Master thesis

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Wandering domains and entire maps of bounded type

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Summary

Complex dynamics is one of the richest and most active branches of dynamical systems. Its goal is to study what happens to analytic functions on the complex plane (or the Riemann sphere) when it is iterated.. In this master thesis the focus is on transcendental dynamics since the assumption is that $f : \mathbb{C} \to \mathbb{C}$ is a transcendental entire function.

The foundations of complex dynamics were laid by Pierre Fatou and Gaston Julia in the 1920s when they defined the Fatou and Julia sets, named after them. Roughly speaking, the Fatou set is the *stable set* since all the points in a neighbourhood have the same behaviour after iteration. Alternatively, the points of the Julia set are those that behave unpredictably after iteration. For that reason the Julia set is also called the *chaotic set.* Both sets are invariant and give a natural partition of the complex plane. The Fatou set is made up of the complementary domains in $\mathbb{C}$ of the Julia set, the Fatou components. Since it is stable a possible Fatou component U can be either periodic (if $f^p(U) = U$ for some $p \in \mathbb{N}$), pre-periodic (if they are periodic eventually) or wandering (if $f^n(U) \cap f^m(U) = \emptyset$ for $m \neq n$).

The study of the existence or absence of wandering domains is one of the most relevant problems in complex dynamics. A remarkable result by D. Sullivan showed that for a rational function there are no wandering domains. However, I. N. Baker showed an example of a transcendental entire function with wandering domains, answering the question whether transcendental entire functions could have wandering domains. Sullivan introduced quasi-conformal analysis to prove this result which is an extremely powerful tool. Using this technique and motivated by the fact that rational maps have a finite number of singularities, Eremenko-Lyubich and Golberg-Keen were able to generalise Sullivan's argument to prove that for transcendental entire functions with a finite singular set there are no wandering domains. Many mathematicians tried to use quasi-conformal surgery to prove results on the existence or non existence of wandering domains for certain transcendental entire functions. It was recently that C. Bishop proved the existence of wandering domains for $f \in \mathcal{B}$, functions whose set of singular values is bounded.

Mathematicians Mihaljević-Rempe used a different tool, hyperbolic geometry, to show that for functions in $\mathcal{B}^*_{\text{real}}$, that is functions in $\mathcal{B}$ which are real and whose set of singular values is also real, that satisfied a certain technical condition there are no wandering domains. They also made use of several previous results. For instance Fatou's result that states that limit functions of the family of iterates of a transcendental entire function over a wandering domain are constant, or Baker's refinement on this result that says that the constant limit functions are in the post singular set. Also, a result by Eremenko-Lyubich is of vital importance since it shows that for $f \in \mathcal{B}$ the family of iterates over a wandering domain can not tend to infinity uniformly. Using all this tools the goal of this master thesis is to give a comprehensive, self-contained and detailed proof of the result by Mihaljević-Rempe which reads as follows: if $f \in \mathcal{B}^*_{\text{real}}$ satisfying a the sector condition, then f has no wandering domains.

Chapter 1

Introduction

Dynamical systems theory, one of the main fields of mathematics, studies the behaviour of mathematical models governed by either ordinary differential equations, partial differential equations or iterates of regular functions as they evolve through time (continuous or discrete). Nowadays, complex dynamics is a very rich branch of dynamical systems, with numerous relevant applications. The goal of complex dynamics is to study what happens to an analytic function in the complex plane $\mathbb{C}$ or in the Riemann sphere, $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, when it is iterated. The history of complex dynamics goes back to the late 1800s, but it is not until the beginning of the twentieth century when the major breakthrough takes place. At the time all the work being done on the subject was mainly centred in France, where mathematicians Pierre Fatou and Gaston Julia achieved to lay the foundations of complex dynamics (for the original papers see [Fat19] and [Jul26]).

The work of Fatou and Julia is based on the concept of normal families, presented by Paul Montel on the early 1900s, which allowed them to define the Julia and Fatou sets, named after them. Here the assumption is that $f : \mathbb{C} \to \mathbb{C}$ is a transcendental entire function but an analogous definition can be given for other type of holomorphic functions. In the sense of Montel, if the family of functions over a neighbourhood U of a point in the complex plane misses two points, then it is a normal family. The Fatou and Julia sets give a natural dynamical partition of the phase space into domains of normality and domains of non-normality, taking a normal family as the sequence of iterates of the function and using Montel's theory to develop those concepts. The relationship between the Fatou and Julia sets of a function and the idea of normal families is that the *Fatou set*, $\mathcal{F}(f)$, is defined as the set of points of the complex plane (or Riemann sphere) where the family of iterates is normal. Alternatively, the *Julia set*, $\mathcal{J}(f)$, is the set of points where the family of iterates $\{f^n\}_{n\geq 0}$ fails to be normal. From this idea of normality it follows a more intuitive approach to those two sets. The Fatou set is the set of points that "behave well" under iteration, which means that if we take a neighbourhood U of points then the iterates of those points have the same behaviour. For that reason, the Fatou set is also called the *stable set*. On the other hand, the Julia set is often refereed as *chaotic set*, since the points in $\mathcal{J}(f)$ have an unpredictable behaviour. Much has been said about the properties of these sets, for instance that both are invariant. Besides, by definition $\mathcal{F}(f)$ is an open set, whereas, $\mathcal{J}(f)$ is a closed set which is known to be infinite. As for the Fatou set, it is made up of the complementary domains in $\mathbb{C}$ of the Julia set, the so-called *Fatou components*. For a more in depth review see [Mil06].

At that moment, the main focus was on the study of rational functions and it was not until a few years later that mathematicians made the leap to transcendental dynamics. Since the dynamics of rational maps is fairy well understood, the idea was to see if some of those results could be showed to be true also for transcendental entire functions. It is known that for a transcendental entire function f, if we consider a component U of the Fatou set $\mathcal{F}(f)$, then $f(U) \subset V$ where V is also in a connected component of $\mathcal{F}(f)$. Consequently Fatou components are either periodic, pre-periodic or wandering. More precisely,

- if $f^p(U) = U$, for some $p \in \mathbb{N}$, then U is *periodic*;
- if U is not periodic but there exists $n \in \mathbb{N}$ such that $f^n(U)$ is periodic, then U is *pre-periodic*;
- finally, if for all $n, m \in \mathbb{N}$ so that $n \neq m$, $f^n(U) \cap f^m(U) = \emptyset$, then U is a *wandering domain*.

Hence, wandering domains are connected components of the Fatou set that evolve after iteration in such a way that they never go back to a domain they have already been in, which means that they "wander". For future proofs, it will be quite interesting to classify wandering domains by their possible (constant) limit functions. Hence, a wandering domain can be either *escaping* (if its iterates tend to infinity), *oscillating* (if one sequence of iterates tends to infinity and another tends to a finite $a \in \mathbb{C}$) or *bounded* (all converging sequences of iterates tend to finite points of the complex plane).

- if $f^n|_W \to \infty$ as $n \to +\infty$, then W is *escaping*,
- if there exist two increasing sequences $\{n_k\}_{k\geq 0}$ and $\{m_k\}_{k\geq 0}$ and a point $a \in \mathbb{C}$, such that $f^{n_k}|_W \to \infty$ and $f^{m_k}|_W \to a$ as $k \to +\infty$, then W is *oscillating*,
- otherwise W is *bounded*.

Notice that these definitions are at dynamical level and they do not refer to the topology of the wandering domains. Thus far, the existence of bounded wandering domains has not been proved. As for periodic components, Fatou gave a classification of all the possibilities. A periodic connected component of the Fatou set either is an *attracting* or *parabolic basin*, a *Siegel disc*, a *Herman ring* or a *Baker domain*.

The main difference between the dynamics of rational maps and transcendental entire functions lies in the role played by their *singularities*, i.e. points of the complex plane for which the inverse function is not well defined. The set of singular values of a function f is denoted by $S(f)$. Moreover, consider the *post singular set* as the set made up by the union of all the orbits of the singular values, i.e. $E = \{f^n(s),\, s \in S(f)\}$. In general, for transcendental dynamics *singular values* can be classified in two types, *critical values* and *asymptotic values*. Given a point $z \in \mathbb{C}$ such that $f'(z) = 0$, then z is a *critical point* and its image, $f(z)$, is a *critical value*. Asymptotic values are refereed as "points coming from infinity". More formally, considering a curve γ going to infinity such that the image of the function tends to a point v along γ, then v is an asymptotic value. In particular, every immediate basin of an attracting or parabolic domain must have a point of $S(f)$ inside.

Rational maps have only critical values, and a finite number of them, but for transcendental entire maps we have also asymptotic values, the fact that $\#S(f)$ might be infinite and that $z = \infty$ is an essential singularity. Recall that in any neighbourhood of an essential singularity the map f covers almost all of $\mathbb{C}$. This is precisely Picard's theorem.

Theorem (Picard's theorem). *Let $f : \mathbb{C} \to \mathbb{C}$ be an analytic function that has an essential singularity at z_0 and U a neighbourhood of z_0. Then, considering the function $f : U \setminus \{z_0\} \to \mathbb{C}$, $f(z)$ takes all the values in $\mathbb{C}$, except at most one value, infinitely many times.*

The exponential map is the example that is systematically studied in transcendental dynamics, analogously to the quadratic family in the rational case. This map is of great interest because it has only one finite singular value, the asymptotic value $z = 0$ and it is omitted by the exponential map.

As stated before, the connected components of the Fatou set are mainly two, (pre)periodic domains and wandering domains. In this paper, since the focus will be on transcendental dynamics where wandering domains are a possibility, we will concentrate on the study of these later domains. From the work of Fatou and Julia in the 1920s to the mid 1980s there were no major breakthroughs in the field, until Denis Sullivan introduced quasi-conformal analysis, a new and extremely powerful tool. Using this concept, Sullivan [Sul85] proved one of the most important theorems in complex dynamics, the No Wandering Domain Theorem. This theorem says that for a rational map every component of the Fatou set is eventually periodic, which particularly means that for rational maps there are no wandering domains.

The question that arises naturally is if this is true in the setting of transcendental dynamics. In 1976 Baker [Bak76] answered that question by giving an example of the existence of wandering domains for a transcendental entire function and a few years later, in 1984, Herman [Her84] proposed a systematic way of constructing wandering domains. However, this opened the floor for debate on the possibility of a theorem about the absence of wandering domains for a class of transcendental entire functions. Motivated by rational maps having a finite number of singularities, mathematicians asked themselves if Sullivan's No Wandering Domain could be true for transcendental entire functions with a finite number of singularities, functions in the so-called *Speiser class*, $\mathcal{S}$. The answer to that question was that for $f \in \mathcal{S}$ there are no wandering domains since the argument given by Sullivan was generalized by Eremenko-Lyubich [EL92] and Golberg-Keen [GK86].

To be able to do a comprehensive approach to the study of wandering domains, it will be necessary to have a background on the results known up to this moment. For that, some concepts of vital importance will be stated and proved throughout this document. It was Fatou [Fat19] himself that proved that the limit functions of a connected component of $\mathcal{F}(f)$ are either constant or $g(z) = e^{i\theta} z$ for some $\theta \in \mathbb{R} \setminus \mathbb{Q}$. He also showed that for wandering domains the only possible limit functions are constant. Several years later, in 1970, Baker [Bak70] gave a stronger version of Fatou's result, showing that any possible constant limit function should belong to $\overline{E} \cup \{\infty\}$ (notice that a has to be in the Julia set, so in fact $a \in \mathcal{J}(f) \cap (\overline{E} \cup \{\infty\})$). What is more, in 1993 Bergweiler-Haruta-Kriete-Meier-Terglane [BHK$^+$93] refined the result as they proved that $a \in \mathcal{J}(f) \cap (E' \cup \{\infty\})$, where E' is the set of finite limit points of E, also known as the *derived set.* With this in mind it can be proved that the exponential map has no wandering domains. If it had a wandering domain, W, the limit functions of a would have to be either ∞ or in E', but since the only finite singular value is $z = 0$ one can see that $E' = \emptyset$. Further in this thesis, there will be examples of the proof of the absence of wandering domains for different functions using arguments based on Baker's theorem.

Once finite type functions were contemplated, the following step was to consider the *Eremenko-Lyubich class*, $\mathcal{B}$, made up by functions whose set of singular values is bounded. Whether a function in class $\mathcal{B}$ can have wandering domains is a question that has remained open until recently. Based on a result by Eremenko and Lyubich [EL92] it has been proved that there are no escaping wandering domains in class $\mathcal{B}$. However, until 2014 there was no proof on the existence or absence of wandering domains for a function in class $\mathcal{B}$, but it was right at that time when Bishop [Bis15] published a paper that gives a constructive way to build functions in $\mathcal{B}$ that have wandering domains, answering the question. Despite this result, in a paper by Mihaljević-Brandt and Rempe-Gillen [MBRG13], they propose a question that remains still open. Consider a function in class $\mathcal{B}$. Then they ask: if the iterates of the singular values tend to infinity uniformly, that is $\lim_{n\to\infty} \inf_{s\in S(f)} |f^n(s)| = \infty$, can the function f have wandering domains? As a matter of fact, they were able to give a partial answer to that question proving that under that hypothesis, together with the fact that f satisfies a certain technical condition, then the function f has no wandering domains. It should be noted that this result is not in contradiction with the example given by Bishop, since the wandering domains he constructed do not satisfy the hypothesis of the theorem.

In that paper, Mihaljević-Brandt and Rempe-Gillen prove another key result in the theory of wandering domains. Set $\mathcal{B}^*_{\text{real}}$ to be the set of functions in class $\mathcal{B}$ which are real, that is $f(\mathbb{R}) \subset \mathbb{R}$, and whose set of singular values is also real. If $f \in \mathcal{B}^*_{\text{real}}$ satisfies a certain geometric condition, the so-called *sector condition*, then f has no wandering domains.

Theorem . *Let $f : \mathbb{C} \to \mathbb{C}$ be a function in $\mathcal{B}^*_{real}$. Then if f satisfies the sector condition it has no wandering domains.*

Since there are no escaping wandering domains for functions in class $\mathcal{B}$ the proof of the theorem will be based on proving the absence of bounded and oscillating wandering domains. For the bounded case, there is a result by Rempe-Gillen and van Strien [RGvS15] that ensures that there are no wandering domains of this type for $f \in \mathcal{B}^*_{\text{real}}$. Finally, the last step is to prove that there are no oscillating wandering domains. To prove that fact the authors had to impose a geometric condition over f, the *sector condition*. Roughly speaking, the sector condition over f means that given R there exists a sector $S = \{x + i\,y,\, x > R' \text{ and } |y| < c\}$ such that if $z \in S$ then $|f(z)| > R$. Hence any oscillating wandering domain should leave S before "returning from infinity". For details see Chapter 4.

The goal of this thesis is to give a comprehensive proof of this last theorem, based on the paper by Mihaljević-Brandt and Rempe-Gillen, which will be given in Chapter 4. Before reaching that stage it will be necessary to lay the foundations by giving some background on complex dynamics and, in particular, on transcendental dynamics, as well as introducing hyperbolic geometry as one of the main tools, which will be done in Chapters 2 and 3. Besides, in order to give a self-contained document, all the results that have been talked about along this lines will be stated and proved throughout the course of this paper.

Chapter 2

Preliminaries

2.1 Basic definitions

In what follows we will consider $f : \mathbb{C} \to \mathbb{C}$ a transcendental entire function. We say that f is a transcendental entire function if it is holomorphic in $\mathbb{C}$ and it has an essential singularity at infinity. Notice that polynomials are entire functions but they are not transcendental since they extend analytically to infinity. Some examples of transcendental entire functions are the exponential map or trigonometric functions. Also, as we will consider the iterates of such functions, we will abbreviate the notation by denoting the n-th iterate by $f^n = f \circ \overset{n)}{\cdots} \circ f$.

It is well-known that to study the dynamics of the discrete dynamical system induced by the iterates of f, namely the family $\{f^n\}_{n \geq 0}$, it is crucial to have a good understanding of the singularities of the inverse map f^{-1}. We will classify transcendental entire functions depending on the number and distribution of those singularities.

Definition. A *critical point* of a function f is a point $c \in \mathbb{C}$ such that $f'(c) = 0$. Its image, $v = f(c)$, is called a *critical value*.

Of course, if v is a critical value then it is a singularity of the inverse function f^{-1}. However, in this setting there are other, non algebraic, singularities of the inverse.

Definition. An *asymptotic value* is a point $c \in \hat{\mathbb{C}}$ such that there exists a curve $\gamma : [a, b] \to \mathbb{C}$ such that $\gamma(t) \to \infty$ and $f(\gamma(t)) \to c$ as $t \to \infty$.

For instance, if $f(z) = e^z$ then $c = 0$ is a finite asymptotic value since $\lim_{t\to\infty} f(\gamma(t)) = 0$ for $\gamma(t) = -t$. Notice that $c = \infty$ is also an asymptotic value of the exponential map since it is also true that $\lim_{t\to\infty} f(\gamma(t)) = 0$ for $\gamma(t) = t$.

Definition. Let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental entire function, then v is a *singular value* if it is either a critical value or an asymptotic value. The set of singular values is denoted by $S(f)$. This set is also called the set of singularities of the inverse function f^{-1}.

The main focus of this paper will be on a specific class of transcendental entire functions, those that have a bounded set of singular values.

Definition 2.1. Let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental entire function. We say that a function is in the *Eremenko-Lyubich class* (or equivalently *class* $\mathcal{B}$ or f is of *bounded type*) if $S(f)$ is bounded. That is, there exists $M > 0$ such that $S(f) \subset B(0, M)$.

Next, we will recall several basic definitions and properties on transcendental dynamics that are important to keep in mind.

Definition. Let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental entire function. We say $q_0 \in \mathbb{C}$ is a p-periodic point if $f^p(q_0) = q_0$, $p \in \mathbb{N}$ and $f^k(q_0) \neq q_0$ for $k < p$. If $p = 1$ we say that q_0 is a *fixed point of* f. Let $\{q_0, q_1, \dots, q_{p-1}\}$ be the *orbit* of q_0 (that is $q_i = f(q_{i-1})$ with $i = 1, \dots, p-1$ and $f(q_{p-1}) = q_0$). We denote by $\lambda := (f^p)'(q_0)$ the *multiplier of the periodic orbit* (notice that by the chain rule the definition of λ does not depend on the point of the orbit).

We say $q \in \mathbb{C}$ is a *pre-periodic point* if there exists $n \in \mathbb{N}$ such that $f^n(q)$ is a periodic point of any period $p \in \mathbb{N}$.

Depending on λ we can give a classification of a periodic point q: it is either attracting (all the points in a neighbourhood of q tend to it after iteration), repelling (it is attracting by the inverse function) or neutral (if it is neither attracting nor repelling).

Theorem 2.2. *Let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental entire function and let λ be the multiplier of an orbit of period $p \in \mathbb{N}$, $\{q_0, q_1, \dots, q_{p-1}\}$, so $\lambda = (f^p)'(q_0)$. Then, the behaviour of the periodic orbit can be characterized by the multiplier,*

- *if $|\lambda| < 1$, q_0 is an attracting point;*
- *if $|\lambda| = 1$, q_0 is a neutral point; rational if $\lambda = e^{2\pi i\theta}$ with θ rational and irrational if θ is irrational;*
- *if $|\lambda| > 1$, then q_0 is a repelling point.*

2.1.1 Normal families. The definition of Julia and Fatou sets

As has been said in the introduction, we know that the Fatou and Julia sets are two invariant subsets that conform the whole complex plane. The points in the Julia set are those that have a "chaotic" behaviour after iteration and those in the Fatou set behave "stably". To be able to give a formal definition of both sets first we need to introduce the concept of normal families. From now on, in general we will consider as a family of functions the family of iterates $\{f^n\}_{n\geq 0}$.

Definition. Let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental entire function and $U \subset \mathbb{C}$. Consider the well-defined family of iterates, $\{f^n|_U\}_{n\geq 0}$. Then $\{f^n|_U\}_{n\geq 0}$ is a *normal family in* U if for any sequence $\{f^{n_k}|_U\}_{k\geq 1}$, there exists a subsequence $\{f^{n_{k_l}}|_U\}_{l\geq 0}$ such that either

$$\{f^{n_{k_l}}|_U\} \overset{l\to\infty}{\longrightarrow} g, \tag{2.1}$$

where $g : U \to \mathbb{C}$ is a holomorphic function, or

$$\{f^{n_{k_l}}|_U\} \overset{l\to\infty}{\longrightarrow} \infty. \tag{2.2}$$

We say that a family of iterates $\{f^n\}_{n\geq 1}$ is *normal at* $z \in \mathbb{C}$ if and only if there exists a neighbourhood U of z such that $\{f^n|_U\}_{\geq 1}$ is normal in U.

Here the notion of convergence is on compact subsets of U, or equivalently, we have locally uniform convergence. With this definition we are in the position to define the Julia and Fatou sets, denoted as $\mathcal{J}(f)$ and $\mathcal{F}(f)$ respectively, which is based on the concept of normal families.

Definition. Let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental entire function and consider $U \subset \mathbb{C}$. Set $z \in U$. We define the *Julia* and *Fatou sets* as,

$$\mathcal{J}(f) = \{z \in \mathbb{C},\ \{f^n|_U\}_{n\geq 1} \text{ is not normal in } U\},$$
$$\mathcal{F}(f) = \{z \in \mathbb{C},\ \{f^n|_U\}_{n\geq 1} \text{ is normal at some } U\}.$$

Hence, by definition, $\mathcal{F}(f) = \mathbb{C} \setminus \mathcal{J}(f)$. Since it is not easy in general to prove when a family of iterates defines a normal family in a certain open set, the following theorem will allow us to give an easier approach to show if a family of iterates is normal.

Theorem 2.3 (Montel's Theorem). *Let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental entire function and $U \subset \mathbb{C}$. Set $z_0 \in U$ and suppose that there exist two points $\{a, b\} \in \mathbb{C}$ such that $f^n(z) \neq \{a, b\}$ for all $z \in U$ and for all $n \geq 1$. Then $\{f^n|_U\}_{n\geq 0}$ is a normal family at z_0, which implies that $z_0 \in \mathcal{F}(f)$.*

This means that if the family of iterates $f^n : U \to \mathbb{C}$ misses at most two points, then $U \subset \mathcal{F}(f)$. To exemplify this concept, consider the map $f(z) = e^{z-2}$ and the unit disc defined as $\mathbb{D} = \{z \in \mathbb{C}, |z| < 1\}$. We claim that if we take the closure of $\mathbb{D}$, $\overline{\mathbb{D}} = \{z \in \mathbb{C}, |z| \leq 1\}$, then the iterates $f^n(\overline{\mathbb{D}})$ remain in $\mathbb{D}$, which implies that $\overline{\mathbb{D}} \subset \mathcal{F}(f)$. To show that claim let $z = e^{i\theta}$ a point in the boundary of $\overline{\mathbb{D}}$, then $|f(e^{i\theta})| = e^{-2} \cdot \left| e^{e^{i\theta}} \right| = e^{-2} \cdot e^{\mathrm{Re}\,(e^{i\theta})} = e^{-2} \cdot e^{\cos\theta} < 1$. Moreover, keeping that in mind and since f is bounded on $\overline{\mathbb{D}}$, the interior of the unit disc maps to itself. Hence $f^n(\overline{\mathbb{D}})$ misses all the points of $\mathbb{C} \setminus \overline{\mathbb{D}}$ and by Montel's theorem we have that $\overline{\mathbb{D}} \in \mathcal{F}(f)$.

Proposition 2.4. *Let $f : \mathbb{C} \to \mathbb{C}$ a transcendental entire function. The following statements hold*

- $\mathcal{J}(f) = \mathbb{C} \setminus \mathcal{F}(f)$.
- *The Fatou set is open and the Julia set is closed. In general $\mathcal{F}(f)$ may have infinitely many connected components. Each one of them is called a Fatou component of f.*
- *The Fatou and Julia sets are completely invariant, i.e. $f(\mathcal{J}) = \mathcal{J}$ and $f(\mathcal{F}) = \mathcal{F}$.*
- *Given a connected component U of the Fatou set, $f(U) \subset V$ where V is also a connected component of the Fatou set.*
- *$\mathcal{J}(f)$ is either nowhere dense or it coincides with $\mathbb{C}$.*
- *The Julia set is unbounded and non-empty.*
- *$\mathcal{J}(f)$ coincides with the closure of repelling periodic points,*

$$\mathcal{J}(f) = \overline{\bigcup_{p\in\mathbb{N}} \text{repelling points of period } p}.$$

What is more, given a connected component $U \in \mathcal{F}(f)$ then U has to be either a *periodic domain* (such that if $f^p(U) = U$, for some $p \in \mathbb{N}$), a *pre-periodic domain* (a domain that eventually becomes periodic) or a *wandering domain.* The main object of study of this document are wandering domains. Intuitively, we understand that a domain of this kind "wanders", which means that the domain evolves on time but does not come back to a place it has already visited.

Definition. Let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental entire function. If $U \subset \mathbb{C}$ is a Fatou component of f such that

$$f^n(U) \cap f^m(U) = \emptyset,$$

for all $n \neq m \in \mathbb{N}$, then U is a *wandering domain.* Conversely, if $f^n(U) = f^m(U)$ for some $n > m \geq 0$ them U is *pre-periodic.* Moreover, if $m = 0$, U is *periodic* and if $n = 1$ it is *fixed.*

In view of this definition, it is interesting to give the classification of periodic Fatou components (notice that given a domain, if we can discard that it is periodic domain then it will have to be a wandering domain).

Theorem 2.5. *Let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental entire function and $U \subseteq \mathcal{F}(f)$ be a p-periodic component, then U can be classified as*

- *Immediate attracting basin, if U contains an attracting $p-$periodic point z_0 and $f^{np}(z) \to z_0$ as $n \to \infty$, for all $z \in U$.*
- *Parabolic basin, if ∂U contains a unique $p-$periodic point z_0 and $f^{np}(z) \to z_0$ as $n \to \infty$, for all $z \in U$. Also, $(f^p)'(z_0) = 1$.*
- *Siegel disc, if there exists a holomorphic homeomorphism $\varphi : U \to \mathbb{D} = \{z \in \mathbb{C}, |z| < 1\}$ such that $(\varphi \circ f^p \circ \varphi^{-1})(z) = e^{2\pi\theta i}z$, for some $\theta \in \mathbb{R} \setminus \mathbb{Q}$.*
- *Herman ring, if there exists $r \in \mathbb{R}$ so that $r > 1$ and a holomorphic homeomorphism $\varphi : U \to \{1 < |z| < r\}$ such that $(\varphi \circ f^p \circ \varphi^{-1})(z) = e^{2\pi\theta i}z$, for some $\theta \in \mathbb{R} \setminus \mathbb{Q}$.*
- *Baker domain, if ∂U contains a point z_0 such that $f^{np}(z) \to z_0$ as $n \to \infty$, for all $z \in U$ but z_0 is an essential singularity, i.e. $f(z_0)$ is not defined.*

2.2 Background on hyperbolic geometry

In what follows we will use many geometric arguments, since they will help us prove and understand better the ideas behind the results that will be introduced. Because of that, we are going to write all the results shown in this document in those terms, to make all the proofs easier and more intuitive. First, it is necessary to give a brief introduction to the basic concepts of hyperbolic geometry.

Given that we are going to introduce geometrical notions, we need to define a distance for this setting. This will allow us to present different concepts, such as surfaces, length, straight lines, etc. Besides, we will set our goal on studying a metric that is invariant under rotations of the complex plane. Because of that we are going to use conformal maps, which are maps that preserve angles locally.

For our purposes we will consider always $U \subset \mathbb{C}$ as the domain of study, which is, in particular, a Riemann surface. Moreover, the focus will be on a particular type of subsets of $\mathbb{C}$ known as *hyperbolic domains.* Given $U \subset \mathbb{C}$, U is a *hyperbolic domain* if there exists a holomorphic covering map $\pi : \mathbb{D} \to U$. If U is simply-connected then π is the Riemann map. Another characterization of hyperbolic domains is that if $U \subset \mathbb{C}$ is a non empty open set, then U is a hyperbolic surface if and only if $\mathbb{C} \setminus U$ contains at least two points.

Going back to the definition of metric, we still need to define a distance. As we already know, for any path $\gamma : [a, b] \to \mathbb{C}$ the Euclidean length is given by

$$\ell(\gamma) = \int_\gamma |dz| = \int_a^b |\gamma'(t)|\, dt,$$

where $|dz| = |\gamma'(t)|dt$ is the element of euclidean length of f on $\mathbb{C}$.

The goal now is to be able to define a *metric* (or *density*) on a hyperbolic domain U. For that purpose we will use the fact that we have a holomorphic covering map $\pi : \mathbb{D} \to U$ and that to have the metric on U we will need to do nothing but to push-forward the map π, i.e. if we have the metric on the unit disc, $\rho_\mathbb{D}(z)$, then the metric on U, $\rho_U(z)$, will be $\pi^*(\rho_\mathbb{D}(z))$. Hence, first have to define a metric on the unit disc and once we reach that stage we will be in the appropriate position to give a metric on a hyperbolic domain U.

Definition. Let $U \subset \mathbb{C}$ be a hyperbolic domain and consider the covering map $\pi : \mathbb{D} \to U$. Then the hyperbolic density on U at a point $z \in \mathbb{D}$ is defined as

$$\rho_U(\pi(z)) = \frac{\rho_\mathbb{D}(z)}{|\pi'(z)|}.$$

In order to give a metric on $\mathbb{D}$, now we are going to put forward some definitions and properties of the hyperbolic geometry defined over the upper half-plane $\mathbb{H} = \{z \in \mathbb{C}, \operatorname{Im}(z) > 0\}$. This is due to the fact that over $\mathbb{H}$ the computation of the conformal metric is simpler, and we can pass easily from $\mathbb{H}$ to $\mathbb{D}$. Furthermore, by the Riemann Mapping Theorem we know that the covering map $\pi : \mathbb{D} \to \mathbb{H}$ is a Riemann map, so that $\mathbb{H}$ is conformally isomorphic to $\mathbb{D}$. Hence, we shall use some results over $\mathbb{H}$ that will yield other results over $\mathbb{D}$, which in turn will give the metric on U.

It is known that the set of automorphisms of the complex plane is given by functions of the type $f(z) = a\, z + b$ where $a, b \in \mathbb{C}$. Imposing the invariance of the real axis, it is easy to show that the set of automorphisms of $\mathbb{H}$ will be of the form $A(z) = az + b$ with $a, b \in \mathbb{R}$ and $a > 0$. What is more, the metric has to be invariant, which means that for an automorphism of $\mathbb{H}$ it has to be satisfied that,

$$\rho_\mathbb{H}(z) = \rho_\mathbb{H}(A(z)) \cdot |A'(z)| = \rho_\mathbb{H}(az + b) \cdot a \quad \forall z \in \mathbb{C}.$$

Next, we take $z = i$ and assume by convention a weight of $\rho_\mathbb{H}(i) = 1$. We can make this choice of the value of z because $i \overset{A}{\mapsto} ai + b$, which can be any point $z \in \mathbb{H}$. Hence,

$$1 = \rho_\mathbb{H}(i) = \rho_\mathbb{H}(ai + b) \cdot a \;\Rightarrow\; \rho_\mathbb{H}(z) = \frac{1}{a} = \frac{1}{\operatorname{Im} z}.$$

As we have stated before, it is easy to go from $\mathbb{D}$ to $\mathbb{H}$ and vice versa. Then, we will be able to pass from the metric on $\mathbb{H}$ to the metric on the disc. For that, consider the Möbius transformation

$$f : \mathbb{D} \to \mathbb{H} \text{ such that } \; z \mapsto i \cdot \frac{1+z}{1-z}. \tag{2.3}$$

Using the fact that we know that the metric on $\mathbb{D}$ is a push-forward of f of the metric on $\mathbb{H}$, the metric on the unit disc will be,

$$\rho_{\mathbb{D}}(z) = \rho_{\mathbb{H}}(f(z)) \cdot |f'(z)| = \frac{1}{\operatorname{Im}(f(x))} \cdot |f'(z)|,$$

$$\operatorname{Im}(f(z)) = \operatorname{Im}\left(i \cdot \frac{1+z}{1-z}\right) = \operatorname{Im}\left(\frac{-2y + i(1-|z|^2)}{|1-z|^2}\right) = \frac{1-|z|^2}{|1-z|^2},$$

$$|f'(z)| = \frac{2}{|1-z|^2}$$

Then,

$$\rho_{\mathbb{D}}(z) = \frac{|1-z|^2}{1-|z|^2} \cdot \frac{2}{|1-z|^2} = \frac{2}{1-|z|^2}.$$

Next, we will find the metric on the unit punctured disc $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$, which will be needed later. Consider the map $g : \mathbb{D}^* \to \mathbb{H}$ such that $z \mapsto -i \cdot \log(z)$, doing the same as for the unit disc we have

$$\rho_{\mathbb{D}^*}(z) = \rho_{\mathbb{H}}(g(z)) \cdot |g'(z)| = \frac{1}{\operatorname{Im}(g(x))} \cdot |g'(z)|,$$

$$\operatorname{Im}(g(z)) = \operatorname{Im}(-i \cdot \log(z)) = \operatorname{Im}(-i \cdot (\log|z| + i \arg z)) = -\log|z|,$$

$$|g'(z)| = \frac{1}{|z|}$$

Gathering all the previous results, we have already proved the following proposition.

Proposition 2.6. *The metric for the half-plane $\mathbb{H}$, the unit disc $\mathbb{D}$ and the punctured unit disc $\mathbb{D}^*$ are given by,*

$$\rho_{\mathbb{H}}(z) = \frac{1}{\operatorname{Im} z}, \qquad \rho_{\mathbb{D}}(z) = \frac{2}{1-|z|^2}, \qquad \rho_{\mathbb{D}^*} = \frac{1}{|z| \cdot |\log|z||}.$$

At this point, we are in the appropriate framework to be able to define a metric for a hyperbolic domain. Consider $U \subset \mathbb{C}$ a hyperbolic domain and a metric $\rho_U : U \to \mathbb{R}$, that is a continuous and (everywhere) positive map. Since the goal is to preserve the metric under rotations we have a *conformal metric*, or also *conformal distortion*, $\rho_U(z)\,|dz|$. This may be interpreted as the element of length over the euclidean metric giving rise to a new element of length $\rho_U(z)\,|dz|$, controlled by scaling the element of euclidean length $|dz|$ at every point $z \in \mathbb{C}$, where we scale by the value given by the density on U, $\rho_U(z)$. It is important to notice that for the case of hyperbolic surfaces the metric will depend on the choice of the coordinate z.

Due to the fact that we have already defined a metric suitable for this setting we are now in the position to define a length and a distance.

Definition. Let $U \subset \mathbb{C}$ a hyperbolic domain and $\gamma : [a, b] \to \mathbb{C}$ a continuous path over U, with $\gamma(a) = z$ and $\gamma(b) = w$. Then, we define the length of γ with respect to the element of length $\rho_U(z)\,|dz|$ as the integral

$$\ell_U(\gamma) = \int_\gamma \rho_U(z)\,|dz|.$$

In a natural way, the notion of length can be used to define a distance by taking the shortest length of all paths between two points.

Definition. Let $U \subset \mathbb{C}$ be a hyperbolic domain and take $z, w \in U$. Consider $\gamma : [a, b] \to U$ a continuous path over U, with $\gamma(a) = z$ and $\gamma(b) = w$. Then, we define $d_U(z, w)$, the *hyperbolic distance between* z *and* w, as the smallest length among all the possible paths γ connecting z and w,

$$d_U(z, w) = \inf_{\gamma} \ell_U(\gamma) = \inf_{\gamma} \int_{\gamma} \rho_U(z)\,|dz|.$$

Bearing this in mind, the following statements will introduce what happens to the metric over a hyperbolic surface when we map it to another hyperbolic domain by a holomorphic function. Let U and V be two hyperbolic domains and consider a holomorphic map between them, $f : U \to V$. As said before, assuming we have a metric on U, $\rho_U(z)$, to get the density in V we need only to do a push-forward of f of the metric in U. Thus, for any $z \in U$,

$$\rho_V(f(z)) = f^*(\rho_U(z)) = \frac{\rho_U(z)}{|f'(z)|}. \tag{2.4}$$

As we are considering hyperbolic domains, the metric defined is automatically conformal invariant, so the following statement holds.

Theorem 2.7. *Let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental entire function and $U \subset \mathbb{C}$ a hyperbolic domain. Consider the conformal metric over U, $\rho_U(z)\,|dz|$. Then the metric is conformal invariant,*

$$\rho_U(z) = \rho_U(f(z))\,|f'(z)|$$

for every automorphism $f : U \to U$.

These last definitions will allow us to introduce the concept of *hyperbolic derivative*, which will play a major role in the proof of several results that will be introduced later on.

Definition 2.8. Let $U \subset \mathbb{C}$ and $V \subset \mathbb{C}$ be two hyperbolic domains and take $z \in U$. Let $f : U \to V$ be a holomorphic function. Then we denote the *hyperbolic derivative* with respect to the metrics on U and V as

$$||Df(z)||_V^U := |f'(z)|\,\frac{\rho_V(f(z))}{\rho_U(z)}. \tag{2.5}$$

If $f : U \to U$, then we write $||Df(z)||_U := ||Df(z)||_U^U$.

2.2.1 Hyperbolic geometry and holomorphic functions

Once we have set the basis of the main concepts of hyperbolic geometry, we are in the position to give some important tools that, not only will be useful to express several concepts in the setting within this framework, but also will allow us to prove those results more easily. Hence, throughout these pages we are going to state and give an intuitive idea of the meaning of these results.

In the first place, we are going to give a major theorem: Pick's Theorem. For the proof of this theorem we will need Schwarz's Lemma and to bear in mind the definitions of metric and hyperbolic distance given before. After that, and since we are trying to give a characterization of holomorphic functions between hyperbolic domains, we shall give bounds for the relative metric between two hyperbolic domains. Finally, we will study what happens to the metric when the hyperbolic distance tends to infinity.

Theorem 2.9 (Schwarz's Lemma). *Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function such that $f(0) = 0$. Then, $|f(z)| \leq |z|$, and $|f'(0)| \leq 1$. Moreover, if $|f(z)| = |z|$ or $|f'(0)| = 1$, then exists $\theta \in \mathbb{R} \pmod{2\pi}$ constant such that $f(z) = e^{i\theta}$.*

The idea behind this theorem is that if we have a function from the disc to itself that fixes 0, then either the orbit of $z \in \mathbb{D}$ under f^n is attracted by the fixed point, or the orbit turns around 0 preserving the modulus under f^n. This fact will be used in the proof of Pick's theorem, that will be stated and proved hereafter.

Theorem 2.10 (The Schwarz-Pick Lemma). *Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic function. Then, for all $z_1, z_2 \in \mathbb{D}$,*

$$d_{\mathbb{D}}(f(z_1), f(z_2)) \leq d_{\mathbb{D}}(z_1, z_2), \text{ or equivalently, } ||D\,f(z)||_{\mathbb{D}} := |f'(z)| \cdot \frac{\rho_{\mathbb{D}}(f(z))}{\rho_{\mathbb{D}}(z)} \leq 1, z \in \mathbb{D}. \quad (2.6)$$

In this case we say that f is a hyperbolic contraction. The equality in (2.6) holds if and only if f is a hyperbolic isometry, i.e.

$$d_{\mathbb{D}}(f(z_1), f(z_2)) = d_{\mathbb{D}}(z_1, z_2) \text{ or equivalently } ||D\,f(z)||_{\mathbb{D}} = 1,\ z \in \mathbb{D}.$$

Proof. Consider g and h two automorphisms of the disc. Notice that g and h are in fact isometries. Let z_1 and z_2 two points in $\mathbb{D}$, without loss of generality we can assume that $g(z_1) = 0$ and $h(f(z_1)) = 0$. Take $F = h \circ f \circ g^{-1}$ which is holomorphic (it is a composition of holomorphic maps).

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{f} & \mathbb{D} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ \mathbb{D} & \xrightarrow[F]{} & \mathbb{D} \end{array}$$

Moreover, it fixes zero, $F(0) = h(f(g^{-1}(0))) = h(f(z_1)) = 0$. Then, by Theorem 2.9 we have that $|F(z)| \leq |z|$ and $|F'(0)| \leq 1$. Combining this with the fact that we know the explicit value of the hyperbolic distance in the disc we have that,

$$d_{\mathbb{D}}(0, F(z)) = \frac{2}{1 - |F(z)|^2} \leq \frac{2}{1 - |z|^2} = d_{\mathbb{D}}(0, z). \quad (2.7)$$

Then,

$$\begin{aligned} d_{\mathbb{D}}(f(z_1), f(z_2)) &\overset{(1)}{=} d_{\mathbb{D}}(h(f(z_1)), h(f(z_2))) \overset{(2)}{=} d_{\mathbb{D}}(F(g(z_1)), F(g(z_2))) = \\ &= d_{\mathbb{D}}(0, F(g(z_2))) \overset{(3)}{\leq} d_{\mathbb{D}}(0, g(z_2)) \overset{(1)}{=} d_{\mathbb{D}}(z_1, z_2) \end{aligned}$$

Above, (1) follows from the fact that g and h are isometries, (2) follows from the functional equation $F = h \circ f \circ g^{-1}$ and (3) is a consequence of (2.7).

Now, using the chain rule on $Fg = hf$ yields $F'(g(z_1))\, g'(z_1) = h'(f(z_1))\, f'(z_1)$. This fact together the result in Theorem 2.7,

$$|F'(g(z_1))| = |F'(0)| = \frac{|h'(f(z_1))|}{|g'(z_1)|}\,|f'(z_1)| = \frac{\rho_{\mathbb{D}}(f(z_1))}{\rho_{\mathbb{D}}(h(f(z_1)))} \cdot \frac{\rho_{\mathbb{D}}(g(z_1))}{\rho_{\mathbb{D}}(z_1)}\,|f'(z_1)|$$
$$= \frac{\rho_{\mathbb{D}}(f(z_1))}{\rho_{\mathbb{D}}(0)} \cdot \frac{\rho_{\mathbb{D}}(0)}{\rho_{\mathbb{D}}(z_1)}\,|f'(z_1)| = \frac{\rho_{\mathbb{D}}(f(z_1))}{\rho_{\mathbb{D}}(z_1)} = ||D\, f(z_1)||_{\mathbb{D}} \leq 1,$$

as we wanted to see since z_1 is an arbitrary point of $\mathbb{D}$. In the case that we have an isometry the equality holds and conversely, by definition, if we have the equality then f is an isometry.

□

The idea now is to give a general version of Pick's theorem that will be valid for hyperbolic domains. Hence, the original Pick's theorem over $\mathbb{D}$ can be extended via the tools of hyperbolic geometry introduced earlier.

Theorem 2.11 (Pick's Theorem). *Let $f : U \to V$ be a holomorphic map and $U, V \subset \mathbb{C}$ two hyperbolic planar domains, then*

a) for all $z, w \in U$

$$d_V(f(z), f(v)) \leq d_U(z, w), \tag{2.8}$$

b) the equality holds if and only if f is a local isometry.

As said before, we are going to express most of the results in terms of hyperbolic geometry. For that it is interesting to give bounds of the metric in a hyperbolic domain $U \subset \mathbb{C}$.

Proposition 2.12. *Let $U \subset \mathbb{C}$ a hyperbolic domain, then*

$$\rho_U(z) \leq \frac{2}{dist(z, \partial U)}. \tag{2.9}$$

If U is simply-connected, then

$$\rho_U(z) \geq \frac{1}{2\, dist(z, \partial U)}. \tag{2.10}$$

Proof. For (2.9) take $z_0 \in U$ and $R = d(z_0, \partial U)$. Now consider the disc $D := D_R(z_0)$ of center z_0 and radius R. Then $D \subseteq U$, so

$$\rho_U(z_0) \leq \rho_D(z_0) = \frac{2\,R}{R^2 - |z_0 - z_0|^2} = \frac{2}{R} = \frac{2}{\mathrm{dist}(z_0, \partial U)}.$$

In the case that U is simply-connected, fix $z \in U$ and consider a conformal map $f : \mathbb{D} \to U$ such that zero goes to z, i.e. $f(0) = z$. Applying Koebe's 1/4 theorem (see for instance [CG93, Theorem 1.4])

$$D\left(f(0), \frac{|f'(0)|}{4}\right) \subset f(\mathbb{D}) \;\Rightarrow\; \mathrm{dist}(z, \partial U) > \frac{|f'(0)|}{4}.$$

Taking into account that $\rho_{\mathbb{D}}(z) = \frac{2}{1-|z|^2}$ and the fact that $\rho_{\mathbb{D}}(z) = \rho_U(f(z))|f'(z)|$,

$$4\,\rho_U(z)\mathrm{dist}(z, \partial U) \geq \rho_U(f(0))|f'(0)| = \rho_{\mathbb{D}}(0) = 2 \;\Rightarrow\; \rho_U \geq \frac{1}{2\,\mathrm{dist}(z, \partial U)}.$$

□

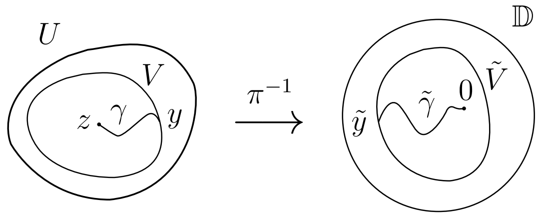
Observe that if V is compactly contained in U, which means that U and V do not share the boundary at any point (that is $\partial U \cap \partial V = \emptyset$), it follows from the bounds given above that $\rho_U(z) < +\infty$ for all $z \in V$. In particular this means that $d_U(z, U \setminus V) < \infty$ for $z \in V$. However, it will be very useful to have estimates on the measure of the relative density of V with respect to U, that is, which is the magnitude of $\rho_V^U(z)$ as we change z in V. In the following proposition we will give sharp bounds of $\rho_V^U(z)$ that will depend on the distance $d_U(z, U \setminus V)$.

Proposition 2.13. *Let $U, V \subset \mathbb{C}$ be two connected hyperbolic domains such that $V \subsetneq U$. Let $z \in V$ and suppose that $R := d_U(z, U \setminus V) < \infty$. Then we have the following estimates*

$$1 < \frac{2e^R}{(e^{2R}-1)\cdot \log \frac{e^R+1}{e^R-1}} \leq \rho_V^U \leq 1 + \frac{2}{e^R - 1}. \tag{2.11}$$

Proof. Since U is a hyperbolic domain there exists a universal covering $\pi : \mathbb{D} \to U$, and consider $\pi(0) = z$. Now, take $\tilde{V}$ the component of $\pi^{-1}(V)$ such that $0 \in \tilde{V}$. Furthermore, by Pick's Theorem as π is a universal covering we have that $d_{\mathbb{D}}(x, y) \geq d_U(\pi(x), \pi(y))$ for $x, y \in \mathbb{D}$. In particular consider $x \in \tilde{V}$ and $y = 0$, then $d_{\mathbb{D}}(x, 0) \geq d_U(\pi(x), z) = R$ with $z \in U$. Hence, $\tilde{V}$ contains the disc $D(0, R)$ (with respect to the metric in $\mathbb{D}$).

Seeing that we have that $R = d_U(z, U \setminus V)$, and this is the hyperbolic distance between the point z in V to the boundary ∂V in U(and this hyperbolic distance is nothing but the infimum of all lengths over paths between a point in V to ∂V with respect to the density in U). Then there must exist $y \in \partial V$ such that the infimum is attained, i.e. $d_U(z, y) = R$. Consider also the geodesic γ that joins both points.



Since geodesics of U lift to geodesics on the disc $\mathbb{D}$ we can consider the lift of γ via the lift π^{-1}. Considering now the metrics in $\mathbb{D}$, as we lift there exists a point $\tilde{y} = \pi^{-1}y$ so that there is a geodesic joining $\pi^{-1} = 0$ to $\tilde{y}$ so that the distance is R, i.e. $d_{\mathbb{D}}(0, \tilde{y}) = R$. We can consider without loss of generality that $\tilde{y} = \tilde{R}$, where $\tilde{R}$ is a positive real number, because we can do a composition with a rotation over the disc $\mathbb{D}$.

Keeping in mind that there is an equivalence between discs in Euclidean geometry and discs in hyperbolic geometric we have

$$D(0, \tilde{R}) \subset \tilde{V} \subset \mathbb{D} \setminus \{\tilde{R}\}$$

Next let's obtain upper and lower bounds for $\rho_V^U(z)$:

$$\begin{aligned} \rho_V^U(z) &= \frac{\rho_V(z)}{\rho_U(z)} = \frac{\rho_V(\pi(0))}{\rho_U(\pi(0))} \overset{(1)}{=} \frac{\rho_{\tilde{V}}(0)}{|\pi'(0)|} \frac{|\pi'(0)|}{\rho_{\mathbb{D}}(0)} \\ &= \frac{\rho_{\tilde{V}}(0)}{\rho_{\mathbb{D}}(0)} \overset{(2)}{\leq} \frac{\rho_{D(0,\tilde{R})}(0)}{\rho_{\mathbb{D}}(0)} = \frac{2}{2d_{\mathbb{D}}(0, D_{\tilde{R}}(0))} = \frac{1}{\tilde{R}}, \end{aligned} \tag{2.12}$$

$$\rho_V^U(z) = \frac{\rho_{\tilde{V}(0)}}{\rho_{\mathbb{D}}(0)} \overset{(3)}{\geq} \frac{\rho_{\mathbb{D}\setminus\{\tilde{R}\}}(0)}{\rho_{\mathbb{D}}(0)} \overset{(4)}{=} \frac{\rho_{\mathbb{D}^*}(\tilde{R})}{\rho_{\mathbb{D}}(\tilde{R})}, \tag{2.13}$$

where the fact that π is a covering map yields (1). Considering the inclusion maps $i : D(0, \tilde{R}) \to \tilde{V}$ and $\hat{i} : \tilde{V} \to \mathbb{D} \setminus \{\tilde{R}\}$ together with Pick's theorem we have that $\rho_{D(0,\tilde{R})}(0) \geq \rho_{\tilde{V}}(0)$ and $\rho_{\tilde{V}}(0) \geq \rho_{\mathbb{D}\setminus\{\tilde{R}\}}(0)$, which respectively yield (2) and (3). Furthermore, we have an isometry between the disc without one point and the punctured disc (the disc without zero), so $\rho_{\mathbb{D}\setminus\{\tilde{R}\}}(0) = \rho_{\mathbb{D}\setminus\{0\}}(\tilde{R})\frac{\rho_{\mathbb{D}}(0)}{\rho_{\mathbb{D}}(\tilde{R})}$, and that gives (4). Also, if the consider the contraction between the punctured disc and the disc, by Pick's theorem, we have that $\frac{\rho_{\mathbb{D}^*}(\tilde{R})}{\rho_{\mathbb{D}}(\tilde{R})} > 1$. That, together with (2.12) and (2.13) gives,

$$1 < \frac{\rho_{\mathbb{D}^*}(\tilde{R})}{\rho_{\mathbb{D}}(\tilde{R})} \leq \rho_V^U(z) \leq \frac{1}{\tilde{R}}. \tag{2.14}$$

To give the bound on (2.11) we will give the explicit expressions in (2.14). As we have seen in Proposition 2.6 we have that the hyperbolic metric in $\mathbb{D}$ is given by $\frac{2}{1-|z|^2}$, hence the distance in $\mathbb{D}$ of the center with respect to $z \in \mathbb{D}$ is given by the infimum of the length of any path γ going from zero to z. Then, consider $0 < r < 1$ and the continuous path $\gamma : [0, r] \to \mathbb{D}$ such that $\gamma(t) = t$. Computing the length

$$\begin{aligned}
\ell_{\mathbb{D}}(\gamma) = \int_\gamma \rho_{\mathbb{D}}(z)\,|dz| &= \int_\gamma \frac{2}{1-|z|^2}\,|dz| \\
&= \int_0^r \frac{2}{1-t^2}|\gamma'(t)|\,dt \\
&= \int_0^r \left(\frac{1}{1+t} + \frac{1}{1-t}\right) dt \\
&= \log\frac{1+r}{1-r}.
\end{aligned}$$

Thus, the hyperbolic distance will be $d_{\mathbb{D}}(z, 0) = \log\frac{1+|z|}{1-|z|}$, and as we already know that $d_D(0, \tilde{R}) = R$, we can find the expression of $\tilde{R}$ and, therefore, the explicit expression of the upper bound.

$$d_{\mathbb{D}}(0, \tilde{R}) = \log\frac{1+\tilde{R}}{1-\tilde{R}} = R \;\Rightarrow\; \tilde{R} = \frac{e^R - 1}{e^R + 1} \;\Rightarrow\; \frac{1}{\tilde{R}} = 1 + \frac{2}{e^R - 1}. \tag{2.15}$$

Equation (2.15) yields the upper bound so we just have to find the lower bound. Using the expression of the metric on the disc and the punctured disc and the expression of $\tilde{R}$ we shall get the lower bound.

$$\frac{\rho_{\mathbb{D}^*}(\tilde{R})}{\rho_{\mathbb{D}}(\tilde{R})} = \frac{1-\tilde{R}^2}{2\tilde{R}\cdot|\log\tilde{R}|} \overset{(1)}{=} \frac{2e^R}{(e^R+1)(e^R-1)}\cdot\frac{1}{|\log\tilde{R}|} \overset{(2)}{=} \frac{2e^R}{(e^{2R}-1)\log\frac{1}{\tilde{R}}} = \frac{2e^R}{(e^{2R}-1)\log\frac{e^R+1}{e^R-1}},$$

since we can apply the explicit expression of $\tilde{R}$ to have (1) and for (2) the fact that $|\log\tilde{R}| = |\log(e^R - 1) - \log(e^R + 1)| = \log(e^R + 1) - \log(e^R - 1) = \log\frac{1}{\tilde{R}}$.

□

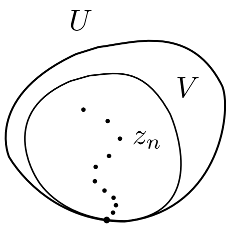
As a consequence of Proposition 2.13 it is interesting to study when the bounds are reached. The following corollary gives a result on what happens when $R = d_U(z, U \setminus V)$ goes to infinity, that is when U and V share a piece of the boundary.

Corollary 2.14. *Let $U \subset \mathbb{C}$ a hyperbolic domain and $V \subsetneq U$ an open subset. Consider $\{z_n\}_{n\geq 1}$ a sequence of points in V. Then*

$$d_U(z_n, U \setminus V) \to \infty \;\Leftrightarrow\; \rho_V^U(z_n) \searrow 1 \text{ as } n \to \infty.$$

It is clear that if $V \subsetneq U$, the only way in which we could have that $d_U(z_n, U \setminus V) \to \infty$, for a sequence $\{z_n\}_{n\geq 1}$ of points in V, is if V and U share a part of the boundary and z_n tends to $\partial U \cap \partial V$. As $\{z_n\}$ approaches those points in $\partial U \cap \partial V$ we would be measuring the hyperbolic distance in U of a point very close to boundary of U, which we know goes to infinity. Then, it makes no sense to consider the case in which $V \subsetneq U$ and $\partial U \cap \partial V = \emptyset$.

The idea behind this Corollary is that if $d_U(z_n, U \setminus V) \to \infty$ that means that z_n is approaching $\partial U \cap \partial V \neq \emptyset$. To be tending to $\partial U \cap \partial V$ is equivalent to saying that the measuring the distance with respect to U is basically the same as doing it with respect to V, which we can translate as $\rho_V^U(z_n) \searrow 1$.



Proof. First assume that $d_U(z_n, U \setminus V) \to \infty$, which in the setting of Proposition 2.13 means that $R \to \infty$. If we make R go to ∞ in (2.11),

$$1 < \rho_V^U(z_n) \leq 1 + \frac{2}{e^{2R} - 1} \xrightarrow{R\to\infty} 1 < \rho_V^U(z_n) \leq 1,$$

we have that $\rho_V^U(z_n)$ decreases to 1 as $n \to \infty$.

On the other hand, assuming $\rho_V^U(z_n) \searrow 1$ as $n \to \infty$ we have that the limit of the lower bound in (2.11) has to tend to 1. In fact, if $R \to \infty$ that holds.

$$\lim_{R\to\infty} \frac{2e^R}{(e^{2R} - 1) \cdot \log \frac{e^R+1}{e^R-1}} \overset{(1)}{=} \lim_{x\to\infty} \frac{2x}{(x^2 - 1) \cdot \log \frac{x+1}{x-1}} = \frac{\infty}{\infty} \overset{(2)}{=} \lim_{x\to\infty} \frac{2}{2x \log \frac{x+1}{x-1} - (x^2 - 1)\frac{-2}{(x^2-1)}} = 1,$$

doing the change of variable $x = e^R$ in (1) and applying L'Hôpital rule in (2).

□

Chapter 3

Constant limit functions and wandering domains

The intention of this chapter is to introduce two key theorems that give a characterization of the limit functions of wandering domains. The first theorem is a result given by Baker [Bak70] that is true for all transcendental entire functions. It says that the limit functions of a wandering domain are in $\overline{E} \cup \{\infty\}$. To be able to prove it, first it shall be convenient to give general results on transcendental dynamics. For that we will have to give a definition of the inverse branch of f^n, after that we will see that if the limit function over a Fatou component U is constant then that limit function is in $\overline{E} \cup \{\infty\}$ and, finally, we will close the argument by proving that the limit functions of wandering domains are constant.

As for the second main result of this chapter, recall first that a function is in class $\mathcal{B}$ if it is a transcendental entire function such that its set of singular values is bounded (for a formal definition see Definition 2.1). It is necessary to take this definition into account since it is one of the hypothesis to prove that a wandering domain of $f \in \mathcal{B}$ can not be escaping.

3.1 Constant limit functions

Let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental entire function and set $U \subset \mathcal{F}(f)$ a domain of the Fatou set. From a previous discussion we know that any sequence $\{f^{n_k}|_U\}_{k\geq 1}$ has a convergent subsequence. Then, the limit function for $\{f^{n_k}|_U\}_{k\geq 1}$ can be either infinity or a holomorphic function $g : U \to \mathbb{C}$. If the limit is either infinity or a constant function, then we say that the convergent subsequence has a *constant limit function.* The question that arises naturally is which are the possible values of the limit functions for a given $U \subset \mathcal{F}(f)$. That will be the aim of this section.

Lemma 3.1. *Let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental entire. Assume that the branches $z = G_n(w)$ of the inverse functions $f^{-n}(w)$ of $w = f^n(z)$ are well-defined in a regular domain D for all $n \geq 1$. Then, the family $\{G_n\}_{n\geq 1}$ is normal in D.*

Proof. Since we want to prove the normality of a family the goal is to check the hypothesis of Montel's theorem. We know that the repelling periodic points are dense in $\mathcal{J}(f)$ (see Proposition 2.4). So there must be two periodic points α_0 and β_0 of periods p and q respectively, such that

$p, q \geq 3$ and $p \neq q$. In fact, there are infinitely many repelling periodic points of any period. Then, by the definition of periodic point,

$$\begin{cases} f^p(\alpha) = \alpha \text{ and } f^k(\alpha) \neq \alpha, \ \forall k < p, \\ f^q(\beta) = \beta \text{ and } f^k(\beta) \neq \beta, \ \forall k < q. \end{cases}$$

Thus, we have two cycles

$$\begin{cases} A = (\alpha_0, \alpha_1, \ldots, \alpha_{p-1}) \text{ with } p \text{ points}, \\ B = (\beta_0, \beta_1, \ldots, \beta_{q-1}) \text{ with } q \text{ points}. \end{cases}$$

Now let's consider a domain D where we can define a certain branch $z = G_n(w)$ of the inverse functions $f^{-n}(w)$ of $w = f^n(z)$. Then it is easy to prove that in $D \setminus A$ we have that $G_n \neq \alpha_i$ for every $i \in \{0, \ldots, p-1\}$. Assume that this is not true, then for $z \in D \setminus A$, $G_n(z) = \alpha_i$ for some $i \in \{0, \ldots, p-1\}$. By definition of the inverse branch, this last statement is equivalent to $f^n(\alpha_i) = z$, and since $f^n(\alpha_i) = \alpha_j \in A$ it implies $z \in A$ which contradicts the fact that $z \in D \setminus A$. The same argument is valid to show that in $D \setminus B$ we have that $G_n \neq \beta_j$ for every $j \in \{0, \ldots, q-1\}$.

Furthermore, it is clear that $A \cap B = \emptyset$ since A and B are two cycles of periodic points of different periods. Now, consider a point $z \in D$, then we have three possibilities: $z \notin A \cup B$, $z \notin A$ or $z \notin B$.

Then, if $z \notin A \cup B$ then there exists a U open neighbourhood of z such that $U \subset D$ and $U \cap (A \cup B) = \emptyset$. By the arguments above, $\{G_n|_U\}_{n \geq 1}$ misses all the points in A and B so, by Montel's theorem, $\{G_n\}_{n \geq 1}$ is normal at z. For the cases $z \notin A$ and $z \notin B$, by an analogous argument it is clear that $\{G_n\}_{n \geq 1}$ is normal at z. Thus, for all $z \in \mathbb{D}$ the family $\{G_n\}_{n \geq 1}$ is normal at z, so $\{G_n\}_{n \geq 1}$ is normal in D as we wanted to see.

□

Remember that $S(f)$ denotes the set of singular values of f, that is the singularities of the inverse function f^{-1}. Consider now the *post singular set* as the set of the iterates of the points of $S(f)$, $E(f) = \{f^n(s), \, s \in S(f)\}$. The following lemma gives a characterization of the points of the post singular set.

Lemma 3.2. *Let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental entire function. A point $z \in E$ is precisely a finite singularity of some inverse of an iterate of f.*

Proof. For this proof we will consider two cases: when the singularity is a critical value and when it is an asymptotic value. Take a critical value β of some inverse function $f^{-n}(z)$. Then, there exists α critical point such that

$$\begin{cases} \beta = f^n(\alpha), \\ (f^n)'(\alpha) = 0. \end{cases} \tag{3.1}$$

But by the chain rule $(f^n)'(\alpha) = (f \circ \overset{n)}{\cdots} \circ f)'(\alpha) = \prod_{i=0}^{n-1} f'(f^i(\alpha))$. So, if $(f^n)'(\alpha) = 0$ that means that there exists $i \in \{1, \ldots, n\}$ such that $f'(f^i(\alpha)) = 0$. Let j the smallest value such that $f'(f^i(\alpha)) = 0$, i.e. $j = \min_{i \in \{0, \ldots, n\}} \{f'(f^i(\alpha)) = 0\}$. Now, consider $s = f(f^j(\alpha)) = f^{j+1}(\alpha)$. Taking into account (3.1), we have that s is a critical value of the inverse of f with $f^j(\alpha)$ as its critical point,

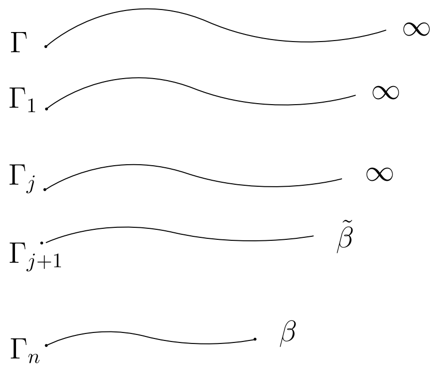
$$\begin{cases} s = f(f^j(\alpha)), \\ f'(f^j(\alpha)) = 0. \end{cases}$$

Therefore, $s \in S(f)$ and

$$\beta = f^n(\alpha) = f^{n-j-1+j+1}(\alpha) = f^{n-j-1}(f^{j+1}(\alpha)) = f^{n-j-1}(s),$$

so β is an iterate of a point in the singular set. Thus, $\beta \in E(f)$.

Consider now that β is an asymptotic value of $f^{-n}(z)$. Intuitively β "comes from infinity", i.e. there exists a curve $\Gamma \subset \mathbb{C}$ tending to infinity such that $f^n(z) \to \beta$ as $z \to \infty$ along Γ. Denote $\Gamma_k = f^k(\Gamma)$. Since Γ_n tends to β and Γ is a curve tending to infinity, at some iterate Γ_k we will stop having an unbounded curve. Consider j the greatest iterate such that $f^j(\Gamma)$ is unbounded. Notice that if $n = j+1$, then it means that Γ_k is unbounded for $k = 0, \ldots, n-1$. Hence, $\beta \in S(f)$ by definition of asymptotic value since Γ_{n-1} is a curve running to infinity and $f(\Gamma_{n-1})$ tends to β as z goes to infinity along Γ_{n-1}. Otherwise, we have that Γ_{j+1} is bounded and tends to $\tilde{\beta}$ and, following the same argument as before, $\tilde{\beta} \in S(f)$. Now, by continuity $\tilde{\beta}$ has to be a root of $f^{n-j-1}(z) = \beta$. Hence, $\beta = f^{n-j-1}(\tilde{\beta})$ which means that $\beta \in E(f)$. □



This two last lemmas will allow us to prove a general result that characterizes the limit functions of a component of a Fatou set for a transcendental entire function.

Theorem 3.3. *Let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental entire function and consider the post singular set E of f. Any constant limit of a sequence $f^n(z)$ in a component U of $\mathcal{F}(f)$ belongs to $L = \overline{E} \cup \{\infty\}$.*

Proof. Suppose this is not true and take $\alpha \notin L$. Since L is a closed set there exists $\delta > 0$ sufficiently small and $z_0 \notin L$ so that $\alpha \in D = D(z_0, \delta)$ and $D \cap E = \emptyset$. Consider U a component of the Fatou set and let $\{n_k\}_{k \geq 1}$ be a sequence such that $\{f^{n_k}|U\}_{k \geq 1}$ converges in compact subsets of U to α. So we have a constant limit $\alpha \notin L$ that is the limit of a sequence in a component of $\mathcal{F}(f)$.

Let $\xi \in U$ and $\alpha_k := f^{n_k}(\xi)$ the iterates of ξ, without loss of generality we may assume that $\alpha_k \in D \; \forall k \geq 1$. More precisely,

$$\begin{cases} \alpha_k = f^{n_k}(z) \to \alpha, \text{ as } l \to \infty, & \text{in } U, \\ \alpha \in D. \end{cases}$$

By construction we can consider the well-defined inverse branch of $z = g_{n_k}(w)$ such that $g^{n_k}(\alpha_k) = \xi$, since $D \cap L = \emptyset$ by hypothesis. Hence we can extend g_{n_k} to a regular function in D. Due to the fact that we are under the hypothesis of Lemma 3.1 we have that the family $\{g_{n_k}\}_{k\geq 1}$ is normal. By definition of normality there exists a subsequence of iterates such that

$$g^{n_{k_l}} \to \Phi \text{ uniformly},$$

where Φ is a holomorphic function. Recalling that $\alpha_k = f^{n_k}(\xi)$ over U and that $g^{n_k}(\alpha_k) = \xi$ we have that $\Phi(\alpha) = \xi$.

Consider another point $\xi_1 \neq \xi$ in U and its iterates $\alpha_k^1 = f^{n_k}(\xi_1)$. Take $K \subset U$ a compact set such that ξ, $\xi_1 \in K$. Since we have considered a compact set we can ensure that for k large enough $f^{n_k}(K) \subset D$. Now, in an analogous way by construction we can consider the well defined branch $g^{n_k}(\overline{w})$ of the inverse such that $g(\alpha_k^1) = \xi^1$. Again we can extend g^{n_k} to a regular function over K and by Lemma 3.1 the family $\{g_{n_k}\}_{k\geq 1}$ is normal. Hence,

$$\begin{cases} g^{n_{k_l}} \to \Phi \text{ uniformly} \\ \alpha_k^1 \to \alpha \\ g^{n_k}(\alpha_k^1) = \xi^1 \end{cases} \Rightarrow \ \Phi(\alpha) = \xi^1.$$

So, we would have that $\Phi(\alpha) = \xi = \xi^1$ by the uniqueness of the limit. However this contradicts the fact that we assumed $\xi \neq \xi^1$. Hence, $\alpha \in L$. □

The next step is to prove that the limit functions of a wandering domain are constant to be able to apply the last result to prove that the constant limit functions of a wandering domain W are in $\overline{E} \cup \{\infty\}$.

Theorem 3.4. *Let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental entire function. Let U be a wandering domain. Any limit function $\{f^{n_k}|_U\}_{k\geq 1}$ is constant.*

Proof. Denote $U_0 = U$ and $U_n = f^n(U)$. Since $\{f^n|_U\}_{n\geq 0}$ is a normal family then there exists a subsequence of iterates $\{n_k\}_{k\geq 0}$ such that either $f^{n_k} \to \infty$ or $f^{n_k} \to g$ a holomorphic function over U. If $f^{n_k} \to \infty$ we already have that the limit function is constant. Hence, assume $f^{n_k} \to g$ a non-constant holomorphic function, so $g' \not\equiv 0$ and there exists $z_0 \in U$ such that $g'(z_0) \neq 0$. Then, by the implicit function theorem we can consider a compact $K \subset U$ containing z_0 such that g is injective. Since $z_0 \in K$ and $g'(z_0) \neq 0$ by the inyectivity of g we have that $g(z_0) \notin g(\partial K)$. Therefore,

$$\varepsilon := \inf_{z\in\partial K} |g(z) - g(z_0)| \ \Rightarrow \ \varepsilon \leq |g(z) - g(z_0)|, \ \forall z \in \partial K. \tag{3.2}$$

Also, we have that $f^{n_k}|_U \to g$, so over compacts we have that the convergence is uniform. Hence, there exists $N \in \mathbb{N}$ such that

$$|f^{n_k}(z) - g(z)| < \epsilon, \ \forall k > N, \forall z \in K. \tag{3.3}$$

Denote $F_k(z) := f^{n_k}(z) - g(z_0)$ and $G(z) := g(z) - g(z_0)$. Then, $\forall z \in \partial K$ and $\forall k > N$

$$|F_k(z)-G(z)| = |f^{n_k}(z)-g(z_0)-g(z)+g(z_0)| = |f^{n_k}(z)-g(z)| \overset{(3.3)}{<} \varepsilon \overset{(3.2)}{\leq} |g(z)-g(z_0)| = |G(z)|.$$

Then, by Rouché's theorem we have that F_k and G have the same number of zeros over K for all $k > N$. We know that G has at least one zero in K since $G(z_0) = 0$ so F_k has at least one zero. Then, for each $k > N$, $\exists z_k \in K$ such that $F_k(z_k) = 0$. But, $F_k(z_k) = f^{n_k}(z_k) - g(z_0) = 0$.

Therefore $f^{n_k}(z_k) = g(z_0)\ \forall k > N$. Hence, since $z_0 \in K \subset U$ and $U_n = f^n(U)$ we have that $g(z_0) \in U_n \cap U_m$ for $n \neq m > N$. As a consequence U can not be a wandering domain, which contradicts the statement of the theorem and gives that the limit has to be constant. □

Taking all the results showed up to this point and joining them in an right way, the proof of the main result of this section comes easily.

Corollary 3.5. *Let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental entire function. Let U a wandering domain. Then any limit function of a sequence $\{f^n|_U\}_{n\geq 0}$ belongs to $\overline{E} \cup \{\infty\}$.*

Proof. By Theorem 3.4 we have that the limit of a sequence $\{f^n\}_{n\geq 0}$ in a wandering domain is constant. Now, since the limit is constant Theorem 3.3 yields that the limit belongs to $\overline{E} \cup \{\infty\}$. □

This corollary follows from Theorem 3.4, which was presented in 1992 by A. Eremenko and M. Yu Lyubich [EL92]. Subsequently, in an article by Bergweiler-Haruta-Kriete-Meier-Terglane in 1993 [BHK+93] they gave a refinement of that theorem that we will not prove in this document. Remember that the *derived set* of A, denoted by A', is the set of finite limit points of A, i.e. $a' \in A'$ if and only if there exists a sequence $\{a_n\}_{n\geq 1}$ such that $a_k \neq a'$ and $a_k \to a'$ as $k \to \infty$.

Theorem 3.6. *Let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental entire function and $U \subset \mathbb{C}$ be a wandering domain of the Fatou set. Consider the derived set of the post singular set E'. Any constant limit function of a sequence $\{f^n|_U\}_{n\geq 0}$ belongs to $E' \cup \{\infty\}$.*

3.2 Constant limit functions and wandering domains in Eremenko-Lyubich class

Up to this point we showed that the constant limit functions of a wandering domain are in $\overline{E} \cup \{\infty\}$. Actually we know a stronger result that says that the limit functions are in $E' \cup \{\infty\}$. The goal of this section is to show that if $f \in \mathcal{B}$, i.e. f is a transcendental entire function such that $S(f)$ is bounded, and U is a wandering domain then it can not be escaping, that is $f^n|_U \not\to \infty$. In other words, wandering domains can not tend uniformly to infinity (notice that this does not imply that there are no oscillating wandering domains because there may be a subsequence $\{n_k\}_{k\geq 1}$ such that $\{f^{n_k}|_U\} \to \infty$). To be able to prove that, it is necessary to state more general results on transcendental dynamics. These concepts will help setting the framework since they characterize the behaviour of transcendental functions over components of the Fatou set. The following two proposition do not need the function to be in class $\mathcal{B}$.

Proposition 3.7. *Let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental entire function, and let $U \subset \mathbb{C}$ be a multi-connected component of $\mathcal{F}(f)$. Then,*

a) $f^n(z) \to \infty$ uniformly on compact subsets of U.

b) For every Jordan curve γ non-homotopic to a point in U the winding number is zero, $ind_0 f^n(\gamma) \neq 0$, for some $n \in \mathbb{N}$ sufficiently large.

Proof. For part a) consider a Jordan curve γ in U, with interior domain D. As U is multi-connected we can take γ such that U contains points of the Julia set. To arrive to contradiction let's assume that $f^n \not\to \infty$ uniformly on compacts in U. Then there exists a subsequence $\{n_k\}_{k\geq 1}$

such that $|f^{n_k}| \leq M$ on γ. Since $|f^{n_k}| \leq M$ on $\gamma := \partial D$, by the maximum principle it also holds in its interior, so $|f^{n_k}| \leq M$ and $|(f^{n_k})'| \leq M|$ in D. However, D contains points in the Julia set and since repelling periodic points are dense in $\mathcal{J}(f)$, D should also contain repelling periodic points. Of course $|(f^{n_k})'(\xi)|$ tends to infinity if ξ is a repelling periodic point. This contradicts the fact that $|(f^{n_k})'| \leq M$ in D. Hence, $f^n \to \infty$ uniformly on compacts in U.

To prove b) assume by contradiction that for a Jordan curve γ non-homotopic to a point in a multi-connected component of $\mathcal{F}(f)$ there exists a subsequence $\{n_k\}_{k\geq 1}$ such that $\operatorname{ind}_0 f^{n_k}(\gamma) = 0$ for all $k \geq 1$. Let D be the interior of γ, then $f^{n_k}(z) \neq 0$ for $z \in D$ (if this were not true then $f^{n_k}(z) = 0$ for some $z \in \mathbb{D}$ and that would mean that $0 \in f^{n_k}(\gamma)$ which contradicts the fact that $\operatorname{ind}_0 f^{n_k}(\gamma) = 0$).

As in the case of a), we know that there are points of the Julia set in D, so by the same argument as before there exists α_0 a repelling periodic point of period p, $p \in \mathbb{N}$. Consider $A = (\alpha_0, \alpha_1, \ldots, \alpha_{p-1})$ the orbit of the periodic point α_0 and set $M > \max_{i=0,\ldots,p-1} |\alpha_i|$. Then all the points in the orbit A are inside the ball $B(0, M)$. On the other hand, by a) we have that $f^{n_k}(\gamma) \to \infty$ uniformly. Hence, eventually for some k we have that $f^{n_k}(\gamma)$ is outside the ball $B(0, M)$. Bearing in mind that $\operatorname{ind}_0 f^{n_k}(\gamma) = 0$, the only situation possible is the one in Figure 3.1.

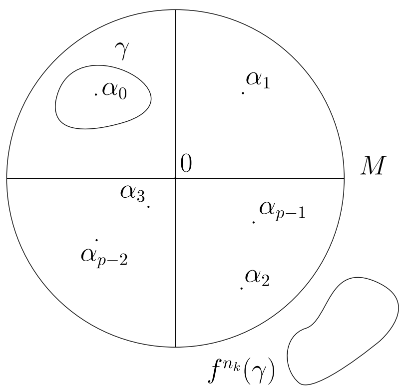


Figure 3.1: The situation if $\operatorname{ind}_0 f^{n_k}(\gamma) = 0$.

Since $f^{n_k}(z) \neq 0$ for $z \in D$ we can apply the minimum modulus principle, which yields

$$\min_{z\in\gamma} |f^{n_k}(z)| = \min_{z\in D} |f^{n_k}(z)|.$$

In particular for $z = \alpha_0 \in D$ we have that $f^{n_k}(\alpha_0) = \alpha_i$ for some $i \in \{0, \ldots, p-1\}$. However, by the minimum modulus principle there exists $w \in \gamma$ such that $f^{n_k}(w)$ attains the minimum modulus of the function. Then we arrive to contradiction,

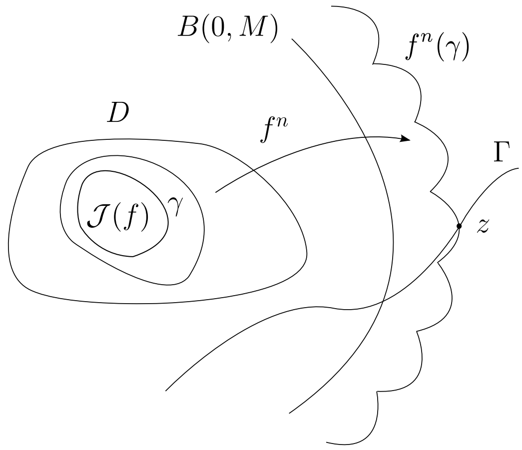
$$M \overset{(1)}{<} |f^{n_k}(w)| \overset{(2)}{\leq} |f^{n_k}(\alpha_0)| = |\alpha_i| \overset{(3)}{<} M.$$

The fact that $f^{n_k}(\gamma)$ is outside the ball $B(0, M)$ gives (1), (2) is a direct implication of the minimum modulus principle and (3) comes from the definition of M. The statement above contradicts the fact that we are under the hypothesis to apply the minimum principle, that is $f^{n_k}(z) \neq 0$ for $z \in D$ which is equivalent to $\operatorname{ind}_0 f^{n_k}(\gamma) = 0$ for all $k \geq 1$.

□

Proposition 3.8. *Let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental entire function bounded on a curve Γ going to infinity. Then all the components of $\mathcal{F}(f)$ are simply-connected.*

Proof. Suppose it is not true, then exists D multi-connected component of the Fatou set. Take γ a Jordan curve in D such that its interior contains points of the Julia set, i.e. it is non-homotopic to a point in D. Since Γ is a curve going to infinity there exists $M > 0$ sufficiently large such that $\partial B(0, M) \cap \Gamma \neq \emptyset$.



Since we are under the hypothesis of Proposition 3.7, then there exists $n_0 \in \mathbb{N}$ such that $\text{ind}_0\, f^n(\gamma) \neq 0$, so we have that $B(0, M) \subset \text{int} f^n(\gamma)$, $\forall n \geq n_0$. So, since Γ goes to infinity there exists $z \in f^n(\gamma) \cap \Gamma$. However, $f|_\Gamma$ is bounded, but by Proposition 3.7 we know that $f^m(z) \to \infty$ uniformly over γ. Hence, we arrive to a contradiction which yields that all the components of $\mathcal{F}(f)$ are simply-connected.

□

3.2.1 Logarithmic coordinates for Eremenko-Lyubich functions

For this part of the section we will assume that f is in the Eremenko-Lyubich class. Remember that $f \in \mathcal{B}$ if it is a transcendental entire function whose set of singular values $S(f)$ is bounded. We want to prove that wandering domains over functions in class $\mathcal{B}$ do not escape to infinity. This is a corollary of a stronger statement that says that for $f \in \mathcal{B}$ the points in the Fatou set do not tend to infinity after iteration. To be able to show this theorem we will use logarithmic coordinates, a tool introduced by Eremenko and Lyubich to study functions in class $\mathcal{B}$. The following theorem describes the geometry of f. For the proof of this result we refer to [DT86].

Theorem 3.9. *Let $f \in \mathcal{B}$ and consider $R > 0$ such that $S(f) \subset D(0, R)$. Let $A := \mathbb{C} \setminus \overline{D(0, R)}$ and $G := f^{-1}(A)$ (over all preimages). The following statement holds,*

a) Any connected component V of $G = f^{-1}A$ is a topological disc whose closure contains ∞.

b) $f : V \to A$ is a universal covering.

This means that V is a simply-connected component of G bounded by a non-closed analytic curve with both ends tending to infinity such that $f : V \subset G \to A$ is a universal covering, as seen in Figure 3.2. These domains are called *exponential tracts* .

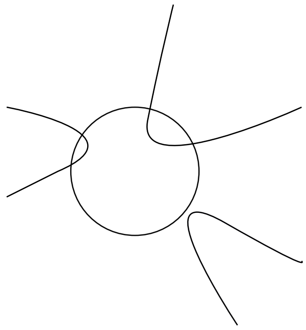


Figure 3.2: Exponential tracts

Notice that the boundary of any connected component of G is given by an unbounded curve whose image by f is bounded. Indeed if $\Gamma = \partial V$, then $|f(\Gamma)| = R$. This fact leads to the following result,

Proposition 3.10. *If $f \in \mathcal{B}$ transcendental then all the components of the Fatou set of f are simply-connected.*

Proof. If $f \in \mathcal{B}$ and $\Gamma = \partial V$ for some connected component of $f^{-1}(A)$ (following the notation in Theorem 3.9 where $A = \mathbb{C} \setminus D(0, R)$ and $S(f) \subset D(0, R)$) then $f|_\Gamma$ is bounded. Thus, by Proposition 3.8 we have that all the components of $\mathcal{F}(f)$ are simply-connected. $\square$

Next, we can set the frame to prove the main result. We choose R sufficiently large such that $f(0) \in D(0, R)$. In particular we have that $f(0) \notin A$ and $0 \notin G$. Since G is composed by exponential tracts, denote U as the logarithm of G, $U = \ln G$, and let W a component of U. Thus, $\exp : W \subset U \to G$. Also, define the half-plane $H := \ln A = \{\xi, \text{Re } \xi > \ln R\}$. Consequently, we have a commutative diagram where $F : U \to H$ is the lift of f. In other words, the functional equation $f \circ \exp = \exp \circ F$ holds.

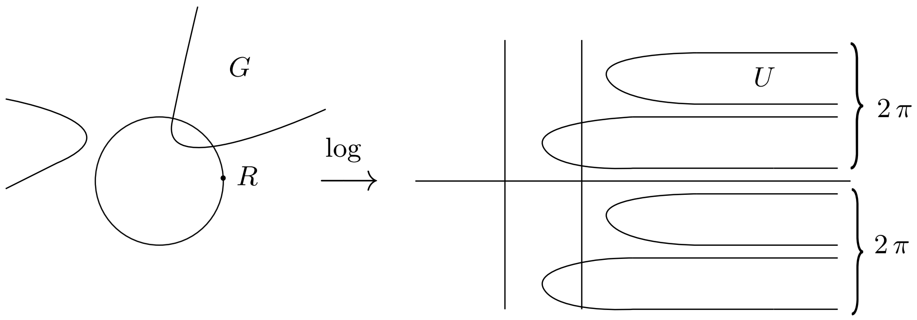
$$\begin{array}{ccc} U & \xrightarrow{F} & H \\ {\scriptstyle \exp}\downarrow & & \downarrow{\scriptstyle \exp} \\ G & \xrightarrow[f]{} & A \end{array}$$

Clearly, the action of F resembles the action of f through the exponential map since F is the lift of f via logarithm coordinates. The advantage of considering the lift is that some explicit computations are easier to do upstairs. In particular we will show that $F|_U$ is expansive.

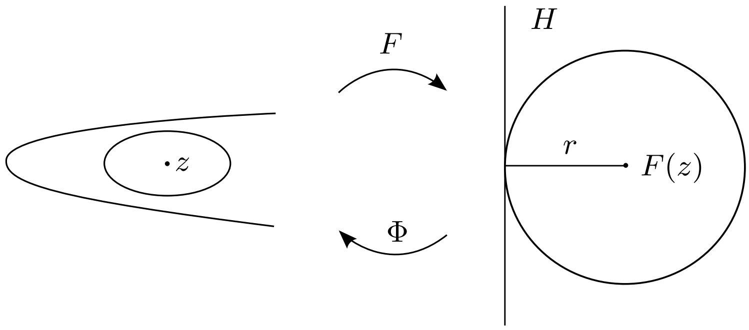
Lemma 3.11. *Let F the lift defined before, then $|F'(z)| \geq \frac{1}{4\pi}(\text{Re } F(z) - \ln R)$.*

Proof. Consider W a connected component of U. As described before, we know that W is bounded by a curve such that both ends tend to infinity. Moreover, we know that W can not contain any vertical segment of length greater of 2π.

Let z be a point in W and consider its image $F(z) \in H$. Denote the inverse of F by Φ so that $\Phi(F(z)) = z$ (notice that a point $w \in H$ has infinitely many preimages, one every 2π, and we choose the preimage that is in the track where W is). Since $F(z) \in H$, we can construct a disc



completely contained in the half-plane H by taking $F(z)$ as the center and $r = |\ln R - \text{Re } F(z)|$ as the radius, that is $D = D(F(z), r)$.



Now, by the Koebe's Quarter Theorem (see for instance [CG93, Theorem 1.4]) we have that

$$D\left(\Phi(F(z)), \frac{|\Phi'(F(z))|\, r}{4}\right) \subset \Phi(D).$$

As stated before, W can't contain any vertical segments of length 2π and $\Phi(D) \subset W$ so the vertical distance between z and the boundary of W is smaller that π. Therefore

$$\frac{|\Phi'(F(z))|\, r}{4} = \frac{|\Phi'(F(z))|\, |\ln R - \text{Re } F(z)|}{4} \leq \pi. \tag{3.4}$$

Applying the chain rule, $(\Phi \circ F)'(z) = \Phi'(F(z))F'(z)$. Then since $\Phi = F^{-1}$ we have,

$$\Phi'(F(z)) = \frac{(\Phi \circ F)'(z)}{F'(z)} = \frac{1}{F'(z)},$$

Substituting in (3.4) we have

$$\frac{|\Phi'(F(z))|\, |\ln R - \text{Re } F(z)|}{4} = \frac{|\ln R - \text{Re } F(z)|}{4\,|F'(z)|} \leq \pi \iff \frac{|\ln R - \text{Re } F(z)|}{4\pi} \leq |F'(z)|,$$

as we wanted to see. □

Now we have all the tools needed to prove the main theorem of this section.

Theorem 3.12. *Let $f \in \mathcal{B}$ and consider $R > 0$ such that $S(f) \subset D(0,R)$. Let $A := \mathbb{C} \setminus \overline{D(0,R)}$ and $G := f^{-1}(A)$. Then if z is a point in the Fatou set of f it does not tend to infinity after iteration, i.e. $\{f^n(z)\}_{n\geq 1} \not\to \infty$.*

Proof. Suppose it's not true. Then there exists $z_0 \in \mathcal{F}(f)$ such that $f^n(z_0) \to \infty$ as $n \to \infty$. Consider $r > 0$ and the disc $B_0 = D(z_0, r) \subset \mathcal{F}(f)$. Therefore, we can construct the sequence $B_n = f^n(B_0)$, such that $z_n = f^n(z_0) \in B_n$. Since we assumed B_n in the Fatou set and that $z_n \to \infty$ we have that $B_n \to \infty$ uniformly. Because of that and taking into account the definition of A, eventually $B_n \subset G$ for some n. Assume without loss of generality that $B_n \subset G$, for all $n \geq 0$. Now we will consider the equivalent sequence in the logarithmic frame. Consider C_0 a connected component of $\ln B_0$ and denote $C_n = F^n(C_0)$. Then, by the commutativity of the diagram defined by f and F,

$$\exp C_n = \exp(F^n(C_0)) = f^n(\exp C_0) = f^n(B_0) = B_n.$$

With that in mind, $C_n \subset \ln B_n \subset G = U$ by definition. As B_n goes to infinity uniformly we claim that Re C_n tends to infinity as well. To prove the claim assume it does not, then Re $C_n \leq M$ for some M. Hence, if we apply the exponential map to C_n then we have that $|B_n| = |\exp(C_n)| = \exp(\text{Re } C_n) \leq M$, but this contradicts the fact that B_n tends to infinity.

Now let's construct a sequence of points $\{\zeta_m\}$ such that $\zeta_0 \in C_0$ and $\zeta_m := F^m\zeta_0 \in C_m$. For each of the points ζ_i consider the biggest disk centred in ζ_i completely contained in C_i, whose radii will be $d_m = \sup_{i\in\mathbb{N}}\{r_i,\, D(\zeta_i, r_i) \subset C_m\}$. Applying Koebe's quarter theorem over the function $F : D(\zeta_m, d_m) \to \mathbb{C}$ we have that

$$D\left(F(\zeta_m), \frac{|F'(\zeta_m)|\, d_m}{4}\right) \subset F(D(\zeta_m, d_m))$$

Then it follows that

$$d_{m+1} \geq \frac{d_m\, |F'(\zeta_m)|}{4},$$

by definition of d_{m+1}. Using Lemma 3.11 and the fact that Re $F^m C_0 \to \infty$ we get

$$d_{m+1} \geq \frac{d_m\, |F'(\zeta_m)|}{4} \geq \frac{d_m\, |\text{Re } F(\zeta_m) - \ln \zeta_m|}{4\pi} = \frac{d_m\, |\text{Re } F^{m+1}(\zeta_0) - \ln \zeta_m|}{4\pi} \to \infty.$$

Therefore, $d_{m+1} \to \infty$ and that means that C_{m+1} contains disks such that their radiis tend to infinity. However, we know that C_m is in the image via the logarithmic change of variable of G, so C_m is in a component of $\ln G$ that can not have a vertical segment greater than 2π, which is a contradiction. Hence, is doesn't exist $z_0 \in \mathcal{F}(f)$ such that $f^m(z_0) \to \infty$ as $m \to \infty$, as we wanted to show. □

The main theorem of this section is now a corollary of the previous result.

Corollary 3.13. *Let f be a function in $\mathcal{B}$ and consider W a wandering domain of f. Then $\{f^n|_W(z)\}_{n\geq 1} \not\to \infty$.*

Proof. It follows from Theorem 3.12 since $W \subset \mathcal{F}(f)$. □

Chapter 4

Absence of wandering domains for certain functions in the Eremenko-Lyubich class

4.1 Introduction

In 1985 D. Sullivan [Sul85] showed that rational functions of degree greater than one have no wandering domains, or equivalently, any Fatou component is eventually periodic. One of the key points used in the proof of this remarkable paper is the fact that rational functions have a finite number of singularities. This suggested that this result could be generalized to the *Speiser class* $\mathcal{S}$. Indeed, a short time after Sullivan's proof, Eremenko and Lyubich [EL92] and Golberg and Keen [GK86] showed, independently, that if $f \in \mathcal{S}$, then f has no wandering domains.

However, in the case of the Eremenko-Lyubich class the question on the existence of wandering domains remained open until very recently, when C. Bishop [Bis15] gave an example of a function in class $\mathcal{B}$ that has wandering domains. In the meantime there were many attempts to show the non existence of wandering domains for certain classes of transcendental functions. In all these cases the idea was to follow Sullivan's approach on quasi-conformal surgery. However, in 2012 Mihaljević-Rempe [MBRG13] used different arguments to study the existence of wandering domains. They considered functions in class $\mathcal{B}$ and assuming some dynamics and distribution of the singular set and using hyperbolic geometry they developed new tools. In that same paper, Mihaljević-Rempe focus on special functions of the Eremenko-Lyubich class, the ones that are real and whose set of singular values are real. Adding more ingredients, they are able to show the non existence of wandering domains for functions of this type. The study of these functions is the main interest of this paper. To be able to state the main theorem of this paper first we need to define all the objects that play an important role to establish the framework.

Definition. The set of all transcendental entire functions that are real, i.e. functions such that $f(\mathbb{R}) \subset \mathbb{R}$, such that $S(f)$ is bounded is denoted by $\mathcal{B}_{\text{real}}$. Moreover, the set of functions in $\mathcal{B}_{\text{real}}$ such that $S(f) \subset \mathbb{R}$ is called $\mathcal{B}^*_{\text{real}}$. In particular, notice that if $f \in \mathcal{B}^*_{\text{real}}$ then $E(f) \subset \mathbb{R}$, where $E(f)$ is, as usual, the post singular set of f.

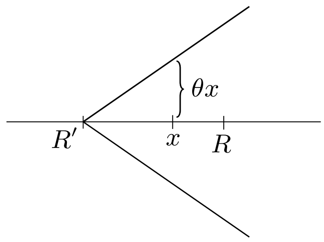
For the proof of the theorem it is necessary to impose one more condition. Functions in $\mathcal{B}^*_{\text{real}}$ have no wandering domains if they satisfy a certain geometrical condition, called the *sector condition.*

Definition 4.1. Let $f \in \mathcal{B}_{\text{real}}$, then $\Sigma_0(f)$ denotes the set of signs, $\{+,-\}$, that indicates if f tends in the positive or the negative direction to $\pm\infty$, i.e.

$$\Sigma_0(f) := \{\sigma \in \{+,-\}\,,\ \lim_{x\to+\infty} |f(\sigma x)| = +\infty\}.$$

Take $\sigma \in \Sigma_0(f)$, then f satisfies the *sector condition at* $\sigma\infty$ if for all $R > 0$, there exist R', $\theta > 0$ such that

$$|f(\sigma x + iy)| > R, \quad \text{for } x > R' \text{ and } |y| < \theta x.$$



Since we are studying the set of singular values we are interested in controlling what happens to those points. Also, in this section we are studying the case $f \in \mathcal{B}_{\text{real}}$. Hence, let us define the set of signs of the real singular values whose limit functions tend to $\pm\infty$.

$$\Sigma(f) := \{\sigma \in \{+,-\}, \exists s \in S(f) \cap \mathbb{R},\, f^{n_j}(s) \underset{j\to\infty}{\longrightarrow} \sigma\infty\} \subset \Sigma_0(f).$$

Definition. Let $f \in \mathcal{B}_{\text{real}}$, then f satisfies the *(real) sector condition* if f satisfies the sector condition at σx for all $\sigma \in \Sigma_0$.

Remark 4.2. It is important to keep into account that this definition is given for a function $f \in \mathcal{B}_{\text{real}}$, it does not require the set of singular values to be real.

Now we are in the appropriate setting to introduce the main theorem, which we will prove at the end of this chapter.

Theorem 4.3. *Let $f \in \mathcal{B}^*_{real}$ that satisfies the sector condition. Then f has no wandering domains.*

4.2 The hyperbolic lemma and auxiliary results

To begin with, the proof of the absence of wandering domains for functions $\mathcal{B}^*_{\text{real}}$ requires a theorem that we are going to prove in this section. In fact, the following theorem is a general statement that applies, in particular, to the case of entire functions. However, since the theorem is very technical, it will be convenient to give an example of how to use the theorem to prove the absence of wandering domains. For that, we are going to show that for the entire function $f(z) = e^z$ there are no wandering domains using the result introduced. Finally, we shall give the formal proof of the theorem using all the tools introduced in previous sections.

Theorem 4.4. *Let $U \subset \mathbb{C}$ be a hyperbolic domain and $U' \subset U$ be an open subset. Assume $f : U' \to U$ is a holomorphic covering map.*

a) Assume there is an open connected set $W \subset U'$ such that $f^n(W) \subset U'$ for all $n \geq 0$. Then, for all $w \in W$,

$$\liminf_{n\to\infty} d_U(f^n(w), U \setminus f^n(W)) > 0. \tag{4.1}$$

b) Let $D \subset U$ be an open subset and consider $V := f^{-1}(D)$. Suppose there exists $\{n_k\}_{k\geq 1}$ a sequence such that $f^{n_k}(w) \in D$ and $d_U(f^{n_k}(w), U \setminus D) \to \infty$. Then,

$$d_U(f^{n_k-1}(w), U \setminus V) \to \infty. \tag{4.2}$$

It is a well-known fact that the exponential map, $f(z) = e^z$, has no wandering domains since $f \in \mathcal{S}$. In fact, it is the case that $\mathcal{J}(e^z) = \mathbb{C}$. However, we will give an alternative proof that illustrates the use of the previous theorem and several previous results on constant limit functions.

Corollary 4.5. *The map $f(z) = e^z$ has no wandering domains.*

Proof. To prove the absence of wandering domains for the exponential map the setting is the following: let $U = \mathbb{C} \setminus [0, \infty)$ and $U' = f^{-1}(U)$. Assume there exists a wandering domain W of f. It is obvious that $f(z) = e^z$ has no critical points since $f'(z) = e^z \neq 0$ for all $z \in C$ and has one finite asymptotic value at $z = 0$.

The first step of this proof is to show that there exists $\{n_k\}_{k\geq 1}$ such that $f|_W^{n_k} \to \infty$ as $k \to \infty$. On the one hand, from Theorem 3.12 we know that $f^{n_k}|_W \not\to \infty$ uniformly and from Corollary 3.5 we have that if $f^{n_k}|_W \to a \in \mathbb{C}$, then a is in the post critical set, i.e. $a \in \{0, e^e, e^{e^e}, \dots\}$. Gathering all this results, we have that $f^{n_k}|_W \to f^j(0)$ for $j \geq 0$.

On the other hand, one can show that there exists a sequence $\{m_k\}_{k\geq 0}$ so that $f^{m_k}|_W \to 0$. On account of that, if there exists sequence such that the wandering domain W accumulates to zero.

Since $z = 0$ is the only singular value, $f : U' \to U$ is a holomorphic covering because $0 \notin U$. Furthermore, $f^n(W) \subset U$ for all $n \geq 0$ (if not that would mean that there is a point $w \in W$ such that $f^n(w) \notin U$, which implies that $f^n(w) \in [0, \infty)$ and this fact means that $f^n(w) \to \infty$ as $n \to \infty$ which is a contradiction). Notice that in particular it also holds that $f^n(W) \subset U'$.

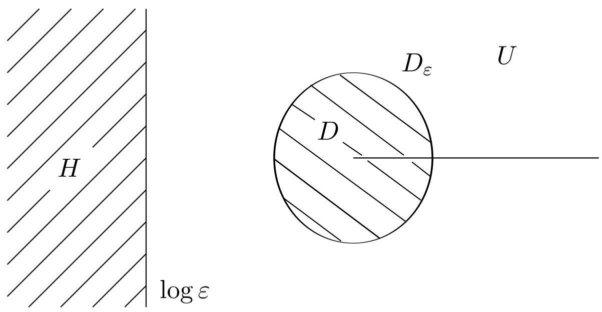
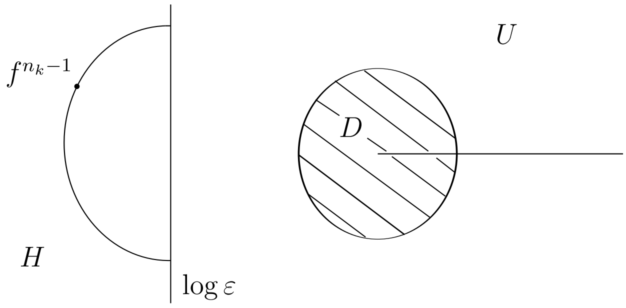


Figure 4.1: Framework for the proof of the absence of wandering domains for the exponential map

Let $D = U \cap D(0,\varepsilon)$ and $V = f^{-1}(D)$. Then, by definition of the exponential map, V is contained in the left half-plane $H = \{\operatorname{Re} z < \ln \varepsilon\}$, as seen in Figure 4.1. By what has been showed before, let $\{n_k\}_{k\geq 0}$ such that $f^{n_k}|_W \to 0$. Hence, if we consider the hyperbolic distance $d_U(f^{n_k}(w), U \setminus D)$, and taking into account that the euclidean distance remains constant in U, this implies $d_U(f^{n_k}(w), U \setminus D) \to \infty$. As we are under the hypothesis of Theorem 4.4 it holds that $d_U(f^{n_k-1}(w), U \setminus V) \to \infty$.

For all $w \in W$ we have that $f^{n_k-1}(w) \in H$. So, we can write the hyperbolic distance to $U \setminus V$ in terms of ∂H, $d_U(f^{n_k-1}(w), U \setminus V) = d_U(f^{n_k-1}(w), \partial H) > +\infty$. The last inequality follows from the fact that the hyperbolic distance from f^{n_k-1} and ∂H is given by the euclidean length of the arc of the circle centred at zero and radius $|f^{n_k-1}(w)|$, which is finite.



Seeing that, the distance $d_U(f^{n_k-1}(w), U \setminus D)$ is finite. Hence, we have a contradiction with Theorem 4.4 which implies that there are no wandering domains for the exponential map. □

Proof of Theorem 4.4. Before starting the proof we shall write the equations (4.1) and (4.2) in terms of hyperbolic density. Setting $R_n := d_U(f^n(w), U \setminus f^n(W))$, it turns out that condition (4.1) can be written as $\liminf_{n\to\infty} R_n > 0$. Now, using the inequalities in Proposition 2.13 this last condition is equivalent to

$$\limsup_{n\to\infty} \rho^U_{f^n(W)}(f^n(w)) < \infty. \tag{4.3}$$

On the other hand, by Corollary 2.14, condition (4.2) is equivalent to

$$\rho^U_V(f^{n_k-1}(w)) \to 1. \tag{4.4}$$

Consequently, the proof of the theorem will consist on proving the equivalent conditions (4.3) and (4.4), instead of (4.1) and (4.2). As in the statement of the theorem let $U \subset \mathbb{C}$ be a hyperbolic domain and let $f : U' \to U$ is a holomorphic covering map. Consider $W \subset U'$ such that $f^n(W) \subset U'$ for all $n \geq 0$. Take $w \in W$ and denote $w_n := f^n(w)$ and $W_n := f^n(W)$. First, it shall be interesting to control the hyperbolic derivatives of f^n (see Definition 2.8) with respect to the metrics in U and W_n respectively.

$$\delta_n := ||Df^n(w)||_U = |(f^n)'(w)| \cdot \frac{\rho_U(w_n)}{\rho_U(w)} \tag{4.5}$$

$$\tilde{\delta}_n := ||Df^n(w)||^W_{W_n} = |(f^n)'(w)| \cdot \frac{\rho_{W_n}(w_n)}{\rho_W(w)} \tag{4.6}$$

By Pick's theorem it follows automatically that $\tilde{\delta}_n \leq 1$. Consider the map $f : f^{-1}(U) \to U$, which is a covering map, then Pick's theorem yields $\rho_{f^{-1}(U)}(z) = \rho_U(f(z)) \cdot |f'(z)|$. This two last results put together imply that $\rho_U(z) < \rho_U(f(z)) \cdot |f'(z)|$. Seeing that, it follows that $\delta_n := |(f^n)'(w)| \cdot \frac{\rho_U(w_n)}{\rho_U(w)} > 1$. Moreover, we can establish a relation between both inequalities.

$$0 < 1 \overset{(1)}{\leq} \delta_n := |(f^n)'(w)| \cdot \frac{\rho_U(w_n)}{\rho_U(w)} = |(f^n)'(w)| \cdot \frac{\rho_U(w_n)}{\rho_{W_n}(w_n)} \frac{\rho_{W_n}(w_n)}{\rho_W(w)} \frac{\rho_W(w)}{\rho_U(w)} \overset{(2)}{=}$$

$$\overset{(2)}{=} \tilde{\delta}_n \cdot \frac{\rho_U(w_n)}{\rho_{W_n}(w_n)} \frac{\rho_W(w)}{\rho_U(w)} \overset{(3)}{\leq} \tilde{\delta}_n \frac{\rho_W(w)}{\rho_U(w)} \overset{(4)}{\leq} \frac{\rho_W(w)}{\rho_U(w)} \overset{(5)}{=} C,$$

where (1) and (2) come directly from (4.5) and (4.6) and since $\tilde{\delta}_n \leq 1$ we have (4). To justify (3) just notice that $W_n \subset U$ and by Pick's theorem $\rho_{W_n}(w_n) > \rho_U(w_n)$, so $\frac{\rho_U(w_n)}{\rho_{W_n}(w_n)} < 1$. Finally, for (5) we use again Pick's theorem to show that $\frac{\rho_W(w)}{\rho_U(w)} = \rho_W^U(w)$ is a constant C, not depending on n. This implies

$$\rho_{W_n}^U(w_n) := \frac{\rho_{W_n}(w_n)}{\rho_U(w_n)} = \frac{\rho_{W_n}(w_n)}{\rho_U(w_n)} \frac{\rho_W(w)}{\rho_U(w)} \frac{\rho_U(w)}{\rho_W(w)} = \rho_W^U(w) \frac{\tilde{\delta}_n}{\delta_n} \leq C,$$

since $\delta_n \geq 1$ and $\tilde{\delta}_n \leq 1$. In particular,

$$\limsup_{n\to\infty} \rho_{W_n}^U(w_n) \leq C < \infty,$$

which implies *a*).

To prove statement *b*) we consider the derivative of f at w_n with respect to the metric in U, $\eta_n := ||Df(w_n)||_U$. Then, applying the chain rule and the definitions of η_n and δ_n it can be proved that $\delta_{n+1} = \eta_n \cdot \delta_n$.

$$\delta_{n+1} = ||Df^{n+1}(w)||_U = |(f^{n+1})'(w)| \cdot \frac{\rho_U(f(w_n))}{\rho_U(w)} = |(f(f^n))'(w)| \cdot \frac{\rho_U(f(w_n))}{\rho_U(w_n)} \frac{\rho_U(w_n)}{\rho_U(w)}$$

$$= |f'(f^n(w))| \cdot \frac{\rho_U(f(w_n))}{\rho_U(w_n)} |(f^n)'(w)| \cdot \frac{\rho_U(w_n)}{\rho_U(w)} = \eta_n \cdot \delta_n.$$

Hence, $\lim_{n\to\infty} \eta_n = \lim_{n\to\infty} \frac{\delta_{n+1}}{\delta_n} = 1$, since δ_n is increasing and bounded.

Consider $D \subset U$ an open set and $V := f^{-1}(D)$ as in the statement of the theorem. Take a sequence $\{n_k\}_{k\geq 0}$ such that the iterates $w_{n_k} = f^{n_k}(w)$ are in D for all $w \in W$. Assume, by hypothesis, that $d_U(f^{n_k}(w), U \setminus D) \to \infty$. By Corollary 2.14 this is equivalent to $\rho_D^U(w_{n_k}) \to 1$ as $k \to \infty$. Then, $w_{n_k - 1} = f^{-1}(w_{n_k}) \in V$. Since $f : V \to D$ is a covering map, by Pick's theorem $||Df(w_{n_k-1})||_D^V = 1$. Using these facts we have,

$$\eta_{n_k-1} := ||Df(w_{n_k-1})||_U = |f'(w_{n_k-1})| \cdot \frac{\rho_U(w_{n_k})}{\rho_U(w_{n_k-1})}$$

$$= |f'(w_{n_k-1})| \cdot \frac{\rho_D(w_{n_k})}{\rho_V(w_{n_k-1})} \frac{\rho_V(w_{n_k-1})}{\rho_U(w_{n_k-1})} \frac{\rho_U(w_{n_k})}{\rho_D(w_{n_k})}$$

$$= ||Df(w_{n_k-1})||_D^V \cdot \frac{\rho_V^U(w_{n_k-1})}{\rho_D^U(w_{n_k})} \overset{(1)}{=} \frac{\rho_V^U(w_{n_k-1})}{\rho_D^U(w_{n_k})}$$

Equality (1) follows from Pick's theorem. Hence, $\rho^U_V(w_{n_k-1}) = \eta_{n_k-1} \cdot \rho^U_D(w_{n_k})$. Because $d_U(f^{n_k}(w), U \setminus D) \to \infty$, we can apply Corollary 2.14 which yields $\rho^U_D(w_{n_k}) \to 1$ as $k \to \infty$. Combining those two facts we have that

$$\rho^U_V(w_{n_k-1}) = \eta_{n_k-1} \cdot \rho^U_D(w_{n_k}) \to 1 \text{ as } k \to \infty, \tag{4.7}$$

and again by Corollary 2.14, this implies $\rho^U_V(f^{n_k-1}(w)) \to 1$, which is equivalent to (4.2) and finishes the proof. □

Once we have proved the main result of this section, and before the proof of Theorem 4.3, we state without proof the following result:

Lemma 4.6. *Let $f \in \mathcal{B}$ and assume there exists $R > 0$ such that the iterates of f over the set of points in the post critical set outside the disc $D(0, R)$, that is $\{z \in E(f), |z| \geq R\}$.*

Let W a wandering domain of f for which ∞ is a limit function. Let $w \in W$, then there are $s \in S(f)$ and a sequence $\{n_k\}$ such that

$$f^{n_k} \to s \text{ and } f^{n_k-1} \to \infty.$$

Notice that if W is a wandering domain for which infinity is a constant limit function then we have as an immediate consequence that there exists a sequence $\{n_k\}_{k\geq 1}$ such that $f^{n_k}|_W \to a \in \mathbb{C}$ with a a point in the post singular set. Hence this last result is an improvement of that statement since it shows that there exists a sequence of iterates of f over W converging to a point in the singular set whose pre-image tends to infinity.

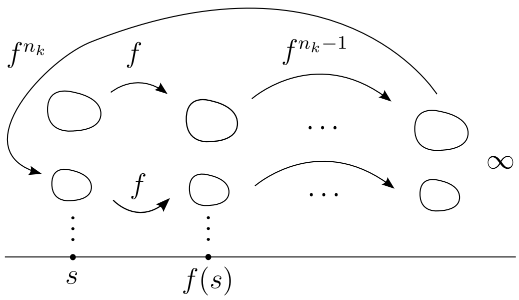


Figure 4.2: Oscillating wandering domain.

4.3 Proof of Theorem 4.3

The full and comprehensive proof of Theorem 4.3 splits in several partial results. The proof of this theorem will be divided in two parts. First we will give a strong result on the absence of bounded wandering domains for functions in $\mathcal{B}^*_{\text{real}}$ and after that we shall give results on the sector condition (see Definition 4.1).

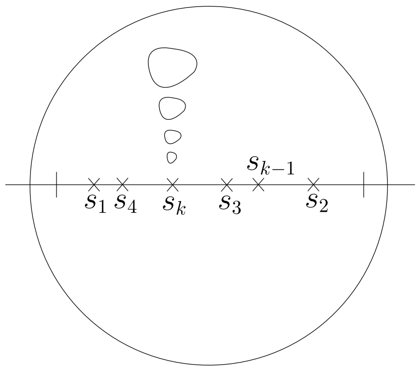
Let f a function in $\mathcal{B}_{\text{real}}$ and let A to be the set of points in $\mathbb{C}$ such that there exists a compact set $K \subset \mathbb{R}$ such that if $f^{n_k}(a) \to a$ as $k \to \infty$ for $z \in A$, then $a \in K$. Under this hypothesis the following theorem states that the interior of U is empty (see [RGvS15]).

Theorem 4.7. *Let $f \in \mathcal{B}_{real}$ and A the set of points $z \in \mathbb{C}$ whose ω-limit set is a compact subset of the real line and which are not contained in an attracting or parabolic basin. Then A has empty interior.*

As a consequence of this theorem, we will be able to prove the absence of wandering domains under some conditions.

Corollary 4.8. *If $f \in \mathcal{B}^*_{real}$, then f has no wandering domains whose set of constant limit functions is bounded.*

Proof. Let $f \in \mathcal{B}^*_{\text{real}}$ and assume there exists U a wandering domain of f such that its set of constant limit functions is bounded. Then, since $f \in \mathcal{B}^*_{\text{real}}$ all of them should belong to a compact set $K \subset \mathbb{R}$. A contradiction with Theorem 4.7 since $W \subset \mathcal{F}(f)$ can not have empty interior. □



This last result is a key point in the proof of Theorem 4.3. On the one hand, considering a wandering domain U for $f \in \mathcal{B}^*_{\text{real}}$ by Theorem 3.12 we know that $f^{n_k}|_W \not\to \infty$ uniformly. On the other hand, by Corollary 4.8, U can not accumulate only to a bounded set. Then, the only way in which a wandering domain could exist is if it were oscillating.

The condition that ensures the non existence of oscillating wandering domains is the (real) sector condition stated at the beginning of the chapter. Morally a function that satisfies this condition has the property that points in $\mathbb{R}$ with sufficiently large absolute value then they belong to the escaping set. More precisely,

Proposition 4.9. *Let $f \in \mathcal{B}_{real}$ satisfying the sector condition and consider $M > 0$ sufficiently large. Then the set*

$$A := \bigcup_{\sigma \in \Sigma(f)} \sigma \cdot [M, \infty), \tag{4.8}$$

satisfies $A \subset I(f)$ and $f(A) \subset A$.

Proof. Let $f \in \mathcal{B}^*_{\text{real}}$, by Ahlfors distortion theorem we have that

$$\liminf_{x \to +\infty} \frac{\log \log |f(\sigma x)|}{\log |x|} \geq \frac{1}{2}.$$

Assume without loss of generality that $\sigma = +$. Then by the inequality above we have,

$$\log\log|f(x)| \geq \log\sqrt{|x|} \;\Rightarrow\; \log|f(x)| \geq \sqrt{|x|} \;\Rightarrow\; |f(x)| \geq e^{\sqrt{|x|}} \overset{(1)}{>} |x|.$$

Above (1) comes from applying l'Hôpital rule to the limit

$$\lim_{x\to\infty} \frac{e^{\sqrt{|x|}}}{|x|} = \frac{\infty}{\infty} = \lim_{x\to\infty} \frac{1}{2\sqrt{|x|}} e^{\sqrt{|x|}} = \infty.$$

Then it is clear that $|f(x)| > |x|$ so the proposition holds. □

Using all the results introduced in this section we will be able to prove that for $f \in \mathcal{B}^*_{\text{real}}$ satisfying the real sector condition have no wandering domains, excluding in that way the case that the wandering domain oscillates. Then if W is a wandering domain for $f \in \mathcal{B}^*_{\text{real}}$ then there must exist two sequences $\{n_k\}_{k\geq 1}$ and $\{m_k\}_{k\geq 1}$ such that $f^{n_k}|_W \to \infty$ and $f^{m_k}|_W \to \in \overline{E}(f)$ as $k \to \infty$ (see Theorem 3.12 and Theorem 3.3). Hence, to prove Theorem 4.3 we just need to discard the possibility of the existence of oscillating wandering domains.

Proof of Theorem 4.3. Assume there exists a wandering domain W of $f \in \mathcal{B}^*_{\text{real}}$. Since the set of singular values is a subset of $\mathbb{R}$ and the function is real, $f(\mathbb{R}) \subset \mathbb{R}$, we have that the post singular set is also a subset of the real line, $E(f) \subset \mathbb{R}$.

Consider $A = \bigcup_{\sigma\in\Sigma(f)} \sigma \cdot [M, \infty)$ and $U = \mathbb{C} \setminus (A \cup E(f))$. We claim that W has to be a subset of U. To see this claim notice that if $W \cap A = \emptyset$ since A is in the escaping set and $W \cap E(f) = \emptyset$ since points in the post singular set either tend to infinity or they are bounded.

We already know by Theorem 3.12 that there are no real images of W escaping to infinity. Besides that, by Theorem 4.8 the limit functions of f can not be all bounded. Then if we apply Proposition 4.6 there exist $s \in S(f)$ and a sequence $\{n_k\}_{k\geq 1}$ such that $f^{n_k}(w) \to s$ and $f^{n_k-1}(w) \to \infty$ for $w \in W$. This behaviour is what was explained intuitively in Figure 4.2.

In addition, we also have the hypothesis that f satisfies the sector condition. Take $R > |s|+\varepsilon$, then by the sector condition, for $\sigma \in \Sigma(f)$, there are $R' > 0$ and $\theta > 0$ such that $|f(\sigma x+iy)| > R$ whenever $x \geq R'$ and $|y| \leq \theta|x|$. Equivalently, if we call $z = \sigma x + iy$, then the condition states that $|f(z)| > R$ whenever $x \geq R'$ and $|z - \sigma x| \leq \theta|x|$. We can assume without loss of generality that $\theta \in (0, 1)$. We denote by L this sector.

Now, take the disc $D_\varepsilon(s)$ and let $D = D_\varepsilon(s) \cap U$. Assume without loss of generality that $f^{n_k}(w) \in D_\varepsilon(s)$. Keeping in mind that $s \in \partial U$ but that the euclidean distance with respect to $U \setminus D$ is constant and equal to ε, we know that f^{n_k} is approaching the boundary of U while keeping constant the distance with respect to $U \setminus D$, as seen in Figure 4.3. In particular, it follows that $d_U(f^{n_k}(w), U \setminus D) \to \infty$. Hence, the hypothesis of Theorem 4.4 are satisfied, which implies that $d(f^{n_k-1}(w), U \setminus V) \to \infty$.

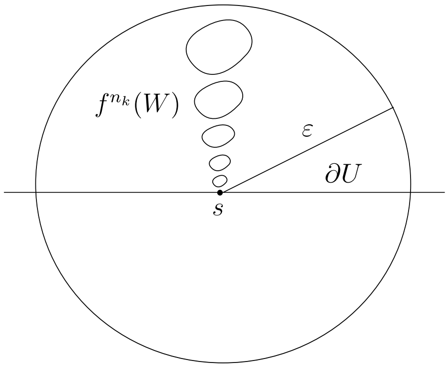


Figure 4.3: Here f^{n_k} is approaching the boundary of U, so $d_U(f^{n_k}(w), U \setminus D) \to \infty$.

Set $V = f^{-1}(D)$ and consider $z_0 \in V$. By construction we have that $V \cap L = \emptyset$ since $|f(z)| > R$ for all $z \in L$. Then, if we consider $z_0 = x + iy$ we will have that $|y| > \theta|x|$.

By the definition of cosinus we have that $|x| = |\cos\alpha||z_0|$, so $|y| > \theta|\cos\alpha||z_0| \geq c \cdot |x|$ with $c \in (0,1)$, as seen in Figure 4.4. Consider a path γ connecting z_0 to the boundary ∂V. Then, for all $z \in \gamma$ we can bound from below the distance between z and ∂V since $|y| > c \cdot |x|$, so $d(z, \partial U) > c \cdot |x|$. Seeing that, we can estimate the hyperbolic distance between z_0 and $U \setminus V$.

$$d_U(z_0, U \setminus V) \overset{(1)}{\leq} \ell_U(\gamma) \overset{(2)}{\leq} |y - \sigma x| \max_{z \in \gamma} \rho_U(\gamma) \overset{(3)}{\leq} |y - \sigma x| \cdot \max_{z \in \gamma} \frac{2}{d(z, \partial U)} \leq \frac{2\,|y - \sigma x|}{|y|} < K,$$

Inequalities (1) and (2) come from the definition of hyperbolic distance. The first one comes from the fact that the hyperbolic distance is the infimum of the length over paths over U to the set $U \setminus V$. For (2) we use that we are measuring the a relation between the euclidean distance, $|y - \sigma x|$, and the metric over U. Finally, (3) follows from Proposition 2.13. Since $z_0 \in V$ was arbitrary we have $d_U(f^{n_k - 1}(w), U \setminus V) < +\infty$, a contradiction with Theorem 4.4. $\square$

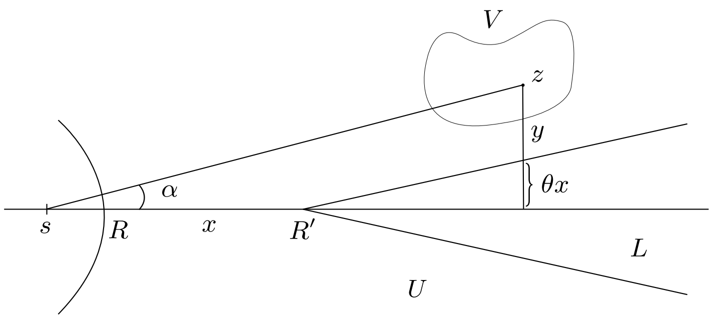


Figure 4.4: Setting to show that $d_U(f^{n_k - 1}(w), U \setminus V) < +\infty$.

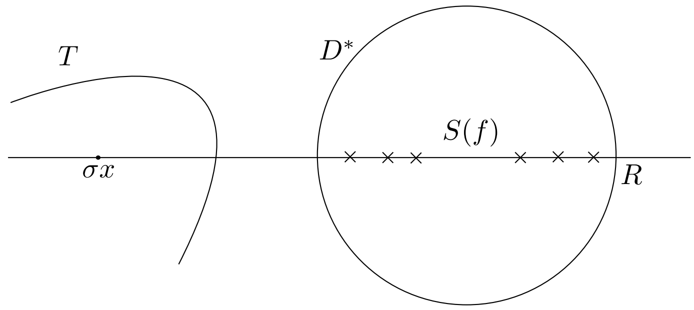
With this last theorem we have showed that there are no wandering domains for functions in $\mathcal{B}^*_{\text{real}}$ that satisfy the sector condition. However, we can go further and give an equivalent theorem to one above by giving an explicit inequality that is equivalent to the sector condition.

Theorem 4.10. *Let $f \in \mathcal{B}_{real}$ and $\sigma \in \Sigma_0(f)$. Then f satisfies the sector condition at $\sigma\infty$ if and only if there exist r, $K > 0$ such that*

$$\frac{|f'(\sigma x)|}{|f(\sigma x)|} \leq K \cdot \frac{\log|f(\sigma x)|}{|x|}, \tag{4.9}$$

where $|x| \geq r$.

Proof. Since the set of singular values is bounded we can consider a disc of radius $R > 1$ big enough so that $S(f) \subset D(0, R)$. Set $D^* := \mathbb{C} \setminus \overline{D(0, R)}$. As we showed in the first chapter $f^{-1}|_{D^*}$ is composed by simply-connected components bounded by a non-closed analytic curve with both ends tend to infinity. Take the tract T that contains σx, called tracts. Furthermore, we know by Theorem 3.9 that $f : T \to D^*$ is a universal covering map.



By Proposition 2.13 we have that

$$\frac{1}{2\text{dist}\,(z, \partial T)} \leq \rho_T(z) \leq \frac{2}{\text{dist}\,(z, \partial T)}. \tag{4.10}$$

We know that there is a homeomorphism between the unit punctured disc to D^* by the map that sends $z \mapsto \frac{R}{z}$. Then by 2.6 we can compute the metric on D^*.

$$\rho_{\mathbb{D}^*}(z) = \frac{1}{|z| \cdot |\log|z||} \;\Rightarrow\; \rho_{\mathbb{D}^*}\left(\frac{R}{z}\right) = \frac{1}{\frac{R}{|z|}\,|\log|\frac{R}{z}||} = \frac{1}{\frac{R}{|z|} \cdot \log\frac{|z|}{R}} \;\Rightarrow\; \rho_{D^*}(z) = \frac{1}{|z| \cdot \log\frac{|z|}{R}}.$$

Since $R > 1$, then $\log R > 0$ which means that $\log|z| > \log|z| - \log R$. Hence,

$$\frac{1}{|z| \cdot \log|z|} \leq \frac{1}{|z| \cdot \log\frac{|z|}{R}} = \rho_{D^*}(z) \tag{4.11}$$

Next, if we take $|z| > R^2$

$$|z| > R^2 \Leftrightarrow |z|^2 > R^2\,|z| \;\Leftrightarrow\; \frac{|z|^2}{R^2} > |z| > 1 \;\Leftrightarrow\; \log\frac{|z|^2}{R^2} > \log|z|$$
$$\Leftrightarrow 2\log\frac{|z|}{R} > \log|z| \;\Leftrightarrow\; \frac{1}{\log\frac{|z|}{R}} < \frac{2}{\log|z|}.$$

which yields

$$\rho_{D^*}(z) \leq \frac{2}{|z| \log \frac{|z|}{R}} \tag{4.12}$$

Combining (4.11) and (4.12) gives

$$\frac{1}{|z| \cdot \log |z|} \leq \rho_{D^*}(z) \leq \frac{2}{|z| \log \frac{|z|}{R}}. \tag{4.13}$$

Besides that, we know that f is a universal covering map over T. Thus, by Pick's theorem $||Df(z)||^T_{D^*} = 1$ so $\rho_T(z) = |f'(z)| \cdot \rho_{D^*}(f(z))$. Joining together (4.10) and (4.13),

$$\frac{|f'(z)|}{|f(z)| \cdot \log |f(z)|} \leq |f'(z)| \cdot \rho_{D^*}(f(z)) = \rho_T(z) \leq \frac{2}{\operatorname{dist}(z, \partial T)},$$
$$\frac{1}{2 \operatorname{dist}(z, \partial T)} \leq \rho_T(z) = |f'(z)| \cdot \rho_{D^*}(f(z)) \leq \frac{2\,|f'(z)|}{|f(z)| \cdot \log |f(z)|}.$$

Using those two inequalities it is easy to get an estimate for $\frac{|f'(z)|}{|f(z)| \cdot \log |f(z)|}$ when $|z| > R^2$.

$$\frac{1}{4 \operatorname{dist}(z, \partial T)} \leq \frac{|f'(z)|}{|f(z)| \cdot \log |f(z)|} \leq \frac{2}{\operatorname{dist}(z, \partial T)}. \tag{4.14}$$

From the inequalities given by (4.14) we are going to prove an equivalent condition for (4.9). Assume now that for some $\varepsilon > 0$ and x sufficiently large so that

$$\operatorname{dist}(\sigma x, \partial T) \geq \varepsilon x. \tag{4.15}$$

We claim that if (4.15) holds then the condition (4.9) is also satisfied. To see the claim we need some computations.

$$\operatorname{dist}(\sigma x, \partial T) \geq \varepsilon x \;\Leftrightarrow\; \frac{1}{\varepsilon x} \geq \frac{1}{\operatorname{dist}(\sigma x, \partial T)} \overset{(4.14)}{\geq} \frac{|f'(z)|}{2\,|f(z)| \cdot \log |f(z)|} \;\Leftrightarrow$$
$$\Leftrightarrow\; \frac{|f'(z)|}{|f(z)|} \leq \frac{2 \log |f(z)|}{\varepsilon \cdot |x|} = K \cdot \frac{\log |f(z)|}{\varepsilon \cdot |x|}.$$

The key point now is to show that (4.15) is equivalent to the sector condition. If we show that this will mean that the sector condition is equivalent to (4.9) and that will terminate the proof.

However, the sector condition states that for all R there exists $R', \theta > 0'$ such that $|f(\sigma x + y)| > R$ with $x \geq R'$ and $|y| \leq \varepsilon x$. Then if we consider R large enough so that $S(f) \subset B(0, R)$ and $\sigma x \in T$ such that $\operatorname{dist}(\sigma x, \partial T) \geq \varepsilon x$ we have that there is a ball completely contained in T of center σ As a result, if we map that ball under f we will have that for $|y| \leq \varepsilon x$ then $|f(\sigma x + y)| > R$ because of the definition of the tract T. Hence, the sector condition is equivalent to (4.15), which finishes the proof.

□

Chapter 5

Examples

5.1 Introduction

In this chapter we would like to present different examples of entire functions over $\mathbb{C}$ having no wandering domains. Until now many tools have been introduced to prove this fact. First, we will give a more theoretical approach based on the theorems by Fatou and Baker that say that for a sequence $\{n_k\}_{k\geq 1}$ the family of iterates over a wandering domain W is such that $f^{n_k}|_W \to a \in \overline{E}$ as $k \to \infty$. If we consider a function in $\mathcal{B}^*_{\text{real}}$, a second approach will be to use the explicit condition given by Theorem 4.10. Thus, we will illustrate how to prove the absence of wandering domains using several methods given throughout this paper.

5.2 First example

The first example is a function used as an example in many papers regarding this topic,

$$f(z) = \frac{\sin z}{z}.$$

To begin with, we are going to show that f is indeed in class $\mathcal{B}$. For that we have to prove that:

1. f is a transcendental entire function.
2. $S(f)$, the set of singular values, is bounded.

That it is a transcendental function is clear, so we just have to show that f is entire. Let z any complex number different from zero. Then

$$f'(z) = \frac{1}{z^2}\,(z\cos z - \sin z).$$

This exists for any $z \neq 0$. However, for the case $z = 0$ using the definition of limit and l'Hôpital we get that the derivative exists for all $z \in \mathbb{C}$.

$$f'(0) = \lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0}\frac{\sin h - h}{h^2} = \frac{0}{0} \overset{(*)}{=} \lim_{h\to 0}\frac{\cos h - 1}{2h} \overset{(*)}{=} \lim_{h\to 0}\frac{-\sin h}{2} = 0.$$

The following step is to prove that the set of singular values is bounded. We already know that there exist two types of singularities: critical and asymptotic. In this case there are no points "coming from infinity", so there are no asymptotic values. Hence, we have to look only for the critical points, i.e. the points $z \in \mathbb{C}$ such that $f'(z) = 0$.

$$f'(z) = \frac{z\,\cos z - \sin z}{z^2} = 0 \;\Leftrightarrow\; z\cos z - \sin z = 0 \;\Leftrightarrow\; z = \tan z.$$

It is easy to prove that all the critical points are real, in fact those are the points of the form $z = \tan x$, for all $x \in \mathbb{R}$.

$$\tan z = \frac{\sin x \cosh y + i\,\cos x \sinh y}{\cos x \cosh y - i\,\sin x \sinh y} = \frac{\sin 2x + i \sinh 2y}{\cosh 2y + \cos 2x} = x + iy. \tag{5.1}$$

Thus we have that the vectors $(\sin 2x, \sinh 2y)$ and (x, y) are proportional. As a consequence, the following determinant has to be zero.

$$\begin{vmatrix} x & y \\ \sin 2x & \sinh 2y \end{vmatrix} = x \sinh 2y - y \sin 2x = 0 \;\Leftrightarrow\; x \sinh 2y = y \sin 2x.$$

Taking into account that $|\sin x| \leq |x|$ and $|\sinh x| \geq |x|$ over the previous equality will yield the result.

$$2|x||y| \leq |x \sinh 2y| = |y \sin 2x| \leq 2|x||y| \;\Rightarrow\; \begin{cases} |y \sin 2x| = 2|x||y| \\ |x \sinh 2y| = 2|x||y| \end{cases} \;\Rightarrow\; \begin{cases} |\sin 2x| = 2|x| \\ |\sinh 2y| = 2|y| \end{cases}.$$

Therefore we get two cases, when $x = 0$ and $y = 0$. Imposing that $y = 0$ gives the solution $x = \tan x$, that has infinite real solutions. Whereas, for $x = 0$ we would have to solve the equation $\tan(iy) = iy$, which yields $y = 0$, so that the only solution is the trivial solution $z = 0$.

$$\tan(iy) = i\,\frac{\sinh 2y}{1 + \cosh 2y} = iy \;\Leftrightarrow\; \tanh 2y = 2y \;\Leftrightarrow\; y = 0.$$

Seeing that, we have that there are infinitely many real critical points. Hence, to show that $f \in \mathcal{B}$ what remains to prove is that the set of critical values associated to those points is bounded. The following inequality shows that $f \in \mathcal{B}$.

$$|f(\tan x)| = \left|\frac{\sin(\tan x)}{\tan x}\right| \leq \left|\frac{\tan x}{\tan x}\right| = 1.$$

Assume now that there exists W a wandering domain. As we saw in Chapter 3, $f^n(W) \not\to \infty$ and the limit function over a wandering domain is constant, so there exists a subsequence $\{n_k\}_{k\geq 1}$ so that $f^{n_k}|_W \to a$. Moreover, (3.5) states that the limit of a wandering domain has to be in $\overline{E} \cup \{\infty\}$, so $a \in \overline{E}$.

In addition, it is easy to see that the limit functions of f over W have to be in the Julia set. Indeed, let $w \in W$ and set $w_k := f^{n_k}(w)$. Since we know that w_k jumps from Fatou component to another, consider the line r_k joining w_k and w_{k+1}. Then, by properties of the Fatou and Julia set, there exists $x_k \in r_k$ such that $x_k \in \mathcal{J}(f)$. If we make k tend to infinity then the wandering domain accumulates to a, and so does x_k. Hence, $a \in \mathcal{J}(f)$.

As a consequence, we have that $a \in \overline{E} \cap \mathcal{J}(f)$. Then if we show that the intersection is empty we will arrive to a contradiction with the existence of W. For that we are going to see that $\overline{E}$ is inside the Fatou set of f. First we look for the fixed points of f. It can be shown that we have a fixed point,

$$f(z) = \frac{\sin z}{z} = z \;\Rightarrow\; z = 0.876726.$$

Let z_0 be the fixed point, since $f'(z_0) < 0$ it is an attracting fixed point. Moreover, we know that $\overline{E} \subset [0, 1]$ and it can be seen that all points in $[0, 1]$ are in the basin of attraction of z_0. Therefore, $\overline{E} \subset \mathcal{F}(f)$ which implies that $\overline{E} \cap \mathcal{J}(f) = \emptyset$. Thus, there do not exist wandering domains for f.

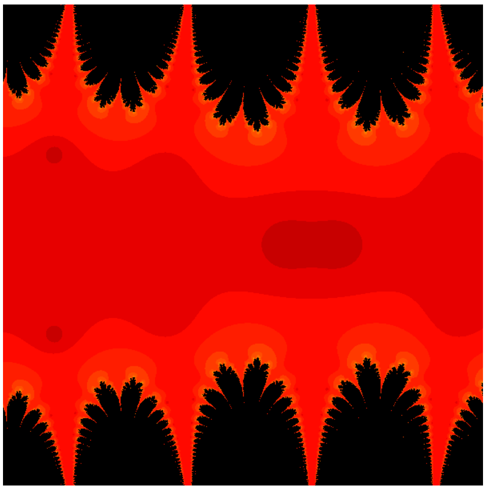


Figure 5.1: Julia and Fatou sets of $\frac{\sin z}{z}$.

In Figure 5.1 we can see the Fatou set in red and the Julia set in black and made up by filaments going from $-\infty$ to $+\infty$. This figure shows that after a few iterations all the points in the real line go to $[0, 1]$, which is in the basin of attraction of z_0. The different shades of red correspond to the speed at which the iterates converge to the attracting point, the darker the red the faster they converge.

5.3 Second example

Secondly we shall study the function

$$f(z) = \frac{\sinh z}{z},$$

which is similar to the previous one. As in the case of $\frac{\sin z}{z}$ we have to see that $f \in \mathcal{B}$. The scheme to show this fact is exactly as before. First we are going to show that f is, in fact, a transcendental entire function and then we are going to express the function in an explicit form that will allow us to prove that the singular set is bounded. Using l'Hôpital's rule we have,

$$
\begin{aligned}
f'(z) &= \frac{z \cdot \cosh z - \sinh z}{z^2}, \ \forall 0 \neq z \in \mathbb{C}, \\
f'(0) &= \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{\frac{\sinh h}{h} - 1}{h} = \lim_{h \to 0} \frac{\sinh h - h}{h^2} = \\
&= \frac{0}{0} = \lim_{h \to 0} \frac{\cosh h - 1}{2h} = \frac{0}{0} = \lim_{h \to 0} \frac{\sinh h}{2} = 0, \text{ for } z = 0.
\end{aligned}
$$

Hence, f is holomorphic over the whole complex plane. This together with the fact that f is trivially transcendental we have that f is a transcendental entire function. Next, to find the critical points of f,

$$
\begin{aligned}
f'(z) &= \frac{z \cdot \cosh z - \sinh z}{z^2} = \Leftrightarrow \ z = \tanh z, \\
\tanh z &= \frac{\sinh x \ \cos y + i \cosh x \ \sin y}{\cosh x \ \cos y + i \sinh x \ \sin y} = \frac{\sinh 2x + i \sin 2y}{\cosh 2x + i \cos 2y}.
\end{aligned}
$$

Then, $(\sinh 2x, \sin 2y)$ and (x, y) are proportional, so

$$
\begin{vmatrix} x & y \\ \sinh 2x & \sin 2y \end{vmatrix} = x \ \sin 2y - y \ \sinh 2x = 0 \ \Leftrightarrow \ x \ \sin 2y = y \ \sinh 2x.
$$

Using the inequalities $|\sin x| \leq |x|$ and $|\sinh x| \geq |x|$ on the previous equation we get that the solutions are of the form $z = i \tan y$, with $y \in \mathbb{R}$.

$$
2\,|x||y| \leq |y \ \sinh 2x| = |x \ \sin 2y| \leq 2\,|x||y| \ \Rightarrow \begin{cases} |\sin 2x| = 2\,|x| \ \Rightarrow \ x = 0, \\ |\sinh 2y| = 2\,|y| \ \Rightarrow \ y = 0. \end{cases}
$$

- If $y = 0 \ \Rightarrow \ \tanh x = x \ \Rightarrow \ x = 0$. Then the only solution for this case is $z = 0$.
- If $x = 0$,

  $$
  \tanh iy = iy = \Leftrightarrow \ i \frac{\sin 2y}{1 + \cos i\,y} \ \Leftrightarrow \ \tan y = y,
  $$

  so there are an infinite number of solutions of the form $z = i \ \tan y$, $y \in \mathbb{R}$.

Seeing this, there are an infinite number of critical points $z = i \ \tan y$, $y \in \mathbb{R}$. As there are no asymptotic values, all the singular points are of the form $f(z)$, with $z = i \ \tan y$, and this set of points is bounded.

$$
|f(i \ \tan y)| = \left| \frac{\sinh(i \ \tan y)}{i \ \tan y} \right| = \left| \frac{i \ \sin(\tan y)}{i \tan y} \right| \leq \left| \frac{\sin(\tan y)}{\tan y} \right| \leq 1.
$$

Thus, the set of singular values is bounded which means that $f \in \mathcal{B}$. This is analogous to the previous example but in this case the singular values are on the imaginary axis of the complex plane. This implies that $f \notin \mathcal{B}^*_{\text{real}}$ so we can not try to use the expression on (4.9) to prove the absence of wandering domains. However, a similar argument as in the case $\frac{\sin z}{z}$ will yield the result we are looking for. As before, let's begin by finding the fixed points of f, that is

$$
\frac{\sinh z}{z} = z \ \Rightarrow \begin{cases} z_1 = 1.31328, \\ z_2 = 2.639. \end{cases}
$$

The point z_1 is an attractor and the second one, z_2, is a repulsor. Moreover, the singular values are in $[0, i]$ and those points are mapped to the real axis, more concretely to $[0, 1]$. Since f is a real function, $f(\mathbb{R}) \subset \mathbb{R}$, the iterates of the singular values will remain forever in the real axis. Also, it can be seen that the interval $[0, 1]$ is in the basin of attraction of z_1.

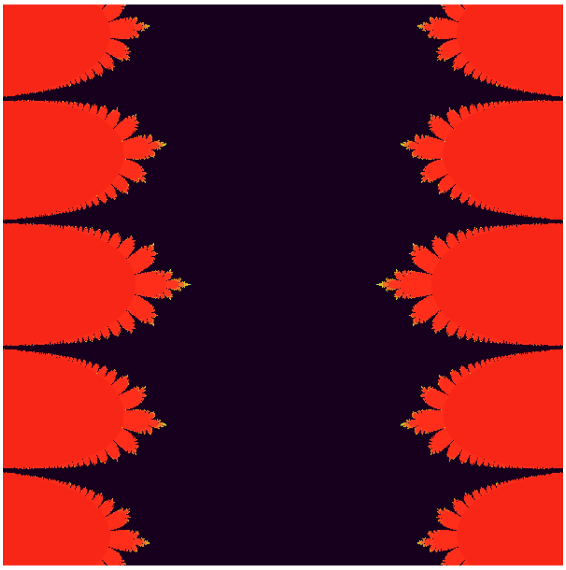


Figure 5.2: Julia and Fatou sets of $\frac{\sinh z}{z}$.

Taking all these facts, we have that this implies that the post singular set $E(f)$ is in the Fatou set of f. Now, assume there is a wandering domain W of f. As we already know, the limit functions of W have to be in $\overline{E} \cap \mathcal{J}$ (the argument followed is exactly the same as in the previous example). With this in mind, as the post singular set is in the Fatou set and $\mathcal{F} \cap \mathcal{J} = \emptyset$ it is clear that $\overline{E} \cap \mathcal{J} = \emptyset$, which contradicts the hypothesis that a wandering domain exists for f. In Figure 5.2 we can see the Julia set in red and the Fatou set in black. It is cleat that the singular set is contained in the Fatou set as we showed earlier.

5.4 Third example

This final example is an explicit function in $\mathcal{B}^*_{\text{real}}$ which satisfies the section condition. Hence, to prove the absence of wandering domains for this case we are going to prove that inequality (4.9) holds. Consider $\{x_i\}$, real numbers such that $0 < x_n \leq x_{n+1}$. Then the function,

$$f(z) = \prod_{n=1}^{\infty} \left(1 + \frac{z}{x_n}\right), \tag{5.2}$$

has zeros at $z = -x_n$ for all $n \geq 1$. Hence, f has infinitely many real zeros. On the one hand, if we impose $\sup_{x<0} |f(x)| < \infty$ then the singular set of critical values will be bounded. On the other hand, by the Denjoy-Carleman-Ahlfors theorem the number of asymptotic values attained by a non-constant entire function of order p on curves going outwards toward infinite absolute value is less than or equal to $2p$. Hence, the set of asymptotic values of f is finite. Then, the singular set $S(f)$ is bounded and $S(f) \subset \mathbb{R}$. It is clear that f is a real function also, which yields $f \in \mathcal{B}^*_{\text{real}}$. Then, we shall see that (4.9) holds.

$$\begin{aligned}
x \cdot \frac{f'(x)}{f(x)} &= x \cdot \frac{\sum_{i=1}^{\infty} \frac{1}{x_i} \prod_{\substack{n=1 \\ n \neq i}} \left(1 + \frac{1}{x_n}\right)}{\prod \left(1 + \frac{x}{x_n}\right)} \\
&= \sum_{i=1}^{\infty} \frac{1}{x_i} \cdot \frac{1}{\left(1 + \frac{x}{x_i}\right)} \\
&= x \cdot \sum_{i=1}^{\infty} \frac{1}{x_i + x} \\
&= \sum_{i=1}^{\infty} \left(1 - \frac{x_i}{x_i + x}\right).
\end{aligned}$$

Consider now the sequence $\frac{x_i + x}{x_i} = 1 + \frac{x}{x_i}$. If we fix x and we make n tend to ∞ then $1 + \frac{x}{x_i} \searrow 1$ since $x_i \to \infty$. Now we shall find $K > 0$ such that (4.9) holds.

$$x \cdot \frac{f'(x)}{f(x)} = \sum_{i=1}^{\infty} \left(1 - \frac{1}{1 + \frac{x}{x_i}}\right) \overset{(1)}{<} \sum_{i=1}^{\infty} \log \left(1 + \frac{x}{x_i}\right) = \prod_{i=1}^{\infty} \left(1 + \frac{x}{x_i}\right) = \log f(x),$$

since $\log t > 1 - \frac{1}{t}$ yields (1). Then, by Theorem 4.10 f satisfies the sector condition which implies, by Theorem 4.3, that f has no wandering domains.

Chapter 6

Conclusions

The question whether domains of the Fatou set that are neither periodic nor pre-periodic exist or not is one of the most relevant questions in complex dynamics. For the case of rational maps the answer was given by D. Sullivan, who proved the absence of wandering domains for functions of that type. In the case of transcendental dynamics this question remains open. However, through the years many mathematicians have published several crucial results on this topic.

In this thesis the main focus has been in the study of wandering domains for transcendental entire functions whose singular set is bounded, the so-called class $\mathcal{B}$. One of the most remarkable results on this topic was the one given by C. Bishop [Bis15] that showed the existence of wandering domains for functions in class $\mathcal{B}$.

The result by D. Sullivan [Sul85] was a major breakthrough not only because of what it proved but also for the techniques he used to do so. He introduced quasi-conformal surgery, a very powerful tool that has been used since then in many other arguments, including the proof of the absence of wandering domains for transcendental entire functions with a finite set of singular values.

However, the paper studied in this thesis by Mihaljević-Rempe [MBRG13] gives a different approach as it uses arguments of hyperbolic geometry instead of quasi-conformal analysis. Using this technique they are able to prove the absence of wandering domains for some functions in class $\mathcal{B}$. For instance the main result of this thesis, that for real functions in $\mathcal{B}$ whose set of singular values is real and that satisfy a geometrical condition there are no wandering domains. Moreover, this leads us to believe that these arguments can be considered in other settings and that hyperbolic geometry is a suitable tool that could be used in many other proofs regarding this topic.

Finally, it has been proved that there exist escaping and oscillating wandering domains. However, a problem of great interest in the mathematical community is if there exist bounded wandering domains. Much has been showed on the existence or absence of wandering domains for transcendental entire maps but there are still plenty open questions that need to be solved and that make this topic very relevant.

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