Geometrical Optics Study of the Magnifying Glass

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The magnifying glass is probably the simplest visual instrument for the observation of small objects. It consists of a converging lens placed between the object and the eye. The mutual distances between the elements and the accommodation state of the eye have an important role in the final magnification obtained. In the present work, we analyze all the relevant details in terms of the geometrical optics theory. Some interesting results are found and explained.

Statement of the problem

In the direct visual observation of small objects, image enlargement is basically facilitated by the reduction of the object–eye distance. This reduction requires the accommodation of the lens of the eye (by the ciliary muscle), a process naturally limited up to a minimum observation distance. Let us call this minimum distance, i.e., the "near point" for the comfortable examination of a small object of height y_0 by the naked eye, d_{min}, with t being the distance from the lens of the eye to the retina, as in Fig. 1. Note that t is, in fact, different for different eyes, but we will see that this distance t is not relevant for our results.

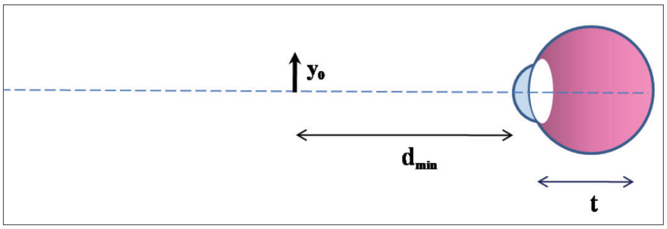

Fig. 1. Observation with the naked eye.

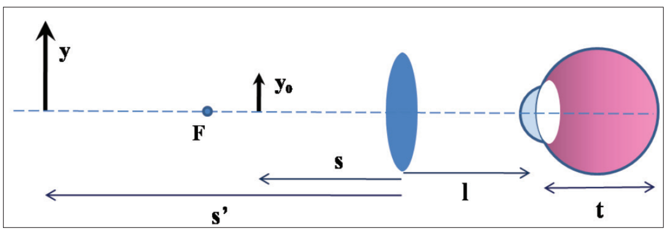

Fig. 2. Observation through a thin lens.

Since d_{min} was the minimum distance for visual observation for the naked eye,

$$y'_{max} = y_0 \frac{t}{d_{min}} \quad (d_{min}, t > 0) \tag{1}$$

is the maximum size of the object on the retina as seen without any visual aid. We will consider an eye having the far point at infinity and the near point at 25 cm to be emmetropic. We label this reference point of 25 cm for an emmetropic eye as d_{ref}.

There is not a universal sign convention in geometrical optics for positions and distances.[1–4] We will use the convention presented in optical engineering books,[2,3] accurately defined in Fig. 2.6 of Ref 2 and Figs. 3.1 and 3.3 of Ref. 3. In short, positions to the left of the lens are negative, and those to the right are positive. Since d_{min} and d_{ref} are reference values, both are taken as positive numbers. We will also consider the lenses to be thin for simplicity.

Thus, the thin-lens equation is written

$$-\frac{1}{s} + \frac{1}{s'} = \frac{1}{f}. \tag{2}$$

If a converging lens (the magnifying glass) of focal length f is placed between the object and the eye, with the object located at s (between the focal object point of the lens F and the lens itself), the layout is as shown in Fig. 2.

The image given by the magnifying glass is at s', with size y. Using Eqs. (1) and (2), we find that the size of the final image on the retina is

$$y' = y\frac{t}{l-s'} = y_0 \frac{s'}{s}\frac{t}{l-s'} = y_0 \frac{f}{f+s}\frac{t}{l-s'}. \tag{3}$$

Note that, besides the signs introduced in Eq. (1), here we have $y_0, y, l, f > 0$ and $s, s' < 0$, according to the usual sign convention mentioned.

Thus, the visual magnification M obtained with the use of the magnifying glass is

$$M = \frac{y_0 \dfrac{f}{f+s}\dfrac{t}{l-s'}}{y_0 \dfrac{t}{d_{min}}} = \frac{d_{min} f}{\left(f + \dfrac{fs'}{f-s'}\right)(l-s')} = \frac{d_{min}(f-s')}{f(l-s')}. \tag{4}$$

This formula compares the maximum size of the image y'_{max} as seen with the naked eye with the size y' when the object is observed through the magnifying glass. Note that when $s' \to -\infty$, i.e., when $s \to -f$, then $M \to d_{min}/f$. Equation (4) is of general validity and can be equally used for $l > f$ or for $l \to 0$. Our purpose is to calculate the magnification attained when using the magnifying glass in several practical cases and, particularly, to find the maximum magnification achievable.

First note that, when using Eq. (4), the image given by the lens (at position s' from the lens) must be farther than d_{min} from the eye to be sharply visible. Thus, the useful positions s where the object can be situated are restricted to those giving (according our previous notation)

$$(l - s') > d_{min}. \tag{5}$$

Second, by computing analytically the partial derivative of Eq. (4) with respect to the distance l, we find that the following inequality holds:

$$\frac{\partial M}{\partial l} = -\frac{d_{min}(f-s')}{f(l-s')^2} < 0. \tag{6}$$

Thus, the optimum observation condition for the greatest magnification is always with $l = 0$.

Third, by computing

$$\frac{\partial M}{\partial s'} = \frac{d_{\min}}{f}\frac{(f-l)}{(l-s')^2}, \quad (7)$$

one immediately sees that

$$\frac{\partial M}{\partial s'} > 0 \Leftrightarrow l < f. \quad (8)$$

Thus, provided $l < f$, the magnification always increases as the distance s' increases, i.e., becomes less negative.

In summary, considering together the implications of Eqs. (5), (6), and (8), the maximum magnification will be obtained with $l = 0$, for the maximum value of s' that allows $-s' > d_{\min} \Rightarrow s' = -d_{\min}$. Note that $l = 0$ is a feasible condition within the assumed thin-lenses approximation. Using this value inside Eq. (4), we see that the maximum magnification is always

$$M_{\max} = \frac{d_{\min}}{f} + 1, \quad (9)$$

obtained when $l = 0$ and the eye is accommodating at its minimum distance $d_{\min}$.

If observation is made accommodating at the minimum distance but with $l \neq 0$, then $d_{\min} = l - s' \Rightarrow s' = l - d_{\min}$, which used in Eq. (4) gives

$$M = \frac{d_{\min}}{f} + 1 - \frac{l}{f} = M_{\max} - \frac{l}{f}, \quad (10)$$

showing a linear decrease of $M_{\max}$ with l, with slope $-1/f$.

Use with an emmetropic eye: The reference case

For standard reference, it is customary to designate several quantities as corresponding to the emmetropic eye. First, one labels the magnification obtained when the emmetropic eye is observing an object placed at the focal object point of a magnifying glass of focal length f as the "conventional magnification" (M_{ref}) (of course, the eye is observing without accommodation). Using Eq. (4), we obtain $M_{\text{ref}} = d_{\text{ref}}/f$. If $d_{\text{ref}}/f = \#$ (e.g., $\# = 3$ or 5, etc.), it is common to designate the magnifying glass as being of type "3X," or "5X," etc.

According to Eq. (9), for the above case the maximum magnification (with $l = 0$) will be $M_{\max} = M_{\text{ref}} + 1$, which is a result included in several references.[1–5] The above explanations precisely show that, indeed, $M_{\max} = M_{\text{ref}} + 1$ is the absolute maximum magnification, accessible only with $l = 0$ and maximum eye accommodation.

Figure 3 summarizes the practical results for an emmetropic eye ($d_{\min} = d_{\text{ref}} = 25$ cm), when using a lens (magnifying glass) with focal length $f = 5$ cm, i.e., with $M_{\text{ref}} = 5$, or 5X magnifier. The x-axis is the position s in centimeters. The left vertical axis plots M (continuous red curve), as given by Eq. (4). Only the object positions s between the front focal point F and the lens are considered, as explained before. The horizontal red lines are M_{ref} (dashed) and $M_{\max} = M_{\text{ref}} + 1$ (dotted), as given by Eq. (9). The final distance $(l - s')$ in terms of s is shown on the right vertical axis (continuous green curve). The horizontal dotted green line is d_{ref}, and the vertical dashed green line is the position s for the object that gives the image y at the distance $l - s' = d_{\text{ref}}$ from the magnifying glass. In fact, we have already taken $l \equiv 0$, as we have mentioned that is always the condition for highest image enlargement.

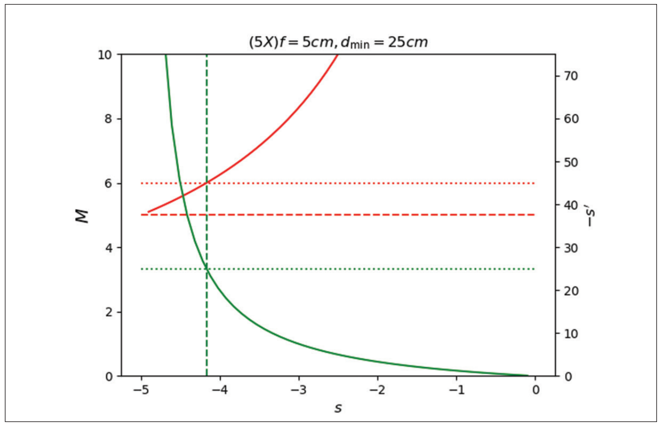

Fig. 3. Magnifications attainable with a 5X lens, for an emmetropic eye.

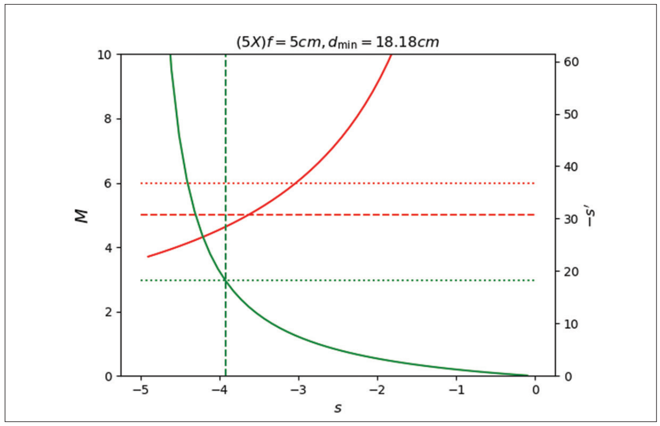

Fig. 4. Magnifications attainable with a 5X lens, for a –1.5 D myopic eye.

The way to take advantage of Fig. 3 is as follows. Given the focal length f of the magnifying glass and the $d_{\min}$ for the observing eye (here $d_{\min} = d_{\text{ref}} = 25$ cm) and restricting ourselves to $l \equiv 0$, we plot $M(s)$ (continuous red curve) using Eq. (4) and also $l - s'(s) = -s'(s)$ (continuous green curve). Since $|s'|$ must be greater than d_{ref} for practical purposes, we plot the horizontal green dotted line corresponding to $|s'| = d_{\text{ref}}$. Next we plot a vertical green dashed line through the crossing point of the continuous and dotted green lines. When this vertical dashed green line intersects the continuous red line, we find the maximum attainable M. The horizontal red lines are only shown for reference. We see that, for the present emmetropic eye, the attainable magnification ranges between $M_{\text{ref}} = 5$ (minimum value, for the object at the focal point F) and $M_{\max} = M_{\text{ref}} + 1 = 6$, for $s = -4.16$ cm (which gives $s' = d_{\min}$).

Other examples: Use with ametropic eyes

Our aim is to illustrate the significance of the above results in unconventional cases: using the magnifying glass with naked eyes, either emmetropic or ametropic. We assume the naked eye condition in what follows because it is more comfortable for the user and, besides, if the user still wears the corrective lens we have in fact a two-lens system in front of

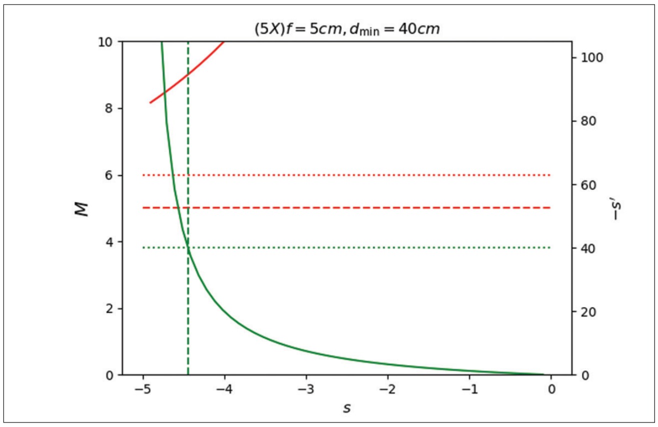

Fig. 5. Magnifications attainable with a 5X lens, for a +1.5 D hyperopic eye.

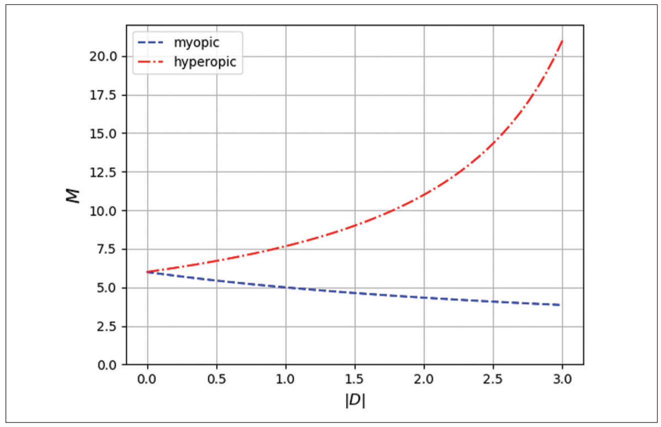

Fig. 6. Maximum magnification attainable with a 5X lens, for –3.0 D to +3.0 D eyes.

the eye. In comparison with the emmetropic eye (far point at infinity and near point at 25 cm), myopic and hyperopic eyes have their near point closer or farther than 25 cm, respectively. Consequently, for ametropic (myopic and hyperopic) eyes, we will assume that the necessary myopia or hyperopia correction simply corresponds to the dioptric power of the thin lens that brings the image of a real object placed at $d_{ref} \equiv 25$ cm to the actual near point d_{min} of the ametropic eye. In addition, we will assume the correcting lens to be in contact with the lens of the eye. This last assumption is very good, except when high dioptric power values are required[4] for eye correction. We will not consider astigmatic eyes.

The prescriptions (spectacle lens correction) for ametropic eyes are given in "diopters" as follows: $D \equiv 1/f$, for a lens of focal length f, expressed in meters. Given the prescription D, we will find the minimum focusing distance (near point) for the naked eye d_{min} as explained just above: the image of an object at d_{ref} is sent to the actual near point d_{min} of the eye by the prescribed lens. Using Eq. (2) with the corresponding signs, the calculation is

$$-\frac{1}{-d_{\text{ref}}}+\frac{1}{-d_{\text{min}}}=D \Rightarrow d_{\text{min}}=\frac{d_{\text{ref}}}{1-d_{\text{ref}}D} \text{ (in meters).} \quad (11)$$

Thus, for a myopic naked eye with prescription –1.5 D (f = –0.66 m), we assume that d_{min} = 0.1818 m = 18.18 cm. For a hyperopic naked eye with +1.5 D (f = +0.66 m) required lens correction, we take d_{min} = 0.4 m = 40.0 cm. Note that hyperopic eyes with a required lens correction above + 4.0 D (f = +0.25 cm) cannot be considered here since they cannot focus real objects with the naked eye.

Figure 4 plots the same quantities as Fig. 3 but for an eye with –1.5 D of myopia. Note that since a myopic eye is able to see at distances closer than 25 cm (in the present case up to d_{min} = 18.18 cm), the final magnification attainable using our 5X magnifying glass is less than before, for the emmetropic eye. In fact, now the maximum is M_{max} = 4.64, for s = –3.92 cm.

Conversely, Fig. 5 shows the results for an eye with +1.5 D of hyperopia. Now, since for this hyperopic eye the minimum distance is 40.0 cm, the final magnification attainable is much higher than for the emmetropic eye. Here this maximum value is M_{max} = 9.00 (for s = –4.44 cm), much more than M_{max} = 6 found for the 5X magnifying glass with the emmetropic eye. In fact the values found for the ametropic eyes above are simply particular cases of using Eq. (9), whose validity is general, as stated. In effect, for the two cases d_{min} = 18.18 and 40.0 cm, with f = 5 cm, Eq. (9) gives M_{max} = 4.64 and 9.00, as indicated.

How the value of M_{max} given by Eq. (9) changes for non-emmetropic eyes is a remarkable point (note that we are assuming the use of the magnifying glass with the naked eye). Figure 6 illustrates the variations of M_{max} for myopic and hyperopic eyes, ranging from –3.0 D to +3.0 D corrective prescription. Clearly, the tendency for hyperopic eyes is quite impressive.

Experiments

Checking the validity of the values of the maximum magnifications presented in Figs. 4 and 5 is easily done by using a camera with the objective focused to d_{min} and comparing the size of a small area on the camera sensor as seen i) directly and ii) with the magnifying glass placed just in front of the camera lens. Figure 7 compares the image of a reticle as given by the camera alone focused to 40 cm (left), with the image taken when a converging lens of +1.5 D (the magnifying glass) is placed in contact with the camera objective (right). The images correspond to the same area of pixels. The camera had a sensor 1/3 in. (1024 × 768 square pixels, each 4.65 μm on a side) and an f = 5 cm focal length objective. By measuring equivalent distances in both images, the magnification found was M = 8.98 ± 0.13, in close agreement with the expected result M = 9.0.

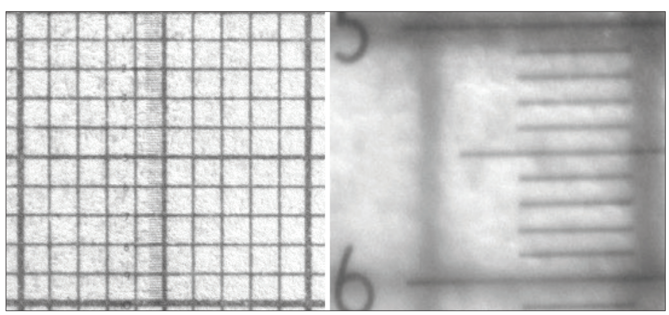

Fig. 7. Experimental results: comparison of images without and with the magnifying glass.

Conclusions

A magnifying glass can be used as a visual aid for the naked eye. The visual magnification is the ratio between sizes at the retina with and without the magnifying glass (with the object placed at the closest comfortable distance) and depends significantly on the relative positions between the object, magnifier lens, and eye. We have deduced the general formula for calculating this visual magnification. With the formula, it is easy to find the best conditions in order to obtain its maximum value. Moreover, the formula also shows that for a fixed conventional magnification $M_{ref} = d_{ref}/f$ of the lens being used, the maximum magnification attainable greatly varies.

Indeed, if l is the distance between magnifying glass and the eye lens:

- The maximum magnification M_{max} is always obtained with $l = 0$.
- With $l = 0$, $M_{max} = \frac{d_{min}}{f} + 1$, with d_{min} being the minimum observation distance with maximum eye accommodation.
- Thus, for emmetropic eyes ($d_{min} = d_{ref} = 25$ cm), we have $M_{max} = M_{ref} + 1$.
- For myopic eyes, M_{max} is always less than for emmetropic ones, decreasing with increasing prescription.
- For hyperopic eyes, M_{max} is always more than for emmetropic ones, increasing quickly with increasing prescription.
- If $l \neq 0$, then M_{max} always decreases linearly with l, with slope $-1/f$.

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