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SMALL PERTURBATIONS IN A HYPERBOLIC STOCHASTIC PARTIAL DIFFERENTIAL EQUATION

by

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1 Introduction

This paper deals with the hyperbolic stochastic partial differential equation

$$\frac{\partial^2 X_{s,t}}{\partial s \partial t} = a_3(X_{s,t})\, \dot{W}_{s,t} + a_4(X_{s,t}) + a_1(s,t)\, \frac{\partial X_{s,t}}{\partial t} + a_2(s,t)\, \frac{\partial X_{s,t}}{\partial s}, \quad (1.1)$$

with initial condition $X_{s,t} = X_0$ if $(s,t) \in T,\ s \cdot t = 0,\ T = [0,1]^2$. The coefficients are measurable functions $a_i : T \longrightarrow \mathbb{R},\ i = 1,2,\ a_i : \mathbb{R} \longrightarrow \mathbb{R}$, $i = 3,4$, $\{\dot{W}_{s,t}, (s,t) \in T\}$ is a white noise on T and X_0 a $\mathcal{F}_{0,0}$-measurable random variable, where $\mathcal{F}_{s,t},\ (s,t) \in T$, is the completion of $\sigma\{W_{u,v},\ (u,v) \le (s,t)\}$.
A solution of (1.1) is a process $X = \{X_{s,t},\ (s,t) \in T\}$, adapted to the filtration $\{\mathcal{F}_{s,t},\ (s,t) \in T\}$, satisfying

$$X_{s,t} = X_0 + \int_{R_{s,t}} \gamma_{s,t}(u,v)\, \Big(a_3(X_{u,v}) dW_{u,v} + a_4(X_{u,v}) du\, dv\Big), \qquad (1.2)$$

where $R_{s,t}$ denotes the rectangle $[0,s] \times [0,t]$ and $\gamma_{s,t}(u,v)$ is the Green function associated with the second order differential operator

$$\mathcal{L}f(s,t) = \frac{\partial^2 f(s,t)}{\partial s \partial t} - a_1(s,t)\, \frac{\partial f(s,t)}{\partial t} - a_2(s,t)\, \frac{\partial f(s,t)}{\partial s}\ .$$

These kind of equations appear in the construction of a Wiener sheet on manifolds (see [8]). We refer the reader to [11] for the details concerning existence and uniqueness of solution for (1.2).
The first problem we analyze in this work is the existence of density $p_z(y)$ for the law of $X_z,\ z \in T \backslash E$, where E denotes the set $\{(s,t) \in T : s \cdot t = 0\}$. We study the properties of $p_z(y)$ as a function of y and z, separately.
This is a continuation of the programme developed in [11], where we have proved the existence of $p_z(y)$ and that $y \longmapsto p_z(y)$ is $\mathcal{C}^\infty$ for any $z \in T\backslash E$. In the above mentioned reference the coefficients $a_i,\ i = 3,4$, are allowed to depend also on (s,t) but, as a counterpart, we impose a non-degeneracy condition of Hörmander's type involving only derivatives of a_3 up to the first order. Here we state this result under more general conditions, actually, the so called Hörmander's condition in the case of diffusions (see (H3) in Section 2).
In a second step we show that $z \longrightarrow p_z(y)$ is Lipschitz continuous for any fixed $y \in \mathbb{R}$. The ideas used to study this problem have been recently developed in [7], where a similar problem for the probability law of the

solution to a parabolic stochastic partial differential equation is studied. The idea consists in proving the estimate

$$\left|E\Big(f(X_z) - f(X_{z'})\Big)\right| \leq C\|F\|_\infty \, |z - z'|, \tag{1.3}$$

for any smooth function f and every $z, z' \in T$, where F denotes the primitive of f.
The stochastic integral in (1.2) does not possess the martingale property. Hence it is not possible to apply an Itô formula to get an expression for the left hand-side of (1.3). However, a Taylor expansion and the integration by parts formula of Malliavin Calculus do the job.
Consider small perturbations of the noise W given by a parameter $\varepsilon \in (0,1]$. The evolution equation driven by this noise is given by

$$X^\varepsilon_{s,t} = X_0 + \int_{R_{s,t}} \gamma_{s,t}(u,v)\Big(\varepsilon\, a_3(X^\varepsilon_{u,v})\, dW_{u,v} + a_4(X^\varepsilon_{u,v}) du\, dv\Big). \tag{1.4}$$

Let $p^\varepsilon_z(y)$ be the density of X^ε_z for $z \in T \backslash E$. As $\varepsilon \to 0$, the solution of (1.4) tends to a deterministic function. So, we expect $p^\varepsilon_z(y)$ to converge to a degenerate density. The second question we study in this paper is the rate of that convergence. Suppose $x := X_0$ is deterministic; then, under general hypoelliptic conditions

$$\lim_{\varepsilon \downarrow 0} \varepsilon^2 \, \log \, p^\varepsilon_z(y) = C\,(x,y). \tag{1.5}$$

The quantity $C(x,y)$ in (1.5) is described in terms of the skeleton of (1.2), a notion which is relevant for the characterization of the topological support for the law of $X = \{X_z, \; z \in T\}$ and to establish a large deviation principle for $X^\varepsilon = \{X^\varepsilon_z, \; z \in T\}$ (see [11], [12]).

Let $\mathcal{H}$ be the Cameron-Martin space associated with the Brownian sheet $W = \{W_{s,t}, \; (s,t) \in T\}$. That means, $\mathcal{H}$ is the set of functions $h : T \longrightarrow \Re$ such that $\|h\|_{\mathcal{H}} := \left(\int_T \left(\frac{\partial^2 h}{\partial s \partial t}\,(s,t)\right)^2 ds\, dt\right)^{1/2} < \infty$. The skeleton of X, with deterministic initial condition x, is the solution to the evolution equation

$$S^h_{s,t} = x + \int_{R_{s,t}} \gamma_{s,t}(u,v)\Big(a_3(S^h_{u,v})\; \dot{h}_{u,v} + a_4(S^h_{u,v})\Big)\, du\, dv, \;\; h \in \mathcal{H}, \tag{1.6}$$

where $\dot{h}_{u,v}$ denotes the derivative $\frac{\partial^2 h}{\partial u \partial v}\,(u,v)$.
Let

$$d^2(x,y) = \inf \{ \, \|h\|^2_{\mathcal{H}} : S^h_z = y \}. \tag{1.7}$$

Then, the limit in (1.5) is

$$C(x,y) = -\frac{1}{2}\, d^2(x,y).$$

For diffussions, (1.5) is the classical Varadhan estimate. Léandre and Russo, in a series of papers, have developed a method, combining large deviations estimates and Malliavin Calculus, which allows to analyze this problem for several examples of stochastic partial differential equations (see, for instance, [1], [3], [4], [6]). In all these examples a non-degeneracy condition of elliptic type is required. We use here this method, but the novelty is that hypoelliptic conditions can also be allowed. This rests on the fact that we are able to check a regularity condition on the skeleton, that is, $\langle D\,S_z^h, D\,S_z^h\rangle_{\mathcal{H}} > 0$ (see Lemma 3.7).
Finally, we study the finiteness of $d^2(x,\,.)$. This question is related to the characterization of the support for the law of X_z, stated in [12].
Here is the index of the paper. Section 2 contains the results concerning existence and properties of the density, section 3 those related with the asymptotic behaviour of p_z^ε. The paper closes with a very short appendix, quoting the properties on the Green function $\gamma_z(\eta)$ used along the paper. As usually, all constants will be denoted by C independently of its value. For all questions concerning Analysis on the Wiener space (notions and notation) we refer the reader to [9].

2 Properties of the density

In this section we assume the following conditions on the coefficients:

(H1) $a_i : T \longrightarrow \mathbb{R}$, $i = 1,2$ are bounded, differentiable, with bounded derivatives,

(H2) $a_i : \mathbb{R} \longrightarrow \mathbb{R}$, $i = 3,4$ are infinitely differentiable with bounded derivatives of any order,

(H3) either $a_3(X_0) \neq 0$ a.s. or there exists an integer $n_0 \geq 1$ such that $a_3^{(n)}(X_0) = 0$ a.s., for any $0 \leq n \leq n_0 - 1$ and $(a_3^{(n_0)} a_4)(X_0) \neq 0$ a.s., where, for $n = 0$, $a_3^{(n)} = a_3$.

Our purpose is to prove the following result

Theorem 2.1 *Under* (H1), (H2) *and* (H3), *for any* $z \in T\backslash E$, *the law of* X_z *solution to* (1.2) *is absolutely continuous with respect to Lebesgue's measure on* $\mathbb{R}$ *and the density,* p_z, *is infinitely differentiable. In addition, for any* $y \in \mathbb{R}$, *the mapping*

$$\begin{array}{ccc} T\backslash E & \longrightarrow & \mathbb{R} \\ z & \longmapsto & p_z(y) \end{array}$$

is Lipschitz.

Remark 2.2 For coefficients a_i, $i = 3, 4$, depending also on time, the existence of a $\mathcal{C}^\infty$ density $p_z(\cdot)$ has been proved in [11] (see Proposition 3.7), under non-degeneracy conditions of type (H3) involving only derivatives of first order. In particular, assuming either $a_3(X_0) \neq 0$ or $a_3'(X_0) = 0$ and $(a_3' a_4)(X_0) \neq 0$ a.s.

The proof of theorem 2.1 needs the property stated in the next Proposition.

Proposition 2.3 *For any* $z = (s,t) \in T$, $h \in (0, 1-s]$, $v \in [0,1]$, *let*

$$Z_{s,t}(h,v) = X_{s,t} + v(X_{s+h,t} - X_{s,t}).$$

Then, if (H1), (H2) *and* (H3) *are satisfied,*

$$\left(\int_T |D_\eta Z_z(h,v)|^2 d\eta\right)^{-1} \in \bigcap_{p\geq 1} L^p(\Omega).$$

Proof. We will prove the following equivalent property: for any $p \in [1, \infty)$ there exists ε_0 such that

$$P\left\{\int_T |D_\eta Z_z(h,v)|^2 d\eta < \varepsilon\right\} \leq \varepsilon^p, \tag{2.1}$$

for any $\varepsilon \leq \varepsilon_0$.
Let $\{Y_z(\eta),\ 0 \leq \eta \leq z \leq (1,1)\}$ be the solution to

$$Y_z(\eta) = \gamma_z(\eta) + \int_{(\eta,z]} \gamma_z(\alpha) Y_\alpha(\eta)\,[a_3'(X_\alpha)dW_\alpha + a_4'(X_\alpha)d\alpha].$$

Then $D_\eta X_z = a_3(X_\eta) Y_z(\eta) 1_{[0,z]}(\eta)$. Set $Y_{s,t}^{h,v}(\eta) = Y_{s,t}(\eta) + v\Big(Y_{s+h,t}(\eta) - Y_{s,t}(\eta)\Big)$, and $\gamma_{s,t}^{h,v}(\eta) = \gamma_{s,t}(\eta) + v\Big(\gamma_{s+h,t}(\eta) - \gamma_{s,t}(\eta)\Big)$.

We first study the case $a_3(X_0) \neq 0$. Fix $\varepsilon, \beta, \delta \in (0,1)$ and define $C^z_{\beta,\delta}(\varepsilon) = (0, \varepsilon^\beta) \times (t - \varepsilon^\delta, t)$ then

$$P\left\{\int_T |D_\eta Z_{s,t}(h,v)|^2 d\eta \leq \varepsilon\right\} \leq q_1(\varepsilon,\beta) + q_2(\varepsilon,\beta),$$

with

$$\begin{aligned} q_1\Big(\varepsilon,\beta\Big) &= P\left\{\int_{C^z_{\beta,\beta}(\varepsilon)} \Big(a_3(X_\eta)\ Y^{h,v}_{s,t}(\eta) - a_3(X_0)\gamma^{h,v}_{s,t}(\eta_1,t)\Big|^2 d\eta > \varepsilon\right\}, \\ q_2(\varepsilon,\beta) &= P\left\{\int_{C^z_{\beta,\beta}(\varepsilon)} a_3^2(X_0)\ \gamma^{h,v}_{s,t}(\eta_1,t)^2 d\eta \leq 4\varepsilon\right\}, \end{aligned}$$

$\eta = (\eta_1, \eta_2)$.
In [11] (see (3.9) and (3.10)) we have proved, for $p \in [1,\infty)$.

$$\sup_{\eta \in C^z_{\beta,\delta}(\varepsilon)} E(|X_\eta - X_0|^{2p}) \leq C\,\varepsilon^{\beta p}, \tag{2.2}$$

$$\sup_{\eta \in C^z_{\beta,\delta}} E(|Y_z(\eta) - \gamma_z(\eta)|^{2p}) \leq C\,\varepsilon^{\delta p}.$$

Thus,

$$\sup_{\eta \in C^z_{\beta,\delta}(\varepsilon)} E(|Y^{h,v}_{s,t}(\eta) - \gamma^{h,v}_{s,t}(\eta)|^{2p}) \leq C\,\varepsilon^{\delta p}, \quad p \in [1,\infty). \tag{2.3}$$

Using the triangle inequality, properties (2.2), (2.3) and the Lipschitz property for $\gamma_z(\cdot)$, one easily obtains

$$\sup_{\eta \in C^z_{\beta,\beta}(\varepsilon)} E\,|a_3(X_\eta)Y^{h,v}_{s,t}(\eta) - a_3(X_0)\,\gamma^{h,v}_{s,t}(\eta_1,t)\,|^{2q} \leq C\,\varepsilon^{\beta q}, \quad q \in [1,\infty).$$

Then, by Chebychev's inequality

$$q_1(\varepsilon,\beta) \leq C\,\varepsilon^{q(3\beta-1)}, \quad \text{for any } q \in [1,\infty).$$

Since $a_3(X_0) \neq 0$, (4.1) ensures

$$\int_{C^z_{\beta,\beta}(\varepsilon)} a_3^2(X_0)\,\gamma^{h,v}_{s,t}(\eta_1,t)^2\,d\eta \geq C\,\varepsilon^{2\beta}.$$

Choose $\beta \in (\frac{1}{3}, \frac{1}{2})$. Then, $q_2(\varepsilon,\beta) = 0$ and (2.1) follows.

Assume now there exists $n_0 \geq 1$ with $a_3^{(n)}(X_0) = 0$ for $0 \leq n \leq n_0 - 1$ and $(a_3^{(n_0)} a_4)(X_0) \neq 0$ a.s., where for $n = 0$, $a_3^{(n)} = a_3$. Then,

$$P\left\{\int_T |D_\eta Z_{s,t}(h,v)|^2 d\eta < \varepsilon\right\} \leq q_1(\varepsilon,\beta,\delta) + q_2(\varepsilon,\beta,\delta)$$

with

$$q_1(\varepsilon,\beta,\delta) = P\left\{\int_{C^z_{\beta,\delta}(\varepsilon)} \left(a_3(X_\eta)\left(Y^{h,v}_{s,t}(\eta) - \gamma^{h,v}_{s,t}(\eta_1,t)\right)\right)^2 d\eta > \varepsilon\right\},$$
$$q_2(\varepsilon,\beta,\delta) = P\left\{\int_{C^z_{\beta,\delta}(\varepsilon)} \left(a_3(X_\eta)\,\gamma^{h,v}_{s,t}(\eta_1,t)\right)^2 d\eta \leq 4\,\varepsilon\right\}.$$

Using (2.3), Chebychev's inequality and the Lipchitz property for $\gamma_z(\cdot)$, one easily obtain,

$$q_1(\varepsilon,\beta,\delta) \leq C\ \varepsilon^{(2\delta+\beta-1)q}, \quad q \in [1,\infty). \tag{2.4}$$

We have

$$q_2(\varepsilon,\beta,\delta) \leq q_{21}(\varepsilon,\beta,\delta) + q_{22}(\varepsilon,\beta,\delta),$$

with

$$\begin{aligned} q_{21}(\varepsilon,\beta,\delta) \leq P\Bigg\{\int_{C^z_{\beta,\delta}(\varepsilon)} \Big(\gamma^{h,v}_{s,t}(\eta_1,t)\Big(a_3(X_\eta) \\ - \frac{1}{n_0!}\,a_3^{(n_0)}(X_0)\,(X_\eta - X_0)^{n_0}\Big)\Big)^2 d\eta > 4\,\varepsilon\Bigg\}, \end{aligned}$$

$$q_{22}(\varepsilon,\beta,\delta) = P\left\{\int_{C^z_{\beta,\delta}(\varepsilon)} \left(\gamma^{h,v}_{s,t}(\eta_1,t)\,\frac{1}{n_0!}\,a_3^{(n_0)}(X_0)\,(X_\eta - X_0)^{n_0}\right)^2 d\eta \leq 16\,\varepsilon\right\}.$$

In the sequel the following inequality will be needed:

$$\sup_{\substack{0\leq\eta_1\leq\varepsilon^\beta \\ 0\leq\eta_2\leq t}} E(|X_\eta - X_0|^p) \leq C\,\varepsilon^{\beta p} \quad p \in [2,\infty). \tag{2.5}$$

The proof of (2.5) uses the property $a_3(X_0) = 0$ and the usual procedure based on Burkholder's, Hölder's and Gronwall's inequalities.
A Taylor expansion for $a_3(X_\eta)$, Chebychev's inequality and (2.5) yield

$$\begin{aligned} q_{21}(\varepsilon,\beta,\delta) &\leq C\,\varepsilon^{(\beta+\delta-1)q} \sup_{\eta\in C^z_{\beta,\delta}(\varepsilon)} E\left(|X_\eta - X_0|^{2q(n_0+1)}\right) \\ &\leq C\,\varepsilon^{(\beta(2n_0+3)+\delta-1)q}. \end{aligned} \tag{2.6}$$

We further decompose $q_{22}(\varepsilon,\beta,\delta)$ as follows,

$$q_{22}(\varepsilon,\beta,\delta) \le q_{221}(\varepsilon,\beta,\delta) + q_{222}(\varepsilon,\beta,\delta),$$

where

$$q_{221}(\varepsilon,\beta,\delta) \le P\left\{ \int_{C^z_{\beta,\delta}(\varepsilon)} \gamma^{h,v}_{s,t}(\eta_1,t)^2 \, \frac{a_3^{(n_0)}(X_0)^2}{(n_0!)^2} \left((X_\eta - X_0)^{n_0} - \left(\int_{R_\eta} \gamma_{0,\eta_2}(0,\xi_2) a_4(X_0) d\xi \right)^{n_0} \right)^2 d\eta > 16\,\varepsilon \right\},$$

$$q_{222}(\varepsilon,\beta,\delta) =$$

$$P\left\{ \int_{C^z_{\beta,\delta}(\varepsilon)} \gamma^{h,v}_{s,t}(\eta_1,t)^2 \, \frac{a_3^{(n_0)}(X_0)^2}{(n_0!)^2} \left(\int_{R_\eta} \gamma_{0,\eta_2}(0,\xi_2) a_4(X_0) d\xi \right)^{2n_0} d\eta \le 64\,\varepsilon \right\}.$$

We examine $q_{221}(\varepsilon,\beta,\delta)$. Assume $n_0 \ge 2$. By the mean value theorem

$$\begin{aligned}
&(X_\eta - X_0)^{n_0} - \Big(\int_{R_\eta} \gamma_{0,\eta_2}(0,\xi_2)\, a_4(X_0) d\xi \Big)^{n_0} \\
&\quad = n_0 \left\{ \lambda(X_\eta - X_0) + (1-\lambda) \int_{R_\eta} \gamma_{0,\eta_2}(0,\xi_2)\, a_4(X_0) d\xi \right\}^{n_0-1} \\
&\qquad \times \Big(X_\eta - X_0 - \int_{R_\eta} \gamma_{0,\eta_2}(0,\xi_2)\, a_4(X_0)\, d\xi \Big), \quad \text{a.s.}
\end{aligned}$$

for some $\lambda \in (0,1)$ depending on ω.
Set

$$A_1(\eta) = \left(E\Big(\Big| \lambda(X_\eta - X_0) + (1-\lambda) \int_{R_\eta} \gamma_{0,\eta_2}(0,\xi_2)\, a_4(X_0) d\xi \Big|^{4q(n_0-1)} \Big) \right)^{1/2},$$

$$A_2(\eta) = \left(E\Big(\Big| X_\eta - X_0 - \int_{R_\eta} \gamma_{0,\eta_2}(0,\xi_2)\, a_4(X_0) d\xi \Big|^{4q} \Big) \right)^{1/2}.$$

Then, Chebychev's and Schwarz's inequalities yield

$$q_{221}(\varepsilon,\beta,\delta) \le C\,\varepsilon^{q(\beta+\delta-1)} \sup_{\eta \in C^z_{\beta,\delta}(\varepsilon)} \Big(A_1(\eta)\, A_2(\eta) \Big). \tag{2.7}$$

We next prove

$$\sup_{\eta \in C^z_{\beta,\delta}(\varepsilon)} A_1(\eta) \le C\,\varepsilon^{2(n_0-1)\beta q}, \tag{2.8}$$

$$\sup_{\eta \in C^z_{\beta,\delta}(\varepsilon)} A_2(\eta) \le C\,\varepsilon^{3\beta q}. \tag{2.9}$$

Indeed,

$$E\Big(\Big|\int_{R_\eta} \gamma_{0,\eta_2}(0,\xi_2)\, a_4(X_0)\, d\xi\Big|^p\Big) \le C\,\varepsilon^{\beta p}, \quad p\in[1,\infty).$$

This inequality, together with (2.5), yields (2.8). In order to prove (2.9) we notice that

$$A_2^2(\eta) \le C\Big(a_{21}(\eta) + a_{22}(\eta) + a_{23}(\eta)\Big),$$

where

$$\begin{aligned} a_{21}(\eta) &= E\Big(\Big|\int_{R_\eta} \gamma_\eta(\xi)\, a_3(X_\xi)\, dW_\xi\Big|^{4q}\Big),\\ a_{22}(\eta) &= E\Big(\Big|\int_{R_\eta} \gamma_\eta(\xi)\Big(a_4(X_\xi) - a_4(X_0)\Big)\, d\xi\Big|^{4q}\Big),\\ a_{23}(\eta) &= E\Big(\Big|\int_{R_\eta} \Big(\gamma_\eta(\xi) - \gamma_{0,\eta_2}(0,\xi_2)\Big)\, a_4(X_0)\, d\xi\Big|^{4q}\Big). \end{aligned}$$

Fix $\eta \in C^z_{\beta,\delta}(\varepsilon)$. Using $a_3(X_0) = 0$ and (2.5), we obtain

$$\sup_{\eta\in C^z_{\beta,\delta}(\varepsilon)} \Big(a_{21}(\eta) + a_{22}(\eta)\Big) \le C\,\varepsilon^{6\beta q}.$$

Moreover, the Lipschitz property of $\gamma_z(\eta)$ implies

$$\sup_{\eta\in C^z_{\beta,\delta}(\varepsilon)} a_{23}(\eta) \le C\,\varepsilon^{8\beta q}.$$

Consequently (2.9) is completely proved.
The inequalities (2.7), (2.8) and (2.9) give

$$q_{221}(\varepsilon,\beta,\delta) \le C\,\varepsilon^{((2n_0+2)\beta+\delta-1)q}. \tag{2.10}$$

For $n_0 = 1$, we also obtain (2.10). Indeed, this follows from (2.9).
The study of the term $q_{222}(\varepsilon,\beta,\delta)$ is based on the positivity property of the function γ given in (4.1). Indeed, since $(a_3^{(n_0)} a_4)(X_0) \neq 0$, (4.1) ensures

$$\begin{aligned} &\Bigg(\int_{C^z_{\beta,\delta}(\varepsilon)} \gamma^{h,v}_{s,t}(\eta_1,t)^2 \Big(\int_{R_\eta} \gamma_{0,\eta_2}(0,\xi_2) d\xi\Big)^{2n_0} d\eta\Bigg) \left(\frac{a_3^{(n_0)}(X_0)\, a_4(X_0)^{n_0}}{n_0!}\right)^2\\ &\ge C \int_0^{\varepsilon^\beta} \int_{t-\varepsilon^\delta}^t (\eta_1\,\eta_2)^{2n_0}\, d\eta_1\, d\eta_2 \ge C(t-\varepsilon^\delta)^{2n_0}\; \varepsilon^\delta\; \varepsilon^{(2n_0+1)\beta}\\ &\ge C\; \varepsilon^{\delta+(2n_0+1)\beta}. \end{aligned}$$

Choose $\beta, \delta > 0$ satisfying

$$1 - \delta - (2n_0 + 1)\beta > 0, \tag{2.11}$$

then $q_{222}(\varepsilon, \beta, \delta) = 0$.
Let $\delta = \frac{1}{2}$, $\beta = \frac{1}{4n_0+3}$. It is easy to check that this choice fulfils (2.11) as well as the following restrictions: $2\delta + \beta - 1 > 0$, $\beta(2n_0 + 2) + \delta - 1 > 0$. Hence, the estimate (2.1) follows from (2.4), (2.6) and (2.10). □

We next quote a lemma from [7] needed in the proof of Theorem 2.1. It follows from the duality between the Malliavin derivative and the Skorohod integral operator δ. Given a random variable X in $\mathbb{D}^{1,2}$, set $\|DX\|^2 = \langle DX, DX \rangle$.

Lemma 2.4 *Let $Z, \xi \in \mathbb{D}^\infty$ satisfying $\|D\xi\|^{-1} \in \bigcap_{p\geq 1} L^p(\Omega)$. Define $H_0(Z, \xi) = Z$ and*

$$H_{n+1}(Z, \xi) = \delta\left(H_n(Z, \xi)\, \frac{D\xi}{\|D\xi\|^2}\right), \quad n \geq 0.$$

Then, for any integers $n, q \geq 1$ and $p \in (1, \infty)$,

$$\|H_n(Z, \xi)\|_{q,p} \leq C\,(\|Z\|_{q+n,4^n p}),$$

where C is a constant depending on the following norms:

$$\|Z\|_{q+2,4^n p}, \ \|\xi\|_{q+2,4^n p} \text{ and } E[(\|D\xi\|^2)^{-1}]^{k(n)}, \text{ with } k(n) \in \mathbb{N}.$$

Proof of Theorem 2.1 Fix $z = (s, t) \in T \backslash E$; Proposition 3.5 of [11] establishes $X_z \in \mathbb{D}^\infty$. In addition, Proposition 2.3 yields, for $v = 0$, $(\int_T |D_\eta X_z|^2 d\eta)^{-1} \in \bigcap_{p\geq 1} L^p(\Omega)$. These two properties ensure the existence of a smooth density for the law of X_z.
Let f be a smooth real function. Denote by F its primitive. We next show, for any $h \in (0, 1 - s)$,

$$\left|E\Big(f(X_{s+h,t}) - f(X_{s,t})\Big)\right| \leq C\, \|F\|_\infty h. \tag{2.12}$$

Indeed, consider the Taylor expansion up to the second order,

$$E\Big(f(X_{s+h,t}) - f(X_{s,t})\Big) = A_1 + A_2, \tag{2.13}$$

with

$$A_1 = E\Big(f'(X_{s,t})(X_{s+h,t} - X_{s,t})\Big),$$
$$A_2 = E\Big((X_{s+h,t} - X_{s,t})^2 \int_0^1 (1-v)\ f''(Z_{s,t}(h,v))\,dv\Big).$$

We decompose A_1 as follows:

$$A_1 = \sum_{i=1}^{3} A_{1i},$$

where

$$A_{11} = E\left(f'(X_{s,t}) \int_{R_{s,t}} (\gamma_{s+h,t}(\eta) - \gamma_{s,t}(\eta))\, a_3(X_\eta)\, dW_\eta\right),$$
$$A_{12} = E\left(f'(X_{s,t})\Big(\int_{R_{s,t}} (\gamma_{s+h,t}(\eta) - \gamma_{s,t}(\eta))\, a_4(X_\eta)\, d\eta + \int_s^{s+h}\int_0^t \gamma_{s+h,t}(\eta)\, a_4(X_\eta)\, d\eta\Big)\right),$$
$$A_{13} = E\left(f'(X_{s,t}) \int_s^{s+h}\int_0^t \gamma_{s+h,t}(\eta)\, a_3(X_\eta)\, dW_\eta\right).$$

Let

$$K^{(1)}_{s,t}(h) = \int_{R_{s,t}} \Big(\gamma_{s+h,t}(\eta) - \gamma_{s,t}(\eta)\Big)\, a_3(X_\eta)\, dW_\eta.$$

The integration by parts applied repeatedly implies

$$A_{11} = E\Big(F(X_{s,t})\ H_2(K^{(1)}_{s,t}(h),\ X_{s,t})\Big) \le \|F\|_\infty\ E\Big(\,|H_2(K^{(1)}_{s,t}(h), X_{s,t})|\,\Big).$$

The random variables $Z := K^{(1)}_{s,t}(h)$, $\xi := X_{s,t}$ satisfy the assumptions of Lemma 2.4, by Proposition 2.3. Consequently,

$$|A_{11}| \le C\ \|F\|_\infty\ \|K^{(1)}_{s,t}(h)\|_{2,16}.$$

Moreover, the Lipschitz property of $\gamma_\cdot(\eta)$ implies

$$\|K^{(1)}_{s,t}(h)\|_{n,p} \le C\,h.$$

Thus,

$$|A_{11}| \le C\ \|F\|_\infty\, h.$$

Let

$$K_{s,t}^{(2)}(h) = \int_{R_{s,t}} \Big(\gamma_{s+h,t}(\eta) - \gamma_{s,t}(\eta)\Big)\, a_4(X_\eta)\, d\eta + \int_s^{s+h} \int_0^t \gamma_{s+h,t}(\eta)\, a_4(X_\eta)\, d\eta.$$

As before,

$$A_{12} = E\Big(F(X_{s,t})\, H_2(K_{s,t}^{(2)}(h), X_{s,t})\Big)$$

and Lemma 2.4 applied to $Z := K_{s,t}^{(2)}(h),\ \xi := X_{s,t}$ yields

$$|A_{12}| \le C\|F\|_\infty\ \|K_{s,t}^{(2)}(h)\|_{2,16} \le C\ \|F\|_\infty\ h.$$

Finally, $f'(X_{s,t})$ and $\int_s^{s+h} \int_0^t \gamma_{s+h,t}(\eta)\ a_3(X_\eta)\, dW_\eta$ are independent random variables. Thus $A_{13} = 0$. This shows

$$|A_1| \le C\ \|F\|_\infty\, h. \tag{2.14}$$

We now study A_2. We have

$$E\Big((X_{s+h,t} - X_{s,t})^2\ f''(Z_{s,t}(h,v))\Big) = E\Big(F(Z_{s,t}(h,v))\ H_3(Z,\xi)\Big)$$

with

$$Z = (X_{s+h,t} - X_{s,t})^2, \quad \xi = Z_{s,t}(h,v).$$

Since the assumptions of Lemma 2.4 are satisfied,

$$|A_2| \le C\ \|F\|_\infty\ \|(X_{s+h,t} - X_{s,t})^2\|_{3,64}.$$

In addition

$$\|(X_{s+h,t} - X_{s,t})^2\|_{n,p} \le C\, h.$$

Indeed, for $n = 0$ see, for instance, Proposition 2.5 in [12]. For $n \ge 1$ this can be checked taking into account that $X_z \in \mathbb{D}^\infty$, for any $z \in T$.
Hence

$$|A_2| \le C\ \|F\|_\infty\, h. \tag{2.15}$$

Thus, the estimate (2.12) follows from (2.13), (2.14) and (2.15). Analogously, for any $h \in (0, 1-t)$

$$|E(f(X_{s,t+h}) - f(X_{s,t}))| \le C\ \|F\|_\infty h. \tag{2.16}$$

Fix $y \in \mathbb{R}$ and consider the delta Dirac function $\delta_{\{y\}}$. Let $\{f_n,\ n \ge 1\}$ be a sequence of smooth real functions converging to $\delta_{\{y\}}$. By passing to the limit the estimates (2.12), (2.16) for f_n, as $n \to \infty$, we obtain the Lipschitz property for the mapping $z \longrightarrow p_z(y)$. This finishes the proof of Theorem 2.1. □

3 Asymptotic behaviour of the density

Consider the process $\{X_z^\varepsilon,\ z \in T\}$, $\varepsilon > 0$, obtained from (1.2) by perturbation of the noise, with deterministic initial condition x. That means, $\{X_z^\varepsilon,\ z \in T\}$ is the solution to the equation

$$X_z^\varepsilon = x + \int_{R_z} \gamma_z(\eta) \left(\varepsilon\, a_3(X_\eta^\varepsilon)\, dW_\eta + a_4(X_\eta^\varepsilon)\, d\eta\right), \quad \varepsilon > 0. \tag{3.1}$$

We assume that assumptions (H1) to (H3) of section 2 are satisfied and, therefore, for any $z \in T \backslash E$, the probability law of X_z^ε has a density p_z^ε, for every $\varepsilon > 0$.
The purpose of this section is to prove the following result.

Theorem 3.1 *Under* (H1), (H2) *and* (H3),

$$\lim_{\varepsilon \downarrow 0} \varepsilon^2 \log p_z^\varepsilon(y) = -\frac{1}{2}\, d^2(x, y),$$

with $d^2(x,y)$ given by (1.7).

In the proof of this theorem we will use the formulation of Léandre's and Russo's method presented in [10] for general Wiener functionals. For the sake of completeness we quote these ingredients.
A random vector $F : \Omega \longrightarrow \mathbb{R}^m$ is said to be nondegenerate if $F \in \mathbb{D}^\infty$ and the Malliavin matrix $\Gamma_F = \langle DF, DF\rangle$ satisfies

$$\det\, \Gamma_F^{-1} \in \bigcap_{p \geq 1} L^p(\Omega).$$

Proposition 3.2 (Proposition 4.4.2 [10]) *Let $\{F^\varepsilon,\ \varepsilon \in (0,1]\}$ be a family of nondegenerate random vectors satisfying*

(a) $\sup_{\varepsilon \in (0,1]} \|F^\varepsilon\|_{k,p} < \infty$, *for each integer $k \geq 1$ and any $p \in (1, \infty)$.*

(b) For any $p \in [1, \infty)$, there exists $N(p) \in [1, \infty)$ such that $\|\Gamma_{F^\varepsilon}^{-1}\|_p \leq \varepsilon^{-N(p)}$.

(c) $\{F^\varepsilon,\ \varepsilon \in (0,1]\}$ obeys a large deviation principle on $\mathbb{R}^m$ with rate function $I(y)$, $y \in \mathbb{R}^m$.

Then, if p^ε denotes the density of F^ε,

$$\limsup_{\varepsilon \downarrow 0} \varepsilon^2 \log p^\varepsilon(y) \leq -I(y).$$

Consider the particular case $\mathcal{H} = L^2(A, \mathcal{A}, \mu)$, for some atomless measure space. Fix a random vector $F : \Omega \to \mathbb{R}^m$ in $L^2(\Omega)$ with Wiener chaos decomposition $F = E(F) + \sum_{n=1}^{\infty} I_n(f_n)$.

Proposition 3.3 *For any $h \in \mathcal{H}$ we assume $\{F(\varepsilon\omega + h),\ \varepsilon \in (0,1]\} \subset L^2(\Omega)$. Set*

$$\Phi(h) = E(F) + \sum_{n=1}^{\infty} \int_{A^n} f_n(s)\, dh_{s_1} \ldots dh_{s_n}, \tag{3.2}$$

$$Z(h) = \sum_{n=1}^{\infty} n \int_{A^n} f_n(s)\, dh_{s_1} \ldots dh_{s_{n-1}}\, dW_{s_n}, \tag{3.3}$$

$s = (s_1, \ldots, s_n)$.
Then

$$L^2 - \lim_{\varepsilon \downarrow 0} \frac{F(\varepsilon\omega + h) - \Phi(h)}{\varepsilon} = Z(h), \tag{3.4}$$

where $Z(h)$ is a Gaussian, zero mean random vector.
Moreover, if Φ is Fréchet differentiable, the covariance matrix of $Z(h)$ coincides with $\Gamma_\Phi^h := \langle D\Phi(h),\ D\Phi(h)\rangle_{\mathcal{H}}$ where D denotes the Fréchet derivative.

Proof. Since $E(|F|^2) = \sum_{n=1}^{\infty} n!\, \|f_n\|^2_{L^2(A^n)} < \infty$. This implies that the series in (3.2) is absolutely convergent. Indeed, Schwarz's inequality yields

$$\left| \int_{A^n} f_n(s)\, dh_{s_1} \ldots dh_{s_n} \right| \leq \|f_n\|_{L^2(A^n)}\, \|h\|^n_{\mathcal{H}}.$$

Set $C = \|h\|_{\mathcal{H}}$. Then

$$\sum_{n=1}^{\infty} \left| \int_{A^n} f_n(s)\, dh_{s_1} \ldots dh_{s_n} \right| \leq \sum_{n=1}^{\infty} C^n\, \|f_n\|_{L^2(A^n)}$$
$$\leq \left(\sum_{n=1}^{\infty} \frac{C^{2n}}{n!} \right)^{1/2} \left(\sum_{n=1}^{\infty} n!\, \|f_n\|^2_{L^2(A^n)} \right)^{1/2} = \left(e^{C^2} - 1 \right)^{1/2} \|F\|_2 < +\infty.$$

The property $F(\omega + h) \in L^2(\Omega)$ can be expressed in terms of the Wiener decomposition of F as follows,

$$\sum_{\ell=0}^{\infty} \ell! \int_{A^\ell} \left(\sum_{n=\ell}^{\infty} \binom{n}{n-\ell} \int_{A^{n-\ell}} f_n(s)\, d\,h_{s_{\ell+1}} \ldots d\,h_{s_n} \right)^2 ds_1 \ldots d_{s_\ell} < \infty. \tag{3.5}$$

Indeed,

$$\begin{aligned} F(\omega + h) &= E(F) + \sum_{n=1}^{\infty} \int_{A^n} f_n(s)\, d\,(\omega + h)_{s_1} \ldots d\,(\omega + h)_{s_n} \\ &= E(F) + \sum_{n=1}^{\infty} \sum_{k=0}^{n} \binom{n}{k} \int_{A^n} f_n(s)\, d\,h_{s_1} \ldots d\,h_{s_k}\, d\,\omega_{s_{k+1}} \ldots d\,\omega_{s_n} \\ &= E(F) + \sum_{\ell=0}^{\infty} \sum_{n=\ell}^{\infty} \binom{n}{n-\ell} \int_{A^\ell} \left(\int_{A^{n-\ell}} f_n(s)\, d\,h_{s_{\ell+1}} \ldots d\,h_{s_n} \right) d\,\omega_{s_1} \ldots d\,\omega_{s_\ell}. \end{aligned} \tag{3.6}$$

Condition (3.5) yields, in particular, that the series in (3.3) converges in $L^2(\Omega)$. In fact, from (3.6) we get

$$F(\omega + h) = \Phi(h) + Z(h) + F_1(\omega),$$

with

$$F_1(\omega) = \sum_{n=2}^{\infty} \sum_{k=0}^{n-2} \binom{n}{k} \int_{A^n} f_n(s)\, d\,h_{s_1} \ldots d\,h_{s_k}\, d\,\omega_{s_{k+1}} \ldots d\,\omega_{s_n} \tag{3.7}$$

and

$$E\left(|F_1|^2\right) = \sum_{\ell=2}^{\infty} \ell! \int_{A^\ell} \left| \sum_{n=\ell}^{\infty} \binom{n}{n-\ell} \int_{A^{n-\ell}} f_n(s)\, d\,h_{s_{\ell+1}} \ldots d\,h_{s_n} \right|^2 d\,s_1 \ldots d\,s_\ell.$$

In order to prove the derivability of $\varepsilon \longmapsto F(\varepsilon\omega + h)$ we notice that

$$F(\varepsilon\omega + h) - \Phi(h) = \varepsilon\, Z(h) + F_1^\varepsilon(\omega),$$

with

$$F_1^\varepsilon(\omega) = \sum_{n=2}^{\infty} \sum_{k=0}^{n-2} \binom{n}{k} \varepsilon^{n-k} \int_{A^n} f_n(s)\, d\,h_{s_1} \ldots d\,h_{s_k}\, d\,\omega_{s_{k+1}} \ldots d\,\omega_{s_n}.$$

Thus, it suffices to check $L^2-\lim_{\varepsilon\to 0} \frac{1}{\varepsilon} F_1^\varepsilon(\omega) = 0$. But, for $k \in \{0, \ldots, n-2\}$, $n-k \geq 2$ and therefore $|\varepsilon^{-1} F_1^\varepsilon(\omega)| \leq \varepsilon\,|F_1(\omega)|$, with F_1 defined in (3.7). Hence, the derivability is proved.
The covariance matrix of $Z(h)$ is given by

$$\mathrm{Var}\,\Big(Z(h)\Big) = \int_A \Big(\sum_{n=1}^{\infty} n \int_{A^n} f_n\,(\sigma, s_1, \ldots, s_{n-1})\, d\,h_{s_1} \ldots d\,h_{s_{n-1}}\Big)\times$$
$$\times\Big(\sum_{n=1}^{\infty} n \int_{A^n} f_n\,(\sigma, s_1, \ldots, s_{n-1})\, d\,h_{s_1} \ldots d\,h_{s_{n-1}}\Big)^t\, d\sigma.$$

Assume Φ is Fréchet differentiable at $h \in \mathcal{H}$. Then, for any $k \in \mathcal{H}$,

$$D\,\Phi(h)\,(k) = \lim_{\varepsilon\downarrow 0} \frac{\Phi\,(h+\varepsilon\,k) - \Phi(h)}{\varepsilon}.$$

Using the expression

$$\Phi\,(h{+}\varepsilon\,k) = E(F){+}\sum_{n=1}^{\infty}\sum_{\ell=0}^{n}\binom{n}{\ell}\,\varepsilon^{n-\ell}\int_{A^n} f_n\,(s)\,d\,h_{s_1}\ldots d\,h_{s_\ell}\,d\,k_{s_{\ell+1}}\ldots d\,k_{s_n},$$

it is easy to check that

$$D\,\Phi\,(h)\,(k) = \sum_{n=1}^{\infty} n \int_{A^n} f_n\,(s)\,d\,h_{s_1}\ldots d\,h_{s_{n-1}}\,d\,k_{s_n}.$$

Consequently,

$$\Gamma_\Phi^h := \langle D\,\Phi\,(h),\; D\,\Phi\,(h)\rangle_{\mathcal{H}} = \mathrm{Var}\,\Big(Z(h)\Big). \quad \square$$

Remark 3.4 Assume $F \in L^{2+\delta}\,(\Omega)$ for some $\delta > 0$; using Girsanov's theorem it is easy to check that the assumption of Proposition 3.3 is satisfied.

For any $y \in \mathbb{R}^m$, let

$$d_R^2(y) = \inf\Big\{\,\|h\|_{\mathcal{H}}^2 : \Phi\,(h) = y,\; \det\,\Big(\langle D\,\Phi\,(h),\; D\,\Phi\,(h)\rangle_{\mathcal{H}}\Big) > 0\Big\}. \qquad (3.8)$$

Proposition 3.5 *Let F be a nondegenerate random vector. Assume that the convergence* (3.4) *stated in Proposition* 3.3 *holds in the topology of* $\mathbb{D}^\infty$ *and that $\Phi(\cdot)$ given in* (3.2) *is Fréchet differentiable. Then, if p^ε denotes the density of the random variable $F^\varepsilon(\omega) = F\,(\varepsilon\,\omega)$, $\varepsilon \in (0,1]$,*

$$\liminf_{\varepsilon\downarrow 0}\, \varepsilon^2\,\log\, p_\varepsilon(y) \geq -\frac{1}{2}\, d_R^2(y). \qquad (3.9)$$

For the proof of this proposition we refer the reader to [10], Proposition 4.4.1. In the next lemmas we state some ingredients used in the proof of theorem 3.1.

Lemma 3.6 *Assume* (H1), (H2) *and* (H3). *For any* $p \geq 1,\ \varepsilon \in (0,1]$,

$$\| (\Gamma_z^\varepsilon)^{-1} \|_p \leq C\, \varepsilon^{-2}, \tag{3.10}$$

where $\Gamma_z^\varepsilon = \int_T |D_r\, X_z^\varepsilon|^2\, d\,r$.

Proof: Let $\{Y_z^\varepsilon(r),\ 0 \leq r \leq z\}$ be the solution of

$$Y_z^\varepsilon(r) = \gamma_z(r) + \int_{(r,z]} \gamma_z(\eta)\, Y_\eta^\varepsilon(r)\, \left[\varepsilon\, a_3'(X_\eta^\varepsilon)\, d\, W_\eta + a_4'(X_\eta^\varepsilon)\, d\,\eta\right].$$

The Malliavin derivative of X_z^ε is given by

$$D_r\, X_z^\varepsilon = 1_{\{r \leq z\}}\ \varepsilon\, a_3(X_r^\varepsilon)\ Y_z^\varepsilon(r)$$

and, consequently, $\Gamma_z^\varepsilon = \varepsilon^2\, Q_z^\varepsilon$, with

$$Q_z^\varepsilon = \int_{R_z} a_3\,(X_r^\varepsilon)^2\ Y_z^\varepsilon(r)^2\, d\,r.$$

The arguments in the proof of the first part of theorem 2.1 show

$$\sup_{0<\varepsilon\leq 1} E\left(|\, (Q_z^\varepsilon)^{-1}|^p \right) \leq C_p, \quad p \in [1,\infty).$$

Therefore (3.10) holds true. □

Assumptions (H1) and (H2) imply that the mapping $\mathcal{H} \ni h \mapsto S_z^h,\ z \in T$, defined in (1.6), is infinitely Fréchet differentiable. Moreover, the Fréchet derivative $D\,S_z^h$ is given by

$$D\,S_z^h\,(k) = \int_{R_z} (D_r\, S_z^h)\ \dot{k}_r\, dr, \quad k \in \mathcal{H}.$$

Consider the equation

$$J_z(r) = \gamma_z(r) + \int_{(r,z]} \gamma_z(\eta)\ J_\eta(r)\ \left[a_3'\,(S_\eta^h)\ \dot{h}_\eta + a_4'\,(S_\eta^h)\right] d\eta, \quad 0 \leq r \leq z. \tag{3.11}$$

As for the Malliavin derivative $D_r X_z^\varepsilon$, it is easy to check that

$$D_r S_z^h = 1_{\{r \le z\}}\, a_3\left(S_r^h\right)\, J_z(r).$$

Set

$$\Gamma_z^h = \int_{R_z} |D_r S_z^h|^2\, dr,$$

that means, Γ_z^h is the analogue of the Malliavin matrix in the deterministic case.

Lemma 3.7 *Assume that* (H1) *to* (H3) *are satisfied. Then, for any $h \in \mathcal{H}$,* $\Gamma_z^h > 0$.

Proof: Suppose $a_3(x) \neq 0$. For any $\varepsilon > 0$ define, as in section 2, $C^z_{\beta,\delta}(\varepsilon) = (0, \varepsilon^\beta) \times (t - \varepsilon^\delta, t)$. Set

$$B_1(\varepsilon) = \int_{C^z_{1,1}(\varepsilon)} a_3^2(x)\ \gamma^2_{s,t}(0,t)\, dr,$$

$$B_2(\varepsilon) = \int_{C^z_{1,1}(\varepsilon)} \left(a_3(S_r^h)\ J_z(r) - a_3(x)\ \gamma_{s,t}(0,t)\right)^2 dr.$$

Obviously

$$\Gamma_z^h \ge \frac{1}{2}\, B_1(\varepsilon) - B_2(\varepsilon). \tag{3.12}$$

Since $a_3(X_0) \neq 0$, (4.1) yields

$$B_1(\varepsilon) \ge C\,\varepsilon^2. \tag{3.13}$$

We will prove

$$B_2(\varepsilon) \le C\,\varepsilon^3. \tag{3.14}$$

Indeed, from (3.11), using Schwarz's inequality and Gronwall's lemma, we can check

$$\sup_{z \in T}\ \sup_{r \le z}\ |J_z(r)\,| \le C \tag{3.15}$$

for some finite constant C.
Consequently, for $r \in C^z_{\beta,\delta}(\varepsilon)$

$$|\, J_z(r) - \gamma_z(r)\,| \le C \int_{(r,z]} |\, J_\eta(r)\,|\ (\dot{h}_\eta + 1)\, d\eta \le C\,\varepsilon^{\delta/2}. \tag{3.16}$$

Moreover,

$$\sup_{z\in T} |S_z^h| \le C \tag{3.17}$$

$$|S_r^h - x| \le C\,(r_1 r_2)^{1/2},\; r = (r_1, r_2), \tag{3.18}$$

for some constant depending on the Lipschitz constant of the coefficients and $\|h\|_{\mathcal{H}}$. The assumptions (H2), the properties of γ and (3.15) yield

$$\Big|\, a_3(S_r^h)\, J_z(r) - a_3(x)\, \gamma_{s,t}(0,t) \,\Big| \le C \Big\{ |J_z(r) - \gamma_z(r)| + |r_1| + |t - r_2| + |S_r^h - x| \Big\}.$$

Hence, (3.14) follows from (3.16) and (3.18). By (3.13) and (3.14) there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, the right hand-side of (3.12) is strictly positive; thus, $\Gamma_z^h > 0$.
Assume now $a_3(x) = 0$. Fix $\beta, \delta, \varepsilon > 0$ to be determined later. Set

$$A_1\,(\varepsilon,\beta,\delta) = \frac{1}{2} \int_{C^z_{\beta,\delta}(\varepsilon)} \gamma_z^2(r)\; a_3^2(S_r^h)\; dr,$$
$$A_2\,(\varepsilon,\beta,\delta) = \int_{C^z_{\beta,\delta}(\varepsilon)} \Big(J_z(r) - \gamma_z(r)\Big)^2\; a_3^2(S_r^h)\, dr.$$

Clearly,

$$\Gamma_z^h \ge \int_{C^z_{\beta,\delta}(\varepsilon)} |D_r\, S_z^h|^2\, dr \ge A_1(\varepsilon,\beta,\delta) - A_2(\varepsilon,\beta,\delta). \tag{3.19}$$

In addition, by (3.16) and (3.17)

$$A_2(\varepsilon,\beta,\delta) \le C\; \varepsilon^{\beta+2\delta}. \tag{3.20}$$

Consider the Taylor expansion of a_3 around x, the initial condition of the evolution equation defining S_z^h,

$$a_3\,(S_r^h) = \frac{1}{n_0\,!}\; a_3^{(n_0)}(x)\; (S_r^h - x)^{n_0} + \frac{1}{(n_0+1)\,!}\; a_3^{(n_0+1)}\,(\bar{x})\; (S_r^h - x)^{n_0+1}, \tag{3.21}$$

where $\bar{x}$ is some point lying between S_r^h and x.

Consider also the decomposition $S_r^h - x = \sum_{j=1}^4 S_r^j$ with

$$S_r^1 = \int_{R_r} \gamma_r(\xi)\, a_3(S_\xi^h)\, \dot{h}_\xi \, d\xi,$$

$$S_r^2 = \int_{R_r} \gamma_r(\xi)\, \Big(a_4(S_\xi^h) - a_4(x)\Big) d\xi,$$

$$S_r^3 = \int_{R_r} \Big(\gamma_r(\xi) - \gamma_{0,r_2}(0,\xi_2)\Big)\, a_4(x)\, d\xi,$$

$$S_r^4 = \int_{R_r} \gamma_{0,r_2}(0,\xi_2)\, a_4(x)\, d\xi.$$

The identity (3.21) and the triangle inequality imply

$$2\, A_1(\varepsilon,\beta,\delta) \geq A_{11}(\varepsilon,\beta,\delta) - A_{12}(\varepsilon,\beta,\delta), \tag{3.22}$$

where

$$A_{11}(\varepsilon,\beta,\delta) = \frac{1}{2^{2n_0}} \int_{C^z_{\beta,\delta}(\varepsilon)} \gamma_z^2(r) \left(\frac{a_3^{(n_0)}(x)}{n_0\,!}\right)^2 (S_r^4)^{2n_0}\, dr,$$

$$A_{12}(\varepsilon,\beta,\delta) = \int_{C^z_{\beta,\delta}(\varepsilon)} \gamma_z^2(r) \left(\frac{1}{2}\left(\frac{a_3^{(n_0)}(x)}{n_0\,!}\right)^2 \Big(\sum_{j=1}^3 S_r^j\Big)^{2n_0} + \left(\frac{a_3^{(n_0+1)}(\bar{x})}{(n_0+1)\,!}\right)^2 \Big(S_r^h - x\Big)^{2(n_0+1)}\right) dr.$$

Moreover,

$$A_{11}(\varepsilon,\beta,\delta) \geq A_{111}(\varepsilon,\beta,\delta) - A_{112}(\varepsilon,\beta,\delta) \tag{3.23}$$

with

$$A_{111}(\varepsilon,\beta,\delta) = \frac{1}{2^{2n_0+1}} \int_{C^z_{\beta,\delta}(\varepsilon)} \gamma_z^2(r_1,t) \left(\frac{a_3^{(n_0)}(x)}{n_0\,!}\right)^2 \Big(S_r^4\Big)^{2n_0} dr,$$

$$A_{112}(\varepsilon,\beta,\delta) = \frac{1}{2^{2n_0}} \int_{C^z_{\beta,\delta}(\varepsilon)} \Big(\gamma_z(r) - \gamma_z(r_1,t)\Big)^2 \left(\frac{a_3^{(n_0)}(x)}{n_0\,!}\right)^2 \Big(S_r^4\Big)^{2n_0} dr.$$

From (4.1), and since $(a_3^{(n_0)}\, a_4)(x) \neq 0$, it follows

$$A_{111}(\varepsilon,\beta,\delta) \geq C \int_{C^z_{\beta,\delta}(\varepsilon)} (r_1\, r_2)^{2n_0}\, dr_1\, dr_2 \geq C\, \varepsilon^{\delta+(2n_0+1)\beta}. \tag{3.24}$$

The rest of the proof consists in finding appropiate upper bounds for $A_{12}(\varepsilon, \beta, \delta)$ and $A_{112}(\varepsilon, \beta, \delta)$, respectively.
We have

$$\int_{C^z_{\beta,\delta}(\varepsilon)} \gamma_z^2(r)\; a_3^{(n_0)}\;(x)^2\;(S_r^1)^{2n_0}\,dr \le C\;\varepsilon^{(3n_0+1)\beta+\delta}. \tag{3.25}$$

Indeed, for any $z=(s,t)\in T$, $\quad |S_z^h - x| \le C\;|st|$. Consequently,

$$|\,S_r^1|^{2n_0} \le C\,(r_1\,r_2)^{3n_0}$$

and this yields (3.25).
Analogously,

$$|\,S_r^2|^{2n_0} \le C\,(r_1\,r_2)^{4n_0}$$

and therefore

$$\int_{C^z_{\beta,\delta}(\varepsilon)} \gamma_z^2(r)\; a_3^{(n_0)}\,(x)^2\;(S_r^2)^{2n_0}\,dr \le C\;\varepsilon^{(4n_0+1)\beta+\delta}. \tag{3.26}$$

By the Lipschitz property of γ,

$$|\,S_r^3|^{2n_0} \le C\;r_1^{4n_0}\,r_2^{2n_0}.$$

Consequently,

$$\int_{C^z_{\beta,\delta}(\varepsilon)} \gamma_z^2(r)\; a_3^{(n_0)}\;(x)^2\;(S_r^3)^{2n_0}\,dr \le C\;\varepsilon^{(4n_0+1)\beta+\delta}. \tag{3.27}$$

We also have

$$\int_{C^z_{\beta,\delta}(\varepsilon)} \gamma_z^2(r)\; a_3^{(n_0+1)}\;(\bar{x})^2\;(S_r^h - x)^{2n_0+2}\,dr \le C\;\varepsilon^{(2n_0+3)\beta+\delta}. \tag{3.28}$$

Then, (3.25) to (3.28) imply

$$A_{12}(\varepsilon,\beta,\delta) \le C\left(\varepsilon^{(3n_0+1)\beta+\delta} + \varepsilon^{(2n_0+3)\beta+\delta}\right). \tag{3.29}$$

Finally,

$$A_{112}(\varepsilon,\beta,\delta) \le C\int_{C^z_{\beta,\delta}(\varepsilon)} |t-r_2|^2\;|r_1\,r_2|^{2n_0}\,dr \le C\;\varepsilon^{(2n_0+1)\beta+3\delta}. \tag{3.30}$$

From (3.19), (3.22), (3.23) it follows

$$\Gamma_z^h \geq \frac{1}{2}\, A_{111}\,(\varepsilon,\beta,\delta) - \frac{1}{2}\, A_{112}\,(\varepsilon,\beta,\delta) - \frac{1}{2}\, A_{12}\,(\varepsilon,\beta,\delta) - A_2\,(\varepsilon,\beta,\delta).$$

Then,(3.24), (3.30), (3.29) and (3.20) ensure

$$\Gamma_z^h \geq C\left(\varepsilon^{(2n_0+1)\beta+\delta} - \varepsilon^{\beta+2\delta} - \varepsilon^{\delta+((3n_0+1)\wedge(2n_0+3))\beta}\right), \tag{3.31}$$

for some positive constant C.
Choose $0 < \beta, \delta$ such that $\delta > 2\,n_0\,\beta$. Then (3.31) yields $\Gamma_z^h > 0$. □

Set $y_z^{\varepsilon,h} = X_z\,(\varepsilon\omega + h)$, $\varepsilon \in [0,1]$, $h \in \mathcal{H}$, $z \in T$. The process $\{y_z^{\varepsilon,h},\ z \in T\}$ satisfies the equation

$$y_z^{\varepsilon,h} = x + \int_{R_z} \gamma_z(\eta)\,\left[\varepsilon\, a_3\,(y_\eta^{\varepsilon,h})\, d\,W_\eta + a_3\,(y_\eta^{\varepsilon,h})\,\dot{h}_\eta\, d\eta + a_4\,(y_\eta^{\varepsilon,h})\, d\eta\right]$$

and, by uniqueness of solution, $y_z^{0,h} = S_z^h$.

Lemma 3.8 *Assume* (H1) *and* $a_i : \mathbb{R} \longrightarrow \mathbb{R}$, $i = 3, 4$, *Lipschitz. Then*

$$\lim_{\varepsilon \downarrow 0}\left(\sup_{z\in T}\, E\,(\,|y_z^{\varepsilon,h} - S_z^h|^p)\right) = 0, \quad p \in [1,\infty).$$

Proof: It is an easy consequence of Gronwall's lemma and the following estimate

$$\sup_{0\leq\varepsilon\leq 1}\ \sup_{z\in T}\ E\,(\,|y_z^{\varepsilon,h}|^p) \leq C, \tag{3.32}$$

for some positive and finite constant C. □

Consider also $\{Z_z^h,\ z \in T\}$ defined by

$$Z_z^h = \int_{R_z} \gamma_z(\eta)\ a_3(S_\eta^h)\ dW_\eta + \int_{R_z} \gamma_z(\eta)\ Z_\eta^h\ \left[a_3'\,(S_\eta^h)\ \dot{h}_\eta + a_4'\,(S_\eta^h)\right] d\eta.$$

Notice that Z_z^h is gaussian, because the Malliavin derivative $D\,Z_z^h$ is deterministic.
Let $\zeta_z^{\varepsilon,h} = \frac{y_z^{\varepsilon,h} - S_z^h}{\varepsilon}$, $\varepsilon \in (0,1]$. Our next purpose is to study the convergence of $\zeta_z^{\varepsilon,h}$, as $\varepsilon \downarrow 0$, for fixed $z \in T$, $h \in \mathcal{H}$, i.e., the derivability of $\varepsilon \longmapsto y_z^{\varepsilon,h}$.

Lemma 3.9 *Assume* (H1) *and*

(H2') $a_i, i = 3, 4$ *are* C^1*-functions with bounded derivatives.*

Then

$$L^p - \lim_{\varepsilon \downarrow 0} (\zeta_z^{\varepsilon,h} - Z_z^h) = 0, \quad p \in [1, \infty),$$

uniformly in $z \in T$.

Proof: Fix $p \in [2, \infty)$. Then,

$$E\left(|\zeta_z^{\varepsilon,h} - Z_z^h|^p\right) \le C\left(A_1(\varepsilon, z) + A_2(\varepsilon, z) + A_3(\varepsilon, z)\right),$$

with

$$A_1(\varepsilon, z) = E\left(\Big|\int_{R_z} \gamma_z(\eta)\left[a_3(y_\eta^{\varepsilon,h}) - a_3(S_\eta^h)\right] dW_\eta\Big|^p\right),$$

$$A_2(\varepsilon, z) = E\left(\Big|\int_{R_z} \gamma_z(\eta)\left[\frac{a_3(y_\eta^{\varepsilon,h}) - a_3(S_\eta^h)}{\varepsilon} - a_3'(S_\eta^h)\, Z_\eta^h\right] \dot{h}_\eta\, d\eta\Big|^p\right),$$

$$A_3(\varepsilon, z) = E\left(\Big|\int_{R_z} \gamma_z(\eta)\left[\frac{a_4(y_\eta^{\varepsilon,h}) - a_4(S_\eta^h)}{\varepsilon} - a_4'(S_\eta^h)\, Z_\eta^h\right] d\eta\Big|^p\right).$$

Burkholder's inequality and the Lipschitz property of a_3 yields

$$\sup_{z \in T} A_1(\varepsilon, z) \le C \sup_{z \in T} E\left(|y_z^{\varepsilon,h} - S_z^h|^p\right). \tag{3.33}$$

By the mean-value theorem $a_i(y_\eta^{\varepsilon,h}) - a_i(S_\eta^h) = a_i'(\chi_\eta^{\varepsilon,h,i})\,(y_\eta^{\varepsilon,h} - S_\eta^h)$, with $\chi_\eta^{\varepsilon,h,i} = \lambda^i y_\eta^{\varepsilon,h} + (1 - \lambda^i) S_\eta^h$, for some $\lambda^i \in (0,1)$ depending on $\omega, i = 3, 4$. Consequently, for $i = 3, 4$,

$$\left|\frac{a_i(y_\eta^{\varepsilon,h}) - a_i(S_\eta^h)}{\varepsilon} - a_i'(S_\eta^h) Z_\eta^h\right| \le C\left\{|\zeta_\eta^{\varepsilon,h} - Z_\eta^h| + |\chi_\eta^{\varepsilon,h,i} - S_\eta^h|\,|Z_\eta^h|\right\}.$$

The typical arguments based on Burkholder's and Hölder's inequalities and Gronwall's lemma ensure

$$\sup_{z \in T} E\left(|Z_z^h|^p\right) \le C, \quad p \in [1, \infty].$$

Moreover, by Lemma 3.8

$$\sup_{z \in T} E\left(|\chi_z^{\varepsilon,h,i} - S_z^h|^p\right) = \sup_{z \in T} E\left((\lambda^i)^p |y_z^{\varepsilon,h} - S_z^h|^p\right) \xrightarrow[\varepsilon \downarrow 0]{} 0. \tag{3.34}$$

Consequently,

$$\sup_{z\in T} E\left(|\zeta_z^{\varepsilon,h} - Z_z^h|^p\right) \le C\left\{a(\varepsilon) + \int_{R_z} \sup_{\xi\le\eta} E\left(|\zeta_\xi^{\varepsilon,h} - Z_\xi^{\varepsilon,h}|^p\right) d\eta\right\}$$

with $\lim_{\varepsilon\downarrow 0} a(\varepsilon) = 0$, because of (3.33) and (3.34). Thus, the statement follows from Gronwall's lemma. □

In the sequel we will need a strengthening of Lemma 3.9 as follows.

Lemma 3.10 *Assume (H1) and (H2). Then*

$$\lim_{\varepsilon\downarrow 0}\left(\zeta_z^{\varepsilon,h} - Z_z^h\right) = 0\,,$$

in the topology of $\mathbb{D}^\infty$.

Proof: Since S_z^h is deterministic, Z_z^h gaussian and, due to Lemma 3.9, it only remains to check

$$\lim_{\varepsilon\downarrow 0} E\left(\left|\int_T \left(\frac{1}{\varepsilon} D_\eta y_z^{\varepsilon,h} - D_\eta Z_z^h\right)^2 d\eta\right|^{p/2}\right) = 0\,, \tag{3.35}$$

$$\lim_{\varepsilon\downarrow 0} E\left(\left|\int_{T^j} \left(\frac{1}{\varepsilon} D_\eta^j y_z^{\varepsilon,h}\right)^2 d\eta\right|^{p/2}\right) = 0\,, \quad j = 2,3,\dots\,, \tag{3.36}$$

$p \in [1,\infty)$.
We have

$$D_\eta y_z^{\varepsilon,h} = 1_{\{\eta\le z\}}\, \varepsilon\, a_3\!\left(y_\eta^{\varepsilon,h}\right) M_z^{\varepsilon,h}(\eta)\,,$$

with

$$M_z^{\varepsilon,h}(\eta) = \gamma_z(\eta) + \int_{(\eta,z]} \gamma_z(\xi) M_\xi^{\varepsilon,h}(\eta)\Big[\varepsilon\, a_3'(y_\xi^{\varepsilon,h}) dW_\xi + a_3'(y_\xi^{\varepsilon,h})\dot{h}_\xi d\xi + a_4'(y_\xi^{\varepsilon,h}) d\xi\Big]\,.$$

Analogously, $D_\eta Z_z^h = 1_{\{\eta\le z\}}\, a_3(S_\eta^h) N_z^h(\eta)$, where

$$N_z^h(\eta) = \gamma_z(\eta) + \int_{(\eta,z]} \gamma_z(\xi) N_\xi^h(\eta)\Big[a_3'(S_\xi^h)\dot{h}_\xi + a_4'(S_\xi^h)\Big]\, d\xi\,.$$

Then

$$
\begin{aligned}
&E\left(\left|\int_T \left(\frac{1}{\varepsilon}D_\eta y_z^{\varepsilon,h} - D_\eta Z_z^h\right)^2 d\eta\right|^{p/2}\right)\\
&= E\left(\left|\int_{R_z} \left(a_3(y_\eta^{\varepsilon,h})M_z^{\varepsilon,h}(\eta) - a_3(S_\eta^h)N_z^h(\eta)\right)^2 d\eta\right|^{p/2}\right)\\
&\le C\left(C_1(\varepsilon,z) + C_2(\varepsilon,z)\right),
\end{aligned}
$$

with

$$
\begin{aligned}
C_1(\varepsilon,z) &= E\left(\left|\int_{R_z} \left(a_3(y_\eta^{\varepsilon,h})(M_z^{\varepsilon,h}(\eta) - N_z^h(\eta))\right)^2 d\eta\right|^{p/2}\right),\\
C_2(\varepsilon,z) &= E\left(\left|\int_{R_z} |y_\eta^{\varepsilon,h} - S_\eta^h|^2 \, |N_z^h(\eta)|^2 d\eta\right|^{p/2}\right).
\end{aligned}
$$

Similar arguments as these used in the proof of Lemma 3.8 show

$$
\sup_{\eta\le z} E\left(|M_z^{\varepsilon,h}(\eta) - N_z^h(\eta)|^p\right) \xrightarrow[\varepsilon\downarrow 0]{} 0, \quad p\in[1,\infty).
$$

In addition $\sup_{\eta\le z} |N_z^h(\eta)| \le C$. These properties together with (3.32) yield (3.35).
The validity of (3.36) is a trivial consequence of the following fact: for $j\in\mathbb{N}-\{1\}$,

$$
D_\eta^j y_z^{\varepsilon,h} = \varepsilon^j \, 1_{\{0\le\eta_1\le\ldots\le\eta_j\le z\}} R_z^{\varepsilon,h}(\eta), \tag{3.37}
$$

$\eta=(\eta_1,\ldots,\eta_j)$, with $\sup_{0\le\eta_1\le\ldots\le\eta_j\le z} E(|R_z^{\varepsilon,h}(\eta)|^p) \le C$, $p\in[1,\infty)$.
Indeed, (3.37) can be easily checked recursively. □

In [11] we have proved that the family $\{X^\varepsilon, \varepsilon>0\}$ of solutions to (3.1), satisfies a large deviation principle, on the space of continuous functions defined on T with value x on the axes. The rate function is given by

$$
I(g) = \inf\left\{\frac{1}{2}\| h \|_{\mathcal{H}}^2 : S^h = g\right\}.
$$

This is a well-know consequence of the following estimate (see [11], Theorem 4.4):
Assume (H1) and

(H2") $a_i : \mathbb{R} \longrightarrow \mathbb{R}$, $i = 3, 4$ are bounded, Lipschitz functions.

For all $h \in \mathcal{H}$, $\delta, R > 0$, $\varepsilon \in [0,1]$, there exists $\alpha > 0$ such that

$$P\left\{ \sup_{z\in T} |X_z^\varepsilon - S_z^h| > \delta, \quad \sup_{z\in T} |\varepsilon W_z - h| < \alpha \right\} \le \exp\left(- \frac{R}{\varepsilon^2} \right). \tag{3.38}$$

As a consequence we obtain the following result.

Proposition 3.11 *Assume (H1) and $a_i := \mathbb{R} \longrightarrow \mathbb{R}$, $i = 3, 4$, Lipschitz. Fix $z \in T\backslash E$. Then, the family of random variables $\{X_z^\varepsilon, \varepsilon \in [0,1]\}$ satisfies a large deviation principle with rate function*

$$I(y) = \inf \left\{ \frac{1}{2} \| h\|_{\mathcal{H}}^2 : S_z^h = y \right\}.$$

Proof: If the coefficients a_i, $i = 3, 4$, are bounded, the result follows by the contraction principle of large deviations. For a_i, $i = 3, 4$, Lipschitz we use a localization procedure as follows. Let

$$T^\varepsilon(\delta) = \inf \left\{ t : \sup_{z\le(t,t)} |X_z^\varepsilon - S_z^h| \ge \delta \right\} \wedge 1, \quad \delta > 0.$$

Set $\tau^\varepsilon(\delta) = \left(T^\varepsilon(\delta), T^\varepsilon(\delta) \right)$. Notice

$$\sup_{z\le\tau^\varepsilon(\delta)} |X_z^\varepsilon| \le \delta + \sup_{z\in T} |S_z^h| \le C. \tag{3.39}$$

The left hand-side of (3.38) coincides with

$$P\left\{ \sup_{z\le\tau^\varepsilon(\delta)} |X_z^\varepsilon - S_z^h| > \delta, \ \sup_{z\in T} |\varepsilon W_z - h| < \alpha \right\}$$

and, because of (3.39), the coefficients a_i, $i = 3, 4$, can be supposed to be bounded. □

We can now give the proof of the main result of this section.
Proof of Theorem 3.1 Let $F^\varepsilon = X_z^\varepsilon$, the solution to (3.1) at a fixed point $z \in T\backslash E$. In [11] we have proved $X_z \in \mathbb{D}^\infty$. The same arguments used to show this fact also prove $\sup_{\varepsilon\in(0,1]} \|X_z^\varepsilon\|_{k,p} < \infty$, for any integer $k \ge 1$,

$p \in (1, \infty)$. This property, together with Lemma 3.6 and Proposition 3.11, ensure the assumptions of Proposition 3.2. Consequently,

$$\limsup_{\varepsilon \downarrow 0} \varepsilon^2 \log p_z^\varepsilon(y) \le -\frac{1}{2} d^2(x, y).$$

Lemma 3.10 yields the validity of the hypothesis of Proposition 3.5, with $\Phi(h) = S_z^h$. In addition, Lemma 3.7 yields that $d_R^2(y)$ defined in (3.8) coincides with $d^2(x, y)$. Hence, by (3.9),

$$\liminf_{\varepsilon \downarrow 0} \varepsilon^2 \log p_z^\varepsilon(y) \ge -\frac{1}{2} d^2(x, y). \quad \square$$

We finish this section by analyzing the finiteness of $d^2(x, y)$. This problem is related with the positivity of the density $p_z^\varepsilon(y)$. Carrying out the programme developed in [5] for equation (1.2) for $x := X_0$ deterministic, one could obtain

$$\left\{ p_z^\varepsilon(y) > 0 \right\} = \left\{ y ;\ \exists\, h \in \mathcal{H} : S_z^h = y \right\}.$$

Therefore,

$$d^2(x, y) < \infty \Leftrightarrow p_z^\varepsilon(y) > 0.$$

Consequently, if $d^2(x, y) = \infty$, Theorem 3.1 is trivial.
By Proposition 4.1.2 in [10],

$$\left\{ p_z^\varepsilon(y) > 0 \right\} = \overbrace{\operatorname{supp} P \circ \left(X_z^\varepsilon\right)^{-1}}^{\circ}, \tag{3.40}$$

where supp $P \circ \left(X_z^\varepsilon\right)^{-1}$ denotes the topological support of the law of X^ε. The set supp $P \circ \left(X_z^\varepsilon\right)^{-1}$ is a closed interval (a closed and connected set, by a result of [2]). Notice that, since $P \circ \left(X_z^\varepsilon\right)^{-1}$ is absolutely continuous, (3.40) ensures $\{p_z^\varepsilon(y) > 0\} \neq \phi$.

Proposition 3.12 *Assume* $a_3(z) \neq 0$, *for any* $z \in \mathbb{R}$. *Then*

$$\{y : d^2(x, y) < \infty\} = \mathbb{R}.$$

Proof: Fix $y \in \mathbb{R}$. We first prove the existence of $k_z \in \mathcal{H}$ such that if

$$f(z) := x + \int_{R_z} \gamma_z(\eta) \dot{k}_z(\eta)\, d\eta,$$

Then $f(z) = y$.
Indeed, (4.1) yields $\int_{R_z} \gamma_z(\eta)^2 d\eta > 0$. Let

$$\alpha(z) = (y - x)\left(\int_{R_z} \gamma_z(\eta)^2 d\eta\right)^{-1}.$$

Then, $\dot{k}_z(\eta) = \alpha(z)\,\gamma_z(\eta)$ satisfies $f(z) = y$.
For any $\xi \in R_z$ set

$$f(\xi) = x + \int_{R_\xi} \gamma_\xi(\eta)\ \dot{k}_z(\eta)\, d\eta \tag{3.41}$$

and for any $\eta \in R_z$ define

$$\dot{h}_z(\eta) = -\frac{a_4(f(\eta)) - \dot{k}_z(\eta)}{a_3(f(\eta))}. \tag{3.42}$$

(3.41) and (3.42) yield

$$f(\xi) = x + \int_{R_\xi} \gamma_\xi(\eta)\Big[a_3(f(\eta))\,\dot{h}_z(\eta) + a_4(f(\eta))\Big]\, d\eta\,, \quad \xi \in R_z\,.$$

Thus, $f(\xi) = S_\xi^{h_z}$, $\xi \in R_z$. In particular $y = S_z^{h_z}$ and hence $d^2(x, y) < \infty$. □

4 Appendix

We collect here the results on the Green function, which have been used in the paper. Their proofs are given in [11]. We recall that the Green function corresponding to the second order differential operator $\mathcal{L}$ defined in section 1 is given as the solution on $\{(u, v) : (0, 0) \le (u, v) \le (s, t)\}$ of

$$(P)\begin{cases} \dfrac{\partial^2 \gamma_{s,t}(u,v)}{\partial u \partial v} + \dfrac{\partial(a_1(u,v)\gamma_{s,t}(u,v))}{\partial v} + \dfrac{\partial(a_2(u,v)\gamma_{s,t}(u,v))}{\partial u} = 0\,. \\ \dfrac{\partial \gamma_{s,t}(u,v)}{\partial u} = -a_1(u,v)\gamma_{s,t}(u,v), \quad \text{when} \quad v = t\,, \\ \dfrac{\partial \gamma_{s,t}(u,v)}{\partial v} = -a_2(u,v)\gamma_{s,t}(u,v), \quad \text{when} \quad u = s\,, \\ \gamma_{s,t}(u,v) = 1, \quad \text{when} \quad u = s \quad \text{and} \quad v = t\,. \end{cases}$$

Here are some properties of γ:

Boundedness:

$$\sup_{(s,t)\in T}\ \sup_{(u,v)\leq(s,t)} |\gamma_{s,t}(u,v)| \leq C\,.$$

Lipschitz property:

$$\sup_{(s,t)\in T} |\gamma_{s,t}(\overline{u},\overline{v}) - \gamma_{s,t}(u,v)| \leq C\left\{|\overline{u}-u| + |\overline{v}-v|\right\},\ (\overline{u},\overline{v}),(u,v) \leq (s,t)\,,$$

$$\sup_{(u,v)\in T} |\gamma_{\overline{s},\overline{t}}(u,v) - \gamma_{s,t}(u,v)| \leq C\left\{|\overline{s}-s| + |\overline{t}-t|\right\},\ (\overline{s},\overline{t}),(s,t) \geq (u,v)\,,$$

Positivity: From (P) we obtain

$$\begin{aligned} \gamma_{s,t}(s,v) &= \exp\left(\int_v^t a_2(s,w)\,dw\right), \quad 0\leq v\leq t\,, \\ \gamma_{s,t}(u,t) &= \exp\left(\int_u^s a_1(r,t)\,dr\right), \quad 0\leq u\leq s\,. \end{aligned} \tag{4.1}$$

Consequently,

$$\inf_{0\leq v\leq t} \gamma_{s,t}(s,v) > 0 \quad \text{and} \quad \inf_{0\leq u\leq s} \gamma_{s,t}(u,t) > 0\,.$$

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