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**ADVANCED MATHEMATICS**
**MASTER'S FINAL PROJECT**

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# KAM THEORY FOR PERIODIC ORBITS

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## Abstract

This project develops a theoretical framework for the convergence of the Newton method in a system with a periodic orbit, proving its existence with an explicit computed error. We adapt the analytical techniques of KAM theory with a small KAM iterative scheme in the sense of a loss of part of the analytic domain while finding the periodic orbit.

More explicitly, we present a conceptual background with norm definitions, analytic results and non-degeneracy results to avoid problems with resonances and to help formulate an iterative lemma for the Newton method to finally present a Convergence Theorem, proving the existence and uniqueness of a periodic orbit if the initial error is small enough.

Therefore, we obtain a theoretical result to prove the existence of periodic orbits when the initial error is small enough. This setup can be applied to Computer Assisted Proof for the existence of periodic orbits.

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# 1 Introduction

To analyze periodic orbits and their behavior, the classical approach method is Floquet theory, which leads to linearization of the system around the periodic orbit to reduce the study of local stability to a linear system with periodic coefficients. Nevertheless, this local method cannot give us a global description of the phase map. Moore presented in 2005 [5] an approach using Floquet theory as a computational tool to construct an efficient Fourier spectral algorithm for approximating periodic orbits of unknown period.

Another approach to compute periodic orbits was published by J. D. García-Saldaña and A. Gasull in 2012 [4], presenting a heuristic approach for finding truncated Fourier series as an approximation to the periodic solutions provided by the Harmonic Balance Method. However, the method is confined to low-order approximations due to complex symbolic operations.

A technique that has thrived is the parametrization method [3], developed to find the exact nonlinear change of variables for the periodic orbit. G. Huguet and R. de la Llave presented in 2013 [2] a quasi-Newton type method for the computation of the change of variables with an analytic result in two dimensions for the existence of the periodic orbit without explicit constants.

In this project we develop a theoretical framework for the proof of the existence of a periodic orbit in an $n$-dimensional system while also computing explicit constants to control the error of the convergence method. We also follow Moore's approach to Floquet theory and his strategy presented in [5] for fixing the phase and extending these ideas to incorporate a normal condition of the periodic orbit. Finally, we present the *Convergence Theorem* in (6.1) of the method with explicit error control that could be applied to computer-assisted proofs for proving the existence of a periodic orbit.

## 2 Overview of the Problem

We consider a dynamical system

$$\dot{x} = X(x) \quad x \in \mathbb{C}^n$$

And denote

- $\mathbb{T}_\rho = \{\theta \in \mathbb{C}/\mathbb{Z} : |Im(\theta)| \leq \rho\}$
- $C^{\omega,r}(\mathbb{T}_\rho, \mathbb{C}^{m\times n})$ as the set of real analytic matrix valued maps in $\overset{\circ}{\mathbb{T}}_\rho$ whose components and their derivatives up to order $r$ are continuous in $\mathbb{T}_\rho$.

We will consider the formal Floquet theory, which transforms linear, periodic differential equations into constant coefficient systems. Because our system might be non-linear, we apply Floquet theory to the linearized system near the periodic orbit. This translates into the study of the change of variables with linear component $K \in \mathcal{C}^{\omega,1}(\mathbb{T}_\rho, \mathbb{C}^{n\times 1})$, normal component $N \in \mathcal{C}^{\omega,1}(\mathbb{T}_\rho, \mathbb{C}^{n\times(n-1)})$, unknown frequency $\omega \in \mathbb{R}$, and a constant Floquet matrix $\Lambda \in \mathbb{R}^{(n-1)\times(n-1)}$. To study these objects, we analyze the behavior of the error of a Newton-type method applied to the associated system:

$$X(K(\theta)) - K'\omega = 0 \tag{2.1}$$

$$DX(K(\theta))N - N'\omega - N\Lambda = 0 \tag{2.2}$$

Because the period of the periodic orbit is an unknown, we have infinite solutions given that $K(\theta) = K(\theta + c)$ for any constant $c$ (they differ by phase). A similar problem will arise for $N$: any linear transformation of the form $\bar{N}(\theta) = N(\theta)B$ where $B$ is a constant invertible matrix, is also a solution of (2.2) with $\bar{\Lambda} = B^{-1}\Lambda B$.

Following the ideas exposed by Moore in [5] we impose the following conditions for uniqueness:

$$\int_0^1 K_*'(\theta)^T K(\theta) = 0 \qquad\qquad \int_0^1 N_*(\theta)^T N(\theta) = I_{n-1}$$

for fixed functions $K_*', N_*$ that will be introduced afterwards.

Throughout the project we will work with the frame $P(\theta) = \begin{pmatrix} K' & N \end{pmatrix}$, which by differentiating (2.1) allows us to rewrite the system as:

$$DX(K(\theta))P - P'\omega - P\begin{pmatrix} 0 & 0 \\ 0 & \Lambda \end{pmatrix} = 0 \tag{2.3}$$

The main idea is to part from an initial approximation $(K_0, \omega_0, N_0, \Lambda_0)$ with initial invariance error $K_0$ and Floquet error $E_N$ defined as:

$$X(K_0) - K_0'\omega_0 = E_{K_0}$$
$$DX(K_0) - N_0'\omega_0 - N_0\Lambda_0 = E_{N_0}$$

Moreover:

$$DX(K(\theta))P_0 - P_0'\omega_0 - P_0\begin{pmatrix} 0 & 0 \\ 0 & \Lambda_0 \end{pmatrix} = \begin{pmatrix} E_{K_0}' \\ E_{N_0} \end{pmatrix} = E_{R_0}$$

And consider a Newton step:

$$\begin{aligned} K_1 &= K_0 + \Delta K \\ N_1 &= N_0 + \Delta N \\ \omega_1 &= \omega_0 + \Delta\omega \\ \Lambda_1 &= \Lambda_0 + \Delta\Lambda \end{aligned}$$

such that the new tuple $(K_1, \omega_1, N_1, \Lambda_1)$ will solve the system up to quadratic error :

$$\begin{aligned} X(K_1) - K_1'\omega_1 &= E_{K_1} \\ DX(K_1)N_1 - N_1'\omega_1 - N_1\Lambda_1 &= E_{N_1} \end{aligned}$$

Actually, studying the expression of the new error in terms of the previous one, we can obtain an explicit formulation of the error.

## 2.1 Error for One Step of the Newton Method

- For the linear equation we have:

$$\begin{aligned} E_{K_1} =& X(K_1) - K_1'\omega_1 = X(K_0 + \Delta K) - (K_0 + \Delta K)'(\omega_0 + \Delta\omega) \\ =& \boxed{X(K_0)} + DX(K_0)\Delta K + \underbrace{\int_0^1 (1-s)D^2X(K_0 + s\Delta K)[\Delta K, \Delta K]\, ds}_{I_K} \\ & \boxed{-K_0'\omega_0} - \Delta K'\omega_0 - K_0'\Delta\omega - \Delta K'\Delta\omega \\ =& E_K + DX(K_0)\Delta K - \Delta K'\omega_0 - K_0'\Delta\omega - \Delta K'\Delta\omega + I_K \end{aligned}$$

We now consider a change of basis to the frame $P$ where we know the system has a more diagonal behavior:

$$\Delta K = P_0\xi^K \quad \text{where} \quad \xi^K = \begin{pmatrix} \xi^{K,L} \\ \xi^{K,N} \end{pmatrix}$$

**Remark**: Notice we will denote as $^L$ the functions multiplying the linear part of the frame, $K'$, to not overload the notation with $^{K'}$.

$$\begin{aligned} E_{K_1} =& E_{K_0} + DX(K_0)P_0\xi^K - (P_0\xi^K)'\omega_0 - K_0'\Delta\omega - \Delta K'\Delta\omega + I_K \\ =& E_{K_0} + DX(K_0)P_0\xi^K - P_0'\xi^K\omega_0 - P_0\xi_K'\omega_0 - K_0'\Delta\omega - \Delta K'\Delta\omega + I_K \\ =& E_{K_0} + \left[DX(K_0)P_0 - P_0'\omega_0\right]\xi^K - P_0\xi_K'\omega_0 - K_0'\Delta\omega - \Delta K'\Delta\omega + I_K \end{aligned}$$

Because that $K_0'\Delta\omega = P_0\begin{pmatrix} 1 \\ 0 \end{pmatrix}\Delta\omega$, we have:

$$E_{K_1} = E_{K_0} + E_{R_0}\xi^K + P\Lambda_0\xi^K - P_0\xi_K'\omega_0 - P_0\begin{pmatrix} \Delta\omega \\ 0 \end{pmatrix} - \Delta K'\Delta\omega + I_K$$

We define $-\eta^K = P_0^{-1} E_{K_0}$ where $\eta^K = \begin{pmatrix} \eta^{K,L} \\ \eta^{K,N} \end{pmatrix}$ and write:

$$E_{K_1} = E_{R_0}\xi^K + P_0\left[-\eta^K + \begin{pmatrix} 0 & 0 \\ 0 & \Lambda_0 \end{pmatrix}\xi^K - (\xi^K)'\omega_0 - \begin{pmatrix} \Delta\omega \\ 0 \end{pmatrix}\right] - \Delta K'\Delta\omega + I_K$$

Obtaining the cohomological equations:

$$\eta_K = \begin{pmatrix} 0 & 0 \\ 0 & \Lambda_0 \end{pmatrix}\xi^K - (\xi^K)'\omega_0 - \begin{pmatrix} \Delta\omega \\ 0 \end{pmatrix} \Longleftrightarrow$$
$$\begin{pmatrix} \eta^{K,L} \\ \eta^{K,N} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \Lambda_0 \end{pmatrix}\begin{pmatrix} \xi^{K,L} \\ \xi^{K,N} \end{pmatrix} - \begin{pmatrix} (\xi^{K,L})' \\ (\xi^{K,N})' \end{pmatrix}\omega_0 - \begin{pmatrix} \Delta\omega \\ 0 \end{pmatrix} \Longleftrightarrow$$

$$\eta^{K,L} = -\Delta\omega - (\xi^{K,L})' \cdot \omega_0 \tag{2.4}$$
$$\eta^{K,N} = \Lambda_0\xi^{K,N} - (\xi^{K,N})' \cdot \omega_0 \tag{2.5}$$

Finding the functions $\xi^{K,L}, \xi^{N,N}$ that solve this equations (notice the average $\langle \xi^{K,L} \rangle$ is a free term that will be linked to the phase condition) we will have the new error in terms of the initial approximation:

$$\boxed{E_{K_1} = E_{R_0}\xi^K - \Delta K'\Delta\omega + \int_0^1 (1-s)D^2X(K_0 + s\Delta K)[\Delta K, \Delta K]\,ds} \tag{2.6}$$

- For the normal equation we do the equivalent computations:

$$\begin{aligned}
E_{N_1} =& DX(K_1)N_1 - N_1'\omega_1 - N_1\Lambda_1 \\
=& DX(K_0+\Delta K)(N_0+\Delta N) - (N_0+\Delta N)'(\omega_0+\Delta\omega) - (N_0+\Delta N)(\Lambda_0+\Delta\Lambda) \\
=& DX(K_0+\Delta K)N_0 + DX\left(K_0+\Delta K\right)\Delta N - N_0'\omega_0 - N_0'\Delta\omega - \Delta N'\omega_0 - \Delta N\Delta\omega \\
& - N_0\Lambda_0 - N_0\Delta\Lambda - \Delta N\Lambda_0 - \Delta N\Delta\Lambda \\
=& \boxed{DX(K_0)N_0} + \underbrace{\int_0^1 D^2X(K_0+s\Delta K)[\Delta K, N_0]\,ds}_{I_{N_1}} \\
& + DX(K_0)\Delta N + \underbrace{\int_0^1 D^2X(K_0+s\Delta K)[\Delta K, \Delta N]\,ds}_{I_{N_2}} \\
& \boxed{-N_0'\omega_0} - N_0'\Delta\omega - \Delta N'\omega_0 - \Delta N\Delta\omega \boxed{-N_0\Lambda_0} - N_0\Delta\Lambda - \Delta N\Lambda_0 - \Delta N\Delta\Lambda \\
=& E_N + DX(K_0)\Delta N - N_0'\Delta\omega - \Delta N'\omega_0 - \Delta N\Delta\omega - N_0\Delta\Lambda - \Delta N\Lambda_0 - \Delta N\Delta\Lambda \\
& + I_{N_1} + I_{N_2}
\end{aligned}$$

We consider again a change of basis for the increment:

$$\Delta N = P_0\xi^N \quad \text{where} \quad \xi^N = \begin{pmatrix} \xi^{N,L} \\ \xi^{N,N} \end{pmatrix}$$

Having:

$$\begin{aligned} E_{N_1} =& E_{N_0} + DX(K_0)\cdot P_0 \cdot \xi^N - N_0'\Delta\omega - (P\xi^N)'\omega_0 - \Delta N'\Delta\omega \\ & - N_0\Delta\Lambda - P_0\xi^N\Lambda_0 - \Delta N\Delta\Lambda + I_{N_1} + I_{N_2} \\ \\ =& E_{N_0} + DX(K_0)P_0\xi^N - N_0'\Delta\omega - P_0'\xi^N\omega_0 - P_0(\xi_N)'\omega_0 \\ & - \Delta N'\Delta\omega - N_0\Delta\Lambda - P_0\xi^N\Lambda_0 - \Delta N\Delta\Lambda + I_{N1} + I_{N_2} \\ \\ =& E_{N_0} + \left[DX(K_0)P_0 - P_0'\omega_0\right]\xi^N - N_0'\Delta\omega - P_0(\xi_N)'\omega_0 \\ & - N_0\Delta\Lambda - P_0\xi^N\Lambda_0 - \Delta N'\Delta\omega - \Delta N\Delta\Lambda + I_{N_1} + I_{N_2} \end{aligned}$$

Notice that: $N_0\Delta\Lambda = P_0\begin{pmatrix}0\\1\end{pmatrix}\Delta\Lambda$ therefore:

$$\begin{aligned} E_{N_1} =& E_{N_0} + P_0\Lambda_0\xi^N + E_{R_0}\xi^N - N_0'\Delta\omega - P_0(\xi_N)'\omega_0 - P_0\begin{pmatrix}0\\\Delta\Lambda\end{pmatrix} \\ & - P_0\xi^N\Lambda_0 - \Delta N'\Delta\omega - \Delta N\Delta\Lambda + I_{N_1} + I_{N_2} \end{aligned}$$

Defining:

$$-\eta^N = P_0^{-1}\left(E_{N_0} - N_0'\Delta\omega + I_{N_1}\right)$$

We have:

$$\begin{aligned} E_{N_1} =& E_{R_0}\xi^N + P_0\left[-\eta^N + \Lambda_0\xi^N - (\xi_N)'\omega_0 - \begin{pmatrix}0\\\Delta\Lambda\end{pmatrix} - \xi^N\Lambda_0\right] \\ & - \Delta N'\Delta\omega - \Delta N\Delta\Lambda + I_{N_2} \end{aligned}$$

And obtain the cohomological equations:

$$\eta^N = \Lambda_0\xi^N - (\xi^N)'\omega_0 - \begin{pmatrix}0\\\Delta\Lambda\end{pmatrix} - \xi^N\Lambda_0 \iff$$

$$\begin{pmatrix}\eta^{N,L}\\\eta^{N,N}\end{pmatrix} = \begin{pmatrix}0 & 0\\0 & \Lambda_0\end{pmatrix}\begin{pmatrix}\xi^{N,L}\\\xi^{N,N}\end{pmatrix} - \begin{pmatrix}(\xi^{N,L})'\\(\xi^{N,N})'\end{pmatrix}\omega_0 - \begin{pmatrix}0\\\Delta\Lambda\end{pmatrix} - \begin{pmatrix}\xi^{N,L}\\\xi^{N,N}\end{pmatrix}\Lambda_0 \iff$$

$$\eta^{N,L} = -(\xi^{N,L})'\omega_0 - \xi^{N,L}\Lambda_0 \tag{2.7}$$

$$\eta^{N,N} = \Lambda_0\xi^{N,N} - (\xi^{N,N})'\omega_0 - \Delta\Lambda - \xi^{N,N}\Lambda_0 \tag{2.8}$$

Finding the solutions $\xi^{N,L}$ and $\xi^{N,N}$ (notice the average $\left\langle\xi^{N,N}\right\rangle$ is also a free term that will be fixed by the normal condition), we obtain the expression for the Floquet error in terms of the previous error:

$$\boxed{E_{N_1} = E_{R_0}\xi_N - \Delta N'\Delta\omega - \Delta N\Delta\Lambda + \int_0^1 D^2X(K_0 + s\Delta K)[\Delta K, \Delta N]\,ds} \tag{2.9}$$

# 3 Conceptual Background

In this section we will introduce the mathematical concepts and tools used in this project along with some results that will be useful later on.

## 3.1 Vector and Matrix Norms

- For a constant vector $v \in \mathbb{C}^m$, we consider the supremum norm:

$$|v| = \max_{1 \leq i \leq m} |v_i|$$

- For a constant matrix $A \in \mathbb{C}^{m \times n}$, we will consider its supremum norm as

$$|A| = \sup_{1 \leq i \leq m} \sum_{1 \leq j \leq n} |a_{i,j}|$$

This norm is compatible with the vector supremum norm and have the following properties:

- For $A \in \mathbb{C}^{m \times n}, v \in \mathbb{C}^n$

$$|Av| \leq |A||v|$$

- For $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{n \times p}$

$$|AB| \leq |A||B|$$

Moreover, to bound inverse matrices we will use:

**Neumann series**: *If a matrix $A \in \mathbb{C}^{m \times m}$ holds $|A| < 1$, then*

$$\left|(Id - A)^{-1}\right| \leq \frac{1}{1 - |A|}$$

**Lemma 1.** *Given an invertible matrix $A_* \in \mathbb{C}^{m \times m}$ with known inverse $A_*^{-1}$ and a matrix $A \in \mathbb{C}^{m \times m}$ such that*

$$|A - A_*| < r_A < (\left|A_*^{-1}\right|)^{-1}$$

*Then, A is invertible and*

$$\left|A^{-1}\right| < C_{A^{-1}} := \frac{\left|A_*^{-1}\right|}{1 - \left|A_*^{-1}\right| r_A}$$

*Proof.* We can write

$$|A| \leq |A_*| + |A - A_*| < r_A + |A_*| < C_{A^{-1}}$$

And the norm of A is bounded. Also,notice:

$$\left|A_*^{-1}(A - A_*)\right| \leq \left|A_*^{-1}\right||A - A_*| < \left|A_*^{-1}\right| r_A < 1$$

Therefore, we can write:

$$A = A_* + (A - A_*) = A_*(Id + A_*^{-1}(A - A_*))$$

And by the previous inequality, we can apply Neumann series:

$$\left|A^{-1}\right| \leq \left|A_*^{-1}\right|\left|(Id + A_*^{-1}(A - A_*))^{-1}\right| \leq \frac{\left|A_*^{-1}\right|}{1 - \left|A_*^{-1}\right||A - A_*|} < \frac{\left|A_*^{-1}\right|}{1 - \left|A_*^{-1}\right| r_A} = C_{A^{-1}}$$

□

- We will also work with the operator $Vec(A) : \mathbb{C}^{m\times n} \longrightarrow \mathbb{C}^{mn}$ as:

$$Vec(A) = \begin{pmatrix} a_{11} \\ a_{21} \\ ... \\ a_{m1} \\ a_{12} \\ ... \\ a_{mn} \end{pmatrix}$$

Because $Vec(A)$ has the structure of a vector, we have:

$$|Vec(A)| = \sup_{1\leq i\leq m, 1\leq j\leq n} |a_{i,j}|$$

And the following result:

**Proposition 1.** *For a matrix $A \in \mathbb{C}^{m\times n}$:*

$$|Vec(A)| \leq |A| \leq n|Vec(A)|$$

*Proof.*

$$\begin{aligned} |A| &= \sup_{1\leq i\leq m} \sum_{j=1}^{n} |a_{ij}| = \sup_{1\leq i\leq m} (|a_{i1}| + ... + |a_{in}|) \leq \sup\ |a_{i1}| + ... + \sup\ |a_{in}| \\ &= \sum_{1\leq j\leq n} \sup_{1\leq i\leq m} |a_{ij}| \leq n \sup_{1\leq i\leq m, 1\leq j\leq n} |a_{ij}| = n|Vec(A)| \end{aligned}$$

Moreover

$$|Vec(A)| = \sup_{1\leq i\leq m, 1\leq j\leq n} |a_{ij}| \leq \sup_{1\leq i\leq m} \sum_{1\leq j\leq n} |a_{ij}| \leq |A|$$

□

- Additionally, we will consider *Kronecker product* presented along its properties and some important results in (pg 24) [1].

$$\begin{aligned} \otimes : \quad \mathbb{C}^{m\times n} \times \mathbb{C}^{p\times q} &\longrightarrow \mathbb{C}^{(mp)\times(nq)} \\ A \otimes B &\longrightarrow \begin{pmatrix} a_{11}B & a_{12}B & ... & a_{1n}B \\ a_{21}B & a_{22}B & ... & a_{2n}B \\ \vdots & \vdots & & \vdots \\ a_{m1}B & a_{m2}B & ... & a_{mn}B \end{pmatrix} \end{aligned}$$

A specific result we will be using is (Lemma 3.2.10) [1] which states that for $X, C \in \mathbb{C}^{m\times m}$:

$$Vec(XC - CX) = (C^T \otimes I_m - I_m \otimes C)Vec(X) \tag{3.1}$$

Moreover, as a bound we will use:

**Proposition 2.** *For $C \in \mathbb{C}^{m\times m}$*

$$\begin{aligned}\left|C^T \otimes I_m\right| &\leq m|C| \\ |I_m \otimes C| &\leq |C|\end{aligned}$$

*Proof.*

$$\left|C^T \otimes I_m\right| \leq m\left|Vec(C^T \otimes I_m)\right| = m|Vec(C)| \leq m|C|$$

$$|I_m \otimes C| \leq \left|\begin{pmatrix} C & 0 & ... & 0 \\ 0 & C & ... & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & ... & C \end{pmatrix}\right| \leq |C|$$

□

### 3.2 Extension to Matrix-Valued Functions

- When having a function $f \in C^{\omega,0}(\mathbb{T}_\rho, \mathbb{C})$, its supremum norm is defined as

$$\|f\|_\rho = \sup_{\theta\in\mathbb{T}_\rho} |f(\theta)|$$

- For a function $A \in C^{\omega,0}(\mathbb{T}_\rho, \mathbb{C}^{m\times n})$, we will consider its supremum norm as:

$$\|A\|_\rho = \sup_{1\leq i\leq m} \sum_{1\leq j\leq n} \|A_{i,j}\|_\rho \quad \text{where} \quad \|A_{i,j}\|_\rho = \sup_{\theta\in\mathbb{T}_\rho} |A_{i,j}(\theta)|$$

  With the property:

  – For $A \in C^{\omega,0}(\mathbb{T}_\rho, \mathbb{C}^{m\times n})$, and $B \in C^{\omega,0}(\mathbb{T}_\rho, \mathbb{C}^{n\times p})$

$$\|AB\|_\rho \leq \|A\|_\rho \|B\|_\rho$$

  **Obs 1.** *Lemma (1) can be also applied for matrix-valued functions by considering the inequality pointwise for all fixed $\theta \in \mathbb{T}_\rho$.*

- To bound derivatives of functions in the imaginary band while loosing some radius we will work with the following main result:

  **Cauchy inequality**: *For any $f \in C^{\omega,1}(\mathbb{T}_\rho, \mathbb{C}^{m\times,n})$:*

$$\|f'\|_{\rho-\delta} \leq \frac{1}{\delta}\|f\|_\rho$$

### 3.3 Previous Lemmas

Having defined the norms we will work with, in this part we see *Russmann* Lemma for bounding a specific sum with Fourier coefficients by the norm of the function. The one-dimensional case is first presented to develop a matrix-valued function result afterwards. These Lemmas will be useful to bound the norms of the solutions to the cohomological equations.

**Lemma 2** (Russmann)**.** *We consider a function $f \in C^{\omega,0}(\mathbb{T}_\rho, \mathbb{C})$. Then*

$$\sum_{k\in\mathbb{Z}} |\widehat{f}_k|^2 e^{4\pi|k|\rho} \leq 2\|f\|_\rho^2$$

*Proof.* We define $\forall |a| \leq \rho$, the analytic function

$$f_a(\theta) = f(\theta + ia)$$

Notice it is continue in $\bar{\mathbb{T}}_{\rho-|a|}$. Therefore, we consider the Fourier transform:

$$\widehat{(f_a)}_k = \int_0^1 f(\theta + ia) e^{-2\pi i k\theta} d\theta$$

Considering Bessel's inequality we have:

$$\sum_{k\in\mathbb{Z}} |\widehat{(f_a)}_k|^2 \leq \int_0^1 |f(\theta + ia)|^2 d\theta \leq \|f\|_\rho^2$$

If we now consider

$$g(a) \longrightarrow \widehat{(f_a)}_k e^{2\pi k a} = \int_0^1 f(\theta + ia) e^{-2\pi i k(\theta+ia)} d\theta$$

By Cauchy integral form, $g(a) = \widehat{f}_k$, meaning $g(a)$ si not depending on $a$, and:

$$\widehat{(f_a)}_k = \widehat{f}_k e^{-2\pi k a}$$

By Bessel's inequality:

$$\sum_{k\in\mathbb{Z}} \left|\widehat{f}_{a_k}\right|^2 = \sum_{k\in\mathbb{Z}} \left|\widehat{f}_k e^{-2\pi k a}\right|^2 = \sum_{k\in\mathbb{Z}} \left|\widehat{f}_k\right|^2 e^{-4\pi k a} \leq \|f\|_\rho^2 \quad \forall |a| \leq \rho$$

And finally, applying this result

$$\sum_{k\in\mathbb{Z}} \left|\widehat{f}_k\right|^2 e^{4\pi|k|\rho} \leq \sum_{k\leq 0} \left|\widehat{f}_k\right|^2 e^{-4\pi k\rho} + \sum_{k\geq 0} \left|\widehat{f}_k\right|^2 e^{4\pi k\rho} \leq \sum_{k\in\mathbb{Z}} \left|\widehat{f}_k\right|^2 e^{-4\pi k\rho} + \sum_{k\in\mathbb{Z}} \left|\widehat{f}_k\right|^2 e^{-4\pi k(-\rho)} \leq 2\|f\|_\rho^2$$

□

**Lemma 3** (Russmann generalized)**.** *We consider a function $A \in C^{\omega,0}(\mathbb{T}_\rho, \mathbb{C}^{m\times n})$ and its supremum norm. Then*

$$\sum_{k\in\mathbb{Z}} |\widehat{A_k}|^2 e^{4\pi|k|\rho} \leq 2nm\|A\|_\rho^2$$

*Proof.* By Lemma 2, we have

$$\sum_{k\in\mathbb{Z}} |(\widehat{a_{i,j}})_k|^2 e^{4\pi|k|\rho} \leq 2\|a_{i,j}\|_\rho^2$$

We write:

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} |\widehat{a_k}|^2 e^{4\pi|k|\rho} &= \sum_{k\in\mathbb{Z}} \left( \max_i \sum_j |(\widehat{a_{i,j}})_k| \right)^2 e^{4\pi|k|\rho} \underbrace{\leq}_{C.S} \sum_{k\in\mathbb{Z}} \max_i \left( \sum_j |(\widehat{a_{i,j}})_k|^2 \sum_j 1^2 \right) e^{4\pi|k|\rho} \\
&= \sum_{k\in\mathbb{Z}} \max_i \left( n \sum_j |(\widehat{a_{i,j}})_k|^2 \right) e^{4\pi|k|\rho} \leq n \sum_{k\in\mathbb{Z}} \sum_i \sum_j |(\widehat{a_{i,j}})_k|^2 e^{4\pi|k|\rho} \\
&\leq n \sum_i \sum_j \sum_{k\in\mathbb{Z}} |(\widehat{a_{i,j}})_k|^2 e^{4\pi|k|\rho} \underbrace{\leq}_{Lemma(2)} 2n \sum_i \sum_j \|a_{i,j}\|_\rho^2 \\
&\leq 2n \sum_i \max_i \sum_j \|a_{i,j}\|_\rho^2 \leq 2n \sum_i \left( \max_i \sum_j \|a_{i,j}\|_\rho \right)^2 \\
&\leq 2n \sum_i \|A\|_\rho^2 \leq 2nm\|A\|_\rho^2 \leq 2nm\|A\|_\rho^2
\end{aligned}$$

□

# 4 Framework

To avoid ressonances with the eigenvalues of the Floquet matrix and be able to solve the cohomological equations (2.4), (2.5), (2.7), (2.8) we must introduce definitions for a non-degenerate tuple to work on.

## 4.1 Definitions of Non-Degeneracy

**Definition 1.** *We consider*

- $K \in \mathcal{C}^{\omega,1}(\mathbb{T}_\rho, \mathbb{C}^{n\times 1})$
- $N \in \mathcal{C}^{\omega,1}(\mathbb{T}_\rho, \mathbb{C}^{n\times(n-1)})$
- $P \in \mathcal{C}^{\omega,0}(\mathbb{T}_\rho, \mathbb{C}^{n\times n}) \quad as \quad P = \begin{pmatrix} K' & N \end{pmatrix}$

*Then, we say* ***the tuple*** $(K, N)$ *is* ***non-degenerate*** *in* $\mathbb{T}_\rho$ *if*

- $P(\theta)$ *is invertible for any* $\theta \in \mathbb{T}_\rho$

**Definition 2.** *We consider*

- $\omega \in \mathbb{R}$
- $\Lambda \in \mathbb{R}^{(n-1)\times(n-1)}$

*Then, we say* ***the tuple*** $(\omega, \Lambda)$ *is* ***non-degenerate in*** $\mathbb{T}_\rho$ *if*

- *Exists* $\Lambda^{-1}$
- $\omega \neq 0$
- $\sqrt{\sum_{k\in\mathbb{Z}^*} |(\Lambda - 2\pi ik\omega\ I_{n-1})^{-1}|^2}$ *is finite*
- $\sqrt{\sum_{k\in\mathbb{Z}^*} \left|\left(I_{n-1}\otimes\Lambda - \Lambda^T\otimes I_{n-1} - 2\pi ik\omega\ I_{(n-1)^2}\right)^{-1}\right|^2}$ *is finite*

**Definition 3.** ***The tuple*** $(K, \omega, N, \Lambda)$ *is* ***non-degenerate*** *if* $(K, N)$ *and* $(\omega, \Lambda)$ *are non-degenerate.*

**Proposition 3.** *If the eigenvalues of* $\Lambda$ *and* $I_{n-1}\otimes\Lambda - \Lambda^T\otimes I_{n-1}$ *are different from* $2\pi ik\omega$ *for all* $k \in \mathbb{Z}$, $\omega \neq 0$ *and exists* $\Lambda^{-1}$ *then* ($\Lambda$,$\omega$) *is non-degenerate with bounds:*

$$\sqrt{\sum_{k\in\mathbb{Z}^*} |(\Lambda - 2\pi ik\omega\ I_{n-1})^{-1}|^2} \leq C_{1,\Sigma}(\omega,\Lambda)$$
$$\sqrt{\sum_{k\in\mathbb{Z}^*} \left|\left(I_{n-1}\otimes\Lambda - \Lambda^T\otimes I_{n-1} - 2\pi ik\omega\ I_{(n-1)^2}\right)^{-1}\right|^2} \leq C_{2,\Sigma}(\omega,\Lambda)$$

*where:*

$$C_{1,\Sigma}(\omega,\Lambda) = C_1(\omega,\Lambda)\frac{1}{2\sqrt{3}|\omega|} \qquad C_1(\omega,\Lambda) = \sup_{k\in\mathbb{Z}^*}\left|\left(\frac{\Lambda}{2\pi ik\omega} - I_{n-1}\right)^{-1}\right|$$
$$C_{2,\Sigma}(\omega,\Lambda) = C_2(\omega,\Lambda)\frac{1}{2\sqrt{3}|\omega|} \qquad C_2(\omega,\Lambda) = \sup_{k\in\mathbb{Z}^*}\left|\left(\frac{I_{n-1}\otimes\Lambda - \Lambda^T\otimes I_{n-1}}{2\pi ik\omega} - I_{(n-1)^2}\right)^{-1}\right|$$

*Proof.*

$$\sqrt{\sum_{k\in\mathbb{Z}^*}|(\Lambda-2\pi ik\omega I_{n-1})^{-1}|^2}\leq\sqrt{\sum_{k\in\mathbb{Z}^*}|(2\pi ik\omega)^{-1}|^2\left|\left(\frac{\Lambda}{2\pi ik\omega}-I_{n-1}\right)^{-1}\right|^2}\leq C_1(\omega,\Lambda)\sqrt{\sum_{k\in\mathbb{Z}^*}\frac{1}{4\pi^2k^2\omega^2}}$$

Recall $\sum_{k=1}^{\infty}\frac{1}{k^2}=\frac{\pi^2}{6}$ and:

$$\sqrt{\sum_{k\in\mathbb{Z}^*}|(\Lambda-2\pi ik\omega I_{n-1})^{-1}|^2}\leq C_1(\omega,\Lambda)\sqrt{\frac{1}{4\pi^2\omega^2}2\frac{\pi^2}{6}}=C_1(\omega,\Lambda)\frac{1}{2\sqrt{3}|\omega|}$$

Similarly, we have:

$$\sqrt{\sum_{k\in\mathbb{Z}^*}\left|\left(I_{n-1}\otimes\Lambda-\Lambda^T\otimes I_{n-1}-2\pi ik\omega\ I_{(n-1)^2}\right)^{-1}\right|^2}\leq C_2(\omega,\Lambda)\frac{1}{2\sqrt{3}|\omega|}=C_{2,\Sigma}(\omega,\Lambda)$$

□

**Remark:** The conditions of non-resonance for proposition (3) are equivalent to satisfying a finite number of conditions. That is because there exists $|k|<k_0$ for which the eigenvalues will be different enough.

For example, conditions

$$\left|(\Lambda-2\pi ik\omega I_{n-1})^{-1}\right|<C_{\Lambda_k^{-1}}$$

for $|k|<k_0$ where $k_0$ holds

$$\frac{|\Lambda|}{2\pi|k_0||\omega|}<\frac{1}{2}$$

Indeed, for $|k|\geq k_0$ the condition is fulfilled without further assumptions:

$$\begin{aligned}\left|(\Lambda-2\pi ik\omega I_{n-1})^{-1}\right|&=\left|\frac{1}{2\pi ik\omega}\left(\frac{\Lambda}{2\pi ik\omega}-I_{n-1}\right)^{-1}\right|\leq\frac{1}{2\pi i|k||\omega|}\left|\left(\frac{\Lambda}{2\pi ik\omega}-I_{n-1}\right)^{-1}\right|\\&\underbrace{\leq}_{Neumann}\frac{1}{2\pi|k||\omega|}\frac{1}{1-\frac{|\Lambda|}{2\pi|k||\omega|}}\leq\frac{|\Lambda|}{2\pi|k||\omega_0|}\frac{2}{|\Lambda|}\leq\frac{1}{|\Lambda|}<\infty\end{aligned}$$

The case for the eigenvalues of $I_{n-1}\otimes\Lambda-\Lambda^T\otimes I_{n-1}$ is analogous.

These definitions of non-degeneracy allows us to state a result for tuples close enough to a non-degenerate tuple to also be non-degenerate.

### 4.2 Lemmas of Non-Degeneracy

**Super-Lemma 1.** *Given a non-degenerate tuple $(K_*, N_*)$ in $\mathbb{T}_{\rho_*}$, we denote*

$$m_{K'_*} = \int_0^1 K'_*(\theta)^T K'_*(\theta) d\theta \qquad\qquad m_{N_*} = \int_0^1 N_*(\theta)^T N_*(\theta) d\theta$$

*And consider constants $r_K, r_{K'}, r_N, r_{N'}$ fulfilling:*

$$r_{K'} < \frac{m_{K'_*}}{\|(K'_*)^T\|_\rho} \qquad\qquad r_N < \frac{\left|m_{N_*}^{-1}\right|^{-1}}{\|(N_*)^T\|_\rho} \qquad\qquad r_P := r_{K'} + r_N < \left\|P_*(\theta)^{-1}\right\|_\rho^{-1}$$

*Then, any $(K, N)$ such that*

$$\|K - K_*\|_\rho < r_K \qquad\qquad \|N - N_*\|_\rho < r_N$$
$$\left\|K' - K'_*\right\|_\rho < r_{K'} \qquad\qquad \left\|N' - N'_*\right\|_\rho < r_{N'}$$

*is non-degenerate in $\mathbb{T}_\rho$ for $\rho < \rho_*$ and*

1. $\|K\|_\rho \le C_K = r_K + \|K_*\|_{\rho_*}$
2. $\left\|K'\right\|_\rho \le C_{K'} = r_{K'} + \|K'_*\|_{\rho_*}$
3. $\|N\|_\rho \le C_N = r_N + \|N_*\|_{\rho_*}$
4. $\left\|N'\right\|_\rho \le C_{N'} = r_{N'} + \|N'_*\|_{\rho_*}$
5. $\left\|P(\theta)^{-1}\right\|_\rho \le C_{P^{-1}} = \dfrac{\left\|P_*(\theta)^{-1}\right\|_\rho}{1 - \|P_*(\theta)^{-1}\|_\rho \, r_P}$
6. $\left|\left(\displaystyle\int_0^1 K'_*(\theta)^T K'(\theta) d\theta\right)^{-1}\right| \le C_{m_{K'_*}} = \dfrac{1}{m_{K'_*} - \|(K'_*)^T\|_\rho r_{K'}}$
7. $\left|\left(\displaystyle\int_0^1 N_*(\theta)^T N(\theta) d\theta\right)^{-1}\right| \le C_{m_{N_*}} = \dfrac{\left|m_{N_*}^{-1}\right|}{1 - \left|m_{N_*}^{-1}\right| \|(N_*)^T\|_\rho r_N}$

*Proof.*

*1.* $\|K\|_\rho = \|K - K_* + K_*\|_\rho \le \|K - K_*\|_\rho + \|K_*\|_\rho < r_K + \|K_*\|_{\rho_*} = C_K$

*2,3,4.* Similarly to *1*

*5.* Proven using Observation (1)

*6.* Notice that:

$$\left|\int_0^1 K'_*(\theta)^T K'(\theta) d\theta - \int_0^1 K'_*(\theta)^T K'_*(\theta) d\theta\right| = \left|\int_0^1 K'_*(\theta)^T (K'(\theta) - K'_*(\theta)) d\theta\right|$$
$$\le \left|(K'_*)^T\right| \left|K' - K'_*\right| \le \left|(K'_*)^T\right| r_{K'} \le \left|m_{K'_*}\right|$$

And we can apply Lemma 1

*7.* Similarly to *6* □

**Super-Lemma 2.** *Given a non-degenerate tuple $(\omega_*, \Lambda_*)$ in $\mathbb{T}_{\rho_*}$, we consider constants $r_\omega, r_\Lambda$ such that*

$$r_\omega < |\omega_*|, \quad r_\Lambda < \left|\Lambda_*^{-1}\right|^{-1}$$

*fulfilling*

$$C_1(\Lambda_*, \omega_*)\left(\frac{r_\Lambda}{2\pi|\omega_*|} + \frac{r_\omega}{|\omega_*|}\right) < 1 \qquad\qquad C_2(\Lambda_*, \omega_*)\left(\frac{n\ r_\Lambda}{2\pi|\omega_*|} + \frac{r_\omega}{|\omega_*|}\right) < 1$$

*Then, any $(\omega, \Lambda)$ satisfying*

$$|\omega - \omega_*| < r_\omega$$
$$|\Lambda - \Lambda_*| < r_\Lambda$$

*is non-degenerate in $\mathbb{T}_\rho$ for $\rho < \rho_*$ and*

1. $|\omega| \leq C_\omega = r_\omega + |\omega_*|$
2. $|\Lambda| \leq C_\Lambda = r_\Lambda + |\Lambda_*|$
3. $\left|\omega^{-1}\right| \leq C_{\omega^{-1}} = \dfrac{1}{|\omega_*| - r_\omega}$
4. $\left|\Lambda^{-1}\right| \leq C_{\Lambda^{-1}} = \dfrac{\left|\Lambda_*^{-1}\right|}{1 - \left|\Lambda_*^{-1}\right|\ r_\Lambda}$
5. $\sqrt{\displaystyle\sum_{k\in\mathbb{Z}^*} \left|(\Lambda - 2\pi ik\omega\ I_{n-1})^{-1}\right|^2} \leq C_{S_1} = \dfrac{C_{1,\Sigma}(\Lambda_*, \omega_*)}{1 - C_1(\Lambda_*, \omega_*)\left(\frac{r_\Lambda}{2\pi|\omega_*|} + \frac{r_\omega}{|\omega_*|}\right)}$
6. $\sqrt{\displaystyle\sum_{k\in\mathbb{Z}^*} \left|(\Lambda^T \otimes I_{n-1} - I_{n-1} \otimes \Lambda - 2\pi ik\omega\ I_{(n-1)^2})^{-1}\right|^2} \leq C_{S_2} = \dfrac{C_{2,\Sigma}(\Lambda_*, \omega_*)}{1 - C_2(\Lambda_*, \omega_*)\left(\frac{nr_\Lambda}{2\pi|\omega_*|} + \frac{r_\omega}{|\omega_*|}\right)}$

*Proof.*

*1.* $|\omega| = |\omega - \omega_* + \omega_*| \leq |\omega - \omega_*| + |\omega_*| < r_\omega + |\omega_*| = C_\omega$

*2.* Similarly to 1

*3,4.* Proven using Lemma 1

*5.* Because $(\omega_*, \Lambda_*)$ is non-degenerate, we know:

$$\sqrt{\sum_{k\in\mathbb{Z}^*} \left|(\Lambda_* - 2\pi ik\omega_* I_{n-1})^{-1}\right|^2} < C_{1,\Sigma}(\Lambda_*, \omega_*)$$

We can write:

$$\begin{aligned}
\Lambda - 2\pi ik\omega\ I_{n-1} &= (\Lambda - \Lambda_*) + \Lambda_* - 2\pi ik\left((\omega - \omega_*) + \omega_*\right)\ I_{n-1} \\
&= \Lambda_* - 2\pi ik\omega_*\ I_{n-1} + (\Lambda - \Lambda_*) - 2\pi ik(\omega - \omega_*)\ I_{n-1} \\
&= (\Lambda_* - 2\pi ik\omega_*\ I_{n-1})\left(I_{n-1} + (\Lambda_* - 2\pi ik\omega_*\ I_{n-1})^{-1}((\Lambda - \Lambda_*) - 2\pi ik(\omega - \omega_*)\ I_{n-1})\right)
\end{aligned}$$

Therefore

$$\sqrt{\sum_{k\in\mathbb{Z}^*}|(\Lambda-2\pi ik\omega\ I_{n-1})^{-1}|^2}$$
$$=\sqrt{\sum_{k\in\mathbb{Z}^*}\left|(\Lambda_*-2\pi ik\omega_*\ I_{n-1})^{-1}\left(I_{n-1}+(\Lambda_*-2\pi ik\omega_*\ I_{n-1})^{-1}((\Lambda-\Lambda_*)-2\pi ik(\omega-\omega_*)\ I_{n-1})\right)^{-1}\right|^2}$$

Notice we can apply Neumann because, by Proposition (3) and considering the hypothesis

$$\left|\left(\frac{\Lambda_*-2\pi ik\omega_*\ I_{n-1}}{2\pi ik\omega_*}\right)^{-1}\frac{(\Lambda-\Lambda_*)-2\pi ik(\omega-\omega_*)\ I_{n-1}}{2\pi ik\omega_*}\right|$$
$$=\left|\left(\frac{\Lambda_*}{2\pi ik\omega_*}-I_{n-1}\right)^{-1}\right|\left|\left(\frac{\Lambda-\Lambda_*}{2\pi ik\omega_*}-\frac{\omega-\omega_*}{\omega_*}I_{n-1}\right)\right|$$
$$\leq C_1(\Lambda_*,\omega_*)\left(\frac{|\Lambda-\Lambda_*|}{2\pi 1|\omega_*|}+\frac{|\omega-\omega_*|}{|\omega_*|}\right)\leq C_1(\Lambda_*,\omega_*)\left(\frac{r_\Lambda}{2\pi|\omega_*|}+\frac{r_\omega}{|\omega_*|}\right)<1$$

Hence:

$$\sqrt{\sum_{k\in\mathbb{Z}^*}|(\Lambda-2\pi ik\omega\ I_{n-1})^{-1}|^2}$$
$$\leq\sqrt{\sum_{k\in\mathbb{Z}^*}|(\Lambda_*-2\pi ik\omega_*\ I_{n-1})^{-1}|^2\left(\frac{1}{1-|(\Lambda_*-2\pi ik\omega_*)^{-1}||(\Lambda-\Lambda_*)-2\pi ik(\omega-\omega_*)I_{n-1}|}\right)^2}$$
$$\leq C_{1,\Sigma}(\Lambda_*,\omega_*)\frac{1}{1-C_1(\Lambda_*,\omega_*)\left(\frac{r_\Lambda}{2\pi|\omega_*|}+\frac{r_\omega}{|\omega_*|}\right)}=\frac{C_{1,\Sigma}(\Lambda_*,\omega_*)}{1-C_1(\Lambda_*,\omega_*)\left(\frac{r_\Lambda}{2\pi|\omega_*|}+\frac{r_\omega}{|\omega_*|}\right)}=C_{S_1}$$

*6.* Considering

$$\widehat{\Lambda}_*=\Lambda_*^T\otimes I_{n-1}-I_{n-1}\otimes\Lambda_*$$
$$\widehat{\Lambda}=\Lambda^T\otimes I_{n-1}-I_{n-1}\otimes\Lambda$$

The proof is analogous to the previous one.

By Proposition (2) we have

$$\left|(\Lambda-\Lambda_*)^T\otimes I_{n-1}\right|\leq(n-1)|\Lambda-\Lambda_*|$$
$$|I_{n-1}\otimes(\Lambda-\Lambda_*)|\leq|\Lambda-\Lambda_*|$$

Therefore:

$$\left|\widehat{\Lambda}-\widehat{\Lambda}_*\right|=n|\Lambda-\Lambda_*|$$

Hence, the condition for invertibility is fulfilled due to the hypothesis and:

$$\sqrt{\sum_{k\in\mathbb{Z}^*}\left|\left(\Lambda^T\otimes I_{n-1}-I_{n-1}\otimes\Lambda-2\pi ik\omega\ I_{(n-1)^2}\right)^{-1}\right|^2}\leq\frac{C_{2,\Sigma}(\Lambda_*,\omega_*)}{1-C_2(\Lambda_*,\omega_*)\left(\frac{nr_\Lambda}{2\pi|\omega_*|}+\frac{r_\omega}{|\omega_*|}\right)}=C_{S_2}$$

□

Note that (1) gives us the the theoretical tools to work with the phase and normal condition. This result is directly applied in *Iterative Lemma* (5.1). Moreover, Lemma (2) gives us the conditions to solve the cohomological equations.

### 4.3 Resolution of Cohomological Equations

**Lemma 4.** *Under the hypothesis of Super-Lemma (2), for all $\eta^{K,L} \in \mathcal{C}^{\omega,0}(\mathbb{T}_\rho, \mathbb{C}^{1\times 1})$ there exists a family of solutions $\xi^{K,L} \in \mathcal{C}^{\omega}(\mathbb{T}_\rho, \mathbb{C})$ and $\nu^{K,L} \in \mathbb{R}$ parametrized by $\langle \xi^{K,L} \rangle \in \mathbb{C}^{1\times 1}$ solving:*

$$\eta^{K,L} = -(\xi^{K,L})'\omega - \nu^{K,L}$$

*given by*

$$\nu^{K,L} = \nu^{K,L}\left(\langle \xi^{K,L} \rangle\right)$$
$$\xi^{K,L}(\theta) = \langle \xi^{K,L} \rangle + \widetilde{\xi^{K,L}}(\theta)$$

*where $\widetilde{\xi^{K,L}}$ is the unique zero-average solution of*

$$\widetilde{\eta^{K,L}} = -(\widetilde{\xi^{K,L}})'\omega$$

*Moreover,*

$$\left|\nu^{K,L}\right| \leq \left|(\widehat{\eta^{K,L}})_0\right|$$
$$\|\widetilde{\xi^{K,L}}\|_\rho \leq \mathcal{C}_\omega^{K,L} \left\|\eta^{K,L}\right\|_\rho \quad \textit{where} \quad \mathcal{C}_\omega^{K,L} = \frac{C_{\omega^{-1}}}{\sqrt{6}}$$

*Proof.* We can write the equation in Fourier series:

$$\sum_{k\in\mathbb{Z}} (\widehat{\eta^{K,L}})_k e^{2\pi i k\theta} = \nu^{K,L} - \sum_{k\in\mathbb{Z}^*} 2\pi i k\omega \ (\widehat{\xi^{K,L}})_k \ e^{2\pi i k\theta}$$

For $k = 0$ we have

$$\nu^{K,L} = -(\widehat{\eta^{K,L}})_0$$

And for $k \neq 0$:

$$(\widehat{\xi^{K,L}})_k = \frac{-1}{2\pi i k\omega} (\widehat{\eta^{K,L}})_k \quad \text{for} \quad k \neq 0$$

And the unique solution for the average-zero equation is:

$$\widetilde{\xi^{K,L}} = \sum_{k\in\mathbb{Z}^*} \frac{-1}{2\pi i k\omega} (\widehat{\eta^{K,L}})_k e^{2\pi i k\theta}$$

We bound the norm:

$$\left\|\widetilde{\xi^{K,L}}\right\|_\rho \leq \frac{1}{2\pi}\frac{1}{|\omega|} \sum_{k\in\mathbb{Z}^*} \frac{1}{|k|} \left|(\widehat{\eta^{K,L}})_k\right| e^{2\pi|k|\rho} \underbrace{\leq}_{C.S} \frac{1}{2\pi}\frac{1}{|\omega|} \sqrt{\sum_{k\in\mathbb{Z}^*} \left(\frac{1}{|k|}\right)^2 \sum_{k\in\mathbb{Z}^*} |(\widehat{\eta^{K,L}})_k|^2 e^{4\pi|k|\rho}}$$
$$\leq \frac{1}{2\pi|\omega|} \sqrt{\sum_{k\in\mathbb{Z}^*} \frac{1}{|k|^2} \sum_{k\in\mathbb{Z}} |(\widehat{\eta^{K,L}})_k|^2 e^{4\pi|k|\rho}}$$

By Russmann Lemma(3) and $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$:

$$\left\|\widetilde{\xi^{K,L}}\right\|_\rho \leq \frac{1}{2\pi|\omega|} \sqrt{2\frac{\pi^2}{6} 2\|\eta^{K,L}\|_\rho^2} \leq \frac{1}{|\omega|}\frac{1}{\sqrt{6}} \left\|\eta^{K,L}\right\|_\rho \leq \frac{C_{\omega^{-1}}}{\sqrt{6}} \|\eta^{K,L}\|_\rho = \mathcal{C}_\omega^{K,L} \|\eta^{K,L}\|_\rho$$

□

**Lemma 5.** *Under the hypothesis of Super-Lemma (2), for all $\eta^{K,N} \in \mathcal{C}^{\omega,0}(\mathbb{T}_\rho, \mathbb{C}^{(n-1)\times 1})$ there exists an unique solution $\xi^{K,N} \in \mathcal{C}^{\omega,1}(\mathbb{T}_\rho, \mathbb{C}^{(n-1)\times 1})$ solving*

$$\eta^{K,N} = \Lambda\xi^{K,N} - (\xi^{K,N})'\omega$$

*Moreover,*

$$\left\|\xi^{K,N}\right\|_\rho \leq \mathcal{C}^{K,N}_{\omega,\Lambda}\left\|\eta^{K,N}\right\|_\rho \quad \textit{where} \quad \mathcal{C}^{K,N}_{\omega,\Lambda} = C_{\Lambda^{-1}} + \sqrt{2}\sqrt{n-1}C_{S_1}$$

*Proof.* We can write

$$\sum_{k\in\mathbb{Z}^*} (\widehat{\eta^{K,N}})_k \; e^{2\pi ik\theta} = \Lambda \sum_{k\in\mathbb{Z}^*} (\widehat{\xi^{K,N}})_k \; e^{2\pi ik\theta} - \sum_{k\in\mathbb{Z}^*} 2\pi ik \;\; (\widehat{\xi^{K,N}})_k \; e^{2\pi ik\theta}\omega \implies$$

$$(\widehat{\eta^{K,N}})_k = \Lambda(\widehat{\xi^{K,N}})_k - 2\pi ik \;\; (\widehat{\xi^{K,N}})_k\omega = (\Lambda - 2\pi ik\omega)(\widehat{\xi^{K,N}})_k \implies$$

$$(\widehat{\xi^{K,N}})_k = (\Lambda - 2\pi ik\omega)^{-1}(\widehat{\eta^{K,N}})_k$$

Having:

$$\xi^{K,N} = \sum_{k\in\mathbb{Z}} (\Lambda - 2\pi ik\omega)^{-1}(\widehat{\eta^{K,N}})_k e^{2\pi ik\theta}$$

For bounding the norm, we separate the average and the oscillatory part:

$$\left\|\langle\xi^{K,N}\rangle\right\|_\rho \leq \left\|\Lambda^{-1}(\widehat{\eta^{K,N}})_0\right\|_\rho \leq \left|\Lambda^{-1}\right|\left\|(\widehat{\eta^{K,N}})_0\right\|_\rho \leq C_{\Lambda^{-1}}\left\|\eta^{K,N}\right\|_\rho$$

$$\left\|\widetilde{\xi^{K,N}}\right\|_\rho \leq \sum_{k\in\mathbb{Z}^*} \left|(\Lambda - 2\pi ik\omega)^{-1}\right|\left|(\widehat{\eta^{K,N}})_k\right| e^{2\pi|k|\rho} \underbrace{\leq}_{C.S} \sqrt{\sum_{k\in\mathbb{Z}^*} \left|(\Lambda - 2\pi ik\omega)^{-1}\right|^2}\sqrt{\sum_{k\in\mathbb{Z}^*}\left|(\widehat{\eta^{K,N}})_k\right|^2 e^{4\pi|k|\rho}}$$

$$\underbrace{\leq}_{\text{Russmann (3)}} \quad C_{S_1}\sqrt{2(n-1)\|\eta^{K,N}\|^2_\rho} = \sqrt{2}\sqrt{n-1}C_{S_1}\left\|\eta^{K,N}\right\|_\rho$$

Finally:

$$\begin{aligned}\left\|\xi^{K,N}\right\|_\rho &= \left\|\langle\xi^{K,N}\rangle + \widetilde{\xi^{K,N}}\right\|_\rho \leq \left\|\langle\xi^{K,N}\rangle\right\|_\rho + \left\|\widetilde{\xi^{K,N}}\right\|_\rho \leq \left(C_{\Lambda^{-1}} + \sqrt{2}\sqrt{n-1}C_{S_1}\right)\|\eta^{K,N}\|_\rho \\ &= \mathcal{C}^{K,N}_{\omega,\Lambda}\|\eta^{K,N}\|_\rho\end{aligned}$$

□

**Lemma 6.** *Under the hypothesis of Super-Lemma (2), for all $\eta^{N,L} \in \mathcal{C}^{\omega,0}(\mathbb{T}_\rho, \mathbb{C}^{1\times(n-1)})$ there exists an unique solution $\xi^{N,L} \in \mathcal{C}^{\omega,1}(\mathbb{T}_\rho, \mathbb{C}^{1\times(n-1)})$ solving*

$$\eta^{N,L} = -\xi^{N,L}\Lambda - (\xi^{N,L})'\omega$$

*Moreover,*

$$\left\|\xi^{N,L}\right\|_\rho \leq \mathcal{C}^{N,L}_{\omega,\Lambda}\left\|\eta^{N,L}\right\|_\rho \quad \textit{where} \quad \mathcal{C}^{N,L}_{\omega,\Lambda} = C_{\Lambda^{-1}} + \sqrt{2}\sqrt{n-1}C_{S_1}$$

*Proof.* We can write:

$$\sum_{k\in\mathbb{Z}}(\widehat{\eta^{N,L}})_k e^{2\pi ik\theta} = -\sum_{k\in\mathbb{Z}}(\widehat{\xi^{N,L}})_k e^{2\pi ik\theta}\Lambda - \sum_{k\in\mathbb{Z}}(\widehat{\xi^{N,L}})_k 2\pi ik\ e^{2\pi ik\theta}\omega = \sum_{k\in\mathbb{Z}} -(\widehat{\xi^{N,L}})_k(\Lambda + 2\pi ik\omega)e^{2\pi ik\theta}$$

And:

$$(\widehat{\xi^{N,L}})_k = -(\widehat{\eta^{N,L}})_k(\Lambda + 2\pi ik\omega)^{-1}$$

Having:

$$\xi^{N,L} = \sum_{k\in\mathbb{Z}} -(\widehat{\eta^{N,L}})_k(\Lambda + 2\pi ik\omega)^{-1}e^{2\pi ik\theta}$$

For bounding the norm, we consider separately the average and the oscillatory part:

$$\left\|\langle\xi^{N,L}\rangle\right\|_\rho \le \left\|(\widehat{\eta^{N,L}})_0\Lambda^{-1}\right\|_\rho \le \left\|(\widehat{\eta^{N,L}})_0\right\|_\rho\left|\Lambda^{-1}\right| \le C_{\Lambda^{-1}}\left\|\eta^{N,L}\right\|_\rho$$

$$\left\|\widetilde{\xi^{N,L}}\right\|_\rho \le \sum_{k\in\mathbb{Z}^*}\left|(\widehat{\eta^{N,L}})_k\right|\left|(\Lambda + 2\pi ik\omega)^{-1}\right|e^{2\pi|k|\rho} \underbrace{\le}_{C.S} \sqrt{\sum_{k\in\mathbb{Z}^*}|(\Lambda + 2\pi ik\omega)^{-1}|^2}\sqrt{\sum_{k\in\mathbb{Z}^*}\left|(\widehat{\eta^{N,L}})_k\right|^2 e^{4\pi|k|\rho}}$$

Notice that, because $k\in\mathbb{Z}$, by symmetry

$$\sum_{k\in\mathbb{Z}^*}\left|(\Lambda + 2\pi ik\omega)^{-1}\right| = \sum_{k\in\mathbb{Z}^*}\left|(\Lambda - 2\pi ik\omega)^{-1}\right|$$

Hence, considering Russmann Lemma (3)

$$\left\|\widetilde{\xi^{N,L}}\right\|_\rho \le C_{S_1}\sqrt{2(n-1)\|\eta^{N,L}\|_\rho^2} = \sqrt{2}\sqrt{n-1}C_{S_1}\left\|\eta^{N,L}\right\|_\rho$$

Therefore:

$$\begin{aligned}\left\|\xi^{N,L}\right\|_\rho &= \left\|\langle\xi^{N,L}\rangle + \widetilde{\xi^{N,L}}\right\|_\rho \le \left\|\langle\xi^{N,L}\rangle\right\|_\rho + \left\|\widetilde{\xi^{N,L}}\right\|_\rho \le \left(C_{\Lambda^{-1}} + \sqrt{2}\sqrt{n-1}C_{S_1}\right)\|\eta^{K,N}\|_\rho \\ &= \mathcal{C}^{N,L}_{\omega,\Lambda}\|\eta^{K,N}\|_\rho\end{aligned}$$

□

**Lemma 7.** *Under the hypothesis of Super-Lemma (2), for all $\eta^{N,N}\in\mathcal{C}^{\omega,0}(\mathbb{T}_\rho,\mathbb{C}^{(n-1)\times(n-1)})$, there exists a family of solutions $\xi^{N,N}\in\mathcal{C}^{\omega,1}(\mathbb{T}_\rho,\mathbb{C}^{(n-1)\times(n-1)})$ and $\nu^{N,N}\in\mathbb{C}^{(n-1)\times(n-1)}$ parametrized by $\langle\xi^{N,N}\rangle\in\mathbb{C}^{(n-1)\times(n-1)}$ solving*

$$\eta^{N,N} = \Lambda\xi^{N,N} - \xi^{N,N}\Lambda - (\xi^{N,N})'\omega - \nu^{N,N}$$

*given by*

$$\begin{aligned}\nu^{N,N} &= \nu^{N,N}\left(\langle\xi^{N,N}\rangle\right)\\ \xi^{N,N}(\theta) &= \langle\xi^{N,N}\rangle + \widetilde{\xi^{N,N}}(\theta)\end{aligned}$$

*where $\widetilde{\xi^{N,N}}$ is the unique zero-average solution of*

$$\widetilde{\eta^{N,N}} = \Lambda\widetilde{\xi^{N,N}} - \widetilde{\xi^{N,N}}\Lambda - (\widetilde{\xi^{N,N}})'\omega$$

*Moreover*

$$\begin{aligned}&\left|\nu^{N,N}\right| \le \|\eta^{N,N}\|_\rho + 2C_\Lambda\left|\langle\xi^{N,N}\rangle\right|\\ &\left\|\widetilde{\xi^{N,N}}\right\|_\rho \le \mathcal{C}^{N,N}_{\omega,\Lambda}\left\|\eta^{N,N}\right\|_\rho \quad \textit{where} \quad \mathcal{C}^{N,N}_{\omega,\Lambda} = (n-1)^2\sqrt{2}C_{S_2}\end{aligned}$$

*Proof.* We write

$$\sum_{k\in\mathbb{Z}}(\widehat{\eta^{N,N}})_k e^{2\pi ik\theta} = \Lambda\sum_{k\in\mathbb{Z}}(\widehat{\xi^{N,N}})_k e^{2\pi ik\theta} - \sum_{k\in\mathbb{Z}}(\widehat{\xi^{N,N}})_k e^{2\pi ik\theta}\Lambda - \sum_{k\in\mathbb{Z}}(\widehat{\xi^{N,N}})_k 2\pi ike^{2\pi ik\theta}\omega$$

For $k=0$ we have:

$$(\widehat{\eta^{N,N}})_0 = \Lambda(\widehat{\xi^{N,N}})_0 - (\widehat{\xi^{N,N}})_0\Lambda - \nu^{N,N}$$

And therefore:

$$\nu^{N,N} = -(\widehat{\eta^{N,N}})_0 + \Lambda\left\langle\xi^{N,N}\right\rangle - \left\langle\xi^{N,N}\right\rangle\Lambda$$

When $k\neq 0$ we have:

$$(\widehat{\eta^{N,N}})_k = (\widehat{\xi^{N,N}})_k\Lambda - \Lambda(\widehat{\xi^{N,N}})_k - 2\pi ik\omega(\widehat{\xi^{N,N}})_k$$

Considering the operator $Vec(\cdot)$, we can write:

$$Vec((\widehat{\eta^{N,N}})_k) = Vec((\widehat{\xi^{N,N}})_k\Lambda - \Lambda(\widehat{\xi^{N,N}})_k) - Vec(2\pi ik\omega(\widehat{\xi^{N,N}})_k)$$

By equation (3.1), we have:

$$\begin{aligned}
&Vec((\widehat{\eta^{N,N}})_k) = \left(\Lambda^T\otimes I_{n-1} - I_{n-1}\otimes\Lambda\right)Vec((\widehat{\xi^{N,N}})_k) - 2\pi ik\omega Vec((\widehat{\xi^{N,N}})_k) \Longrightarrow\\
&Vec((\widehat{\xi^{N,N}})_k) = (\Lambda^T\otimes I_{n-1} - I_{n-1}\otimes\Lambda - 2\pi ik\omega I_{(n-1)^2})^{-1}\cdot Vec((\widehat{\eta^{N,N}})_k)
\end{aligned}$$

And finally:

$$Vec(\widetilde{\xi^{N,N}}) = \sum_{k\in\mathbb{Z}^*}(\Lambda^T\otimes I_{n-1} - I_{n-1}\otimes\Lambda - 2\pi ik\omega I_{(n-1)^2})^{-1}\cdot Vec((\widehat{\eta^{N,N}})_k)e^{2\pi ik\theta}$$

For the bounds of the norms:

$$\left|\nu^{N,N}\right| \leq \left|(\widehat{\eta^{N,N}})_0\right| + 2|\Lambda|\left|\left\langle\xi^{N,N}\right\rangle\right| \leq \|\eta^{N,N}\|_\rho + 2C_\Lambda\left|\left\langle\xi^{N,N}\right\rangle\right|$$

And

$$\begin{aligned}
\left\|Vec(\widetilde{\xi^{N,N}})\right\|_\rho &\leq \sum_{k\in\mathbb{Z}^*}\left|(\Lambda^T\otimes I_{n-1} - I_{n-1}\otimes\Lambda - 2\pi ik\omega I_{(n-1)^2})^{-1}\right|\left\|Vec((\widehat{\eta^{N,N}})_k)\right\|e^{2\pi|k|\rho}\\
&\leq \sum_{k\in\mathbb{Z}^*}\left|(\Lambda^T\otimes I_{n-1} - I_{n-1}\otimes\Lambda - 2\pi ik\omega I_{(n-1)^2})^{-1}\right|\left\|Vec((\widehat{\eta^{N,N}})_k\right\|)e^{2\pi|k|\rho}\\
&\underbrace{\leq}_{C.S}\sqrt{\sum_{k\in\mathbb{Z}^*}\left|(\Lambda^T\otimes I_{n-1} - I_{n-1}\otimes\Lambda - 2\pi ik\omega I_{(n-1)^2})^{-1}\right|^2}\sqrt{\sum_{k\in\mathbb{Z}^*}\left\|Vec((\widehat{\eta^{N,N}})_k)\right\|^2 e^{4\pi|k|\rho}}\\
&\underbrace{\leq}_{\text{Lemma (3)}} C_{S_2}\sqrt{2(n-1)^2\|Vec(\eta^{N,N})\|_\rho^2} = (n-1)\sqrt{2}C_{S_2}\left\|Vec(\eta^{N,N})\right\|_\rho
\end{aligned}$$

Therefore:

$$\left\|\widetilde{\xi^{N,N}}\right\|_\rho \leq (n-1)\left\|Vec(\widetilde{\xi^{N,N}})\right\|_\rho \leq (n-1)^2\sqrt{2}C_{S_2}\left\|\eta^{N,N}\right\|_\rho = \mathcal{C}_{\omega,\Lambda}^{N,N}\left\|\eta^{N,N}\right\|_\rho$$

□

# 5 Iterative Lemma

All the previous results allow us to formulate an iterative Lemma for the computation of the next-step error on the Newton method with an explicit formulation depending on the initial error.

## 5.1 Statement of the Lemma

**Iterative Lemma.** *Given a non-degenerate tuple* $(K_*, \omega_*, N_*, \Lambda_*)$*, an open domain* $\mathcal{D} = \left\{x \in \mathbb{C}^n : \exists \theta \in \mathbb{T}_{\rho_*} \quad s.t. \quad \|x - K_*\|_{\rho_*} < r_K\right\}$*, and a real analytic vector field* $X : \mathcal{D} \longrightarrow \mathbb{C}^n$ *such that*

$$\sup_{x\in\mathcal{D}}|DX(x)| \leq C_{DX}, \quad \sup_{x\in\mathcal{D}}\left|D^2X(x)\right| \leq C_{D^2X}$$

*Given values* $r_{K'}, r_{K'}, r_N, r_{N'}, r_\omega, r_\Lambda$ *under the hypothesis of Super-Lemma (1) and (2) and a tuple* $(K_0, \omega_0, N_0, \Lambda_0)$ *such that:*

$$\|K_0 - K_*\|_\rho < r_K \qquad \|N_0 - N_*\|_\rho < r_N \qquad |\omega_0 - \omega_*| < r_\omega$$
$$\left\|K_0' - K_*'\right\|_\rho < r_{K'} \qquad \left\|N_0' - N_*'\right\|_\rho < r_{N'} \qquad |\Lambda_0 - \Lambda_*| < r_\Lambda$$

*and verifies the imposed phase and normal conditions:*

$$\int_0^1 K_*'(\theta)^T K_0(\theta)d\theta = 0 \qquad \int_0^1 N_*(\theta)^T N_0(\theta)d\theta = I_{n-1}$$

*We define the initial invariant error* $E_{K_0}$ *and Floquet error* $E_{N_0}$ *as:*

$$X(K_0) - K_0'\omega_0 = E_{K_0} \tag{5.1}$$
$$DX(K_0) - N_0'\omega_0 - N_0\Lambda_0 = E_{N_0} \tag{5.2}$$

*In addition, there exist explicit constants* $C_{\Delta K}, C_{\Delta\omega}, C_{\Delta N}, C_{\Delta\Lambda}$ *such that:*

$$\frac{C_{\Delta K}\|E_{K_0}\|_\rho}{r_K - \|K_* - K_0\|_\rho} < 1 \qquad \frac{C_{\Delta N}\max\left\{\|E_{K_0}\|_\rho, \|E_{N_0}\|_\rho\right\}}{r_N - \|N_0 - N_*\|_\rho} < 1$$

$$\frac{C_{\Delta K}\|E_{K_0}\|_\rho}{\delta\left(r_{K'} - \|K_0' - K_*'\|_\rho\right)} < 1 \qquad \frac{C_{\Delta N}\max\left\{\|E_{K_0}\|_\rho, \|E_{N_0}\|_\rho\right\}}{\delta\left(r_{N'} - \|N_0' - N_*'\|_\rho\right)} < 1$$

$$\frac{C_{\Delta\omega}\|E_{K_0}\|_\rho}{r_\omega - |\omega_0 - \omega_*|} < 1 \qquad \frac{C_{\Delta\Lambda}\max\left\{\|E_{K_0}\|_\rho, \|E_{N_0}\|_\rho\right\}}{r_\Lambda - |\Lambda_0 - \Lambda_*|} < 1$$

*Then the tuple* $(K_1, \omega_1, N_1, \Lambda_1)$ *defined as a Newton step:*

$$K_1 = K_0 + \Delta K \qquad \omega_1 = \omega_0 + \Delta\omega$$
$$N_1 = N_0 + \Delta N \qquad \Lambda_1 = \Lambda_0 + \Delta\Lambda$$

*also satisfies the Moore conditions and holds the Super-Lemma (1) and (2) hypothesis in* $\mathbb{T}_{\rho-\delta}$ *where* $\delta > 0$*. Moreover, the errors of the new system* $E_{K_1}, N_1$ *are quadratic w.r.t the initial error with the following expression with explicit constants* $Q_K(\delta), Q_N(\delta)$*:*

$$\|E_{K_1}\|_{\rho-\delta} \leq Q_K(\delta)\|E_{K_0}\|_\rho \max\left\{\frac{\|E_{K_0}\|_\rho}{\delta}, \|E_{N_0}\|_\rho\right\}$$

$$\|E_{N_1}\|_{\rho-\delta} \leq Q_N(\delta)\max\left\{\|E_{K_0}\|_\rho, \|E_{N_0}\|_\rho\right\}\max\left\{\frac{\|E_{K_0}\|_\rho}{\delta}, \|E_{N_0}\|_\rho\right\}$$

## 5.2 Proof of Iterative Lemma

Because we are in a non-degenerate set up, we can find an explicit expresion for the next step error of the method finding the unique function that solves the cohomological equations:

### 5.2.1 Solutions to the Cohomological Equations

By Lemmas (4),(5),(6),(7), we have unique functions $\widetilde{\xi^{K,L}}, \xi^{K,N}, \xi^{N,L}, \widetilde{\xi^{N,N}}$ satisfying the Cohomological Equations (2.4),(2.5),(2.7),(2.8), moreover, by Moore conditions, we have:

- For $\left\langle \xi^{K,L} \right\rangle$ we fix

$$\int_0^1 K_*'(\theta)^T K_1(\theta) d\theta = 0$$

Considering:

$$\xi^{K,L} = \left\langle \xi^{K,L} \right\rangle + \widetilde{\xi^{K,L}}$$

We have:

$$\Delta K = P_0 \xi^K = K_0' \xi^{K,L} + N_0 \xi^{N,L} = K_0' \left\langle \xi^{K,L} \right\rangle + K_0' \widetilde{\xi^{K,L}} + N_0 \xi^{K,N} = K_0' \left\langle \xi^{K,L} \right\rangle + \widetilde{\Delta K}$$

And we can develop:

$$\begin{aligned} 0 &= \int_0^1 K_*'(\theta)^T K_1(\theta) d\theta = \int_0^1 K_*'(\theta)^T \left( K_0(\theta) + K_0'(\theta) \left\langle \xi^{K,L} \right\rangle + \widetilde{\Delta K}(\theta) \right) d\theta \\ &= 0 + \left( \int_0^1 K_*'(\theta)^T K_0(\theta) d\theta \right) \left\langle \xi^{K,L} \right\rangle + \int_0^1 K_*'(\theta)^T \widetilde{\Delta K}(\theta) d\theta \end{aligned}$$

Finding:

$$\begin{aligned} \left\langle \xi^{K,L} \right\rangle &= - \left( \int_0^1 K_*'(\theta)^T K_0'(\theta) d\theta \right)^{-1} \int_0^1 K_*'(\theta)^T \widetilde{\Delta K}(\theta) d\theta \\ &= - \left( \int_0^1 K_*'(\theta)^T K_0'(\theta) d\theta \right)^{-1} \int_0^1 K_*'(\theta)^T \left( K_0'(\theta) \widetilde{\xi^{K,L}}(\theta) + N_0(\theta) \xi^{K,N}(\theta) \right) d\theta \end{aligned}$$

- For $\left\langle \xi^{N,N} \right\rangle$ we fix:

$$\int_0^1 N_*(\theta)^T N_1(\theta) d\theta = I_{n-1}$$

Recall:

$$\Delta N = P_0 \xi^N = K_0' \xi^{N,L} + N_0 \xi^{N,N} = K_0' \xi^{N,L} + N_0 \left\langle \xi^{N,N} \right\rangle + N_0 \widetilde{\xi^{N,N}} = N_0 \left\langle \xi^{N,N} \right\rangle + \widetilde{\Delta N}$$

Therefore:

$$\begin{aligned} I_{n-1} &= \int_0^1 N_*(\theta)^T N_1(\theta) d\theta = \int_0^1 N_*(\theta)^T \left( N_0(\theta) + N_0(\theta) \left\langle \xi^{N,N} \right\rangle + \widetilde{\Delta N}(\theta) \right) d\theta \\ &= I_{n-1} + \int_0^1 N_*(\theta)^T N_0(\theta) \left\langle \xi^{N,N} \right\rangle d\theta + \int_0^1 N_*(\theta)^T \widetilde{\Delta N}(\theta) d\theta \end{aligned}$$

Having:

$$\begin{aligned}\left\langle \xi^{N,N} \right\rangle &= -\left(\int_0^1 N_*(\theta)^T N_0(\theta) d\theta\right)^{-1} \int_0^1 N_*(\theta)^T \widetilde{\Delta N}(\theta) d\theta \\ &= -\left(\int_0^1 N_*(\theta)^T N_0(\theta) d\theta\right)^{-1} \int_0^1 N_*(\theta)^T \left(K_0'(\theta)\xi^{N,L}(\theta) + N_0 \widetilde{\xi^{N,N}}(\theta)\right) d\theta\end{aligned}$$

Hence, for this fixed functions, we can consider the expressions of the errors found in (2.6) and (2.9)

### 5.2.2 Error Bounds

To find an upper bound of the new error, we need to bound all the appearing terms.

A. Bounds for the Terms of the New Error

For the terms in the first equation:

- Because we defined $-\eta^K = P_0^{-1} E_{K_0}$, we have:

$$\begin{aligned}\left\|\eta^{K,L}\right\|_\rho &\leq \left\|P_0^{-1}\right\|_\rho \|E_{K_0}\|_\rho \leq C_{P^{-1}} \|E_{K_0}\|_\rho = C_{\eta^K} \|E_{K_0}\|_\rho \\ \left\|\eta^{K,N}\right\|_\rho &\leq \left\|P_0^{-1}\right\|_\rho \|E_{K_0}\|_\rho \leq C_{P^{-1}} \|E_{K_0}\|_\rho = C_{\eta^K} \|E_{K_0}\|_\rho\end{aligned}$$

- By Lemma 4 and 5:

$$\begin{aligned}\left\|\widetilde{\xi^{K,L}}\right\|_\rho &\leq \mathcal{C}_\omega^{K,L} \left\|\eta^{K,L}\right\|_\rho \leq \mathcal{C}_\omega^{K,L} C_{\eta^K} \|E_{K_0}\|_\rho = C_{\widetilde{\xi^{K,L}}} \|E_{K_0}\|_\rho \\ \left\|\xi^{K,N}\right\|_\rho &\leq \mathcal{C}_{\omega,\Lambda}^{K,N} \left\|\eta^{K,N}\right\|_\rho \leq \mathcal{C}_{\omega,\Lambda}^{K,N} C_{\eta^K} \|E_{K_0}\|_\rho = C_{\xi^{K,N}} \|E_{K_0}\|_\rho\end{aligned}$$

- For the average $\left\langle \xi^{K,L} \right\rangle$:

$$\begin{aligned}\left|\left\langle \xi^{K,L} \right\rangle\right| &= \left|\left(\int_0^1 K_*'(\theta)^T K_0'(\theta) d\theta\right)^{-1} \int_0^1 K_*'(\theta)^T \left(K_0'(\theta)\widetilde{\xi^{K,L}}(\theta) + N_0(\theta)\xi^{K,N}(\theta)\right) d\theta\right| \\ &\leq C_{m_{K_*'}} \left\|(K_*')^T\right\|_\rho \left(\left\|K_0'\right\|_\rho \left\|\widetilde{\xi^{K,L}}\right\|_\rho + \|N_0\|_\rho \left\|\xi^{K,N}\right\|_\rho\right) \\ &\leq C_{m_{K_*'}} \left\|(K_*')^T\right\|_\rho \left(r_{K'} C_{\widetilde{\xi^{K,L}}} + r_N C_{\xi^{K,N}}\right) \|E_{K_0}\|_\rho = C_{\langle \xi^{K,L} \rangle} \|E_{K_0}\|_\rho\end{aligned}$$

- Therefore:

$$\left\|\xi^{K,L}\right\|_\rho \leq \left|\left\langle \xi^{K,L} \right\rangle\right| + \left\|\widetilde{\xi^{K,L}}\right\|_\rho \leq \left(C_{\langle \xi^{K,L} \rangle} + C_{\widetilde{\xi^{K,L}}}\right) \|E_{K_0}\|_\rho = C_{\xi^{K,L}} \|E_{K_0}\|_\rho$$

- For the increment of $K$:

$$\Delta K = P_0 \xi^K = \begin{pmatrix} K_0' & N_0 \end{pmatrix} \begin{pmatrix} \xi^{K,L} \\ \xi^{K,N} \end{pmatrix} = K_0' \xi^{K,L} + N_0 \xi^{K,N}$$

Hence:

$$\|\Delta K\|_\rho \leq \left\|K_0'\right\|_\rho \left\|\xi^{K,L}\right\|_\rho + \|N_0\|_\rho \left\|\xi^{K,N}\right\|_\rho \leq \left(C_{K'} C_{\xi^{K,L}} + C_N C_{\xi^{K,N}}\right) \|E_{K_0}\|_\rho = C_{\Delta K} \|E_{K_0}\|_\rho$$

- We know by Lemma (4) that for the increment of $\omega$:

$$|\Delta\omega| = |(\widehat{\eta_K^L})_0| \leq \|\eta^{K,L}\|_\rho \leq C_{\eta^K}\|E_{K_0}\|_\rho = C_{\Delta\omega}\|E_{K_0}\|_\rho$$

For the terms in the Floquet equation:

- Because we defined

$$\eta^N = -P_0^{-1}\left(E_{N_0} - N_0'\Delta\omega + \int_0^1 D^2X(K_0 + s\Delta K)[\Delta K, N_0]\,ds\right)$$

we have

$$\begin{aligned}
\|\eta^N\|_\rho &\leq \left\|P_0^{-1}\right\|_\rho \left(\|E_{N_0}\|_\rho + \left\|N_0'\right\|_\rho|\Delta\omega| + \int_0^1 C_{D^2X}\|\Delta K\|_\rho\|N_0\|_\rho ds\right) \\
&\leq C_{P^{-1}}\left(\|E_{N_0}\|_\rho + C_{\Delta\omega}C_{N'}\|E_{K_0}\|_\rho + C_{D^2X}C_{\Delta K}C_N\|E_{K_0}\|_\rho\right) \\
&\leq C_{P^{-1}}\left(1 + C_{\Delta\omega}C_{N'} + C_{D^2X}C_{\Delta K}C_N\right)\max\left\{\|E_{K_0}\|_\rho, \|E_{N_0}\|_\rho\right\} \\
&= C_{\eta^N}\max\left\{\|E_{K_0}\|_\rho, \|E_{N_0}\|_\rho\right\}
\end{aligned}$$

And we can bound:

$$\begin{aligned}
\|\eta^{N,L}\|_\rho &\leq C_{\eta^N}\max\left\{\|E_{K_0}\|_\rho, \|E_{N_0}\|_\rho\right\} \\
\|\eta^{N,N}\|_\rho &\leq C_{\eta^N}\max\left\{\|E_{K_0}\|_\rho, \|E_{N_0}\|_\rho\right\}
\end{aligned}$$

- We bound the function of the frame by Lemmas (6) and (7):

$$\begin{aligned}
\|\xi^{N,L}\|_\rho &\leq \mathcal{C}_{\omega,\Lambda}^{N,L}\|\eta^{N,L}\|_\rho \leq \mathcal{C}_{\omega,\Lambda}^{N,L}C_{\eta^N}\max\left\{\|E_{K_0}\|_\rho, \|E_{N_0}\|_\rho\right\} = C_{\xi^{N,L}}\max\left\{\|E_{K_0}\|_\rho, \|E_{N_0}\|_\rho\right\} \\
\|\widetilde{\xi^{N,N}}\|_\rho &\leq \mathcal{C}_{\omega,\Lambda}^{N,N}\left\|\eta^{N,N}\right\|_\rho \leq \mathcal{C}_{\omega,\Lambda}^{N,N}C_{\eta^N}\max\left\{\|E_{K_0}\|_\rho, \|E_{N_0}\|_\rho\right\} = C_{\widetilde{\xi^{N,N}}}\max\left\{\|E_{K_0}\|_\rho, \|E_{N_0}\|_\rho\right\}
\end{aligned}$$

- For the average $\langle\xi^{N,N}\rangle$:

$$\begin{aligned}
\left|\langle\xi^{N,N}\rangle\right| =& \left|\left(\int_0^1 N_*(\theta)^T N_0(\theta)d\theta\right)^{-1}\int_0^1 N_*(\theta)^T K_0'(\theta)\xi^{N,L}(\theta) + N_*(\theta)^T N_0(\theta)\widetilde{\xi^{N,N}}(\theta)d\theta\right| \\
\leq& \left|\left(\int_0^1 N_*(\theta)^T N_0(\theta)d\theta\right)^{-1}\right|\left\|N_*^T\right\|_\rho\left(\left\|K_0'\right\|_\rho\left\|\xi^{N,L}\right\|_\rho + \|N_0\|_\rho\left\|\widetilde{\xi^{N,N}}\right\|_\rho\right) \\
\leq& C_{m_{K_*'}}\left\|N_*^T\right\|_\rho\left(C_{K'}C_{\xi^{N,L}} + C_N C_{\widetilde{\xi^{N,N}}}\right)\max\left\{\|E_{K_0}\|_\rho, \|E_{N_0}\|_\rho\right\} \\
=& C_{\langle\xi^{N,N}\rangle}\max\left\{\|E_{K_0}\|_\rho, \|E_{N_0}\|_\rho\right\}
\end{aligned}$$

- Therefore:

$$\begin{aligned}
\left\|\xi^{N,N}\right\|_\rho = \left\|\langle\xi^{N,N}\rangle + \widetilde{\xi^{N,N}}\right\|_\rho &\leq \left\|\langle\xi^{N,N}\rangle\right\|_\rho + \left\|\widetilde{\xi^{N,N}}\right\|_\rho \\
&\leq \left(C_{\langle\xi^{N,N}\rangle} + C_{\widetilde{\xi^{N,N}}}\right)\max\left\{\|E_{K_0}\|_\rho, \|E_{N_0}\|_\rho\right\} \\
&= C_{\xi^{N,N}}\max\left\{\|E_{K_0}\|_\rho, \|E_{N_0}\|_\rho\right\}
\end{aligned}$$

- Recall
$$\Delta N = P_0 \xi^N = \begin{pmatrix} K_0' & N_0 \end{pmatrix} \begin{pmatrix} \xi^{N,L} \\ \xi^{N,N} \end{pmatrix} = K_0' \xi^{N,L} + N_0 \xi^{N,N}$$
Hence:
$$\begin{aligned} \|\Delta N\|_\rho &\leq \|K_0'\|_\rho \|\xi^{N,L}\|_\rho + \|N_0\|_\rho \|\xi^{N,N}\|_\rho \leq \left(C_{K'} C_{\xi^{N,L}} + C_N C_{\xi^{N,N}}\right) \max\left\{\|E_{K_0}\|_\rho, \|E_{N_0}\|_\rho\right\} \\ &= C_{\Delta N} \max\left\{\|E_{K_0}\|_\rho, \|E_{N_0}\|_\rho\right\} \end{aligned}$$
- For $\Delta\Lambda$, by Lemma (7) we have:
$$\begin{aligned} |\Delta\Lambda| &\leq \|\eta^{N,N}\|_\rho + 2|\Lambda| \left|\langle \xi^{N,N} \rangle\right| \leq \left(C_{\eta^N} + 2 C_\Lambda C_{\langle \xi^{N,N} \rangle}\right) \max\left\{\|E_{K_0}\|_\rho, \|E_{N_0}\|_\rho\right\} \\ &= C_{\Delta\Lambda} \max\left\{\|E_{K_0}\|_\rho, \|E_{N_0}\|_\rho\right\} \end{aligned}$$

B. Bounds on the New Errors

We recover the expression of $E_{K_1}(\theta)$ from equation (2.6). We bound it using triangular inequality:
$$\begin{aligned} \|E_{K_1}(\theta)\|_{\rho-\delta} \leq & \|E_{K_0}'\|_{\rho-\delta} \|\xi^{K,L}\|_{\rho-\delta} + \|E_{N_0}\|_{\rho-\delta} \|\xi^{K,N}\|_{\rho-\delta} + \|\Delta K'\|_{\rho-\delta} |\Delta\omega| \\ & + \int_0^1 \sup |D^2 X(K_0 + s\Delta K)| \|\Delta K\|_{\rho-\delta} \|\Delta K\|_{\rho-\delta} ds \end{aligned}$$
And by terms, applying Cauchy inequality:
- $\|E_{K_0}'\|_{\rho-\delta} \|\xi^{K,L}\|_{\rho-\delta} \leq \dfrac{1}{\delta} \|E_{K_0}\|_\rho \|\xi^{K,L}\|_\rho \leq \|E_{K_0}\|_\rho C_{\xi^{K,L}} \dfrac{\|E_{K_0}\|_\rho}{\delta}$
- $\|E_{N_0}\|_{\rho-\delta} \|\xi^{K,N}\|_{\rho-\delta} \leq \|E_{N_0}\|_\rho C_{\xi^{K,N}} \|E_{K_0}\|_\rho$
- $\|\Delta K'\|_{\rho-\delta} |\Delta\omega| \leq \dfrac{1}{\delta} \|\Delta K\|_\rho |\Delta\omega| \leq \dfrac{\|E_{K_0}\|_\rho}{\delta} C_{\Delta K} C_{\Delta\omega} \|E_{K_0}\|_\rho$
- $\displaystyle\int_0^1 \sup |D^2 X(K_0 + s\Delta K)| \|\Delta K\|_{\rho-\delta} \|\Delta K\|_{\rho-\delta} ds$
$\displaystyle\leq \int_0^1 C_{D^2 X} C_{\Delta K} \|E_{K_0}\|_\rho C_{\Delta K} \|E_{K_0}\|_\rho \leq \|E_{K_0}\|_\rho \frac{C_{D^2 X} C_{\Delta K}^2}{2} \|E_{K_0}\|_\rho$

Therefore we have:
$$\|E_{K_1}\|_{\rho-\delta} \leq \left(C_{\xi^{K,L}} + C_{\xi^{K,N}} + C_{\Delta K} C_{\Delta\omega} + \frac{C_{D^2 X} C_{\Delta K}^2 \delta}{2}\right) \|E_{K_0}\|_\rho \max\left\{\frac{\|E_{K_0}\|_\rho}{\delta}, \|E_{N_0}\|_\rho\right\}$$
$$\boxed{\|E_{K_1}\|_{\rho-\delta} \leq Q_K(\delta) \|E_{K_0}\|_\rho \max\left\{\frac{\|E_{K_0}\|_\rho}{\delta}, \|E_{N_0}\|_\rho\right\}} \tag{5.3}$$

Recall the expression of $E_{N_1}(\theta)$ from equation (2.9) bounded by the triangular inequality:
$$\begin{aligned} \|E_{N_1}\|_{\rho-\delta} \leq & \|E_{K_0}'\|_{\rho-\delta} \|\xi^{N,L}\|_{\rho-\delta} + \|E_{N_0}\|_{\rho-\delta} \|\xi^{N,N}\|_{\rho-\delta} + \|\Delta N'\|_{\rho-\delta} |\Delta\omega| \\ & + \|\Delta N\|_{\rho-\delta} |\Delta\Lambda| + \int_0^1 \|D^2 X(K_0 + s\Delta K)\|_{\rho-\delta} \|\Delta K\|_{\rho-\delta} \|\Delta N\|_{\rho-\delta} ds \end{aligned}$$
By terms

- $\|E'_{K_0}\|_{\rho-\delta}\|\xi^{N,L}\|_{\rho-\delta} \leq \frac{1}{\delta}\|E_{K_0}\|_\rho C_{\xi^{N,L}} \max\left\{\|E_{K_0}\|_\rho, \|E_{N_0}\|_\rho\right\}$
- $\|E_{N_0}\|_{\rho-\delta}\|\xi^{N,N}\|_{\rho-\delta} \leq \|E_{N_0}\|_\rho C_{\xi^{N,N}} \max\left\{\|E_{K_0}\|_\rho, \|E_{N_0}\|_\rho\right\}$
- $\|\Delta N'\|_{\rho-\delta}|\Delta\omega| \leq \frac{1}{\delta}C_{\Delta N} \max\left\{\|E_{K_0}\|_\rho, \|E_{N_0}\|_\rho\right\} C_{\Delta\omega}\|E_{K_0}\|_\rho$
- $\|\Delta N\|_{\rho-\delta}|\Delta\Lambda| \leq C_{\Delta N} \max\left\{\|E_{K_0}\|_\rho, \|E_{N_0}\|_\rho\right\} C_{\Delta\Lambda} \max\left\{\|E_{K_0}\|_\rho, \|E_{N_0}\|_\rho\right\}$
- $\int_0^1 \|D^2X(K_0 + s\Delta K)\|_{\rho-\delta}\|\Delta K\|_{\rho-\delta}\|\Delta N\|_{\rho-\delta}ds$
  $\leq C_{D^2X}C_{\Delta K}C_{\Delta N}\|E_{K_0}\|_\rho \max\left\{\|E_{K_0}\|_\rho, \|E_{N_0}\|_\rho\right\}$

**Obs 2.**

$$\begin{aligned}
\max\left\{\|E_{K_0}\|_\rho, \|E_{N_0}\|_\rho\right\} &= \max\{\delta\frac{\|E_{K_0}\|_\rho}{\delta}, \|E_{N_0}\|_\rho\} \\
&\leq \max\left\{\delta \max\left\{\frac{\|E_{K_0}\|_\rho}{\delta}, \|E_{N_0}\|_\rho\right\}, \max\left\{\frac{\|E_{K_0}\|_\rho}{\delta}, \|E_{N_0}\|_\rho\right\}\right\} \\
&= \max\{\delta, 1\} \max\left\{\frac{\|E_{K_0}\|_\rho}{\delta}, \|E_{N_0}\|_\rho\right\}
\end{aligned}$$

Therefore, by Observation (2) we have:

$$\begin{aligned}
\|E_{N_1}\|_{\rho-\delta} \leq \Bigg( & C_{\xi^{N,L}}\frac{\|E_{K_0}\|_\rho}{\delta} + C_{\xi^{N,N}}\|E_{N_0}\|_\rho + C_{\Delta N}C_{\Delta\omega}\frac{\|E_{K_0}\|_\rho}{\delta} \\
& + C_{\Delta N}C_{\Delta\Lambda} \max\left\{\|E_{K_0}\|_\rho, \|E_{N_0}\|_\rho\right\} \\
& + C_{D^2X}C_{\Delta K}C_{\Delta N}\|E_{K_0}\|_\rho\Bigg) \max\left\{\|E_{K_0}\|_\rho, \|E_{N_0}\|_\rho\right\}
\end{aligned}$$

$$\begin{aligned}
\leq & \left(C_{\xi^{N,L}} + C_{\xi^{N,N}} + C_{\Delta N}C_{\Delta\omega} + \max\{\delta, 1\}C_{\Delta N}C_{\Delta\Lambda} + C_{D^2X}C_{\Delta K}C_{\Delta N}\delta\right) \\
& \cdot \max\left\{\|E_{K_0}\|_\rho, \|E_{N_0}\|_\rho\right\} \max\left\{\frac{\|E_{K_0}\|_\rho}{\delta}, \|E_{N_0}\|_\rho\right\}
\end{aligned}$$

$$\boxed{\|E_{N_1}\|_{\rho-\delta} \leq Q_N(\delta) \max\left\{\|E_{K_0}\|_\rho, \|E_{N_0}\|_\rho\right\} \max\left\{\frac{\|E_{K_0}\|_\rho}{\delta}, \|E_{N_0}\|_\rho\right\}}$$

### 5.2.3 Super-Lemma Hypothesis

The hypothesis of the *Super-Lemma* (1) and (2) hold (the tuple is non-degenerate):

- For $K_1$:

$$\begin{aligned}\|K_1 - K_*(\theta)\|_\rho \leq \|K_1 - K_0\|_\rho + \|K_0 - K_*\|_\rho = \|\Delta K\|_\rho + \|K_0 - K_*\|_\rho \\ \leq C_{\Delta K}\|E_{K_0}\|_\rho + \|K_0 - K_*\|_\rho < r_K\end{aligned}$$

- For $K_1'$:

$$\begin{aligned}\|K_1' - K_*'\|_{\rho-\delta} = \|K_0' + \Delta K' - K_*'\|_{\rho-\delta} \leq \|\Delta K'\|_{\rho-\delta} + \|K_0' - K_*'\|_{\rho-\delta} \\ \leq \frac{1}{\delta}\|\Delta K\|_\rho + \left\|K_0' - K_*'\right\|_\rho \leq \frac{1}{\delta}C_{\Delta K}\|E_{K_0}\|_\rho + \left\|K_0' - K_*'\right\|_\rho < r_{K'}\end{aligned}$$

- For $N_1$:

$$\begin{aligned}\|N_1 - N_*\|_\rho = \|N_0 + \Delta N - N_*\|_\rho \leq \|\Delta N\|_\rho + \|N_0 - N_*\|_\rho \\ \leq C_{\Delta N}\max\left\{\|E_{K_0}\|_\rho, \|E_{N_0}\|_\rho\right\} < r_N\end{aligned}$$

- For $N_1'$:

$$\begin{aligned}\|N_1' - N_*'\|_{\rho-\delta} = \|N_0' + \Delta N' - N_*'\|_{\rho-\delta} \leq \|\Delta N'\|_{\rho-\delta} + \|N_0' - N_*'\|_{\rho-\delta} \\ \leq \frac{1}{\delta}\|\Delta N\|_\rho + \left\|N_0' - N_*'\right\|_\rho \leq \frac{1}{\delta}C_{\Delta N}\max\left\{\|E_{K_0}\|_\rho, \|E_{N_0}\|_\rho\right\} \\ + \left|N_0' - N_*'\right| < r_{N'}\end{aligned}$$

- For $\omega_1$:

$$|\omega_1 - \omega_*| = |\omega_0 + \Delta\omega - \omega_*| \leq |\Delta\omega| + |\omega_0 - \omega_*| \leq C_{\Delta\omega}\|E_{K_0}\|_\rho + |\omega_0 - \omega_*| < r_\omega$$

- For $\Lambda_1$:

$$|\Lambda_1 - \Lambda_*| = |\Lambda_0 + \Delta\Lambda - \Lambda_*| \leq |\Delta\Lambda| + |\Lambda_0 - \Lambda_*| \leq C_{\Delta\Lambda}\max\left\{\|E_{K_0}\|_\rho, \|E_{N_0}\|_\rho\right\} + |\Lambda_0 - \Lambda_*| < r_\Lambda$$

- Notice $P_1$ is already non-degenerate given that:

$$\|P_1 - P_*\|_\rho = \left\|\begin{pmatrix}K_0' + \Delta K' - K_*' \\ N_0 + \Delta N - N_*\end{pmatrix}\right\|_\rho = \max\{\left\|K_1' - K_*'\right\|_\rho, \|N_1 - N_*\|_\rho\} < r_{K'} + r_N$$

# 6 Convergence Theorem

The idea for writing the *Convergence Theorem* is to apply the *Iterative Lemma* infinitely steps so the error of the method converges to zero. In this case, each time we do a step, we loose some different band of the analytic domain. If we define an initial band $\rho_0$ with a defined first step $\rho_1 = \rho_0 - \delta_0$ and choose a goal band of size $\rho_\infty$ where the solution will be defined, we have:

$$\begin{cases} \rho_1 = \rho_0 - \delta_0 \\ \rho_2 = \rho_1 - \delta_1 = \rho_0 - \delta_0 - \delta_1 \\ .... \\ \rho_j = \rho_{j-1} - \delta_{j-1} = \rho_0 - \sum_{i=0}^{j-1} \delta_i \\ ... \\ \rho_\infty = \rho_0 - \sum_{i=0}^{\infty} \delta_i \end{cases}$$

Therefore, the sequence $(\delta_j)_j$ must be decreasing and converging to zero to lose less domain each step and arrive to the final chosen band $\rho_\infty$.

To choose the step of the sequence, we define it as $\alpha = \frac{\delta_{j-1}}{\delta_j}$ obtaining the equality:

$$\delta_j = \frac{\delta_{j-1}}{\alpha} = \frac{\delta_{j-1}}{\alpha^2} = ... = \frac{\delta_0}{\alpha^j}$$

Therefore, we have the following equivalence:

$$\rho_\infty = \rho_0 - \sum_{i=0}^{\infty} \delta_i = \rho_0 - \sum_{i=0}^{\infty} \frac{\delta_0}{\alpha^i} = \rho_0 - \delta_0 \frac{1}{1 - \frac{1}{\alpha}} = \rho_0 - \delta_0 \frac{\alpha}{\alpha - 1}$$

And we can find a explicit definition of $\alpha$ depending on the fixed bands $\rho_0, \rho_1, \rho_\infty$

$$\begin{cases} \rho_1 = \rho_0 - \delta_0 \\ \rho_\infty = \rho_0 - \delta_0 \frac{\alpha}{\alpha - 1} \end{cases} \implies \begin{cases} \rho_0 - \rho_1 = \delta_0 \\ \rho_0 - \rho_\infty = \delta_0 \frac{\alpha}{\alpha - 1} \end{cases} \implies \frac{\rho_0 - \rho_\infty}{\rho_0 - \rho_1} = \frac{\alpha}{\alpha - 1} \implies \frac{\rho_0 - \rho_1}{\rho_0 - \rho_\infty} = 1 - \frac{1}{\alpha} \implies$$

$$\frac{1}{\alpha} = 1 - \frac{\rho_0 - \rho_1}{\rho_0 - \rho_\infty} \implies \frac{1}{\alpha} = \frac{\rho_1 - \rho_\infty}{\rho_0 - \rho_\infty} \implies \alpha = \frac{\rho_0 - \rho_\infty}{\rho_1 - \rho_\infty}$$

**Example**(typical choice): We consider an initial band $\rho_0$ and a first cut of $\delta_0 = \frac{\rho_0}{4}$. We want to find the solution in a reduced band $\rho_\infty = \frac{\rho_0}{2}$, therefore, we have:

$$\alpha = \frac{\rho_0 - \rho_0/2}{(\rho_0 - \rho_0/4) - \rho_0/2} = \frac{1/2}{(4 - 1 - 2)/4} = 2$$

Meaning we would be loosing half of the band each step.

**Remark:** Notice that, for the sum to be convergent, we need $\alpha > 1$.

With all this previous set up, we can write the *Convergence Theorem*.

### 6.1 Statement of the Theorem

**Convergence Theorem.** *Given $\rho_* > \rho_0 > \rho_1 > \rho_\infty > 0$, a non-degenerate tuple $(K_*, \omega_*, N_*, \Lambda_*)$ in $\mathbb{T}_{\rho_*}$ and constants*

$$r_K, \quad r_{K'}, \quad r_N, \quad r_{N'}, \quad r_\omega < |\omega_*|, \quad r_\Lambda < \left|\Lambda_*^{-1}\right|^{-1}$$

*such that:*

$$r_{K'} + r_N < \left\|P_*(\theta)^{-1}\right\|_\rho^{-1} \qquad C_1(\Lambda_*, \omega_*)\left(\frac{r_\Lambda}{2\pi|\omega_*|} + \frac{r_\omega}{|\omega_*|}\right) < 1 \qquad C_2(\Lambda_*, \omega_*)\left(\frac{nr_\Lambda}{2\pi|\omega_*|} + \frac{r_\omega}{|\omega_*|}\right) < 1$$

*Given an open domain $\mathcal{D} = \left\{x \in \mathbb{C}^n : \exists \theta \in \mathbb{T}_{\rho_*} \quad s.t. \quad \|x - K_*\|_{\rho_*} < r_K\right\}$ and a real analytic vector field $X : \mathcal{D} \longrightarrow \mathbb{C}^n$ verifying:*

$$\sup_{x\in\mathcal{D}}|DX(x)| \le C_{DX} \qquad \sup_{x\in\mathcal{D}}\left|D^2X(x)\right| \le C_{D^2X}$$

*Given also and an initial tuple $(K_0, \omega_0, N_0, \Lambda_0)$ in $\mathbb{T}_{\rho_0}$ holding:*

$$\|K_0 - K_*\|_{\rho_0} < r_K \qquad \|N_0 - N_*\|_{\rho_0} < r_N \qquad |\omega_0 - \omega_*| < r_\omega$$
$$\left\|K_0' - K_*'\right\|_{\rho_0} < r_{K'} \qquad \left\|N_0' - N_*'\right\|_{\rho_0} < r_{N'} \qquad |\Lambda_0 - \Lambda_*| < r_\Lambda$$

*and:*

$$\int_0^1 K_*'(\theta)^T K_0(\theta) d\theta = 0 \qquad \int_0^1 N_*(\theta)^T N_0(\theta) d\theta = I_{n-1}$$

*We define the initial invariant error $E_{K_0}$ and Floquet error $E_{N_0}$ as:*

$$X(K_0) - K_0'\omega_0 = E_{K_0} \tag{6.1}$$
$$DX(K_0) - N_0'\omega_0 - N_0\Lambda_0 = E_{N_0} \tag{6.2}$$

*We also define $\delta_0 = \rho_0 - \rho_1$ and the loss of band at each step $\alpha = \frac{\delta_{j-1}}{\delta_j}$ such that:*

$$\alpha = \frac{\rho_0 - \rho_\infty}{\rho_1 - \rho_\infty} > 1$$

*Also for explicit constants $C_{\Delta K}, C_{\Delta N}, C_{\Delta\omega}, C_{\Delta\Lambda}$:*

$$\mathcal{E}_0 = \max\left\{\frac{\|E_{K_0}\|_\rho}{\delta_0}, \|E_{N_0}\|_\rho\right\}, \quad Q_\mathcal{E} = \max\{\alpha Q_K, Q_N, \delta_0 Q_N\}$$

*where $Q_K = Q_K(\delta_0), Q_N = Q_N(\delta_0)$ are the constants of the Iterative Lemma (5.1) evaluated in $\delta_0$*

*We suppose:*

$$Q_\mathcal{E}\mathcal{E}_0 \le \mu < 1$$

*and for explicit constants $C_{\Delta K}, C_{\Delta N}, C_{\Delta\omega}, C_{\Delta\Lambda}$:*

$$\frac{C_{\Delta K}\delta_0\Sigma_{-1}(\alpha,\mu)}{r_K - \|K_0 - K_*\|_\rho}\mathcal{E}_0 < 1 \qquad \frac{1}{\delta_0}\frac{C_{\Delta N}\max\{1,\delta_0\}\Sigma_1(\alpha,\mu)}{r_{N'} - \|N_0' - N_*'\|_\rho}\mathcal{E}_0 < 1$$
$$\frac{C_{\Delta K}\Sigma_0(\alpha,\mu)}{r_{K'} - \|K_0' - K_*'\|_\rho}\mathcal{E}_0 < 1 \qquad \frac{C_{\Delta\omega}\delta_0\Sigma_{-1}(\alpha,\mu)}{r_\omega - \|\omega_0 - \omega_*\|_\rho}\mathcal{E}_0 < 1$$
$$\frac{C_{\Delta N}\max\{1,\delta_0\}\Sigma_0(\alpha,\mu)}{r_N - \|N_0 - N_*\|_\rho}\mathcal{E}_0 < 1 \qquad \frac{C_{\Delta\Lambda}\max\{1,\delta_0\}\Sigma_0(\alpha,\mu)}{r_\Lambda - \|\Lambda_0 - \Lambda_*\|_\rho}\mathcal{E}_0 < 1$$

*where*

$$\Sigma_a(\alpha,\mu)=\sum_{i=0}^{\infty}\alpha^{ai}\mu^{2^i-1}$$

*Then there exists a non-degenerate tuple* $(K_\infty,\omega_\infty,N_\infty,\Lambda_\infty)$ *in* $\mathbb{T}_{\rho^*}$ *at distances:*

$$\|K_\infty-K_0\|_{\rho_\infty}\leq C_{\Delta K}\delta_0\Sigma_{-1}(\alpha,\mu)\mathcal{E}_0 \qquad \|N_\infty-N_0\|_{\rho_\infty}\leq C_{\Delta N}\max\{\delta_0,1\}\Sigma_0(\alpha,\mu)\mathcal{E}_0$$
$$\|K'_\infty-K'_0\|_{\rho_\infty}\leq C_{\Delta K}\Sigma_0(\alpha,\mu)\mathcal{E}_0 \qquad \|N'_\infty-N'_0\|_{\rho_\infty}\leq \frac{\max\{\delta_0,1\}}{\delta_0}C_{\Delta N}\Sigma_0(\alpha,\mu)\mathcal{E}_0$$
$$|\omega_\infty-\omega_0|\leq C_{\Delta\omega}\delta_0\Sigma_{-1}(\alpha,\mu)\mathcal{E}_0 \qquad |\Lambda_\infty-\Lambda_0|\leq C_{\Delta\Lambda}\max\{\delta_0,1\}\Sigma_0(\alpha,\mu)\mathcal{E}_0$$

*such that:*

$$X(K_\infty)-K'_\infty\omega_\infty=0$$
$$DX(K_\infty)N_\infty-N'_\infty\omega_\infty-N_\infty\Lambda_\infty=0$$

*Moreover, if there exists another periodic orbit* $\overline{K}_\infty,\overline{\omega}_\infty$ *also solving*

$$X(\overline{K}_\infty)-\overline{K}'_\infty\overline{\omega}_\infty=0$$

*and verifying for an explicit* $Q_\Delta$*:*

$$\frac{Q_\Delta C_{\Delta K}\delta_0\Sigma_{-1}(\alpha,\mu)}{1-Q_\Delta\|\overline{K}_\infty-K_0\|_{\rho_\infty}}\mathcal{E}_0<1 \qquad \frac{Q_\Delta C_{\Delta\omega}\delta_0\Sigma_{-1}(\alpha,\mu)}{1-Q_\Delta|\overline{\omega}_\infty-\omega_0|}\mathcal{E}_0<1$$

*Then*

$$\overline{K}_\infty=K_\infty,\quad \overline{\omega}_\infty=\omega_\infty$$

*and the new periodic orbit is actually the one already found.*

## 6.2 Proof of Existence

Assume we have done $j-1$ iterations of the Newton method, we have a tuple $(K_{j-1},\omega_{j-1},N_{j-1},\Lambda_{j-1})$ in $\mathbb{T}_{\rho_{j-1}}$ verifying the hypothesis of the *Super-Lemma* (1) and (2):

$$\|K_{j-1}-K_*\|_{\rho_{j-1}}<r_K \qquad \|N_{j-1}-N_*\|_{\rho_{j-1}}<r_N \qquad |\omega_{j-1}-\omega_*|<r_\omega$$
$$\left\|K'_{j-1}-K'_*\right\|_{\rho_{j-1}}<r_{K'} \qquad \left\|N'_{j-1}-N'_*\right\|_{\rho_{j-1}}<r_{N'} \qquad |\Lambda_{j-1}-\Lambda_*|<r_\Lambda$$

Hence the error for the $j$'s step is bounded by:

$$\|E_{K_j}\|_{\rho_j}\leq Q_K(\delta_{j-1})\|E_{K_{j-1}}\|_{\rho_{j-1}}\max\left\{\frac{\|E_{K_{j-1}}\|_{\rho_{j-1}}}{\delta_{j-1}},\|E_{N_{j-1}}\|_{\rho_{j-1}}\right\}$$
$$\|E_{N_j}\|_{\rho_j}\leq Q_N(\delta_{j-1})\max\left\{\|E_{K_{j-1}}\|_{\rho_{j-1}},\|E_{N_{j-1}}\|_{\rho_{j-1}}\right\}\max\left\{\frac{\|E_{K_{j-1}}\|_{\rho_{j-1}}}{\delta_{j-1}},\|E_{N_{j-1}}\|_{\rho_{j-1}}\right\}$$

If we consider:

$$\mathcal{E}_j=\max\left\{\frac{\|E_{K_j}\|_{\rho_j}}{\delta_j},\|E_{N_j}\|_{\rho_j}\right\}$$

We can write:

$$\mathcal{E}_j = \max\left\{ Q_K(\delta_{j-1}) \frac{\|E_{K_{j-1}}\|_{\rho_{j-1}}}{\delta_j} \max\left\{ \frac{\|E_{K_{j-1}}\|_{\rho_{j-1}}}{\delta_{j-1}}, \|E_{N_{j-1}}\|_{\rho_{j-1}} \right\}, \right.$$
$$\left. Q_N(\delta_{j-1}) \max\left\{ \|E_{K_{j-1}}\|_{\rho_{j-1}}, \|E_{N_{j-1}}\|_{\rho_{j-1}} \right\} \max\left\{ \frac{\|E_{K_{j-1}}\|_{\rho_{j-1}}}{\delta_{j-1}}, \|E_{N_{j-1}}\|_{\rho_{j-1}} \right\} \right\}$$

$$= \max\left\{ Q_K(\delta_{j-1}) \frac{\delta_{j-1}}{\delta_j} \frac{\|E_{K_{j-1}}\|_{\rho_{j-1}}}{\delta_{j-1}} \mathcal{E}_{j-1}, Q_N(\delta_{j-1}) \max\left\{ \|E_{K_{j-1}}\|_{\rho_{j-1}}, \|E_{N_{j-1}}\|_{\rho_{j-1}} \right\} \mathcal{E}_{j-1} \right\}$$

$$\leq \max\left\{ Q_K(\delta_{j-1}) \alpha \frac{\|E_{K_{j-1}}\|_{\rho_{j-1}}}{\delta_{j-1}}, Q_N(\delta_{j-1}) \|E_{K_{j-1}}\|_{\rho_{j-1}}, Q_N(\delta_{j-1}) \|E_{N_{j-1}}\|_{\rho_{j-1}} \right\} \mathcal{E}_{j-1}$$

$$\leq \max\left\{ Q_K(\delta_{j-1}) \alpha \mathcal{E}_{j-1}, Q_N(\delta_{j-1}) \mathcal{E}_{j-1} \delta_{j-1}, Q_N(\delta_{j-1}) \mathcal{E}_{j-1} \right\} \mathcal{E}_{j-1}$$

$$= \max\left\{ Q_K(\delta_{j-1}) \alpha, Q_N(\delta_{j-1}) \delta_{j-1}, Q_N(\delta_{j-1}) \right\} (\mathcal{E}_{j-1})^2$$

because we are considering $(\delta_j)_j$ a decreasing sequence, we will consider as an upper bound $\delta = \delta_0$ and therefore, we have:

$$\mathcal{E}_j \leq \max\left\{ Q_K \alpha, Q_N, Q_N \delta_0 \right\} (\mathcal{E}_{j-1})^2 = Q_{\mathcal{E}} (\mathcal{E}_{j-1})^2$$

If we iterate this inequality:

$$\mathcal{E}_j \leq Q_{\mathcal{E}} \left( {Q_{\mathcal{E}}}^2 (\mathcal{E}_{j-2})^{2^2} \right) \leq Q_{\mathcal{E}} {Q_{\mathcal{E}}}^2 \left( {Q_{\mathcal{E}}}^4 (\mathcal{E}_{j-1})^{2^3} \right) \leq ... \leq Q_{\mathcal{E}}^{\sum_{i=0}^{j-1} 2^i} {\mathcal{E}_0}^{2^j}$$

Notice that

$$\sum_{i=0}^{j-1} 2^i = \frac{1 - 2^j}{1 - 2} = 2^j - 1$$

Therefore

$$\mathcal{E}_j \leq Q_{\mathcal{E}}^{2^j - 1} {\mathcal{E}_0}^{2^j} = (Q_{\mathcal{E}} \mathcal{E}_0)^{2^j - 1} \mathcal{E}_0 \leq \mu^{2^j - 1} \mathcal{E}_0$$

Moreover, the next iteration $(K_j, \omega_j, N_j, \Lambda_j)$ falls under the same hypothesis:

- For $K_j$, notice we can write:

$$\|K_j - K_*\|_{\rho_j} \leq \sum_{i=0}^{j-1} \|K_{i+1} - K_i\|_{\rho_i} + \|K_0 - K_*\|_{\rho_0} = \sum_{i=0}^{j-1} \|\Delta K_i\|_{\rho_i} + \|K_0 - K_*\|_{\rho_0}$$

And

$$\sum_{i=0}^{j-1} \|\Delta K_i\|_{\rho_i} \leq \sum_{i=0}^{j-1} C_{\Delta K} \|E_{K_i}\|_{\rho_i} \leq \sum_{i=0}^{j-1} C_{\Delta K} \delta_i \underbrace{\frac{\|E_{K_i}\|_{\rho_i}}{\delta_i}}_{\leq \mathcal{E}_i} \leq \sum_{i=0}^{j-1} C_{\Delta K} \delta_i \mathcal{E}_i \leq \sum_{i=0}^{j-1} C_{\Delta K} \frac{\delta_0}{\alpha^i} \mu^{2^i - 1} \mathcal{E}_0$$

$$\leq C_{\Delta K} \delta_0 \mathcal{E}_0 \sum_{i=0}^{\infty} \alpha^{-i} \mu^{2^i - 1} = C_{\Delta K} \delta_0 \Sigma_{-1}(\alpha, \mu) \mathcal{E}_0$$

By hypothesis:

$$\|K_j - K_*\|_{\rho_j} \leq C_{\Delta K}\delta_0\Sigma_{-1}(\alpha,\mu)\mathcal{E}_0 + \|K_0 - K_*\|_{\rho_0} < r_K$$

**Remark**: Notice that for $j = \infty$ we have:

$$\|K_\infty - K_0\|_{\rho_\infty} = \sum_{i=0}^{\infty}\|K_{i+1} - K_i\|_{\rho_i} \leq C_{\Delta K}\delta_0\Sigma_{-1}(\alpha,\mu)\mathcal{E}_0$$

- For $K'_j$, we have similarly:

$$\|K'_j - K'_*\|_{\rho_j} \leq \sum_{i=0}^{j-1}\|K'_{i+1} - K'_i\|_{\rho_i} + \|K'_0 - K'_*\|_{\rho_0} = \sum_{i=0}^{j-1}\|\Delta K'_i\|_{\rho_{i+1}} + \|K'_0 - K'_*\|_{\rho_0}$$

And

$$\begin{aligned}\sum_{i=0}^{j-1}\|\Delta K'_i\|_{\rho_{i+1}} &\leq \sum_{i=0}^{j-1}\frac{\|\Delta K_i\|_{\rho_i}}{\delta_i} \leq \sum_{i=0}^{j-1} C_{\Delta K}\frac{\|E_{K_i}\|_{\rho_i}}{\delta_i} \leq \sum_{i=0}^{j-1} C_{\Delta K}\mathcal{E}_i \\ &\leq \sum_{i=0}^{j-1} C_{\Delta K}\mu^{2^i-1}\mathcal{E}_0 \leq C_{\Delta K}\sum_{i=0}^{\infty}\mu^{2^i-1}\mathcal{E}_0 \leq C_{\Delta K}\Sigma_0(\alpha,\mu)\mathcal{E}_0\end{aligned}$$

By hypothesis:

$$\|K'_j - K'_*\|_{\rho_j} \leq C_{\Delta K}\Sigma_0(\alpha,\mu)\mathcal{E}_0 + \|K'_0 - K'_*\|_{\rho_0} < r_{K'}$$

**Remark**: Notice that for $j = \infty$ we have:

$$\|K'_\infty - K'_0\|_{\rho_\infty} \leq C_{\Delta K}\Sigma_0(\alpha,\mu)\mathcal{E}_0$$

- For $\omega_j$

$$\sum_{i=0}^{j-1}|\Delta\omega| \leq \sum_{i=0}^{j-1} C_{\Delta\omega}\|E_{K_i}\|_{\rho_i} \leq \sum_{i=0}^{j-1} C_{\Delta\omega}\delta_i\mathcal{E}_i \leq C_{\Delta\omega}\delta_0\sum_{i=0}^{\infty}\alpha^{-i}\mu^{2^i-1}\mathcal{E}_0 \leq C_{\Delta\omega}\delta_0\Sigma_{-1}(\alpha,\mu)\mathcal{E}_0$$

And by hypothesis:

$$|\omega_j - \omega_*| \leq \sum_{i=0}^{j-1}|\Delta\omega| + |\omega_0 - \omega_*| \leq C_{\Delta\omega}\delta_0\Sigma_{-1}(\alpha,\mu)\mathcal{E}_0 + |\omega_0 - \omega_*| < r_\omega$$

**Remark**: Notice that for $j = \infty$ we have:

$$\|\omega_\infty - \omega_0\|_{\rho_\infty} \leq C_{\Delta\omega}\delta_0\Sigma_{-1}(\alpha,\mu)\mathcal{E}_0$$

- For $\Lambda_j$

$$\begin{aligned}\sum_{i=0}^{j-1}|\Delta\Lambda_i| &\leq \sum_{i=0}^{j-1} C_{\Delta\Lambda}\max\left\{\|E_{K_i}\|_{\rho_i}, \|E_{N_i}\|_{\rho_j}\right\} \underbrace{\leq}_{\text{Obs (2)}} \sum_{i=0}^{j-1} C_{\Delta\Lambda}\max\{\delta_i, 1\}\mathcal{E}_i \\ &\leq C_{\Delta\Lambda}\max\{\delta_0, 1\}\sum_{i=0}^{\infty}\mu^{2^i-1}\mathcal{E}_0 \leq C_{\Delta\Lambda}\max\{\delta_0, 1\}\Sigma_0(\alpha,\mu)\mathcal{E}_0\end{aligned}$$

And by hypothesis:

$$|\Lambda_{j-1} - \Lambda_*| \leq \sum_{i=0}^{j-1} |\Delta\Lambda_i| + |\Lambda_0 - \Lambda_*| \leq C_{\Delta\Lambda} \max\{\delta_0, 1\} \Sigma_0(\alpha, \mu) \mathcal{E}_0 + |\Lambda_0 - \Lambda_*|$$

**Remark**: Notice that for $j = \infty$ we have:

$$|\Lambda_\infty - \Lambda_0| \leq C_{\Delta\Lambda} \max\{\delta_0, 1\} \Sigma_0(\alpha, \mu) \mathcal{E}_0$$

- For $N_j$

$$\begin{aligned}
\sum_{i=0}^{j-1} \|\Delta N_i\|_{\rho_i} &\leq \sum_{i=0}^{j-1} C_{\Delta N} \max\left\{ \|E_{K_i}\|_{\rho_i}, \|E_{N_i}\|_{\rho_j} \right\} \underbrace{\leq}_{\text{Obs (2)}} \sum_{i=0}^{j-1} C_{\Delta N} \max\{\delta_i, 1\} \max\left\{ \frac{\|E_{K_i}\|_{\rho_i}}{\delta_i}, \|E_{N_i}\|_{\rho_i} \right\} \\
&\leq \sum_{i=0}^{j-1} C_{\Delta N} \max\{\delta_0, 1\} \mathcal{E}_i \leq \sum_{i=0}^{j-1} C_{\Delta N} \max\{\delta_0, 1\} \mu^{2^i - 1} \mathcal{E}_0 \\
&\leq C_{\Delta N} \max\{\delta_0, 1\} \sum_{i=0}^{\infty} \mu^{2^i - 1} \mathcal{E}_0 \leq C_{\Delta N} \max\{\delta_0, 1\} \Sigma_0(\alpha, \mu) \mathcal{E}_0
\end{aligned}$$

And by hypothesis

$$\|N_j - N_*\|_{\rho_j} \leq \sum_{i=0}^{j-1} \|\Delta N_i\|_{\rho_i} + \|N_0 - N_*\|_{\rho_0} \leq C_{\Delta N} \max\{\delta_0, 1\} \Sigma_0(\alpha, \mu) \mathcal{E}_0 + \|N_0 - N_*\|_{\rho_0} < r_N$$

**Remark**: Notice that for $j = \infty$ we have:

$$\|N_\infty - N_0\|_{\rho_\infty} \leq C_{\Delta N} \max\{\delta_0, 1\} \Sigma_0(\alpha, \mu) \mathcal{E}_0$$

- For $N'_j$

$$\begin{aligned}
\sum_{i=0}^{j-1} \|\Delta N'_i\|_{\rho_{i+1}} &\leq \sum_{i=0}^{j-1} \frac{1}{\delta_i} C_{\Delta N} \max\left\{ \|E_{K_i}\|_{\rho_i}, \|E_{N_i}\|_{\rho_j} \right\} \underbrace{\leq}_{\text{Obs (2)}} \sum_{i=0}^{j-1} \frac{1}{\delta_i} C_{\Delta N} \max\{\delta_i, 1\} \mathcal{E}_i \\
&\leq \frac{\max\{\delta_0, 1\}}{\delta_0} C_{\Delta N} \sum_{i=0}^{\infty} \alpha^i \mu^{2^i - 1} \mathcal{E}_0 \leq \frac{\max\{\delta_0, 1\}}{\delta_0} C_{\Delta N} \Sigma_1(\alpha, \mu) \mathcal{E}_0
\end{aligned}$$

And by hypothesis:

$$\left\|N'_j - N'_*\right\|_{\rho_j} \leq \sum_{i=0}^{j-1} \|\Delta N'_i\|_{\rho_{i+1}} + \|N'_0 - N'_*\|_{\rho_0} \leq \frac{C_{\Delta N} \max\{\delta_0, 1\}}{\delta_0} \Sigma_1(\alpha, \mu) \mathcal{E}_0 + \|N'_0 - N'_*\|_{\rho_0} < r_{N'}$$

**Remark**: Notice that for $j = \infty$ we have:

$$\|N'_\infty - N'_0\|_{\rho_\infty} \leq \frac{\max\{\delta_0, 1\}}{\delta_0} C_{\Delta N} \Sigma_0(\alpha, \mu) \mathcal{E}_0$$

Therefore, we have have a sequence $(K_j, \omega_j, N_j, \Lambda_j)$ and functions $K_\infty \in \mathcal{C}^{\omega,1}(\mathbb{T}_\rho, \mathbb{C}^{n\times 1})$, $\omega_\infty \in \mathbb{R}$, $N_\infty \in \mathcal{C}^{\omega,1}(\mathbb{T}_\rho, \mathbb{C}^{n\times(n-1)})$, $\Lambda_\infty \in \mathbb{R}^{(n-1)\times(n-1)}$ such that:

$$(K_j, \omega_j, N_j, \Lambda_j) \longrightarrow (K_\infty, \omega_\infty, N_\infty, \Lambda_\infty)$$

and:

$$\begin{array}{lll} \|K_\infty - K_*\|_\rho < r_K & \|N_\infty - N_*\|_\rho < r_N & |\omega_\infty - \omega_*| < r_\omega \\ \left\|K'_\infty - K'_*\right\|_\rho < r_{K'} & \left\|N'_\infty - N'_*\right\|_\rho < r_{N'} & |\Lambda_\infty - \Lambda_*| < r_\Lambda \end{array}$$

Given that $\mu < 1$ we have also $\mathcal{E}_j \leq \mu^{2^j-1}\mathcal{E}_0 \longrightarrow 0$ when $j \longrightarrow \infty$, meaning:

$$\begin{aligned} \|E_{K_j}\|_{\rho_j} \leq \delta_j \mathcal{E}_j \longrightarrow 0 \quad \text{when} \quad j \longrightarrow \infty \\ \|E_{N_j}\|_{\rho_j} \leq \mathcal{E}_j \longrightarrow 0 \quad \text{when} \quad j \longrightarrow \infty \end{aligned}$$

Therefore

$$\begin{aligned} X(K_\infty) - K'_\infty \omega_\infty = 0 \\ DX(K_\infty)N_\infty - N'_\infty \omega_\infty - N_\infty \Lambda_\infty = 0 \end{aligned}$$

### 6.3 Proof of Uniqueness

We suppose a new different periodic orbit $(\overline{K}_\infty, \overline{\omega}_\infty)$ such that:

$$\left\|\overline{K}_\infty - K_*\right\|_{\rho_\infty} < r_K \qquad \left\|\overline{K}'_\infty - K'_*\right\|_{\rho_\infty} < r_{K'} \qquad |\overline{\omega}_\infty - \omega_*| < r_\omega$$

and with phase condition:

$$\int_0^1 K'_*(\theta)^T \overline{K}_\infty(\theta) d\theta = 0$$

We consider:

$$\begin{aligned} \Delta K = \overline{K}_\infty - K_\infty \\ \Delta\omega = \overline{\omega}_\infty - \omega_\infty \end{aligned}$$

And have:

$$\begin{aligned} 0 =& X(\overline{K}_\infty) - \overline{K}'_\infty \overline{\omega}_\infty = X(K_\infty + \Delta K) - (K_\infty - \Delta K)'(\omega_\infty - \Delta\omega) \\ =& \boxed{X(K_\infty)} + DX(K_\infty)\Delta K + \int_0^1 (1-s) D^2X(K_\infty + s\Delta K)[\Delta K, \Delta K] ds \\ & \boxed{-K'_\infty \omega_\infty} - \Delta K' \omega_\infty - K'_\infty \Delta\omega - \Delta K' \Delta\omega \\ =& DX(K_\infty)\Delta K - \Delta K' \omega_\infty - K'_\infty \Delta\omega - \Delta K' \Delta\omega + \int_0^1 (1-s) D^2X(K_\infty + s\Delta K)[\Delta K, \Delta K] ds \end{aligned}$$

Hence:

$$DX(K_\infty)\Delta K - \Delta K'\omega_\infty - K'_\infty\Delta\omega = \underbrace{\Delta K'\Delta\omega - \int_0^1 (1-s)D^2X(K_\infty + s\Delta K)[\Delta K, \Delta K]ds}_{-E}$$

Notice we have a similar situation as when bounding the error of the following step of the Newton method.

Recall that we also have:

$$DX(K_\infty)P_\infty - P'_\infty(\theta) - P_\infty \begin{pmatrix} 0 & 0 \\ 0 & \Lambda_\infty \end{pmatrix} = 0$$

Considering the frame $\Delta K = P_\infty \xi^K$ where $\xi^K = \begin{pmatrix} \xi^{K,L} \\ \xi^{K,N} \end{pmatrix}$

$$DX(K_\infty)P_\infty\xi^K - P'_\infty\xi^K\omega_\infty - P_\infty(\xi^K)'\omega_\infty - K'_\infty\Delta\omega = -E$$

$$\left[DX(K_\infty)P_\infty - P'_\infty\omega_\infty\right]\xi^K - P_\infty(\xi^K)'\omega_\infty - K'_\infty\Delta\omega = -E$$

$$P_\infty \begin{pmatrix} 0 & 0 \\ 0 & \Lambda_\infty \end{pmatrix}\xi^K - P_\infty(\xi^K)'\omega_\infty - K'_\infty\Delta\omega = -E$$

If we denote:

$$\eta^K = -(P_\infty)^{-1}E$$

We have:

$$P_\infty\left[\begin{pmatrix} 0 & 0 \\ 0 & \Lambda_\infty \end{pmatrix}\xi^K - (\xi^K)'\omega_\infty - \begin{pmatrix} \Delta\omega \\ 0 \end{pmatrix} - \eta^K\right] = 0$$

And by lemmas (4) and (5) we have unique solutions $\widetilde{\xi^{K,L}}$ and $\xi^{K,N}$ for the cohomological equations:

$$\begin{aligned} \eta^{K,L} &= (\xi^{K,L})'\omega_\infty - \Delta\omega \\ \eta^{K,N} &= \Lambda_\infty\xi^{K,N} - (\xi^{K,N})'\omega_\infty \end{aligned}$$

The average $\left\langle \xi^{K,L} \right\rangle$ is fixed by the condition:

$$\int_0^1 K'_*(\theta)^T\overline{K}_\infty(\theta)d\theta = 0$$

So, if we denote

$$\xi^{K,L} = \left\langle \xi^{K,L} \right\rangle + \widetilde{\xi^{K,L}}$$

We have

$$\begin{aligned} \Delta K &= K'_\infty\xi^{K,L} + N_\infty\xi^{N,L} = K'_\infty\left\langle \xi^{K,L} \right\rangle + K'_\infty\widetilde{\xi^{K,L}} + N_\infty\xi^{K,N} \\ &= K'_\infty\left\langle \xi^{K,L} \right\rangle + \widetilde{\Delta K} \end{aligned}$$

And we can write:

$$\begin{aligned}0 &= \int_0^1 K_*'(\theta)^T \overline{K}_\infty(\theta)d\theta = \int_0^1 K_*'(\theta)^T \left(K_\infty(\theta) + K_\infty'(\theta)\left\langle \xi^{K,L}\right\rangle + \widetilde{\Delta K}(\theta)\right) d\theta \\ &= 0 + \left(\int_0^1 K_*'(\theta)^T K_\infty(\theta)\right)\left\langle \xi^{K,L}\right\rangle + \int_0^1 K_*'(\theta)^T \widetilde{\Delta K}(\theta)\end{aligned}$$

Finding:

$$\begin{aligned}\left\langle \xi^{K,L}\right\rangle &= -\left(\int_0^1 K_*'(\theta)^T K_\infty'(\theta)d\theta\right)^{-1} \int_0^1 K_*'(\theta)^T \widetilde{\Delta K}(\theta)d\theta \\ &= -\left(\int_0^1 K_*'(\theta)^T K_\infty'(\theta)d\theta\right)^{-1} \int_0^1 K_*'(\theta)^T K_\infty'(\theta)\widetilde{\xi^{K,L}}(\theta) + K_*'(\theta)^T N_\infty(\theta)\xi^{K,N}(\theta)\end{aligned}$$

Now we study the behavior of the distance between these two solutions considering the unique solutions of the cohomological equations $\xi^{K,L}, \xi^{K,N}$. We bound all the appearing terms:

Notice that we can write $\Delta K'$ as:

$$\begin{aligned}\Delta K' &= \overline{K}_\infty' - K_\infty' = \frac{1}{\overline{\omega}_\infty} X(\overline{K}_\infty) - \frac{1}{\omega_\infty} X(K_\infty) \\ &= \left(\frac{1}{\overline{\omega}_\infty} - \frac{1}{\omega_\infty}\right) X(\overline{K}_\infty) + \frac{1}{\omega_\infty}\left(X(\overline{K}_\infty) - X(K_\infty)\right) \\ &= \frac{\omega_\infty - \overline{\omega}_\infty}{\overline{\omega}_\infty \omega_\infty} X(\overline{K}_\infty) + \frac{1}{\omega_\infty}\int_0^1 DX(K_\infty + s\Delta K)\Delta K ds \\ &= -\frac{\Delta\omega}{\omega_\infty}\overline{K}_\infty' + \frac{1}{\omega_\infty}\int_0^1 DX(K_\infty + s\Delta K)\Delta K ds\end{aligned}$$

avoiding in this way applying Cauchy estimates and loose domain and bound it by:

$$\begin{aligned}\|\Delta K'\|_{\rho_\infty} &\leq \frac{|\Delta\omega|\|\overline{K}_\infty'\|_{\rho_\infty}}{|\omega_\infty|} + \frac{1}{|\omega_\infty|}\|DX(K_\infty + s\Delta K)\|_{\rho_\infty}\|\Delta K\|_{\rho_\infty} \\ &\leq C_{K'}C_{\omega^{-1}}|\Delta\omega| + C_{\omega^{-1}}C_{DX}\|\Delta K\|_{\rho_\infty}\end{aligned}$$

Considering this, for $\eta^K$ we have:

$$\begin{aligned}\|\eta^K\|_{\rho_\infty} &= \|P_\infty^{-1}\left(\Delta K'\Delta\omega - \int_0^1 (1-s)D^2X(K_\infty + s\Delta K)[\Delta K, \Delta K]ds\right)\|_{\rho_\infty} \\ &\leq \|P_\infty^{-1}\|_{\rho_\infty}\left(\|\Delta K'\|_{\rho_\infty}|\Delta\omega| + \frac{\|D^2X(K_\infty + s\Delta K)\|_{\rho_\infty}\|\Delta K\|_{\rho_\infty}\|\Delta K\|_{\rho_\infty}}{2}\right) \\ &\leq C_{P^{-1}}\left(C_{K'}C_{\omega^{-1}}|\Delta\omega|^2 + C_{\omega^{-1}}C_{DX}|\Delta\omega|\|\Delta K\|_{\rho_\infty} + \frac{C_{D^2X}}{2}\|\Delta K\|_{\rho_\infty}^2\right) \\ &\leq C_{P^{-1}}\left(C_{K'}C_{\omega^{-1}} + C_{\omega^{-1}}C_{DX} + \frac{C_{D^2X}}{2}\right)\max\left\{|\Delta\omega|, \|\Delta K\|_{\rho_\infty}\right\}^2 \\ &= C_{\eta_\infty^K}\max\left\{|\Delta\omega|, \|\Delta K\|_{\rho_\infty}\right\}^2\end{aligned}$$

Because of the Lemmas (4) and (5), we know the solutions of the equations are bounded by

$$\|\widetilde{\xi^{K,L}}\|_{\rho_\infty} \leq \mathcal{C}_\omega^{K,L}\|\eta^K\|_{\rho_\infty}$$
$$\|\xi^{K,N}\|_{\rho_\infty} \leq \mathcal{C}_{\omega,\Lambda}^{K,N}\|\eta^K\|_{\rho_\infty}$$

And for the average, because we are under the same hypothesis as in *Iterative Lemma*, the computations are the same as in (A.), obtaining:

$$\left|\left\langle \xi^{K,L}\right\rangle\right| \leq C_{\langle \xi^{K,L}\rangle}\|\eta^K\|_{\rho_\infty}$$

Having:

$$\begin{aligned}\|\xi^{K,L}\|_{\rho_\infty} &\leq \left|\left\langle \xi^{K,L}\right\rangle\right| + \|\widetilde{\xi^{K,L}}\|_{\rho_\infty} \leq \left(C_{\langle \xi^{K,L}\rangle} + \mathcal{C}_\omega^{K,L}\right)\|\eta^K\|_{\rho_\infty} \\ &\leq C_{\xi^{K,L}} C_{\eta_\infty^K} \max\left\{|\Delta\omega|, \|\Delta K\|_{\rho_\infty}\right\}^2 = C_{\xi_\infty^{K,L}} \max\left\{|\Delta\omega|, \|\Delta K\|_{\rho_\infty}\right\}^2\end{aligned}$$

$$\|\xi^{K,N}\|_{\rho_\infty} \leq \mathcal{C}_{\omega,\Lambda}^{K,N}\|\eta^K\|_{\rho_\infty} \leq \mathcal{C}_{\omega,\Lambda}^{K,N} C_{\eta_\infty^K} \max\left\{|\Delta\omega|, \|\Delta K\|_{\rho_\infty}\right\}^2 = C_{\xi_\infty^{K,N}} \max\left\{|\Delta\omega|, \|\Delta K\|_{\rho_\infty}\right\}^2$$

Bounding $\|\Delta K\|_{\rho_\infty}$ and $|\Delta\omega|$ we have:

$$\begin{aligned}\|\Delta K\|_{\rho_\infty} &= \|K'_\infty \xi^{K,L} + N_\infty \xi^{K,N}\|_{\rho_\infty} \leq \|K'_\infty\|_{\rho_\infty}\|\xi^{K,L}\|_{\rho_\infty} + \|N_\infty\|_{\rho_\infty}\|\xi^{K,N}\|_{\rho_\infty} \\ &\leq \left(C_{K'} C_{\xi_\infty^{K,L}} + C_N C_{\xi_\infty^{K,N}}\right) \max\left\{|\Delta\omega|, \|\Delta K\|_{\rho_\infty}\right\}^2 = C_{\Delta K_\infty} \max\left\{|\Delta\omega|, \|\Delta K\|_{\rho_\infty}\right\}^2\end{aligned}$$

$$|\Delta\omega| = \left|(\widehat{\eta^{K,L}})_0\right| \leq \|\eta^K\|_{\rho_\infty} \leq C_{\eta_\infty^K} \max\left\{|\Delta\omega|, \|\Delta K\|_{\rho_\infty}\right\}^2 = C_{\Delta\omega_\infty} \max\left\{|\Delta\omega|, \|\Delta K\|_{\rho_\infty}\right\}^2$$

And we can write the recursive inequality:

$$\max\left\{|\Delta\omega|, \|\Delta K\|_{\rho_\infty}\right\} \leq \max\{C_{\Delta K_\infty}, C_{\Delta\omega_\infty}\} \max\left\{|\Delta\omega|, \|\Delta K\|_{\rho_\infty}\right\}^2 = Q_\Delta \max\left\{|\Delta\omega|, \|\Delta K\|_{\rho_\infty}\right\}^2$$

Notice we have

$$\begin{aligned}Q_\Delta|\Delta\omega| = Q_\Delta|\overline{\omega}_\infty - \omega_\infty| &\leq |\overline{\omega}_\infty - \omega_0| + |\omega_\infty - \omega_0| \\ \underbrace{\leq}_{\text{Proof of existence}} \quad & Q_\Delta|\overline{\omega}_\infty - K_0| + Q_\Delta C_{\Delta\omega}\delta_0\Sigma_{-1}(\alpha,\mu)\mathcal{E}_0 < 1\end{aligned}$$

$$\begin{aligned}Q_\Delta\|\Delta K\|_{\rho_\infty} = Q_\Delta\|\overline{K}_\infty - K_\infty\|_{\rho_\infty} &\leq \|\overline{K}_\infty - K_0\|_{\rho_\infty} + \|K_\infty - K_0\|_{\rho_\infty} \\ \underbrace{\leq}_{\text{Proof of existence}} \quad & Q_\Delta\|\overline{K}_\infty - K_0\|_{\rho_\infty} + Q_\Delta C_{\Delta K}\delta_0\Sigma_{-1}(\alpha,\mu)\mathcal{E}_0 < 1\end{aligned}$$

And considering the following result:

**Lemma 8.** *For $x \in \mathbb{R}^+$ and a constant $q$ such that $x \leq qx^2$ and $xq < 1$. Then $x = 0$*

*Proof.* If $x \neq 0$, then we have $1 \leq qx$, in contradiction with $qx < 1$. □

We obtain $\max\left\{|\Delta\omega|, \|\Delta K\|_{\rho_\infty}\right\} = 0$ and indeed $\overline{K}_\infty = K_\infty, \overline{\omega}_\infty = \omega_\infty$.

**Remark**: Once having the uniqueness of the periodic orbit, we can fix Floquet matrix and the normal component, having uniqueness for the complete system.

# References


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## Annex: Summary of Constants

Here we have a recap of all the defined variables in this project which depend on the fixed initial non-degenerate tuple $(K_*, \omega_*, N_*, \Lambda_*)$, the distance between any point and the initial tuple $r_K, r_{K'}, r_\omega, r_N, r_{N'}, r_\Lambda$, the vector field $X$ and for some variables, the band loss $\delta$.

| Table of constants |
|---|
| $C_K = r_K + \|K_*\|_{\rho_*}$ |
| $C_{K'} = r_{K'} + \|K'_*\|_{\rho_*}$ |
| $C_N = r_N + \|N_*\|_{\rho_*}$ |
| $C_{N'} = r_{N'} + \|N'_*\|_{\rho_*}$ |
| $C_\omega = r_\omega + \|\omega_*\|$ |
| $C_{\Lambda^{-1}} = \dfrac{\left\|\Lambda_*^{-1}\right\|}{1 - \left\|\Lambda_*^{-1}\right\| r_\Lambda}$ |
| $C_{\omega^{-1}} = \dfrac{\left\|\omega_*^{-1}\right\|}{1 - \left\|\omega_*^{-1}\right\| r_\omega}$ |
| $C_{P^{-1}} = \dfrac{\left\|P_*(\theta)^{-1}\right\|_\rho}{1 - \|P_*(\theta)^{-1}\|_\rho \, r_P}$ |
| $C_1(\omega_*, \Lambda_*) = \sup\limits_{k \in \mathbb{Z}^*} \left\| \left( \dfrac{\Lambda_*}{2\pi i k \omega_*} - I_{n-1} \right)^{-1} \right\|$ |
| $C_2(\omega_*, \Lambda_*) = \sup\limits_{k \in \mathbb{Z}^*} \left\| \left( \dfrac{I_{n-1} \otimes \Lambda_* - \Lambda_*^T \otimes I_{n-1}}{2\pi i k \omega_*} - I_{(n-1)^2} \right)^{-1} \right\|$ |

| |
|---|
| $C_{1,\Sigma}(\omega_*, \Lambda_*) = C_1(\omega_*, \Lambda_*)\frac{1}{2\sqrt{3}\omega_*}$ |
| $C_{2,\Sigma}(\omega_*, \Lambda_*) = C_2(\omega_*, \Lambda_*)\frac{1}{2\sqrt{3}\omega_*}$ |
| $C_{S_1} = \frac{C_{1,\Sigma}(\Lambda_*, \omega_*)}{1 - C_1(\Lambda_*, \omega_*)\left(\frac{r_\Lambda}{2\pi\lvert\omega_*\rvert} + \frac{r_\omega}{\lvert\omega_*\rvert}\right)}$ |
| $C_{S_2} = \frac{C_{2,\Sigma}(\Lambda_*, \omega_*)}{1 - C_2(\Lambda_*, \omega_*)\left(\frac{nr_\Lambda}{2\pi\lvert\omega_*\rvert} + \frac{r_\omega}{\lvert\omega_*\rvert}\right)}$ |
| $m_{K'_*} = \int_0^1 K'_*(\theta)^T K'_*(\theta) d\theta$ |
| $m_{N_*} = \int_0^1 N_*(\theta)^T N_*(\theta) d\theta$ |
| $C_{m_{K'_*}} = \frac{1}{\lvert m_{K'_*}\rvert - \lVert (K'_*)^T \rVert_\rho r_{K'}}$ |
| $C_{m_{N_*}} = \frac{\lvert m_{N_*}^{-1}\rvert}{1 - \lvert m_{N_*}^{-1}\rvert \lVert (N_*)^T \rVert_\rho r_N}$ |
| $\mathcal{C}_\omega^{K,L} = \frac{C_{\omega^{-1}}}{\sqrt{6}}$ |
| $\mathcal{C}_{\omega,\Lambda}^{K,N} = C_{\Lambda^{-1}} + \sqrt{2}\sqrt{n-1}C_{S_1}$ |
| $\mathcal{C}_{\omega,\Lambda}^{N,L} = C_{\Lambda^{-1}} + \sqrt{2}\sqrt{n-1}C_{S_1}$ |
| $\mathcal{C}_{\omega,\Lambda}^{N,N} = (n-1)^2\sqrt{2}C_{S_2}$ |

| |
|---|
| $C_{DX} \geq \sup_{x \in \mathcal{D}} \lvert DX(x) \rvert$ |
| $C_{D^2X} \geq \sup_{x \in \mathcal{D}} \left\lvert D^2X(x) \right\rvert$ |
| $C_{\eta^K} = C_{P^{-1}}$ |
| $C_{\widetilde{\xi^{K,L}}} = \mathcal{C}_{\omega}^{K,L} C_{\eta^K}$ |
| $C_{\xi^{K,N}} = \mathcal{C}_{\omega,\Lambda}^{K,N} C_{\eta^K}$ |
| $C_{\langle \xi^{K,L} \rangle} = C_{m_K} \left\lVert (K'_*)^T \right\rVert_{\rho} \left( r_{K'} C_{\widetilde{\xi^{K,L}}} + r_N C_{\xi^{K,N}} \right)$ |
| $C_{\xi^{K,L}} = C_{\langle \xi^{K,L} \rangle} + C_{\widetilde{\xi^{K,L}}}$ |
| $C_{\Delta K} = C_{K'} C_{\xi^{K,L}} + C_N C_{\xi^{K,N}}$ |
| $C_{\Delta\omega} = C_{\eta^K}$ |
| $C_{\eta^N} = C_{P^{-1}} \left( 1 + C_{\Delta\omega} C_{N'} + C_{D^2X} C_{\Delta K} C_N \right)$ |
| $C_{\xi^{N,L}} = \mathcal{C}_{\omega,\Lambda}^{N,L} C_{\eta^N}$ |
| $C_{\widetilde{\xi^{N,N}}} = \mathcal{C}_{\omega,\Lambda}^{N,N} C_{\eta^N}$ |
| $C_{\langle \xi^{N,N} \rangle} = C_{m_K} \left\lVert N_*^T \right\rVert_{\rho} \left( C_{K'} C_{\xi^{N,L}} + C_N C_{\widetilde{\xi^{N,N}}} \right)$ |
| $C_{\xi^{N,N}} = \left( C_{\langle \xi^{N,N} \rangle} + C_{\widetilde{\xi^{N,N}}} \right)$ |

| |
|---|
| $C_{\Delta N} = C_{K'} C_{\xi^{N,L}} + C_N C_{\xi^{N,N}}$ |
| $C_{\Delta\Lambda} = C_{\eta^N} + 2 C_\Lambda C_{\langle \xi^{N,N} \rangle}$ |
| $Q_K(\delta) = C_{\xi^{K,L}} + C_{\xi^{K,N}} + C_{\Delta K} C_{\Delta\omega} + \dfrac{C_{D^2 X} C_{\Delta K}^2 \delta}{2}$ |
| $Q_N(\delta) = C_{\xi^{N,L}} + C_{\xi^{N,N}} + C_{\Delta N} C_{\Delta\omega} + \max\{\delta, 1\} C_{\Delta N} C_{\Delta\Lambda} + C_{D^2 X} C_{\Delta K} C_{\Delta N} \delta$ |
| $\mathcal{E}_0 = \max\left\{ \dfrac{\|E_{K_0}\|_\rho}{\delta_0}, \|E_{N_0}\|_\rho \right\}$ |
| $Q_{\mathcal{E}} = \max\{\alpha Q_K, Q_N, \delta_0 Q_N\}$ |
| $\Sigma_a(\alpha, \mu) = \sum_{i=0}^{\infty} \alpha^{-ai} \mu^{2^i - 1}$ |
| $C_{\eta_\infty^K} = C_{P^{-1}} \left( C_{K'} C_{\omega^{-1}} + C_{\omega^{-1}} C_{DX} + \dfrac{C_{D^2 X}}{2} \right)$ |
| $C_{\xi_\infty^{K,L}} = C_{\xi^{K,L}} C_{\eta_\infty^K} C_{\xi_\infty^{K,N}} = mathcal C_{\omega,\Lambda}^{K,N} C_{\eta_\infty^K}$ |
| $C_{\Delta K_\infty} = C_{K'} C_{\xi_\infty^{K,L}} + C_N C_{\xi_\infty^{K,N}}$ |
| $C_{\Delta\omega_\infty} = C_{\eta_\infty^K}$ |
| $Q_\Delta = \max\{C_{\Delta K_\infty}, C_{\Delta\omega_\infty}\}$ |