n have the same number of zeros in $B(a, r)$ (counting multiplicities).*

Theorem 1.1.6 (Identity principle). *Let Ω be a connected open set and let $f : \Omega \to \mathbb{C}$ be a holomorphic function. Then the following are equivalent statements:*

1. *$f \equiv 0$,*

2. *there is a point z_0 in Ω such that $f^{(n)}(z_0) = 0$ for each $n \geq 0$,*

3. *$\{z \in \Omega \mid f(z) = 0\}$ has a limit point in Ω.*

Theorem 1.1.7 (Rouche's theorem). *Let $\omega \subset \mathbb{C}$ be an open set and $f, g \in M(\Omega)$. Let γ be a simple closed path homologous to 0 in Ω. Suppose that f and g has neither zeros nor poles in γ^*, where γ^* is the image of γ. If $|f(z) - g(z)| < |g(z)|$ for $z \in \gamma^*$, then $Z_f - P_f = Z_g - P_g$, where Z_f and Z_g denote the number of zeros with multiplicity of f and g in the interior of γ, and P_f and P_g denote the number of poles of f and g in the interior of γ.*

Theorem 1.1.8 (Koebe 1/4-theorem). *Let $f \in S$, where S denotes the class of all univalent functions f on the unit disk $\mathbb{D}$ such that $f(0) = 0$ and $f'(0) = 1$. Then, $f(\mathbb{D}) \supseteq \{\varepsilon \in \mathbb{C} \mid |\varepsilon| < 1/4\}$.*

In particular, this implies that $r_{n+1} \geq \frac{1}{4}|f'(z_n)|r_n$, where r_n and r_{n+1} are radius related to the behaviour of f.

Lemma 1.1.9 (Injectivity lemma). *Let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic function and $z_0 \in \mathbb{D}$ be a point such that $f'(z_0) \neq 0$. Then, there exists a neighbourhood U of z_0 with the property that the restriction $f_{|U} : U \to \mathbb{C}$ is injective.*

Lemma 1.1.10 (Maximum modulus principle). *Let $\Omega \subset \mathbb{C}$ be open and bounded, and let $f \in H(\Omega) \cap C(\overline{\Omega})$. Then, the maximum of $|f|$ is attained on the boundary $\partial\Omega$.*

1.2 The extended complex plane

In this section, we introduce the extended complex plane and its associated metric. When working with polynomials or rational maps, infinity can be treated as any other point in the complex plane. The *extended complex plane* is defined as the union of $\mathbb{C}$ and infinity, i.e., $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.

We can represent $\widehat{\mathbb{C}}$ as a sphere. Let S be a sphere in $\mathbb{R}^3$ given by

$$S = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \ | \ x_1^2 + x_2^2 + x_3^2 = 1\}$$

and let us identify $\mathbb{C}$ with the horizontal plane

$$\{(x_1, x_2, x_3) \in \mathbb{R}^3 \ | \ x_3 = 0\}.$$

For any point $z \in \mathbb{C}$, we can trace a straight line which connects it to the top point of S, i.e., the point $N = (0, 0, 1)$. We denote these lines by l_z. For each $z \in \mathbb{C}$, its corresponding line l_z intersects S at a point z^* different from N. The function $\pi : z \to z^*$ is known as the *stereographic projection* of $\mathbb{C}$ into S. We observe that if $|z|$ is large, then z^* is near N. Consequently, we define the projection of ∞ to be N. Thus, π is a bijective function from $\widehat{\mathbb{C}}$ to S. This explains why $\widehat{\mathbb{C}}$ is also referred to as the *Riemann sphere*.

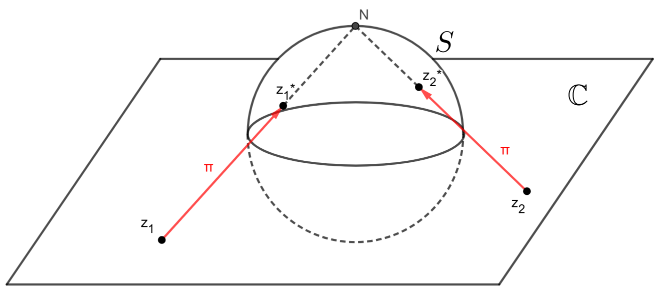


Figure 1.1: The Riemann Sphere and stereographic projections

Furthermore, $\widehat{\mathbb{C}}$ is a topological compactification of $\mathbb{C}$ by the addition of the point at infinity. We conclude this section with two additional definitions.

Definition 1.2.1 (Chordal metric). *Given two points $z_1, z_2 \in \widehat{\mathbb{C}}$, we define the* chordal metric *and denote it as $\sigma(z_1, z_2) = |\pi(z_1) - \pi(z_2)| = |z_1^* - z_2^*|$. Therefore, we obtain the following expressions:*

$$\sigma(z_1, z_2) = \frac{2|z_1 - z_2|}{(1 + |z_1|^2)^{1/2}(1 + |z_2|^2)^{1/2}}, \ \ if \ \ z_1, z_2 \in \mathbb{C}$$

$$\sigma(z_1, \infty) = \frac{2}{(1 + |z_1|^2)^{1/2}}, \ \ if \ \ z_1 \in \mathbb{C}$$

An alternative metric on $\widehat{\mathbb{C}}$ is the *spherical metric*, which is equivalent to the chordal metric. Although we will not explore the spherical metric in this project, this and additional details, can be found in [3, Section 2.1].

Definition 1.2.2 (Spherical length). *If* $\gamma : [0,1] \to \widehat{\mathbb{C}}$ *is a curve in* $\widehat{\mathbb{C}}$, *the* spherical length *of* γ *is defined as*

$$l_\gamma = \int_\gamma \frac{2|dz|}{1+|z|^2} = \int_0^1 \frac{2|\gamma'(t)|}{1+|\gamma(t)|^2} dt.$$

1.3 Normal families

In this section, we introduce the concept of normal families. This notion was formulated by P. Montel in 1911 and it was crucial in the work of Fatou and Julia on the field of complex iteration theory. The content can be found in references [3] and [22].

Definition 1.3.1 (Normal family). *Let* $\mathcal{F} = \{f_m : \Omega \to \widehat{\mathbb{C}},\ m \in \mathbb{N}\}$ *be a family of holomorphic functions, where* $\Omega \subset \widehat{\mathbb{C}}$. *We say that* $\mathcal{F}$ *is a normal family in* Ω *if, for any sequence* $\{f_n\}_n$, $f_n \in \mathcal{F}$, *there exits a subsequence* $\{f_{n_k}\}_k$ *that converges uniformly on compact sets of* Ω *to either a holomorphic function* $f : \Omega \to \widehat{\mathbb{C}}$ *or to* $g \equiv \infty$ *(where* $g : \Omega \to \widehat{\mathbb{C}}$ *is also a holomorphic function).*

Theorem 1.3.2 (Great Montel's theorem). *Let* $\mathcal{F}$ *be a family of holomorphic functions defined on a domain* $\Omega \subset \widehat{\mathbb{C}}$. *If there exists three different fixed values* $a, b, c \in \widehat{\mathbb{C}}$ *such that, for every* $f \in \mathcal{F}$, $f(\Omega) \subset \widehat{\mathbb{C}} \setminus \{a, b, c\}$, *then* $\mathcal{F}$ *is a normal family in* Ω.

As a consequence of Great Montel's theorem we can state the following theorem:

Theorem 1.3.3 (Simple Montel's theorem). *Let* $\mathcal{F}$ *be a family of functions that are holomorphic and uniformly bounded in a domain* $\Omega \subset \widehat{\mathbb{C}}$. *Then,* $\mathcal{F}$ *is a normal family in* Ω.

1.4 Singular values

In this section, we introduce the ideas of critical points and critical values. These concepts will be important for studying the escaping set for polynomials. Additional details on these topics can be found in references [3], [14], [15] and [25]. We begin by defining two related notions: regular and singular values.

Definition 1.4.1 (Regular value and (Local) branch). *Let* $f : U \subset \mathbb{C} \to \mathbb{C}$ *be a holomorphic function. We say that* $v_0 \in \mathbb{C}$ *is a* regular value *if there exists a neighbourhood* V *of* v_0 *such that for all components* U *of* $f^{-1}(V)$, $f : U \to V$ *is a homeomorphism.*

Every function $\varphi : V \to U$ *such that* $f \circ \varphi = id$, *where* $id : V \to V$ *is the identity function, is called a* (local) branch *of* f^{-1}.

These (local) branches possess an important property, which we formalize in the following proposition:

Proposition 1.4.2. *All (local) branches* $\varphi : V \to U$ *of* f^{-1} *are univalent.*

Definition 1.4.3 (Singular value). *Let* $U \subset \mathbb{C} \to \mathbb{C}$ *be a holomorphic function. We say that* $v_0 \in \mathbb{C}$ *is a* singular value *if it is not regular, i.e., for any neighbourhood* V *of* v_0, *not all branches of* f^{-1} *are well defined.*

In chapter four, we will introduce *asymptotic values*, a specific type of singular value that will be essential in the study of the escaping set for transcendental functions. For the moment, however, we focus on critical values.

Definition 1.4.4 (Critical point and Critical value). *Let f be a holomorphic function. We say that a point $z_0 \in \mathbb{C}$ is a* critical point *of f if $f'(z_0) = 0$. Then, $v_0 = f(z_0)$ is called a critical value.*

As a consequence of the definition and the inverse function theorem, v_0 is a singular value.

To properly use the properties of critical points, we need to understand the concept of holomorphic proper maps.

Definition 1.4.5 (Holomorphic proper map). *Let $f : U \to V$ be a holomorphic function. The map f is said to be proper of degree $k \geq 1$ if every point in V has exactly k preimages in U counted with multiplicity. This will be expressed by writing*

$$f : U \xrightarrow{k:1} V$$

Theorem 1.4.6. *Equivalently, a holomorphic function $f : U \to V$ is a proper map of degree $k \geq 1$ if, and only if, $f(\partial U) = \partial V$.*

Note 1. *If f is a rational function, then for all $V \subset \widehat{\mathbb{C}}$ open and U a connected component of $f^{-1}(V)$, $f : U \to V$ is proper.*

With this, we can state the following property.

Proposition 1.4.7. *Let z_0 be a critical point of f, a holomorphic function, and let k be such that $f'(z_0) = \cdots = f^{k-1}(z_0) = 0$ but $f^k(z_0) \neq 0$. Then, for all neighbourhoods V of $v_0 = f(z_0)$, there exists a neighbourhood U of z_0 such that $f : U \to V$ is proper of degree k.*

An additional property is that if $\frac{p(z)}{q(z)}$ is a rational function, then all singular values are also critical values, implying that there can only exist a finite number of them. Moreover, a rational function $\frac{p(z)}{q(z)}$ of degree $d > 0$ has at most $2d - 2$ critical points in $\widehat{\mathbb{C}}$, while a polynomial $r(z)$ of degree $d > 0$ has at most $d - 1$ critical points in $\mathbb{C}$.

1.5 Covering maps

In this section, we introduce the notion of covering maps. This concept will be necessary for studying a key characteristic of the escaping set of the transcendental entire functions which the set of singular values of their inverse function is bounded. The content and further details can be found in [10] and [12].

Definition 1.5.1 (Covering map). *Let $f : U \to V$ be a continuous function between two topological spaces U and V. The function f is said to be a* covering map *if, for every point $v \in V$, there exists a neighbourhood W of v such that each component U_i of $f^{-1}(W)$ is open and $f : U_i \to W$ is a homeomorphism for each of these components U_i. Additionally, if U is simply connected, then f is called an* universal covering map.

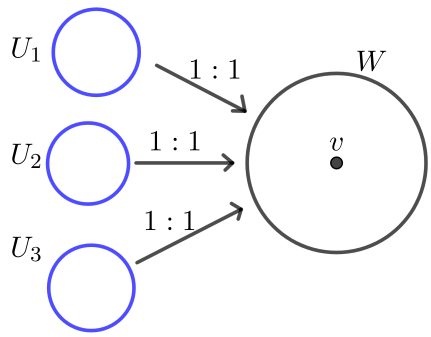


Figure 1.2: Visual representation of a covering map

We will also state two lemmas related to covering maps, which will be required for a important proof of the third section of chapter four.

Lemma 1.5.2. ***([4, Lemma 2.2])*** *Let $U \subset \widehat{\mathbb{C}}$, let $\mathbb{D}$ be the unit disk and $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$.*

a. If f is a holomorphic covering from $U \to \mathbb{D}$, then U is simply connected and f is univalent.

b. If f is a holomorphic covering from $U \to \mathbb{D}^*$, then either U is biholomorphic to $\mathbb{D}^*$ and f is equivalent to z^d, or U is simply connected and f is the universal covering, hence equivalent to the exponential map.

Lemma 1.5.3. *Let $G \subset \mathbb{C}$ and let U be a component of $f^{-1}(G)$. If G has no singular values, then $f : U \to G$ is a covering map.*

This last lemma can be consider as a consequence of the definition of the local branches of f^{-1} and singular values, both of which were introduced in the previous section.

Chapter 2

Complex iteration

The primary objective of this project is to study the escaping set for specific types of holomorphic functions. To achieve this, we must first comprehend some of the main ideas in iteration theory. In this chapter, we study the dynamical behaviour of a given holomorphic function f.

The first section of the chapter focuses on the local theory, covering periodic points, their classification, conformal conjugacy and the linearization problem. The second section expands to a broader perspective, exploring the Julia and Fatou sets along with their fundamental properties, and providing a brief classification of Fatou components.

Through this project, we will be working within both the complex plane and the extended complex plane. Additionally, we will always utilize holomorphic functions; specifically, entire functions when operating in the complex plane and rational functions when operating in the extended complex plane.

We begin by selecting an initial point $z_0 \in \mathbb{C}$ and applying a function f repeatedly. This process generates a sequence of new points $z_0,\ \ z_1 = f(z_0),\ \ z_2 = f(z_1) = f \circ f(z_0) = f^2(z_0), \ldots$. More precisely, it constructs the *(forward) orbit* of z_0 under the function f, which we denote and define as

$$Orb^+(z_0) = \{z_n = f^n(z_0)\}_{n\in\mathbb{N}}.$$

2.1 Local theory

Our main concern is to understand the behaviour of these orbits or sequences in terms of their initial condition z_0, that is, we aim to analyse their asymptotic behaviour as n tends to ∞. We turn our attention to the case when z_0 is either a fixed point or a periodic point, which we define below. In certain cases, it is also possible to determine the behaviour of points in the neighbourhood of a fixed or periodic point when they are iterated under the same function f.

Definition 2.1.1 (Periodic points and fixed points). *Let $\Omega \subset \mathbb{C}$ be an open set and let $f : \Omega \to \mathbb{C}$ be a holomorphic function on Ω. A point $z_0 \in \mathbb{C}$ is a* periodic point *of f with minimal period p if $f^p(z_0) = z_0$ and $f^k(z_0) \neq z_0$ for all $0 < k < p$, where $p \in \mathbb{N}$ and $p \geq 1$. Additionally, the sequence $\{z_0,\ \ldots,\ f^{p-1}(z_0)\}$ is referred to as a periodic orbit. If $p = 1$, meaning $f(z_0) = z_0$, then $z_0 \in \mathbb{C}$ is called a* fixed point *of f .*

Occasionally, depending on the function we are working with, it can be challenging to fully grasp the dynamics of neighbouring points of periodic or fixed points. To address this issue, we might attempt to study the point using a simpler, well-known function. However, not all simple functions offer an optimal solution; what we require is a conformal conjugacy of the original function.

Definition 2.1.2 (Conformal conjugacy). *Let $U, V \subset \mathbb{C}$ be open sets. We say that a function $f : U \to U$ is* conformally conjugate *to a function $g : V \to V$ if there exists a conformal map $\varphi : U \to V$ such that $g = \varphi \circ f \circ \varphi^{-1}$, i.e. $\varphi(f(z)) = g(\varphi(z))\ \forall z \in U$.*

The equation $\varphi(f(z)) = g(\varphi(z))$, along with its variants, is known as *Schröder's equation.* We can see how the following diagram commutes:

$$\begin{array}{ccc} U & \xrightarrow{f} & U \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ V & \xrightarrow[g]{} & V \end{array}$$

In fact, conjugacy is an equivalence relation. Furthermore, we can also obtain several immediate consequences from the definition:

1. If f and g are conjugate, then $deg(f) = deg(g)$.
2. Conjugacy respects iteration: if $g = \varphi \circ f \circ \varphi^{-1}$, then $g^n = \varphi \circ f^n \circ \varphi^{-1}$ and $g^{-1} = \varphi \circ f^{-1} \circ \varphi^{-1}$.
3. Conjugacy also respects periodic points: if $g = \varphi \circ f \circ \varphi^{-1}$, then $\varphi(z_0)$ is a periodic point of g with period p if and only if z_0 is a periodic point of f with period p.

Thanks to these properties, we do not need to distinguish between conjugate functions in a dynamical sense.

2.1.1 The classification of periodic points

To study the asymptotic behaviour of an orbit, we must first examine its initial condition. We begin by classifying fixed points since fixed points are simply periodic points of period $p = 1$. Suppose z_0 is a fixed point of a holomorphic function $f : \Omega \to \mathbb{C}$, where $\Omega \subset \mathbb{C}$. According to $\lambda = f'(z_0)$, which is called the *multiplier* of f at z_0, we can classify z_0 as follows:

a. **attracting** if $|\lambda| < 1$ and $\lambda \neq 0$.

b. **superattracting** if $\lambda = 0$.

c. **Repelling** if $|\lambda| > 1$.

d. **Rationally indifferent or parabolic** if $|\lambda| = 1$ and $\lambda^n = 1$ for some integer n.

e. **Irrationally indifferent** if $|\lambda| = 1$ but λ^n is never equal to 1.

This classification can be extrapolated to periodic points of period $p > 1$, as a periodic point z_p is a fixed point of f^p.

Now, let us understand why (super)attracting periodic points are named as such by considering the fixed point case. Let z_0 be an attracting (or superattracting) fixed point. From the derivation of $f(z)$, we already know that

$$\left|\frac{f(z)-f(z_0)}{z-z_0}\right| \underset{z\to z_0}{\to} |f'(z_0)| = |\lambda| < 1.$$

Following from this, for $\varepsilon > 0$, there exists $|\lambda| < \rho < 1$ such that if $|z - z_0| < \varepsilon$, then $\left|\frac{f(z)-f(z_0)}{z-z_0}\right| < \rho$. Thus $|f(z) - f(z_0)| \leq \rho|z - z_0|$. By induction, we obtain

$$|f^n(z) - f^n(z_0)| \leq \rho^n |z - z_0|.$$

And because z_0 is a fixed point, meaning that $f^n(z_0) = z_0$, and $\rho < 1$, it follows that

$$|f^n(z) - z_0| \leq \rho^n |z - z_0| \underset{n\to\infty}{\to} 0.$$

This implies that the iterates $f^n(z)$ converge uniformly to z_0. This justifies the name (super)attracting.

Definition 2.1.3 (Basin of attraction of a periodic orbit). *Let z_0 be a periodic point of period p of an rational function f and let $< z_0 >= \{z_0, \ldots, z_{p-1}\}$ be its periodic orbit. The* basin of attraction of $< z_0 >$ *is then defined as*

$$A(< z_0 >) = \{z \in \widehat{\mathbb{C}} \mid f^{np}(z) \to z_i \text{ as } n \to \infty \text{ for some } 0 \leq i \leq p-1\}.$$

The connected component of $A(< z_0 >)$ containing the cycle is called the immediate basin of attraction of $< z_0 >$ and is denoted by $A^*(< z_0 >)$.

Observation 2.1.4. $A(< z_0 >)$ is an open subset due to coinciding with the union of the backward iterates $f^{-n}(\mathbb{D}(z_0, \varepsilon))$ for $\varepsilon > 0$ small.

If $p = 1$, we define the *basin of attraction* of an attracting fixed point z_0 of a holomorphic function f as

$$A(z_0) = \{z \in \mathbb{C} \mid f^n(z) \to z_0\}.$$

The connected component of $A(z_0)$ containing z_0 is called the *immediate basin of attraction* of z_0 and is denoted by $A^*(z_0)$.

It is important to note that conformal conjugation preserves the basin of attraction, i.e., the basin of attraction of an attracting fixed point z_0 of f is mapped by a conjugacy φ onto the basin of attraction of the attracting fixed point $\varphi(z_0)$ of g.

2.1.2 The linearization problem

After classifying fixed points, the next step is to consider if it is always possible to conjugate a function f to a simpler function g, specifically, to its linear part. The answer, however, depends on the multiplier of f. In this project, we are going to prove that conjugation is possible in the cases of repelling, attracting and superattracting fixed points.

The cases of irrationally indifferent and rationally indifferent fixed points fall outside the scope of this work, but we can offer a very brief overview. If z_0 is an indifferent fixed point, it implies that $|\lambda| = 1$, allowing us to write λ as $\lambda = e^{2\pi i\theta}$. Since the dynamics near the fixed point are local and invariant under translation, we can assume $z_0 = 0$. If the multiplier $\lambda = e^{2\pi i\theta}$ has a rational θ, i.e., $\lambda = e^{2\pi ip/q}$, then λ is a root of unity and $\lambda^q = 1$. Let $f(z) = \lambda z + az^2 + \cdots$ be a holomorphic function, then $f^q(z)$ rotates a point around the origin back to its starting position and behaves like the identity near z_0. However, $f(z)$ is not conjugated to its linear part unless $f^q(z) = Id$ due to the higher-order terms in $f(z)$. On the other hand, if θ is irrational, the existence of a conjugacy depends of the closeness of λ to a root of unity, that means, it depends on the arithmetics of θ.

We begin with the existence of a conjugacy in the case of attracting fixed points whose multipliers are non-zero.

Theorem 2.1.5 (Koenigs' linearization theorem). *Suppose f has an attracting fixed point at z_0, with multiplier λ such that $0 < |\lambda| < 1$. Then, there exists a conformal map $\zeta = \varphi(z)$ of a neighbourhood of z_0 onto a neighbourhood of 0 which conjugates $f(z)$ to the linear function $g(\zeta) = \lambda\zeta$. The conjugating function is unique up to multiplication by a non-zero scale factor. This last sentence means that, if ψ is another function such that $\psi(f(z)) = \lambda \cdot \psi(z)$, then $\psi = c \cdot \varphi$ for some constant $c \neq 0$.*

Proof. We will only prove the existence part. The proof of the uniqueness assertion can be found in [21, Theorem 8.2].

Assume $z_0 = 0$ is an attracting fixed point. We define a sequence of functions $\varphi_n(z) = \lambda^{-n} \cdot f^n(z) = z + \cdots$. Notice that φ_n satisfies $\varphi_n \circ f = \lambda^{-n} \cdot f^{n+1} = \lambda \cdot \varphi_{n+1}$. Therefore, if φ_n converges to some non-constant φ, then $\varphi \circ f = \lambda \cdot \varphi$, implying that $\varphi \circ f \circ \varphi^{-1} = \lambda \cdot \zeta$, which shows that φ is a conjugation.

So our aim is to prove that φ_n converges to some non-constant, conformal function φ. To do so, we can observe that for a small $\delta > 0$

$$|f(z) - \lambda z| \leq C|z|^2, \quad |z| \leq \delta,$$

for some constant $C > 0$. Thus, $|f(z)| \leq |\lambda||z| + C|z|^2 \leq (|\lambda| + C\delta)|z|$. By induction, with $|\lambda| + C\delta < 1$, we obtain

$$|f^n(z)| \leq (|\lambda| + C\delta)^n |z|, \quad |z| \leq \delta.$$

Choosing $\delta > 0$ so small that $\rho = (|\lambda| + C\delta)^2/\lambda < 1$, we get

$$|\varphi_{n+1}(z) - \varphi_n(z)| = \left| \frac{f(f^n(z)) - \lambda f^n(z)}{\lambda^{n+1}} \right| \leq \frac{C|f^n(z)|^2}{|\lambda|^{n+1}} \leq \frac{C((|\lambda| + C\delta)^n |z|)^2}{|\lambda|^{n+1}} = \frac{\rho^n C|z|^2}{|\lambda|},$$

$|z| \leq \delta$. And since $\rho < 1$,

$$|\varphi_{n+1}(z) - \varphi_n(z)| \leq \frac{\rho^n C|z|^2}{|\lambda|} \to 0 \text{ when } n \to \infty, \quad |z| \leq \delta.$$

Therefore, $\varphi_n(z)$ converges uniformly for $|z| \leq \delta$, thus proving the existence of the conjugation $\varphi(z)$. Now we only need to prove that φ is non-zero and conformal.

We can observe that each $\varphi_n(z)$ is defined as $\varphi_n(z) = z + \cdots$, which implies that $\varphi_n'(0) = 1$. Since $\varphi_n(z)$ is holomorphic and converges uniformly to $\varphi(z)$ on $|z| \leq \delta$, by Weierstrass' theorem, $\varphi(z)$ is holomorphic on $|z| \leq \delta$ and $\varphi'(0) = 1 \neq 0$, implying that $\varphi(z)$ is non-constant and hence non-zero. Furthermore, since $\varphi'(0) = 1 \neq 0$ on $|z| < \delta$, by Lemma 1.1.9, $\varphi(z)$ is locally invertible and its inverse is also holomorphic. Thus, $\varphi(z)$ is conformal. □

The existence of a conjugating function for a repelling fixed point follows immediately from the attracting case. The proof is straightforward.

Proof. Suppose $f(z) = z_0 + \lambda(z - z_0) + \cdots$, where $|\lambda| > 1$. Then, the inverse function $f^{-1}(z) = z_0 + (z - z_0)/\lambda + \cdots$ has an attracting fixed point at z_0. Any function that conjugates $f^{-1}(z)$ to ζ/λ, also conjugates $f(z)$ to $\lambda\zeta$. □

We now conclude this subsection with the existence of a conjugation for the superattracting case. The following theorem due to L.E. Böttcher (1904) establishes this result.

Theorem 2.1.6 (Böttcher's linearization theorem). *Suppose f has a superattracting fixed point at z_0, and f is such that $f(z) = z_0 + a_p(z - z_0)^p + \cdots$, where $a_p \neq 0$ and $p \geq 2$. Then there exists a conformal map $\zeta = \varphi(z)$ of a neighbourhood of z_0 onto a neighbourhood of 0 which conjugates $f(z)$ to ζ^p. Moreover, the conjugating function is unique up to multiplication by a $(p-1)$-th root of unity.*

Proof. Like Koenigs' theorem, we will not prove the uniqueness statement here. And like the previous theorem, its proof can be found complete in [21, Theorem 9.1].

Assume $z_0 = 0$ is a superattracting fixed point. For $|z|$ small, there is a constant $C > 1$ such that $|f(z)| \leq C|z|^p$. This follows from the fact that, since z_0 is a superattracting fixed point, the behaviour of $f(z)$ near 0 is dominated by the term $a_p(z - z_0)^p = a_p z^p$, as $z_0 = 0$. Since $p \geq 2$, by induction, we find that $|f^n| \leq (C|z|)^{p^n}$, $|z| \geq \delta$. This shows that $f^n(z) \to 0$ very fast as $n \to \infty$.

If we change variables by setting $w = cz$, where $c^{p-1} = \frac{1}{a_p}$, then we have conjugated f to the form $f(w) = w^p + \cdots$. This allows us to assume $a_p = 1$, simplifying f. Now, our goal is to find a conjugating function $\varphi(z) = z + \cdots$ such that $\varphi(f(z)) = \varphi(z)^p$. This is equivalent to find a φ that satisfies the equation $\varphi \circ f \circ \varphi^{-1} = \zeta^p$. Let us define

$$\varphi_n(z) = f^n(z)^{p^{-n}} = (z^{p^n} + \cdots)^{p^{-n}} = z(1 + \cdots)^{p^{-n}},$$

which is well-defined in a neighbourhood of the origin because, for sufficiently small $|z|$, the term $(1 + \cdots)^{p^{-n}}$ converges. Therefore, $\varphi_n(z)$ is holomorphic near $z = 0$. The sequence φ_n satisfies

$$\varphi_{n-1} \circ f = (f^{n-1} \circ f)^{p^{-n+1}} = \varphi_n^p,$$

So, if $\varphi_n \to \varphi$, then φ satisfies the desired condition $\varphi \circ f \circ \varphi^{-1} = \zeta^p$.

To prove that $\{\varphi_n\}$ converges, note that

$$\frac{\varphi_{n+1}(z)}{\varphi_n(z)} = \left(\frac{f(f^n(z))^{\frac{1}{p}}}{f^n(z)}\right)^{p^{-n}} = \left(\frac{(f^n(z)^p + O(f^n(z)^p)^{\frac{1}{p}})}{f^n(z)}\right)^{p^{-n}} = \left(\frac{f^n(z) + O(f^n(z)^p)}{f^n(z)}\right)^{p^{-n}}$$

Thus, for small $|z|$,

$$\frac{\varphi_{n+1}(z)}{\varphi_n(z)} = (1+O(f^n(z)^{p-1}))^{p^{-n}} = (1+O(|f^n(z)|))^{p^{-n}} = 1+O(p^{-n})O(|z|^{p^n}C^{p^n}) = 1+O(p^{-n})$$

if $|z| \leq c < 1/C$. Hence, the product

$$\prod_{n=1}^{\infty} \frac{\varphi_{n+1}}{\varphi_n} = \lim_{n\to\infty} \frac{\varphi_n}{\varphi_1}$$

converges uniformly for $|z| \leq c < 1/C$, implying that $\{\varphi_n\}$ converges. Therefore, the conjugacy $\varphi(z)$ exists.

Now, we need to prove that $\varphi(z)$ is non-zero and conformal. The proof follows exactly as in Koenig's case. Since each $\varphi_n(z)$ is defined as $\varphi_n(z) = z(1+\cdots)^{p^{-n}}$, then $\varphi_n'(0) = 1$. By Weierstrass' theorem, $\varphi(z)$ is holomorphic on $|z| \leq c < 1/C$ and $\varphi'(0) = 1 \neq 0$, implying that $\varphi(z)$ is non-constant and hence non-zero. Furthermore, since $\varphi'(0) = 1 \neq 0$ on $|z| < c$, by Lemma 1.1.9, $\varphi(z)$ is locally invertible and its inverse is also holomorphic. Thus, $\varphi(z)$ is conformal.

□

2.2 Julia and Fatou sets

Up to this point, we have been focusing on a local approach, but now we transition to the global study of holomorphic dynamics. The foundations of this field were laid by French mathematicians Pierre Fatou and Gaston Julia around 1918. They introduced the core idea in iteration theory: the division of the complex plane into the Fatou and Julia sets.

The definitions of both sets are based on the concept of normal families previously introduced in definition 1.3.1.

Definition 2.2.1 (Fatou and Julia sets). *Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be an holomorphic function. The* Fatou set *of f consists of all the points $z \in \widehat{\mathbb{C}}$ such that there exists a neighbourhood of z where $\{f^n\}_{n=1}^{\infty}$ is a normal family. We denote this set by $F(f)$.*

The Julia set *of f is defined as the complement of the Fatou set of f in $\widehat{\mathbb{C}}$, and is denoted by $J(f)$, i.e., $J(f) = \widehat{\mathbb{C}} \setminus F(f)$.*

The term "Fatou set" is not as universally recognized as the term "Julia set", and thus alternative names such as "the stable set" or "the set of normality" are sometimes used, especially in older papers. These names arise from the fact that points in the Fatou set are by definition associated with stable dynamics, where small perturbations of initial conditions lead to similar long-term behaviour. In contrast, the Julia set consists of points where the iterates of f display chaotic behaviour, meaning that small changes in the initial condition can lead to dramatically different outcomes.

To illustrate the Fatou and Julia sets, we will take a look at these sets for the fundamental example $f(z) = z^2$.

Example 2.2.2. Consider $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, where $f(z) = z^2$. First, we seek its fixed points by solving $f(z) = z^2 = z$, and we find that they are: 0, 1 and ∞. The next step is to classify these points based on their multiplier. In this case, $f'(z) = 2z$, so 1 is a repelling fixed point, while 0 and ∞ are superattracting. If we write z as $z = re^{i\theta}$, we can observe the following:

- if $|z| < 1$, i.e., $z \in \mathbb{D}$, then $f^n(z) \to 0$, where $f^n(z) = z^{2^n} = r^{2^n} e^{i2^n\theta}$;
- if $|z| = 1$, then $z \in S^1$;
- and if $|z| > 1$, i.e., $z \in \widehat{\mathbb{C}} \setminus \bar{\mathbb{D}}$, then $f^n(z) \to \infty$.

Let us now determine if $\{f^n\}_n$ is normal in these sets. For $z \in \mathbb{D}$ the subsequence $\{f^{n_k}\}_{n_k}$ converges uniformly to the constant function $z \to 0$. Similarly, for $z \in \widehat{\mathbb{C}} \setminus \bar{\mathbb{D}}$, $\{f^{n_k}\}_{n_k}$ converges uniformly to ∞, indicating that $\{f^n\}_n$ is a normal family in both sets. However, if $z \in S^1$, $\{f^{n_k}\}_{n_k}$ doesn't converge uniformly. We show this by contradiction. Suppose that $\{f^{n_k}\}_{n_k}$ converges uniformly to a function g. By theorem 1.1.3, g is a holomorphic function, which implies that g is continuous. But this is impossible, as g exhibits a jump discontinuity as we cross the unit circle. Therefore, $\{f^{n_k}\}_{n_k}$ does not converge uniformly, and consequently, $\{f^n\}_n$ is not a normal family on this set. So we can conclude that $J(f) = S^1$ and $F(f) = \mathbb{D} \cup \widehat{\mathbb{C}} \setminus \bar{\mathbb{D}} = \widehat{\mathbb{C}} \setminus S^1$.

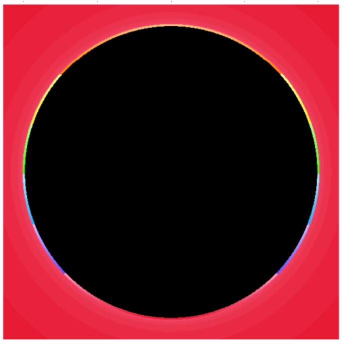


Figure 2.1: Representation of $f(z) = z^2$ illustrating the behaviour of its orbits. Points with orbits converging to 0 are shown in black, those with orbits converging to ∞ are in red, and the points forming the Julia set of the polynomial are represented in a gradient of rainbow colours.

Both Julia and Fatou sets possess a compelling collection of properties; here we introduce some basic ones in the form of upcoming lemmas. It is important to state in advance that these properties are under the hypothesis of f being a rational function when we are on the extended complex plane or an entire function when we are on the complex plane. In both cases $deg(f) \geq 2$.

Lemma 2.2.3. *$F(f)$ is an open set and $J(f)$ is a closed one.*

We will not prove it since it is already proved by the definition itself, as $F(f)$ is the biggest open set of normality and $J(f)$ is its complement.

Lemma 2.2.4 (Invariance lemma). *$F(f)$ and $J(f)$ are completely invariant. That is, $z_0 \in F(f)$ if and only if $f(z_0) \in F(f)$, and the same holds for $J(f)$.*

Proof. ($\Rightarrow$)Assume that $z_0 \in F(f)$. By definition, there exists a neighbourhood U of z_0 such that $\{f^n\}_n$ is normal in U, where U is an open subset of $\mathbb{C}$. Then, $f(z_0) \in f(U)$, and since f is open, $f(U)$ is an open neighbourhood of $f(z_0)$. Let $\{f^{n_k}\}_k$ be a sequence in U, then $\{f^{n_k-1}\}_k$ is a sequence in $f(U)$. Therefore, there exists a subsequence $f^{n_{k_j}}$ converging to a function g in U, which implies that $f^{n_{k_j}-1}$ converges to $f \circ g$ in $f(U)$. Thus, $\{f^n\}_n$ is normal in $f(U)$, meaning $f(z_0) \in F(f)$.

($\Leftarrow$)Now suppose that $f(z_0) \in F(f)$. By definition, there exists a neighbourhood V of $f(z_0)$ such that $\{f^n\}_n$ is normal in V, where V is an open subset of $\mathbb{C}$. Since f is open, there exists a neighbourhood $U \subseteq f^{-1}(V)$ of z_0 with $f(U) \subset V$. Let $\{f^n\}_n$ be a restricted family to U. Then, for each subsequence $\{f^{n_k}\}_k$ in U, the corresponding $\{f^{n_k-1}\}_k$ is a subsequence in V. Therefore, there exists a sub-subsequence $f^{n_{k_j}-1}$ converging to a function g in V, which implies that $f^{n_{k_j}}$ converges to $f^{-1} \circ g$ in U.Thus, $\{f^n\}_n$ is normal in U, meaning $z_0 \in F(f)$.

$F(f)$ being completely invariant implies that its complement, $J(f)$, is also completely invariant. □

Lemma 2.2.5 (Iteration lemma). *For any $k > 0$, $F(f) = F(f^k)$ and $J(f) = J(f^k)$.*

Proof. ($\subseteq$) Assume that $\{f^n\}_n$ is normal in a neighbourhood U of z_0. Therefore, we know that any subsequence has a converging sub-subsequence. Our objective is to see if $\{f^{nk}\}_n$ also has a converging subsequence. And it has it, because the subsequence $f^{n_i k}$ is also a subsequence of $\{f^n\}_n$.

($\supseteq$) Now, suppose that $\{f^{nk}\}_n$ is normal in a neighbourhood U of z_0. Then, for $1 \leq j \leq k-1$, $\{f^{nk+j}\}_n$ are normal in U due to they are a shifted subsequence of $\{f^{nk}\}_n$, and they cover all iterates of f. Thus, any subsequence f^{n_i} must contain infinite elements within at least one of these $\{f^{nk+j}\}_n$ subsequence, so a convergent sub-subsequence must exist.

$F(f) = F(f^k)$ implies that $J(f) = J(f^k)$. □

Lemma 2.2.6. *Let $z_0 \in J(f)$ and U be a neighbourhood of z_0. Then, $\bigcup_{n\geq 0} f^n(U) \supset \widehat{\mathbb{C}} \setminus \{a, b\}$, $a, b \in \widehat{\mathbb{C}}$*

Proof. By Montel's theorem, if $\bigcup_{n\geq 0} f^n(U) \subset \widehat{\mathbb{C}} \setminus \{a, b, c\}$, then $\{f^n\}_n$ is normal in U which contradicts the assumption that $z_0 \in J(f)$. □

Lemma 2.2.7. *$J(f)$ is not empty.*

Proof. We will prove this statement only for rational functions, the proof of the entire case can be found in [5, Theorem 3].

Assume $J(f)$ is empty. Then, the family $\{f^n\}_n$ is normal on $\widehat{\mathbb{C}}$, and there exists a subsequence $\{f^{n_k}\}$ such that $f^{n_k} \to g$, where g is an holomorphic function on $\widehat{\mathbb{C}}$. By Hurwitz's theorem, on any compact set of $\widehat{\mathbb{C}}$, there exists $N \in \mathbb{N}$ such that for $n \geq N$, f^{n_k} has the same number of zeros as g, counting with multiplicity. However, g can only have a finite number of zeros, while f^{n_k} have d^{n_k} zeros counting with multiplicity as $n \to \infty$. This leads to a contradiction, thus proving that $J(f)$ can not be empty. □

Lemma 2.2.8. *If $J(f)$ contains an interior point, then $J(f) = \widehat{\mathbb{C}}$.*

Proof. Assume z_0 is an interior point of $J(f)$, then there exists an open neighbourhood U of z_0 such that $U \subset J(f)$. By lemma 2.2.6, we have $\bigcup_{n\geq 0} f^n(U) \supset \widehat{\mathbb{C}} \setminus \{a, b\}$. Since $J(f)$ is invariant, it follows that $\bigcup_{n\geq 0} f^n(U) \subset J(f)$, which implies that $\widehat{\mathbb{C}} \setminus \{a, b\} \subset J(f)$. And since $J(f)$ is closed, we have $J(f) \supset \overline{\widehat{\mathbb{C}} \setminus \{a, b\}} = \widehat{\mathbb{C}}$. Hence, we conclude that $J(f) = \widehat{\mathbb{C}}$. □

Lemma 2.2.9 (Basins and repelling points). *We can divide this lemma in three statements:*

a. *Every attracting periodic orbit and their basin of attraction are contained in the Fatou set of f, i.e. $A(< z_0 >) \subset F(f)$.*

b. *However, every repelling periodic orbit is contained in the Julia set. Moreover, periodic points are dense in $J(f)$.*

c. *The topological boundary of the basin of attraction is equal to the entire Julia set, i.e. $\partial A(< z_0 >) = J(f)$.*

Proof. Since a periodic point of f is just a fixed point of some iterate f^n, we can prove this lemma for fixed points and generalize it to the periodic points using the iteration lemma.

a. First, we will prove the first statement. We have already seen and proved in the justification of the name (super)attracting fixed point, subsection 2.1.1, that if $\lambda < 1$, then the iterates $f^n(z)$ converge to z_0 in a neighbourhood $|z - z_0| < \varepsilon$ for $\varepsilon > 0$, that is, the successive iterates of f, when restricted to a small neighbourhood, converge uniformly to the constant function $g(z) = z_0$. Hence, $\{f^n\}$ is normal. Given $\mathbb{D}(z_i, \varepsilon_i)$, and due to the iterates $\{f^n\}$ being uniformly bounded in $\mathbb{D}(z_i, \varepsilon_i)$, by Theorem 1.3.3, it follows that $\mathbb{D}(z_i, \varepsilon_i) \subset F(f)$. Thus, by invariance, $A(< z_0 >) \subset F(f)$ since every point must eventually enter $\mathbb{D}(z_i, \varepsilon_i)$.

b. We will prove only the first part of the second statement, and we will prove it by contradiction. Suppose that a repelling fixed point z_0 is contained in the Fatou set. By definition, there exists a neighbourhood U of z_0 such that $\{f^n\}_n$ is normal in U. Then, by Theorem 1.1.3, $\{f^{n_k}\}_{n_k} \to g$ and $\{(f^{n_k})'\}_{n_k} \to g'$, where g is holomorphic. However, this is not possible since $(f^{n_k})'(z_0) = \lambda^{n_k} \to \infty$.

c. For the third statement, we begin by proving that $\partial A(< z_0 >) \supset J(f)$. Suppose that $z \in J(f)$ and let U be a neighbourhood of z. By lemma 2.2.6, $\bigcup_{n \geq 0} f^n(U) \supset \widehat{\mathbb{C}} \setminus \{a, b\},\ a, b \in \widehat{\mathbb{C}}$. Hence, $\bigcup_{n \geq 0} f^n(U) \cap A(< z_0 >) \neq \varnothing$. Therefore, there exists $w \in U$ such that for $N > 0$, $f^N(w) \in A(< z_0 >)$. Since $z \notin A(< z_0 >)$, it follows that $z \in \partial A(< z_0 >)$.

   Now we aim to prove that $\partial A(< z_0 >) \subset J(f)$. Suppose that $z \in \partial A(< z_0 >)$ and let U be a neighbourhood of z. If $\{f^n\}_n$ is normal in U, then $\{f^{n_k}\}_{n_k} \to g$, where g is a holomorphic function. On $U \cap \partial A(< z_0 >)$, g is equivalent to z_i for some i since the limit function g on $U \cap \partial A(< z_0 >)$ should approximate the behaviour of points being attracted to $< z_0 >$. But $\{f^{n_k}\}_{n_k} \nrightarrow z_i$ for any i due to boundary points of $A(< z_0 >)$ do not converge to z_i. Therefore, $z \in J(f)$.

□

The main focus of this last part is to prove that the Julia set is perfect, that is, it is closed and has no isolated points. For this, we need to introduce a few new concepts related to orbits, particularly the notion of the exceptional set, which will be crucial to the proof.

Definition 2.2.10 (Grand orbit). *Given a rational function $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, the* grand orbit *of a point $z_0 \in \widehat{\mathbb{C}}$ is defined as the set consisting of all points $z \in \widehat{\mathbb{C}}$ whose orbits eventually intersect the orbit of z_0. More precisely,*

$$GO(z, f) = \{z \in \widehat{\mathbb{C}} \mid f^n(z_0) = f^m(z) \text{ for some } n \geq 0,\ m \geq 0\}.$$

Definition 2.2.11 (Exceptional points). *Given a rational function $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, a point $z_0 \in \widehat{\mathbb{C}}$ is called* exceptional *(or* grand-orbit-finite*) under f if its grand orbit $GO(z, f) \subset \widehat{\mathbb{C}}$ is a finite set.*

The set of the exceptional points is referred to as the *exceptional set* and is denoted $\mathcal{E}(f)$. For instance, for polynomials, infinity is a superattracting fixed point and belongs to the exceptional set, as we will see next.

Lemma 2.2.12 (Finite Grand Orbits). *If $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a rational function of degree $d \geq 2$, then the set $\mathcal{E}(f)$ can have at most two elements. These exceptional points, if they exists, must always be superattracting periodic points of f and thus they must belong to the Fatou set.*

We are not going to prove this here; its complete proof can be found in [21, Lemma 4.9]. It follows immediately from the definition of $\mathcal{E}(f)$ and the preceding lemma that if $z_0 \notin \mathcal{E}$, then $J(f)$ is adherent to the inverse orbit $\bigcup_{n\geq 1} f^{-n}(z_0)$ of z_0. Since $\mathcal{E}(f)$ is disjoint from $J(f)$, we obtain the important result:

Lemma 2.2.13. *The backward iterates of any $z \in J(f)$ are dense in $J(f)$.*

With this, we can conclude this chapter by proving our upcoming and final property.

Lemma 2.2.14 ($J(f)$ is perfect). *If f has degree 2 or more, then $J(f)$ has no isolated points.*

Proof. We will prove this lemma only for the rational case. Suppose $z_0 \in J(f)$ and let U be an open neighbourhood of z_0. Our objective is to show that there exists another point in U that also belongs to $J(f)$. To do this, we have to study the situation based on whether z_0 is a periodic point or not.

Assume first that z_0 is not periodic. In this case, choose z_1 such that $f(z) = z_0$. Then $f^n(z_0) \neq z_1$ $\forall n$. Since $z_1 \in J(f)$, the backward iterates of z_1 are dense in $J(f)$, which means that there exists $z_2 \in U$ such that $f^m(z_2) = z_1$. Hence, $z_2 \in J(f) \cap U$ and $z_2 \neq z_0$.

Now suppose that z_0 is a periodic point of period n. For the sake of contradiction, we assume that z_0 is the only solution of $f^n(z_0) = z_0$. This implies that z_0 exhibits highly attractive behaviour in its neighbourhood, making z_0a superattracting periodic point. By Lemma 2.2.12, this would imply that $z_0 \in F(f)$, which contradicts the assumption that $z_0 \in J(f)$. Therefore, there exists $z_1 \neq z_0$ such that $f^n(z_0) = z_1$. Moreover, $f^i(z_0) \neq z_1$ $\forall i$ since, otherwise, there would exist some $0 \leq i < n$ (by periodicity) for which $f^i(z_0) = f^{n+i}(z_0) = f^n(z_1) = z_0$, contradicting that z_0 is a periodic point of period n. As before, z_1 must have a preimage in $J(f) \cap U$ which cannot be z_0. □

We conclude this section by introducing the concept of Fatou components and providing a brief classification of their dynamical behaviour.

Definition 2.2.15 (Fatou component). *A* Fatou component *is a maximal connected subset of the Fatou set.*

Assume we are still under the hypothesis used to state the previous properties. The image of any component $\mathcal{U}$ of the $F(f)$ under f is also a component of $F(f)$, since the Fatou set is completely invariant. Moreover, a Fatou component $\mathcal{U}$ under f can only be:

a. **Periodic:** if $f^p(\mathcal{U}) = \mathcal{U}$ for some minimal $p \geq 1$. If $p = 1$, it is a *fixed component.*

b. **Preperiodic:** if $f^n(\mathcal{U})$ is periodic for some $n \geq 1$.

c. **Wandering:** if all iterates $\{f^k(\mathcal{U})\}$ are distinct.

Furthermore, Fatou components can be considered as a generalization version of the local theory seen in the previous section.

Theorem 2.2.16. ***([9, Theorem 2.1])*** *Let $f : \Omega \to \Omega$ be an holomorphic function of deg$(f) \geq 2$ in which $\Omega \in \{\mathbb{C}, \widehat{\mathbb{C}}\}$. If $\mathcal{U}$ is a periodic component of $F(f)$, then exactly one of the following holds:*

a. $\mathcal{U}$ is a (super)attracting domain*, i.e., there exists a (super)attracting periodic point $z_0 \in \mathcal{U}$ of period p such that $f^{np}(z) \to z_0$ as $n \to \infty$, $\forall z \in \mathcal{U}$.*

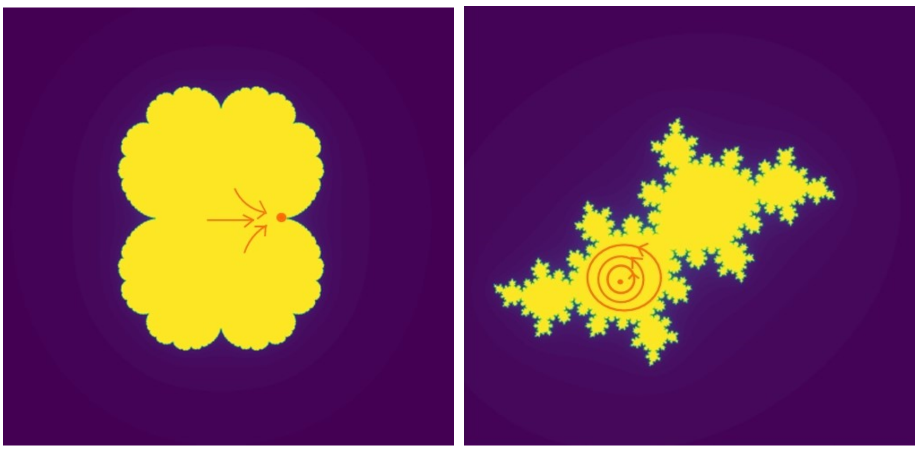
b. $\mathcal{U}$ is a parabolic domain*, i.e., there exists a rationally indifferent periodic point $z_0 \in \partial\mathcal{U}$ of period p such that $f^{np}(z) \to z_0$ as $n \to \infty$, $\forall z \in \mathcal{U}$.*

c. $\mathcal{U}$ is a Siegel disk*, i.e., $\mathcal{U}$ is simply connected and some f^n is conjugate to an irrational rotation in $\mathcal{U}$.*

d. $\mathcal{U}$ is a Herman ring*, i.e., $\mathcal{U}$ is doubly connected and f^n is conjugate to either an irrational rotation on an annulus or to an irrational rotation followed by an inversion in $\mathcal{U}$.*

e. $\mathcal{U}$ is a Baker domain*, i.e., there exists $z_0 \in \partial\mathcal{U}$ such that $f^{np}(z) \to z_0$ as $n \to \infty$, $\forall z \in \mathcal{U}$, but $f^p(z_0)$ is not defined.*

In the first section of chapter four, we will explore an example of a Baker domain, showing its dynamics and behaviour within the context of transcendental entire functions. Furthermore, in Figure 2.2 we can observe examples of a parabolic domain and a Siegel disk.



(a) $p(z) = z^2 + 0.25$ (b) $p(z) = z^2 + (-0.73737 - 0.67549i)z$

Figure 2.2: Julia sets of two polynomials. The yellow region represents the points whose orbits remain bounded under iteration of p: (a) is a parabolic domain while (b) is a Siegel disk. The boundary between the yellow region and the purple region (escaping orbits) is the Julia set of p.

Chapter 3

The escaping set for polynomials

In this chapter, we explore the dynamics of polynomial functions, focusing on their escaping set. We begin by defining the escaping set and examining its relationship with the Julia set and the basin of attraction of infinity. We then prove important properties of the basin of attraction of infinity which will be used to describe the connectivity of the basin and the escaping set. Finally, we conclude the chapter by proving that the basin of attraction of infinity is either simply connected or infinitely connected, proving Theorem A presented in the introduction.

It is easy to misinterpret that, given an initial condition z_0, their iteration will be simpler to comprehend due to polynomials being easier compared to other functions. In reality, however, most of them have a rather chaotic behaviour. We begin by defining the escaping set for polynomials.

Definition 3.0.1 (Set of escaping points). *Given a complex polynomial $p(z) = a_0 + a_1 z + \cdots + a_d z^d$ of degree $d \geq 2$, where $a_i \in \mathbb{C}$. The* set of escaping points *for p is then defined as*

$$I(p) = \{z \in \mathbb{C} | \lim_{n\to\infty} p^n(z) = \infty\}.$$

From this definition, we obtain an alternative interpretation of the Julia set that does not relay on normal families, but rather the set of escaping points. We define and denote the *filled Julia set of* p as $K(p) = \mathbb{C} \backslash I(p)$. Consequently, the Julia set of p is the boundary of the filled Julia set of p, i.e., $J(p) = \partial K(p)$. This statement will be justified later in this chapter.

Note 2. *Since $K(p)$ and $I(p)$ are complements of each other in $\mathbb{C}$, the Julia set of p is also the boundary of $I(p)$, i.e. $J(p) = \partial I(p)$.*

An important result that we previously stated without proving it in the last chapter is that infinity is a superattracting fixed point for all complex polynomials $p : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree $d \geq 2$. We will prove this here due to its significance for this section.

Proposition 3.0.2. *Let $p(z) = a_0 + a_1 z + \cdots + a_d z^d$ be a complex polynomial of degree $d \geq 2$, where $a_i \in \mathbb{C}$ and $a_d \neq 0$. Then, $z = \infty$ is a superattracting fixed point of p.*

Proof. First of all, we prove see that infinity is a fixed point of $p(z)$:

$$p(\infty) = \lim_{z\to\infty} p(z) = \lim_{z\to\infty} a_0 + a_1 z + \cdots + a_d z^d = \infty.$$

The next step is to show that it is superattracting. To do this, we will seek a function q that is conformally conjugated to p in a neighbourhood of ∞ due to conjugacy respecting fixed points. The conformal map for this is $\varphi(z) = 1/z$. So let $q(z)$ be the function defined as

$$q(z) = \varphi \circ p \circ \varphi^{-1}(z) = \frac{1}{p(\frac{1}{z})}.$$

Since

$$\varphi(\infty) = \lim_{z\to\infty} \varphi(z) = \lim_{z\to\infty} \frac{1}{z} = 0,$$

it follows that $z = 0$ is the corresponding point of $z = \infty$ when working with $q(z)$. If we develop our expressions, then we have

$$p\left(\frac{1}{z}\right) = a_0 + \frac{a_1}{z} + \cdots + \frac{a_d}{z^d} = \frac{1}{z^d}(a_0 z^d + a_1 z^{d-1} + \cdots + a_d),$$

hence

$$q(z) = \frac{1}{p(\frac{1}{z})} = \frac{1}{\frac{1}{z^d}(a_0 z^d + a_1 z^{d-1} + \cdots + a_d)} = \frac{z^d}{a_0 z^d + a_1 z^{d-1} + \cdots + a_d}$$

Now, given that $a_d \neq 0$, we can divide the numerator and denominator by a_d. Denoting $\frac{a_i}{a_d}$ as a_i' for $i = 0, \ldots, d$. Therefore, we can express $q(z)$ as

$$q(z) = \frac{\frac{1}{a_d} z^d}{a_0' z^d + a_1' z^{d-1} + \cdots + a_{d-1}' z + 1} = \frac{1}{a_d} z^d \frac{1}{a_0' z^d + a_1' z^{d-1} + \cdots + a_{d-1}' z + 1}$$

If we consider ε as $\varepsilon = a_0' z^d + a_1' z^{d-1} + \cdots + a_{d-1}' z$, we can observe that $\varepsilon \approx 0$ for $z \approx \varepsilon$. Thus,

$$q(z) = \frac{1}{a_d} z^d \frac{1}{\varepsilon + 1} = \frac{1}{a_d} z^d (1 + O(z))$$

It follows that $q(0) = 0$, which means $z = 0$ is a fixed point of q, reaffirming that $z = \infty$ is a fixed point of p. So we only need to prove that $z = 0$ is a superattracting fixed point of q. We prove it using its multiplier.

$$q'(z) = \frac{d}{a_d} z^{d-1}(1 + O(z)).$$

Evaluating at $z = 0$, since the degree is $d \geq 2$, $|q'(0)| = 0$. Thus, $z = 0$ is a superattracting fixed point of $q(z)$, which implies that $z = \infty$ is a superattracting fixed point of $p(z)$. □

In chapter two, we also introduced the concept of the basin of attraction of an attracting fixed point. In the same vein, we define and denote the *basin of attraction of infinity* as

$$A_p(\infty) = \{z \in \mathbb{C} \mid p^n(z) \to \infty, \text{ as } n \to \infty\},$$

where $p : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a polynomial of degree $d \geq 2$. From this definition we can observe a very important result.

Theorem 3.0.3. *Let $p : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a polynomial of degree $d \geq 2$, then $A_p(\infty) = I(p)$.*

Proof. Both concepts share the same definition. □

This implies the following three important equalities:

1. $K(p) = \mathbb{C} \setminus A_p(\infty) = \{z \in \mathbb{C} \mid p^n(z) \text{ is bounded when } n \to \infty\}$,
2. $J(p) = \partial K(p) = \partial A_p(\infty)$, which we "promised" in Definition 3.0.1,
3. $F(p) = A_p(\infty) \cup int(K(p))$, as a consequence of the previous two equalities.

Proposition 3.0.4. *$A_p(\infty)$ is an open set for polynomials $p : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree $d \geq 2$.*

Proof. Our goal is to show that for every $z \in A_p(\infty)$ there exists an open neighbourhood of z which is also contained in $A_p(\infty)$.

Let $p(z) = a_0 + a_1 z + \cdots + a_d z^d$ be a polynomial of degree $d \geq 2$. For large values of $|z|$, we have

$$|p(z)| = |a_0 + a_1 z + \cdots + a_d z^d| = |z||a_0 z^{-1} + a_1 + \cdots + a_d z^{d-1}| > \lambda |z|, \ \ \lambda > 0,$$

This implies that there exists $r > 0$ such that if $|z| > r$, then $p^n \to \infty$. Thus, $A_p(\infty)$ contains all points with large modulus $|z| > r$ and any points that eventually satisfy $|p^n(z)| > r$ for sufficiently large n.

Let $z_0 \in A_p(\infty)$. This means there exists an integer N such that $|p^n(z_0)| > r$ for all $n \geq N$. Due to the continuity of p, there exists a neighbourhood U of z_0 such that for all $z \in U$, the iterates $p^n(z)$ will remain close to $p^n(z_0)$ for $n = 0, \ldots, N$. Since $|p^n(z_0)| > r$, we can choose U such that $|p^n(z)| > r$ for all $z \in U$. This implies that every point $z \in U$ will converge to infinity under further iteration of p. Therefore, $U \subset A_p(\infty)$. □

From this proposition, it follows immediately that the filled Julia set $K(p)$ is a compact set, while the escaping set $I(p)$ is an open set. Moreover, since $J(p) = \partial K(p)$, it follows that the Julia set $J(p)$ is bounded.

The next proposition is the first statement in Theorem A.

Proposition 3.0.5. *$A_p(\infty)$ is a connected set for polynomials $p : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree $d \geq 2$.*

Proof. We will prove it by contradiction. Given that the degree of p is $d \geq 2$, it follows from the previous chapter that $J(p)$ is non-empty and invariant. Suppose there exists a subset $C \subset J(p)$ such that its interior, $Int(C)$, lies within $A_p(\infty)$. Since $J(p)$ is bounded and invariant, it follows that $|f^n(C)| \leq M \ \forall n$, where $M > 0$ is a constant. By the maximum modulus principle, we have $|f^n(Int(C))| \leq M \ \forall n$. Thus, $Int(C)$ is not contained in $A_p(\infty)$. Therefore, $A_p(\infty)$ is connected. □

Based on the work we have done thus far, we can conclude that $A_p(\infty)$ surrounds $J(p)$, since $J(p) = \partial A_p(\infty)$ is bounded and $A_p(\infty)$ is an open set as in Figure 3.1.

After observing that $A_p(\infty)$ surrounds $J(p)$, the next question to consider is how many holes exist in $A_p(\infty)$. Therefore, the main objective of this final section is to prove that $A_p(\infty)$ is either simply connected or infinitely connected. To achieve this, we first need to state some preliminary results.

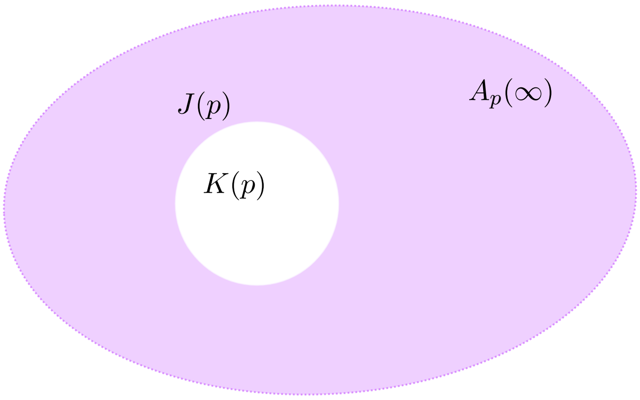


Figure 3.1: $A_p(\infty)$ surrounding $J(p)$

Lemma 3.0.6 (Extension of $|\varphi|$). *If $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ has a superattracting fixed point z_0 with basin $A_f(z_0)$, then the function $z \to |\varphi(z)|$ of Theorem 2.1.6 extends uniquely to a continuous map $|\varphi| : A_f(z_0) \to [0, 1)$ which satisfies the identity $|\varphi(f(z))| = |\varphi(z)|^p$.*

Proof. Set $|\varphi(z)|$ equal to $|\varphi(f^k(z))|^{1/p^k}$ for large k. □

The reason for this approach lies in the fact that it is not possible to extend the local map $\varphi(z)$, obtained from Theorem 2.1.6, throughout $A_p(\infty)$ as a holomorphic function $\varphi : A_p(\infty) \to \mathbb{C}$ such that $\varphi(z) = \varphi(p^k(z))^{1/n^k}$ for large k. This is because we would have to resolve $\sqrt[n]{\varphi(p^n(z))}$, which would not be possible at other critical points of p in $A_p(\infty)$. However, thanks to Lemma 3.0.6, we can instead construct $|\varphi| : A_p(\infty) \to \mathbb{R}_{\geq 0}$ such that $|\varphi| = |\varphi(p^k(z))|^{1/n^k}$ for large k.

Furthermore, if we apply the logarithmic function to the equation $|\varphi(f(z))| = |\varphi(z)|^p$ of Lemma 3.0.6, we obtain the functional equation $\log|\varphi(f(z))| = p \cdot \log|\varphi(z)|$. Thus, $\log|\varphi(z)|$ can also be extended to the entire basin of attraction $A_f(z_0)$ of z_0.

Consider now a polynomial $p(z) = az^d + \cdots$ with $d \geq 2$ and $a \neq 0$. As previously proved, it has a superattracting fixed point at ∞. Replacing 0 by ∞ in Böttcher's theorem, we can see that the polynomial $p(z)$ is conjugate to ζ^d near ∞. This conjugating function takes the form $\varphi(z) = cz + O(1)$, with a simple pole at ∞. As observed earlier, the equation $\log|\varphi(p(z))| = d \cdot \log|\varphi(z)|$ for $|z| > R$ allows us to extend $\log|\varphi(z)|$ to $A_p(\infty)$, which we already know it is a connected set. Moreover, the extended function $\log|\varphi(z)|$ is harmonic on $A_p(\infty)$ and satisfies that $\log|\varphi(z)| \to 0$ as $z \to \partial A_p(\infty)$. Therefore,

$$\log|\varphi(z)| = \log|z| + \log|c| + o(1), \quad |z| \to \infty.$$

Consequently, $\log|\varphi(z)|$ coincides with the *Green's function* for $A_p(\infty)$ with a pole at ∞, and we denote it as $G(z) = \log|\varphi(z)|$.

A Green's function is formally defined as follows:

Definition 3.0.7 (Green's function). *Let Ω be a region in the plane and let $z_0 \in \Omega$. A* Green's function of Ω with a singularity at z_0 *is a function $G : \Omega \setminus \{z_0\} \to \mathbb{R}$ with the properties:*

1. *G is harmonic in $\Omega \setminus \{z_0\}$, i.e., G has continuous second partial derivatives and*

$$\frac{\partial^2 G}{\partial x^2} + \frac{\partial^2 G}{\partial y^2} = 0;$$

2. *$H(z) = G(z) + \log|z - z_0|$ is harmonic in a disk about z_0;*

3. *$\lim_{z \to w} G(z) = 0$ for each w in $\partial\Omega$, including points at infinity.*

It is important to note that for a given domain Ω and point z_0, the Green's function is unique.

Since $\log|\varphi(p(z))| = d \cdot \log|\varphi(z)|$, it follows that $G(p(z)) = d \cdot G(z)$ for $z \in A_p(\infty)$. The Green's function $G(z)$ measures the rate of escape to infinity. Therefore, p maps level curves of G to level curves of G, increasing the value of G by a factor of d. The exterior of the level curve $\{G(z) = r\}$, denoted by $\{G(z) > r\}$, is invariant under p, and p maps it $d : 1$ onto the exterior $\{G(z) > rd\}$. For sufficiently large r, $\varphi(z)$ is defined on the exterior $\{G(z) > r\}$ and maps it conformally onto $\mathbb{C} \setminus \overline{\mathbb{D}}_{e^r}$. The equation $\varphi(z) = (\varphi(p(z)))^{1/d}$ allows us to extend $\varphi(z)$ to the exterior $\{G(z) > r/d\}$, provided no critical point of p belong to this domain.

Before introducing the main theorem, observe that $A_p(\infty)$ is infinitely connected if and only if $K(p)$ has infinitely many connected components. Consequently, $A_p(\infty)$ is infinitely connected if and only if $J(p)$ has infinitely many connected components, since $J(p)$ is the boundary of both sets.

The next theorem completes the statement of Theorem A in the introduction.

Theorem 3.0.8. *Let $p(z) = az^d + \cdots$ be a polynomial with $d \geq 2$ and $a \neq 0$.*

a. *$J(p)$ is connected if and only if there is no finite critical point of p in $A_p(\infty)$, that is, if and only if the forward orbit of each critical point is bounded.*

b. *If all critical points of p are in $A_p(\infty)$, then $J(p)$ is totally disconnected.*

Proof. a. To prove the first statement of the theorem, we will use the Green's function which we showed to be $G(z) = \log|\varphi(z)|$. Consider the following two cases:

Case 1: There is no critical point of p in $A_p(\infty)$.
If there is no critical point of p in $A_p(\infty)$, then $\varphi : A_p(\infty) \to \mathbb{C} \setminus \overline{\mathbb{D}}$ conformally. In fact, $K(p) = \bigcap_{n=1}^{\infty} \{G(z) < \frac{1}{n}\}$ is connected since is the intersection of infinitely connected sets. Hence, $A_p(\infty)$ is simple connected, and the Julia set $J(p) = \partial A_p(\infty)$ is connected.

Case 2 :$A_p(\infty)$ contains a critical point of p.
If $A_p(\infty)$ contains a critical point of p, we can extend φ until reaching the level curve $\{G(z) = r\}$, which contains the critical point of p. The domain $\{G(z) > r\}$ is connected and is conformally mapped onto $\overline{\mathbb{D}}_{e^r}$. However, $\{G(z) > r\}$ forms several cusps at the critical point, and $\varphi(z)$ approaches different values as z approaches the critical point through different cusps (see Figure 3.2). Thus, the level curve $\{G(z) = r\}$ consists of at least two simple closed curves that meet at the critical point. Within each of these curves there are points of $J(p)$, because otherwise G would be constant within the curve. Hence, $J(p)$ is disconnected. Moreover, $J(p)$ has uncountably many connected components. This can be seen by noting that the

critical points of G are the critical points of p and all their inverse iterates, and by following the splitting of level curves at each such critical point. Consequently, both $K(p)$ and $A_p(\infty)$ have uncountably many connected components.

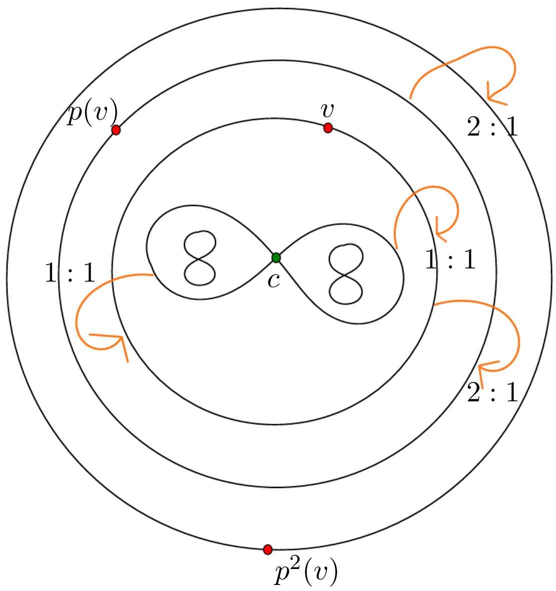


Figure 3.2: Level curves of Green's function when $A_p(\infty)$ has a critical point of p. The polynomial is of degree $d = 2$. The critical point is denoted as c and its critical value is $p(c) = v$.

b. Assume that all critical points of p lie in $A_p(\infty)$. Let $\mathbb{D}$ be a large open disk containing $J(p)$ such that $p(\widehat{\mathbb{C}} \setminus \mathbb{D}) \subset \widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. We can choose a N large enough so p^N maps the critical points of p to $\widehat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. Since the critical points of p^n are the critical points of p and its first $n-1$ iterates, there are no critical points of p^n in $p^{-n}(\overline{\mathbb{D}})$ for $n \geq N$. This implies that there are no critical values of p^n in $\overline{\mathbb{D}}$. Since critical values are singular values, all inverse branches p^{-n} are well defined and map $\overline{\mathbb{D}}$ conformally into $\mathbb{D}$. Let $z_0 \in J(p)$, by the Invariance lemma (Lemma 2.2.4), $p^n(z_0) \in J(p)$. We define f_n as the inverse branch of p^n which maps $p^n(z_0)$ to z_0. These f_n functions are uniformly bounded on a neighbourhood of $\overline{\mathbb{D}}$, therefore, by the Simple Montel's theorem (Theorem 1.3.3), they form a normal family in $\overline{\mathbb{D}}$. Since $f_n(z)$ accumulates on $J(p)$ for $z \in \mathbb{D} \cap A_p(\infty)$, any limit function f maps $\mathbb{D} \cap A_p(\infty)$ into $J(p)$. Since $J(p) \neq \widehat{\mathbb{C}}$, by Lemma 2.2.8, $J(p)$ contains no open sets. Thus, f must be constant. Therefore, the diameter of $f_n(\overline{\mathbb{D}})$ tends to 0. Since $f_n(\partial\mathbb{D})$ is disjoint from $J(p)$, $\{z_0\}$ must be a connected component of $J(p)$. Thus, $J(p)$ is totally disconnected.

□

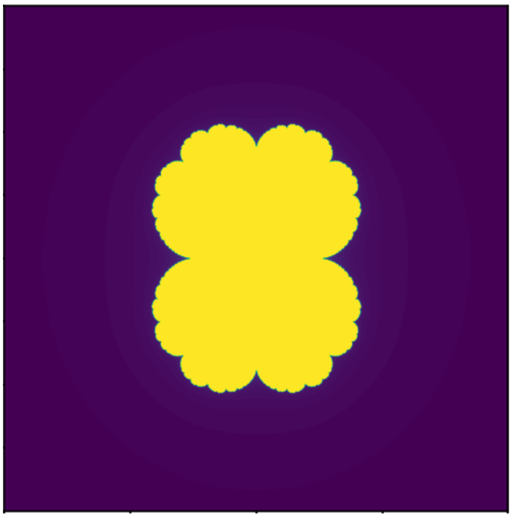
Remark 1. *When $K(p)$ is totally disconnected and the degree of the polynomial is $d = 2$, $K(p) = J(p)$ is a Cantor set, meaning its components are points. While if the degree is $d > 2$, some of the components of $K(p)$ may not be points (see Figure 3.3c and Figure 3.3d respectively).*

With this theorem, we can conclude that if the filled Julia set $K(p)$ contains all finite critical points of the polynomial p, then $A_p(\infty)$ contains only a single hole which is $K(p)$

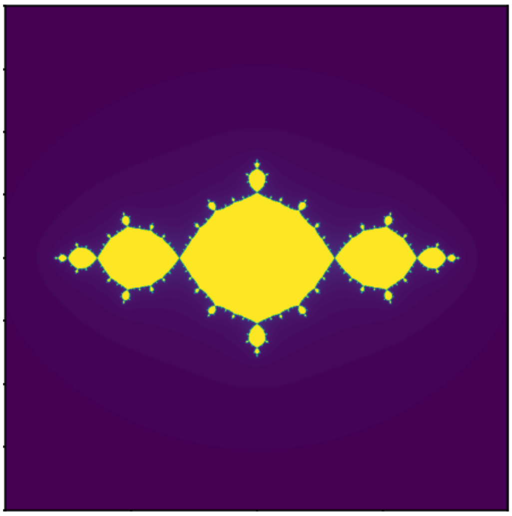
in $\mathbb{C}$, making $A_p(\infty)$ simply connected in $\widehat{\mathbb{C}}$. However, if at least one critical point of p lies in $A_p(\infty)$, then $A_p(\infty)$ have uncountably many connected components.

Let us notice here how these are dynamical conditions which determine the topology of the escaping set, as the escaping set coincides with the basin of attraction of infinity, as previously established.

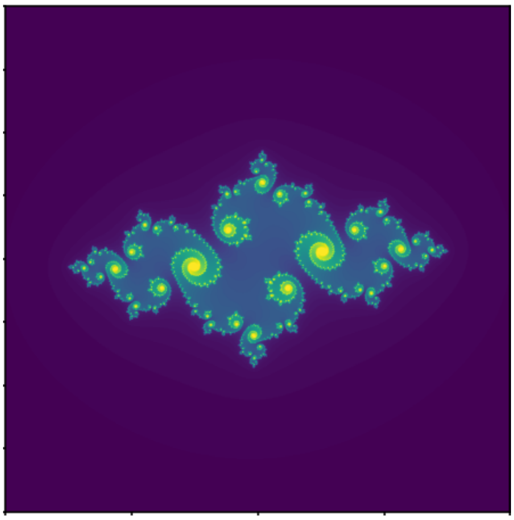
As a closure for this chapter, in Figure 3.3 we present a series of pictures of Julia sets of polynomials generated using a Python program detailed in Appendix 1.



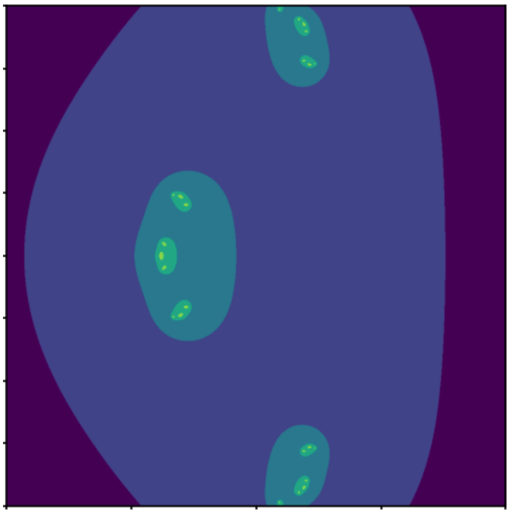
(a) $p(z) = z^2 + 0.25$



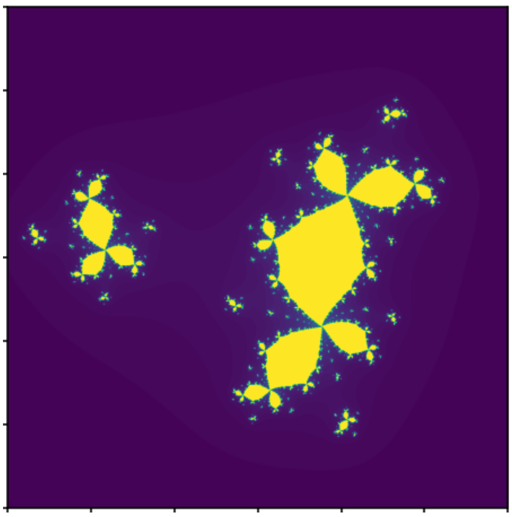
(b) $p(z) = z^2 - 1$



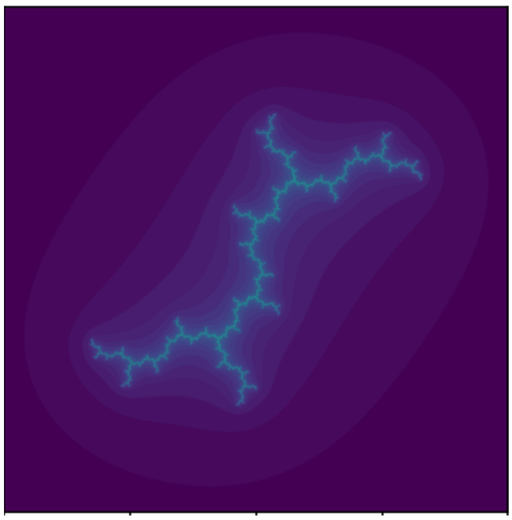
(c) $p(z) = z^2 - 0.765 + 0.12i$



(d) $p(z) = z^3 + 3z + 2$



(e) $p(z) = z^3 - 0.48z + 0.706 + 0.503i$



(f) $p(z) = z^2 + i$

Figure 3.3: Julia sets of polynomials $p(z)$ of degree 2 and 3. The yellow points represent the points whose orbits remain bounded under iteration of p. The boundary between the yellow region (bounded orbits) and the purple region (escaping orbits) is the Julia set of p.

Chapter 4

The escaping set for transcendental entire functions

In this final chapter, we concentrate on the escaping set for transcendental entire functions. This chapter is divided into three sections: the first provides a brief characterization of the transcendental entire functions, the second is focuses on proving that the escaping set for such functions is non-empty, and the third establishes that, for a specific class of transcendental entire functions, the interior of the escaping set is also non-empty.

4.1 Characterizing transcendental entire functions

Transcendental entire functions are holomorphic on the whole complex plane but have $z = \infty$ as an essential singularity. This marks the main difference between their dynamics and the dynamics of polynomials, where $z = \infty$ was a superattracting fixed point and $I(p) = A_p(\infty)$.

Unlike polynomials, transcendental functions are iterated in the complex plane rather than the Riemann sphere, as they cannot be defined or extended at $z = \infty$. Another important contrast is the nature of singular values. While polynomials only have critical points and critical values, transcendental functions also have asymptotic values, a second type of singular value which it is defined as it follows:

Definition 4.1.1 (Asymptotic value). *Let $f : \mathbb{C} \to \mathbb{C}$ be an transcendental entire function. A point $v_0 \in \mathbb{C}$ is said to be an* asymptotic value *if there exists a curve $\gamma \subset \mathbb{C}$ tending to infinity such that $f(z) \to v_0$ as $z \to \infty$ along γ.*

This means that there is at least one branch of the inverse function which cannot be defined in v_0. For example, $z = 0$ is an asymptotic value of $f(z) = e^z$.

Since our objective is to analyse their escaping set, we first need to understand the dynamical behaviour of $z = \infty$ as essential singularity of an transcendental entire function. To comprehend it, we rely on *Great Picard's theorem.*

Theorem 4.1.2 (Great Picard's theorem). *Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function with a essential singularity at $z_0 \in \mathbb{C}$. Then in any neighbourhood U of z_0, $f(z)$ assumes each complex number, with one possible exception, an infinite number of times.*

We will not prove it here since the proof escapes of the scope of this project; however, its proof can be found in [10, chapter XII, section 4].

This theorem implies that for every transcendental entire function $f : \mathbb{C} \to \mathbb{C}$, the images of points in a neighbourhood of $z = \infty$ cover almost the entire complex plane, except for at most one point. This observation leads to the first question we need to address: if there exist $z \in \mathbb{C}$ such that $f^n \to \infty$ as $n \to \infty$. In other words, we want to determine if the $f^n(z)$ iterations for large $n \geq N$ can escape to infinity rather than remain within the complex plane. To formalize this, we define the escaping set as follows:

Definition 4.1.3 (Set of escaping points). *Let $f : \mathbb{C} \to \mathbb{C}$ be an transcendental entire function. The* set of escaping points for f *is defined as*

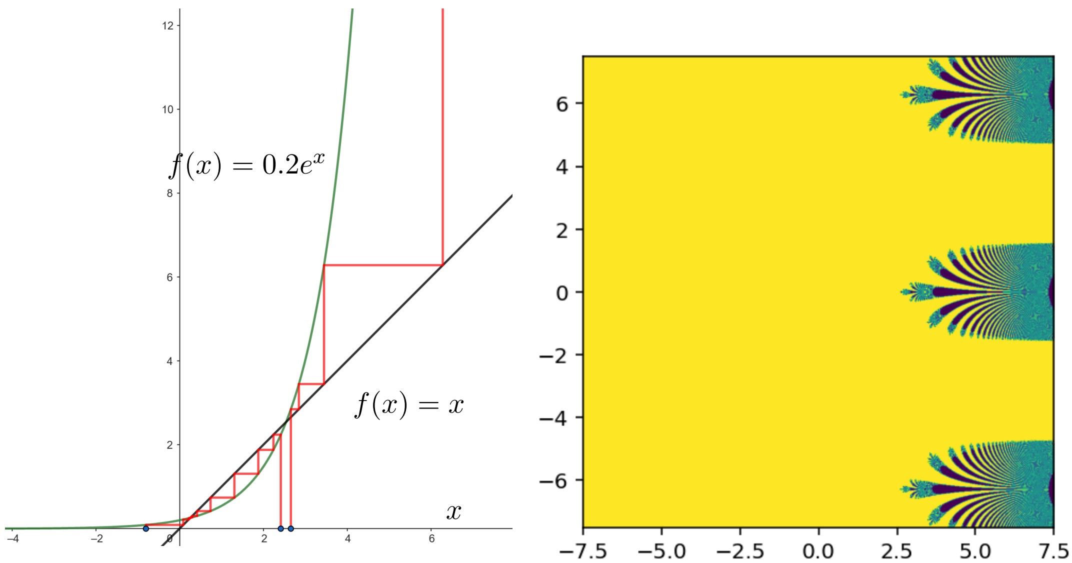
$$I(f) = \{z \in \mathbb{C} | \lim_{n\to\infty} f^n(z) = \infty\}.$$

Our first objective is to investigate if $I(f) \neq \varnothing$. For instance, we can begin by studying the exponential family, which is a classical example of transcendental entire functions.

Example 4.1.4 ($f(z) = 0.2e^z$). Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = 0.2e^z$. By iterating the function, we have

$$z, \; f(z) = 0.2e^z, \; f^2(z) = f(0.2e^z) = 0.2e^{0.2e^z}, \; f^3(z) = f(f^2(z)) = 0.2e^{0.2e^{0.2e^z}}, \ldots$$

We can observe that from a specific $x \in \mathbb{R}$, the sequence of iterates $x, \; 0.2e^x, \; 0.2e^{0.2e^x}, \ldots$ diverges to infinity as $n \to \infty$. Thus, for this specific transcendental entire function, $I(f) \neq \varnothing$. The escaping behaviour of points $x \in \mathbb{R}$ as they escape to infinity is illustrated in Figure 4.1a. Furthermore, the interior of $I(f)$ is empty. This function is an example of a *Cantor bouquet*, as we can observe in Figure 4.1b.



(a) The graphic representation of $f(x) = 0.2e^x$ reveals two fixed points: $x \approx 2.5426$, which is repelling, and $x \approx 0.2591$, which is attracting. From the repelling point, $x \in \mathbb{R}$ escape to infinity under iteration of f.

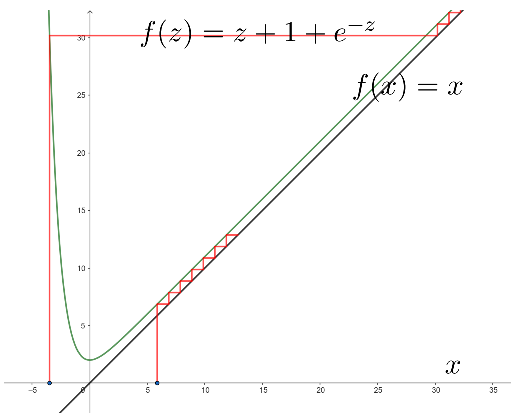
(b) This figure shows the Julia set of $f(z) = 0.2e^z$. Points in yellow belong to the basin of attraction of the attracting fixed point, while points in purple constitute the Julia set of f.

Figure 4.1: Study of the behaviour of the function $f(z) = 0.2e^z$

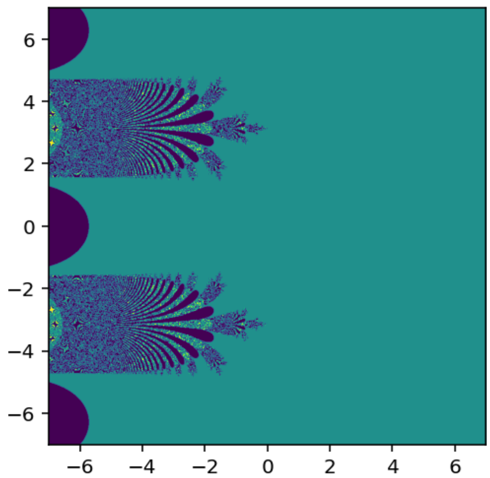
Example 4.1.5 ($f(z) = z + 1 + e^{-z}$)**.** Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = z + 1 + e^{-z}$. By iterating the function, we have

$$z,\ \ f(z) = z + 1 + e^{-z},\ \ f^2(z) = f(z + 1 + e^{-z}) = (z + 1 + e^{-z}) + 1 + e^{-(z+1+e^{-z})}, \ldots$$

Similar to the previous example, the sequence of iterates $x,\ \ x+1+e^{-x},\ \ (x+1+e^{-x})+1+e^{-(x+1+e^{-x})}, \ldots$ diverges to infinity as $n \to \infty$, but this time for every $x \in \mathbb{R}$, as shown in Figure 4.2a. Thus, for this specific transcendental entire function, $I(f)$ is also non-empty. In contrast to the first example, however, $I(f)$ contains half of the plane, as seen in Figure 4.2b. Therefore, the interior of $I(f)$ is non-empty. This follows from the fact that $|e^z| = e^{Re(z)}$, and thus $|e^{-z}| = e^{-Re(z)} \to 0$ as $Re(z) \to \infty$, which implies that these points escape to infinity as $z \to z+1$. This function serves as an example of a *Baker domain* (see Theorem 2.2.16).



(a) In the graphic representation of$f(z) = z+1+e^{-z}$, we can observe that each $x \in \mathbb{R}$ escapes to infinity under iteration of f. Additionally, we can see that there there are no real fixed points, all fixed points are purely imaginary: $f(z) = z + 1 + e^{-z} = z \Leftrightarrow e^{-z} = -1 \Leftrightarrow e^{-z} = e^{i\pi + 2\pi i n} \Leftrightarrow z = -i\pi(1+2n),\ \ n \in \mathbb{Z}$.



(b) The figure shows the Julia set of $f(z) = z+1+e^{-z}$. Points in blue belong to the the escaping set, which covers the entire right half-plane. Points in purple constitute the Julia set of f. The apparent interior of the purple region is a result of numerical errors.

Figure 4.2: Study of the behaviour of the function $f(z) = z + 1 + e^{-z}$

4.2 The escaping set is non-empty

With the previous examples, we have established the existence of at least two transcendental entire functions for which $I(f) \neq \varnothing$. The main goal of this section is to prove that $I(f) \neq \varnothing$ holds for all transcendental entire functions (Theorem B in the introduction). This result was originally proved by Alexandre Eremenko. However, to fully understand the proof, we first need to introduce a key result known as the Wiman-Valiron theorem.

Let $f : \mathbb{C} \to \mathbb{C}$ be an transcendental entire function represented by its power series expansion: $f(z) = \sum_{n=0}^{\infty} c_n z^n$. For any $r > 0$, the sequence $|c_n| r^n$ tends to zero, therefore it contains a maximal term. We denote the index of this term by $N(r)$. If there are several maximal terms, we take the largest index among them. This increasing function $N(r)$, called the *central index*, identifies the term of the power series most important to the corresponding r. And $N(r)$ tends to infinity as $r \to \infty$, since the biggest monomial always wins. Additionally, we define the maximum for each r as

$$M(r) = \max_{|z|=|r|} |f(z)|, \quad r > 0.$$

For each $r > 0$, we can take a point $w(r)$ such that $M(r) = |f(w(r))|$ and $|w(r)| = r$. At this point $w(r)$, the function behaves almost like a monomial. This behaviour is formalized in Theorem 4.2.1, which provides a more precise statement. With these definitions in place, we can now state the Wiman-Valiron theorem, which we will not prove as it is beyond the extent of this project.

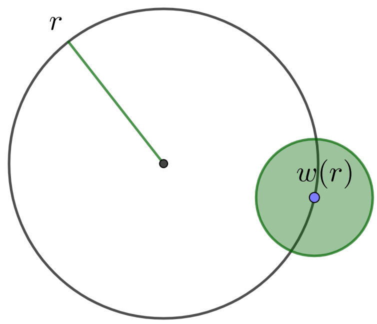


Figure 4.3: The central index $N(r)$ dominates the green disk centred at $w(r)$. Furthermore, within this disk, the function satisfies the approximation $f(z) \sim c_{N(r)} z^{N(r)}$. The precise definition of this marked disk is provided in the statement of the following theorem.

Theorem 4.2.1 (Wiman-Valiron theorem). *Let $f : \mathbb{C} \to \mathbb{C}$ be an transcendental entire function, and let $\alpha > \frac{1}{2}$. If $|z - w(r)| < r(N(r))^{-\alpha}$, then*

$$f(z) = \left(\frac{z}{w(r)}\right)^{N(r)} f(w(r))(1 + \varepsilon_1),$$

which is the precise formula for the approximation of the monomial. Its derivative satisfies

$$f'(z) = N(r)\left(\frac{z}{w(r)}\right)^{N(r)} f(w(r))(w(r))^{-1}(1 + \varepsilon_2)$$

where, for $i \in \{1, 2\}$, $\varepsilon_i = \varepsilon_i(r, z)$ converges uniformly to zero with respect to z as $r \to \infty$ and $r \notin \mathcal{E}(f)$. The exceptional set $\mathcal{E}(f)$, which depends of f and α, has a finite logarithmic measure. This means that

$$Lm(\mathcal{E}(f)) = \int_{\mathcal{E}(f)} \frac{dt}{t} < \infty.$$

To prove our main theorem, we still need to state an additional essential result. This theorem is deduced from Wiman-Valiron's theorem using Rouche's theorem. It states the following:

Theorem 4.2.2. ***[6, Theorem 2.3].*** *Let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental entire function, and let $\mathcal{E}(f)$, $N(r)$ and $w(r)$ be as in the statement of the Wiman-Valiron theorem. For each $\beta > 1$, there exists an $\alpha \in (1/2, 1)$ such that, if $r \notin \mathcal{E}(f)$ is sufficiently large, then*

$$\left\{ z \in \mathbb{C} \mid \frac{|f(w(r))|}{\beta} \leq |z| \leq \beta |f(w(r))| \right\} \subset f\left(\mathbb{D}\left(w(r), \frac{r}{N(r)^{\alpha}}\right)\right).$$

With these results in place, we can now proceed to prove our central theorem.

Theorem 4.2.3 ($I(f)$ **is non-empty).** *For every transcendental entire function $f : \mathbb{C} \to \mathbb{C}$, the set $I(f)$ is non-empty.*

Proof. We will prove the theorem by constructing a nested sequence of domains and using the branches of f^{-1} in them to find a point whose orbit escapes to infinity.

Our first objective is to create the image of the disk $|z - w(r)| < r(N(r))^{-\alpha}$ described in the Wiman-Valiron theorem, where $\alpha > 1/2$. We can choose a radius $r_1 > 2$ such that $r_1 \notin \mathcal{E}(f)$, and so large that it satisfies the following properties:

$$M(r) > 4r, \;\; r \geq r_1,$$

$$|\log(1+\varepsilon_i)| < 1, \;\; r \geq r_1, \;\; r \notin \mathcal{E}(f),$$
$$Lm(\mathcal{E}(f) \cap [r_1, \infty)) < 1,$$
$$N(r_1) > 10^4.$$

The first property implies that we need the function to grow very fast, which is guaranteed since f is transcendental. Next, we select a point $w(r_1)$ such that $M(r_1) = |f(w(r_1))|$ and $|w(r_1)| = r_1$. By the Wiman-Valiron theorem, for $\alpha > 1/2$, we can approximate the behaviour of $f(z)$ within the disk $|z - w(r_1)| < r_1(N(r_1))^{-\alpha}$ as

$$f(z) \approx \left(\frac{z}{w(r_1)}\right)^{N(r_1)} f(w(r_1)).$$

This implies that $f(z)$ behaves like the dominant monomial of it series expansion near $w(r_1)$. Furthermore, the logarithmic measure condition on $\mathcal{E}(f)$ ensures that for the majority of $r > 0$, $r \notin \mathcal{E}(f)$

By Bergweiler, Rippon and Stallard's theorem, for $\beta > 1$ and $\alpha \in (1/2, 1)$, we have:

$$\left\{ z \in \mathbb{C} \;\middle|\; \frac{|f(w(r_1))|}{\beta} \leq |z| \leq \beta |f(w(r_1))| \right\} \subset f\left(\mathbb{D}\left(w(r_1), \frac{r_1}{N(r_1)^\alpha}\right)\right).$$

This result implies that the image of the disk $|z - w(r_1)| < r_1(N(r_1))^{-\alpha}$ under f contains a large annulus in $\mathbb{C}$. Therefore, f maps neighbourhoods of $w(r_1)$ to regions where $|f(z)|$ grows, significantly. We can denote this annulus as A_1. The reason of why A_1 has to be large is to avoid the set $\mathcal{E}(f)$.

Since A_1 avoids $\mathcal{E}(f)$, we can find a circle within the annulus with radius $r_2 \notin \mathcal{E}(f)$. We can take a point $w(r_2)$ such that $M(r_2) = |f(w(r_2))|$ and $|w(r_2)| = r_2$. Thus, the disk $|z - w(r_2)| < r_2(N(r_2))^{-\alpha}$ is contained within the image of the disk $|z - w(r_1)| < r_1(N(r_1))^{-\alpha}$ under f. Repeating the previous argument, this time we obtain that the image of the disk $|z - w(r_2)| < r_2(N(r_2))^{-\alpha}$ under f contains a larger annulus than A_1, which we can denote as A_2.

Moreover, by construction,

$$\mathbb{D}\left(w(r_2), \frac{r_2}{N(r_2)^\alpha}\right) \subset f\left(\mathbb{D}\left(w(r_1), \frac{r_1}{N(r_1)^\alpha}\right)\right).$$

This shows that $|z - w(r_1)| < r_1(N(r_1))^{-\alpha}$ contains the preimage of the disk $|z - w(r_2)| < r_2(N(r_2))^{-\alpha}$.

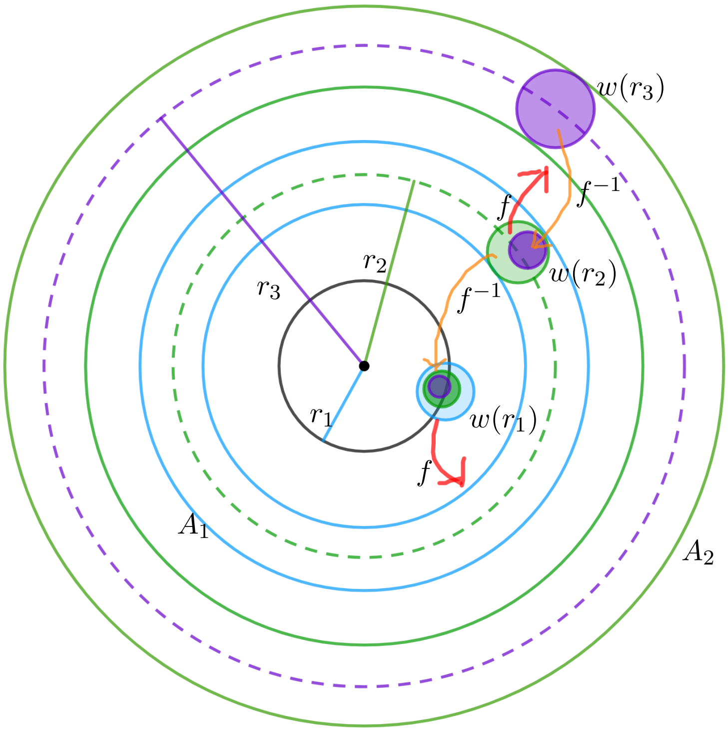


Figure 4.4: This illustration shows how the image of the disk $|z-w(r_1)| < r_1(N(r_1))^{-\alpha}$ under f contains the annulus A_1, and how the image of the disk $|z - w(r_2)| < r_2(N(r_2))^{-\alpha}$ under f contains the annulus A_2. Additionally, the disk centred at $w(r_2)$ includes the preimage of the disk centred at $w(r_3)$, and the disk centred at $w(r_1)$ contains the preimage of the disk centred at $w(r_2)$, as well as the preimage of the preimage (contained in the disk centred at $w(r_2)$) of the disk centred at $w(r_3)$.

By iterating this process, we can construct a sequence of sets

$$\left\{D_1 = \mathbb{D}\left(w(r_1), \frac{r_1}{N(r_1)^{\alpha}}\right),\ D_2 = f(D_1), \dots, D_n = f^n(D_1)\right\}$$

which is nested since $D_1 \subseteq D_2 \subseteq \cdots D_n$ and the maximum modulus of $f^n(D_1)$ tends to infinity as $n \to \infty$. Therefore, the first disk will have the preimages of all the subsequent disks $|z-w(r_i)| < r_i(N(r_i))^{-\alpha}$ for $i > 1$, as we can see in Figure 4.4. So, by the nested set property, there exists at least one point $z_0 \in \mathbb{D}\left(w(r_1), \frac{r_1}{N(r_1)^{\alpha}}\right)$ whose orbit $\{f^n(z_0)\}_{n=1}^{\infty}$ not only remains within these nested sets but also escapes to infinity. This implies that $z_0 \in I(f)$, therefore, $I(f)$ is non-empty as we wanted to prove. □

4.3 The escaping set of bounded-type functions

If we analyse both examples from section one again, we can observe a key difference regarding the interior of their escaping set. In the first example, the interior is empty, while in the second one, the interior is non-empty. Comparing the two functions, we can find the following distinctions:

- The first function, $f(z) = 0.2e^z$, has only one singular value, namely the asymptotic value $z = 0$. Hence, the set of singular values is bounded.

- On the other hand, the second function, $f(z) = z + 1 + e^{-z}$, has infinitely many singular values that are not bounded. We can see this statement is true, for instance, by examining its critical values:

$$f'(z) = 1 - e^{-z} = 0 \Leftrightarrow e^{-z} = 1 \Leftrightarrow e^{-z} = e^{2\pi ik} \Leftrightarrow z_k = -2\pi ik, \quad k \in \mathbb{Z}$$

$$f(z_k) = f(-2\pi ik) = -2\pi ik + 1e^{-2\pi ik} = -2\pi ik + 1 + 1 = -2\pi ik + 2, \quad k \in \mathbb{Z}.$$

These singular values are unbounded, as $|-2\pi ik + 2| \to \infty$ as $k \to \pm\infty$.

From this comparison, we can hypothesize that the state of the interior of $I(f)$ depends on whether the singular values of the function are bounded or not. For a given function f, we denote the set of singular values of f as $Sing(f)$. Using this, we define the following:

Definition 4.3.1 (**B set**). *The set B consists of all entire functions f such that the set $Sing(f)$ is bounded, that is*

$$B = \{f : \mathbb{C} \to \mathbb{C} \text{ entire} \mid Sing(f) \text{ is bounded}\}.$$

Functions belonging to this set are called bounded-type functions.

A direct consequence of this definition is that if $f \in B$, then, by Lemma 1.5.3, f is a covering map over the set $\{z \in \mathbb{C} \mid |z| > R\}$ for sufficiently large R, where the disk $\{z \in \mathbb{C} \mid |z| \leq R\}$ contains all singular values of f. The domain of this covering map f is a component of the preimage of the set $\{z \in \mathbb{C} \mid |z| > R\}$.

The main goal of this section is to prove that the interior of the escaping set of every transcendental entire function f such that $f \in B$ is empty (Theorem C in the introduction). To achieve this, we first need to state and prove several preliminary results, culminating in a important theorem originally proved by Eremenko and Lyubich:

Theorem 4.3.2. ***[13, Therorem 2.1]*** *Let $f \in B$ be a transcendental entire function. If $z \in F(f)$, then the orbit $\{f^n(z)\}_{n=0}^{\infty}$ does not tend to ∞.*

To prove this theorem, we begin by stating a pair of propositions concerning the connectedness of all components of $F(f)$. While we will not provide the proof of the first proposition, we will use it to prove the second one. Both proofs can be found in [13].

Proposition 4.3.3. *Let an entire function f be bounded on a curve γ tending to infinity. Then, all components of $F(f)$ are simply connected.*

Proposition 4.3.4. *If $f \in B$ is transcendental, then all components of $F(f)$ are simply connected.*

Proof. Let $f \in B$ be a transcendental function such that $Sing(f) \subset D(0, R/2)$ for a sufficiently large R, where $D(0, R/2) = \{z \in \mathbb{C} \mid |z - 0| < R/2\}$. Define $A = \mathbb{C} \setminus \overline{D(0, R)}$ and $G = f^{-1}(A)$. Each component V of G is homeomorphic to A, hence, $f : V \to A$ is an universal covering map for all components V. Since $A = \mathbb{C} \setminus \overline{D(0, R)}$ is topologically equivalent to the punctured disk $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$, by Lemma 1.5.2, each component V is simply connected. Furthermore, f maps the boundary of each V to the boundary of A, which is the circle $\{|z| = R\}$. Since f is transcendental, $f^{-1}(\{|z| = R\})$ forms a collection of unbounded curves. These curves do not close because f has an essential singularity at ∞, implying that both ends of each curve tend to infinity. Therefore, by the previous proposition, all components of $F(f)$ are simply connected. □

Continuing with the notation of the proof, let R be chosen sufficiently large such that $|f(0)| < R$. Then $0 \notin G$, and $\exp : W \to G$ is a conformal isomorphism for any component W of the set $U = \ln G$. Let us consider the half-plane $H = \ln A = \{\varepsilon \in \mathbb{C} \mid Re(\varepsilon) > \ln R\}$. Thus, we obtain the following commutative diagram:

$$\begin{array}{ccc} U & \xrightarrow{\mathcal{F}} & H \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow[f]{} & A \end{array}$$

Here, $\mathcal{F}$ is a conformal isomorphism of each connected component of U into H, obtained from f via the logarithmic change of variables $z = e^w$ in a neighbourhood of ∞, where w is the new variable. With this setup, we can state the following lemma, which is relevant for proving Eremenko and Lyubich theorem.

Lemma 4.3.5. $|\mathcal{F}'(z)| \geq \frac{1}{4\pi}(Re(\mathcal{F}(z)) - \ln R)$.

Proof. Let W be a connected component of U. Since the exponential function is univalent in W, the component W does not contain vertical segments of length 2π. Let $\varphi : H \to W$ be the inverse function of F. The disk $D(\mathcal{F}(z), Re(\mathcal{F}(z)) - \ln R)$ is contained in H. By applying the Koebe 1/4-theorem, see Theorem 1.1.8, to the function φ on this disk, we obtain the inequality:

$$\frac{1}{4}|\varphi'(\mathcal{F}(z))| \cdot |Re(\mathcal{F}(z)) - \ln R| \leq \pi.$$

If we expand this expression, we obtain:

$$\frac{1}{4}|\varphi'(\mathcal{F}(z))| \cdot |Re(\mathcal{F}(z)) - \ln R| \leq \pi \Leftrightarrow \frac{1}{|\varphi'(\mathcal{F}(z))|} \geq \frac{1}{4\pi}|Re(\mathcal{F}(z)) - \ln R|.$$

Since $\mathcal{F}'(z) = \frac{1}{\varphi'(\mathcal{F}(z))}$, it follows that:

$$|\mathcal{F}'(z)| \geq \frac{1}{4\pi}|Re(\mathcal{F}(z)) - \ln R|.$$

This prove the lemma. □

Using this result, we can now prove the theorem of Eremenko and Lyubich.

Proof of Theorem 4.3.2. We are going to prove this by contradiction, using the previously established notation. Suppose that the orbit $\{f^n(z_0)\}_{n=0}^{\infty}$ of $z_0 \in F(f)$ tends to infinity. Then there exists a disk $B_0 = D(z_0, r)$, $r > 0$ such that the sequence $\{f^n(z_0)\}$ tends uniformly to infinity. As a result, all $B_n = f^n(B_0)$ are contained in G except for a finite number. Recall that G is the preimage of A, the complement of the closed disk of radius R containing the singular values of f.

Let C_0 be a component of the set $\ln B_0$ and let $C_n = \mathcal{F}^n(C_0)$. This implies that $\exp C_n = B_n$. As a consequence, $C_n \subset U = \ln G$, and $Re(\mathcal{F}^n(z))$ tends to $+\infty$ uniformly in C_0, as we can see in Figure 4.5. Let $\varepsilon_0 \in C_0$, so $\varepsilon_n = \mathcal{F}^n(\varepsilon_0) \in C_n$.

We denote d_n as the maximum radius of the disks centred at ε_n contained in C_n. By the Koebe 1/4-theorem, we have that $d_{n+1} \geq \frac{1}{4}d_n|\mathcal{F}'(\varepsilon_n)|$. By the previous lemma, we know that $|\mathcal{F}'(z)| \geq \frac{1}{4\pi}|Re(\mathcal{F}(z)) - \ln R|$, and since $Re(\mathcal{F}(\varepsilon_n)) \to +\infty$, we obtain that $|\mathcal{F}'(\varepsilon_n)| \to \infty$. Consequently, $d_n \to \infty$. But this is a contradiction, as $C_n \subset U$, and U does not contain vertical segments of length 2π. Hence, the orbit $\{f^n(z)\}_{n=0}^{\infty}$ does not tend to ∞. □

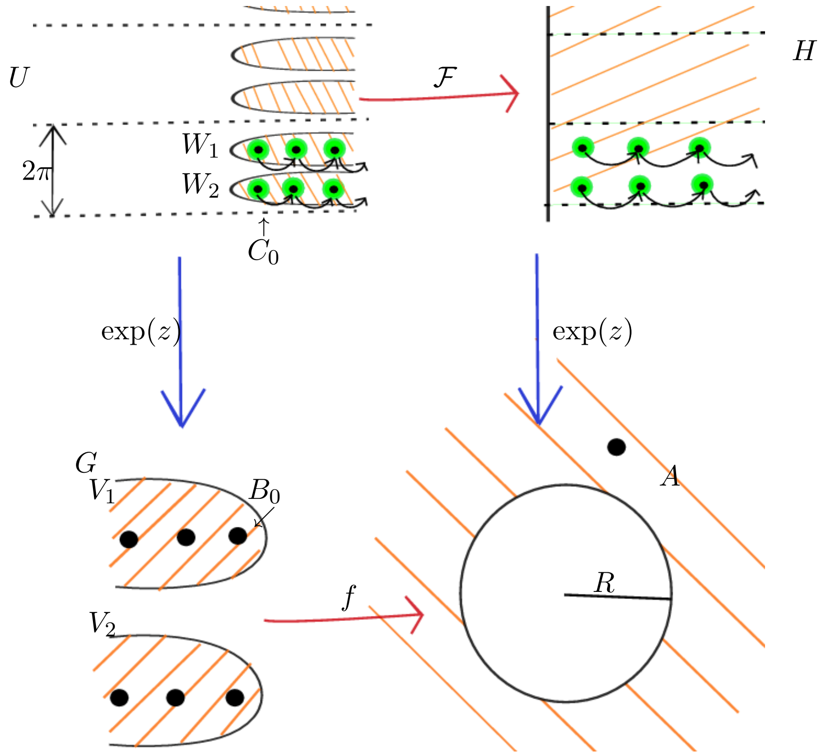


Figure 4.5: Graphic representation of the proof of Theorem 4.3.2.

Corollary 4.3.6. *Let $f \in B$ be a transcendental entire function. Then, $int(I(f))$ is empty.*

Proof. By the theorem of Eremenko and Lyubich, we know that $I(f) \subset J(f)$, since $J(f)$ is the complement of $F(f)$. From Lemma 2.2.8, we already know that the interior of $J(f)$ is either empty or the entire complex plane. If the interior is empty, the result follows immediately.

Suppose instead that $int(J(f)) = \mathbb{C}$. By Lemma 2.2.9, we know that periodic points are dense in $J(f)$, meaning they are not escaping. Therefore, $int(I(f))$ is empty, since any open set $U \subset I(f)$ would intersect $J(f)$ and necessarily contain periodic points, which is a contradiction. □

Conclusions

Finally, I provide an overview of the goals achieved in this project, presented in a more personal and informal tone compared to the discussion in the dissertation.

In this project, I aimed to study the escaping set of polynomials and transcendental entire functions. In order to accomplish this, I first delved deeply into complex dynamics and built a solid foundation in complex analysis. Along the way, I encountered concepts such as normal families, covering maps, holomorphic proper maps, and the Green's function, which are very useful and widely used concepts of complex analysis. While these concepts weren't overly difficult to understand, they were entirely new to me since they weren't covered in my degree program. Additionally, I had the opportunity to learn and program in Python, a highly practical language that turned out to be incredibly useful for visualizing mathematical concepts, making the project even more engaging.

For polynomials, I was able to define their escaping as the basin of attraction of infinity. This connection not only simplified the study of the escaping set but also helped relate it to the Julia and Fatou sets. I also established the importance of the critical points regarding the connectivity of both the basin of attraction of infinity and the Julia set.

On the other hand, the study of the escaping set of transcendental entire functions presented a greater challenge due to the nature of infinity as an essential singularity. This complexity required a deeper exploration of singular values and their influence on the global dynamics of these functions. By using Eremenko's theorem, as well as Bergweiler, Rippon and Stallard's theorem, I was able to prove the non-emptiness of the escaping set for any function within this class using core principles of function theory. Additionally, proving that the interior of the escaping set is empty for all transcendental entire functions whose singular values form a bounded set left me with more questions about the structure of the escaping set under other restrictions or in more general scenarios.

Appendix 1. Code to generate pictures of Julia sets for polynomials

This appendix includes the Python code used to generate the pictures of Julia sets presented in chapter three. The program is specifically designed for the case of polynomials. For each image, the only modification to the code is the definition of $p(z)$ in the fifth line.

```
1    import numpy as np
2    import matplotlib.pyplot as plt
3
4    def p(z): # we'll plot the julia set of this polynomial
5        return z**2+c
6
7    def color(z):
8        R=10
9        n=100
10       for i in range(n):
11           z=p(z)
12           if np.abs(z)>=R:
13               return i
14       return n
15
16   def main():
17       L=2
18       N=1000
19       M=np.zeros((N,N))
20       coord=np.linspace(-L, L, N)
21       for i in range(N):
22           for j in range(N):
23               z=coord[i]+coord[j]*1j
24               M[j,i]=color(z)
25       plt.imshow(M, cmap="viridis", extent=[-L, L, -L, L],
             aspect="equal")
26       plt.show()
27   main()
```

Appendix 2. Code to generate pictures of Julia sets for the exponential family

This appendix includes the Python code used to generate the pictures of Julia sets presented in chapter four. The program is specifically designed for the case of exponential functions. For the function $f(z) = 1 + z + e^{-z}$, an additional condition was included in color(z) function to identify and distinguish escaping points. This condition is included as commented lines in the code below.

```
import numpy as np
import matplotlib.pyplot as plt

def f(z): # we'll plot the julia set of this exponential
    return 1+z+np.exp(-z)

def color(z):
    R=300
    n=150
    for i in range(n):
        z=f(z)
        if np.real(z)>=R:
            return 0
        #elif np.real(z)>10:
            #return 0.5
    return 1

def main():
    L=7.5
    N=1000
    M=np.zeros((N,N))
    coord=np.linspace(-L, L, N)
    for i in range(N):
        for j in range(N):
            z=coord[i]+coord[j]*1j
            M[j,i]=color(z)
    plt.imshow(M, cmap="viridis", extent=[-L, L, -L, L], aspect="equal")
    plt.show()
main()
```

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