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Research Article

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A_∞-structures for Bott-Chern and Aeppli cohomologies

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Abstract: In this article, we introduce certain arity-3 operations on complex manifolds arising from homotopy transfer theory. Such operations are related to the triple *ABC*-Massey products from Bott-Chern to Aeppli cohomology. We present a package for computing such products as well as other cohomological and homotopical invariants for complex nilmanifolds and give some examples.

Keywords: complex manifolds, pluripotential homotopy theory, A-infinity structures, Massey products, Bott-Chern cohomology, Aeppli cohomology

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1 Introduction

Given a cochain complex defined over a field, there is always a homotopy transfer diagram connecting it with its cohomology. When the cochain complex carries additionally an algebra structure, this diagram induces an A_∞-structure on its cohomology, which is unique up to A_∞-isomorphism [6]. This A_∞-structure is strongly related to Massey products and encodes the homotopy type of the original differential-graded algebra. This set-up has proven to be particularly useful in rational or real homotopy theory, when considering the algebra of piece-wise linear forms of a topological space, or the de Rham algebra of a smooth manifold, especially for formality considerations.

In the case of complex manifolds, it is natural to include in this set-up the additional geometric information coming from the decomposition of the complexified de Rham algebra into (p, q)-differential forms. In [4], the first and third authors develop a homotopy transfer theory for filtered algebras. When applied to the complexified de Rham algebra of a complex manifold, together with the Hodge filtration, this leads to new computations of geometric-topological invariants arising from the multiplicative structures of the Frölicher spectral sequence. Further applications to Massey products of complex manifolds are discussed in [3]. In this approach, the symmetry between ∂ and $\bar{\partial}$ is broken by the choice of the Hodge (column) filtration.

Recent work of Angella and Tomassini [1] shows that the consideration of symmetric Massey-like triple products for complex manifolds, treating the two components $\bar{\partial}$ and ∂ on equal footing, give interesting and new biholomorphic invariants. For instance, in contrast with ordinary Massey products, such products do not necessarily vanish for compact Kähler manifolds [8]. The Massey products of [1], called *ABC-Massey products*, take as input three cohomology classes in Bott-Chern cohomology, and give as output a set of cohomology classes in Aeppli cohomology. Such products are shown to be obstructions to the notion of *strong formality* introduced in [7], where Milivojevic and Stelzig extend *ABC*-Massey products to higher arities.

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Strong formality is governed by the class of weak equivalences defined by those maps inducing isomorphisms in both Bott-Chern and Aeppli cohomologies. This may be understood as the finest class one can choose, while keeping the homotopical flavor (for instance, an isomorphism on Bott-Chern and Aeppli is always an isomorphism on Dolbeault and de Rham cohomologies). The homotopy theory of double complexes with respect to such class of weak equivalences has been recently developed by Stelzig [11]. The theory naturally gives the equation

$$[\partial, [\bar{\partial}, h]] = f - g$$

for the notion of *pluripotential homotopy* between two morphisms of double complexes f and g.

The currently established results serve as an invitation to develop a theory of pluripotential operadic calculus, including a pluripotential homotopy theory of operads in double complexes, a Koszul duality theory, pluripotential bar and cobar resolutions, and homotopy transfer theory with pluripotential homotopy equivalences, among others. This is the main content of the forthcoming PhD thesis of the third author. In this article, we explain the first steps, related to the study of the lower arity operations arising from pluripotential homotopy transfer theory for complex manifolds. This produces new algebraic invariants that detect pluripotential homotopy types. We then exhibit the relation between these operations and the triple ABC-Massey products. In particular, the vanishing of our arity-3 operations implies the vanishing of ABC-Massey products. This shows evidence that pluripotential homotopy transfer is bound to be a new useful tool to study strong formality. Wait and see!

2 Pluripotential homotopy transfer

Associated with a double complex $(K, \partial, \bar{\partial})$, there are various functorial cohomologies: aside from the ∂- and $\bar{\partial}$-cohomologies

$$H_\partial := \frac{\operatorname{Ker}(\partial)}{\operatorname{Im}(\partial)} \quad \text{and} \quad H_{\bar{\partial}} := \frac{\operatorname{Ker}(\bar{\partial})}{\operatorname{Im}(\bar{\partial})},$$

we also have the Bott-Chern H_{BC} and Aeppli H_A cohomologies, given, respectively, by

$$H_{BC} := \frac{\operatorname{Ker}(\bar{\partial}) \cap \operatorname{Ker}(\partial)}{\operatorname{Im}(\bar{\partial}\partial)} \quad \text{and} \quad H_A := \frac{\operatorname{Ker}(\bar{\partial}\partial)}{\operatorname{Im}(\bar{\partial}) + \operatorname{Im}(\partial)}.$$

Note that the identity induces maps

$$\begin{tikzcd} H_{BC} \arrow[r] \arrow[d] & H_\partial \arrow[d] \\ H_{\bar{\partial}} \arrow[r] & H_A \end{tikzcd}$$

which are isomorphisms if and only if the $\bar{\partial}\partial$-lemma is satisfied. Note that usually the $\bar{\partial}\partial$-lemma refers to the vertical map on the left being an isomorphism, but both conditions turn out to be equivalent.

Consider the natural transformation induced by the identity $H_{BC} \to H_A$, and let $\tilde{H}_{BC}$ and $\tilde{H}_A$ be the functors that send a double complex K to the kernel and cokernel of $H_{BC}(K) \to H_A(K)$, respectively. Explicitly, this gives

$$\tilde{H}_{BC} = \frac{\operatorname{Im}\partial \cap \operatorname{Ker}\bar{\partial} + \operatorname{Im}\bar{\partial} \cap \operatorname{Ker}\partial}{\operatorname{Im}\partial\bar{\partial}},$$
$$\tilde{H}_A = \frac{\operatorname{Ker}\partial\bar{\partial}}{\operatorname{Im}\partial + \operatorname{Im}\bar{\partial} + \operatorname{Ker}\partial \cap \operatorname{Ker}\bar{\partial}}.$$

The *ABC-cohomology* of a double complex $(K, \partial, \bar{\partial})$ is the double complex defined by

$$H_{ABC}(K) := H_{BC}(K) \oplus \tilde{H}_A(K),$$

together with the two differentials ∂ and $\bar\partial$ induced from those of K. Note that this is a double complex, but it is *pluripotentially minimal*, in the sense that $\partial\bar\partial \equiv 0$.

Theorem 2.1. (Homotopy transfer data) *Let* $(K, \partial, \bar\partial)$ *be a double complex. There is a diagram*

$$\begin{tikzcd}(K,\partial,\bar\partial) \arrow[loop left, "h"] \arrow[r, shift left, "\rho"] & (H_{ABC}(K),\partial,\bar\partial) \arrow[l, shift left, "\iota"]\end{tikzcd}$$

where ρ *and* ι *are the morphisms of double complexes such that* $\rho\iota = 1$, *and* h *is a pluripotential homotopy from the identity to* $\iota\rho$:

$$1 - \iota\rho = [\partial, [\bar\partial, h]].$$

Proof. We sketch a proof. A detailed account will appear in [10]. By Corollary 1.4 in [11], we can decompose (non-uniquely) every double complex K as

$$K = K_{sq} \oplus K_{zig}.$$

Moreover, by Lemma 1.21 in [11], we have an isomorphism of double complexes

$$K_{zig} \cong H_{ABC}(K).$$

Therefore, we may define $\rho : K \twoheadrightarrow K_{zig} \cong H_{ABC}(K)$ and $\iota : H_{ABC}(K) \cong K_{zig} \hookrightarrow K$. We then let

$$h : K_{sq} \oplus K_{zig} \to K_{sq} \oplus K_{zig}$$

be defined as follows. Given a pair $(a, b) \in K_{sq} \oplus K_{zig}$, we let $h(a, b) = (a', 0)$ if $a = \partial\bar\partial a'$ and zero otherwise. One then checks that $1 - \iota\rho = [\partial, [\bar\partial, h]]$ as desired. □

Let now $(K, \wedge, \partial, \bar\partial)$ be a bidifferential bigraded algebra (such as the complexified de Rham algebra of a complex manifold). The product $\wedge$ induces an algebra structure in Bott-Chern cohomology, as well as an H_{BC}-module structure in Aeppli cohomology.

A main result of [10] endows the double complex $H_{ABC}(K)$ with a pluripotential version of the notion of A_∞-algebra, unique up to certain A_∞-isomorphism. In this article, we describe the first operations, of lower arities. By Theorem 2.1, we may choose homotopy transfer data (ρ, ι, h). We then define

$$\mu_2 : H^{*,*}_{ABC}(K) \otimes H^{*,*}_{ABC}(K) \longrightarrow H^{*,*}_{ABC}(K)$$

to be the product given by

$$\mu_2(a, b) := \rho(\iota(a) \wedge \iota(b)).$$

This operation defines a product structure on H_{ABC}, which is graded-commutative but only associative up to homotopy. Indeed, let

$$\mu_3 : H^{*,*}_{ABC}(K) \otimes H^{*,*}_{ABC}(K) \otimes H^{*,*}_{ABC}(K) \longrightarrow H^{*-1,*-1}_{ABC}(K)$$

be the operation defined by

$$\mu_3(x, y, z) := \rho(h(\iota(x) \wedge \iota(y)) \wedge \iota(z) - \iota(x) \wedge h(\iota(y) \wedge \iota(z))).$$

The following relation is satisfied:

$$[\partial, [\bar\partial, \mu_3]] = \mu_2(\mu_2, 1) - \mu_2(1, \mu_2).$$

Therefore, we may interpret μ_3 as a pluripotential homotopy for the associator.

Remark 2.2. (Classical A_∞-structures) Let $\mu_{\bar\partial}$ be the operation of arity 3 and bidegree $(-1, 0)$ defined by

$$\mu_{\bar\partial} := \bar\partial\mu_3 - \mu_3\bar\partial.$$

It satisfies

$$[\partial, \mu_{\bar\partial}] = \mu_2(\mu_2, 1) - \mu_2(1, \mu_2).$$

In particular, $\mu_{\bar\partial}$ acts as the ternary operation of an A_∞-algebra with respect to the differential ∂. By symmetry, the same holds for $\bar\partial$ and the operation given by

$$\mu_\partial \coloneqq -\partial\mu_3 + \mu_3\partial.$$

Remark 2.3. The μ_3-product satisfies the following relations:

(1) As a consequence of graded commutativity,

$$\mu_3(z, y, x) = -(-1)^{|x||y|+|x||z|+|y||z|}\mu_3(x, y, z).$$

(2) If K is the complexified de Rham algebra of a compact complex manifold and the homotopy h satisfies $h(\bar x) = -\overline{h(x)}$, then

$$\mu_3(\bar x, \bar y, \bar z) = -\overline{\mu_3(x, y, z)}.$$

This will always be the case in the examples of Section 3.

As in the classical case, the A_∞-operation μ_3 is strongly related to triple Massey products. In this case, the comparison needs to be made with the ABC-Massey products of [1], as we next explain.

Definition 2.4. Let $[x], [y], [z] \in H_{BC}(K)$ be three Bott-Chern cohomology classes such that $[x]\cdot[y] = [y]\cdot[z] = 0$. The *triple ABC-Massey product* $\langle [x], [y], [z]\rangle$ is defined as the following set of Aeppli classes:

$$\langle [x], [y], [z]\rangle \coloneqq \{[x \wedge v - u \wedge z] \mid \partial\bar\partial u = x \wedge y,\ \partial\bar\partial v = y \wedge z\} \subset H_A(K).$$

The triple ABC-Massey product is said to be *trivial* if $0 \in \langle [x], [y], [z]\rangle$.

Let us remark that in [1], a different sign convention is used. We are sticking to the sign convention in [7] instead. Also, in these references, the triple product is defined as an element in the coset space

$$\frac{H_A(K)}{[x]\cdot H_A(K) + [z]\cdot H_A(K)}$$

rather than as set of Aeppli classes. Both viewpoints are equivalent. In particular, a triple ABC-Massey product is trivial if and only if it is zero in the aforementioned coset space.

Proposition 2.5. *Given* $[x], [y], [z] \in H_{BC}(K)$ *such that* $[x]\cdot[y] = [y]\cdot[z] = 0$ *and homotopy transfer data* (ρ, ι, h), *we have*

$$-[\iota\mu_3([x], [y], [z])]_A \in \langle [x], [y], [z]\rangle,$$

where $[-]_A$ *denotes taking the Aeppli cohomology class.*

Proof. Note first that there is an inclusion $H_{BC}(K) \subseteq H_{ABC}(K)$ so we may view such classes in ABC-cohomology. Write $x = \iota([x])$, $y = \iota([y])$ and $z = \iota([z])$. Since $[x]\cdot[y] = [y]\cdot[z] = 0$, we may choose an element u such that

$$\partial\bar\partial u = x \wedge y \quad \text{and} \quad h(x \wedge y) = u.$$

In the same way, we can choose v such that

$$\partial\bar\partial v = y \wedge z \quad \text{and} \quad h(y \wedge z) = v.$$

This gives the following:

$$\mu_3([x], [y], [z]) = \rho(h(x \wedge y) \wedge z - x \wedge h(y \wedge z)) = \rho(u \wedge z - x \wedge v).$$

Thus, we obtain

$$-[\iota\mu_3([x],[y],[z])]_A = [x \wedge v - u \wedge z]_A \in \langle [x],[y],[z]\rangle. \qquad \square$$

Corollary 2.6. *Let $[x],[y],[z] \in H_{BC}(K)$ such that $[x]\cdot[y] = [y]\cdot[z] = 0$. If $\mu_3([x],[y],[z]) = 0$, then the triple ABC-Massey product $\langle [x],[y],[z]\rangle$ is trivial.*

Remark 2.7. This reflects the situation observed in the ordinary case. By Corollary 3.3 in [2], the triple products of A_∞-structures arising from homotopy transfer data, as described in Theorem 1.2 in loc.cit., always detect Massey products, i.e.,

$$\pm\mu_3(a,b,c) \in \langle a,b,c\rangle.$$

Consequently, if $\mu_3(a,b,c) = 0$, the corresponding Massey product must also vanish.

Remark 2.8. As with the triple *ABC*-Massey products, for μ_3 to be non-zero, it is necessary that $\partial\bar\partial \neq 0$, which corresponds to having $K_{sq} \neq 0$ in the decomposition $K = K_{sq} \oplus K_{zig}$. For instance, for the Kodaira-Thurston manifold with its standard complex structure, the complex algebra of left-invariant forms is given by

$$\Lambda(a,\bar a,b,\bar b), \quad \text{with } db = a\bar a.$$

One verifies that $\partial\bar\partial = 0$, and consequently, μ_3 is trivial in this case.

3 Examples

For the following examples, we have used the code written by the second author [5]. This Sage script takes as input any real Lie algebra together with a complex structure and computes the arity-3 operations described in this article, for the associated Chevalley-Eilenberg algebra together with the induced (p,q)-decomposition.

The chosen homotopy h satisfies, in both examples, the additional equation $h(\bar x) = -\overline{h(x)}$. Hence, in the following tables, some redundant information is omitted, according to Remark 2.3. Namely, the value of μ_3 at bidegrees (p,q) with $p > q$, and at (z,y,x) (provided that $\mu_3(x,y,z)$ appears in the table). All other products that do not occur in the tables vanish.

3.1 Iwasawa manifold

Consider the Iwasawa nilmanifold $\mathbb{I}_3 = \mathbb{H}(3;\mathbb{Z}[i])\backslash\mathbb{H}(3;\mathbb{C})$. The group $\mathbb{H}(3;\mathbb{C})$ admits a left invariant coframe of (1,0)-forms

$$a = dz_1, \ b = dz_2, \ c = dz_3 - z_1 dz_2,$$

which gives rise to a left-invariant complex structure on $\mathbb{I}_3$. The complex algebra of left-invariant forms is given by

$$\mathcal{A} = \Lambda(a,b,c,\bar a,\bar b,\bar c) \quad \text{with } d(c) = -ab, d(\bar c) = -\bar a\bar b.$$

A basis for the *ABC*-cohomology is given by the following table:

$\overline{abc}$	$a\overline{abc}$, $b\overline{abc}$, $c\overline{abc}$	$ab\overline{abc}$, $ac\overline{abc}$, $bc\overline{abc}$	$abc\overline{abc}$
$\overline{ab}$, $\overline{ac}$, $\overline{bc}$	$a\overline{ab}$, $a\overline{ac}$, $a\overline{bc}$, $b\overline{ab}$, $b\overline{ac}$, $b\overline{bc}$, $c\overline{ac}$, $c\overline{bc}$	$ab\overline{ac}$, $ab\overline{bc}$, $ac\overline{ab}$, $ac\overline{ac}$, $ac\overline{bc}$, $bc\overline{ab}$, $bc\overline{ac}$, $bc\overline{bc}$	$abc\overline{ab}$, $abc\overline{ac}$, $abc\overline{bc}$
$\bar a$, $\bar b$, $\bar c$	$a\bar a$, $a\bar b$, $a\bar c$, $b\bar a$, $b\bar b$, $b\bar c$, $c\bar a$, $c\bar b$	$ab\bar a$, $ab\bar b$, $ac\bar a$, $ac\bar b$, $ac\bar c$, $bc\bar a$, $bc\bar b$, $bc\bar c$	$abc\bar a$, $abc\bar b$, $abc\bar c$
1	a, b, c	ab, ac, bc	abc

We define homotopy transfer data

$$\begin{tikzcd} (\mathcal{A},\partial,\overline{\partial}) \arrow[loop left, "h"] \arrow[r, shift left, "\rho"] & (H_{ABC}(\mathcal{A}),\partial,\overline{\partial}) \arrow[l, shift left, "\iota"] \end{tikzcd}$$

as follows: the map ι is induced by the identity on all elements. The map ρ is induced by the identity, except for the following elements, which belong to a square and for which ρ is zero:

$$\begin{tikzcd} c\overline{a}\overline{b} \arrow[r] & ab\overline{a}\overline{b} \\ c\overline{c} \arrow[u] \arrow[r] & ab\overline{c} \arrow[u] \end{tikzcd}$$

The homotopy h is given by

$$h(ab\overline{a}\overline{b}) = -c\overline{c}$$

and zero otherwise.

The μ_3-product is concentrated in the following bidegrees (of $H_{ABC}^{\otimes 3}$):

$$(p,q) \in \{(2,\ 3),\quad (3,2),\quad (2,\ 4),\quad (3,\ 3),\quad (4,\ 2),\quad (3,\ 4),\quad (4,\ 3)\}.$$

More specifically, the nontrivial products are recorded in the following tables.

Total bidegree (2, 3)

x	y	z	$\mu_3(x,y,z)$	x	y	z	$\mu_3(x,y,z)$
$ab\overline{a}$	$\overline{b}$	$\overline{a}$	$c\overline{a}\overline{c}$	$ab\overline{a}$	$\overline{b}$	$\overline{b}$	$c\overline{b}\overline{c}$
$ab\overline{b}$	$\overline{a}$	$\overline{a}$	$-c\overline{a}\overline{c}$	$ab\overline{b}$	$\overline{a}$	$\overline{b}$	$-c\overline{b}\overline{c}$
ab	$\overline{a}\overline{b}$	$\overline{a}$	$c\overline{a}\overline{c}$	ab	$\overline{a}\overline{b}$	$\overline{b}$	$c\overline{b}\overline{c}$
$a\overline{a}\overline{b}$	b	$\overline{a}$	$c\overline{a}\overline{c}$	$a\overline{a}\overline{b}$	b	$\overline{b}$	$c\overline{b}\overline{c}$
$b\overline{a}\overline{b}$	a	$\overline{a}$	$-c\overline{a}\overline{c}$	$b\overline{a}\overline{b}$	a	$\overline{b}$	$-c\overline{b}\overline{c}$
$a\overline{a}$	$b\overline{b}$	$\overline{a}$	$-c\overline{a}\overline{c}$	$a\overline{a}$	$b\overline{b}$	$\overline{b}$	$-c\overline{b}\overline{c}$
$b\overline{a}$	$a\overline{b}$	$\overline{a}$	$c\overline{a}\overline{c}$	$b\overline{a}$	$a\overline{b}$	$\overline{b}$	$c\overline{b}\overline{c}$
$a\overline{b}$	$b\overline{a}$	$\overline{a}$	$c\overline{a}\overline{c}$	$a\overline{b}$	$b\overline{a}$	$\overline{b}$	$c\overline{b}\overline{c}$
$b\overline{b}$	$a\overline{a}$	$\overline{a}$	$-c\overline{a}\overline{c}$	$b\overline{b}$	$a\overline{a}$	$\overline{b}$	$-c\overline{b}\overline{c}$
a	$b\overline{a}\overline{b}$	$\overline{a}$	$c\overline{a}\overline{c}$	a	$b\overline{a}\overline{b}$	$\overline{b}$	$c\overline{b}\overline{c}$
b	$a\overline{a}\overline{b}$	$\overline{a}$	$-c\overline{a}\overline{c}$	b	$a\overline{a}\overline{b}$	$\overline{b}$	$-c\overline{b}\overline{c}$
$\overline{a}\overline{b}$	ab	$\overline{a}$	$c\overline{a}\overline{c}$	$\overline{a}\overline{b}$	ab	$\overline{b}$	$c\overline{b}\overline{c}$
$\overline{a}$	$ab\overline{a}$	$\overline{b}$	$-c\overline{a}\overline{c}$	$\overline{a}$	$ab\overline{b}$	$\overline{a}$	$2\quad c\overline{a}\overline{c}$
$\overline{a}$	$ab\overline{b}$	$\overline{b}$	$c\overline{b}\overline{c}$	$\overline{b}$	$ab\overline{a}$	$\overline{b}$	$-2\quad c\overline{b}\overline{c}$

The third product in the aforementioned table recovers the nontrivial ABC-Massey product computed in [1]:

$$\langle[ab],[\overline{a}\overline{b}],[\overline{a}]\rangle = [-c\overline{a}\overline{c}] \in \frac{H_A(\mathcal{A})}{[\overline{a}]\cdot H_A(\mathcal{A})}.$$

Total bidegree (2, 4)

x	y	z	$\mu_3(x,y,z)$	x	y	z	$\mu_3(x,y,z)$
$\bar{a}\bar{b}$	$ab\bar{a}$	$\bar{b}$	$c\bar{a}\bar{b}\bar{c}$	$\bar{a}\bar{b}$	$ab\bar{b}$	$\bar{a}$	$-c\bar{a}\bar{b}\bar{c}$
$ab\bar{a}$	$\bar{b}$	$\bar{a}\bar{b}$	$-c\bar{a}\bar{b}\bar{c}$	$ab\bar{b}$	$\bar{a}$	$\bar{a}\bar{b}$	$c\bar{a}\bar{b}\bar{c}$
ab	$\bar{a}\bar{b}$	$\bar{a}\bar{b}$	$-c\bar{a}\bar{b}\bar{c}$	$a\bar{a}\bar{b}$	b	$\bar{a}\bar{b}$	$-c\bar{a}\bar{b}\bar{c}$
$b\bar{a}\bar{b}$	a	$\bar{a}\bar{b}$	$c\bar{a}\bar{b}\bar{c}$	$a\bar{a}$	$b\bar{b}$	$\bar{a}\bar{b}$	$c\bar{a}\bar{b}\bar{c}$
$b\bar{a}$	$a\bar{b}$	$\bar{a}\bar{b}$	$-c\bar{a}\bar{b}\bar{c}$	$a\bar{b}$	$b\bar{a}$	$\bar{a}\bar{b}$	$-c\bar{a}\bar{b}\bar{c}$
$b\bar{b}$	$a\bar{a}$	$\bar{a}\bar{b}$	$c\bar{a}\bar{b}\bar{c}$	a	$b\bar{a}\bar{b}$	$\bar{a}\bar{b}$	$-c\bar{a}\bar{b}\bar{c}$
b	$a\bar{a}\bar{b}$	$\bar{a}\bar{b}$	$c\bar{a}\bar{b}\bar{c}$				

Total bidegree (3, 3)

x	y	z	$\mu_3(x,y,z)$	x	y	z	$\mu_3(x,y,z)$
$a\bar{a}$	$ab\bar{a}$	$\bar{b}$	$-ac\bar{a}\bar{c}$	$a\bar{a}$	$ab\bar{b}$	$\bar{a}$	$ac\bar{a}\bar{c}$
$b\bar{a}$	$ab\bar{a}$	$\bar{b}$	$-bc\bar{a}\bar{c}$	$b\bar{a}$	$ab\bar{b}$	$\bar{a}$	$bc\bar{a}\bar{c}$
$a\bar{b}$	$ab\bar{a}$	$\bar{b}$	$-ac\bar{b}\bar{c}$	$a\bar{b}$	$ab\bar{b}$	$\bar{a}$	$ac\bar{b}\bar{c}$
$b\bar{b}$	$ab\bar{a}$	$\bar{b}$	$-bc\bar{b}\bar{c}$	$b\bar{b}$	$ab\bar{b}$	$\bar{a}$	$bc\bar{b}\bar{c}$
$a\bar{a}$	ab	$\bar{a}\bar{b}$	$-ac\bar{a}\bar{c}$	$b\bar{a}$	ab	$\bar{a}\bar{b}$	$-bc\bar{a}\bar{c}$
$a\bar{b}$	ab	$\bar{a}\bar{b}$	$-ac\bar{b}\bar{c}$	$b\bar{b}$	ab	$\bar{a}\bar{b}$	$-bc\bar{b}\bar{c}$
$a\bar{a}$	$a\bar{a}\bar{b}$	b	$-ac\bar{a}\bar{c}$	$a\bar{a}$	$b\bar{a}\bar{b}$	a	$ac\bar{a}\bar{c}$
$b\bar{a}$	$a\bar{a}\bar{b}$	b	$-bc\bar{a}\bar{c}$	$b\bar{a}$	$b\bar{a}\bar{b}$	a	$bc\bar{a}\bar{c}$
$a\bar{b}$	$a\bar{a}\bar{b}$	b	$-ac\bar{b}\bar{c}$	$a\bar{b}$	$b\bar{a}\bar{b}$	a	$ac\bar{b}\bar{c}$
$b\bar{b}$	$a\bar{a}\bar{b}$	b	$-bc\bar{b}\bar{c}$	$b\bar{b}$	$b\bar{a}\bar{b}$	a	$bc\bar{b}\bar{c}$
$ab\bar{a}$	$\bar{b}$	$a\bar{a}$	$ac\bar{a}\bar{c}$	$ab\bar{a}$	$\bar{b}$	$b\bar{a}$	$bc\bar{a}\bar{c}$
$ab\bar{a}$	$\bar{b}$	$a\bar{b}$	$ac\bar{b}\bar{c}$	$ab\bar{a}$	$\bar{b}$	$b\bar{b}$	$bc\bar{b}\bar{c}$
$ab\bar{b}$	$\bar{a}$	$a\bar{a}$	$-ac\bar{a}\bar{c}$	$ab\bar{b}$	$\bar{a}$	$b\bar{a}$	$-bc\bar{a}\bar{c}$
$ab\bar{b}$	$\bar{a}$	$a\bar{b}$	$-ac\bar{b}\bar{c}$	$ab\bar{b}$	$\bar{a}$	$b\bar{b}$	$-bc\bar{b}\bar{c}$
ab	$\bar{a}\bar{b}$	$a\bar{a}$	$ac\bar{a}\bar{c}$	ab	$\bar{a}\bar{b}$	$b\bar{a}$	$bc\bar{a}\bar{c}$
ab	$\bar{a}\bar{b}$	$a\bar{b}$	$ac\bar{b}\bar{c}$	ab	$\bar{a}\bar{b}$	$b\bar{b}$	$bc\bar{b}\bar{c}$
$a\bar{a}\bar{b}$	b	$a\bar{a}$	$ac\bar{a}\bar{c}$	$a\bar{a}\bar{b}$	b	$b\bar{a}$	$bc\bar{a}\bar{c}$
$a\bar{a}\bar{b}$	b	$a\bar{b}$	$ac\bar{b}\bar{c}$	$a\bar{a}\bar{b}$	b	$b\bar{b}$	$bc\bar{b}\bar{c}$
$b\bar{a}\bar{b}$	a	$a\bar{a}$	$-ac\bar{a}\bar{c}$	$b\bar{a}\bar{b}$	a	$b\bar{a}$	$-bc\bar{a}\bar{c}$
$b\bar{a}\bar{b}$	a	$a\bar{b}$	$-ac\bar{b}\bar{c}$	$b\bar{a}\bar{b}$	a	$b\bar{b}$	$-bc\bar{b}\bar{c}$
$a\bar{a}$	$a\bar{a}$	$b\bar{b}$	$ac\bar{a}\bar{c}$	$a\bar{a}$	$b\bar{a}$	$a\bar{b}$	$-ac\bar{a}\bar{c}$
$a\bar{a}$	$a\bar{b}$	$b\bar{a}$	$-ac\bar{a}\bar{c}$	$a\bar{a}$	$b\bar{b}$	$b\bar{a}$	$-bc\bar{a}\bar{c}$
$a\bar{a}$	$b\bar{b}$	$a\bar{b}$	$-ac\bar{b}\bar{c}$	$a\bar{a}$	$b\bar{b}$	$b\bar{b}$	$-bc\bar{b}\bar{c}$
$b\bar{a}$	$a\bar{a}$	$b\bar{b}$	$bc\bar{a}\bar{c}$	$b\bar{a}$	$b\bar{a}$	$a\bar{b}$	$-bc\bar{a}\bar{c}$
$b\bar{a}$	$a\bar{b}$	$a\bar{b}$	$ac\bar{b}\bar{c}$	$b\bar{a}$	$a\bar{b}$	$b\bar{b}$	$bc\bar{b}\bar{c}$
$a\bar{b}$	$a\bar{a}$	$b\bar{b}$	$ac\bar{b}\bar{c}$	$a\bar{b}$	$b\bar{a}$	$b\bar{b}$	$bc\bar{b}\bar{c}$

Total bidegree (3, 4)

x	y	z	$\mu_3(x,y,z)$	x	y	z	$\mu_3(x,y,z)$
$a\bar{a}\bar{b}$	$ab\bar{a}$	$\bar{b}$	$ac\bar{a}\bar{b}\bar{c}$	$a\bar{a}\bar{b}$	$ab\bar{b}$	$\bar{a}$	$-ac\bar{a}\bar{b}\bar{c}$
$b\bar{a}\bar{b}$	$ab\bar{a}$	$\bar{b}$	$bc\bar{a}\bar{b}\bar{c}$	$b\bar{a}\bar{b}$	$ab\bar{b}$	$\bar{a}$	$-bc\bar{a}\bar{b}\bar{c}$
$a\bar{a}\bar{b}$	ab	$\bar{a}\bar{b}$	$ac\bar{a}\bar{b}\bar{c}$	$b\bar{a}\bar{b}$	ab	$\bar{a}\bar{b}$	$bc\bar{a}\bar{b}\bar{c}$
$a\bar{a}\bar{b}$	$a\bar{a}\bar{b}$	b	$ac\bar{a}\bar{b}\bar{c}$	$a\bar{a}\bar{b}$	$b\bar{a}\bar{b}$	a	$-ac\bar{a}\bar{b}\bar{c}$
$b\bar{a}\bar{b}$	$a\bar{a}\bar{b}$	b	$bc\bar{a}\bar{b}\bar{c}$	$b\bar{a}\bar{b}$	$b\bar{a}\bar{b}$	a	$-bc\bar{a}\bar{b}\bar{c}$
$a\bar{a}\bar{b}$	$a\bar{a}$	$b\bar{b}$	$-ac\bar{a}\bar{b}\bar{c}$	$a\bar{a}\bar{b}$	$b\bar{a}$	$a\bar{b}$	$ac\bar{a}\bar{b}\bar{c}$
$a\bar{a}\bar{b}$	$a\bar{b}$	$b\bar{a}$	$ac\bar{a}\bar{b}\bar{c}$	$a\bar{a}\bar{b}$	$b\bar{b}$	$a\bar{a}$	$-ac\bar{a}\bar{b}\bar{c}$
$b\bar{a}\bar{b}$	$a\bar{a}$	$b\bar{b}$	$-bc\bar{a}\bar{b}\bar{c}$	$b\bar{a}\bar{b}$	$b\bar{a}$	$a\bar{b}$	$bc\bar{a}\bar{b}\bar{c}$
$b\bar{a}\bar{b}$	$a\bar{b}$	$b\bar{a}$	$bc\bar{a}\bar{b}\bar{c}$	$b\bar{a}\bar{b}$	$b\bar{b}$	$a\bar{a}$	$-bc\bar{a}\bar{b}\bar{c}$
$ab\bar{a}$	$\bar{b}$	$a\bar{a}\bar{b}$	$-ac\bar{a}\bar{b}\bar{c}$	$ab\bar{a}$	$\bar{b}$	$b\bar{a}\bar{b}$	$-bc\bar{a}\bar{b}\bar{c}$
$ab\bar{b}$	$\bar{a}$	$a\bar{a}\bar{b}$	$ac\bar{a}\bar{b}\bar{c}$	$ab\bar{b}$	$\bar{a}$	$b\bar{a}\bar{b}$	$bc\bar{a}\bar{b}\bar{c}$
ab	$\bar{a}\bar{b}$	$a\bar{a}\bar{b}$	$-ac\bar{a}\bar{b}\bar{c}$	ab	$\bar{a}\bar{b}$	$b\bar{a}\bar{b}$	$-bc\bar{a}\bar{b}\bar{c}$
$a\bar{a}\bar{b}$	a	$b\bar{a}\bar{b}$	$ac\bar{a}\bar{b}\bar{c}$	$a\bar{a}\bar{b}$	b	$a\bar{a}\bar{b}$	$-2ac\bar{a}\bar{b}\bar{c}$
$a\bar{a}\bar{b}$	b	$b\bar{a}\bar{b}$	$-bc\bar{a}\bar{b}\bar{c}$	$b\bar{a}\bar{b}$	a	$b\bar{a}\bar{b}$	$2bc\bar{a}\bar{b}\bar{c}$

3.2 A compact $\partial\bar{\partial}$-manifold

The following procedure, explained in [9], gives a $\partial\bar{\partial}$-manifold by constructing an orbifold of global quotient type out of the Iwasawa nilmanifold.

Consider the action $\sigma : \mathbb{C}^3 \to \mathbb{C}^3$ by a finite group of biholomorphisms of order 4, given by $\sigma(z_1, z_2, z_3) = (iz_1, iz_2, -z_3)$. This gives a well-defined action on the Iwasawa manifold. The resulting quotient $M = \mathbb{I}_3/\langle\sigma\rangle$ is a compact orbifold with 16 isolated singular points. By Theorem 7.1 of [9], the orbifold M admits a resolution into a smooth $\partial\bar{\partial}$-compact manifold. The sub-algebra of $\mathbb{I}_3$ of σ-invariant differential forms $\mathcal{A}_\sigma(M)$ is generated by the elements

$$a\bar{a}, b\bar{b}, c\bar{c}, a\bar{b}, b\bar{a}, c\bar{a}\bar{b}, ab\bar{c}, abc, \bar{a}\bar{b}\bar{c}.$$

The only non-trivial differentials are

$$\partial(c\bar{c}) = -ab\bar{c}, \ \ \partial(c\bar{a}\bar{b}) = -ab\bar{a}\bar{b}$$

and their complex conjugates. This algebra satisfies the $\partial\bar{\partial}$-condition and so in particular, all cohomologies (including Dolbeault, Bott-Chern, Aeppli and ABC) coincide. These are given by

$\overline{abc}$			$abc\overline{abc}$
		$ac\overline{ac}, ac\overline{bc}, bc\overline{ac}, bc\overline{bc}$	
	$a\bar{a}, a\bar{b}, b\bar{a}, b\bar{b}$		
1			abc

The μ_3-product is non-trivial only when all three inputs have bidegree (1, 1). In particular, all triple ABC-Massey products vanish outside this range. In such cases, we obtain

x	y	z	$\mu_3(x,y,z)$
$a\bar{a}$	$a\bar{a}$	$b\bar{b}$	$-a\bar{a}c\bar{c}$
$a\bar{a}$	$b\bar{b}$	$b\bar{b}$	$b\bar{b}c\bar{c}$
$a\bar{a}$	$b\bar{b}$	$a\bar{b}$	$c\bar{c}a\bar{b}$
$a\bar{a}$	$b\bar{b}$	$b\bar{a}$	$c\bar{c}b\bar{a}$
$a\bar{a}$	$a\bar{b}$	$b\bar{a}$	$a\bar{a}c\bar{c}$
$a\bar{a}$	$b\bar{a}$	$a\bar{b}$	$a\bar{a}c\bar{c}$
$b\bar{b}$	$a\bar{a}$	$a\bar{b}$	$c\bar{c}a\bar{b}$
$b\bar{b}$	$a\bar{a}$	$b\bar{a}$	$c\bar{c}b\bar{a}$
$b\bar{b}$	$a\bar{b}$	$b\bar{a}$	$b\bar{b}c\bar{c}$
$b\bar{b}$	$b\bar{a}$	$a\bar{b}$	$b\bar{b}c\bar{c}$
$a\bar{b}$	$a\bar{b}$	$b\bar{a}$	$c\bar{c}a\bar{b}$
$a\bar{b}$	$b\bar{a}$	$b\bar{a}$	$-c\bar{c}b\bar{a}$

Note that for all of the aforementioned products, the corresponding Massey sets are singletons and we have $\langle x, y, z\rangle = \{-\mu_3(x, y, z)\}$.

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