Universitat de Barcelona

Facultat de Matemàtiques i Informàtica

ADVANCED MATHEMATICS

MASTER'S FINAL PROJECT

Introduction to contact topology

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Abstract

This master's thesis provides an introduction to contact topology, with the primary objective of proving Martinet's Theorem, which asserts that every closed, connected 3-manifold admits a contact structure. The proof heavily relies on the Lickorish-Wallace Theorem, which states that any such 3-manifold can be obtained from S^3 via a finite sequence of Dehn surgeries. The thesis explores key concepts in contact topology, such as contact structures, Darboux's Theorem, and Gray stability. A complete proof of the Lickorish-Wallace Theorem is given before focusing on the detailed proof of Martinet's Theorem, highlighting the ubiquity of contact structures in 3-manifolds.

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1 Introduction

Contact topology is a rich and intriguing field of study within differential topology that deals with contact structures on manifolds. A contact structure on a smooth manifold is a completely non-integrable hyperplane distribution in its tangent bundle. In three dimensions, this can be visualized as a collection of planes that twist in a specific way, preventing the existence of smooth surfaces tangent to the planes everywhere. Contact structures are essential in the study of dynamical systems, geometric topology, and even theoretical physics. This sibling relationship underpins much of the theory, where many techniques and ideas in symplectic topology have contact analogues.

Definition 1.1. *Let M be a manifold of odd dimension $2n+1$. A contact structure $\xi \subset TM$ is a hyperplane field given (maybe locally) as the kernel of a contact form, i.e., $\xi = \mathrm{Ker}(\alpha)$. A differential form α on M is called a contact form if*

$$\alpha \wedge (d\alpha)^n \neq 0,$$

at any point, where "$\neq 0$" means that the form does not vanish.

This project aims to prove the following theorem:

Theorem 1.2 (Martinet)**.** *Every closed, orientable 3–manifold M admits a contact structure.*

This theorem shows how common contact structures are in 3-manifolds and forms the main focus of this work.

To prove Martinet's Theorem, we first need to understand two key results in contact topology: *Darboux's Theorem* and *Gray Stability*. Much like the symplectic version of Darboux's theorem, Darboux's theorem in contact topology ensures that contact structures are locally the same, meaning that there are no local invariants distinguishing them up to a smooth change of coordinates. This is a key fact that distinguishes it from Riemannian geometry, where powerful local invariants, such as curvature, play a central role in describing the geometry of a space. The lack of such local invariants in contact geometry centres the study on more global phenomena.

To prove Martinet's Theorem, we rely on a key topological result: the *Lickorish and Wallace Theorem*:

Theorem 1.3 (Lickorish and Wallace)**.** *Any closed, connected, orientable 3-manifold M can be obtained from S^3 by a finite collection of Dehn surgeries.*

Dehn surgery is a fundamental technique in 3-manifold topology. It involves removing a tubular neighbourhood of a knot in a 3-manifold and gluing it back differently, specified by a pair of coprime integers. This process alters the topology of the manifold in a controlled manner, allowing one to construct new 3-manifolds from simpler ones.

To prove Martinet's Theorem, we can use Lickorish and Wallace's theorem to reduce the problem to S^3 with surgery along some knots. The goal is to construct a contact structure on S^3, which is relatively straightforward, and then extend this structure to the neighborhoods of the knots where Dehn surgery is performed. Fortunately, we can deform the knots to make them transverse to the hyperplane field. This, alongside Darboux's Theorem, allows us to construct a model for the neighborhoods of knots, enabling us to extend the structure from S^3.

2 Contact manifolds

In this section, we introduce the foundational concepts of contact manifolds, which are essential for understanding geometric structures in higher dimensions. We begin by defining contact structures and outlining their fundamental properties. This is followed by a series of illustrative examples that demonstrate their occurrence in various settings. Finally, we establish key results, such as the Darboux theorem and Gray's stability theorem, which highlight the local behavior and invariance of contact structures.

2.1 Contact structures

A contact structure is a field of hyperplanes on a smooth manifold that is maximally non-integrable. Equivalently, the hyperplane field may be given (at least locally) as the kernel of a differential one-form and maximal non-integrability becomes a certain non-degeneracy condition on this form.

We first start by defining a field of hyperplanes:

Definition 2.1. *Let M be a differential manifold and TM its tangent bundle. A field of hyperplanes on M is a smooth sub-bundle of TM of codimension 1.*

By the following Lemma we can choose locally, a differential 1-form, called a contact form, whose kernel is the hyperplane field.

Lemma 2.2. *Locally, ξ can be written as the kernel of a differential 1-form α. It is possible to write $\xi = \ker\alpha$ with a 1-form α defined globally on all of M if and only if ξ is coorientable, which by definition means that the quotient line bundle TM/ξ is trivial.*

Proof. As ξ is a locally trivial vector bundle, then the quotient bundle TM/ξ and its dual bundle are locally trivial. Hence, over small neighborhoods U, one can define a differential 1-form α_U as the pull-back to the cotangent bundle T^*M of a non-zero section of $(TM/\xi)^*|_U$ under the bundle projection $\pi : TM \to TM/\xi$.

This 1-form clearly satisfies $\mathrm{Ker}(\alpha_U) = \xi|_U$.

If ξ is coorientable, then $(TM/\xi)^*$ admits a global section, and the above construction yields a global 1-form α defining ξ. Conversely, a global 1-form α defining $\xi = \ker\alpha$ induces a global non-zero section of $(TM/\xi)^*$. □

Definition 2.3. *A hyperplane field ξ on a manifold M is called integrable if for any point $p \in M$ one can find a codimension 1 manifold $N \subset M$ such that its tangent space is the hyperplane field $TN = \xi$*

At first glance, one may think that if a hyperplane field is smooth, as in our case, the regularity could imply somehow the integrability condition. Nevertheless, the following

Theorem will help us classify integrability and prove that not all hyperplane fields are integrable.

Theorem 2.4 (Frobenius integrability condition). *A hyperplane field is $\xi = \operatorname{Ker}\alpha$ integrable precisely when:*

$$\alpha \wedge d\alpha \equiv 0$$

We are now ready to define a contact structure, which is a maximally non-integrable hyperplane field.

Definition 2.5. *Let M be a manifold of odd dimension $2n+1$. A contact structure is a hyperplane field $\xi = ker\alpha$ such that*

$$\alpha \wedge (d\alpha)^n \neq 0$$

at any point.

This definition of a contact structure does not depend on the specific choice of the contact form. While the 1-form α that gives rise to $\xi = \operatorname{Ker}(\alpha)$ defines the contact structure locally, it is not unique. In fact, any other 1-form defining the same contact structure must differ from α by multiplication with a nowhere-vanishing smooth function $\lambda : M \to \mathbb{R} \setminus \{0\}$. More precisely, if α' is another form such that $\operatorname{Ker}(\alpha') = \xi$, then there exists a smooth function λ such that $\alpha' = \lambda\alpha$.

To see why the definition does not depend on α, let α and α' be two 1-forms defining the same contact structure, i.e., $\operatorname{Ker}(\alpha) = \operatorname{Ker}(\alpha') = \xi$. Since their kernels coincide, it follows that $\alpha' = \lambda\alpha$ for some nowhere-vanishing smooth function λ.

Now, assuming that α is a contact form, meaning that $\alpha \wedge (d\alpha)^n \neq 0$, we will show that α' is also a contact form. m, i.e. it satisfies $\alpha \wedge (d\alpha)^n \neq 0$ let's see α' is also a contact form:

$$\begin{aligned} \alpha' \wedge (d\alpha')^n &= \lambda\alpha \wedge (d(\lambda\alpha))^n \\ &= \lambda^{n+1}\alpha \wedge (d\alpha)^n \neq 0, \end{aligned}$$

as λ^{n+1} is nowhere zero.

What do we mean by maximally non-integrable? To answer this question, we need to define isotropic submanifolds.

Definition 2.6. *Let (M, ξ) be a contact manifold, a submanifold $L \subseteq M$ is called an isotropic submanifold if for every point $p \in L$, the tangent space of L is inside the hyperplane field $T_pL \subseteq \xi_p$.*

A submanifold $L \subseteq M$ is called an integrable submanifold if for every point $p \in L$, the tangent space of L is precisely the distribution at p, i.e., $T_pL = \xi_p$.

In a sense, isotropic submanifolds generalize integrable submanifolds. Now, when talking about maximally non-integrable, what we really mean is that the dimension of all the isotropic submanifolds is bounded as much as possible. The following proposition gives the bound on the dimension of isotropic submanifolds.

Proposition 2.7. *Let (M, ξ) be a contact manifold of dimension $2n+1$ and let $L \subseteq (M, \xi)$ be an isotropic submanifold. Then* $\dim L \leq n$.

Proof. Consider the inclusion $i : L \to M$ and let α be a contact form that defines ξ. Now as L is isotropic $i^*\alpha \equiv 0$ and thus $i^* d\alpha \equiv 0$. So for each point $p \in L$, $(\xi_p, d\alpha|_{\xi_p})$ is a symplectic vector space. Now $T_pL \subseteq \xi_p$ satisfies that $T_pL \subset (T_pL)^{\perp}$, where $\perp$ is the symplectic orthogonal complement [1]. Using the dimension formula for a subspace U of a symplectic vector space (V, ω):

$$\dim U + \dim U^{\perp} = \dim V$$

it is clear that $\dim T_pL \leq \frac{\dim \xi_p}{2} = n$. □

One could ask, is it enough for a 1-form to satisfy the Frobenius integrability condition for the hyperplane field to be maximally non-integrable? In dimension, 3 the contact condition and Frobenius integrability condition are equivalent.But in higher dimension the answer is, at least in general, no. To prove this fact, we may simply give a counterexample. The following example gives a 1-form α such that $\alpha \wedge d\alpha \neq 0$ but $\alpha \wedge (d\alpha)^n = 0$ and we will see an integrable submanifold of dimension higher than n.

Example 2.8. Define the 1-form α on $\mathbb{R}^5$ by:

$$\alpha = x_1\, dx_2 + x_3\, dx_4.$$

Computing, we see:

$$\alpha \wedge d\alpha \neq 0$$
$$\alpha \wedge (d\alpha)^2 = 0.$$

The hyperplane field is spanned by the following vector fields:

$$\mathrm{Ker}(\alpha) = \mathrm{Span}\left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_5}, -x_3 \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_4} \right\}.$$

And so given $C_1, C_2 \in \mathbb{R}$, all the submanifolds defined by the global charts $(a, b, c) \mapsto (a, C_1, b, C_2), c$ are isotropic of dimension $3 = n + 1$.

[1] Check [3] or any symplectic geometry book for more details

Now, we are ready to introduce the Reeb vector field. Given a contact form α, we can uniquely associate a vector field, denoted as R_α. This vector field proves to be invaluable, as it provides a natural flow that simplifies the analysis of the manifold's geometry. By reducing higher-dimensional structures into more manageable components.

Lemma 2.9 (Definition of Reeb vector field)**.** *Given any contact form α, there is a unique vector field called Reeb vector field R_α defined by:*

1. $d\alpha(R_\alpha, -) \equiv 0$,
2. $\alpha(R_\alpha) \equiv 1$.

Proof. If we take a point $p \in M$ then the 2-form $d\alpha$ is a skew symmetric matrix and by the contact condition we know it has maximal rank $2n$. So the kernel of $d\alpha$ at that point is one-dimensional, so we have a smooth line field where R_α lives. Then as α is non-zero in that line field by the contact condition $\alpha(R_\alpha) \equiv 1$ specifies a non-trivial section of it. □

Another fundamental concept in contact geometry is the notion of isomorphisms between contact manifolds, which are referred to as contactomorphisms.

Definition 2.10 (Contactomorphism)**.** *Two contact manifolds (M_1, ξ_1) and (M_2, ξ_2) are said to be contactomorphic if there is a diffeomorphism $f : M_1 \to M_2$ with*

$$Tf(\xi_1) = \xi_2.$$

If $\xi_i = \operatorname{Ker} \alpha_i$, $i = 1, 2$, this is equivalent to saying that α_1 and $f^*\alpha_2$ determine the same hyperplane field, and hence equivalent to the existence of a nowhere zero function $\lambda : M_1 \to \mathbb{R}\backslash\{0\}$ such that

$$f^*\alpha_2 = \lambda\alpha_1.$$

Given two contact forms of two contact manifolds (M_1, α_1) and (M_2, α_2) a strict contactomorphism is a contactomorphism such that

$$f^*\alpha_2 = \alpha_1.$$

Usually contactomorphisms are good enough as the contact form does not enrich the contact structure.

2.2 Examples

Now we will introduce some simple examples of contact manifolds to build intuition and provide foundational context. We will begin with the standard contact structure on $\mathbb{R}^{2n+1}$. This example is particularly important because it serves as the canonical local model for contact structures.

Example 2.11. [Standard contact structure on $\mathbb{R}^{2n+1}$] The contact form is:

$$\alpha_1 = dz + \sum_{j=0}^{n} x_j dy_j$$

Where we have taken the Cartesian coordinates $(x_1, y_1 \ldots, x_n, y_n, z)$

It is an easy exercise to see it is in fact a contact form:

$$d\alpha_1 = \sum_{j=0}^{n} dx_j \wedge dy_j$$
$$(d\alpha_1)^n = dx_1 \wedge dy_1 \wedge \cdots \wedge x_n \wedge dy_n$$
$$\alpha_1 \wedge (d\alpha_1)^n = dz \wedge dx_1 \wedge dy_1 \wedge \cdots \wedge x_n \wedge dy_n \neq 0$$

Also, we can compute the Reeb vector field R_{α_1} lets write it in coordinates:

$$R_{\alpha_1} = R_z \partial_z + \sum_{j=1}^{n} R_{x_i} \partial_{x_j} + \sum_{j=1}^{n} R_{y_i} \partial_{y_j}$$

So let's impose the first condition $d\alpha(R_{\alpha_1}, -) = 0$:

$$d\alpha(R_{\alpha_1}, \partial_z) = (\sum_{j=0}^{n} dx_j \wedge dy_j)(R_{\alpha_1}, \partial_z) \equiv 0 = 0,$$
$$d\alpha(R_{\alpha_1}, \partial_{x_i}) = (\sum_{j=0}^{n} dx_j \wedge dy_j)(R_{\alpha_1}, \partial_{x_i}) = R_{y_i} = 0,$$
$$d\alpha(R_{\alpha_1}, \partial_{y_i}) = (\sum_{j=0}^{n} dx_j \wedge dy_j)(R_{\alpha_1}, \partial_{x_i}) = R_{x_i} = 0.$$

Where we get $R_{\alpha_1} = R_z \partial_z$, but using $\alpha_1(R_{\alpha_1}) = 1$ we get

$$R_{\alpha_1} = \partial_z.$$

Now, the following structure will prove to be very useful when proving Martinet's theorem, as it will serve as the model we consider on the 3-sphere.

Example 2.12. [Standard contact structure in S^{2n+1}] The standard contact structure on the unit sphere S^{2n+1} in $\mathbb{R}^{2n+2}$ (with Cartesian coordinates $(x_1, y_1, \ldots, x_{n+1}, y_{n+1})$) is defined by the following 1-form restricted to S^{2n+1}

$$\alpha_0 = \sum_{j=1}^{n+1} \left(x_j \, dy_j - y_j \, dx_j \right).$$

To see it's a contact form in S^{2n+1} we consider $r = \sum_j x_j y_j$ and we only need to see

$$\alpha_0 \wedge (d\alpha_0)^n \wedge r \, dr \neq 0 \quad \text{for} \quad r = 1,$$

as S^{2n+1} is a level surface of r^2 and rdr does not vanish in S^{2n+1} which is just a matter of calculation.

It can be proven that once a point $p \in \mathbb{S}^{2n+1}$ is removed, the punctured sphere $\mathbb{S}^{2n+1} \setminus p$, equipped with the given contact structure, is contactomorphic to $\mathbb{R}^{2n+1}$ with its standard contact structure. This is why we also call this structure "standard". Further details can be found in Proposition 2.13 of the section on contact geometry in the Handbook of Differential Geometry [1].

Example 2.13. [Standard overtwisted contact structure of $\mathbb{R}^3$] On $\mathbb{R}^3$ with cylindrical coordinates (r, ϕ, z) the contact form is:

$$\alpha_{\text{ot}} = \cos(r)dz + r\sin(r)d\phi.$$

The form α_{ot} is smooth, since $r^2 d\phi$ is a smooth form and the function

$$r \mapsto \begin{cases} \frac{\sin r}{r} & \text{for } r \neq 0, \\ 1 & \text{for } r = 0 \end{cases}$$

is also smooth. Let's check it is a contact form

$$\begin{aligned} d\alpha_{\text{ot}} &= -\sin(r)dr \wedge dz + (\sin(r) + r\cos(r))dr \wedge d\phi \\ \alpha_{\text{ot}} \wedge d\alpha_{\text{ot}} &= \cos(r)(\sin(r) + r\cos(r))dz \wedge dr \wedge d\phi \\ &\quad - r\sin^2(r)d\phi \wedge dr \wedge dz = \\ &= (r + \sin r \cos r)\, dr \wedge d\phi \wedge dz \neq 0 \end{aligned}$$

Now let's compute the Reeb vector field. We start as before by writing it in coordinates:

$$R_{\text{ot}} = R_x \partial_x + R_x \partial_y + R_z \partial_z.$$

Let's impose the first condition:

$$\begin{aligned} d\alpha(R_{\text{ot}}, \partial_r) &= -(-\sin(r)R_z + (\sin(r) + r\cos(r))R_\phi) = 0 \\ d\alpha(R_{\text{ot}}, \partial_\phi) &= (\sin(r) + r\cos(r))R_r) = 0 \\ d\alpha(R_{\text{ot}}, \partial_z) &= -\sin(r)R_r = 0 \end{aligned}$$

So $R_r = 0$. Now imposing $\alpha(R_{\text{ot}}) = 1$ we get

$$\begin{cases} \cos(r)\, R_z + r\sin(r)\, R_\phi & = 1 \\ \sin(r)\, R_z - (\sin(r) + r\cos(r))\phi & = 0. \end{cases}$$

It is a fundamental and challenging result that $\xi_{OT} = \operatorname{Ker} \alpha_{OT}$ is not isomorphic to the standard structure on $\mathbb{R}^3$.

This result is significant in the classification of contact structures, as it demonstrates that certain contact structures cannot be globally modelled by the standard contact structure on $\mathbb{R}^3$. This insight reveals the richness and diversity of contact structures, highlighting the need for more refined methods in their classification and study.

Example 2.14. [3-dimensional torus] We consider $T = S^1 \times S^1 \times S^1$, and take the coordinates x, y, z for each circle, i.e., $x, y, z \in [0,1]/\sim$, where $0 \sim 1$, so that $[0,1]/\sim = S^1$.

$$\alpha_n = \sin(2\pi nz)dx + \cos(2\pi nz)dy.$$

We first verify α_n is in fact a contact form for each $n \in \mathbb{N} \setminus \{0\}$:

$$\begin{aligned} d\alpha_n &= 2\pi n\cos(2\pi nz)dz \wedge dx - 2\pi n\sin(2\pi nz)dz \wedge dy \\ \alpha_n \wedge d\alpha_n &= -2\pi n\sin^2(2\pi nz)dx \wedge dz \wedge dy + 2\pi n\cos^2(2\pi nz)dy \wedge dz \wedge dx = \\ &= 2\pi n dx \wedge dy \wedge dz \neq 0 \end{aligned}$$

Thus, $\xi_n = \mathrm{Ker}(\alpha_n)$ is indeed a contact structure. Moreover, we can compute the Reeb vector field R_n associated with α_n. Expressing this in coordinates:

$$R_n = R_x\partial_x + R_y\partial_y + R_z\partial_z$$

Then:

$$\begin{aligned} d\alpha_n(R_n, \partial_x) &= 2\pi n\cos(2\pi nz)R_z = 0, \\ d\alpha_n(R_n, \partial_y) &= -2\pi n\sin(2\pi nz)R_z = 0, \\ d\alpha_n(R_n, \partial_z) &= -(2\pi n\cos(2\pi nz)R_x - 2\pi n\sin(2\pi nz)R_y) = 0. \end{aligned}$$

From the first two equations we conclude $R_z = 0$ and from the last one $R_x\cos(2\pi nz) = \sin(2\pi nz)R_y$. The second condition implies that:

$$\begin{aligned} \alpha_n(R_n) &= \sin(2\pi nz)R_x + \cos(2\pi nz)R_y = \\ &= \frac{\sin^2(2\pi nz)}{\cos(2\pi nz)}R_y + \cos(2\pi nz)R_y = 1, \\ R_y &= \cos(2\pi nz), \\ R_x &= \sin(2\pi nz). \end{aligned}$$

So the Reeb vector field is:

$$R_n = \sin(2\pi nz)\partial_x + \cos(2\pi nz)\partial_y.$$

It is not trivial to see that the contact structures $\xi_n = \mathrm{Ker}\,\alpha_n$ are pairwise non-isomorphic.

2.3 Gray stability

A fundamental result in contact geometry is gray stability. This result provides insight into the structure and behaviour of contact manifolds under certain perturbations. Gray stability refers to the idea that contact structures remain stable (up to isotopy) under small perturbations of their defining contact forms. This notion is crucial for understanding the rigidity and flexibility of contact structures.

To prove Gray's stability, we first introduce a classical result in differential geometry: the Cartan formula for time-dependent vector fields. This lemma provides a fundamental relationship between Lie derivatives, interior products, and differential forms, and will be a key tool in establishing the stability result.

Lemma 2.15 (Cartan formula for time dependent vector fields)**.** *Let X_t, $t \in \mathbb{R}$, be a time-dependent vector field on a manifold M. Then*

$$\mathcal{L}_{X_t} = d \circ i_{X_t} + i_{X_t} \circ d \tag{1}$$

To proceed with the proof of Gray's stability, we also introduce another technical lemma that will be essential in establishing how contact structures vary under isotopies and time-dependent flows.

Lemma 2.16. *Let ω_t, $t \in [0,1]$, be a smooth family of differential k-forms on a manifold M, and $(\psi_t)_{t\in[0,1]}$ an isotopy of M. Define a time-dependent vector field X_t on M by $X_t \circ \psi_t = \dot{\psi}_t$, where the dot denotes the derivative with respect to t (so that ψ_t is the flow of X_t). Then*

$$\frac{d}{dt}(\psi_t^* \omega_t) = \psi_t^* \left(\dot{\omega}_t + \mathcal{L}_{X_t}\omega_t\right)$$

These two lemmas are proved in Appendix B of [3].

We proceed to state Gray's stability

Theorem 2.17. *Let ξ_t be a smooth family of contact structures on a closed manifold M. Then there is an isotopy $(\psi_t)_{t\in[0,1]}$ of M such that*

$$T\psi(\xi_0) = \xi_t \tag{2}$$

for each $t \in [0,1]$.

Proof. The proof's main idea is to use Moser's trick, which assumes that ϕ_t is the flow of a time-dependent vector field X_t. Using this we will translate (2) into an equation for X_t. Once we have this equation we will solve for X_t and as M is closed integrating X_t will give us the desired ϕ_t.

Let α_t be a smooth family of contact forms so that $\xi_t = \mathrm{Ker}(\alpha_t)$, we can do this because each α_t is defined uniquely up to multiplication by a smooth nowhere zero function. Then (2) becomes:

$$\psi_t^* \alpha_t = \lambda_t \alpha_0$$

for some smooth family of functions $\lambda_t : M \to \mathbb{R}$. And if we take the derivative with respect to t and using Lemma 2.16

$$\psi_t^* \left(\dot{\alpha}_t + \mathcal{L}_{X_t}\alpha\right) = \dot{\lambda}_t \alpha_0 = \tag{3}$$

Now if we define $\mu_t := \frac{d}{dt}(\log \lambda_t) \circ \psi^{-1}$ and use Cartan's formula (1)

$$\psi_t^* \left(\dot{\alpha}_t + d(\alpha_t(X_t)) + i_{X_t} d\alpha_t\right) = \psi_t^*(\mu_t \alpha_t)$$

If we choose $X_t \in \xi_t$, this equation will be satisfied if

$$\dot{\alpha}_t + i_{X_t} d\alpha_t = \mu_t \alpha_t \tag{4}$$

Now if we evaluate this equation at the Reeb vector field R_t we get

$$\dot{\alpha}(R_t) = \mu_t$$

So now we are basically done as we can define $\mu_t := \dot{\alpha}(R_t)$, so (4) becomes

$$i_{X_t} d\alpha_t = \dot{\alpha}(R_t)\alpha_t - \dot{\alpha}_t$$

Now as $d\alpha_i|_\eta$ is non-degenerate and also $R_t \in \mathrm{Ker}(\mu_t \alpha_t - \dot{\alpha}_t)$ this ensures that we have a unique solution for X_t. Now as we said before, M is closed and the flow ϕ_t of X_t is defined everywhere. □

Remark 2.18. Later on, we will use the fact that for every point $p \in M$ where $\dot{\alpha}_t(p)$ vanishes for all t, the time-dependent vector field also vanishes $X_t(p) \equiv 0$. Consequently, such points remain fixed under the isotopy ψ_t.

2.4 Darboux theorem

An essential result in contact geometry is Darboux's theorem, which provides a local normal form for contact structures. This theorem is a cornerstone in the study of contact manifolds because it reveals that contact structures are locally equivalent to a standard, canonical form. In other words, near any point on a contact manifold, a suitable choice of local coordinates can always put the contact form into a standard expression.

Darboux's theorem is analogous to the Darboux normal form in symplectic geometry, demonstrating the local "flatness" of contact structures under appropriate coordinates. This is a key distinction that sets contact geometry apart from Riemannian geometry.

Theorem 2.19 (Darboux). *Let α be a contact form on the $(2n+1)$-dimensional manifold M and p a point on M. Then there are coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n, z$ on a neighbourhood $U \subset M$ of p such that $p = (0, \ldots, 0)$ and*

$$\alpha|_U = dz + \sum_{j=1}^{n} x_j \, dy_j.$$

Proof. As the nature of this Theorem is local it suffices to prove it in the Euclidean space $\mathbb{R}^{2n+1}$ with $p = 0$. We now choose linear coordinates $x_1, \ldots, x_n, y_1, \ldots, y_n, z$ on $\mathbb{R}^{2n+1}$ such that they fulfil the following conditions on $T_0\mathbb{R}^{2n+1}$. First we choose z using that $\operatorname{Ker} \alpha$ has dimension $2n$:

$$\alpha(\partial z) = 1, \quad \iota_{\partial z} d\alpha = 0,$$

Then we choose $x_1 \ldots, x_n, y_1, \ldots, y_n$ such that

$$\partial x_j, \partial y_j \in \operatorname{Ker} \alpha$$

And, as $d\alpha$ is a skew-symmetric form on $T_0\mathbb{R}^{2n+1}$ we can choose a symplectic basis, thereby expressing $d\alpha$ as

$$d\alpha = \sum_{j=1}^{n} dx_j \wedge dy_j.$$

Now set $\alpha_0 = dz + \sum_j x_j dy_j$ and consider the family of 1-forms

$$\alpha_t = (1-t)\alpha_0 + t\alpha, \quad t \in [0,1],$$

on $\mathbb{R}^{2n+1}$. Our choice of coordinates ensures that $\alpha_t = \alpha$, $d\alpha_t = d\alpha$ at the origin therefore, there exists a neighbourhood of zero where α_t is a contact form for each $t \in [0,1]$.

Now using, just as before, Moser's trick to find an appropriate isotopy ϕ_t such that $\psi_t^* \alpha_t = \alpha_0$ in a small enough neighbourhood. Let's assume such ϕ_t exists and is the flow of X_t a time dependent vector field to attain the necessary conditions to construct such isotopy.

Now if we differentiate $\psi_t^* \alpha_t = \alpha_0$

$$\psi_t^* (\dot{\alpha}_t + \mathcal{L}_{X_t} \alpha_t) = 0,$$

So X_t must satisfy

$$\dot{\alpha}_t + d(\alpha_t(X_t)) + \iota_{X_t} d\alpha_t = 0. \tag{5}$$

Now using the Reeb vector R_t field of α_t we can write $X_t = H_t R_t + Y_t$ with $Y_t \in \operatorname{Ker} \alpha_t$. Plugging R_t in (5) we get

$$\dot{\alpha}_t(R_t) + d(\alpha_t(X_t))((R_t)) + d\alpha_t(X_t, R_t) = 0$$
$$\dot{\alpha}_t(R_t) + dH_t(R_t) = 0.$$

Now by means of integration we can find a smooth family of functions H_t on neighbourhood of the origin small enough such that none of the R_t has any closed orbits there.

Since $\dot{\alpha}_t$ is zero at the origin, we may require that $H_t(0) = 0$ and $dH_t|_0 = 0$ for all $t \in [0, 1]$. Once H_t has been chosen, Y_t is defined uniquely by

$$\dot{\alpha}_t + dH_t + \iota_{Y_t} d\alpha_t = 0.$$

Notice that with our assumptions on H_t, we have $X_t(0) = 0$ for all t. Now define ψ_t to be the local flow of X_t. This local flow fixes the origin, so there it is defined for all $t \in [0, 1]$. Since the domain of definition in $\mathbb{R} \times M$ of a local flow on a manifold M is always open, then ψ_t is defined for all $t \in [0, 1]$ on a sufficiently small neighbourhood of the origin in $\mathbb{R}^{2n+1}$. □

2.5 Liouville vector field

Liouville vector fields are fundamental in the study of symplectic and contact geometry. A Liouville vector field on a symplectic manifold (W, ω) is a vector field that preserves the symplectic form up to scaling, satisfying $\mathcal{L}_Y \omega = \omega$, where $\mathcal{L}$ denotes the Lie derivative. This property allows Liouville vector fields to generate contact structures on hypersurfaces that are transverse to the flow of the field. Such hypersurfaces are known to be of contact type.

In contact geometry, Liouville vector fields are crucial for constructing contact forms on hypersurfaces within symplectic manifolds. This connection is formalized in Lemma 2.21, which shows that the 1-form $\alpha = \iota_Y \omega$, where ι_Y is the interior product, defines a contact form on any hypersurface transverse to the Liouville vector field. This result provides a key technique for generating contact structures from symplectic manifolds. We then see a concrete case, Example 2.22 that using standard Liouville vector field on $\mathbb{R}^{2n}$ constructs contact structures on stared spheres giving a wide range of contact structures in S^{2n-1}.

Definition 2.20. *A Liouville vector field Y on a symplectic manifold (W, ω) is a vector field satisfying the equation $\mathcal{L}_Y \omega = \omega$, where $\mathcal{L}$ denotes the Lie derivative.*

The following lemma gives a way to construct contact structures on hypersurfices of symplectic manifolds

Lemma 2.21. *Let Y be a Liouville vector field as before then, the 1-form $\alpha := \iota_Y \omega := \omega(Y, -)$ is a contact form on any hypersurface M transverse to Y (that is, with Y nowhere tangent to M). Such hypersurfaces are said to be of contact type.*

Proof. Using Cartan's formula, we write the condition $\mathcal{L}_Y \omega = \omega$ as

$$\begin{aligned} \omega =& \mathcal{L}_Y \omega = d \circ \iota_Y \omega + \iota_Y \circ d\omega \\ =& d(\iota_Y \omega) \end{aligned}$$

using that ω is closed. Now let $2n$ be the dimension of W let's see if α is a contact form:

$$\begin{aligned} \alpha \wedge (d\alpha)^{n-1} &= \iota_Y \omega \wedge (d(\iota_Y \omega))^{n-1} \\ &= \iota_Y \omega \wedge \omega^{n-1} \\ &= \frac{1}{n} \iota_Y (\omega^n). \end{aligned}$$

Now as ω^n is a volume form as ω is symplectic. Now as $\alpha \wedge (d\alpha)^{n-1}$ is a volume form then so is the restriction to the tangent bundle of any hypersurface transverse to Y □

The following example is just an aplication of the lemma

Example 2.22. Let $(\mathbb{R}^{2n}, \omega_0)$ the standard symplectic structure, then the radial vector field

$$\mathcal{X} = \frac{1}{2} \sum_i x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i}$$

is Liouville. If $i : S^{2n-1} \hookrightarrow \mathbb{R}^{2n}$ is the canonical injection, then $i^*(\iota_{\mathcal{X}} \omega_0) = \alpha_{\text{st}}$. Now let the image of

$$\begin{aligned} e : S^{2n-1} &\longrightarrow \mathbb{R}^{2n} \\ (x_i, y_i) &\longmapsto f(x_i, y_i)(x_i, y_i) \end{aligned}$$

be a starred sphere where f is a strictly positive smooth function. Then:

(i) $\mathcal{X}$ is transverse to the starred sphere $e(S^{2n-1})$

(ii) the contact form $\alpha := e^*(\iota_{\mathcal{X}} \omega_0)$ gives rise to the standard contact structure: $\text{Ker}(\alpha) = \text{Ker}(\alpha_{\text{st}})$

Proof. Firstly, let's check $\mathcal{X}$ i Liouville:

$$\mathcal{L}_{\mathcal{X}}(\omega_0) = d(\iota_{\mathcal{X}}\omega_0) + \iota_{\mathcal{X}} d\omega_0 = d(\iota_{\mathcal{X}}\omega_0) =$$
$$d(\frac{1}{2}\sum_i x_i dy_i - y_i dx_i) = \frac{1}{2}\sum_i dx_i \wedge dy_i - dy_i \wedge dx_i = \omega_0$$

It is indeed

(i) Now, let us verify that $\mathcal{X}$ is transverse to the embedded sphere $e(S^{2n-1})$. We begin by adopting spherical coordinates (r, φ_i), where r denotes the radius, and φ_i are the angular coordinates. The change of variables is the usual:

$$(x_1, x_2, \ldots, x_{2n}) = \begin{cases} r\cos(\varphi_1) \\ r\sin(\varphi_1)\cos(\varphi_2) \\ \vdots \\ r\sin(\varphi_1)\sin(\varphi_2)\cdots\sin(\varphi_{2n-2})\sin(\varphi_{2n-1}) \end{cases}.$$

Now define the smooth function,

$$F : \mathbb{R}^{2n} \setminus \{0\} \to \mathbb{R}$$
$$(r, \varphi_i) \mapsto r - f(\varphi_i).$$

Note that $dF = dr - \sum_{i=1}^{2n-1} \frac{\partial F}{\varphi_i} d\varphi_i$ so 0 is a regular value and F is a submersion $F^{-1}(0) = e(S^{2n-1})$. Now by the Regular Value Theorem the tangent space of the starred sphere is precisely the kernel of dF and trivially $dF(\frac{\partial}{\partial r}) = 1$ the radial field $\mathcal{X} = \frac{1}{2}\frac{\partial}{\partial r}$ is transverse to the starred sphere.

(ii) Now doing the pullback:

$$\alpha = e^*(\iota_{\mathcal{X}}\omega_0) = e^*\left(\frac{1}{2}\sum_i x_i dy_i - y_i dx_i\right) =$$
$$\frac{1}{2}\sum_i f x_i e^*(dy_i) - f y_i e^*(dx_i) =$$
$$\frac{1}{2}\sum_i f x_i \left(f dy_i + \sum_j y_i \frac{\partial f}{\partial x_j} dx_j + y_i \frac{\partial f}{\partial y_j} dy_j \right)$$
$$- f y_i \left(f dx_i + \sum_j x_i \frac{\partial f}{\partial x_j} dx_j + x_i \frac{\partial f}{\partial y_j} dy_j \right) =$$
$$f^2 \frac{1}{2}\sum_i x_i dy_i - y_i dx_i = f^2 \alpha_{\text{st}}$$

Since f^2 is a positive smooth function on S^{2n-1}, the hyperplane fields defined by α and α_{st} coincide. □

3 Lickorish and Wallace Theorem

We now begin preparing for the proof of Martinet's Theorem. This construction relies heavily on the fact that every closed, connected, orientable 3-manifold can be obtained by performing Dehn surgery along some knots. This result is known as the Lickorish-Wallace Theorem. In this section, we provide a complete proof.

The main idea of the proof is to decompose the manifold into two handlebodies, U and V, this is called a Heegaard splitting. By performing Dehn twists on the boundary of U, we aim to obtain S^3 when gluing U and V back together.

To extend the compositions of Dehn twists on ∂U to a homeomorphism of U, it becomes necessary to remove certain solid tori. When these excavated solid tori are filled back in, the gluing map will be modified slightly, requiring us to perform 1-surgeries at those locations.

3.1 Dehn Surgery

Dehn surgery is a fundamental operation in 3-manifold topology, allowing the modification of a 3-manifold by altering its structure along a knot. Let M be an oriented 3-manifold, recall that a knot is an embedded copy of S^1 in M.Take a closed neighbourhood of K that is diffeomorphic to the solid torus $S^1 \times D^2$. Denote this neighbourhood by νK. Note that νK always exists by the Tubular neighbourhood Theorem.

Definition 3.1. *A Dehn surgery along K is obtained by removing the neighbourhood νK and gluing it back in, that is, considering the closure of $M \setminus \nu K$ and using a diffeomorphism of the two-torus to glue back in the solid torus*

$$\phi : \partial(S^1 \times D^2) \to \partial(\nu K).$$

So the resulting manifold will be $\overline{M \setminus \nu K} \cup_\phi S^1 \times D^2$

This seemingly straightforward surgery fundamentally transforms the manifold. In this section, we will prove a powerful result by Lickorish and Wallace, which states that any closed, connected, orientable 3-manifold can be constructed through a series of Dehn surgeries on the 3-sphere.

Our aim is now to classify Dehn surgeries. Ultimately, we will see that Dehn surgery around a given knot is completely determined by the image of the meridian under the gluing homeomorphism. This observation will lead us to define the *surgery coefficient* $p/q \in \mathbb{Q}$, where p and q are coprime integers.

To begin with, it is evident that for a knot K, if we have two gluing morphisms ϕ and ϕ' that are isotopic, then the manifolds resulting from Dehn surgery along K are homeomorphic:

Lemma 3.2. *Suppose that U and V are (compact) 3-manifolds with homeomorphic boundaries, and that $h_0 : \partial U \to \partial V$ and $h_1 : \partial U \to \partial V$ are isotopic homeomorphisms. Then, $U \cup_{h_0} V$ and $U \cup_{h_1} V$ are homeomorphic.*

Proof. Choose a collar neighbourhood[2] C of ∂U in U. C is a neighbourhood of ∂U homeomorphic to $\partial U \times [0,1]$, with ∂U identified with $\partial U \times \{0\}$. A homeomorphism $f : U \cup_{h_0} V \to U \cup_{h_1} V$ can be constructed by defining f to be the identity on $(U - C) \cup V$ and on C defining $f(x,t) = (h_1 \circ h_0^{-1})(x,t)$. □

Naturally, we want now to study the fundamental group $\pi_1(\mathbb{T}) \cong \mathbb{Z} \oplus \mathbb{Z}$, as isotopic gluing maps yield the same manifold. Take unit complex coordinates in each S^1 we can define two curves:

$$L : e^{i\theta} \to (e^{i\theta}, 1) \qquad \text{The longitude}$$
$$M : e^{i\theta} \to (1, e^{i\theta}) \qquad \text{The meridian}$$

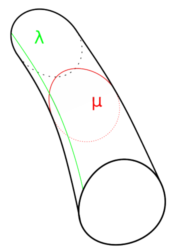


Figure 1: Longitude meridian

Note that $[L]$ and $[M]$ form a basis of $\pi_1(\mathbb{T}) \cong H_1(\mathbb{T})$. The functor π_1 converts a homeomorphism h of the torus $\mathbb{T}$ into a group automorphism h^* of $\pi_1(\mathbb{T}) = \mathbb{Z} \oplus \mathbb{Z}$. This induces a homomorphism from the group of self-homeomorphisms of $\mathbb{T}$, denoted $\mathrm{Aut}(\mathbb{T})$, to the group of 2×2 integral matrices invertible over $\mathbb{Z}$, $\mathrm{GL}(2, \mathbb{Z})$:

$$s : \mathrm{Aut}(\mathbb{T}) \to \mathrm{GL}(2, \mathbb{Z}).$$

The following theorem establishes that this map becomes an isomorphism when considering isotopy classes of homeomorphisms. This result is a crucial step toward classifying Dehn surgeries.

[2]The existence of collar neighborhoods on compact manifolds with boundaries is quite standard, for example check Proposition 3.42 of[5]

Theorem 3.3. *The group of self-homeomorphisms of $\mathbb{T}$, modulo ambient isotopy, is isomorphic to $GL(2,\mathbb{Z})$. Thus, two homeomorphisms of $\mathbb{T}$ are ambient isotopic if and only if they have the same matrix, i.e., they are homotopic as maps.*

Proof. In order to prove the theorem, we will see s is a bijection. We first start with surjectivity. We define the two following "twist" homeomorphisms and as π_1 is a functor we get two isomorphisms of $\mathbb{Z} \times \mathbb{Z}$:

$$h_L(e^{i\theta}, e^{i\phi}) = (e^{i(\theta+\phi)}, e^{i\phi}) \text{ the "longitudinal twist" with matrix } \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

$$h_M(e^{i\theta}, e^{i\phi}) = (e^{i\theta}, e^{i(\theta+\phi)}) \text{ the "meridinal twist" with matrix } \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

We will call these homeomorphisms and their inverses twists. As $GL(2,\mathbb{Z})$ is generated by $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ it is clear s is surjective.

To see injectivity we need only show that the kernel of s is precisely the subgroup of homeomorphisms of $\mathbb{T}$ which are isotopic to the identity.

Since homotopic maps $\mathbb{T} \to \mathbb{T}$ induce the same homomorphism $\pi_1(\mathbb{T}) \to \pi_1(\mathbb{T})$, it is clear that any automorphism of $\mathbb{T}$ isotopic to the identity is in the kernel of s. Conversely, given an automorphism $h : \mathbb{T} \to \mathbb{T}$ such that $s(h)$ is the identity matrix. Our goal is to construct an explicit isotopy between h and the identity map.

The proof of surjectivity is somewhat lengthy and can be found in of Rolfsen's work [11, Lemma 3]. □

Now we can do a small topological argument to conclude that surgery along a knot is completely determined by the image of the meridian.

Theorem 3.4. *Let K be a knot in a 3-manifold M, the manifold obtained by doing Dehn surgery along K with the gluing map $f : \mathbb{T} \to \mathbb{T}$ is completely determined by the image of the meridian, more precisely by its homotopy class.*

Proof. In the longitude-meridian bases on the two tori, (λ_1, μ_1) and (λ_2, μ_2), the homeomorphism f corresponds to a matrix

$$A = \begin{pmatrix} -q & s \\ p & r \end{pmatrix} \text{ where } qr + ps = 1.$$

First we take a bite of the doughnut, $D^2 \times J$ where J is a small segment of S^1 and attach it.The entire solid torus can be represented as

$$D^2 \times S^1 = (D^2 \times J) \cup D^3.$$

Then to attach the rest of the torus we attach the 3-ball D^3 along its boundary $\partial D^3 = S^2$.

Since all orientation-preserving homeomorphisms of S^2 are isotopic to the identity, which was proven by Smale [12], the attachment is well-defined, which completes the proof. □

We have now achieved our goal. First we have to choose an orientation of the longitude and meridian, as if only one of them is reversed the surgery could lead to different results. As the manifold M where we are performing the surgery is oriented, we choose the orientations of λ and μ so that the triple $\{\lambda, \mu, n\}$ is positively oriented, where n is the outward facing normal vector o boundary of νK. Recall that p and q are coprime integers, and the pairs (p, q) and $(-p, -q)$ define the same curve c, as the orientation of c is irrelevant in our context. The surgery along a knot is completely determined by the *surgery coefficient* p/q.

To complete this set, we include $1/0 = \infty$, which corresponds to the identity. Performing $1/0$-surgery on any knot $K \subset S^3$ yields S^3 again.

As we have seen, surgery coefficient determines the manifold obtained. Determining whether two 3-manifolds obtained by Dehn surgery are diffeomorphic is a significant and challenging problem in 3-manifold topology. While there are many tools and techniques to study this, the problem is not fully solved in general, particularly in complex cases.

3.2 Dehn Twist

Having introduced the basic notions of Dehn surgery, we now delve into specific techniques necessary for the proof of the Lickorish-Wallace theorem. The first technique we examine is the Dehn twist.

Let F be a connected, compact, oriented surface, possibly with a non-empty boundary. We will define certain homeomorphisms called Dehn twists. Compositions of these simple homeomorphisms will later be seen to transform almost any curve into any other (up to isotopy).

Definition 3.5 (Dehn Twist). *Let C be a simple closed curve embedded in F, and let A be an annular neighbourhood of C, which we identify with $S^1 \times [0,1]$ with the canonical orientation.*[3] *A* Dehn twist *about C is any homeomorphism isotopic to the homeomorphism $f : F \to F$ defined as follows:*

- $f|_{F \setminus A}$ *is the identity.*
- *Parametrizing A as $S^1 \times [0,1]$ in an orientation-preserving manner, $f|_A$ is given by*

$$f(e^{i\theta}, t) = (e^{i(\theta + 2\pi t)}, t).$$

[3] If we take coordinates (θ, x), the orientation is given by the oriented basis $\left\{ \frac{\partial}{\partial \theta}, \frac{\partial}{\partial x} \right\}$.

Using this twist note that the isotopy class of C remains intact, but other curves may be changed by the twist:

Example 3.6. Take the torus $\mathbb{T}$ consider the longitude, by taking an annulus that loops "around the hole" as in Figure 2. When doing the twist around the annulus, the longitude becomes one longitude plus one meridian.

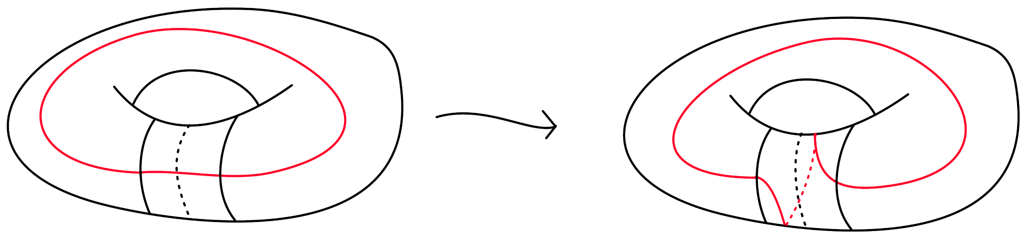


Figure 2: Dehn twist on the torus

As mentioned earlier, these transformations can be used to transform each isotopy class of curves into any other. To establish this fact, we begin by defining an equivalence relation: two curves are said to be Twist equivalent if one can be converted into the other using a finite sequence of Dehn twists.

Definition 3.7 (Twist equivalent curves)**.** *Two oriented simple closed curves p and q contained in the interior of a surface F are called twist-equivalent, written $p \sim_T q$, if $h(p) = q$ for some homeomorphism h of F that is in the group of homeomorphisms generated by all Dehn twists of F.*

The set of all Dehn twists on a surface, together with homeomorphisms isotopic to the identity, forms a group under composition.

We begin with the following two lemmata and use them to progressively extend to the fully general case.

Lemma 3.8. *Let p and q be oriented simple closed curves in the interior of the surface F, intersecting transversely at precisely one point. Then $p \sim_T q$.*

Proof. Let C_1 be a simple closed curve that runs parallel to, and is slightly displaced from q and C_2 be a simple closed curve that runs parallel to, and is slightly displaced from p. First we consider a small annulus around $C1$ and τ_1 a twist about this annulus. This twists p around q as in the second picture of Figure 3.8. The following twist is τ_2 a twist about a small annulus around C_2 then $\tau_2\tau_1 p$ is isotopic to q as we can retract the part that wraps around p using a homeomorphism isotopic to the identity.

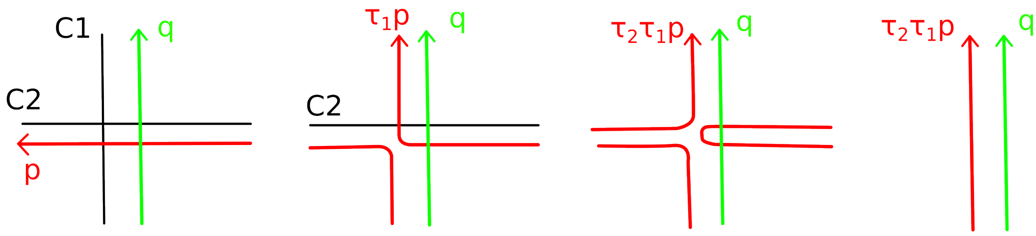


Figure 3: Dehn twists taknig p to q

□

Lemma 3.9. *Let p and q be disjoint oriented simple closed curves in the interior of the surface F and that neither separates F (that is, $[p] \neq 0 \neq [q]$ in $H_1(F, \partial F)$). Then $p \sim_T q$.*

Proof. Our goal is to construct a simple oriented closed curve r that intersects p transversely at exactly one point and q at exactly one point as using the previous Lemma $p \sim_T r \sim_T q$. To achieve this, consider the surface obtained by cutting F along $p \cup q$. We will analyse two cases based on whether this resulting surface is connected or not:

- **Case 1:** The surface remains connected. In this case as the surface is arc-connected there exists a curve r_1 from a point in p^+ to a point inq^- and another curve r_2 connecting the equivalent points in q^+ and p^-, we can assume that both r_1 and r_2 do not self intersect and also that they intersect p and q transversely when gluing back the surface.

  First assume r_1 and r_2 don't intersect, then concatenating them we obtain the knot r we wanted as shown in the left image of 4. If they intersect one another we leave r_1 intact and redefine R_2 in the following manner: Start by following r_2, just before it intersects r_1 for the first time follow r_1 closely without intersecting it. Then just before hitting q go around it, finally follow r_1 from the other side until arriving near the last intersection of the original r_2 and r_2 there follow r_2 until the end. This can be seen in the right of Figure 4, then again concatenating the old r_1 and the new r_2 we obtain the desired r

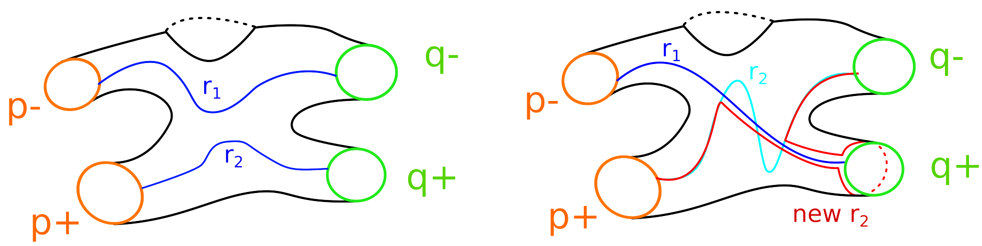


Figure 4: Case 1 without intersections and with intersections

- **Case 2:** In the second case, the cut surface has only two connected components. If it had more, either p or q would separate F. Moreover, both connected components must have both p and q as their boundary. In this scenario, we construct r_1 from p^+ to q^- and r_2 from q^+ to p^-, just as before (see Figure 5). Since r_1 and r_2 have no intersections their concatenation is the knot r we want.

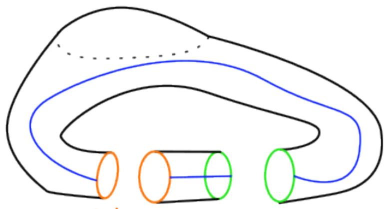


Figure 5: Proof Case 1 and Case 2

□

We are now prepared to generalize to the case where the curves p and q can intersect:

Proposition 3.10. *Let p and q be oriented simple closed curves in the interior of the surface F so that neither separates F. Then $p \sim_T q$.*

Proof. Note that we can assume they intersect transversely at a finite number of points j by taking a homeomorphism of F that is isotopic to the identity (for further detail on transversality, check [4]). We will proceed by induction on j, the number of points. Given the previous lemmata 3.8 and 3.9 we start induction assuming $j \geq 2$ and the result is true for less than j points. Let A and B be consecutive points along p of $p \cap q$. We will prove case by case, depending on if q enters p at B on the same side as it does in A.

- **Case 1:** q enters p at B on the same side as in A. Let r be a simple closed curve in F that starts near A, follows close to p until near B, and then returns to its start in a neighbourhood of q as shown in Figure 6. If r is close enough to p or q (depending on which curve it follows at each time) we can make it intersect p at less than j points so by induction $p \sim_T r$ but also as r intersects q only at C by Lemma 3.8 $r \sim_T q$ so $p \sim_T q$ as we wanted.

- **Case 2:** q enters p at B on the opposite side as in A. Let r_1 and r_2 be the two simple closed curves shown in the second diagram of Figure 6. r_1 starts near A, follows p until B and follows in the direction of q until it A. Similarly r_2 starts near A, follows p until B and follows in the opposite direction of q until it A. Now in $H_1(F, \partial F)$, $[r_1] - [r_2] = [q]$, and hence at least one of r_1 and r_2 does not separate (as $[q] \neq 0$). Now we define r to be $r_1 - r_2$, as r is disjoint from q by Lemma 3.9 $r \sim_T q$ and as $p \cap r$ has at most $n-2$ points by induction $p \sim_T r$ $p \sim_T q$ as we wanted.

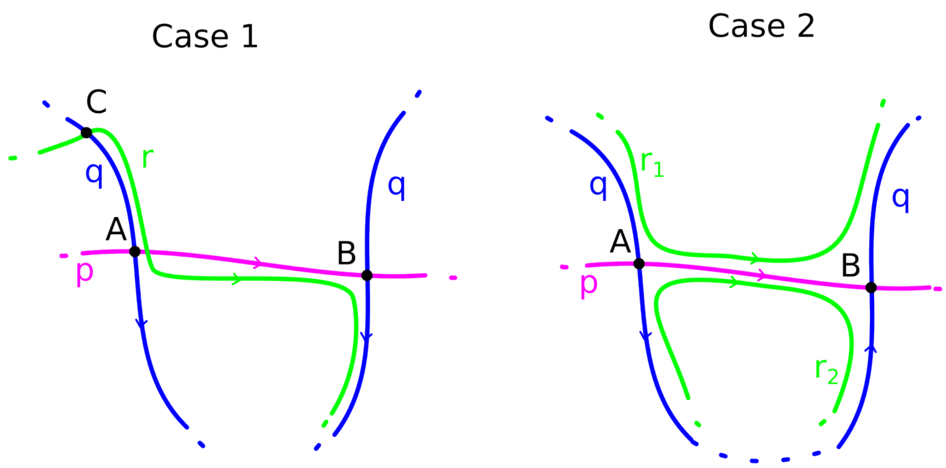


Figure 6

□

Corollary 3.11. *Let $p_1, p_2, \ldots, p_n$ be disjoint simple closed curves in the interior of F such that their union does not separate F. Let $q_1, q_2, \ldots, q_n$ be another set of curves with the same properties. Then there exists a homeomorphism h of F that is in the group generated by twists, such that $h(p_i) = q_i$ for each $i = 1, 2, \ldots, n$.*

Proof. Suppose inductively that such a homeomorphism h can be found so that $h(p_i) = q_i$ for each $i = 1, 2, \ldots, n-1$. Apply the previous Proposition to $h(p_n)$ and q_n in F cutting along $q_1 \cup q_2 \cup \cdots \cup q_{n-1}$. □

We have no

3.3 Heegaard splitting

Now we will introduce the Heegaard splitting, and we will prove every closed connected orientable 3-manifold has a Heegaard splitting. This fact, together with Dehn twists, will give the proper theory to prove Lickorish and Wallace's theorem.

Definition 3.12. *A handlebody is any space obtained from the 3-ball D^3 (called a 0-handle) by attaching g distinct copies of $D^2 \times [-1, 1]$ (called 1-handles) with homeomorphisms mapping the 2-disks $D^2 \times \{\pm 1\}$ onto $2g$ disjoint 2-disks on ∂D^3, all in such a way that the resulting 3-manifold is orientable. The integer g is called the genus.*

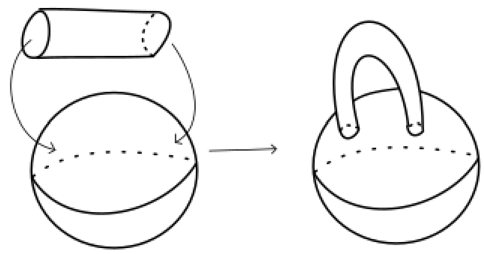


Figure 7: Handlebody of genus 1

As the boundary of a handlebody of genus g is clearly a compact oriented surface of genus g it's homeomorphic to the connected sum of g copies of a torus. So it is not surprising that a handlebody of genus g is homeomorphic to the boundary connected sum of g solid tori.

Definition 3.13. *A Heegaard splitting of a (closed, connected, orientable) 3-manifold M is a pair of handlebodies X and Y contained in M such that $X \cup Y = M$ and $X \cap Y = \partial X = \partial Y$.*

Note that as $\partial X = \partial Y$ X and Y have the same genus, and thus they are homeomorphic.

Example 3.14. Consider the hyperplane in $\mathbb{R}^4$ given by $x_4 = 0$. The intersection of S^3 with this hyperplane gives:

$$S^3 \cap \{x_4 = 0\} = \{(x_1, x_2, x_3, 0) \in \mathbb{R}^4 \mid x_1^2 + x_2^2 + x_3^2 = 1\},$$

which is precisely S^2, a 2-sphere of unit radius embedded in $\mathbb{R}^3$.

The hyperplane $x_4 = 0$ divides S^3 into two regions:

- Let H_1 be the set of points in S^3 for which $x_4 \geq 0$.
- Let H_2 be the set of points in S^3 for which $x_4 \leq 0$.

Each of these regions H_1 and H_2 is homeomorphic to a 3-dimensional ball (a 3-disk D^3). To see this, consider the stereographic projection from the North Pole $(0, 0, 0, 1)$. This projection maps points in the upper hemisphere of S^3 (where $x_4 \geq 0$) to points inside D^3, and similarly, the lower hemisphere is homeomorphic to D^3.

From the previously established facts, we observe that the 3-sphere, S^3, can be constructed as the union of its northern and southern hemispheres. Explicitly, S^3 is the union of two 3-balls, with their boundaries identified via an orientation-reversing diffeomorphism.

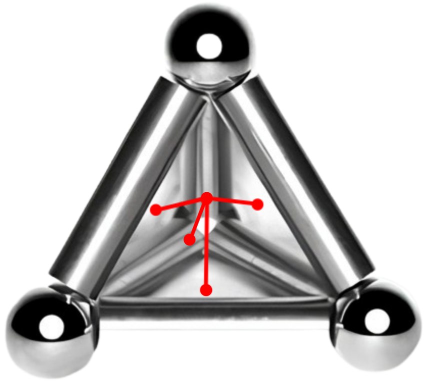
This construction provides the simplest example of a Heegaard splitting, where the two handlebodies are the two 3-balls, and the gluing map is the orientation-reversing diffeomorphism. The resulting splitting is a Heegaard splitting of genus 0. Later, we will see that it is possible to construct Heegaard splittings of any genus.

The example of S^3 demonstrates how Heegaard splittings provide a powerful framework for understanding 3-manifolds. We now generalize this concept and establish that every closed, connected, orientable 3-manifold admits a Heegaard splitting. This result is formalized in the following lemma:

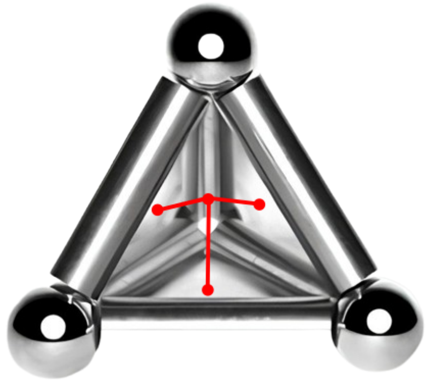
Lemma 3.15. *Any closed connected orientable 3-manifold M has a Heegaard splitting.*

Proof. It is well known that every 3-manifold can be triangulated [10]. We will use this triangulation to construct a handlebody N_1, the complement $N_2 := M \setminus N_1$ will be the other handlebody.

To begin, we will fatten the edges and vertices: that is, we replace each edge (the 1-skeleton) with a small enough cylinder, and each vertex (the 0-skeleton) with a small enough ball. This can be seen in Figure 8a



(a) Construction of N_1



(b) Graph associated to N_2

Figure 8: Constructions in each 3-simplex

Now, we must verify that this construction results in a handlebody. Since the edges and vertices form a graph, we can take a maximal tree that includes all the vertices (commonly known as a spanning tree). To construct the handlebody, we start at the base of the tree by taking a 3-ball, and then we attach handles along the branches of the tree one by one.

A similar argument can be applied to $N_2 := M \setminus N_1$, where, using barycentric subdivision, one can find a graph as in Figure 8b. Now simply construct a handlebody using the maximal tree as before, by construction this handlebody is homeomorphic to N_2 thus we have proven M has a Heegaard splitting.

□

Example 3.16. We now consider a more intricate example: we will construct a Heegaard splitting of S^3 of any genus. Recall that S^3 can be expressed as $S^3 = B_1 \cup B_2$, where B_1 and B_2 are 3-balls glued along their boundaries. Now, let α be an unknotted, embedded arc in B_1 with its endpoints in ∂B_1. Define the handlebodies as follows:

$$H_1 = B_1 \setminus \text{int}(N(\alpha)), \quad H_2 = B_2 \cup N(\alpha),$$

where $N(\alpha)$ is a tubular neighbourhood of α.

Observe that H_1 is obtained by removing an open neighbourhood of an unknotted arc from a 3-ball and thus H_1 is a solid torus. Similarly, H_2 is constructed by adding a 1-handle to a 3-ball, which also results in a solid torus.

Therefore, S^3 is the union of two solid tori glued along their boundary. If we want a Heegaard splitting of genus g, we simply take g unknotted arcs α_i in B_1 that are unknotted and unlinked. Let $U = B_1 \setminus \bigcup_{i=1}^{g} \text{int}(N(\alpha_i))$ and $V = B_2 \cup \bigcup_{i=1}^{g} N(\alpha_i)$. Then U and V are handlebodies.

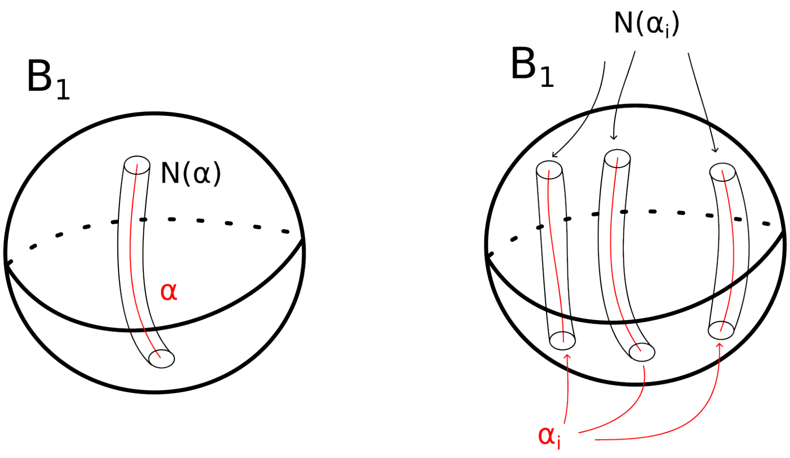


Figure 9: Heegaard splitting genus 1 and Heegaard splitting genus g

These Heegaard splittings of S^3 are called the standard Heegaard splittings of S^3. Moreover, this splitting can be constructed in the following way. Let U and V be two handlebodies of genus g. Consider disjoint simple closed curves $p_1, p_2, \ldots, p_g$ on ∂U that bound disjoint discs in U, and let $q_1, q_2, \ldots, q_g$ be disjoint simple closed curves on ∂V, with each q_i encircling a "hole" of the handlebody V, as shown in Figure 10. The gluing morphism is any morphism that takes p_i to q_i for each i.

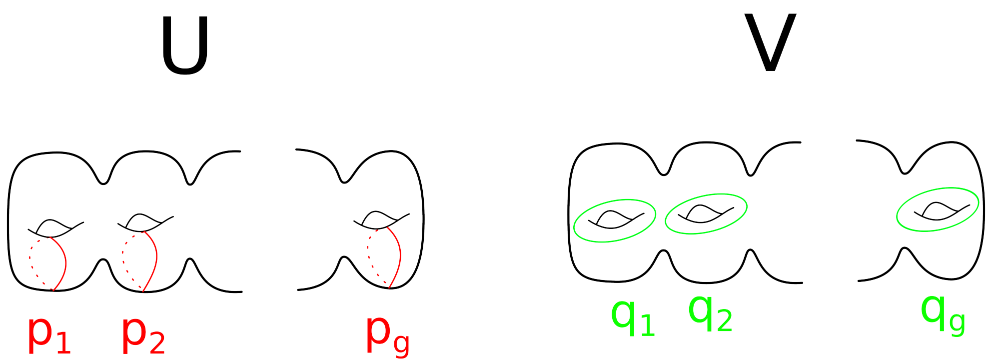


Figure 10: Handlebodies U and V

The following theorem uses the Heegaard splitting of S^3 to prove a result that will lead us to the desired theorem as a corollary. The theorem asserts that, except for the removal of some solid tori, every closed, connected, orientable 3-manifold is homeomorphic to S^3.

The idea behind the proof is straightforward: First, consider a Heegaard splitting of the manifold M. If the gluing map aligns each curve p_i with the corresponding curve q_i (as defined in Figure 10), the result is S^3. However, if this alignment does not occur, we use Dehn twists on the boundary of V to take p_i to q_i. Finally, we extend this homeomorphism to the entire V. However, this cannot always be done in general, so some solid tori need to be removed. In short, we split M into U and V, perform a homeomorphism on V (excluding the tori), and then glue U back in.

Theorem 3.17. *Let M be a closed, connected, orientable 3-manifold. There exist finite sets of disjoint solid tori $T_1, T_2, \dots, T_w$ in M and $T'_1, T'_2, \dots, T'_w$ in S^3 such that $M \setminus \bigcup_{i=1}^{n} \mathrm{Int}(T_i)$ and $S^3 \setminus \bigcup_{i=1}^{m} \mathrm{Int}(T'_i)$ are homeomorphic.*

Proof. By Lemma 3.15, we take a Heegaard splitting of M with two handlebodies U and V of genus g, and a homeomorphism $h : \partial U \to \partial V$ between their boundaries, such that $U \cup_h V = M$.

Now define the simple closed curves p_i in U and q_i in V as in Figure 10. Denote $p'_j := h(p_j)$, the images under the gluing homeomorphism. By Corollary 3.11, there exists a composition of Dehn twists ϕ in ∂V such that $\phi(p_i) = q_i$. As shown in Example 3.16, two handlebodies glued via a homeomorphism taking p_i to q_i are homeomorphic to S^3.

The idea now is to extend ϕ to the entirety of V. We first assume that ϕ is a single Dehn twist. To generalize to the case where ϕ is a composition of Dehn twists, the process will be repeated with slight modifications.

Let $\phi = \tau$, where τ is a Dehn twist supported on an annulus $A \subset \partial V$. Consider a collar neighbourhood[4] $\partial V \times [0,1] \subset V$. The extended homeomorphism will be the identity over V, except on the subset $A \times [0,1] \subset \partial V \times [0,1]$ of the collar neighbourhood. We divide this subset into two regions: the lower half $\partial V \times [1/2, 1]$, which we remove (this constitutes the excavated torus), and the upper half, where we extend the Dehn twist vertically:

$$\tau \times \mathrm{Id} : \partial V \times [0, 1/2] \to \partial V \times [0, 1/2],$$

$$(u, x) \mapsto (\phi(u), x).$$

Since Dehn twists are, by definition, the identity on the boundary of the annulus A, the extension $e : V \setminus \partial V \times [1/2, 1] \to V \setminus \partial V \times [1/2, 1]$ is well-defined. The following figure illustrates this procedure:

[4]The collar neighbourhood is a subset of V, but for simplicity, we identify it directly with $\partial V \times [0, 1]$.

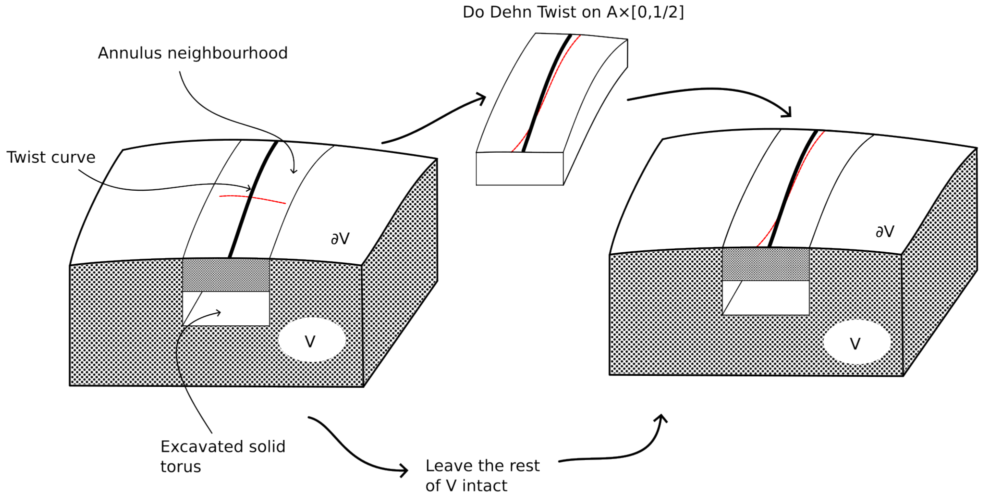


Figure 11: Excavated tunnel

Thus, e restricts to the identity everywhere except above the excavated torus, where it restricts to the map described. To obtain S^3, take $M = U \cup_h V$, remove the excavated torus $T_1 := \partial V \times [1/2, 1] \subset V$, apply e on $V \setminus T_1$, and glue U back in. This yields S^3 minus the excavated torus, as desired.

To generalize this process to multiple Dehn twists, repeat the construction iteratively. A potential complication arises if the supports of two Dehn twists intersect, leading to overlapping excavated tori. To address this, excavate increasingly deeper tunnels.

Specifically, if n twists $\tau_1, \ldots, \tau_n$ are required to ensure the correct gluing maps, modify the collar neighbourhood model to $\partial V \times [0, n]$. For the first twist τ_1, supported on the annulus A_1, remove the solid torus $\partial V \times [n - 1/2, n]$ and apply the twist to the remaining region $\partial V \times [0, n - 1/2]$. By iterating this procedure for each twist, the homeomorphism is achieved. □

Finally, as a corollary, we obtain the following fundamental result, where we simply fill the holes left by the tori.

Theorem 3.18 (Lickorish and Wallace)**.** *Any closed, connected, orientable* 3*-manifold M can be obtained from S^3 by a collection of* ± 1*-surgeries.*

Proof. Using the last theorem, the result is almost trivial; the only remaining step is to fill in the solid tori. Using the previous proof's notation, e maps the boundary of each T_i to itself. In fact, it maps the meridian to a curve that is homologous to one meridian plus or minus one longitude, depending on the orientation of the curve and the Dehn twist. By performing a ± 1-surgery on each torus, the theorem is proved. □

4 Martinet's construction

Martinet's theorem is a cornerstone result in contact topology, asserting that every closed, orientable 3-manifold admits a contact structure. The proof of this theorem relies on a combination of topological and geometric arguments.

Given the Lickorish-Wallace Theorem, Martinet's construction proceeds by explicitly defining a contact structure on S^3 and extending it through the neighborhoods of the knots where the Dehn surgeries are performed. The central challenge is to ensure that the contact structure remains well-defined. This requires careful adjustments in the surgery process to preserve the contact condition.

The starting point of the construction is the standard contact structure on S^3, which can be described in terms of the standard contact form as in Example 2.12:

$$\alpha_{\text{st}} = x_1 dy_1 - y_1 dy_1 + x_2 dy_2 - y_2 dy_2$$

in $\mathbb{R}^4$, restricted to the unit sphere S^3.

For each knot along which a Dehn surgery is performed, Martinet showed that it is possible to adapt the contact structure locally within a neighbourhood of the knot. By Darboux's theorem, the contact structure can always be brought to a standard form locally; this fact makes it possible to give a model for neighborhoods of knots. This model makes it easy to extend the contact structure to the neighborhoods where we do the surgery.

Martinet's construction not only establishes the existence of contact structures on all closed, orientable 3-manifolds but also highlights the deep interplay between contact geometry and the topology of 3-manifolds, bridging results from differential topology, surgery theory, and geometric structures.

4.1 Legendrian and transverse knots

Legendrian and transverse knots are fundamental concepts in the study of contact geometry. A Legendrian knot is a curve that is tangent to a contact structure, while a transverse knot, as the name suggests, intersects it transversely.

Definition 4.1. *A* ***Legendrian knot*** *in a contact 3–manifold* (M, ξ) *is a Legendrian embedding* $\gamma : S^1 \to M$*, that is, an embedding satisfying* $\gamma'(\theta) \in \xi_{\gamma(\theta)}$ *for all* $\theta \in S^1$*.*

A ***transverse knot*** *in* (M, ξ) *is an embedding* $\gamma : S^1 \to M$ *that is everywhere transverse to* ξ*, i.e., we require* $\gamma'(\theta) \notin \xi_{\gamma(\theta)}$ *for all* $\theta \in S^1$*. If* $\xi = \operatorname{Ker} \alpha$ *is cooriented, one speaks of a positively or negatively transverse knot depending on whether* $\alpha(\gamma'(\theta)) > 0$ *or* $\alpha(\gamma'(\theta)) < 0$ *for all* $\theta \in S^1$*.*

The goal will be to construct a model for neighborhoods of the knots that is simple enough to work with. Strictly speaking, the model given does not preserve the knot and some C^0-small isotopies will be involved, however this will have no effect as Dehn surgery only depends on the isotopy of the knot. The main idea will be the following: First we will approximate a generic knot by a Legendrian one, then we will construct a model for neighborhoods of Legendrian knots. This model then can be used to approximate the Legendrian knot with a transverse one, and finally we will give a model for transverse knots. For clarity, we will first present the results concerning neighborhoods, followed by those related to approximations.

4.1.1 Models for neighbourhood of Legendrian knots

Let's start by giving a model for Legendrian knots. First we will prove that all Legendrian knots have contactomorphic neighborhoods and then, as a corollary, we will give the model we want. The proof will consist in having a good bundle isomorphism on TM restricted to the knot and then extending this isomorphism to a neighbourhood of the knot. To make the proof simpler, we start by defining the canonical symplectic bundle structure on $E \oplus E^*$ where $E \to B$ is any vector bundle:

$$\omega_b(X + \eta, X' + \eta') := \eta(X') - \eta'(X)$$

for $X, X' \in E$ and $\eta, \eta' \in E^*$. To see it is a symplectic form is a simple check of the definition.

Lemma 4.2. *The symplectic form ω_b is canonical in the sense that it is invariant under isomorphisms.*

Proof. Consider a vector bundle isomorphism, $\Phi : E \to F$ the dual bundle E^* is naturally isomorphic to the dual bundle F^* via the induced map Φ^*. For clarity, let ω_E be the canonical symplectic structure in $E \oplus E^*$ and ω_F be the other one:

$$\begin{aligned}
\omega_F(\Phi(X + \eta), \Phi(X' + \eta')) &= \omega_b(\Phi(X) + \Phi^*(\eta), \Phi(X') + \Phi^*(\eta')) \\
&= \Phi^*(\eta)(\Phi(X')) - \Phi^*(\eta')(\Phi(X)) \\
&= \eta(\Phi^{-1}(\Phi(X'))) - \eta'(\Phi^{-1}(\Phi(X))) \\
&= \eta(X') - \eta'(X) \\
&= \omega_E(X + \eta, X' + \eta').
\end{aligned}$$

□

Theorem 4.3. *Let (M_i, ξ_i) two contact 3-manifolds $(i = 0, 1)$ and $L_i \subset M_i$ two Legendrian knots, then they have contactomorphic neighborhoods*

Proof. The first part of the proof will consist on constructing a bundle isomorphism between $TM_0|_{L_0}$ and $TM_1|_{L_1}$ that pulls back α_1 to α_0 and $d\alpha_1$ to $d\alpha_0$. Then we will use tubular neighborhoods and the Gray stability Theorem to extend the isomorphism to suitable neighborhoods. Let's start with the first part.

Let R_{α_i} the Reeb vector field of α_i (where α_i is the contact form of M_i) and let $\langle R_{\alpha_i} \rangle$ the vector bundle spanned by R_{α_i} then

$$TM_i|_{L_i} = \langle R_{\alpha_i} \rangle \oplus \xi$$

Let $\phi : L_0 \to L_1$ be a diffeomorphism between the knots and

$$\Phi_R : \langle R_{\alpha_0} \rangle \to \langle R_{\alpha_1} \rangle$$

be the obvious bundle isomorphism defined by requiring that $R_{\alpha_0}(p)$ maps to $R_{\alpha_1}(\phi(p))$. Now the following map:

$$\begin{aligned} \psi_i : \xi_i &\longrightarrow T^*L_i \\ X &\longmapsto \iota_X d\alpha \end{aligned}$$

is also a bundle isomorphism, as $d\alpha$ is symplectic in ξ. Now, by the previous Lemma (4.2) we have the following symplectic bundle isomorphism where Ω_{L_i} are the canonical symplectic structure:

$$T\phi \oplus (\phi^*)^{-1} : (TL_0 \oplus T^*L_0, \Omega_{L_0}) \to (TL_1 \oplus T^*L_1, \Omega_{L_1})$$

So we can construct

$$\tilde{\Phi} : NL_0 \longrightarrow NL_1$$

a bundle isomorphism (covering ϕ) defined by

$$\tilde{\Phi} = \Phi_R \oplus \Psi_1^{-1} \circ (\phi^*)^{-1} \circ \Psi_0.$$

so the bundle map $T\phi \oplus \tilde{\Phi} : TM_0|_{L_0} \to TM_1|_{L_1}$ by construction pulls back α_1 to α_0 and $d\alpha_1$ to $d\alpha_0$ so the first part of the proof is done.

Now our objective will be to use this map and extend it to neighborhoods of L_0 and L_1.

By the Tubular Neighbourhood Theorem (see [7, Theorem 6.17] for a proof) we take a tubular neighbourhood of $L_i \subset M_i$, i.e. an embedding $\tau_i : NL_i \to M_i$ such that:

- $\tau \circ 0_{NL_i}$ is the embedding $L_i \hookrightarrow M_i$ and 0_{NL_i} is the zero section.
- There exist some $U_i \subseteq NL_i$ and $V_i \subseteq M_i$ with $0_{NL_i} \subseteq U_i$ and $L_i \subseteq V_i$ such that $\tau|_{U_i} : U_i \to V_i$ is a diffeomorphism.

Then $\tau_1 \circ \tilde{\Phi} \circ \tau_0^{-1} : V_0 \to V_1$ is a diffeomorphism of neighborhoods V_i of L_i that induces the bundle map

$$T\phi \oplus \tilde{\Phi} : TM_0|_{L_0} \longrightarrow TM_1|_{L_1}.$$

and by the first part of the proof α_0 and $(\tau_1 \circ \tilde{\Phi} \circ \tau_0^{-1})^*(\alpha_1)$ are contact forms on V_0 that coincide on $TM_0|_{L_0}$, and so do their differentials.

Now consider the family of 1–forms

$$\beta_t = (1-t)\alpha_0 + t((\tau_1 \circ \tilde{\Phi} \circ \tau_0^{-1})^*(\alpha_1)), \quad t \in [0,1].$$

On $TM|_{L_0}$, we have $\beta_t \equiv \alpha_0$ and $d\beta_t \equiv d\alpha_0$. Since the contact condition $\alpha \wedge (d\alpha)^n \neq 0$ is an open condition, shrinking $V_0 \subset M_0$ if necessary, β_t is a contact form on V_0 for all $t \in [0,1]$. Now using Gray stability Theorem 2 and the Remark 2.18 that follows there is an isotopy ψ_t of V_0, fixing L_0, such that $\psi_t^* \beta_t = \lambda_t \alpha_0$ for some smooth family of smooth functions $\lambda_t : V_0 \to \mathbb{R}^+$.

Observe that the last thing that remains is to remove the λ_t, to do this we will proceed as in the proof of Darboux (Theorem 2.19). We can use Moser's trick, the equations that the flow X_t has to satisfy are exactly the same, so we can find an isotopy ψ_t such that $\psi_t^* \beta_t = \alpha_0$. The only inconvenience is that the neighbourhood V is not closed, so the flow may be local, however that imposes no problem as the flow is stationary on L_0 so we can shrink again if necessary V_0 so that ϕ_t is a global flow. The proof is now finished, as if we compose with ϕ_1 we find the contactomorphism

$$(\tau_1 \circ \tilde{\Phi} \circ \tau_0^{-1}) \circ \phi_1 : U_0 \subset M_0 \to U_1 \subset M_1$$

.

□

If all Legendrian knots have contactomorphic neighborhoods, we can simply choose a model:

Corollary 4.4. *Let $L \subset (M, \xi)$ be a Legendrian knot in a contact 3-manifold. A model for the neighbourhood of L knot is given by*

$$\left(S^1 \times \mathbb{R}^2, \alpha = \cos\theta\, dx - \sin\theta\, dy\right)$$

where the S^1-coordinate is θ and Cartesian coordinates by (x, y)

Proof. $S^1 \subset S^1 \times \mathbb{R}^2$ is clearly a Legendrian knot as $\alpha|_{S^1}(\partial_\theta) = 0$ □

4.1.2 Models for neighbourhood of transverse knots

Theorem 4.5. *Consider $K \subset M$ a transverse knot in a contact manifold (M, ξ_0) given by an embedding $k : S^1 \hookrightarrow (M, \xi_0)$ transverse to ξ_0. Then there exists a neighbourhood $W \subset M$ of K such that (W, ξ_0) is contactomorphic to*

$$\left(S^1 \times \mathbb{R}^{2n}, \xi_1 = \operatorname{Ker}(\alpha_1) = \operatorname{Ker}\left(d\theta + \sum_{j=1}^{n} (x_j dy_j - y_j dx_j) \right) \right),$$

where θ denotes the S^1–coordinate.

Proof. Throughout the proof it will be useful to think of $(K, \alpha_0|_{TK})$ as a dimension 1 contact manifold, and so the usual contact condition $\alpha \wedge d\alpha^n \neq 0$ degenerates just to $\alpha_0 \neq 0$ at every point in K and the hyperplane field is the zero section.

Now as $k : S^1 \hookrightarrow K \subset M$ is a diffeomorphism, consider $(k^{-1})^*(d\theta)$. rescaling α_0 using a smooth function[5], we can get a new α_0 such that $\alpha_0((k^{-1})^*(d\theta)) = 1$ in K, in other words, $R_{\alpha_0'} := (k^{-1})^*(d\theta)$ is the Reeb vector field of $(K, \alpha_0|_{TK})$.

We now further scale α_0 to force the Reeb vector field $R_{\alpha_0'}$ of K and the Reeb vector field R_{α_0} of M to coincide in K. So we want to find a smooth function $f : M \to \mathbb{R}^+$ such that $f|_K = 1$ and $\iota_{R_{\alpha_0'}} d(f\alpha_0) \equiv 0$ on $TM|_K$. If we impose the second equation:

$$0 = \iota_{R_{\alpha_0'}} d(f\alpha_0) = \iota_{R_{\alpha_0'}} (df \wedge \alpha_0 + f d\alpha_0) = -df + i_{R_{\alpha_0'}} d\alpha_0 \text{ on } TM|_K. \tag{6}$$

As $\beta := \iota_{R_{\alpha_0'}} d\alpha_0$ vanishes under the pullback of the inclusion $i : K \to M$. By relative Poincaré Lemma[6] [3, Corollary A.4] β is an exact form in a neighbourhood $U \subset K$. This means there exists a smooth function g on U such that $\beta = dg$, this g is unique up to a constant function. Moreover, $i_{R_{\alpha_0'}} d\alpha_0|_{TK} = i_{R_{\alpha_0'}} d\alpha_0' \equiv 0$ so we know g is constant in K so we can choose it to be 1 in K by adding a constant. Now simply observe that $f = g$ is our desired function as it is 1 in K and also 6 is achieved as $df = i_{R_{\alpha_0'}} d\alpha_0$.

On the other hand, we can see that on $S^1 \times \mathbb{R}^{2n}$ we don't need to rescale the contact form, as both Reeb vector fields are $R_s = \frac{\partial}{\partial \theta}$.

Having this, we have a contactomorphism $k^{-1} : K \to S^1$ and an isomorphism of symplectic bundles $\Phi : (\xi_1|_K, d\alpha_0) \to (\xi_1|_{S^1}, d\alpha_1)$ that covers k^{-1}. Note that as the contact submanifolds K and S^1 have both dim 1 their respective hyperplane fields are precisely their normal bundle ($TM|_K = TK \oplus \xi|_K$), therefore we can construct the following bundle map:

$$T(k^{-1}) \oplus \Phi : TM|_K \longrightarrow T(S^1 \times \mathbb{R}^{2n})|_{S_1}$$

[5]The function would be $1/\alpha_0((k^{-1})^*(d\theta))$

[6]Also referred to as the generalized Poincaré Lemma.

By construction, this maps pulls back α_1 to α_0 and $d\alpha_1$ to $d\alpha_0$ so proceeding in exactly the same way as in the second part of the proof of Theorem (4.3) we attain the desired contactomorphism.

□

4.1.3 Approximation theorems

Legendrian knots can be easily approximated by transverse knots using the model for neighborhoods of Legendrian knots. So if we are able to approximate a general knot by a Legendrian one, then it can also be approximated by a transverse one as in Lemma 4.6.

To approximate general knots with Legendrian ones, we break the knot into pieces, each covered by a Darboux chart. This reduces the problem to the standard contact structure on $\mathbb{R}^3$. By compactness, a finite number of such charts suffices, and the local approximations can then be combined to approximate the entire knot.

A key tool in these constructions is the Lagrangian projection, which maps a knot in $(\mathbb{R}^3, \xi_{st})$ to a curve in the (x, y)-plane. If the knot is Legendrian, the projection allows us to recover the z coordinate of the curve by integrating the contact condition as we will later see.

Lemma 4.6. *Let $\gamma : S^1 \to (M, \xi)$ be a Legendrian knot in a contact 3-manifold. Then γ can be C^0-approximated by a positively as well as a negatively transverse knot isotopic to γ.*

Proof. By Corollary 4.4 we only need to prove the result for $(S^1 \times \mathbb{R}^2, \alpha = \cos\theta dx - \sin\theta dy)$.

On the torus given by $\{x^2 + y^2 = \delta\}$, for some $\delta \in \mathbb{R}^+$ there are two transverse curves. These curves are defined as

$$\gamma_\pm(\theta) := (\theta, x = \pm\delta\sin\theta, y = \pm\delta\cos\theta), \quad \theta \in S^1.$$

We refer to $\gamma_\pm$ as the positive or negative transverse push-off of γ, respectively. And taking δ sufficiently small, we can construct our approximation.

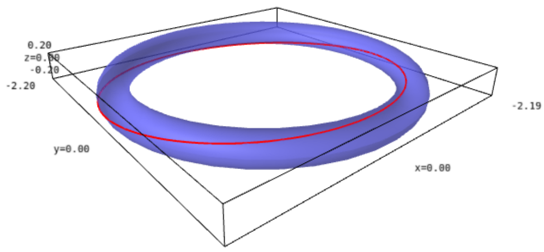


Figure 12: transverse push-off of γ

□

Now to approximate arbitrary knots by Legendrian ones we will first prove it in Darboux charts so in the standard contact structure of $\mathbb{R}^3$ given by $\alpha_{\text{st}} = dz + xdy$. We will write $\gamma(s) = (x(s), y(s), z(s))$ for a curve $\gamma : (a,b) \to \mathbb{R}^3$ and the Legendrian condition becomes the equation

$$\alpha_{\text{st}}(\gamma') = z' + xy' = 0.$$

The Lagrangian projection of a curve is

$$\gamma_L(s) = (x(s), y(s)).$$

Note that if γ is Legendrian, then by integrating the Legendrian condition we can recover the original curve just by integrating the contact condition **??**. The following lemma gives the key facts and details for the reconstruction of Legendrian curves.

Lemma 4.7. *Let $\gamma : (a,b) \to (\mathbb{R}^3, \xi_{st})$ be a Legendrian immersion. Then its Lagrangian projection $\gamma_L(s) = (x(s), y(s))$ is also an immersed curve. The curve γ is recovered from γ_L via*

$$z(s_1) = z(s_0) - \int_{s_0}^{s_1} x(s)y'(s)\, ds. \tag{7}$$

for any $s_0, s_1 \in (a,b)$

A Legendrian immersion $\gamma : S^1 \to (\mathbb{R}^3, \xi_{st})$ has a Lagrangian projection that encloses zero area. Moreover, γ is embedded if and only if every loop in γ_L (except, in the closed case, the full loop γ_L) encloses a non-zero oriented area.

Any curve (defined on an interval) immersed in the (x,y)–plane is the Lagrangian projection of a Legendrian curve in $(\mathbb{R}^3, \xi_{st})$, unique up to translation in the z–direction.

Proof. The Legendrian condition $z' + xy'$ implies that if $y' = 0$ then $z' = 0$, and hence, since γ is an immersion, $x' \neq 0$. So γ_L is an immersion.

The formula (7) is obtained simply by integrating between s_1 and s_0 the Legendrian condition in $\mathbb{R}^3$, $z' + xy' = 0$. Notice that for a simple closed curve, the integral corresponds precisely to the signed area enclosed by the curve, which we will denote as R. This can be easily seen using Green's theorem:

$$\oint_\gamma xdy = \iint 1dA = R$$

So if we have a Lagrangian immersion of S^1 and we parametrize S^1 using an interval (a, b), the condition that the endpoints meet and the integral is zero is the same.

Moreover, for a Legendrian immersion to be embedded, we only need to take care of the self-intersections in γ_L. By the same logic as before, if γ_L has a self-intersection at $(x(s_0), y(s_0)) = (x(s_1), y(s_1))$, then the area of the curve between s_0 and s_1 must not be zero as that would imply $z(s_0) \neq z(s_1)$. The last fact becomes ovbious if we look at 7, to construct the Legendrian curve simply pick a starting z coordinate at $s_0 = a$ and use the formula to construct the curve. □

Now we will present a theorem that states that any knot can be C^0 approximated by a Lagrangian one. To make the proof more digestible, we will first start with a simple example:

Example 4.8. Consider the curve $\gamma : S^1 \to (\mathbb{R}^3, \xi_{std})$, where S^1 is the standard inclusion in the (x, y)-plane. This curve can be C^0-approximated by a Legendrian knot.

A naive way to approximate the curve γ is to use its Lagrangian projection γ_L and apply (7) to reconstruct a Legendrian approximation. This method ensures that the x and y coordinates match, but the z coordinate is not approximated at all. Moreover, this naive approximation fails to produce a knot since the endpoints do not meet, as can be seen in Figure 13, where the original knot S^1 is in blue and the naive reconstruction is in red. Our key idea to improve the approximation is to add small loops to γ_L so that, when using the reconstruction formula, they correct the path of the z coordinate.

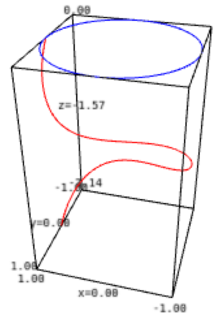


Figure 13: Lagrangian approximation of S^1

The process is as follows. Let $\epsilon > 0$, and suppose γ_L is parameterized by t. Use (7) to obtain the new z coordinate as in the naive case. We follow the new $z(t)$ until t_1 where the distance from the original z is ϵ. At this point t_1, we add a small loop of area ϵ to our approximation, bringing us back to the (x, y)-plane. The process is repeated finely many times until we reach the end of the knot. If, at the end, the z coordinates of the approximation differ by some amount, we add a final loop of the appropriate signed area to close the knot [7]

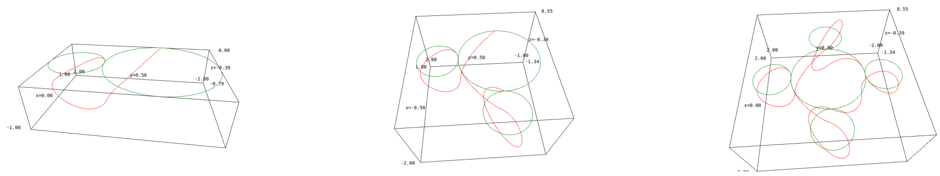


Figure 14: Adding loops step by step

Now, to ensure this is a valid approximation, we verify that each coordinate is well approximated. The x and y coordinates are clearly well approximated, as the small loops become arbitrarily small as ϵ decreases (since they have area ϵ by construction). The z coordinate is approximated by construction: each time z deviates by ϵ, a loop is introduced to correct it. One potential issue is that the loop might temporarily take z more than ϵ away from its original value. However, a careful computation of the z coordinate for each loop shows that the domain of the deviation becomes arbitrarily small as ϵ is decreased.

[7] In the illustrations, the endpoints are made to meet ad hoc to simplify the drawings.

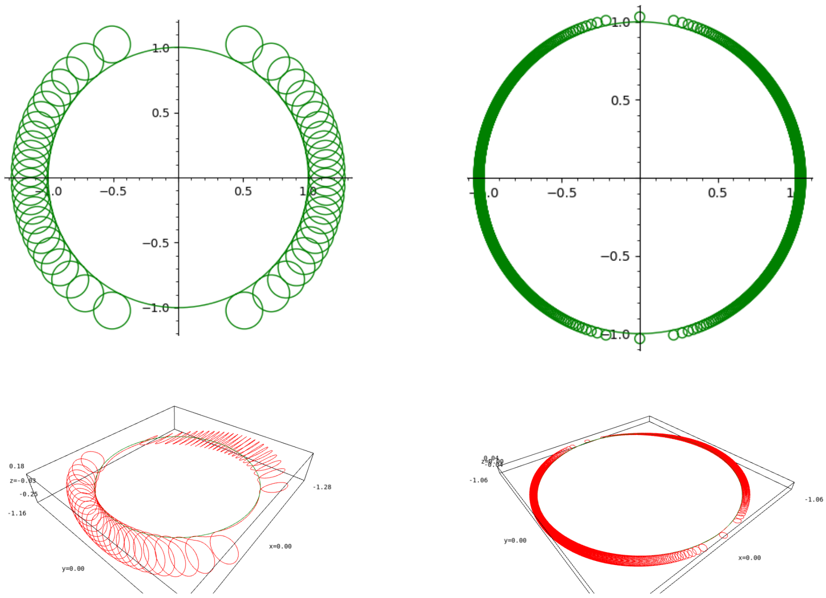


Figure 15: Better Lagrangian aproximations

We have now successfully constructed an immersed curve that approximates γ. By applying the Lemma again, we see that this is, in fact, an embedding, as the intersections in the front projection of our approximation (shown in green) enclose non-zero area.

Theorem 4.9. *Let $\gamma : S^1 \to (M, \xi)$ be a knot in a contact 3–manifold. Then γ can be C^0–approximated by a Legendrian knot isotopic to γ.*

Proof. Using Darboux coordinates, we can cover the knot by Darboux coordinates and by compactness we can refine the cover to a finite one, so the problem can be simplified to the standard contact structure of $\mathbb{R}^3$ if the endpoints of each chart meet appropriately, so the proof will be almost identical as in the Example.

So let $\gamma : (a, b) \to (\mathbb{R}^3, \xi_{st})$ be an immersion and γ_L its Lagrangian projection, we ought to approximate this curve by $\tilde{\gamma}_L$ such that:

- If we use Lemma 4.7 to construct a Legendrian curve $\tilde{\gamma}$ the z coordinate of $\tilde{\gamma}$ approximates the original z
- $\tilde{\gamma}$ has no self-intersections

First we assume γ_L has no singularities, we will deal with them later on.

As before, let $\epsilon > 0$ and let γ_L be parametrized by t. We first start at $x(a), y(a), z(a)$ and follow γ_L until the z coordinate of $\tilde{\gamma}_L$ is too far, so at that point we add a clockwise or anticlockwise small loop to correct the path and follow that loop as many times as necessary to stay close to the original curve. Then we repeat the process until we have successfully constructed a curve $\tilde{\gamma}$ that is at most at distance ϵ to γ. We can also assume that the curves coincide at the last part, if they don't, simply add a loop just before of the correct signed area so that the z coordinates coincide. This is what allows us to glue all the local pieces back together.

Finally, we might have to deal with γ_L having a singularity, when that happens γ must go in the z direction vertically: let's assume γ_L has a singularity in $[c, d]$, then i.e. $x'(s) = y'(s) = 0$ and so $z'(s) \neq 0$ for all $s \in [c, d]$ as γ is an embedding. To construct $\tilde{\gamma}$ in this interval we need to go up (or down depending on the sign of $z'(s)$)in the z-direction so we will add small loops as before. It is beter illustrated with the following example:

Let γ be defined by:

$$x(t) = \begin{cases} t & \text{if } t < 0, \\ 0 & \text{if } t \geq 0, \end{cases}, \qquad y(t) = \begin{cases} t & \text{if } t > 1, \\ 1 & \text{if } t \leq 1, \end{cases}, \qquad z(t) = t.$$

Note that for $t \in (0, 1)$ the curve exhibits a singularity. To counteract this, we add a number of sufficiently small loops so that the approximation remains close enough, as shown in Figure 16. On the left, we see the Lagrangian projection of the curve and the loop we add, which winds around three times in this case. On the right, we observe how the curve rises vertically, allowing us to ascend while staying close to the original curve.

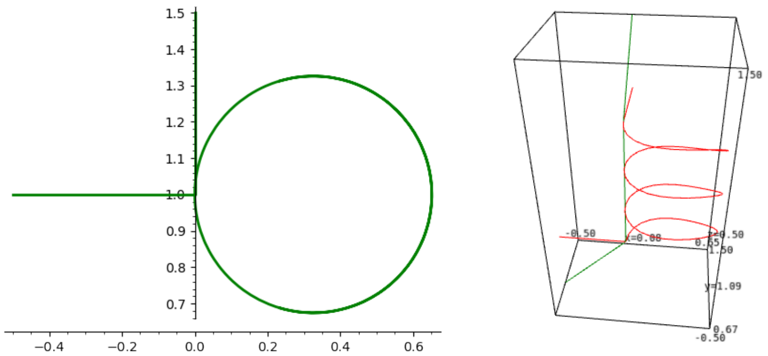


Figure 16: Singularities

To sum up, the construction of the aproximation is simply achieved by using the formula 4.7 to reconstruct the z coordinate and adding small loops when necessary to correct the path. □

Corollary 4.10. *Let $\gamma : S^1 \to (M, \xi)$ be a knot in a contact 3–manifold with ξ cooriented, γ can be C^0–approximated by a positively as well as a negatively transverse knot isotopic to γ.*

Proof. Just approximate by a Legendrian knot using Theorem 4.9 and then use Lemma 4.6 to approximate that knot with a transverse one. □

4.2 Martinet's theorem

Now we have the necessary theory to prove the theorem.

Theorem 4.11 (Martinet)**.** *Every closed, orientable 3–manifold M admits a contact structure.*

As stated in the Lickorish and Wallace Theorem 3.18, every closed, orientable, and connected 3-manifold can be obtained from S^3 by performing a finite sequence of Dehn surgeries. The idea then is to take the standard contact structure in S^3 (Given in Example 2.12) and use the following theorem to extend the structure on the solid tory where Dehn surgery is performed. So to prove Martinet's theorem, it suffices to prove the following more general result, which uses the approximation methods and models we have introduced to simplify the proof substantially.

Theorem 4.12. *Let ξ_0 be a contact structure on a 3–manifold M_0. Let M be the manifold obtained from M_0 by a Dehn surgery along a knot K. Then M admits a contact structure ξ which coincides with ξ_0 outside the neighbourhood of K where the surgery is performed.*

Proof. By Theorem 4.10 we can assume the knot is transverse to the contact structure ξ. If the approximation is close enough, the isotopy class of the knot is unchanged and the approximated knot will give rise to the same Dehn surgery. This allows us to apply Theorem 4.5 in dimension 3.

Specifically, we can find a neighbourhood νK of K that is contactomorphic to

$$(S^1 \times D^2_{\delta_0}, \xi_0 = \mathrm{Ker}(d\overline{\theta} + \overline{r}^2\, d\overline{\varphi})),$$

where $D^2_{\delta_0}$ is a disk of radius $\delta_0 > 0$. Here, $(\overline{\varphi}, \overline{r})$ are polar coordinates on $D^2_{\delta_0}$, and $\overline{\theta}$ represents the angular coordinate on S^1.

Next, we perform Dehn surgery. To ensure precise control of the contact structure near the surgery, we will perform the surgery on a smaller disk

$$S^1 \times D^2_{\delta} \subset S^1 \times D^2_{\delta_0}, \quad \text{where} \quad \delta < \delta_0.$$

Consider the boundary of $S^1 \times D^2_\delta$ and denote by μ denote the meridian $* \times \partial D^2_\delta$, and λ the longitude $S^1 \times *$.

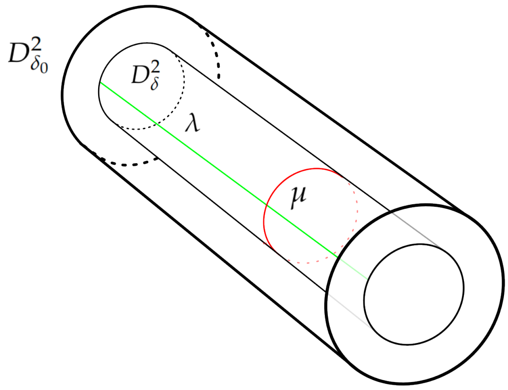


Figure 17: Torus contact structure

Then we glue the boundary of $S^1 \times D^2_\delta$ to itself with a map described by:

$$\mu \longrightarrow p\mu + q\lambda, \quad \lambda \longrightarrow m\mu + n\lambda$$

with some $\begin{pmatrix} p & m \\ q & n \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z})$ adequate for the surgery we are performing.

Now let $(\theta; r, \phi)$ be the coordinates on the copy of $S^1 \times D^2_\delta$ that we are gluing back. Then the gluing map can be described explicitly by

$$\begin{aligned} \psi : S^1 \times \partial D^2_\delta &\longrightarrow S^1 \times \partial D^2_\delta \\ (\theta; r, \phi) &\longmapsto (n\theta + q\phi; r, m\theta + p\phi) \end{aligned}$$

So the contact form $d\overline{\theta} + \overline{r}^2 d\overline{\phi}$ given on $S^1 \times D^2_{\delta_0} \subset M$ pulls back using ψ to

$$d(n\theta + q\phi) + r^2 d(m\theta + p\phi) \tag{8}$$

If we find a contact form on $S^1 \times D^2_\delta$ that coincides with this form on the boundary $S^1 \times \partial D^2_\delta$ we are done as we will extend the contact structure on the knots where we perform Dehn surgery. The following Lemma 4.13 completes the proof. □

Lemma 4.13. *There is a contact form on $S^1 \times D^2_\delta$ that coincides with $(n + mr^2)d\theta + (q + pr^2)d\phi$ near $r = \delta$ and with $\pm d\theta + r^2 d\phi$ near $r = 0$.*

Proof. We first assume the solution will be of the form

$$\alpha = h_1(r)d\theta + h_2(r)d\phi$$

with smooth functions $h_1(r)$, $h_2(r)$ note that this makes it rotationally symmetric, which is somewhat reasonable to assume.

Now we impose the contact condition:

$$d\alpha = h_1'(r)dr \wedge d\theta + h_2'(r)dr \wedge d\phi$$

so the contact condition becomes

$$\alpha \wedge d\alpha = \begin{vmatrix} h_1 & h_2 \\ h_1' & h_2' \end{vmatrix} d\theta \wedge dr \wedge d\phi = 0.$$

So to satisfy the contact condition $\alpha \wedge d\alpha \neq 0$ and to make the contact form coincide with (8), all we have to do is find a parametrized curve

$$r \mapsto (h_1(r), h_2(r)), \quad 0 \leq r \leq \delta,$$

in the plane, satisfying the following conditions:

1. $h_1(r) = \pm 1$ and $h_2(r) = r^2$ near $r = 0$,
2. $h_1(r) = n + mr^2$ and $h_2(r) = q + pr^2$ near $r = \delta$,
3. $(h_1(r), h_2(r))$ is never parallel to $(h_1'(r), h_2'(r))$ for $r \neq 0$.

Conditions 1 and 2 are easily achieved using partitions of unity, meaning you start and end with meeting the conditions, and you smoothly connect pieces. What is not that clear is if condition 3 can be satisfied.

Figure 18 shows curves for the two cases $np - mq = \pm 1$ meaning the third condition, as they curve everywhere (the geodesic curvature does not vanish).

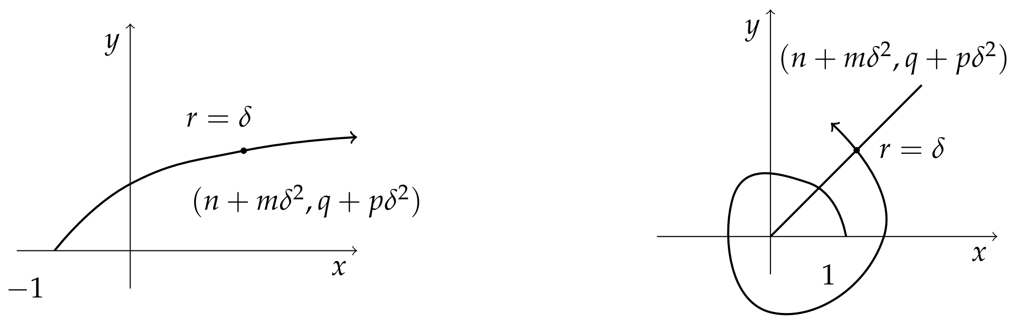


Figure 18: Solutions

□

References

[1] Franki J.E. Dillen and Leopold C.A. Verstraelen, editors. *Handbook of Differential Geometry, Volume 2*. North Holland, 2006.

[2] Siddhartha Gadgil. Heegaard splittings, 2008. Accessed: 2024-12-28.

[3] Hansjörg Geiges. *An Introduction to Contact Topology*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 2008.

[4] Victor Guillemin and Alan Pollack. *Differential Topology*. Prentice-Hall, 1974.

[5] Allen Hatcher. *Algebraic topology*. Cambridge University Press, 2002.

[6] Joren Husic. The lickorish-wallace theorem, 2019. Accessed: 2024-12-28.

[7] John M. Lee. *Introduction to Smooth Manifolds*, volume 218 of *Graduate Texts in Mathematics*. Springer, second edition, 2013.

[8] W.B. Raymond Lickorish. *An Introduction to Knot Theory*. Graduate Texts in Mathematics. Springer, 1997.

[9] Dusa McDuff and Dietmar Salamon. *Introduction to Symplectic Topology*. Oxford University Press, 2nd edition, 1998.

[10] Edwin E. Moise. Affine structures in 3-manifolds: V. the triangulation theorem and hauptvermutung. *Annals of Mathematics*, 56, 1952.

[11] Dale Rolfsen. *Knots and Links*. Publish or Perish, 1976.

[12] Stephen Smale. Diffeomorphisms of the 2-sphere. *Proceedings of the American Mathematical Society*, 10(4), 1959.