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Representation Theory and Structure of Semisimple Lie Algebras

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Abstract

This work focuses on Lie algebras and representation theory, a field of mathematics that was born in the 19th century and remains relevant today in areas such as differential geometry, number theory, and physics.

This dissertation covers several topics. We introduce matrix Lie groups, a specific type of Lie groups, as well as the concept of the Lie algebra associated with them and, more generally, the Lie algebra of an arbitrary Lie group. Next, we introduce the concept of representation. We study and prove the structure of finite-dimensional, complex, irreducible representations of the Lie algebra $\mathfrak{sl}(2; \mathbb{C})$, which is necessary for understanding the structure of semisimple Lie algebras. After introducing Cartan subalgebras, roots and root space decomposition, we state and prove the main theorem of this work. This theorem states that every finite-dimensional, complex, semisimple Lie algebra admits a simple system of roots, a fundamental result in the structure theory of semisimple Lie algebras.

Resum

Aquest treball se centra en les àlgebres de Lie i la teoria de la representació, una àrea matemàtica nascuda al segle XIX i amb rellevància en l'actualitat en àmbits com la geometria diferencial, la teoria de nombres i la física.

Aquesta dissertació cobreix diversos temes. Introduïm els grups de Lie de matrius, una classe de grups de Lie, així com el concepte d'àlgebra de Lie associada a ells i, més en general, l'àlgebra de Lie d'un grup arbitrari. A continuació, introduïm el concepte de representació. Estudiem i demostrem l'estructura de les representacions de dimensió finita, complexes, irreductibles de l'àlgebra de Lie $\mathfrak{sl}(2; \mathbb{C})$, que és necessària per entendre l'estructura de les àlgebres de Lie semisimples.

Un cop introduïdes les subàlgebres de Cartan, les arrels i la descomposició en espais d'arrels, enunciem i demostrem el teorema principal d'aquest treball. Aquest teorema afirma que tota àlgebra de Lie semisimple, complexa i de dimensió finita admet un sistema simple d'arrels, un resultat fonamental en la teoria de l'estructura de les àlgebres de Lie semisimples.

Agraïments

L'elaboració d'aquest treball no hauria estat possible sense la presència i el suport de múltiples persones.

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Introduction

Sophus Lie (1842-1899) was a Norwegian mathematician whose research covered a number of topics such as differential equations, contact geometry, minimal surfaces, and the Erlangen program. Initially prompted by his work on differential equations, he eventually focused on categorizing what he called 'closed systems of commutative transformations'. As mathematicians recognized this as a division of abstract algebra, the area became known as 'continuous transformation groups', which we now call Lie groups.

We now introduce the notion of these structures. Groups, for example $GL(n; \mathbb{R})$, can be endowed with a geometric structure by viewing the matrix entries as coordinates in the space. This makes them a Lie group.

From another perspective, groups also act on spaces. For instance, $GL(n; \mathbb{C})$ acts on $\mathbb{C}^n$. Thus, groups can be viewed as transformation groups. This is precisely the viewpoint adopted by Lie, leading to the development of the theory of transformation groups. He related these to infinitesimal transformation groups, which are now known as Lie algebras.

The notion of manifold was not fully formalized at that time; nevertheless, Lie's work established many important results that later became key to manifold theory. Indeed, a Lie group can be viewed as a manifold endowed with a compatible group structure.

Another key mathematician in the development of Lie's theory was Friedrich Engel (1861-1941). As Lie's student, he contributed significantly to the theory of transformation groups.

Simultaneously, Wilhelm Killing (1847-1923) also discovered what are now known as Lie algebras, unaware of Lie's work due to the limited access to the Scandinavian journal where it had been published. His main goal was ambitious: to classify real finite-dimensional Lie algebras.

Although it negatively affected Lie and Engel's relationship, Engel communicated Lie's ideas to Killing. As a consequence, Killing's work brought new insight into Lie's theory and was responsible for the classification of semisimple Lie algebras. He also introduced root systems, which are essential for this classification.

Other important contributors to Lie algebras and representation theory were Élie Cartan (1869-1951), who reworked much of Killing's study, and Hermann Weyl (1885-1955), who further developed Cartan's ideas. The work on algebraic groups by Chevalley (1909-1984), Grothendieck (1928-2014) and Demazure (1937-present) also helped bring Lie theory into its modern form.

Lie algebra and representation theory is a central area in modern mathematics. It is used in differential geometry for symmetric spaces and other homogeneous spaces, see [2]. It also appears in number theory in the study of automorphic forms, since these are closely related to automorphic representations, see [3]. In physics, elementary particles correspond to representation theory of Lie algebras, see [6].

This work focuses on Lie algebras and representation theory.

First, we introduce a specific type of Lie groups, i.e., matrix Lie groups. We restrict ourselves to these groups because they are easier to understand, since matrices are more familiar objects and make the theory more accessible compared to the general definition of Lie group, which requires substantial knowledge of manifold theory. In addition, the Lie groups considered in the rest of the work are matrix Lie groups. Then we introduce the concept of the Lie algebra of a matrix Lie group and, more generally, the Lie algebra of an arbitrary Lie group.

Next, we define the concept of representation. We study and prove the structure of finite-dimensional, complex, irreducible representations of the Lie algebra $\mathfrak{sl}(2; \mathbb{C})$, which also yield representations of $\mathfrak{su}(2)$. This is necessary for understanding the structure of semisimple Lie algebras.

We work with finite-dimensional, complex, semisimple Lie algebras. This assumption is made because Lie algebra theory relies on working over an algebraically closed field of characteristic zero. The dimension needs to be finite in order to give a solid root space decomposition and classification theory for semisimple Lie algebras. Some results presented in this work may hold in greater generality; however, they are restricted here to the Lie algebras on which we focus.

Once Cartan subalgebras, roots and root space decomposition are introduced, we state and prove the main theorem of this work, Theorem 3.43.

This result states that every finite-dimensional, complex, semisimple Lie algebra admits a simple system of roots, and it plays a fundamental role in the structure theory of semisimple Lie algebras. Indeed, one can show that for a given finite-dimensional, complex, semisimple Lie algebra, all choices of simple system are equivalent and that each simple system relates to a Dynkin diagram. One can see that two Dynkin diagrams are equivalent if and only if the corresponding semisimple Lie algebras are isomorphic. Since there exist only finitely many Dynkin diagrams, this correspondence leads to the complete classification

of finite-dimensional, complex, semisimple Lie algebras in terms of their root systems.

Chapter 1

Matrix Lie Groups and Lie Algebras

In this chapter we introduce some basic definitions and results. In particular, we focus on matrix Lie groups and their corresponding Lie algebras.

For a more extensive treatment of this topic, consult Chapter 1, Matrix Lie Groups, and Chapter 2, Lie Algebras and the Exponential Mapping, of [7].

1.1 Matrix Lie Groups

We define matrix Lie groups, present some examples and discuss their main properties. In addition, we define homomorphism of matrix Lie groups and give an idea of the definition of Lie group.

As usual, we denote by $GL(n; \mathbb{C})$ the **general linear group over $\mathbb{C}$** . That is the group of all $n \times n$ invertible matrices with complex entries.

Definition 1.1. *A **matrix Lie group** is any subgroup H of $GL(n; \mathbb{C})$ such that, if A_n is any sequence of matrices in H and A_n converges to some matrix A , then either $A \in H$ or A is not invertible.*

This is equivalent to saying that H must be a closed subgroup of $GL(n; \mathbb{C})$.

We now consider some classic examples of matrix Lie groups.

Example 1.2. The general linear groups $GL(n; \mathbb{C})$ and $GL(n; \mathbb{R})$

It is clear that $GL(n; \mathbb{C})$ is a closed subgroup over itself. In addition, $GL(n; \mathbb{R})$ is a subgroup of $GL(n; \mathbb{C})$ and given a sequence $A_n \in GL(n; \mathbb{R})$ such that converges to A , then all entries in A are real.

On the other hand, we can consider an example of subgroup of $GL(n; \mathbb{C})$ that it is not closed, and thus it is not a matrix Lie group.

Consider the set H of $n \times n$ invertible matrices with rational entries. This is a subgroup of $GL(n; \mathbb{C})$. We can take, for instance, the sequence

$$A_n = \begin{pmatrix} (1 + \frac{1}{n})^n & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

that converges to

$$A = \begin{pmatrix} e & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

$A \notin H$, thus H is not a matrix Lie group.

Example 1.3. The special linear groups $SL(n; \mathbb{C})$ and $SL(n; \mathbb{R})$

The special linear group over $\mathbb{C}$ or $\mathbb{R}$ is the group of $n \times n$ invertible matrices with determinant 1 and entries in $\mathbb{C}$ or $\mathbb{R}$, respectively. Both are subgroups of $GL(n; \mathbb{C})$. They are closed over $GL(n; \mathbb{C})$ because any sequence A_n of matrices with determinant one, converges to a matrix A with determinant one.

Example 1.4. The orthogonal and special orthogonal groups $O(n)$ and $SO(n)$

The elements of the orthogonal group $O(n)$ are $n \times n$ real matrices A whose column vectors are orthogonal. Thus $\det A = \pm 1$. It is a subgroup of $GL(n; \mathbb{C})$ and the limit of a sequence of orthogonal matrices is orthogonal, because the limit preserves the relation $A^T A = I$ that characterizes orthogonal matrices. The special orthogonal group $SO(n)$ is the group of orthogonal matrices with determinant one. It is a subgroup of $GL(n; \mathbb{C})$ and it is closed given that limits preserve both orthogonality and the property of having determinant one.

Example 1.5. The unitary and special unitary groups $U(n)$ and $SU(n)$

The unitary group $U(n)$ is the group of $n \times n$ unitary matrices, these are matrices whose column vectors are orthonormal. Since $U(n)$ is a subgroup of $GL(n; \mathbb{C})$ and the limit of unitary matrices is unitary, we get that $U(n)$ is a matrix Lie group.

The special unitary group $SU(n)$ is the group of unitary matrices with determinant one. Once again, given that the property of having determinant one is preserved by limits, and so is being unitary, $SU(n)$ is a matrix Lie group.

1.2 Properties of Matrix Lie Groups, Homomorphisms and Isomorphisms

Definition 1.6. A matrix Lie group G is said to be **compact** if it satisfies the following conditions:

1. For any sequence A_n in G such that converges to a matrix A , then $A \in G$.
2. There exists a constant C such that for all $A \in G$, $|A_{ij}| \leq C$ for all $1 \leq i, j \leq n$.

In other words, G is compact if it is closed and bounded in $\mathbb{C}^{n^2}$

For example, the orthogonal and special orthogonal groups $O(n)$ and $SO(n)$ are both compact taking $C = 1$. However, $GL(n; \mathbb{C})$ and $GL(n; \mathbb{R})$ are not compact because we can find a sequence with limit a non-invertible matrix. For instance, $A_k = \text{diag}(1/k, 1, \dots, 1)$ converges to $A = \text{diag}(0, 1, \dots, 1)$.

Definition 1.7. A matrix Lie group G is said to be **connected** if for any given matrices $A, B \in G$, there exists a continuous path with initial point A and final point B .

A non-connected matrix Lie group can be decomposed as a union of several connected pieces called **components**.

For example, $GL(n; \mathbb{C})$ is connected for all $n \geq 1$, but $GL(n; \mathbb{R})$ is not connected for any n . This follows because all matrices in $GL(n; \mathbb{R})$ have $\det(A) \neq 0$. Thus, we can define $GL^+(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) : \det(A) > 0\}$ and $GL^-(n, \mathbb{R}) = \{A \in GL(n, \mathbb{R}) : \det(A) < 0\}$. They are disjoint, then $GL(n; \mathbb{R})$ is not connected.

Definition 1.8. We say that a matrix Lie group is **simply connected** if every loop in G can be shrunk continuously to a point in G .

For example, $SU(2)$ is simply connected, but $U(n)$ is not for any n . For instance, if we consider $\gamma : [0, 1] \rightarrow U(n)$ such that $\gamma(t) = \text{diag}(e^{2\pi it}, 1, \dots, 1)$, we have a non-contractible loop.

Definition 1.9. Let G and H be matrix Lie groups. A map $\varphi : G \rightarrow H$ is called a **Lie group homomorphism** if it is a group homomorphism and continuous. In addition, if it is one-to-one and onto and the inverse map is continuous, then we say it is a **Lie group isomorphism**.

For any G and H matrix Lie groups, we say that they are **isomorphic** $G \cong H$ if there exists a Lie group isomorphism between them.

1.3 Lie Groups

A Lie group can be thought of as a structure such that is simultaneously a group and a differential manifold.

In this work we focus on matrix Lie groups, a specific kind of Lie groups.

This decision is, partially, brought about by a desire to make it easier to understand the concept of Lie algebra. However, the main reason for this simplification is the fact that the Lie groups we focus on are precisely matrix Lie groups.

For those interested in the comprehension of the theory of differential manifolds needed to define Lie group, consult [14].

Definition 1.10. A *Lie group* is a differential manifold G which is also a group, and such that the group product $G \times G \rightarrow G$ and the inverse map, $g \rightarrow g^{-1}$ for all $g \in G$, are differentiable.

Theorem 1.11. As suggested by the terminology, every matrix Lie group is a Lie group. Nevertheless, not all Lie groups correspond to a matrix Lie group.

Proof. For a discussion on this topic, consult [7, Section 1.8]. □

1.4 Lie Algebras of Matrix Lie Groups

In this section we define the concept of Lie algebra of a matrix Lie group, see some examples corresponding to the matrix Lie groups seen earlier and give some basic properties.

Definition 1.12. Let G be a matrix Lie group. Then the *Lie algebra of G* is the set of matrices X such that $e^{tX} \in G$ for all $t \in \mathbb{R}$. We denote it by $\mathfrak{g}$.

For a complete discussion on the matrix exponential used in the definition, consult [10, Chapter 10].

1.4.1 Examples

We now present examples of the Lie algebras of matrix Lie subgroups introduced earlier.

Example 1.13. The general linear groups

Given a $n \times n$ complex matrix, then e^{tX} is invertible. Thus, the Lie algebra of $GL(n; \mathbb{C})$ is the space of all $n \times n$ complex matrices, denoted $\mathfrak{gl}(n; \mathbb{C})$. Analogously, the space of all $n \times n$ real matrices $\mathfrak{gl}(n; \mathbb{R})$ is the Lie algebra of $GL(n; \mathbb{R})$.

Example 1.14. The special linear groups

The Lie algebra of $SL(n; \mathbb{C})$ is the space of all $n \times n$ complex matrices with trace zero and it is denoted by $sl(n; \mathbb{C})$. This follows from the identity $\det e^X = e^{\text{trace} X}$, thus, if $\text{trace} X = 0$, then $\det e^{tX} = 1$ and $e^{tX} \in SL(n; \mathbb{C})$ for all t real. On the other hand, if X is such that $e^{tX} \in SL(n; \mathbb{C})$ for all $t \in \mathbb{R}$, then $e^{(t)\text{trace} X} = 1$ for all $t \in \mathbb{R}$, which means that $(t)\text{trace} X$ is an integer multiple of $2\pi i$, thus $\text{trace} X = 0$. Analogously we define $sl(n; \mathbb{R})$ Lie algebra of $SL(n; \mathbb{R})$.

Example 1.15. The unitary groups

The Lie algebra of $U(n)$ is $u(n)$, the space of $n \times n$ complex matrices X such that the conjugate transpose X^* satisfies $X^* = -X$. Indeed, $U \in U(n)$ if and only if $U^* = U^{-1}$. Hence, $e^{tX} \in U(n)$ if and only if $(e^{tX})^* = e^{tX^*} = (e^{tX})^{-1} = e^{-tX}$, thus, if and only if $X^* = -X$. For $SU(n)$ we can see that its Lie algebra is $su(n)$, the space of $n \times n$ complex matrices X such that the conjugate transpose X^* satisfies $X^* = -X$ with trace zero.

1.4.2 Properties of Lie Algebras

We now establish the basic properties of the Lie algebras of a matrix Lie group.

Theorem 1.16. Take G a matrix Lie group with $\mathfrak{g}$ its Lie algebra and $X, Y \in \mathfrak{g}$. Then

1. $sX \in \mathfrak{g}$ for all $s \in \mathbb{R}$.
2. $X + Y \in \mathfrak{g}$
3. $XY - YX \in \mathfrak{g}$

Proof. Consult [7, Theorem 2.18]. □

Definition 1.17. Given two $n \times n$ matrices A, B , the **bracket or commutator** of A and B is defined by

$$[A, B] = AB - BA$$

In particular, by Theorem 1.16, the Lie algebra of any matrix Lie group is closed under the bracket. We then use the terminology **Lie bracket**.

The following theorem shows that given a homomorphism between matrix Lie groups, it raises to a map between the corresponding Lie algebras.

Theorem 1.18. *Let G and H be matrix Lie groups with $\mathfrak{g}$ and $\mathfrak{h}$ the corresponding Lie algebras. Suppose that there exists a Lie group homomorphism $\phi : G \rightarrow H$. Then there exists a unique real map $\tilde{\phi} : \mathfrak{g} \rightarrow \mathfrak{h}$ such that*

$$\phi(e^X) = e^{\tilde{\phi}(X)} \quad \text{for all } X \in \mathfrak{g}$$

It has the following properties:

1. $\tilde{\phi}(AXA^{-1}) = \phi(A) \tilde{\phi}(X) \phi(A)^{-1}$ for all $X \in \mathfrak{g}$, $A \in G$
2. $\tilde{\phi}([X, Y]) = [\tilde{\phi}(X), \tilde{\phi}(Y)]$ for all $X, Y \in \mathfrak{g}$
3. $\tilde{\phi}(X) = \left. \frac{d}{dt} \right|_{t=0} \phi(e^{tX})$ for all $X \in \mathfrak{g}$

If G, H and K are matrix Lie groups and $\phi : H \rightarrow K$ and $\rho : G \rightarrow H$ are Lie group homomorphisms, then

$$\widetilde{\phi \circ \rho} = \tilde{\phi} \circ \tilde{\rho}$$

Proof. Consult [7, Theorem 2.21]. □

Notice that we can compute $\tilde{\phi}$, given a homomorphism ϕ and a basis of $\mathfrak{g}$, using Property 3.

Definition 1.19. *Let G be a matrix Lie group with Lie algebra $\mathfrak{g}$. Then for each $A \in G$ we define a linear map $\text{Ad}A : \mathfrak{g} \rightarrow \mathfrak{g}$ such that*

$$\text{Ad}A(X) = AXA^{-1}$$

*We denote Ad , the map $A \mapsto \text{Ad}A$. This map is called **the adjoint mapping**.*

Theorem 1.20. *Given G a matrix Lie group with Lie algebra $\mathfrak{g}$, $\text{Ad} : G \rightarrow \text{GL}(n; \mathbb{C})$ is a group homomorphism.*

Proof. Consult [7, Proposition 2.23]. □

1.5 Lie Algebras

Now that we have familiarized ourselves with the idea of Lie algebra of a matrix Lie group, we define Lie algebra for any Lie group.

Definition 1.21. *A **finite-dimensional real or complex Lie algebra** is a finite-dimensional real or complex vector space $\mathfrak{g}$ endowed with a map $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ such that satisfies the following*

1. $[\cdot, \cdot]$ is bilinear
2. **Skew-symmetry:** $[x, y] = -[y, x]$ for all $x, y \in \mathfrak{g}$
3. **Jacobi identity:** $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in \mathfrak{g}$

In particular, the second condition implies that $[x, x] = 0$ for all $x \in \mathfrak{g}$.

Throughout this work all Lie algebras are assumed to be complex and finite-dimensional, even if it is not being said explicitly. This assumption is essential to develop the structure theory of semisimple Lie algebras.

Definition 1.22. A Lie algebra $\mathfrak{g}$ is called *abelian* if its Lie bracket is identically zero, i.e.

$$[x, y] = 0 \quad \text{for all } x, y \in \mathfrak{g}.$$

Proposition 1.23. Both $\mathfrak{gl}(n; \mathbb{C})$ and $\mathfrak{gl}(n; \mathbb{R})$ are Lie algebras with respect to the bracket operation $[A, B] = AB - BA$.

Let V be a finite-dimensional real or complex vector space with $\mathfrak{gl}(V)$ the space of linear maps of V into itself. Then $\mathfrak{gl}(V)$ endowed with the bracket operation $[a, b] = ab - ba$ is a Lie algebra.

Proof. The only non-trivial step is proving the Jacobi identity, which can be established by direct computation. $\square$

Definition 1.24. A *subalgebra* of a real or complex Lie algebra $\mathfrak{g}$ is a subspace $\mathfrak{h} \subseteq \mathfrak{g}$ such that $[h_1, h_2] \in \mathfrak{h}$ for all $h_1, h_2 \in \mathfrak{h}$.

If $\mathfrak{g}$ is a complex Lie algebra and $\mathfrak{h}$ is a real subspace of $\mathfrak{g}$ closed under brackets, then $\mathfrak{h}$ is said to be a *real subalgebra* of $\mathfrak{g}$.

Definition 1.25. If $\mathfrak{g}$ and $\mathfrak{h}$ are Lie algebras, then a linear map $\phi : \mathfrak{g} \rightarrow \mathfrak{h}$ is called a *Lie algebra homomorphism* if it satisfies $\phi([x, y]) = [\phi(x), \phi(y)]$ for all $x, y \in \mathfrak{g}$. In addition, if ϕ is one-to-one and onto, then it is called a *Lie algebra isomorphism*. A Lie algebra isomorphism with itself is a *Lie algebra automorphism*.

The next remarkable results support our decision of focusing in matrix Lie groups.

Proposition 1.26. The Lie algebra $\mathfrak{g}$ of a matrix Lie group G is a real Lie algebra.

Proof. Consult [7, Proposition 2.39]. $\square$

Theorem 1.27 (Ado). *Every finite-dimensional real Lie algebra is isomorphic to a subalgebra of $\mathfrak{gl}(n; \mathbb{R})$. The statement holds for $\mathbb{C}$.*

This theorem is proved in Varadarajan and is too complex to prove it in this dissertation. As a matter of fact, full thesis have been written on this topic, consult [15].

Proposition 1.26 together with Ado's Theorem 1.27 support our decision to focus on matrix Lie groups. They imply that all finite-dimensional Lie algebras of matrix Lie groups can be thought of as subalgebras of $\mathfrak{gl}(n; \mathbb{R})$, which are easier to comprehend.

Chapter 2

Basic representation theory

2.1 Motivation to Study Representations

In this chapter we study the concept of representation of matrix Lie groups and representations of Lie algebras.

A representation can be seen as an action of a matrix Lie group or Lie algebra over a vector space.

For example, they are useful in situations where we are given a system that is symmetric. Then, the set of symmetries forms a group and understanding the representations of this group allows us to be able to apply symmetries to simplify the problem.

For instance, consider a differential equation in three-dimensional space which has a rotational symmetry. The space of solutions is invariant under the action of rotations. Therefore, the space of solutions generates a representation of the rotation group $SO(3)$. Consequently, knowledge of the representations of $SO(3)$ often simplifies finding solutions to the equation.

In addition, understanding the representations of a group can also provide information about the said group. Similarly, with the representations of Lie algebras.

Our focus is on finite-dimensional, irreducible, complex representations of matrix Lie groups. We see that this problem is almost equivalent to the problem of computing finite-dimensional, irreducible, complex representations of the corresponding Lie algebras. After establishing the main notions and results, we consider some examples of representations. Finally, we study the finite-dimensional, complex, irreducible representations of $\mathfrak{su}(2)$. This is going to be fundamental to understand semisimple Lie algebras and the notion of roots that are presented in the following chapter.

For a more extensive treatment of basic representation theory, consult Chapter

4, Basic Representation Theory, of [7].

2.2 Representations

In this next section we give some basic definitions regarding representation theory and, to end with, two relevant propositions.

The first proposition allows us to give a relation between a representation of a matrix Lie group and a representation of the corresponding Lie algebra. The second proposition characterizes, given a representation of a real Lie algebra $\mathfrak{g}$, the corresponding representation in the complexification of $\mathfrak{g}$.

Definition 2.1. A *finite-dimensional complex representation of a given G matrix Lie group* is a Lie group homomorphism

$$\Pi : G \rightarrow \mathrm{GL}(n; \mathbb{C}) (n \geq 1)$$

Realize that we could also say, more generally,

$$\Pi : G \rightarrow \mathrm{GL}(V)$$

Where V is a finite-dimensional complex vector space such that $\dim(V) \geq 1$.

We define a *finite-dimensional real representation of G* similarly. Instead of $\mathbb{C}$, we consider $\mathbb{R}$, or, more generally, take V finite-dimensional real vector space.

Definition 2.2. A *finite-dimensional complex representation of a given $\mathfrak{g}$ Lie algebra* is a Lie algebra homomorphism

$$\pi : \mathfrak{g} \rightarrow \mathfrak{gl}(n; \mathbb{C}) (n \geq 1)$$

More generally,

$$\pi : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$$

Here V is a finite-dimensional real vector space such that $\dim(V) \geq 1$. A *finite-dimensional real representation* is defined analogously, but V is a finite-dimensional real vector space and we take $\mathfrak{gl}(n; \mathbb{R})$.

Definition 2.3. We say that a representation is *faithful* if the homomorphism is one-to-one.

As said previously, a representation can be thought of as the action of a group or Lie algebra on a vector space. Thus, $\Pi(g)$ with $g \in G$ or $\pi(g)$ with $g \in \mathfrak{g}$, depending on whether we are working with groups or Lie algebras, is the operator related to this action.

As for terminology, we say, for instance, " Π is a representation of the matrix Lie group G acting on the vector space V ". We also use the notation (Π, V) to denote such representation and, analogously, for Lie algebras.

Finally, even if $\mathfrak{g}$ is a real Lie algebra, we usually consider complex representations of $\mathfrak{g}$.

Definition 2.4. A subspace W of a complex vector space V is called **invariant** for a (real or complex) representation Π of a matrix Lie group G over V , if

$$\Pi(A)w \in W \quad \forall w \in W, \forall A \in G$$

Definition 2.5. An invariant subspace W of V is **non-trivial** if $W \neq 0$ and $W \neq V$.

Definition 2.6. An **irreducible** representation has no non-trivial invariant subspaces.

For Lie algebras, the terms **invariant**, **non-trivial** and **irreducible** are defined analogously.

Definition 2.7. Consider Π a representation of G , matrix Lie group, acting on the vector space V and Σ a representation of G over the vector space W . A linear map $\phi : V \rightarrow W$ is a **morphism of representations** if

$$\phi(\Pi(A)v) = \Sigma(A)\phi(v) \quad \forall A \in G, \forall v \in V$$

We can define analogously the morphism of representations of a Lie algebra.

Definition 2.8. An **isomorphism** of representations ϕ is a morphism of representations that is also invertible. Two vector spaces, V and W , are said to be **isomorphic** or **equivalent** if there exists an isomorphism between them.

If we are given two isomorphic representations, we can consider them "the same one". A typical problem in representation theory is determining, up to isomorphism, all the irreducible representations of a certain group or Lie algebra. That is, precisely, the culmination of this chapter. We define the structure of all finite-dimensional, complex, irreducible representations of $\mathfrak{su}(2)$.

Proposition 2.9. Consider G a matrix Lie group with Lie algebra $\mathfrak{g}$. Let Π be a representation of G acting on V , a vector space. Then, there exists a unique representation of $\mathfrak{g}$ acting on V , that we denote by π , such that

$$\Pi(e^X) = e^{\pi(X)} \quad \forall X \in \mathfrak{g}.$$

In addition, π can be computed as

$$\pi(X) = \left. \frac{d}{dt} \right|_{t=0} \Pi(e^{tX})$$

and satisfies

$$\pi(AXA^{-1}) = \Pi(A) \pi(X) \Pi(A)^{-1} \quad \forall X \in \mathfrak{g}, \forall A \in G$$

Proof. By Theorem 1.18, for each homomorphism of Lie groups $\phi : G \rightarrow H$, there exists an associated homomorphism of Lie algebras $\tilde{\phi} : \mathfrak{g} \rightarrow \mathfrak{h}$. This association is precisely given by $\phi(e^X) = e^{\tilde{\phi}(X)}$. In particular, we can take $H = \mathrm{GL}(V)$ and $\phi = \Pi$. Since the Lie algebra of $\mathrm{GL}(V)$ is $\mathfrak{gl}(V)$, because the exponential of any operator is invertible, we know that $\tilde{\phi} = \pi$. Accordingly, π goes from $\mathfrak{g}$ to $\mathfrak{gl}(V)$ and is a representation of $\mathfrak{g}$. In addition, it satisfies the stated properties. This follows from the fact that given ϕ , the associated $\tilde{\phi}$ satisfies such properties. $\square$

The following proposition is going to become a key ingredient in our study in the following section. It shows the relation between representations of real Lie algebras and representations of the complexification of such Lie algebras.

The complexification of a real Lie algebra is the complex Lie algebra obtained by extending the scalars from $\mathbb{R}$ to $\mathbb{C}$.

Proposition 2.10. *Consider a real Lie algebra $\mathfrak{g}$ and its complexification $\mathfrak{g}_{\mathbb{C}}$. Then every finite-dimensional representation π of $\mathfrak{g}$ can be uniquely extended to a complex linear representation of $\mathfrak{g}_{\mathbb{C}}$. We also call this extension π and it satisfies*

$$\pi(x + iy) = \pi(x) + i\pi(y) \quad \forall x, y \in \mathfrak{g}$$

Proof. It follows from the **universal property** of the complexification of a real Lie algebra. Let $\mathfrak{g}$ be a real Lie algebra, $\mathfrak{g}_{\mathbb{C}}$ its complexification, and $\mathfrak{h}$ be an arbitrary complex Lie algebra. Then, every real Lie algebra homomorphism from $\mathfrak{g}$ to $\mathfrak{h}$ extends uniquely to a complex Lie algebra homomorphism from $\mathfrak{g}_{\mathbb{C}}$ to $\mathfrak{h}$.

For further information on the universal property, consult [13, Chapter 1, Section 3]. The proof to this proposition can be found in [7, Proposition 3.39]. $\square$

Definition 2.11. *A **unitary representation** of G , a matrix Lie group, is a homomorphism $\Pi : G \rightarrow \mathrm{U}(\mathbb{H})$ that satisfies the following continuity condition*

$$\text{If } A_n \rightarrow A, \text{ with } A_n, A \in G \Rightarrow \Pi(A_n)v \rightarrow \Pi(A)v \quad \forall v \in \mathbb{H} \quad (2.1)$$

where $\mathbb{H}$ is a Hilbert space and $\mathrm{U}(\mathbb{H})$ is the group of unitary operators on $\mathbb{H}$.

Recall that a Hilbert space is a vector space provided with an inner product such that is complete, therefore, all Cauchy sequences converge. A unitary operator on a Hilbert space is an operator such that is surjective, bounded, and preserves the inner product. The continuity condition (2.1) is called **strong continuity**.

Defining an analogous result for Lie algebras has proved to be too technically challenging, according to literature.

2.3 Examples of Representations

This section allows us to obtain a better understanding of what a representation is, beyond the definition. Some relevant and basic representations are presented. Some of them will be useful for further sections.

2.3.1 The Trivial Representation

The most straightforward representation to define is the **trivial representation** of G , any matrix Lie group, over the 1-dimensional complex vector space $\mathbb{C}$. It is given by

$$\Pi : G \rightarrow \text{GL}(1; \mathbb{C})$$

$$\Pi(A) = 1 \quad \forall A \in G$$

It is an irreducible representation, given that $\mathbb{C}$ has no non-trivial subspaces and, therefore, no non-trivial invariant subspaces.

Similarly, we can define the **trivial representation of a Lie algebra** $\mathfrak{g}$

$$\pi : \mathfrak{g} \rightarrow \mathfrak{gl}(1; \mathbb{C})$$

$$\pi(x) = 0 \quad \forall x \in \mathfrak{g}$$

With an argument analogous to the one we have used for matrix Lie groups, this representation is irreducible.

2.3.2 The Standard Representation

By definition, a matrix Lie group G is a subspace of $\text{GL}(n; \mathbb{C})$ or $\text{GL}(n; \mathbb{R})$ for some integer n . Hence, we can take the inclusion map and it is, in fact, a representation of G given by

$$\Pi(A) = A \quad \forall A \in G$$

This representation is called the **standard representation** of G .

Analogously, the **standard representation** of $\mathfrak{g}$, Lie algebra, is given by the inclusion map from $\mathfrak{g}$ to $\mathfrak{gl}(n; \mathbb{R})$ or $\mathfrak{gl}(n; \mathbb{C})$.

2.3.3 The Adjoint Representation

This representation will be relevant in the upcoming sections and chapter.

To define the **adjoint representation** of a matrix Lie group G with corresponding $\mathfrak{g}$ Lie algebra, we use the adjoint map defined in Definition 1.19

$$\text{Ad} : G \rightarrow \text{GL}(\mathfrak{g})$$

$$\text{Ad}A(X) = AXA^{-1}$$

The adjoint map is a Lie group homomorphism; therefore, it defines a representation of G over $\mathfrak{g}$ called the **adjoint representation**.

Now, if we consider $\mathfrak{g}$, a Lie algebra, and the following linear map

$$\text{ad} : \mathfrak{g} \rightarrow \mathfrak{gl}(\mathfrak{g})$$

$$\text{ad}x(y) = [x, y]$$

This map is a Lie algebra homomorphism; consequently, it defines a representation. This is also called the **adjoint representation**.

Proposition 2.12. *Let $\mathfrak{g}$ be a Lie algebra. Then the following is true*

$$\text{ad}([x, y]) = [\text{ad} x, \text{ad} y]$$

Proof. It suffices to do a direct computation using the Jacobi identity. □

Note that Ad and ad are related as follows, by Proposition 2.9,

$$\text{Ad}(e^X) = e^{\text{ad}(X)} \quad \forall X \in \mathfrak{g}$$

The representations presented in this section need not be different from each other.

For instance, in the case of $\text{SO}(3)$, the adjoint representation and the standard representation are both 3-dimensional and, as a matter of fact, they are equal. This is left as an exercise in [7, Chapter 5, 10 Exercises].

2.4 The Irreducible Representations of $\mathfrak{su}(2)$

We now arrive at the main goal of this chapter: the description of finite-dimensional, irreducible, complex representations of $\mathfrak{su}(2)$.

There are multiple reasons why this computation is relevant.

Mainly, it allows us to understand the structure of semisimple Lie algebras, which we will see in the next chapter.

In addition, the study of the irreducible, finite-dimensional, complex representations of $\mathfrak{su}(2)$ shows how commutators can determine representations of Lie algebras.

From an applied perspective, since $\mathfrak{su}(2) \cong \mathfrak{so}(3)$, we would be able to also study representations of $\mathfrak{so}(3)$ and these are useful, for example, in fields such as quantum physics.

We denote $\mathfrak{sl}(2; \mathbb{C})$, the complexification of $\mathfrak{su}(2)$.

As seen previously, in Proposition 2.10, we can uniquely extend every finite-dimensional, complex representation of $\mathfrak{su}(2)$ to a representation of $\mathfrak{sl}(2; \mathbb{C})$.

Recall that π is a complex irreducible representation of $\mathfrak{su}(2)$ acting on V , complex space, if there exists no non-trivial invariant complex subspace W of V .

Proposition 2.13. *A complex representation π of $\mathfrak{su}(2)$ is irreducible if and only if π , seen as the extension in $\mathfrak{sl}(2; \mathbb{C})$, is an irreducible representation of $\mathfrak{sl}(2; \mathbb{C})$.*

Proof. We give an idea of each implication separately.

For the left-to-right direction, it can be proven using that $\mathfrak{su}(2) \subseteq \mathfrak{sl}(2; \mathbb{C})$.

Consider now the other direction. If there was a non-trivial subspace W invariant for π in $\mathfrak{su}(2)$, then it would be invariant for $\pi(X + iY) = \pi(X) + i\pi(Y)$ for any X, Y in $\mathfrak{su}(2)$. Thus, we reach a contradiction because, by hypothesis, π is an irreducible representation of $\mathfrak{sl}(2; \mathbb{C})$. For a more complete proof, consult [7, Proposition 4.6.]. $\square$

This result is important because it allows us to work in the complexified Lie algebra $\mathfrak{sl}(2; \mathbb{C})$, which simplifies computations.

In the remainder of this section, we define a basis of $\mathfrak{sl}(2; \mathbb{C})$ and, with the relations given by commutators in a complex vector space V , we characterize a representation of $\mathfrak{sl}(2; \mathbb{C})$.

In addition, following from this representation, we define all finite-dimensional, complex, irreducible representations of $\mathfrak{sl}(2; \mathbb{C})$. This is equivalent to defining all finite-dimensional, complex, irreducible representations of $\mathfrak{su}(2)$, by Proposition 2.13.

Define the following basis of $\mathfrak{sl}(2; \mathbb{C})$:

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Using the Lie bracket or commutator, we can define the following relations:

$$\begin{aligned} [H, X] &= 2X \\ [H, Y] &= -2Y \\ [X, Y] &= H \end{aligned}$$

From now until the end of this chapter, we will use the notation X, Y, H to refer to this basis of $\mathfrak{sl}(2; \mathbb{C})$. This allows us to define a representation of $\mathfrak{sl}(2; \mathbb{C})$.

In consequence, we are able to state and prove the main theorem of this chapter, which characterizes all irreducible representations of $\mathfrak{sl}(2; \mathbb{C})$.

Now, let V be a finite-dimensional, complex vector space. Throughout the chapter, V refers to such vector space.

Take A, B and C in $\mathfrak{gl}(V)$ that satisfy the following relations:

$$\begin{aligned} [A, B] &= 2B \\ [A, C] &= -2C \\ [B, C] &= A \end{aligned}$$

Then we can define a linear map $\pi : \mathfrak{sl}(2; \mathbb{C}) \rightarrow \mathfrak{gl}(V)$ such that $\pi(H) = A$, $\pi(X) = B$, $\pi(Y) = C$.

Proposition 2.14. *The linear map π that satisfies the previous relations is a representation of $\mathfrak{sl}(2; \mathbb{C})$*

Proof. This proof is based on the bilinearity and skew-symmetry of the commutator.

It might not be technically challenging to prove this statement. However, it is interesting to see how this representation is built to get a better understanding of it, because it is going to be key to develop the rest of our theory regarding irreducible representations of $\mathfrak{sl}(2; \mathbb{C})$

To check that π is a representation we have to see that it satisfies $\pi([x, y]) = [\pi(x), \pi(y)]$ for any x, y in $\mathfrak{sl}(2; \mathbb{C})$. In other words, we have to check that π is a homomorphism of Lie algebras.

It is enough to prove it for the elements in the basis we have defined for $\mathfrak{sl}(2; \mathbb{C})$, i.e., for H , X and Y .

$$\begin{aligned} [\pi(X), \pi(Y)] &= [B, C] = A = \pi(H) = \pi([X, Y]) \\ \frac{1}{2}[\pi(H), \pi(X)] &= \frac{1}{2}[A, B] = B = \pi(X) = \frac{1}{2}\pi([H, X]) \\ -\frac{1}{2}[\pi(H), \pi(Y)] &= -\frac{1}{2}[A, C] = C = \pi(Y) = -\frac{1}{2}\pi([H, Y]) \end{aligned}$$

The result follows. $\square$

Our goal is to be able to study the finite-dimensional, irreducible representations of $\mathfrak{sl}(2; \mathbb{C})$ and understand them.

Let π be an irreducible representation of $\mathfrak{sl}(2; \mathbb{C})$ acting on a finite-dimensional, complex space V . Given that we are working on an algebraically closed field, $\mathbb{C}$, we know that $\pi(H)$ has at least one eigenvector.

Lemma 2.15. *Let u be an eigenvector of the operator $\pi(H)$ on V , with eigenvalue α . Then the following is true.*

$$\begin{aligned} \pi(H)(\pi(X))u &= (\alpha + 2)\pi(X)u \\ \pi(H)(\pi(Y))u &= (\alpha - 2)\pi(Y)u \end{aligned}$$

Proof. These equalities are deduced using that $\pi(H)u = \alpha u$ and the commutator relations of H , X and Y shown earlier. A full explanation may be found in [5, Section 11.1.]. $\square$

From now on, we will call $\pi(X)$ the "rising operator" and $\pi(Y)$ the "lowering operator".

As established earlier, $\pi(H)$ must have, at least, one non-zero eigenvector u with some complex eigenvalue α . This implies that

$$\pi(H)\pi(X)^n u = (\alpha + 2n)\pi(X)^n u, \quad \forall n \geq 0.$$

and $\pi(X)^n u$ is either an eigenvector for $\pi(H)$ with eigenvalue $\alpha + 2n$ or is 0. Since $\pi(H)$ is an operator on a finite-dimensional space, V , it has finitely-many distinct eigenvalues. Since the eigenvalues $\alpha + 2n$ are all distinct, if all vectors $\pi(X)^n u$ were nonzero, then $\pi(H)$ would have infinitely many distinct eigenvalues. As a result, not all $\pi(X)^n u$ are different from 0. Thus, there exists $N > 0$ such that

$$\pi(X)^N u \neq 0$$

but

$$\pi(X)^{N+1} u = 0$$

Let $V_\alpha = \{v \in V \mid (\pi(H) - \alpha I)^k v = 0 \text{ for some } k > 0\}$. Then, $\pi(X) : V_\alpha \rightarrow V_{\alpha+2}$, by the definition of the rising operator $\pi(X)$ given in Lemma 2.15. Note that, by definition, all eigenvalues of the form $\alpha + 2n$ with $n \geq 0$ are different. It follows that all sets $V_{\alpha+2n}$ with $n \geq 0$ are different, so the action of $\pi(X)$ is not periodic. Together with V being finite-dimensional, this implies that we eventually reach $V_{\alpha+2(N+1)} = \{0\}$, for some N .

Define $u_0 = \pi(X)^N u$ and $\lambda = \alpha + 2N$. Then, we have $\pi(H)u_0 = \lambda u_0$ and $\pi(X)u_0 = 0$. Consider $u_k = \pi(Y)^k u_0$ with $k \geq 0$. This implies that, by applying Lemma 2.15, we obtain $\pi(H)u_k = (\lambda - 2k)u_k$. Given that $\pi(H)$ has finitely many eigenvalues, not all u_k can be different from 0.

Lemma 2.16. *With the previous definitions, we have:*

$$\begin{aligned}\pi(X)u_k &= [k\lambda - k(k-1)]u_{k-1} \quad (k > 0) \\ \pi(X)u_0 &= 0\end{aligned}$$

Proof. Proven by induction on k . Seen in [5, Section 11.1.]. □

We are now in a position to prove the following theorem, which is the culmination of this chapter.

Theorem 2.17. *For each integer $m \geq 0$, there exists an irreducible representation of $\mathfrak{sl}(2; \mathbb{C})$ over a finite-dimensional complex vector space with dimension $m + 1$. If we take any two irreducible representations of $\mathfrak{sl}(2; \mathbb{C})$ over finite-dimensional complex vector spaces with the same dimension, then they are equivalent.*

Proof. We consider π , an irreducible representation of $\mathfrak{sl}(2, \mathbb{C})$ over a finite-dimensional complex vector space V . As established earlier, since we are working in an algebraically closed field, $\pi(H)$ must have at least one eigenvector u with eigenvalue α .

In addition, we have proved the following

$$\begin{aligned}\pi(H)(\pi(X))u &= (\alpha + 2)\pi(X)u \\ \pi(H)(\pi(Y))u &= (\alpha - 2)\pi(Y)u\end{aligned}$$

Consequently, we can define, as seen previously, the succession $u_0 = \pi(X)^N u$, for some $N > 0$, and $u_k = \pi(Y)^k u_0$ with $k \geq 0$. Recall that we take N such that $\pi(X)^{N+1} u = 0$. We have also shown that these vectors are eigenvectors of $\pi(H)$. Indeed u_0 has eigenvalue $\lambda = \alpha + 2N$ and the vectors u_k have eigenvalue $\lambda - 2k$, respectively.

$\pi(H)$ has finitely many eigenvalues, it follows that there must exist $m > 0$ integer, such that

$$u_k \neq 0 \quad \forall k \leq m$$

but

$$u_{m+1} = 0$$

Applying now the last lemma

$$0 = \pi(H)u_{m+1} = [(m+1)\lambda - m(m+1)]u_m = (m+1)(\lambda - m)u_m$$

Given that, by definition, $u_m \neq 0$ and $m+1 \neq 0$, we have that $\lambda - m = 0$, so, $\lambda = m$.

This is relevant because we have just proved that λ , which is the eigenvalue of $\pi(H)$ associated to u_0 , is an integer, and, by construction, so are the other eigenvalues of $\pi(H)$ associated to the eigenvectors u_k , with $k < m$. Thus, we use m instead of λ in the rest of the chapter.

In summary, the vectors $u_0, \dots, u_m$ satisfy the following

$$\begin{aligned} \pi(H)u_k &= (m - 2k)u_k \\ \pi(Y)u_k &= u_{k+1} \quad \text{for } k < m \\ \pi(Y)u_m &= 0 \\ \pi(X)u_k &= [km - k(k-1)]u_{k-1} \quad \text{for } k > 0 \\ \pi(X)u_0 &= 0 \end{aligned} \tag{2.2}$$

Given that $u_0, \dots, u_m$ are eigenvectors of $\pi(H)$ with different eigenvalues, hence, they are linearly independent and span a $(m+1)$ -dimensional space that is invariant under the action of $\pi(H), \pi(Y), \pi(X)$. This space, that is closed under the action of $\pi(Z)$ for any Z in $\mathfrak{sl}(2; \mathbb{C})$, must be V . Indeed, by definition, π is an irreducible representation.

We deduce that any irreducible representation has the form (2.2). What is left to check is that everything of that form is an irreducible representation.

Once that is proved we are done. The reasons are the following.

On one hand, given an integer m , if something of the form (2.2) is an irreducible representation of $\mathfrak{sl}(2; \mathbb{C})$, it is an $(m+1)$ -dimensional representation. This would prove the first statement of the theorem.

On the other hand, if we receive two irreducible representations of $\mathfrak{sl}(2; \mathbb{C})$ with the same dimension, we know that both have the form (2.2). Let them be π_1 and π_2 acting on V_1 , with basis $u_0, \dots, u_m$, and V_2 , with basis $\tilde{u}_0, \dots, \tilde{u}_m$, respectively. Proving that something with the form (2.2) is an irreducible representation allows us to define an isomorphism of representations which sends each u_k to $\tilde{u}_k$.

Hence, we will have proved the second statement of the theorem, because both representations will be equivalent.

Then, we now prove that any structure of the form (2.2) defines an irreducible representation of $\mathfrak{sl}(2; \mathbb{C})$.

We define $\pi(H)$, $\pi(X)$ and $\pi(Y)$ such that they satisfy the relations in (2.2), with the vectors u_k basis elements of a $(m+1)$ -dimensional complex vector space V .

First, we have to check that they satisfy the commutation relations needed to be a representation:

$$\begin{aligned} [\pi(H), \pi(X)] &= 2\pi(X) \\ [\pi(H), \pi(Y)] &= -2\pi(Y) \\ [\pi(X), \pi(Y)] &= \pi(H) \end{aligned}$$

We verify each of these equalities.

$$\begin{aligned} [\pi(H), \pi(X)]u_k &= \pi(H)\pi(X)u_k - \pi(X)\pi(H)u_k \\ &= [km - k(k-1)]\pi(H)u_{k-1} - \pi(X)(m-2k)u_k \\ &= [km - k(k-1)](m-2k+2)u_{k-1} - (m-2k)[km - k(k-1)]u_{k-1} \\ &= 2[km - k(k-1)]u_{k-1} \\ &= 2\pi(X)u_k \quad \forall k \end{aligned}$$

$$\begin{aligned} [\pi(H), \pi(Y)]u_k &= \pi(H)\pi(Y)u_k - \pi(Y)\pi(H)u_k \\ &= \pi(H)u_{k+1} - \pi(Y)(m-2k)u_k \\ &= (m-2(k+1))u_{k+1} - (m-2k)u_{k+1} \\ &= -2u_{k+1} \\ &= -2\pi(Y)u_k, \quad k < m \end{aligned}$$

$$\begin{aligned} [\pi(H), \pi(Y)]u_m &= \pi(H)\pi(Y)u_m - \pi(Y)\pi(H)u_m \\ &= 0 - (m-2m)\pi(Y)u_m \\ &= 0 \\ &= -2\pi(Y)u_m, \quad k = m \end{aligned}$$

$$\begin{aligned}
[\pi(X), \pi(Y)]u_k &= \pi(X)\pi(Y)u_k - \pi(Y)\pi(X)u_k \\
&= \pi(X)u_{k+1} - \pi(Y)(km - k(k-1))u_{k-1} \\
&= ((k+1)m - (k+1)k)u_k - (km - k(k-1))u_k \\
&= (m - 2k)u_k \\
&= \pi(H)u_k, \quad k < m
\end{aligned}$$

$$\begin{aligned}
[\pi(X), \pi(Y)]u_m &= \pi(X)\pi(Y)u_m - \pi(Y)\pi(X)u_m \\
&= 0 - \pi(Y)(m^2 - m(m-1))u_{m-1} \\
&= -mu_m \\
&= (m - 2m)u_m \\
&= \pi(H)u_m, \quad k = m
\end{aligned}$$

Hence, this defines a representation.

We are only left to prove irreducibility. To do so we have to see that any non-zero invariant subspace W of V is, in fact, equal to V .

Suppose that W is a non-zero invariant subspace of V . Then we can take $w \in W$ and uniquely write it as

$$w = \alpha_0 u_0 + \cdots + \alpha_m u_m$$

with at least some $\alpha_i \neq 0$. Take k_0 such that it is the biggest subindex that satisfies $\alpha_{k_0} \neq 0$. Since $\pi(X)$ lowers the index by one, because $\pi(X)u_k = [km - k(k-1)]u_{k-1}$, and $\alpha_{k_0} \neq 0$, it follows that $\pi(X)^{k_0}w$ is a non-zero multiple of u_0 . Using that W is invariant under the action of $\pi(X)$, we know that $u_0 \in W$. Now, by construction, each u_k is $u_k = \pi(Y)^k u_0$ with $k \geq 0$. Given that $u_0 \in W$ and W is invariant under $\pi(X)$ and $\pi(Y)$, we have that $u_k \in W$ for all $k \leq m$. Since the vectors u_k give a basis for V , we conclude that $W = V$, as we wanted to prove.

Finally, we have shown that anything of the form (2.2) is an irreducible representation of $\mathfrak{sl}(2; \mathbb{C})$. As we have reasoned before, this concludes the proof of the theorem.

□

Chapter 3

Semisimple Lie algebras, Cartan subalgebras and Roots

We now reach the culminating chapter of this work.

The main result of this chapter is a central theorem in Lie algebra theory related to roots and root spaces. This theorem states that for every complex, finite-dimensional semisimple Lie algebra $\mathfrak{h}$, one can find a basis of $\mathfrak{h}^*$ such that every root can be expressed as a linear combination of the elements in the basis with all coefficients non-negative or non-positive.

To reach this result, we present the concept of semisimple Lie algebra. We focus on complex, finite-dimensional semisimple Lie algebras. We consider semisimple Lie algebras defined over $\mathbb{C}$ because the construction of these structures requires an algebraically closed field with characteristic zero. Finite dimensionality is required to ensure a well-defined root space decomposition.

We use what we have studied in Section 2.4, in particular, we use that all irreducible representations of $\mathfrak{sl}(2; \mathbb{C})$ can be expressed in the form (2.2) to study the said Lie algebras.

In addition, we define the concept of Cartan subalgebra of a semisimple Lie algebra and see some results that we need to gain a deeper understanding of the structure theory of semisimple Lie algebras. To do so, we previously introduce the Killing form.

Also, we define the idea of root of the adjoint action of a given Lie algebra over its Cartan subalgebra. We see how this relates to the decomposition of the given Lie algebra.

Although the theory of representations and structure of semisimple Lie algebras is much deeper, the main theorem of this chapter is essential to develop it. It is thanks to this result that we would be able to see that all finite-dimensional, complex, semisimple Lie algebras can be classified using their corresponding Dynkin

diagrams. Once classified, the properties of such Lie algebras can be studied deeply. In addition, we would be able to understand the irreducible representations of semisimple Lie algebras.

For a more extensive treatment of the topics covered in this chapter, consult [12].

3.1 Semisimple Lie Algebras

We have previously introduced and discussed the concept of Lie algebra of a matrix Lie group and Lie algebra of any Lie group, see Section 1.4 and Section 1.5.

From this point onward, we will talk about Lie algebras and not Lie algebras of matrix Lie groups. We do so in order to develop a more general theory; however, all definitions and results hold for Lie algebras of matrix Lie groups. In fact, most of the examples given will be Lie algebras of matrix Lie groups.

In this section, we focus on a specific type of Lie algebras, semisimple Lie algebras.

We give two possible definitions of semisimple Lie algebra that will prove to be equivalent in the next sections. Plus, we state some important results regarding semisimplicity.

Definition 3.1. Let $\mathfrak{g}$ be a Lie algebra, then a subspace $I \subseteq \mathfrak{g}$ is said to be an *ideal* if $[x, y] \in I, \quad \forall x \in I, \forall y \in \mathfrak{g}$.

Much of the study of semisimple Lie algebras is done via its ideals.

For instance, given a Lie algebra $\mathfrak{g}$ we can see that $[\mathfrak{g}, \mathfrak{g}]$ is an ideal. We have that $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}$, thus $[[\mathfrak{g}, \mathfrak{g}], \mathfrak{g}] \subseteq [\mathfrak{g}, \mathfrak{g}]$.

Definition 3.2. A Lie algebra $\mathfrak{g}$ is said to be *simple* if it has no non-trivial ideals (i.e different from $\{0\}$ and $\mathfrak{g}$) and $\dim \mathfrak{g} \geq 2$.

Note that simple Lie algebras cannot be abelian. Indeed, take $\mathfrak{g}$ simple and abelian, then $[x, y] = 0$ for all $x, y \in \mathfrak{g}$. It follows that any $V \subseteq \mathfrak{g}$ is an ideal of $\mathfrak{g}$ with $\{0\} \subsetneq V \subsetneq \mathfrak{g}$. Given that $\mathfrak{g}$ is simple, this would only be possible if $\dim \mathfrak{g} = 1$, this is a contradiction.

Definition 3.3. A *semisimple* Lie algebra is a Lie algebra $\mathfrak{g}$ such that it can be expressed as a direct sum of simple Lie algebras.

Note that, for example, any simple Lie algebra is a semisimple Lie algebra.

In this chapter we consider finite-dimensional, semisimple Lie algebras over the complex numbers. Some of the complex Lie algebras we have previously seen are, actually, semisimple. For instance, $\mathfrak{sl}(n; \mathbb{C})$, $\mathfrak{so}(n; \mathbb{C})$ with $n \geq 3$ and

$\mathfrak{sp}(n; \mathbb{C})$. As a matter of fact, all of these are simple, except $\mathfrak{so}(4; \mathbb{C})$, because $\mathfrak{so}(4; \mathbb{C}) \cong \mathfrak{sl}(2; \mathbb{C}) \oplus \mathfrak{sl}(2; \mathbb{C})$. For further information on this isomorphism, check [4].

We establish some definitions to be able to give an alternative, but equivalent, definition to the concept of semisimple Lie algebra.

Definition 3.4. Define the *derived series* as a sequence of ideals of $\mathfrak{g}$ given by $\mathfrak{g}^{(0)} = \mathfrak{g}$, $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}]$, $\mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}]$, ..., $\mathfrak{g}^{(i)} = [\mathfrak{g}^{(i-1)}, \mathfrak{g}^{(i-1)}]$. Then we say that a Lie algebra $\mathfrak{g}$ is *solvable* if $\mathfrak{g}^{(n)} = 0$ for some n .

An example of solvable Lie algebras are abelian Lie algebras. Take $\mathfrak{g}$ abelian, then $[\mathfrak{g}, \mathfrak{g}] = 0$, therefore, $\mathfrak{g}^{(1)} = 0$. As a result, it is solvable.

However, any simple Lie algebra must be non-solvable. Indeed, if $\mathfrak{g}$ were simple and solvable, then $\mathfrak{g}^{(n)} = 0$ for some n . We know that $[\mathfrak{g}, \mathfrak{g}]$ is an ideal of $\mathfrak{g}$, hence, $[\mathfrak{g}, \mathfrak{g}] = \{0\}$ or $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. Since all simple Lie algebras are non-abelian, necessarily $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. Hence, there is no such n .

Theorem 3.5 (Lie's Theorem). Let $\mathfrak{g}$ be a solvable Lie algebra. Consider a representation π of $\mathfrak{g}$ on a finite-dimensional vector space V over an algebraically closed field $\mathbb{F}$ such that $\text{char } \mathbb{F} = 0$. Then there exists a basis of V for which matrices $\pi(\mathfrak{g})$ are upper triangular.

Proof. For a full proof, consult [12, Exercise. 5.3]. □

This theorem is key for proving some results in the upcoming chapters.

The following proposition allows us to define the radical and gives an alternative definition of semisimple Lie algebra.

Proposition 3.6. Consider $\mathfrak{g}$ a Lie algebra. Then

- All subalgebras and homomorphic images of $\mathfrak{g}$ solvable are solvable.
- If I is a solvable ideal of $\mathfrak{g}$ such that $\mathfrak{g}/I$ is solvable, then $\mathfrak{g}$ is solvable.
- If I, J are solvable ideals of $\mathfrak{g}$, then $I + J$ is also a solvable ideal of $\mathfrak{g}$.

Proof. For a complete proof, consult [11, Section 3.1.] □

Definition 3.7. The unique maximal solvable ideal of a Lie algebra $\mathfrak{g}$ is said to be the *radical* of $\mathfrak{g}$, $\text{Rad}(\mathfrak{g})$.

We are now in a position to give an alternative definition of semisimple Lie algebra.

Definition 3.8. A Lie algebra $\mathfrak{g}$ is said to be *semisimple* if $\text{Rad}(\mathfrak{g}) = 0$.

One could prove that both Definition 3.3 and Definition 3.8 are equivalent. In order to do so, though, one would need some of the upcoming results in this chapter such as Cartan's Criterion for Solvability and Cartan's Criterion for Semisimplicity.

Definition 3.9. Define the *descending central series* as a sequence of ideals of $\mathfrak{g}$ given by $\mathfrak{g}^0 = \mathfrak{g}$, $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$, $\mathfrak{g}^2 = [\mathfrak{g}, \mathfrak{g}^1]$, ..., $\mathfrak{g}^i = [\mathfrak{g}, \mathfrak{g}^{i-1}]$. Then we say that a Lie algebra $\mathfrak{g}$ is *nilpotent* if $\mathfrak{g}^n = 0$ for some n .

See some examples of nilpotent algebras.

Any abelian subalgebra is nilpotent. Recall that $\mathfrak{g}$ abelian, then $[\mathfrak{g}, \mathfrak{g}] = 0$. Thus, by the definition of descending central series, $\mathfrak{g}^1 = 0$, we can take $n = 1$. This implies that $\mathfrak{g}$ is nilpotent.

For any n , η_n , the set of strictly upper triangular $n \times n$ matrices, is also a nilpotent algebra.

Recall that $\mathfrak{g}^{(i)} = [\mathfrak{g}^{(i-1)}, \mathfrak{g}^{(i-1)}]$. By definition, $\mathfrak{g}^{(i)} \subseteq \mathfrak{g}^i$ for all i . Hence, any nilpotent subalgebra is also solvable.

The converse, however, is false. Consider $\mathfrak{b}_n$, the set of upper triangular matrices. One can see that this is a solvable algebra, but it is not nilpotent.

3.2 Killing form

As we discussed in the introduction of this chapter, to be able to establish and prove some key properties of Cartan subalgebras and roots, we need to define a symmetric bilinear form on $\mathfrak{g}$, a finite-dimensional, complex, semisimple Lie algebra, called the Killing form.

Definition 3.10. The Killing form is the bilinear symmetric form defined by

$$k(x, y) = \text{tr}(\text{ad } x \text{ ad } y) \quad \text{for all } x, y \in \mathfrak{g}.$$

Observe that, by its definition, it is also associative, i.e., $k([x, y], z) = k(x, [y, z])$ for each $x, y, z \in \mathfrak{g}$.

Note that if a given Lie algebra $\mathfrak{g}$ is nilpotent, then $k \equiv 0$.

We will use the following definition in the upcoming sections.

Definition 3.11. We say that a Killing form is *non-degenerate* if its radical is zero, i.e., $S = \{x \in \mathfrak{g} \mid k(x, y) = 0 \forall y \in \mathfrak{g}\}$ is the zero ideal.

Definition 3.12. We call the *orthogonal space* of $\mathfrak{h} \subseteq \mathfrak{g}$ for k the space given by $\mathfrak{h}^\perp = \{x \in \mathfrak{g} \mid k(x, y) = 0 \forall y \in \mathfrak{h}\}$.

Notice that the radical of the Killing form in $\mathfrak{g}$ is $\mathfrak{g}^\perp$.

It can be proved via a short computation that given an ideal I of a Lie algebra $\mathfrak{g}$, then $I^\perp$ is also an ideal.

Now we present some results involving the Killing form that are key in the development of the structure theory of semisimple Lie algebras. They give us a useful characterization for semisimplicity and solvability. The interested reader can find the proof and more details on [11, Section 4.3, Section 5.1].

Theorem 3.13 (Cartan's Criterion for Solvability). *Given a finite-dimensional, complex Lie algebra $\mathfrak{g}$, then it is solvable if and only if $k(\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]) = 0$.*

Theorem 3.14 (Cartan's Criterion for Semisimplicity). *Given a finite-dimensional, complex Lie algebra $\mathfrak{g}$, then it is semisimple if and only if k is non-degenerate.*

As mentioned before, this results are used to prove that Definition 3.3 and Definition 3.8 of semisimple Lie algebra are equivalent. For further information on this topic, read [11, Section 5.2.].

For our goals, we use the Definition 3.3 for semisimple Lie algebra, that was presented first in this chapter.

3.3 Cartan subalgebras

In this section we define Cartan subalgebra.

Understanding the Cartan subalgebras of finite-dimensional, complex, semisimple Lie algebras allows us not only to define roots, but also to give a decomposition of any given Lie algebra. These structures are key to develop a solid structure theory of semisimple Lie algebras.

Definition 3.15. *Consider $\mathfrak{g}$ a finite-dimensional, complex, semisimple Lie algebra. Then we say that a subspace $\mathfrak{h}$ of $\mathfrak{g}$ is a **Cartan subalgebra** if it satisfies the following:*

1. $\mathfrak{h}$ is abelian. Equivalently, given any $h_1, h_2 \in \mathfrak{h}$ they satisfy $[h_1, h_2] = 0$.
2. $\mathfrak{h}$ is a maximal abelian. That is, given $x \in \mathfrak{g}$ such that $[h, x] = 0$ for all $h \in \mathfrak{h}$, then $x \in \mathfrak{h}$.
3. For all $h \in \mathfrak{h}$, $\text{ad } h : \mathfrak{g} \rightarrow \mathfrak{g}$ is diagonalizable.

This is one possible definition of the concept of Cartan subalgebra. However, it is not the most common in the literature related to this topic. We will present an alternative and more standard definition later in this section.

We first introduce Cartan subalgebras through Definition 3.15, as it provides a clearer insight into their main properties.

Note the following. Since $\mathfrak{h}$ is abelian, then for all $h_1, h_2 \in \mathfrak{h}$, $[h_1, h_2] = 0$, thus $[\text{ad } h_1, \text{ad } h_2] = \text{ad}[h_1, h_2] = 0$. Hence, given that, by definition of a Cartan subalgebra, for all $h \in \mathfrak{h}$, $\text{ad } h$ is diagonalizable, we have that all $\text{ad } h$ are simultaneously diagonalizable. This follows from a result in linear algebra that states that a commuting family of diagonalizable operators is simultaneously diagonalizable. For a complete proof, consult [1, Result 5.76].

Definition 3.16. Consider an algebra $\mathfrak{g}$ and a subalgebra $\mathfrak{h}$. We say that the **normalizer of $\mathfrak{h}$** is the following set $N_{\mathfrak{g}}(\mathfrak{h}) = \{x \in \mathfrak{g} \mid [x, \mathfrak{h}] \subseteq \mathfrak{h}\}$.

By the Jacobi identity, $N_{\mathfrak{g}}(\mathfrak{h})$ is also a subalgebra of $\mathfrak{g}$.

If $N_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$, then we say that $\mathfrak{h}$ is **self-normalizing**.

Now we are in a position to give the alternative and most common definition of Cartan subalgebra.

Definition 3.17. Consider $\mathfrak{g}$ a semisimple Lie algebra. Then we say that a subspace $\mathfrak{h}$ of $\mathfrak{g}$ is a **Cartan subalgebra** if it satisfies the following:

1. $\mathfrak{h}$ is nilpotent;
2. $\mathfrak{h}$ is self-normalizing. That is, $N_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$.

We will see in the next section that both definitions are equivalent for finite-dimensional, complex, semisimple Lie algebras.

See some examples of Cartan subalgebras.

Example 3.18. Consider $\mathfrak{g} = \mathfrak{sl}(3; \mathbb{C})$. Define

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad ; \quad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

One can prove that if $\mathfrak{h}$ is the space spanned by H_1 and H_2 , then it is a Cartan subalgebra.

Example 3.19. Consider $\mathfrak{g} = \mathfrak{sl}(2; \mathbb{C})$ and take

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Then the space $\mathfrak{h}$ spanned by H is a Cartan subalgebra.

The following theorem gives us key characteristics of Cartan subalgebras in semisimple Lie algebras.

Theorem 3.20. *Consider $\mathfrak{g}$ a finite-dimensional, complex, semisimple Lie algebra. Then*

1. $\mathfrak{g}$ always admits a Cartan subalgebra $\mathfrak{h}$.
2. Given $\mathfrak{h}_1$ and $\mathfrak{h}_2$ Cartan subalgebras, there exists an automorphism $\varphi \in \text{Aut}(\mathfrak{g})$ such that $\varphi(\mathfrak{h}_1) = \mathfrak{h}_2$. (Chevalley's theorem)
3. Any two Cartan subalgebras have the same dimension.

Proof. The first property is fully proved in [11, Section 15.3].

The second property is also known as Chevalley's theorem. A complete proof can be found in [12, Theorem 9.1 (Chevalley)].

To end with, the last property is a consequence of the second one. $\square$

Definition 3.21. *The **rank** of a finite-dimensional, complex, semisimple Lie algebra $\mathfrak{g}$ is the dimension of a Cartan subalgebra.*

For example, one could see that in $\mathfrak{sl}(n; \mathbb{C})$, the space of diagonal matrices with trace zero is a Cartan subalgebra. Thus, the rank of $\mathfrak{sl}(n; \mathbb{C})$ is $n - 1$.

Note that the examples of Cartan subalgebras for $\mathfrak{sl}(2; \mathbb{C})$ and $\mathfrak{sl}(3; \mathbb{C})$ that we have given earlier are, precisely, the spaces of diagonal matrices with trace zero for the corresponding dimension. Hence, $\dim \mathfrak{sl}(2; \mathbb{C}) = 1$ and $\dim \mathfrak{sl}(3; \mathbb{C}) = 2$.

Proposition 3.22. *Consider a finite-dimensional, complex, semisimple Lie algebra $\mathfrak{g}$. Given $\mathfrak{h}$ a Cartan subalgebra, the Killing form in $\mathfrak{g}$ restricted to $\mathfrak{h}$ is non-degenerate.*

Proof. To prove this statement we use the properties seen in both definitions of Cartan subalgebra.

It suffices to show that if $x \in \mathfrak{h}$ satisfies $k(x, \mathfrak{h}) = 0$, then $x = 0$.

As discussed earlier, the operators $\text{ad } h$ with $h \in \mathfrak{h}$ are simultaneously diagonalizable. It follows that

$$\mathfrak{g} = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda \quad \text{with} \quad V_\lambda = \{ y \in \mathfrak{g} \mid [h, y] = \lambda(h) y \text{ for all } h \in \mathfrak{h} \}$$

where $\lambda \in \mathfrak{h}^*$ is a linear functional on $\mathfrak{h}$.

In particular,

$$V_0 = \{ y \in \mathfrak{g} \mid [h, y] = 0 \ \forall h \in \mathfrak{h} \}$$

Since $\mathfrak{h}$ is abelian, $\mathfrak{h} \subseteq V_0$. For the other inclusion, if $y \in V_0$, then $[y, h] = 0$ for all $h \in \mathfrak{h}$, hence in particular $[y, \mathfrak{h}] \subseteq \mathfrak{h}$. This shows that y belongs to the normalizer $N_{\mathfrak{g}}(\mathfrak{h})$. But $\mathfrak{h}$ is self-normalizing, it follows that $y \in \mathfrak{h}$. Thus, $V_0 = \mathfrak{h}$.

Take $\lambda = 0$, then $k(x, V_0) = k(x, \mathfrak{h}) = 0$, by assumption.

Now let $\lambda \neq 0$, let $y \in V_\lambda$ and choose $h \in \mathfrak{h}$ such that $\lambda(h) \neq 0$. Then

$$k(x, y) = \frac{1}{\lambda(h)} k(x, [h, y])$$

Recall that by invariance of the Killing form, $k(x, [h, y]) = k([x, h], y)$ and, since $\mathfrak{h}$ is abelian and $x, h \in \mathfrak{h}$, $[x, h] = 0$. Thus, $k(x, y) = 0$.

As a result, since $\mathfrak{g} = \bigoplus_\lambda V_\lambda$ and x is orthogonal for all V_λ , $k(x, \mathfrak{g}) = 0$. Recall that, since $\mathfrak{g}$ is semisimple, by Cartan's Criterion for Semisimplicity 3.14, the Killing form is non-degenerate in $\mathfrak{g}$. Hence, $x = 0$ and the Killing form is non-degenerate in $\mathfrak{h}$. $\square$

In the upcoming sections, once weights and roots are introduced, we will see that the decomposition of $\mathfrak{g}$ used in this proof is closely related to the root space decomposition.

Definition 3.23. Take $\mathfrak{h}$ a Cartan subalgebra. We denote as $\mathfrak{h}^*$ the **dual space of $\mathfrak{h}$** . This is the space of linear functionals over $\mathfrak{h}$.

3.4 Roots, Weights and Root System

In the following section we define and study the concept of root, which reminds us of eigenvalues, and how it relates to the concept of weight.

In addition, we state and prove some key properties of Cartan subalgebras and roots, which are necessary to understand the main theorem of this dissertation.

Definition 3.24. Let $\mathfrak{g}$ be a finite-dimensional, complex, semisimple Lie algebra and $\mathfrak{h}$ a Cartan subalgebra. An element $\alpha \in \mathfrak{h}^*$ is said to be a **root** for $\mathfrak{g}$ with respect to $\mathfrak{h}$ if α is non-zero and there exists $z \neq 0$ in $\mathfrak{g}$ such that

$$[h, z] = \alpha(h) z \quad \text{for all } h \in \mathfrak{h}$$

Then z is said to be a **root vector** corresponding to α and the set of all such $z \in \mathfrak{g}$ is called the **root space** corresponding to α . We denote this space $\mathfrak{g}_\alpha$.

We denote the set of all roots by Δ .

To continue, we define the concept of weight and how it relates to roots.

Definition 3.25. Take (π, V) a finite-dimensional, complex, linear representation of a Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$. Then, we say that $\mu \in \mathfrak{h}^*$ is a **weight** if there exists $v \neq 0$ in V such that

$$\pi(h)v = \mu(h)v \quad \forall h \in \mathfrak{h}$$

We then say that v is a **weight vector** for the weight μ .

Consider $\mathfrak{g}$ a finite-dimensional, complex, semisimple Lie algebra. Now that we have defined what a weight is, we can see that a root in $\mathfrak{g}$ is a non-zero weight taking as representation the adjoint representation.

We are now in a position to prove that both definitions of Cartan subalgebra seen in the previous section are equivalent.

Proposition 3.26. Definition 3.15 of Cartan subalgebra is equivalent to Definition 3.17 of Cartan subalgebra, for finite-dimensional, complex, semisimple Lie algebras.

Proof. Consider Definition 3.15 of Cartan subalgebra that states that $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ finite-dimensional, complex, semisimple Lie algebra if

1. $\mathfrak{h}$ is abelian. Equivalently, given any $h_1, h_2 \in \mathfrak{h}$ they satisfy $[h_1, h_2] = 0$.
2. $\mathfrak{h}$ is a maximal abelian. That is, given $x \in \mathfrak{g}$ such that $[h, x] = 0$ for all $h \in \mathfrak{h}$, then $x \in \mathfrak{h}$.
3. For all $h \in \mathfrak{h}$, $\text{ad } h : \mathfrak{g} \rightarrow \mathfrak{g}$ is diagonalizable.

We want to see that this implies Definition 3.17: $\mathfrak{h}$ is a Cartan subalgebra of a semisimple Lie algebra $\mathfrak{g}$ if

1. $\mathfrak{h}$ is nilpotent;
2. $\mathfrak{h}$ is self-normalizing. That is, $N_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$.

Since $\mathfrak{h}$ is abelian, it is nilpotent. Thus, it satisfies the first property established in the Definition 3.17.

On the other hand, $\mathfrak{h} \subseteq N_{\mathfrak{g}}(\mathfrak{h})$ because if we take any $h \in \mathfrak{h}$, then $[h, \mathfrak{h}] = 0$. Indeed, by definition, $\mathfrak{h}$ is abelian. Therefore, $[h, \mathfrak{h}] \in \mathfrak{h}$, so $h \in N_{\mathfrak{g}}(\mathfrak{h})$.

Let us prove the inclusion $N_{\mathfrak{g}}(\mathfrak{h}) \subseteq \mathfrak{h}$. Consider $x \in N_{\mathfrak{g}}(\mathfrak{h})$, then $[x, h] \in \mathfrak{h}$ for all $h \in \mathfrak{h}$. By abelianity of $\mathfrak{h}$, $[h, [x, h]] = 0$, then $-[h, [h, x]] = 0$. It follows that $(\text{ad } h)^2(x) = 0$ for all $h \in \mathfrak{h}$ and, since $T = \text{ad } h$ is diagonalizable, then $\text{Ker}(T^2) = \text{Ker}(T)$. Hence, $[x, h] = 0$ and, by the second property in Definition 3.15, $x \in \mathfrak{h}$.

We now prove that Definition 3.17 implies Definition 3.15.

First, we prove that all $h \in \mathfrak{h}$ is ad-semisimple. In other words, $\text{ad } h$ is semisimple for any $h \in \mathfrak{h}$, which is equivalent to being diagonalizable.

We state that, if $x \in \mathfrak{g}$, then there exist x_n and x_s such that $x = x_s + x_n$, $[x_s, x_n] = 0$ and $\text{ad } x_s$ is semisimple and $\text{ad } x_n$ is nilpotent. A full proof of this statement can be found in [12, Theorem 12.1.].

Consider now, for a given $h \in \mathfrak{h}$, $\text{ad } h$. Then, we can express it as $\text{ad } h = \text{ad } h_n + \text{ad } h_s$.

Note that $\text{ad } h(\mathfrak{h}) = [h, \mathfrak{h}] \subseteq 0$, so $\text{ad } h_n(\mathfrak{h}) \subseteq 0$. Thus, $h_n \in N_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$. This last equality is given by Definition 3.17 of Cartan subalgebra.

We know that $\mathfrak{h}$ is nilpotent, which implies that $\mathfrak{h}$ is solvable. Together with the fact that we work over $\mathbb{C}$ and that ad is a finite-dimensional representation, we can use Lie's Theorem 3.5. Thus, each adjoint action of an element in $\mathfrak{h}$ can be expressed, in a good enough basis, in upper triangular form. In particular, since $\text{ad } h_n$ is nilpotent, it must be strictly upper triangular.

Since $\text{ad } h_n$ is strictly upper triangular, we have that $k(\mathfrak{h}, h_n) = 0$, by definition of Killing form. Since the Killing form is non-degenerate in $\mathfrak{h}$, by Proposition 3.22, we must have that $h_n = 0$. Hence, $h = h_s$ and $\text{ad } h = \text{ad } h_s$. We can then affirm that $\text{ad } h$ is semisimple, this is the second property in Definition 3.15.

It is direct to see that the fact that $\mathfrak{h}$ is self-normalizing implies the second characteristic of Cartan subalgebras seen in the Definition 3.15. Indeed, take $x \in \mathfrak{g}$ such that $[x, \mathfrak{h}] = 0$, then in particular, $[x, \mathfrak{h}] \subseteq \mathfrak{h}$. Thus, $x \in N_{\mathfrak{g}}(\mathfrak{h})$ and, by self-normalization of $\mathfrak{h}$, $x \in \mathfrak{h}$.

Finally, we prove that the first defining property of Definition 3.15 holds and that $\mathfrak{h}$ is abelian.

By Definition 3.17, $\mathfrak{h}$ is nilpotent. Recall that nilpotency implies solvability, as we have discussed earlier. This implies that $\mathfrak{h}$ is solvable.

Since $\mathfrak{h}$ is an algebra, because it is a subalgebra of an algebra, by Theorem 3.13, i.e., Cartan's Criterion for Solvability, we have that $k(\mathfrak{h}, [\mathfrak{h}, \mathfrak{h}]) = 0$.

Applying the non-degeneracy of k in $\mathfrak{h}$, we get that $[\mathfrak{h}, \mathfrak{h}] = 0$. This implies that $\mathfrak{h}$ must be abelian. □

The next definition is justified by the fact that the Killing form restricted to the Cartan subalgebra $\mathfrak{h}$ is non-degenerate, thus we can identify $\mathfrak{h}$ with $\mathfrak{h}^*$.

Definition 3.27. We denote t_α the unique element in $\mathfrak{h}$ corresponding to a root $\alpha \in \mathfrak{h}^*$; which means that it satisfies $\alpha(h) = k(t_\alpha, h)$ for all $h \in \mathfrak{h}$.

Definition 3.28. Given $\alpha, \beta \in \Delta$, we define $(\alpha, \beta) := k(t_\alpha, t_\beta)$. It follows that, $(\alpha, \beta) = \alpha(t_\beta) = \beta(t_\alpha)$.

Proposition 3.29. *Let $\mathfrak{g}$ be a finite-dimensional, complex, semisimple Lie algebra and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$. Then the following is true.*

1. *Given $\alpha, \beta \in \Delta$ such that $\alpha + \beta \neq 0$, then $k(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$*
2. *The Killing form restricts to a perfect pairing $\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}$. In particular, α is a root if and only if $-\alpha$ is a root and $\dim \mathfrak{g}_\alpha = \dim \mathfrak{g}_{-\alpha}$*
3. *Let $\alpha \in \mathfrak{h}^*$ be a root. Then, for all $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$, $[x, y] = k(x, y)t_\alpha$.*
4. *Let $\alpha \in \mathfrak{h}^*$ be a root. Then, $\alpha(t_\alpha) \neq 0$.*

Proof. We prove separately each one of the statement.

1. Take $h \in \mathfrak{h}$ such that $(\alpha + \beta)(h) \neq 0$, $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_\beta$. Then, by associativity of the Killing form, it follows that

$$\alpha(h)k(x, y) = k([h, x], y) = -k([x, h], y) = -k(x, [h, y]) = -\beta(h)k(x, y)$$

Equivalently, $(\alpha + \beta)(h)k(x, y) = 0$. This forces $k(x, y) = 0$, because, by hypothesis, $\alpha + \beta \neq 0$. Hence, $k(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$, as we wanted to prove.

2. Take $x \in \mathfrak{g}_\alpha$ and suppose that

$$k(x, y) = 0 \quad \forall y \in \mathfrak{g}_{-\alpha}.$$

This together with 1. implies that we have $k(x, \mathfrak{g}_\beta) = 0$ for all $\beta \in \Delta$.

Recall that, as seen in the proof of Proposition 3.22, since the operators $\text{ad } h$ with $h \in \mathfrak{h}$ are simultaneously diagonalizable, it follows that

$$\mathfrak{g} = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda \quad \text{with} \quad V_\lambda = \{ y \in \mathfrak{g} \mid [h, y] = \lambda(h)y \text{ for all } h \in \mathfrak{h} \}$$

where $\lambda \in \mathfrak{h}^*$ is a linear functional on $\mathfrak{h}$.

Thus, $k(x, \mathfrak{g}_\beta) = 0$ for all $\beta \in \Delta$ implies that $k(x, \mathfrak{g}) = 0$. By non-degeneracy of the Killing form in $\mathfrak{g}$, $x = 0$.

Therefore, the restriction $k|_{\mathfrak{g}_\alpha \times \mathfrak{g}_{-\alpha}}$ is non-degenerate in the first variable. By symmetry, it is also non-degenerate in the second variable.

Since the spaces are finite-dimensional, the induced map

$$\Phi : \mathfrak{g}_\alpha \longrightarrow \mathfrak{g}_{-\alpha}^*, \quad \Phi(x)(y) = k(x, y),$$

is injective. The same argument applied to $-\alpha$ gives an injective map

$$\mathfrak{g}_{-\alpha} \longrightarrow \mathfrak{g}_\alpha^*.$$

Hence

$$\dim \mathfrak{g}_\alpha = \dim \mathfrak{g}_{-\alpha},$$

and Φ is an isomorphism. Thus k defines a perfect pairing between $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$.

3. Take a root $\alpha \in \mathfrak{h}^*$, $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$. Let $h \in \mathfrak{h}$. Then, applying associativity of the Killing form we have the following

$$k(h, [x, y]) = k([h, x], y) = \alpha(h)k(x, y)$$

Recall that, by definition of t_α , $\alpha(h) = k(h, t_\alpha)$.

Putting everything together, we get that $k(h, [x, y]) - k(h, t_\alpha)k(x, y) = 0$. Hence, since k is non-degenerate, $[x, y] = k(x, y)t_\alpha$, as we wanted to prove.

4. We prove this statement by contradiction. Suppose that $\alpha(t_\alpha) = 0$. Then, by definition of t_α , we have that $(\alpha, \alpha) = 0$.

Take now $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$ such that $k(x, y) = 1$. By 3., $[x, y] = k(x, y)t_\alpha$, thus, we have that $[x, y] = t_\alpha$.

Recall that $t_\alpha \in \mathfrak{h}$. Since $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$, it follows that $[t_\alpha, x] = \alpha(t_\alpha)x = 0$ and $[t_\alpha, y] = -\alpha(t_\alpha)y = 0$. The space $S = \langle x, y, t_\alpha \rangle$ is nilpotent, so it is also solvable. This implies that the adjoint action of any element in S must be upper triangular, by Lie's theorem 3.5. Since $t_\alpha \in [S, S]$, t_α is strictly upper triangular. Therefore, $\text{ad } t_\alpha$ is nilpotent and diagonalizable, since all elements in $\mathfrak{h}$ are semisimple and $t_\alpha \in \mathfrak{h}$ is an element of $\mathfrak{h}$. Hence, $t_\alpha = 0$, which implies that $\alpha = 0$ which contradicts the fact that α is a root.

Given that we have reached a contradiction, we have proved that $\alpha(t_\alpha) \neq 0$.

□

3.5 Generalized eigenspace decomposition of a semisimple Lie algebra

In this section we see that we are able to decompose a given complex, finite-dimensional, semisimple Lie algebra $\mathfrak{g}$ in terms of the Cartan subalgebra $\mathfrak{h}$ and the roots. Thus, we get a better understanding of the structure of such Lie algebras.

Proposition 3.30. Consider a linear operator A on a finite-dimensional vector space V over an algebraically closed field $\mathbb{K}$. Let $\lambda_1, \dots, \lambda_n$ be eigenvalues of A with their corresponding multiplicities $m_1, \dots, m_n$. Then V can be decomposed with the following generalized eigenspace decomposition:

$$V = \bigoplus_{i=1}^n V_{\lambda_i} \quad \text{where } \dim V_{\lambda_i} = m_i$$

Proof. A full discussion of this result can be found in [1, Section 8B]. $\square$

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and (π, V) a representation with V finite-dimensional over an algebraically closed field $\mathbb{K}$ of characteristic 0. In our case we choose $\mathbb{C}$. Then, we have the following generalized eigenspace decomposition for a fixed $a \in \mathfrak{g}$:

$$V = \bigoplus_{\lambda \in F} V_{\lambda}^a, \quad \mathfrak{g} = \bigoplus_{\alpha \in F} \mathfrak{g}_{\alpha}^a$$

where

$$V_{\lambda}^a = \left\{ v \in V \mid (\pi(a) - \lambda I)^N v = 0 \text{ for some } N \in \mathbb{N} \right\}$$

$$\mathfrak{g}_{\alpha}^a = \left\{ g \in \mathfrak{g} \mid (\text{ad } a - \alpha I)^N g = 0 \text{ for some } N \in \mathbb{N} \right\}$$

Note that the decomposition of $\mathfrak{g}$ is a particular case of the decomposition given for any finite-dimensional vector space V corresponding to a representation of $\mathfrak{g}$, in the case of the adjoint representation.

Theorem 3.31. Using the previous notation, $\pi(\mathfrak{g}_{\alpha}^a) V_{\lambda}^a \subseteq V_{\lambda+\alpha}^a$

Proof. This proof is fully discussed in [12, Lecture 6]. $\square$

Definition 3.32. Consider a Lie algebra $\mathfrak{g}$ with a representation (π, V) a representation with V a finite-dimensional vector space over $\mathbb{C}$. Let $\lambda \in \mathfrak{g}^*$ be a functional over $\mathfrak{g}$. Then the **generalized weight space** of $\mathfrak{g}$ in V corresponding to λ is

$$V_{\lambda}^{\mathfrak{g}} = \left\{ v \in V \mid (\pi(g) - \lambda(g)I)^N v = 0 \text{ for some } N = N(g), \forall g \in \mathfrak{g} \right\}$$

The following result shows that if we are given a Lie algebra with a nilpotent subalgebra, then it can be expressed as a direct sum of its generalized weight spaces corresponding to a given representation.

Theorem 3.33. Consider $\mathfrak{g}$ a finite-dimensional Lie algebra with a representation (π, V) , where V is a finite-dimensional vector space over $\mathbb{C}$. Let $\mathfrak{h}$ be a nilpotent subalgebra of $\mathfrak{g}$. Then the following holds

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda^{\mathfrak{h}}$$

$$\pi(\mathfrak{g}_\alpha^{\mathfrak{h}}) V_\lambda^{\mathfrak{h}} \subseteq V_{\lambda+\alpha}^{\mathfrak{h}}$$

Proof. A full proof of this theorem can be found in [12, Lecture 6]. $\square$

Note that, in particular, if we consider the adjoint representation of a finite-dimensional, complex, semisimple Lie algebra $\mathfrak{g}$ we get that

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha^{\mathfrak{h}}, \quad [\mathfrak{g}_\alpha^{\mathfrak{h}}, \mathfrak{g}_\beta^{\mathfrak{h}}] \subseteq \mathfrak{g}_{\alpha+\beta}^{\mathfrak{h}}$$

Our goal now is to relate the decomposition in weight spaces of a given finite-dimensional, complex, semisimple Lie algebra $\mathfrak{g}$ with a Cartan algebra $\mathfrak{h}$ of $\mathfrak{g}$.

Let $\mathfrak{g}$ be a finite-dimensional, complex, semisimple Lie algebra of dimension d . Consider the characteristic polynomial of $\text{ad } x$ for some $x \in \mathfrak{g}$, $\det_{\mathfrak{g}}(\text{ad } x - \lambda I) = (-\lambda)^d + c_{d-1}(-\lambda)^{d-1} + \cdots + \det(\text{ad } x)$. This is a polynomial of degree d . The constant term vanishes because $\text{ad } x(x) = [x, x] = 0$, thus $\det(\text{ad } x) = 0$.

Each c_j is a homogeneous polynomial on $\mathfrak{g}$ of degree $d-j$.

The smallest positive integer r such that $c_r(x)$ is not zero in $\mathfrak{g}$ is called the **rank** of $\mathfrak{g}$.

An element $a \in \mathfrak{g}$ is said to be **regular** if $c_r(a) \neq 0$.

Theorem 3.34. Let $\mathfrak{g}$ be a finite-dimensional, semisimple Lie algebra over $\mathbb{C}$. Since $\mathbb{C}$ is infinite, there exists at least one regular element $x \in \mathfrak{g}$. Let

$$\mathfrak{g} = \bigoplus_{\lambda \in \mathbb{C}} \mathfrak{g}_\lambda^x$$

be the generalized space decomposition of $\mathfrak{g}$ with respect to $\text{ad } x$. Then $\mathfrak{h} = \mathfrak{g}_0^x$ is a Cartan subalgebra.

Proof. It suffices to prove that $\mathfrak{g}_0^x$ is a subalgebra and that it satisfies the defining properties of Cartan subalgebra. $\square$

A full discussion on this topic, introducing the concept of Zarinski open sets, can be found in [12, Lecture 6].

Theorem 3.35 (Engel's Theorem). If all elements of a given finite-dimensional Lie algebra $\mathfrak{g}$ are *ad-nilpotent*, then $\mathfrak{g}$ is nilpotent.

This is one of the various formulations of this theorem. We have chosen the interpretation that is more useful for proving the next result.

Proof. A full proof of Engel's Theorem can be found in [11, Section 3.3]. $\square$

We are now ready to present and prove the result that allows us to relate the generalized weight space decomposition of a finite-dimensional, complex, semisimple Lie algebra $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.

To simplify the notation, we consider $\mathfrak{g}_\lambda = \mathfrak{g}_\lambda^{\mathfrak{h}}$ for all $\lambda \in \mathfrak{h}^*$.

Proposition 3.36. *Let $\mathfrak{g}$ be a finite-dimensional, semisimple Lie algebra over $\mathbb{C}$ with a Cartan subalgebra $\mathfrak{h}$.*

Consider the generalized weight space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$

$$\mathfrak{g} = \bigoplus_{\lambda \in \mathfrak{h}^*} \mathfrak{g}_\lambda$$

Then $\mathfrak{h} = \mathfrak{g}_0$.

Proof. The inclusion $\mathfrak{h} \subseteq \mathfrak{g}_0$ follows from Engel's theorem, since for all $h \in \mathfrak{h}$, $\text{ad } h|_{\mathfrak{h}}$ is nilpotent. We now prove the opposite inclusion.

By definition, for all $h \in \mathfrak{h}$, $\text{ad } h|_{\mathfrak{g}_0}$ is nilpotent. Hence, $\text{ad } h|_{\mathfrak{g}_0/\mathfrak{h}}$ is also nilpotent for all $h \in \mathfrak{h}$.

Suppose, by contradiction, that there exists $b \in \mathfrak{g}_0/\mathfrak{h}$ non-zero such that is annihilated for all $\text{ad } h|_{\mathfrak{g}_0/\mathfrak{h}}$. Then, $[b, \mathfrak{h}] \subseteq \mathfrak{h}$. Thus, $b \in N_{\mathfrak{g}}(\mathfrak{h})$, but $b \notin \mathfrak{h}$, so $\mathfrak{h} \neq N_{\mathfrak{g}}(\mathfrak{h})$, which contradicts $\mathfrak{h}$ being a Cartan subalgebra.

Hence, $\mathfrak{g}_0 \subseteq \mathfrak{h}$.

We have proved both inclusions, this implies that $\mathfrak{h} = \mathfrak{g}_0$. $\square$

To sum up, for a given finite-dimensional, complex, semisimple Lie algebra $\mathfrak{g}$ we can give its generalized weight space decomposition or root decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\substack{\lambda \in \mathfrak{h}^* \\ \lambda \neq 0}} \mathfrak{g}_\lambda \right)$$

where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$.

In fact, since we already have defined the set of roots Δ , we can rewrite it as

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\lambda \in \Delta} \mathfrak{g}_\lambda \right)$$

3.6 Structure of finite-dimensional, complex, semisimple Lie algebras

We reach the final section of the chapter. Its culmination is the main theorem of this work.

In this section our goal is to use what we have learned about irreducible representations of $\mathfrak{sl}(2; \mathbb{C})$ and roots of finite-dimensional, complex, semisimple Lie algebras to study the structure of such Lie algebras.

Recall that we had defined the following basis of $\mathfrak{sl}(2; \mathbb{C}) = \langle H, X, Y \rangle$ such that

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

that satisfy the following relations

$$\begin{aligned} [H, X] &= 2X \\ [H, Y] &= -2Y \\ [X, Y] &= H \end{aligned}$$

Consider a finite-dimensional, complex, semisimple Lie algebras $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Take a root $\alpha \in \Delta$, that is a non-zero weight for the adjoint action of $\mathfrak{h}$ over $\mathfrak{g}$.

Take non-zero $e \in \mathfrak{g}_\alpha$ and $f \in \mathfrak{g}_{-\alpha}$ such that $k(e, f) = \frac{2}{\alpha(t_\alpha)}$. We define $h = [e, f] = \frac{2}{\alpha(t_\alpha)} t_\alpha$. Hence, $\alpha(h) = \frac{2}{\alpha(t_\alpha)} \alpha(t_\alpha) = 2$. The following relations are satisfied:

$$\begin{aligned} [h, e] &= 2e \\ [h, f] &= -2f \\ [e, f] &= h \end{aligned}$$

Take $S = \langle e, f, h \rangle$. It is clear that $S \cong \mathfrak{sl}(2, \mathbb{C})$.

Such notation and terminology are used throughout this section.

This is key in a further study of the structure of semisimple Lie algebras because we have already studied the irreducible representations of $\mathfrak{sl}(2; \mathbb{C})$ and we have proved that they have the form (2.2). Therefore, our understanding of the structure of finite-dimensional, complex, semisimple Lie algebras is supported by the study we have done for $\mathfrak{sl}(2; \mathbb{C})$.

First, we need the definition of simple system of roots.

Definition 3.37. A set of roots $\{\alpha_1, \dots, \alpha_l\}$ is said to be a **simple system** or **basis** if they satisfy:

1. $\{\alpha_1, \dots, \alpha_l\}$ is a vector space basis of $\mathfrak{h}^*$ over $\mathbb{C}$.

2. Every root $\alpha \in \Delta$ can be expressed as a linear combination of elements in $\{\alpha_1, \dots, \alpha_l\}$

$$\alpha = \sum_{i=1}^{\ell} n_i \alpha_i$$

with each n_i integer and such that the n_i 's are either all non-negative or all non-positive. Then α is said to be **positive** or **negative**, respectively.

Definition 3.38. Given $\alpha, \beta \in \Delta$, we call **Cartan integer** $\frac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z}$ with $(\alpha, \beta) = k(t_\alpha, t_\beta)$.

Theorem 3.39. Consider a complex, finite-dimensional, semisimple Lie algebra $\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h}$. The root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$ and the set of roots Δ satisfy the following:

1. Let $\alpha, \beta \in \Delta$. We call the α -string through β , the finite connected string $\{\beta - p'\alpha, \beta - (p' - 1)\alpha, \dots, \beta + q\alpha\} \subseteq \Delta$ with $p', q \in \mathbb{Z}_{\geq 0}$. Then, $p' - q = \frac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z}$.
2. For each root $\alpha \in \Delta$, the corresponding root space $\mathfrak{g}_\alpha$ is one-dimensional.
3. If $\{\alpha, \beta, \alpha + \beta\} \subseteq \Delta$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\beta+\alpha}$.
4. $\{\alpha, c\alpha\} \subseteq \Delta$ if and only if $c = 1$ or $c = -1$.

Proof. 1. Let q be the largest integer such that $\beta + q\alpha \in \Delta \cup \{0\}$. Then $q \geq 0$.

Take non-zero $v \in \mathfrak{g}_{\beta+q\alpha}$.

Recall that at the beginning of this section we have shown that $S = \langle e, f, h \rangle \cong \mathfrak{sl}(2; \mathbb{C})$, thus we can use what we know for representations of $\mathfrak{sl}(2; \mathbb{C})$.

This implies that $\text{ad } e(v) = 0$, because it lies in $\mathfrak{g}_{\beta+(q+1)\alpha}$, and $\text{ad } h(v) = (\beta + q\alpha)(h)v = (\beta(h) + 2q)v = (\beta(\frac{2}{\alpha(t_\alpha)}t_\alpha) + 2q)v = (\frac{2(\alpha, \beta)}{(\alpha, \alpha)} + 2q)v$.

Hence, by the study we have done on representations of $\mathfrak{sl}(2; \mathbb{C})$, $\lambda := \frac{2(\alpha, \beta)}{(\alpha, \alpha)} + 2q \in \mathbb{Z}_{\geq 0}$ and $(\text{ad } f)^j v \in \mathfrak{g}_{\beta+(q-j)\alpha}$ are non-zero vectors for $0 \leq j \leq \lambda$. So, $\{\beta + q\alpha, \beta + (q-1)\alpha, \dots, \beta + (q-\lambda)\alpha\} \in \Delta \cup \{0\}$. Let $p = -(q - \lambda) = q + \frac{2(\alpha, \beta)}{(\alpha, \alpha)}$.

We use an analogous reasoning for p' , the largest integer such that $\beta - p'\alpha \in \Delta \cup \{0\}$, with non-zero $v' \in \mathfrak{g}_{\beta-p'\alpha}$, $\lambda' := \frac{2(\alpha, \beta)}{(\alpha, \alpha)} - 2p' \in \mathbb{Z}_{\geq 0}$ and

$\{\beta - p'\alpha, \beta - (p'-1)\alpha, \dots, \beta - (p' + \lambda')\alpha\} \in \Delta \cup \{0\}$.

Define $q' = -(p' + \lambda') = p' - \frac{2(\alpha, \beta)}{(\alpha, \alpha)}$.

By definition of q, p' we have that $\frac{2(\alpha, \beta)}{(\alpha, \alpha)} = p' - q'$ and $\frac{2(\alpha, \beta)}{(\alpha, \alpha)} = p - q$, showing that $p - q = p' - q'$. Notice that $(p' - p) + (q - q') = (p' - q') - (p - q) = \frac{2(\alpha, \beta)}{(\alpha, \alpha)} - \frac{2(\alpha, \beta)}{(\alpha, \alpha)} = 0$. Since $q \geq q'$ and $p' \geq p$, then $p' - p = 0$ and $q - q' = 0$. Thus, we get $p = p', q = q', q, p \in \mathbb{Z}_{\geq 0}$. Hence, $p' - q = \frac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z}$.

2. We prove this by contradiction.

Suppose that $\dim \mathfrak{g}_\alpha \geq 2$. Then, by Proposition 3.29, $\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}$ is a perfect pairing, so $\dim \mathfrak{g}_{-\alpha} \geq 2$.

Consider now $e \in \mathfrak{g}_\alpha$ and $k(\cdot, e) : \mathfrak{g}_{-\alpha} \rightarrow \mathbb{C}$. Observe that $\dim \mathfrak{g}_{-\alpha} \geq 2$ and $\dim \mathbb{C} = 1$ implies that there exists $v \in \mathfrak{g}_{-\alpha}$ non-zero such that $k(v, e) = 0$.

Therefore, the following is true

$$[e, v] = k(e, v) t_\alpha = 0, \quad [h, v] = -\alpha(h) v = -2v$$

Take the action of $\mathfrak{sl}(2; \mathbb{C})$ over v . In particular, $v, (\text{ad } f)v, (\text{ad } f)^2v, (\text{ad } f)^3v, \dots$ are linear independent vectors and have negative weights $-2, -4, -6, \dots$ with respect to $\text{ad } h$, by the same construction done in Theorem 2.17. Since all weights are distinct, we get infinite linearly independent vectors.

Hence, $\dim \mathfrak{g} = \infty$ and we have reached a contradiction. We have proven that $\dim \mathfrak{g}_\alpha = 1$ for each $\alpha \in \Delta$.

3. Take the largest integers p and q such that $\beta - p\alpha, \beta + q\alpha \in \Delta \cup \{0\}$ and a non-zero vector $v \in \mathfrak{g}_{\beta - p\alpha}$.

By Theorem 3.33

$$\text{ad } f(v) = [f, v] \in [\mathfrak{g}_{-\alpha}, \mathfrak{g}_{\beta - p\alpha}] \subseteq \mathfrak{g}_{-\alpha + (\beta - p\alpha)} = \mathfrak{g}_{\beta - (p+1)\alpha}.$$

Thus, by definition of p , $\text{ad } f(v) = 0$.

Now, since $v \in \mathfrak{g}_{\beta - p\alpha}$, we have that $\text{ad } h(v) = [h, v] = (\beta - p\alpha)(h)v = (\beta(h) - 2p)v = (\beta(\frac{2}{\alpha(t_\alpha)} t_\alpha) - 2p)v = (\frac{2(\alpha, \beta)}{(\alpha, \alpha)} - 2p)v$.

By the study we have done in $\mathfrak{sl}(2; \mathbb{C})$, $(\text{ad } e)^j(v) \neq 0$ for $0 \leq j \leq 2p - \frac{2(\alpha, \beta)}{(\alpha, \alpha)} = p + q$. But $q \geq 1$ since $\alpha + \beta \in \Delta$, so $(\text{ad } e)^{p+1}(v) \neq 0$. This corresponds to the root $\alpha + \beta$, and $(\text{ad } e)^p(v) \in \mathfrak{g}_\beta$, so $[e, \mathfrak{g}_\beta] \neq 0$. Using that $\dim \mathfrak{g}_{\alpha + \beta} = 1$, we have that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha + \beta}$, what we wanted to see.

4. We have already proved earlier that both $\alpha, -\alpha$ are roots because $\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}$ are a perfect pairing by Proposition 3.29. We are left to prove the other direction.

Take $\beta = n\alpha$ with $n \neq 0$. We have then that $\frac{2(\alpha, \beta)}{(\beta, \beta)} = \frac{2(\alpha, n\alpha)}{(n\alpha, n\alpha)} = \frac{2n}{n^2} = \frac{2}{n} \in \mathbb{Z}$.

Hence, if $\beta \in \Delta$, then the only candidates for n are $n = 2, 1, -1, -2$.

Recall that, by 2., $\dim \mathfrak{g}_\alpha = 1$. Then we can find some non-zero $x \in \mathfrak{g}$ such that $\mathfrak{g}_\alpha = \langle x \rangle$ and, consequently, $[x, x] = 0$. This implies that $[\mathfrak{g}_\alpha, \mathfrak{g}_\alpha] = 0$.

If 2α were a root then, by 3., $[\mathfrak{g}_\alpha, \mathfrak{g}_\alpha] = \mathfrak{g}_{2\alpha}$. This contradicts $[\mathfrak{g}_\alpha, \mathfrak{g}_\alpha] = 0$ because, by 2., $\dim \mathfrak{g}_{2\alpha} = 1$. The same happens for -2α root. Therefore, we are left with $n = 1, -1$, proving the remaining direction. $\square$

Theorem 3.40. *Consider finite-dimensional, complex, semisimple Lie algebra $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha)$. Then the following holds:*

1. Δ spans $\mathfrak{h}^*$ over $\mathbb{C}$.
2. $(\cdot, \cdot)$ is $\mathbb{Q}$ -valued in Δ and gives positive quadratic form in Δ .

Proof. We prove each statement individually.

We use that given some $h, h' \in \mathfrak{h}$, by $\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha)$, we have $k(h, h') = \sum_{\beta \in \Delta} \beta(h) \beta(h')$.

1. We prove this statement by contradiction.

Recall that Δ spans $\mathfrak{h}^*$ over $\mathbb{C}$ if and only if the only $h \in \mathfrak{h}$ that is annihilated by every root is $h = 0$.

Suppose that there exists some non-zero $h \in \mathfrak{h}$ such that $\alpha(h) = 0$ for all $\alpha \in \Delta$. It follows that $\sum_{\beta \in \Delta} \beta(h) \beta(h') = 0$ for all $h' \in \mathfrak{h}$.

However, using that $\mathfrak{g} = \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha)$ and the definition of the Killing form, we get that $k(h, h) = 0$. This is a contradiction, since the Killing form is non-degenerate in $\mathfrak{h}$ by Proposition 3.22.

2. Let $\alpha, \beta \in \Delta$.

We know that $(\alpha, \alpha) = k(t_\alpha, t_\alpha) = \sum_{\lambda \in \Delta} \lambda(t_\alpha) \lambda(t_\alpha) = \sum_{\lambda \in \Delta} (\alpha, \lambda)^2$. If $(\alpha, \alpha) = 0$, then $(\alpha, \lambda) = 0$ for all $\lambda \in \Delta$. Thus, by non-degeneracy of the Killing form restricted to $\mathfrak{h}$ and since Δ spans $\mathfrak{h}^*$ over $\mathbb{C}$, $\alpha = 0$, which contradicts the definition of root.

Consider $\frac{4}{(\alpha, \alpha)} = \frac{4(\alpha, \alpha)}{(\alpha, \alpha)^2} = \sum_{\lambda \in \Delta} \frac{4(\alpha, \lambda)^2}{(\alpha, \alpha)^2} = \sum_{\lambda \in \Delta} \left(\frac{2(\alpha, \lambda)}{(\alpha, \alpha)} \right)^2 \in \mathbb{N}_{>0}$. Then, $(\alpha, \alpha) \in \mathbb{Q}_{>0}$. This proves that $(\cdot, \cdot)$ is a positive quadratic form in Δ .

Because $\frac{2(\alpha, \beta)}{(\alpha, \alpha)} \in \mathbb{Z}$ and $\frac{(\alpha, \alpha)}{2} \in \mathbb{Q}_{>0}$, then, $(\alpha, \beta) = \frac{2(\alpha, \beta)}{(\alpha, \alpha)} \cdot \frac{(\alpha, \alpha)}{2} \in \mathbb{Q}$. $\square$

As a matter of fact, $(\cdot, \cdot)$ is semipositive definite and non-degenerate on the span of Δ over $\mathbb{Q}$, then it is positive definite in $\langle \Delta \rangle_{\mathbb{Q}}$.

We call E the extension of $\langle \Delta \rangle_{\mathbb{Q}}$ to $\langle \Delta \rangle_{\mathbb{R}}$. Since $(\cdot, \cdot)$ extends naturally to E and positive definiteness is preserved under scalar extension, E is a real euclidean space.

We are now ready to accomplish our goal of finding a simple system of roots.

Definition 3.41. Take $\lambda \in E$ such that $(\lambda, \alpha) \neq 0$ for all $\alpha \in \Delta$. Then we can define $\Delta_+ = \{\alpha \in \Delta \mid (\lambda, \alpha) > 0\}$ and $\Delta_- = \{\alpha \in \Delta \mid (\lambda, \alpha) < 0\}$. They satisfy $-\Delta_+ = \Delta_-$ and $\Delta = \Delta_+ \sqcup \Delta_-$.

Definition 3.42. We say that $\alpha \in \Delta_+$ is *decomposable* if there exist $\beta, \gamma \in \Delta_+$ such that $\alpha = \beta + \gamma$.

The set of indecomposable roots is denoted by $\Gamma \subseteq \Delta_+$.

We can now state the main theorem of our dissertation.

Theorem 3.43. Consider $\mathfrak{g}$ a finite-dimensional, complex, semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra and Δ the set of roots. Then a simple system of roots $\{\alpha_1, \dots, \alpha_l\}$ exists.

Proof. We prove that the set of indecomposable roots Γ is a simple system of roots.

Recall that we have to see that Γ is a basis of $\mathfrak{h}^*$ and that every root $\alpha \in \Delta$ can be expressed as a linear combination of elements in $\Gamma = \{\alpha_1, \dots, \alpha_l\}$

$$\alpha = \sum_{i=1}^{\ell} n_i \alpha_i$$

with each n_i integer and such that the n_i 's are either all non-negative, then α is said to be positive, or all non-positive, α is said to be negative.

We see, first, that Γ is a basis of $E = \langle \Delta \rangle_{\mathbb{R}}$.

Recall that $\beta - \frac{2(\alpha, \beta)}{(\alpha, \alpha)}\alpha \in \Delta$ by Theorem 3.39 and that the α -string through β is uninterrupted. Hence, if $(\alpha, \beta) < 0$, then $\alpha + \beta \in \Delta$. In addition, if $(\alpha, \beta) > 0$, then $\alpha - \beta, -\alpha + \beta \in \Delta$. Hence, if $\alpha, \beta \in \Gamma$, $(\alpha, \beta) \leq 0$.

We can see that all $\alpha \in \Delta_+$ is a $\mathbb{N}$ -linear combination of elements in Γ . We do this by induction. Suppose that α is the element in Δ_+ such that minimizes $(\lambda, \alpha) > 0$. If there would exist $\gamma, \beta \in \Delta_+$ such that $\alpha = \beta + \gamma$, then $(\lambda, \alpha) = (\lambda, \gamma) + (\lambda, \beta)$, which contradicts the minimality of α .

For the induction step, if $\alpha \in \Gamma$, we are done. Otherwise, there would exist $\gamma, \beta \in \Delta_+$ such that $\alpha = \beta + \gamma$ and we can apply the induction hypothesis.

Hence, $\Delta \subseteq \langle \Gamma \rangle_{\mathbb{Z}}$. Since $E = \langle \Delta \rangle_{\mathbb{R}}$, Γ spans E .

We are left with seeing that the elements in Γ are linearly independent.

Suppose that there exists a non-trivial linear combination of roots in Γ such that $\sum_i k_i \alpha_i = 0$.

To follow with, we split this sum into positive and negative parts and move the negative terms to the other side. Thus, we get $\gamma := \sum_i a_i \alpha_i = \sum_i b_i \alpha_i$ where all a_i and b_i are nonnegative.

We know that $\gamma \neq 0$, otherwise $a_i = b_i = 0$ for all i , thus, $k_i = 0$ for all i . This contradicts our hypothesis of $\sum_i k_i \alpha_i = 0$ being a non-trivial linear combination of simple roots.

Notice that $a_i b_i = 0$ for all i because if $k_i > 0$, then $a_i > 0$ and $b_i = 0$, if $k_i < 0$, then $a_i = 0$ and $b_i < 0$ and if $k_i = 0$, then $a_i = 0$ and $b_i = 0$.

Since $\gamma \in E$, by positive definiteness, $(\gamma, \gamma) > 0$.

Recall that, since for all i $\alpha_i \in \Gamma$, then for all $i \neq j$ $(\alpha_i, \alpha_j) \leq 0$. Thus, $(\gamma, \gamma) = \sum_{i,j} a_i b_j (\alpha_i, \alpha_j) \leq 0$. We have then reached contradiction. Hence, the elements in Γ are linearly independent.

We have finished proving that Γ is a basis of E . Recall that $E = \langle \Delta \rangle_{\mathbb{R}}$.

Since $\mathfrak{g}$ is semisimple, as shown previously, the Killing form is non-degenerate in $\mathfrak{h}$, thus, there is an isomorphic correspondence between $\mathfrak{h}$ and $\mathfrak{h}^*$. Hence, $\mathfrak{h}^*$ corresponds to the complexification of E and Γ is basis of $\mathfrak{h}^*$ because expanding the scalars from $\mathbb{R}$ to $\mathbb{C}$ does not affect the elements of Γ being linear independent nor spanning $\mathfrak{h}^*$. Thus, Γ is a basis of $\mathfrak{h}^*$.

The only thing we have to prove to be able to affirm that Γ is a simple system of roots is that for every $\alpha \in \Delta$, the coefficients of the expression of α as a linear combination of the elements in Γ are, either, all non-negative or all non-positive.

But $\alpha \in \Delta$ and $\Delta = \Delta_+ \sqcup \Delta_-$. Hence, $\alpha \in \Delta_+$ or $\alpha \in \Delta_-$. As established before, every element in Δ_+ can be expressed as a $\mathbb{N}$ -linear combination of elements in Γ , hence all coefficients are non-negative. From this we can also deduce that all elements in $\Delta_- = -\Delta_+$ are $-\mathbb{N}$ -linear combination of elements in Γ and that all coefficients must be non-positive.

In summary, we have proved that Γ is a simple system of roots. The statement follows.

□

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