

Claims Manipulations Discriminate between Egalitarian Rules

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Abstract

We provide axiomatic characterizations of the constrained equal awards and constrained equal losses rules by imposing immunity principles such as non-manipulability via merging (splitting) and strong non-manipulability via merging (splitting). We show that the constrained equal awards rule is the unique rule satisfying strong non-manipulability via merging and contraction independence. Replacing contraction independence with the weaker condition of independence of redundant claims yields a new characterization. Moreover, strong non-manipulability via merging in the above two axiomatizations can be substituted by consistency and non-manipulability via merging. By duality, we achieve parallel characterizations of the constrained equal losses rule.

Keywords:

Claims problems, Constrained equal awards rule, Constrained equal losses rule, Non-manipulability via merging, Non-manipulability via splitting, Contraction independence.

1. Introduction

Consider problems in which a limited resource must be divided among a group of agents. Each agent is characterized by her claim to the resource, but the total resource is insufficient to satisfy all claims. Such problems arise in many real-life situations, such as the distribution of a firm's liquidation value among its creditors or the allocation of scarce medical supplies, such as ventilators or vaccines, to patients based on need. Such situations are usually called claims problems or bankruptcy problems. This type of problems traces back to the 2,000-year-old Babylonian Talmud, where some recommendations are hinted at. O'Neill (1982) is the first to formally study solutions to such problems from an axiomatic perspective. Solutions or rules are functions that propose a desirable division of the resource for each claims problem.

Two widely recognized and extensively studied rules are the constrained equal awards rule (CEA) and the constrained equal losses rule (CEL). The first one was explained by Maimonides in his 12th-century book *Laws of Lending and Borrowing*. To determine the allocation under the CEA, the resource is distributed equally among all agents until some

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reach their full claims. These agents then take their allocations and exit the process, while the remaining resource is redistributed in a recursive way among the others in the same manner. The allocation method of the CEL is similar; however, it focuses on equalizing losses, the difference between an agent’s claim and her award.

These rules have been extensively analyzed in the literature using axiomatic methods. Dagan (1996) is the first to characterize CEA through the property of *claims truncation invariance*, along with other properties. Claims truncation invariance ensures that the awards to agents do not change when claims are truncated at the estate. Similarly, Kıbrıs (2012) characterizes a family of rules containing CEA imposing the stronger property of *contraction independence* in conjunction with other properties. Contraction independence requires that the awards vector remains unchanged when the claim of each agent is contracted to any amount between her claim and her award. Kıbrıs and Kıbrıs (2013) point out that CEA is the only rule in the aforementioned family of rules that satisfies equal treatment of equals. Both claims truncation invariance and contraction independence belong to the group of invariance properties, principles requiring that the awards vector selected by a rule should remain unchanged under certain transformations applied to the parameters of a problem. These axioms have been used to characterize other rules. For more information, see Hokari and Thomson (2003), Bergantiños et al. (2011), Stovall (2014), Flores-Szwagrzak (2015) and Harless (2017).

In this paper we focus on characterizations of CEA and CEL by means of immunity axioms. Immunity principles convey the key idea that any agent could not take advantage by misrepresenting her claim, either via splitting the claim of an agent into several claims or via merging her claim and others’ into one. When, under the application of a rule, the agent cannot benefit from such manipulations, we say the rule is *non-manipulable via splitting or via merging*, respectively. *Non-manipulability*¹ is the combination of both and has been used by O’Neill (1982), Chun (1988), de Frutos (1999) and Ju et al. (2007) to characterize the proportional to claims distribution. *Strong non-manipulability via merging or splitting* prevents any agent not involved in the manipulation from being harmed and it has been analyzed in Curiel et al. (1987) and Moreno-Ternero (2006). Not much attention has been paid to the study of (strong) non-manipulability via merging and splitting separately. Thus, this research line could be interesting in environments where some legal or practical constraints make it difficult to merge or split. We find some efforts in de Frutos (1999), Ju (2003), Ju and Moreno-Ternero (2011) and Calleja et al. (2023). In the claims problems literature, it is typically assumed that agents’ claims are observable, as is the case in real-life applications such as bankruptcy. Even under this natural assumption, however, it is still reasonable in many contexts to consider the possibility that agents may manipulate outcomes by merging or splitting their claims. The NM and NS axioms (together with their variants) are therefore particularly relevant, as they capture precisely these fundamental forms of manipulation that remain possible despite the observability of claims.

It is well known that CEA satisfies (strong) non-manipulability via merging, while CEL satisfies (strong) non-manipulability via splitting. However, to the best of our knowledge, these principles have not been yet employed to characterize these two rules. We therefore en-

¹Introduced in O’Neill (1982) under the name of strategy-proofness.

deavor to contribute by identifying them combining immunity principles as well as invariance properties.

In Section 2 we introduce the main notations and definitions. In Section 3 we formally define the properties and principles we use in the paper. In Section 4 we present our characterizations, among which we highlight that CEA is the only rule meeting strong non-manipulability via merging and contraction independence. Furthermore, refining this result, we provide a new axiomatization imposing the weaker non-manipulability via merging and independence of redundant claims at the price of additionally requiring consistency. Finally, by duality, we obtain parallel characterizations for the CEL. In Section 5 we conclude. All the proofs and the discussion on the logical independence of the properties are relegated to the Appendixes.

2. Claims problems and rules

Let us denote by $\mathbb{N}$ the set of natural numbers which we identify with the universe of potential agents. The family of all finite subsets of $\mathbb{N}$ is denoted by $\mathcal{N}$. Given a finite subset of agents $N = \{1, 2, \dots, n\} \in \mathcal{N}$, a *claims problem* is a triplet $(N, E, c) \in \mathcal{N} \times \mathbb{R}_+ \times \mathbb{R}_+^n$ where $E \geq 0$ is the estate to be allocated among the agents whose claims are depicted in the vector $c = (c_1, c_2, \dots, c_n)$, being $c_i \geq 0$ the legitimate claim of agent i , and $\sum_{i \in N} c_i \geq E$ (scarcity condition). Let $\mathcal{C}$ be the set of all claims problems.

An allocation or *awards vector* for $(N, E, c) \in \mathcal{C}$ is a vector $x \in \mathbb{R}^n$, where x_i denotes the amount awarded to agent i , such that $\sum_{i \in N} x_i = E$, a condition called *efficiency*. Given $x \in \mathbb{R}^n$ and $T \subset N$, $x|_T$ denotes the restriction of x to the members of T . A solution concept or a *rule* is a function F that associates to any claims problem $(N, E, c) \in \mathcal{C}$ a unique allocation $x \in \mathbb{R}^n$ such that $c_i \geq F_i(N, E, c)$ for all $i \in N$ (*claim boundedness*) and $F_i(N, E, c) \geq 0$ for all $i \in N$ (*non-negativity*).

The *constrained equal awards* and the *constrained equal losses* rules are two prominent methods used to allocate resources when there is a shortfall. The CEA focuses on equalizing awards among claimants, subject to no one receiving more than her claim; that is, for all $(N, E, c) \in \mathcal{C}$,

$$CEA_i(N, E, c) = \min \{c_i, \lambda\} \quad \text{for all } i \in N,$$

where λ is chosen so that $\sum_{i \in N} \min \{c_i, \lambda\} = E$.

The CEL aims to equalize losses among claimants, subject to nobody receiving a negative amount; that is, for all $(N, E, c) \in \mathcal{C}$,

$$CEL_i(N, E, c) = \max \{0, c_i - \lambda\} \quad \text{for all } i \in N,$$

where λ is chosen so that $\sum_{i \in N} \max \{0, c_i - \lambda\} = E$.

The CEA and the CEL rules are dual to each other, meaning that CEA divides what is available, E , in the same way as CEL divides the total deficit, $L(N, E, c) = \sum_{i \in N} c_i - E$ and vice versa. If no confusion arises we denote $L = L(N, E, c)$. The notion of duality was first introduced by Aumann and Maschler (1985). Formally, F and F^d are dual rules if $F(N, E, c) = c - F^d(N, L, c)$. A rule is said to be self-dual if its dual is itself, $F = F^d$. An example of a self-dual rule is the proportional rule, P , which allocates the available amount

of resource in proportion to claims. That is, for all $(N, E, c) \in \mathcal{C}$,

$$P_i(N, E, c) = \frac{c_i}{\sum_{j \in N} c_j} E \text{ for all } i \in N.$$

3. Axioms

In this section, we introduce some basic well-known axioms in the literature. Three broadly used properties are *resource monotonicity*, *equal treatment of equals* and *consistency*. Formally, a rule F satisfies:

- *Resource monotonicity (RM)*: if for all $(N, E, c) \in \mathcal{C}$ and $(N, E', c) \in \mathcal{C}$ such that $E' > E$, then

$$F(N, E', c) \geq F(N, E, c).$$

- *Equal treatment of equals (ETE)*: if for all $(N, E, c) \in \mathcal{C}$ and $i, j \in N$ such that $c_i = c_j$ then

$$F_i(N, E, c) = F_j(N, E, c).$$

- *Consistency (CONS)*: if for all $(N, E, c) \in \mathcal{C}$ and all $\emptyset \neq T \subseteq N$, it holds that

$$F(N, E, c)|_T = F(T, E - \sum_{i \in N \setminus T} F_i(N, E, c), c|_T).$$

Resource monotonicity says that if the endowment increases, each agent receives at least as much as initially. Equal treatment of equals reflects the central idea that any two agents with equal claims should receive equal amounts. Consistency is an internal stability condition requiring that when some creditors leave with their awards, then the rule reallocates the remaining resource among the rest of creditors as initially.

The strategic incentives of the agents to manipulate the outcome either by merging or splitting their claims with the objective to achieve better allocations have been intensively studied for claims problems. Non-manipulability (either by merging or by splitting) is introduced by O'Neill (1982). In their papers, de Frutos (1999), Ju (2003) and Ju and Moreno-Ternero (2011) study those incentives separately. Formally, a rule F satisfies:

- *Non-manipulability via merging (NM)*: if for all $(N, E, c) \in \mathcal{C}$, all $\emptyset \neq S \subseteq N$, and all $i \in S$,

$$F_i((N \setminus S) \cup \{i\}, E, c') \leq \sum_{j \in S} F_j(N, E, c),$$

where $c'_i = \sum_{j \in S} c_j$ and $c'_k = c_k$ for all $k \in N \setminus S$.

- *Non-manipulability via splitting (NS)*: if for all $(N, E, c) \in \mathcal{C}$, all $\emptyset \neq S \subseteq N$, and all $i \in S$,

$$F_i((N \setminus S) \cup \{i\}, E, c') \geq \sum_{j \in S} F_j(N, E, c),$$

where $c'_i = \sum_{j \in S} c_j$ and $c'_k = c_k$ for all $k \in N \setminus S$.

Non-manipulability via merging says that no group of agents should have incentives to merge their claims and act as a single claimant. Non-manipulability via splitting requires that no single agent can get more by splitting her demand into the demands of different agents. Non-manipulability states that agents, merging or splitting claims, should be paid exactly the same amount as initially. So, it is the combination of non-manipulability via merging and non-manipulability via splitting.

These two properties are dual (de Frutos, 1999), meaning that when a rule satisfies non-manipulability via merging, as CEA does, its dual rule satisfies non-manipulability via splitting, as CEL does. Formally, two axioms A and A^d are *dual* if whenever a solution F holds A then F^d meets A^d . An axiom is self-dual if $A = A^d$. Resource monotonicity, equal treatment of equals and consistency are self-dual.

Strong non-manipulability via merging (splitting) says that when some agents merge (split), all agents not involved in the operation should not be, one by one, worse off than initially. A solution concept F satisfies:

- *Strong non-manipulability via merging (SNM)*: if for all $(N, E, c) \in \mathcal{C}$, all $\emptyset \neq S \subseteq N$, and all $i \in S$,

$$F_k((N \setminus S) \cup \{i\}, E, c') \geq F_k(N, E, c), \text{ for all } k \in N \setminus S,$$

where $c'_i = \sum_{j \in S} c_j$ and $c'_k = c_k$ for all $k \in N \setminus S$.

- *Strong non-manipulability via splitting (SNS)*: if for all $(N, E, c) \in \mathcal{C}$, all $\emptyset \neq S \subseteq N$, and all $i \in S$,

$$F_k((N \setminus S) \cup \{i\}, E, c') \leq F_k(N, E, c) \text{ for all } k \in N \setminus S,$$

where $c'_i = \sum_{j \in S} c_j$ and $c'_k = c_k$ for all $k \in N \setminus S$.

Strong non-manipulability, first introduced by Curiel et al. (1987), is the conjunction of both strong non-manipulability via merging and strong non-manipulability via splitting, requiring that agents not involved in the manipulation are paid as initially. Under efficiency, the strong version of each of the properties implies the weak one. Moreover, Moreno-Ternero (2006) shows that non-manipulability is equivalent to strong non-manipulability. However, Calleja et al. (2023) emphasizes that neither non-manipulability via merging implies strong non-manipulability via merging nor strong non-manipulability via splitting is a consequence of non-manipulability via splitting. However, under consistency and resource monotonicity, non-manipulability via merging (splitting) implies strong non-manipulability via merging (splitting). As CEA (CEL) satisfies the aforementioned properties, it also satisfies strong non-manipulability via merging (splitting). They also proved that strong non-manipulability via merging and strong non-manipulability via splitting are dual properties.

The objective of this paper is to propose axiomatic characterizations for CEA and CEL making use of such immunity principles. To do it, we additionally need contraction independence, a property first introduced by Kıbrıs (2012) which is inspired by independence of irrelevant alternatives as defined by Nash (1950) for bargaining problems and it is satisfied by CEA. It says that the awards vector should remain the same after truncating all claims in such a way that the initial awards vector is still available. Formally, a rule F satisfies:

- *Contraction independence (CI)*: if for all $(N, E, c) \in \mathcal{C}$ and $F(N, E, c) \leq c' \leq c$, it holds that

$$F(N, E, c') = F(N, E, c).$$

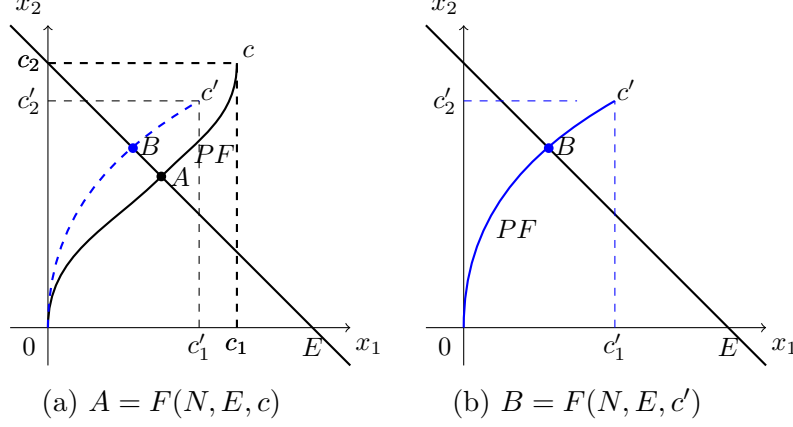


Figure 1: A rule that does not satisfy Contraction Independence. In picture (a), we represent the path of awards PF of a rule F when the claims are c . The intersection of this curve with the budget line ($x_1 + x_2 = E$) gives the allocation $A = F(N, E, c)$. In the same picture, we select a claims point c' , with $A \leq c' \leq c$, which contracts the feasible set of allocations but still contains A . In picture (b), we represent the path of awards PF of the same rule F for the reduced claims vector c' , obtaining the allocation $B = F(N, E, c')$. In picture (a), we overlap the two pictures to show that F does not satisfy the property, since $A \neq B$.

Kıbrıs (2012) shows that CEA satisfies contraction independence. We provide its dual property and name it *Partial Rewards First*.² It states that the awards vector from a rule remains the same by either directly applying the rule to the problem or following a two-step procedure. At the first step, each agent receives an arbitrary partial payment of the initial commitment by the rule, and then, the remaining resource is distributed according to the rule. Formally, a rule F satisfies:

- *Partial rewards first (PRF)*: if for all $(N, E, c) \in \mathcal{C}$ and $0 \leq m \leq F(N, E, c)$, it holds that

$$F(N, E, c) = m + F(N, E - \sum_{i \in N} m_i, c - m).$$

²which is first explained in words by Thomson (2019).

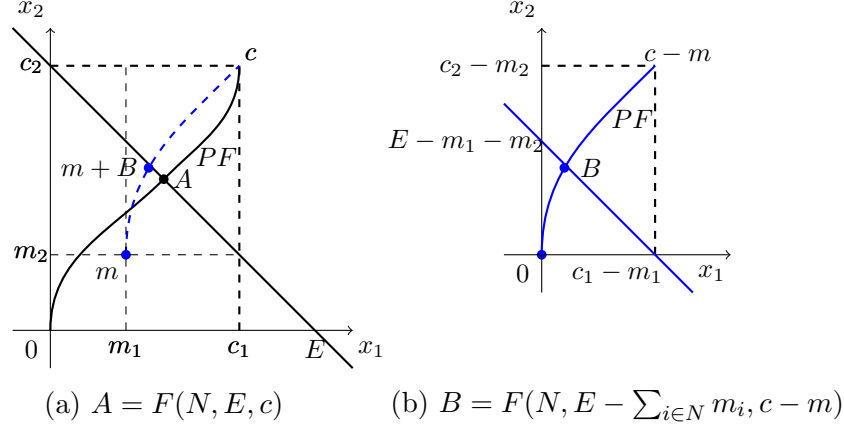


Figure 2: A rule that does not satisfy Partial Reward First. In picture (a), we represent the path of awards PF of a rule F when the claims are c . At the intersection of this curve with the budget line $(x_1 + x_2 = E)$, we find the allocation $A = F(N, E, c)$. In the same picture, we select a point m , with $m \leq A$, representing a partial reward to the players from A . In picture (b), we represent the path of awards PF of the same rule F for the reduced claims $(c_1 - m_1, c_2 - m_2)$, and the allocation B proposed by F when the estate is $E - m_1 - m_2$. Finally, in picture (a), we overlap the two pictures to show that F does not satisfy the property, i.e., $A \neq m + B$.

Proposition 1. Contraction independence and partial rewards first are dual properties.

4. Axiomatic characterizations of the CEA and the CEL rules

We now provide axiomatic characterizations of CEA and CEL rules. In Subsection 4.1, we obtain characterizations imposing strong non-manipulability via merging (splitting). In Subsection 4.2, we propose characterizations weakening strong non-manipulability via merging (splitting) to non-manipulability via merging (splitting).

4.1. Strong non-manipulability via merging

To characterize the CEA imposing strong non-manipulability via merging, we also require contraction independence. We first present that equal treatment of equals is implied by strong non-manipulability via merging (splitting).

Proposition 2. Strong non-manipulability via merging (splitting) implies equal treatment of equals.

The next theorem provides our first characterization.

Theorem 1. The constrained equal awards is the unique rule that satisfies contraction independence and strong non-manipulability via merging.

Contraction independence is a strong property in the sense that no matter how much the claims are truncated, as long as the initial awards vector is still available, the rule selects the same allocation in the truncated problem. In order to make Theorem 1 tighter, we consider possibilities of making contraction independence weaker. The first one, claims truncation

invariance,³ has been used by Dagan (1996) to characterize CEA. It requires invariance of the rule when claims are truncated at the estate. However, the proportional to truncated-claims rule⁴ satisfies claims truncation invariance and strong non-manipulability via merging. The second one, independence of redundant claims, is a new property. It says that the awards vector proposed by a rule is not affected if we truncate the claims endogenously at the highest amount awarded to an agent by the rule. Formally, a solution F satisfies:

- *Independence of redundant claims (IRC)*: if for all $(N, E, c) \in \mathcal{C}$,

$$F(N, E, c) = F(N, E, c^T),$$

where $c_i^T = \min \{c_i, \max_{j \in N} F_j(N, E, c)\}$ for all $i \in N$.

Let us remark that according to the two variants of truncation the initial awards vector is still feasible in the truncated problem. It is well known that contraction independence implies claims truncation invariance (see Thomson, 2019). Our next proposition shows that it also implies independence of redundant claims.

Proposition 3. Contraction independence implies independence of redundant claims.

Therefore, CEA satisfies independence of redundant claims and claims truncation invariance. However, these two axioms are unrelated.⁵ Using the above Proposition, we present our Theorem 2.

Theorem 2. The constrained equal awards is the unique rule that satisfies independence of redundant claims and strong non-manipulability via merging.

To characterize CEL, we take the duality approach. Proposition 1 shows that partial rewards first is the dual of contraction independence. The dual property of independence of redundant claims says that each agent should be paid at least what is left after assuming, not the total deficit (as suggested by minimal rights first)⁶, but just the maximal loss of an agent. Formally, a rule F satisfies:

- *Minimal claims first (MCF)*: if for all $(N, E, c) \in \mathcal{C}$,

$$F(N, E, c) = mc + F\left(N, E - \sum_{i \in N} mc_i, c - mc\right),$$

where $mc_i = \max \left\{ 0, c_i - \max_{j \in N} \{c_j - F_j(N, E, c)\} \right\}$ for all $i \in N$.

³For all $(N, E, c) \in \mathcal{C}$, $F(N, E, c) = F(N, E, (\min \{c_i, E\})_{i \in N})$

⁴Proportional to truncated-claims rule is defined: for all $(N, E, c) \in \mathcal{C}$, $P_i(N, E, c) = \frac{\min\{c_i, E\}}{\sum_{j \in N} \min\{c_j, E\}} \cdot E$, for all $i \in N$.

⁵Notice that for two-person problems, the concede-and-divide rule (see Thomson, 2019) satisfies claims truncation invariance but not IRC. However, assuming that $N = \{i, j\}$ and $c_i \leq c_j$, the rule g defined by setting $g(N, E, c) = CEA(N, E, c)$, if $E < c_j$, and $g(N, E, c) = (E - c_j, c_j)$, if $E \geq c_j$, satisfies IRC but not claims truncation invariance.

⁶Minimal rights first is the dual of claims truncation invariance and it is defined as follows: for all $(N, E, c) \in \mathcal{C}$, $F(N, E, c) = mr + F\left(N, E - \sum_{i \in N} m_i(N, E, c), c - mr\right)$ where $mr_i = \max \left\{ E - \sum_{N \setminus \{i\}} c_j, 0 \right\}$ for all $i \in N$.

The following proposition establishes the logical relation between independence of redundant claims and minimal claims first.

Proposition 4. Independence of redundant claims and minimal claims first are dual properties.

From Proposition 1 and Proposition 4, as well as from the fact that strong non-manipulability via merging and strong non-manipulability via splitting are dual properties, we therefore state the following characterizations of the CEL.

Theorem 3. On the domain of claims problems,

- (i) the constrained equal losses rule is the unique rule that satisfies partial rewards first and strong non-manipulability via splitting.
- (ii) the constrained equal losses rule is the unique rule that satisfies minimal claims first and strong non-manipulability via splitting.

4.2. Non-manipulability via merging

In this section, we look back to the axiomatizations presented so far weakening strong non-manipulability via merging (splitting) into non-manipulability via merging (splitting) at the price of additionally requiring consistency. In fact, Calleja et al. (2023) shows that under consistency and resource monotonicity, both are equivalent, which allows to rewrite the theorems presented so far. However, resource monotonicity is redundant. A key tool for the proofs is that consistency and non-manipulability via merging (splitting) imply equal treatment of equals, a fact first noted by Ju and Moreno-Ternero (2011). Regarding CEA, we present the next characterizations.

Theorem 4. On the domain of claims problems,

- (i) the constrained equal awards rule is the unique rule that satisfies contraction independence, consistency, and non-manipulability via merging.
- (ii) the constrained equal awards rule is the unique rule that satisfies independence of redundant claims, consistency, and non-manipulability via merging.

Adopting the duality approach, we present the following characterizations of the CEL.

Theorem 5. On the domain of claims problems,

- (i) the constrained equal losses rule is the unique rule that satisfies partial rewards first, consistency, and non-manipulability via splitting.
- (ii) the constrained equal losses rule is the unique rule that satisfies minimal claims first, consistency, and non-manipulability via splitting.

5. Concluding Remarks

This paper characterizes the constrained equal awards rule with strong non-manipulability via merging and contraction independence (or independence of redundant claims). We also find that when strong non-manipulability via merging is weakened to non-manipulability via merging, the above characterization holds if consistency is imposed. We leave it as an open question for future research to test whether consistency can be implied by non-manipulability via merging and contraction independence (or independence of redundant claims).

Additionally, we find that strong non-manipulability via merging implies equal treatment of equals. When strong non-manipulability via merging is weakened, non-manipulability via merging together with consistency imply equal treatment of equals as proved by Ju and Moreno-Ternero (2011). Although there are solutions that satisfy consistency but not equal treatment of equals, it remains open whether non-manipulability via merging alone implies equal treatment of equals. It is worth mentioning that in all the results of this paper (strong) non-manipulability via merging (splitting) could be replaced by its pairwise version, where only manipulations by pairs are allowed. Taking the duality approach, we extend our conclusions to the CEL.

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Appendix A. Proofs

The proofs are not presented in the same order as the results in the main text.

Proposition 1 CI and PRF are dual properties.

Proof. Let $(N, E, c) \in \mathcal{C}$ be a problem, F be a rule satisfying CI, F^d be its dual and $m \in \mathbb{R}^n$ with $0 \leq m \leq F^d(N, E, c)$. Let $i \in N$. Define $c'_i = c_i - m_i$. Since by claim boundedness $c_i \geq F_i^d(N, E, c)$, then

$$0 \leq c'_i \leq c_i.$$

Moreover,

$$c'_i = c_i - m_i \geq c_i - F_i^d(N, E, c) = F_i(N, L, c),$$

where the second equality is by duality. Hence, we get

$$\begin{aligned} F_i^d(N, E, c) &= c_i - F_i(N, L, c) = c_i - F_i(N, L, c') = c_i - c'_i + F_i^d\left(N, \sum_{j \in N} c'_j - L, c'\right) \\ &= c_i - c'_i + F_i^d\left(N, E - \left(\sum_{j \in N} c_j - \sum_{j \in N} c'_j\right), c'\right) = m_i + F_i^d\left(N, E - \sum_{i \in N} m_i, c - m\right), \end{aligned}$$

where the first and third equalities are from duality, the second equality is from CI and the last equality is from definition of c' . Observe that

$\sum_{j \in N} c'_j - L = E - (\sum_{j \in N} c_j - \sum_{j \in N} c'_j) = E - \sum_{j \in N} m_j \geq E - \sum_{j \in N} F_j^d(N, E, c) = 0$ and $c'_i \geq 0$. Thus, all the claims problems introduced so far are well defined. We conclude that F^d satisfies PRF. $\square$

We first aim to prove Theorem 2 for which we need Propositions 2 and 3. In order to prove Proposition 2, we also need the following Lemma.

Lemma 1. Let F be a rule satisfying SNM. Let $\rho = (N, E, c) \in \mathcal{C}$, $i \in N$, $k \in N \setminus N$ and $\rho^* = ((N \setminus \{i\}) \cup \{k\}, E, c^*)$ be such that $c_k^* = c_i$ and $c_\ell^* = c_\ell$, for all $\ell \in N \setminus \{i\}$. Then, $F_i(\rho) = F_k(\rho^*)$ and $F_\ell(\rho) = F_\ell(\rho^*)$ for all $\ell \in N \setminus \{i\}$.

Proof. Let F be a rule satisfying SNM. Let ρ and ρ^* be defined as above. First, we show that $F_i(\rho) = F_k(\rho^*)$.

To this aim, define $\rho^1 = (N \cup \{k\}, E, c^1)$ with $c_k^1 = 0$ and $c_\ell^1 = c_\ell$ for all $\ell \in N$. Notice that by non-negativity and claim boundedness, we get $F_k(\rho^1) = 0$. Now, let ℓ be any agent in N . From ρ^1 merge agent k with ℓ under the name of ℓ to go back to ρ . By SNM, $F_\ell(\rho) \leq F_\ell(\rho^1) + F_k(\rho^1) = F_\ell(\rho^1)$. Hence, $F_\ell(\rho) \leq F_\ell(\rho^1)$ for all $\ell \in N$. Then, by efficiency, $E = \sum_{\ell \in N} F_\ell(\rho^1) + F_k(\rho^1) = \sum_{\ell \in N} F_\ell(\rho^1) \geq \sum_{\ell \in N} F_\ell(\rho) = E$. Hence,

$$F_\ell(\rho) = F_\ell(\rho^1) \text{ for all } \ell \in N. \quad (1)$$

In particular,

$$F_i(\rho) = F_i(\rho^1). \quad (2)$$

Next define the problem $\rho^2 = (N \cup \{k\}, E, c^2)$ with $c_i^2 = 0$, $c_k^2 = c_i = c_k^*$ and $c_\ell^2 = c_\ell = c_\ell^*$ for all $\ell \in N \setminus \{i\}$. Notice that by claim boundedness and non-negativity, we have $F_i(\rho^2) = 0$. Now, let $\ell \in (N \cup \{k\}) \setminus \{i\}$. From ρ^2 merge i with ℓ under the name of ℓ to go to ρ^* . Following the same argument as before, by SNM, we get

$$F_\ell(\rho^*) = F_\ell(\rho^2) \text{ for all } \ell \in (N \cup \{k\}) \setminus \{i\}. \quad (3)$$

In particular,

$$F_k(\rho^*) = F_k(\rho^2). \quad (4)$$

Next, to continue, assume that from ρ^1 agent i and k merge into agent k to go to ρ^* . Then, by SNM, $F_k(\rho^1) = 0$ and (2), we have

$$F_k(\rho^*) \leq F_i(\rho^1) + F_k(\rho^1) = F_i(\rho). \quad (5)$$

Moreover, assume that from ρ^2 agent i and k merge into agent i to go to problem ρ , then, by SNM, $F_i(\rho^2) = 0$ and (4), we have

$$F_i(\rho) \leq F_i(\rho^2) + F_k(\rho^2) = F_k(\rho^*). \quad (6)$$

Thus, from (5) and (6),

$$F_i(\rho) = F_k(\rho^*). \quad (7)$$

To conclude, we show that $F_\ell(\rho) = F_\ell(\rho^*)$ for all $\ell \in N \setminus \{i\}$. Assume that from ρ^1 agent i and k merge into k to go to ρ^* . By SNM and (1), $F_\ell(\rho^*) \geq F_\ell(\rho^1) = F_\ell(\rho)$ for all $\ell \in N \setminus \{i\}$. Furthermore, by (7), $F_i(\rho) = F_k(\rho^*)$, and by efficiency,

$$E = \sum_{\ell \in N \setminus \{i\}} F_\ell(\rho) + F_i(\rho) \leq \sum_{\ell \in N \setminus \{i\}} F_\ell(\rho^*) + F_k(\rho^*) = E.$$

As a consequence, $F_\ell(\rho) = F_\ell(\rho^*)$ for all $\ell \in N \setminus \{i\}$. $\square$

Proposition 2 SNM (SNS) implies ETE.

Proof. Let F be a rule satisfying SNM and $\rho = (N, E, c) \in \mathcal{C}$ be such that $c_i = c_j$ for some $i, j \in N$. Let $k \in \mathbb{N} \setminus N$ and define $\rho^1 = ((N \setminus \{i\}) \cup \{k\}, E, c^1)$ such that $c_k^1 = c_i$ and $c_\ell^1 = c_\ell$ for all $\ell \in N \setminus \{i\}$. By Lemma 1,

$$F_i(\rho) = F_k(\rho^1). \quad (8)$$

Now, let $\rho^2 = ((N \setminus \{j\}) \cup \{k\}, E, c^2)$, where $c_k^2 = c_j = c_j^1$ and $c_\ell^2 = c_\ell = c_\ell^1$ for all $\ell \in (N \setminus \{j\})$. By Lemma 1, when comparing ρ^1 with ρ^2 ,

$$F_k(\rho^2) = F_k(\rho^1). \quad (9)$$

Applying again Lemma 1 when comparing ρ with ρ^2 ,

$$F_k(\rho^2) = F_j(\rho). \quad (10)$$

Hence, by (8), (9) and (10), we conclude $F_i(\rho) = F_k(\rho^1) = F_k(\rho^2) = F_j(\rho)$.

To prove that SNS implies ETE, it is enough to note that SNM and SNS are dual to each other while ETE is self-dual. $\square$

Proposition 3 CI implies IRC.

Proof. Let $\rho = (N, E, c) \in \mathcal{C}$ and $\rho^T = (N, E, c^T) \in \mathcal{C}$, where $c_i^T = \min \left\{ c_i, \max_{j \in N} F_j(\rho) \right\}$ for all $i \in N$. Notice that for all $i \in N$, $c_i^T \geq F_i(\rho)$, where the inequality holds since $c_i \geq F_i(\rho)$ and $\max_{j \in N} F_j(\rho) \geq F_i(\rho)$. Therefore, since $F(\rho) \leq c^T \leq c$, by CI, we conclude $F(\rho) = F(\rho^T)$. $\square$

To prove Theorem 2, we additionally need the following Lemma.

Lemma 2. Let F be a rule satisfying NM. Let $(N, E, c) \in \mathcal{C}$, $i \in N$, $k \in \mathbb{N} \setminus N$ and $(N \cup \{k\}, E, c') \in \mathcal{C}$ be such that $c_i = c'_i + c'_k$ and $c_\ell = c'_\ell$ for all $\ell \in N \setminus \{i\}$.

If $F_i(N, E, c) = c_i$ then $F_i(N \cup \{k\}, E, c') = c'_i$ and $F_k(N \cup \{k\}, E, c') = c'_k$.

Proof. Let (N, E, c) and $(N \cup \{k\}, E, c')$ be defined as above. Then,

$$c'_k + c'_i = c_i = F_i(N, E, c) \leq F_k(N \cup \{k\}, E, c') + F_i(N \cup \{k\}, E, c'),$$

where the first inequality is from NM. By claim boundedness, we conclude $F_i(N \cup \{k\}, E, c') = c'_i$ and $F_k(N \cup \{k\}, E, c') = c'_k$. $\square$

Theorem 2 The CEA is the unique rule that satisfies IRC and SNM.

Proof. The CEA satisfies SNM and from Proposition 3, it also satisfies IRC.

To prove uniqueness, let F be a rule satisfying IRC and SNM. By Proposition 2, this rule also satisfies ETE. Let $\rho = (N, E, c) \in \mathcal{C}$ and take $i^* \in N$ such that $F_{i^*}(\rho) = \max_{j \in N} F_j(\rho)$. We prove the following two claims.

Claim 1: for all $j \in N$ such that $c_j \geq F_{i^*}(\rho)$, we have $F_j(\rho) = F_{i^*}(\rho)$.

To prove Claim 1, define a new problem $\rho^T = (N, E, c^T) \in \mathcal{C}$ with $c_\ell^T = \min\{c_\ell, F_{i^*}(\rho)\}$ for all $\ell \in N$. Let $j \in N$ be such that $c_j \geq F_{i^*}(\rho)$, then $c_j^T = F_{i^*}(\rho)$. Moreover, by claim boundedness, $c_{i^*}^T = F_{i^*}(\rho)$. By ETE, since $c_j^T = c_{i^*}^T$,

$$F_{i^*}(\rho^T) = F_j(\rho^T). \quad (11)$$

By IRC,

$$F_{i^*}(\rho^T) = F_{i^*}(\rho) \text{ and } F_j(\rho) = F_j(\rho^T). \quad (12)$$

Thus, from (11) and (12), we get $F_j(\rho) = F_{i^*}(\rho)$, which proves Claim 1.

Claim 2: for all $j \in N$ such that $c_j < F_{i^*}(\rho)$, we have $F_j(\rho) = c_j$.

To prove Claim 2, define $\rho^T = (N, E, c^T) \in \mathcal{C}$ as before. Let $j \in N$ be such that $c_j < F_{i^*}(\rho)$, then $c_j^T = c_j$. Moreover, by IRC,

$$c_{i^*}^T = F_{i^*}(\rho) = F_{i^*}(\rho^T). \quad (13)$$

Now, from ρ^T , split agent i^* into two agents i^* and $k \in N \setminus N$ to get a new problem $\rho' = (N \cup \{k\}, E, c') \in \mathcal{C}$ with $c'_{i^*} = c_j^T$, $c'_k = c_{i^*}^T - c_j^T$ and $c'_\ell = c_\ell^T$ for all $\ell \in N \setminus \{i^*\}$. Observe that $c'_k = c_{i^*}^T - c_j^T = F_{i^*}(\rho) - c_j > 0$. By SNM and from (13), Lemma 2 yields

$$F_{i^*}(\rho') = c'_{i^*}. \quad (14)$$

Moreover, by definition $c'_{i^*} = c_j^T = c'_j$, and thus, by ETE and (14),

$$F_j(\rho') = F_{i^*}(\rho') = c'_{i^*} = c_j^T. \quad (15)$$

Furthermore, by SNM, (15) and $c_j^T = c_j$,

$$F_j(\rho^T) \geq F_j(\rho') = c_j^T = c_j. \quad (16)$$

By IRC, (16) and claim boundedness,

$$F_j(\rho) = F_j(\rho^T) = c_j^T = c_j,$$

which proves Claim 2.

We conclude, from Claim 1 and Claim 2, that $F_j(N, E, c) = \min\{c_j, F_{i^*}(\rho)\}$ for all $j \in N$ and therefore that F is the CEA rule. $\square$

Theorem 1 is a direct result of Theorem 2

Theorem 1 The CEA is the unique rule that satisfies CI and SNM.

Proof. By Theorem 2 and Proposition 3, and as the CEA satisfies CI, we obtain Theorem 1.

In the proof of Theorem 4, we use a result from Ju and Moreno-Ternero (2011) which states that NM and CONS imply ETE.

Theorem 4 On the domain of claims problems,

- (i) the CEA rule is the unique rule that satisfies CI, CONS and NM.
- (ii) the CEA rule is the unique rule that satisfies IRC, CONS and NM.

Proof. Observe that CEA satisfies all the properties. We only prove the second statement, because if the second statement holds, then by Proposition 3, the first statement also holds.

Let F be a rule satisfying IRC, CONS and NM. Then, by Ju and Moreno-Ternero (2011), F also satisfies ETE. We first show that for two-person problems, $F = CEA$. Consider a two-person problem $\rho = (N, E, c) \in \mathcal{C}$ and, w.l.o.g, take $N = \{1, 2\}$ and assume $c_1 \leq c_2$. Consider the following three cases.

Case 1. $F_1(\rho) = F_2(\rho) = \lambda$. Then, $F_1(\rho) = \min\{c_1, \lambda\}$ and $F_2(\rho) = \min\{c_2, \lambda\}$ due to claim boundedness. Therefore, we get that $F(\rho) = CEA(\rho)$.

Case 2. $F_1(\rho) > F_2(\rho)$. Then, truncating all claims by $F_1(\rho)$, we get $\rho^T = (N, E, c^T) \in \mathcal{C}$ where, since $F_1(\rho) \leq c_1 \leq c_2$, $c_1^T = c_2^T = F_1(\rho)$. Hence, by ETE, $F_1(\rho^T) = F_2(\rho^T)$. By IRC, $F_1(\rho) = F_1(\rho^T) = F_2(\rho^T) = F_2(\rho)$, reaching contradiction with the assumption of case 2. Therefore, this case is impossible.

Case 3. $F_1(\rho) < F_2(\rho)$. We distinguish three subcases.

Subcase 3.1. $F_2(\rho) \leq c_1$. Truncating all claims at $F_2(\rho)$, we get $\rho^T = (N, E, c^T) \in \mathcal{C}$ where $c_2^T = \min\{c_2, F_2(\rho)\} = F_2(\rho)$ and $c_1^T = \min\{c_1, F_2(\rho)\} = F_2(\rho)$. By ETE and IRC, $F_1(\rho) = F_2(\rho)$ contradicting the assumption of Case 3. Therefore, this case is impossible.

Subcase 3.2. $F_2(\rho) > c_1$ and $F_1(\rho) = c_1$. In this case, notice that $F_1(\rho) = \min\{c_1, \lambda\}$ and $F_2(\rho) = \min\{c_2, \lambda\}$ where $\lambda = F_2(\rho)$, which means that $F(\rho) = CEA(\rho)$.

Subcase 3.3. $F_2(\rho) > c_1$ and $F_1(\rho) < c_1$. Then, truncating all claims at $F_2(\rho)$, we get $\rho^T = (N, E, c^T) \in \mathcal{C}$ where $c_1^T = \min\{c_1, F_2(\rho)\} = c_1$ and $c_2^T = \min\{c_2, F_2(\rho)\} = F_2(\rho)$. Moreover, under IRC,

$$c_2^T = F_2(\rho) = F_2(\rho^T). \quad (17)$$

Now, from ρ^T , split agent 2 into agent 2 and agent 3 to go to $\rho' = (\{1, 2, 3\}, E, c') \in \mathcal{C}$ with $c'_1 = c_1^T$, $c'_2 = c_2^T$ and $c'_3 = c_2^T - c_1^T$. Observe that by (17), Lemma 2 yields

$$F_2(\rho') = c'_2 \text{ and } F_3(\rho') = c'_3. \quad (18)$$

Moreover,

$$F_1(\rho') = F_2(\rho') = c'_2 = c_1^T = c_1 > F_1(\rho) = F_1(\rho^T). \quad (19)$$

where the first equality is from ETE, the second equality is from (18), the fourth equality is from truncation, the first inequality is from the assumption in Subcase 3.3 and the last equality is from IRC. Hence, $E = F_1(\rho') + F_2(\rho') + F_3(\rho') = F_1(\rho') + c'_2 + c'_3 > F_1(\rho^T) + F_2(\rho^T) = E$,

where the second equality is from (18) and the inequality is from (17) and (19), getting a contradiction.

Combining all cases, we conclude that $F = CEA$ for all two-agent problems. Since F is consistent and the CEA satisfies RM, an application of Proposition 10.1 in Thomson (2019) delivers the CEA for all $(N, E, c) \in \mathcal{C}$. $\square$

Proposition 4 IRC and MCF are dual properties.

Proof. Let F and F^d be two dual rules, F satisfying IRC. We now prove that F^d satisfies MCF. Let $\rho = (N, E, c) \in \mathcal{C}$ and $i \in N$,

$$\begin{aligned} F_i^d(N, E, c) &= c_i - F_i(N, L, c) = c_i - F_i\left(N, L, \left(\min\left\{c_k, \max_{j \in N} F_j(N, L, c)\right\}\right)_{k \in N}\right), \\ &= c_i - \min\left\{c_i, \max_{j \in N} F_j(N, L, c)\right\} \\ &\quad + F_i^d\left(N, \sum_{k \in N} \min\left\{c_k, \max_{j \in N} F_j(N, L, c)\right\} - L, \min\left\{c_k, \max_{j \in N} F_j(N, L, c)\right\}\right)_{k \in N}, \end{aligned} \tag{20}$$

where the first and the last equalities follow from duality and the second one is because F satisfies IRC.

Notice that for all $k \in N$,

$$\begin{aligned} \min\left\{c_k, \max_{j \in N} F_j(N, L, c)\right\} &= \min\left\{0, \max_{j \in N} F_j(N, L, c) - c_k\right\} + c_k \\ &= c_k - \max\left\{0, c_k - \max_{j \in N} F_j(N, L, c)\right\} = c_k - \max\left\{0, c_k - \max_{j \in N} \{c_j - F_j^d(N, E, c)\}\right\}, \end{aligned}$$

where the last equality follows from duality and $E = \sum_{k \in N} c_k - L$. Developing (20), we have

$$\begin{aligned} F_i^d(N, E, c) &= \max\left\{0, c_i - \max_{j \in N} \{c_j - F_j^d(N, E, c)\}\right\} + \\ &F_i^d\left(N, E - \sum_{k \in N} \max\left\{0, c_k - \max_{j \in N} \{c_j - F_j^d(N, E, c)\}\right\}, \left(c_k - \max\left\{0, c_k - \max_{j \in N} \{c_j - F_j^d(N, E, c)\}\right\}\right)_{k \in N}\right) \\ &= mc_i + F_i^d\left(N, E - \sum_{k \in N} mc_k, (c_k - mc_k)_{k \in N}\right). \end{aligned}$$

Therefore, we conclude that F^d satisfies MCF.

Observe that $(N, E - \sum_{k \in N} mc_k, (c_k - mc_k)_{k \in N})$ is well defined. Clearly, as $E \leq \sum_{i \in N} c_i$, then $E - \sum_{k \in N} mc_k \leq \sum_{k \in N} (c_k - mc_k)$. Next, we show that claims are nonnegative. Indeed, for all $k \in N$,

$$\begin{aligned} c_k - \max\left\{0, c_k - \max_{j \in N} \{c_j - F_j(N, E, c)\}\right\} &= \min\left\{c_k, \max_{j \in N} \{c_j - F_j(N, E, c)\}\right\} \\ &\geq \min\{c_k, 0\} = 0. \end{aligned}$$

where the inequality follows from claim boundedness. Finally,

$$\begin{aligned}
E - \sum_{k \in N} mc_k &= E - \sum_{k \in N} \max \left\{ 0, c_k - \max_{j \in N} \{c_j - F_j(N, E, c)\} \right\} \\
&\geq E - \sum_{k \in N} \max \{0, c_k - c_k + F_k(N, E, c)\} \\
&\geq E - \sum_{k \in N} \max \{0, F_k(N, E, c)\} = 0,
\end{aligned}$$

where the last inequality follows from non-negativity and efficiency. $\square$

To finish, we prove Theorem 3 and Theorem 5 by duality.

Theorem 3 On the domain of claims problems,

- (i) the CEL rule is the unique rule satisfying PRF and SNS.
- (ii) the CEL rule is the unique rule satisfying MCF and SNS.

Proof. Notice that CI is dual to PRF (Proposition 1), IRC is dual to MCF (Proposition 4) and, SNM is dual to SNS. By duality, Theorem 3 is a direct consequence of Theorem 1 and Theorem 2. $\square$

Theorem 5 On the domain of claims problems,

- (i) the CEL rule is the unique rule that satisfies PRF, CONS and NS.
- (ii) the CEL rule is the unique rule that satisfies MCF, CONS and NS.

Proof. Observe additionally that CONS is self-dual and NM is dual to NS. By duality, Theorem 5 is a direct consequence of Theorem 4. $\square$

Appendix B. Logical independence

In this section, we analyze the logical independence of the axioms we employed in our results. To this aim, we first introduce a particular weighted constrained equal awards rule, CEA^w , which is defined by setting $CEA^w(N, E, c) = \min \{c_i, \lambda w_i\}$ where $w_i = i$ for all $i \in N$ and λ is such that $\sum_{i \in N} \min \{c_i, \lambda w_i\} = E$, for all $(N, E, c) \in \mathcal{C}$.

To show the logical independence of the properties in Theorems 1, 2 and 3, observe that P satisfies SNM and SNS but not CI, nor IRC, PRF, or MCF. On the other hand, it is not difficult to check that CEA^w satisfies CI and IRC but not SNM and its dual satisfies PRF and MCF but not SNS.

Regarding the logical independence of the properties in Theorems 4 and 5, observe that P satisfies CONS, in addition to the aforementioned properties, as well as NM and NS. Moreover, it is not difficult to check that CEA^w also satisfies CONS but not NM, while its dual satisfies PRF and MCF, but not NS. However, it remains an open question if CONS is consequence of NM (NS) and CI or IRC (PRF or MCF).

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