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Spatial Competition Models

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Abstract

Hotelling's spatial competition model defines a two-stage non-cooperative game in a duopoly. First, each seller simultaneously chooses a location where to operate on a segment. Then, each firm simultaneously selects the price it wants to charge for its products. In the first part of this study, we present in detail this model — announced in 1929 by Harold Hotelling in his seminal paper *Stability of Competition* — , we identify its limitations and we introduce a variation using a different transportation cost function proposed by d'Aspremont et al. in 1979. In addition, we analyse an extension of the model to a circular market announced in 1979 and known as Salop's circle model. The second part of this project is devoted to study the existence of Nash equilibria in games in which firms do not compete on price but only on location. Particularly, we follow Fournier and Scarsini (2019). Finally, we briefly touch upon the problem of inefficiency of Nash equilibria.

Resum

El model de competència espacial de Hotelling defineix un joc no cooperatiu de dues etapes en un duopoli. En la primera etapa, cada empresa tria simultàniament una ubicació on localitzar-se en un segment. En la segona, cada venedor selecciona simultàniament el preu que vol cobrar pels seus productes. A la primera part d'aquest treball, presentem detalladament aquest model — anunciat el 1929 per Harold Hotelling en el seu article *Stability of Competition* — , identifiquem les seves limitacions i introduïm una versió modificada, utilitzant una funció de cost de transport diferent proposada per d'Aspremont et al. el 1979. A més, també analitzem una extensió del model a un mercat circular anunciada el 1979 i coneguda com a model circular de Salop. La segona part d'aquest projecte està dedicada a estudiar l'existència d'equilibris Nash en jocs en què les empreses no competeixen en preus, sinó que només ho fan en localització. En particular, seguim els arguments de Fournier and Scarsini (2019). Finalment, tractem breument el problema de la ineficiència dels equilibris de Nash.

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Introduction

Oligopolies are a type of market structure that has been analysed by leading economic thinkers throughout history. They are characterized by a few large companies, each competing for a higher market share. Because the number of competitors is low, each firm cannot afford to ignore the rivals' likely actions when deciding its own strategy. This is the reason why Game Theory — the study of strategic interaction among decision-makers — is a good tool to better understand the players' behaviour in this market structure.

Throughout history, the problem of competition amongst two or more firms has been described as a mathematical model in order to formalize it. Generally, when referring to oligopoly competition, it is assumed that there is a fixed number of firms producing an undifferentiated product with constant marginal costs. Firms, which do not cooperate between each other, compete in order to increase their market share. Since the product is homogeneous, consumers want to buy the good from the company that offers it at a lower price. Under these assumptions, we provide below a brief overview of the most relevant oligopolistic competition models.

Note that the earliest models were introduced before 1944 — when the modern language of Game Theory was established by von Neumann and Morgenstern [19] —, and therefore, many of the notions we know today were not defined yet. However, due to their relevance, they have been strengthened over the years. This is why, nowadays, we can explain them using current terminology.

Cournot (1838) [3], describes an economic model in which companies offering an identical product simultaneously and independently choose a quantity to produce. That is, quantity supplied is the strategic variable for each firm. Then, the market determines the price at which the product is sold. According to Cournot, the unique equilibrium outcome of the market is when all suppliers have the same marginal costs of production, and the inverse demand function is linear. In addition, he argues that market price is above the marginal cost.

Bertrand (1883) [1], criticizes Cournot's model and suggests considering price, rather than quantity, as the strategy variable. In this case, each firm independently and simultaneously decides at which price charges a homogeneous good. Note that the company that chooses the lowest price serves the entire market. Therefore, firms will decrease prices to the point they start losing. So, assuming all marginal costs are the same for all firms in the market, the equilibrium outcome occurs when price equals the marginal costs.

Note that, in both models, only identical and standardized products are considered. However, in real life, it is challenging to find truly homogeneous products. In fact, even when goods are physically equal, there are subjective reasons that make products different from a consumer's point of view. Within this context, Harold Hotelling, in his seminal paper *Stability of Competition* from 1929 [12], introduces duopoly models with differentiated products. That is, he considers some characteristics that make consumers to buy certain products even when they are more expensive than other similar options — such as the store's proximity, the customer service or the relationship with the seller, among others.

The most popular model described by Hotelling is the one in which two firms which sell a homogeneous product compete on location and price in a market represented by a line segment. Consumers make their buying decision based on transportation costs and product prices. Hotelling states that, when consumers are uniformly distributed along the segment, if the cost of transportation is a linear function of the distance travelled, firms maximize profits when both players locate their stores in the middle of the segment. In fact, he affirms that a principle of minimum differentiation holds. Fifty years later, in 1979, d'Aspremont, Gabszewicz and Thisse [4] analyse Hotelling's model and show that the principle of minimum differentiation is not generally true since there is no price equilibrium solution when the two stores are located too close to each other. In addition, they present a modified version of Hotelling's model using quadratic transport costs. In this case, there is always a price equilibrium solution, but firms maximize profits when they are as far as possible from each other.

Over time, many different variations of Hotelling's model have been constructed. The most remarkable is the one introduced by Salop in 1979 [17], in which consumers are located along a circle instead of a line. This overcomes the difficulties of the end points in the original model.

Several variations of the model assume that firms do not compete on price but only on location. In this case, price is assumed to be the same for all sellers. In real life, these models apply to shops that sell products the price of which is exogenously determined, such as pharmacies or newsstands. One of the most renowned papers that follows this approach is that by Eaton and Lipsey, from 1975 [7]. They adapt Hotelling's model to a game with an arbitrary number of players located across different spatial configurations. In particular, they prove that location games on a circle always admit Nash equilibria, and the same applies to location games on a segment as long as the number of players is not 3.

Continuing with the idea of competition based only on sellers' location, several researches consider that the firms are located on a network. This is the case of Pálvölgyi [16] and Fournier and Scarsini [10]. Particularly, they study the existence of equilibria in location games on a graph, given an arbitrary number of players. Specifically, they show that if the number of players is big enough, there are always Nash equilibria. Furthermore, Fournier and Scarsini add an exhaustive analysis of how efficient these equilibria are in terms of the Price of Anarchy and the Price of Stability.

Most of the literature on spatial competition assumes a uniform distribution of consumers. However, in real life, the concentration of households and population are not equally distributed — density is higher in urban areas, and equally varies from district to district. In that direction, some investigators analyse games in which the distribution of consumers over the market is not uniform or symmetric. For example, in 1995, Tabuchi and Thisse [18], study the equilibrium locations when consumers are concentrated around a market center.

Hotelling's model not only is applied to explain competition on sellers' location, but it can be extrapolated to explain competition on any other characteristic that distinguishes a product — the distance between firms represents the difference between product features.

Note that, in real life, there tends to be multiple features that make products different. This is the reason why some researchers have extended Hotelling's original model to multidimensional spaces. In 1986, Economides [8] studies the analogue of Hotelling's model when products are differentiated by two characteristics. In 1998, Irmen and Thisse [13], propose an n-dimensional differentiation model to factor in all characteristics that may make products different.

Hotelling's framework has been used extensively to understand product differentiation and pricing strategies in competitive settings. However, it has applications in many different fields, such as political science.

In particular, political competition can be explained by the variations of the Hotelling's model in which firms compete only on location. The segment in Hotelling's original model represents the political spectrum. Therefore, candidates play the role of sellers and the voters the role of the consumers. In this case, each voter supports the party that is closest to his or her political ideology.

In this area, one of the most well-known studies is the thesis of Anthony Downs from 1957 — *An Economic Theory of Democracy* [6] — that uses Hotelling's model to explain some aspects of a two-party system. He supposes that all voters of a country are ordered on a segment according to their ideology — where the farthest left voter is located on the left-most extreme and the farthest right voter on the right-most one. Despite the fact that in this case it is not possible to assume that voters are uniformly distributed, the analogous result of Hotelling's model applies: the only real possibility that parties have to govern is applying centralizing politics, and therefore, using similar electoral programs. However, when the political system is formed of more than just two parties, they move to more radical positions.

Note that the above can be used not only to explain party's ideology, but it can also be applied to other aspects in politics — such as parties' thoughts on religion or independence or abortion rights. So, although the model works well to explain individual political concepts, it would be a fairer reflection of reality if it was extended to more dimensions.

Finally, it is important to remark that, when modelling decision-making situations, for the sake of mathematical simplicity, sometimes several assumptions that move away from reality are needed — as it happens in the studies explained above. Despite this, the results of these models help individuals to better understand the problem being addressed — in this case, the competitive scenarios.

About This Work

The main part of this project is devoted to study the existence of Nash equilibria in different games in which geographic location is a strategic variable. In Chapter 1, we outline some basic concepts of Game Theory focused on non-cooperative games that we use in the following chapters. In Chapter 2 we present Hotelling's model — of key importance in our study. We identify its limitations and we introduce the version of Hotelling's original model with quadratic transport costs proposed by d'Aspremont, Gabszewicz and Thisse. In Chapter 3 we review an extension of Hotelling's model to a circular market, which is known as Salop's circle model. Chapter 4 focuses on games in which players compete on location. Particularly, we follow Fournier and Scarsini's arguments to show the existence of equilibria in these location games where the market is represented as a segment, a star network or a circle. Finally, in Chapter 5, we provide a brief explanation of the efficiency of Nash equilibria.

Chapter 1

Preliminaries

Game Theory is the mathematical study that analyses optimal decision making in the context of strategic interaction between agents. The ideas behind Game Theory have appeared multiple times throughout history. However, its modern analysis began with John von Neumann and Oskar Morgenstern in 1944, when they published their book *Theory of Games and Economic Behavior* [19] — which defines the majority of terminology that is still in use today. Since then, the framework has been continuously strengthened with the contributions of many developers, most notably that of John Forbes Nash. Nowadays, Game Theory is utilised across several sciences — like Economics and, in particular, Microeconomics — to study decision making processes that involve various entities whose decisions are influenced by the decisions they expect from others.

In this chapter, the reader will find some basic concepts of Game Theory. We will follow Gibbons (1992) [11] and Jehle and Reny (2011) [14]. These notions are used in the next chapters, since they rely in the analysis of behavior of different agents, buyers and sellers. We hope these notions would help to better understand our study.

1.1 Non-cooperative Games

A *non-cooperative game*, in terms of Game Theory, is defined to be any situation in which there is a group of individuals — called *players* — that individually have to make a decision from a set of options — known as *strategies*. By individually, we mean that no agreement is established between players — that is, each agent acts in his or her self-interest. The strategies chosen by each player determine the outcome of the game. Associated to each possible outcome there is a collection of numerical payoffs, one for each player. These *payoffs* represent the utility that the outcome or result gives to each agent. We assume these utilities are personal and generally speaking non-comparable across agents. There are no binding agreements between agents, each one looks after his or her utility.

We assume that there is a finite number of players n, with $n \geq 2$. Let $N = \{1, \ldots, n\}$ be the set of all players and let $i \in N$ denote an arbitrary player. Let S_i be the set of strategies available to player i, which is called strategy space. Let $s = (s_1, \ldots, s_n) \in S_1 \times \cdots \times S_n = S$ denote a strategy profile, where s_i is the strategy chosen by player i. Let $u_i \colon S \to \mathbb{R}$ denote player i's payoff based on the strategies chosen by all players — that is, $u_i = u_i(s_1, \ldots, s_n)$.

There are two ways in which games are represented: the *normal form* and the *extensive* *form.* Note that any game can be represented in either normal or extensive form, although for some games one of the two forms is more convenient to analyze.

In this project, we will consider *complete and imperfect information games.* By imperfect, we mean that players are unaware of the actions chosen by other agents. However, players know who the other players are, their possible strategies and their payoffs. Therefore, information about other players is complete.

Simultaneous-move games are usually represented by the normal form, which is formally defined as follows.

Definition 1.1. The *normal form representation* of a game G is described by a triple $G = (N, S, u)$, where N is the players' set, $S = S_1 \times \cdots \times S_n$ is the set of strategy combinations and $u = (u_i)_{i\in N}$ is the payoff function.

Whenever the strategy set for any player is finite, a natural way to represent games is via a table, especially for two players. The cells of the table are the payoff for all players in any combination of strategies. Let us clarify it through a classic example.

Example 1.1. *The Prisoners' Dilemma.* Two individuals are arrested and charged with a crime. The police have enough evidence to convict them of that crime, so that each would spend a year in jail given existing sentencing rules. However, the police also suspect that the two individuals have committed another crime, but they lack hard evidence to convict them, unless at least one confesses. To force them to speak, the suspects are separated and advised of their options. If both claim they are innocent of the second crime, each of them gets only one year of prison. If both of them confess, they both go to prison for 6 years. But if one of them accuses the other, whilst the other claims he or she is innocent, the first one gets immunity while the second is sentenced 9 years in prison.

In the situation above, there are two players (the two prisoners) and each one has two strategies available: to confess or to not confess. The payoffs for the two players when a particular pair of strategies is chosen are given in the appropriate cell of the following table:

1/2	Confess	Not Confess
Confess	$(-6, -6)$	$(0, -9)$
Not Confess	$(-9, 0)$	$(-1, -1)$

Each row corresponds to a possible action for prisoner 1, each column corresponds to a possible action for prisoner 2, and each cell corresponds to one of the possible payoffs (within each payoff pair, the first one corresponds to prisoner 1 and the second to prisoner 2).

Observe that if each prisoner thinks about his or her own benefit, the best payoff is when he or she confesses and the other prisoner remains silent. On the other hand, if prisoners seek the best outcome as a group, the best they can do is to not confess.

1.2 Nash Equilibrium

The concept of *Nash equilibrium* was introduced in 1951 by John F. Nash (see [15]) as a generalized solution to $n-$players game.

Informally, a strategy profile $s^* = (s_1^*, \ldots, s_n^*)$ and its corresponding payoffs represent a Nash equilibrium if no player can increase his or her payoff by changing his or her strategy, as long as the other players keep their strategies unchanged. That is, if no player has the incentive to deviate from their chosen strategy. Formally, the Nash equilibrium is defined as follows.

Definition 1.2. Given a normal form game $G = (N, S, u)$, a strategy profile $s^* = (s_1^*, \ldots, s_n^*) \in S_1 \times \cdots \times S_n$ is a *Nash equilibrium* of G if for each player $i \in N$,

$$u_i(s_1^*, \ldots, s_{i-1}^*, s_i^*, s_{i+1}^*, \ldots, s_n^*) \geq u_i(s_1^*, \ldots, s_{i-1}^*, s_i, s_{i+1}^*, \ldots, s_n^*),$$

for all $s_i \in S_i$.

Notice that the number of Nash equilibria in a game depends on the characteristics of each game. In fact, there are games without any Nash equilibrium and others that have an infinite amount.

Strictly speaking, the above concept is called pure Nash equilibrium because each agent plays one of the available strategies from his or her strategy space — this concept is known as pure strategy. That is, a player i chooses a strategy defined on S_i. It is worth mentioning that playing pure strategies is not the only option — instead, an agent can play a mixed strategy, which is a probability distribution over all possible pure strategies. We refer to this as mixed extension of the game.

Example 1.2. Let us find the Nash equilibrium of the Prisoners' Dilemma game explained in Example 1.1. Observe that prisoner 1 spends less time in jail if he or she confesses, regardless of whether prisoner 2 confesses or remains silent. The analogous situation applies to prisoner 2. Therefore, if both prisoners confess, there is no incentive for players to change their strategy. Thus, (Confess, Confess) is the unique Nash equilibrium of the game.

It is interesting to remark that the Nash equilibrium of the Prisoners' Dilemma is not the best outcome prisoners can get as a group, since if neither of them confesses they get a better payoff as a pair. However, when both prisoners look after his or her own interest and confess, they get a worse payoff as a group.

1.3 Subgame Perfect Equilibrium

In this section, we consider games in which players make choices in sequence — we assume that the moves in all previous stages are known before the next stage begins, and we allow simultaneous moves within each stage. We should introduce the concept of subgame — a stage within a game which begins at any point where a player has to make a decision.

These games are better represented by the extensive form. Let us give an informal description of this term.

The extensive form representation of a game specifies: (1) the players in the game; (2) what moves are possible for each player, the order in which they have to play and the information available to them from others' previous moves; and (3) the payoffs received by each player for each possible combination of moves.

Frequently, extensive form games are represented graphically by a tree diagram, in which each node represents the decision of one of the players, each edge represents a possible action, and the leaves represent final outcomes over which player has a utility function. To represent the knowledge available at each stage we will use the concept of *information set*. We will indicate that a collection of decision nodes constitutes an information set by connecting the nodes by a dotted line. The nodes in an information set are indistinguishable to the agent, so all have the same set of actions.

Let us explain how to get the normal form representation from extensive form games. Given a set of players N, a player's strategy can be defined as the chosen action on each decision node, or information set, regardless if he or she plays at that node. The total number of strategies available to a player can be calculated by multiplying the number of options to choose from on each node. The payoff for each player is shown on the final nodes. Knowing how to go from extensive to normal form is very useful to analyze Nash equilibria.

Example 1.3. Consider again the Prisoners' Dilemma game explained in Example 1.1. It can be represented as in the following tree diagram.

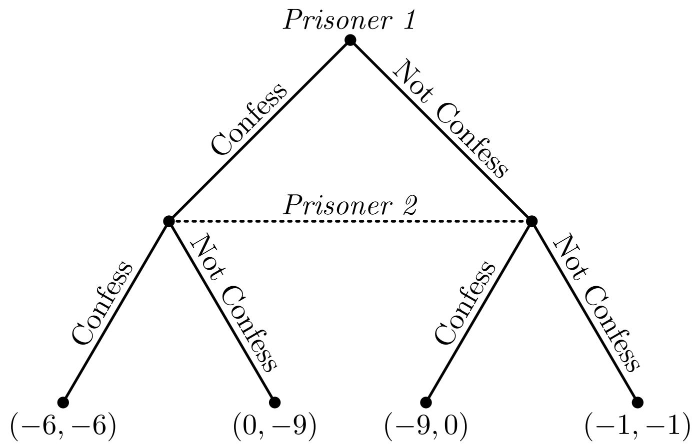

Figure 1.1: Prisoners' Dilemma in extensive form.

Note that we may switch the roles of Prisoner 1 and Prisoner 2, since when they make a decision they do not know what the other player's decision is.

Let us formalize definitions of some of the concepts introduced above.

Definition 1.3. The *extensive form representation* of a game is described by a tuple $\Gamma = (N, A, X, E, \pi, \alpha, \mathcal{I}, (u_i)_{i \in N})$, where:

1. $N = \{1, \dots, n\}$ is a finite set of players.
2. X is a set of non end nodes, where there is just one initial node, x_0.
3. E is a set of end nodes.
4. A is a set of actions, which includes all possible actions that might potentially be taken at some point in the game. Let $A(x) = \{a \in A \,|\, (x, a) \in X\}$ denote the set of actions available to the player whose turn it is to move after the node $x \in X \backslash \{x_0\}$.
5. π is a probability distribution on $A(x_0) \subseteq A$ to describe the role of chance in the game.

6. $\alpha\colon X\backslash(E \cup \{x_0\}) \to N$ is a function that indicates whose turn it is at each decision node in X.

7. $\mathcal{I}$ is a partition which divides the set of decision nodes. $X\backslash(E \cup \{x_0\})$, into information sets. That is, if x and x' are in the same element of the partition, then $A(x) = A(x')$ and $\alpha(x) = \alpha(x')$.

8. $u = (u_1, \ldots, u_n)$, where $u_i\colon E \to \mathbb{R}$ is the payoff to player i.

Definition 1.4. Given an extensive form game Γ, the *subgame* of Γ rooted at node x, Γ_x, is the restriction of Γ to the descendants of x. Whenever y is a decision node following x, and z is in the information set containing y, then z also follows x.

Applying the concept of Nash equilibrium on each subgame of an extensive form game, we get the notion of a *subgame perfect equilibrium.*

Definition 1.5. A strategy profile s is a *pure subgame perfect equilibrium* of the extensive form game Γ if s induces a Nash equilibrium in every subgame of Γ.

Since Γ is in particular its own subgame, every subgame perfect equilibrium is also a Nash equilibrium. That is, subgame perfect Nash equilibrium is a refinement of Nash equilibrium. They use that a rational player, confronted to any stage of the game, will select only a Nash equilibrium. Therefore, any equilibrium which involves unbelievable threats should be discarded.

Let us illustrate all the information above with an example.

Example 1.4. There are two firms competing in a single industry. One is currently producing (the incumbent), and the other is not (the entrant). In a first stage, the entrant must decide whether to enter the industry or to stay out. The best outcome for the incumbent is that the entrant stays out of the market. If it happens, the status quo prevails and the game ends. If the entrant enters, then there is a second stage, where the two firms simultaneously choose between two competition behaviours: fight or accommodate.

The best outcome for both firms is when they decide to accommodate each other, since in this case firms are willing to share the market, causing no change in the market price. Instead, if both firms decide to fight against the other, a price war arises, and as a consequence, the market price deceases and both firms lose profits. Finally, if one firm decides to accommodate its competitor, while the other chooses to fight, the one that cooperates loses all customers, and so its profits will be null. The firm that chooses to compete aggressively — even though it gains customers — also will see its profits reduced because of the costs of the fight.

The extensive form representation of this game (using arbitrary numbers) is depicted in Figure 1.2.

Strategies for Entrant are given by {(Entry, Accommodate), (Entry, Fight), (Stay Out, Accommodate), (Stay Out, Fight)}, whereas strategies for Incumbent are {Accommodate, Entry}.

The part of the game that starts at the node where Incumbent is defines a subgame. Consider now the subgame as a game in its own right. Observe that it has the same structure as the Prisoners' Dilemma but with different payoffs. So, if we apply the Nash

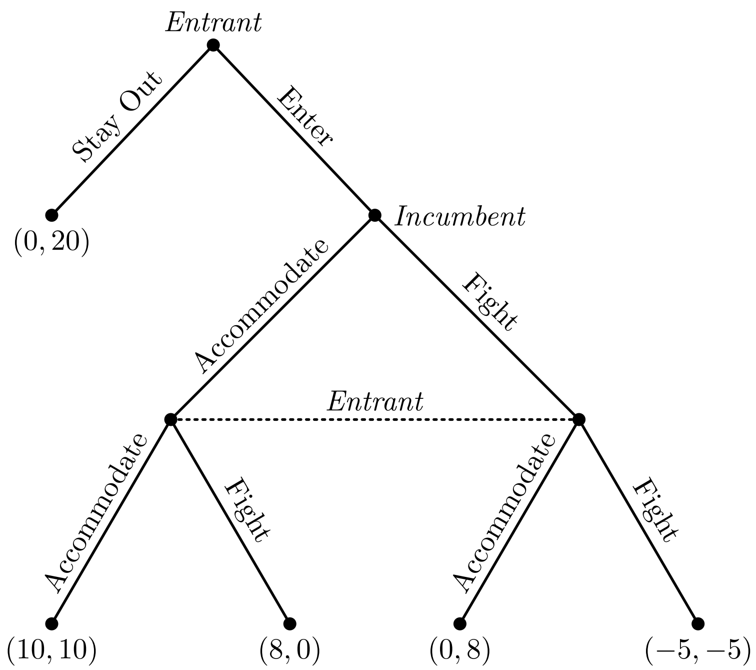

Figure 1.2: Entry game in extensive form.

equilibrium concept to this subgame as we did before, we get that Accommodate is the best option for each of the two firms, regardless of what the other player chooses. That is, (Accommodate, Accommodate) is the Nash equilibrium in the simultaneous move game that follows entry.

Then, assuming that equilibrium in the subgame is (Accommodate, Accommodate), Entrant prefers enter the industry, since it gets 10 rather than 0. Therefore, ((Entry, Accommodate), Accommodate) is the subgame perfect equilibrium, which is unique in this case.

Finally, we represent this example using the normal form as follows:

Entrant/Incumbent	Accommodate	Fight
Enter, Accommodate	$\underline{(10, 10)}$	$(0, 8)$
Enter, Fight	$(8, 0)$	$(-5, -5)$
Stay Out, Accommodate	$(0, 20)$	$\underline{(0, 20)}$
Stay Out, Fight	$(0, 20)$	$\underline{(0, 20)}$

In order to determine the Nash equilibria of the representation above, we highlight in blue the response which leads to the best outcome for Entrant, taking Incumbent's strategies as given. The best actions for Incumbent, given what Entrant is doing, are in green. Therefore, there are 3 Nash equilibria of the game (the payoffs underlined above). However, ((Stay Out, Accommodate), Fight) and ((Stay Out, Fight), Fight) are not subgame perfect equilibria because Entrant has no interest in playing Stay Out on the first node, and so they are not Nash equilibria of the whole game. That is, Stay Out cannot be considered as a credible threat.

Chapter 2

Hotelling's Model

Harold Hotelling, in 1929, in his article *Stability of Competition* (see [12]), argues that if a seller rises substantially the price of a product, its customers will buy the same product from competitors. However, if the increase in price is insignificant, many customers will still prefer to buy from that seller because there are other aspects — such as a store's proximity, customer service or product quality, among others — that influence buying decisions.

In particular, he studies how the location of a store affects the demand for a product. For that purpose, he formulates a model to help each of the two companies under a duopoly choose the best location and product prices.

Notice that the original model of Hotelling was developed many years before Game Theory was established by Von Neumann and Morgenstern, in 1944 [19]. But if we use Game Theory language, the model can be defined as a two-stage game — first, each seller simultaneously selects a location where to operate. And then, based on the location of both stores, each seller simultaneously decides the price to charge. The goal is to find the location of one store relative to the other which maximizes profits for both brands.

Hotelling uses the backward induction process to solve the model and obtain the subgame-perfect Nash equilibrium — first, the equilibrium of the price competition is determined, and then this is used to calculate the profit maximizing location.

In 1979, d'Aspremont, Gabszewicz and Thisse, in their article *On Hotelling's "Stability in competition"* (see [4]), show that Hotelling's argument is not correct and they present a slightly modified version of Hotelling's model for which a different equilibrium exists.

2.1 Original Model

In this section, Hotelling's original model from 1929 [12] is described. The notation used for the model will be different from the original one. Nevertheless we state the same results given in Hotelling's model.

Let A and B be the two stores. Suppose that they sell a homogeneous good produced at zero cost. The market is represented by a line segment of length l which, without loss of generality, can be simplified as $l = 1$. Then this segment is given by $[0, 1]$. Let a and b[1] be the length of the segments $[0, A]$ and $[0, B]$, respectively, where A and B are the

[1] Note that in Hotelling's original model [12], b is the distance of the segment [B,1].

location of stores A and B. We suppose, without loss of generality, that seller A is situated to the left of seller B, so that $0 \leq a \leq b \leq 1$. Figure 2.1 represents this situation:

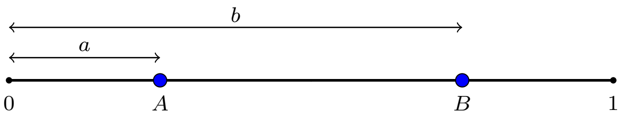

Figure 2.1: Hotelling's model.

Consumers are uniformly distributed along the line segment. They are represented by a mass of 1. Each customer buys a single unit of this good per unit of time, irrespective of its price. In addition, each consumer pays travel cost which is proportional to distance. Let $c(z) = cz \geq 0$ be the cost function, where z is the distance from the consumer to the seller[2], and $c > 0$ a multiplicative constant. Since the product is homogeneous, each consumer will buy from the store where he or she can get the good at the lowest cost — considering both mill price and travel costs.

Let p_A and p_B denote, respectively, the price of the product of sellers A and B. On the other hand, q_A and q_B represent the piece of the market buying the good from seller A and B, respectively. In other words, the demand of the firms.

We shall recall that there is no production cost. Hence, the profit function of each seller will be $\pi_A = p_A q_A$ and $\pi_B = p_B q_B$, respectively. So, let's take a look at the quantity demanded for each seller — the quantity demanded for seller A is given by consumers for whom buying from company A is more worthwhile. And the same applies to seller B. Thus, it is important to know the location of those consumers for whom it is indifferent to buy from seller A or B. Let x be the location of the indifferent consumer. Obviously, once the locations of the stores A and B are fixed, the location of the indifferent consumer depends on the price of each product. In fact, $x \in [0, 1]$ will be indifferent if

$$p_A + c|x - a| = p_B + c|x - b|.$$

Depending on the value of p_A and p_B, three different cases must be studied to determine the possible locations of x with respect to the location of the stores.

(i) If $0 \leq x \leq a$, then $|x - a| = a - x$ and $|x - b| = b - x$. Thus,

$$p_A + c(a - x) = p_B + c(b - x) \Rightarrow p_A - p_B = c(b - a).$$

When the equality above is reached by some p_A and p_B, all consumers are indifferent, so the demand of the market segment $[0, a]$ will be shared equally among sellers A and B. On the other hand, if the equality is not satisfied, there are no indifferent customers between $[0, a]$.

(ii) If $a < x < b$, then $|x-a| = x-a$ and $|x-b| = b-x$. Thus, $p_A+c(x-a) = p_B+c(b-x)$ and consequently

$$x = \frac{p_B - p_A}{2c} + \frac{a + b}{2}. \tag{2.1}$$

And $x \in (a, b)$ as long as

$$a < x < b \Leftrightarrow a < \frac{p_B - p_A}{2c} + \frac{a + b}{2} < b \Leftrightarrow 2ac < p_B - p_A + c(a + b) < 2bc$$
$$\Leftrightarrow c(a - b) < p_B - p_A < c(b - a) \Leftrightarrow |p_B - p_A| < c(b - a).$$

[2] The distance in the line is given by the absolute value of the difference of coordinates.

So, if some p_A and p_B satisfy the inequality above, x separates the segment of length 1 in two. Those consumers located to the left of x will buy from seller A, while those located to the right of x will buy from B. That is, seller A will serve the market segment $[0, x]$, while seller B will serve the market segment $[x, 1]$.

(iii) If $b \leq x \leq 1$ then $|x - a| = x - a$ and $|x - b| = x - b$. Thus,

$$p_A + c(x - a) = p_B + c(x - b) \Rightarrow p_B - p_A = c(b - a).$$

Similarly to case (i), when the equality above is satisfied by some p_A and p_B, all the consumers are indifferent, so demand of the market segment $[b, 1]$ will be divided equally among sellers A and B. Contrarily, there are no indifferent customers between $[b, 1]$.

With the information above, it is possible to know the share of the market for each seller, that is, the quantity demanded.

• Firstly, if $|p_A - p_B| < c(b - a)$, recall case (ii), then demand for firm A is the market segment $[0, x]$ and demand for firm B is $[x, 1]$, with x given by (2.1). That is,

$$q_A = d(0, a) + d(a, x) = a + |x - a| = x$$
$$q_B = d(x, b) + d(b, 1) = |x - b| + |b - 1| = 1 - x,$$

where x is given by (2.1).

• On the other hand, if $|p_A - p_B| = c(b - a)$, two cases must be studied:

(a) If $p_A > p_B$, then $p_A - p_B = c(b - a)$, and as it is explained in case (i), the demand of the market segment $[0, a]$ will be shared equally between sellers A and B, and the market segment $[a, 1]$ will be served by seller B. That is,

$$q_A = \frac{a}{2} \quad \text{and} \quad q_B = 1 - \frac{a}{2}.$$

(b) If $p_A < p_B$, then $p_B - p_A = c(b - a)$, and as it is explained in case (iii), the demand of the market segment $[b, 1]$ will be shared equally between sellers A and B and the market segment $[0, b]$ will be served by seller A. That is,

$$q_A = \frac{1 + b}{2} \quad \text{and} \quad q_B = \frac{1 - b}{2}.$$

• Finally, if $|p_A - p_B| > c(b - a)$, another two cases must be considered:

(a) If $p_A > p_B$, then $p_A > p_B + c(b - a)$. So, customers on the market segment $[a, b]$ will buy from seller B since p_B plus the cost of going from firm A to firm B is lower than p_A. Let's suppose now that customers are located on the market segment $[0, a]$. In particular, the customer located on 0 is the one farthest from both stores. Despite this,

$$p_B + cd(0, b) \leq p_B + cd(0, a) + cd(a, b) = p_B + cd(0, a) + c(b - a) < p_A + cd(0, a),$$

and so, customers between $[0, a]$ will buy from seller B too. In a similar way, since

$$p_B + cd(b, 1) \leq p_B + cd(b, a) + cd(a, 1) = p_B + c(b - a) + cd(a, 1) < p_A + cd(a, 1),$$

customers between $[b, 1]$ will also buy from seller B. That is,

$$q_A = 0 \quad \text{and} \quad q_B = 1.$$

(b) If $p_A < p_B$, then $p_B > p_A + c(b-a)$ and, using a similar argument to that of the previous case,

$$q_A = 1 \quad \text{and} \quad q_B = 0.$$

With the demand of both firms defined, it is possible to define the profit functions as follows:

$$\pi_A(p_A, p_B) = p_A q_A = \begin{cases} p_A & \text{if } \; p_A < p_B - c(b-a), \\ \dfrac{1}{2}\, p_A(1+b) & \text{if } \; p_A = p_B - c(b-a), \\ \dfrac{1}{2c}\left(p_A p_B - p_A^2 + c p_A(a+b)\right) & \text{if } \; |p_A - p_B| < c(b-a), \\ \dfrac{1}{2}\, a p_A & \text{if } \; p_A = p_B + c(b-a), \\ 0 & \text{if } \; p_A > p_B + c(b-a), \end{cases}$$

and

$$\pi_B(p_A, p_B) = p_B q_B = \begin{cases} p_B & \text{if } \; p_B < p_A - c(b-a), \\ \dfrac{1}{2}\, p_B(2-a) & \text{if } \; p_B = p_A - c(b-a), \\ p_B - \dfrac{1}{2c}\left(p_A p_B - p_B^2 - c p_B(a+b)\right) & \text{if } \; |p_A - p_B| < c(b-a), \\ \dfrac{1}{2}\, p_B(1-b) & \text{if } \; p_B = p_A + c(b-a), \\ 0 & \text{if } \; p_B > p_A + c(b-a). \end{cases}$$

A particular feature of π_A and π_B is that they have two discontinuities at the price where a whole group of buyers is indifferent between the two sellers.

All the information above describes a two-person game with players firm A and firm B, strategies $p_A \in [0, \infty)$, and $p_B \in [0, \infty)$, and payoff functions given by the profit functions.

Strategy p_A of seller A is the best reply against a strategy of seller B when it maximizes $\pi_A(\cdot\,, p_B)$ on $[0, \infty)$ for a given p_B. And the same applies to player B. So, the Nash equilibrium point is a pair (p_A^*, p_B^*) such that p_A^* is the best reply against p_B^* and vice versa. Once the price equilibrium is reached, differentiating respect to variables a and b, the location maximizing profit will be determined.

Using the backward induction previously explained, Hotelling claimed that the most profitable location is next to a competitor in the middle of a geographic line. This tendency of business to cluster is known as the *Principle of Minimum Differentiation.*

2.2 Against Hotelling's Conclusion

The paper *On Hotelling's "Stability in competition"* [4] aims to prove that the *Principle of Minimum Differentiation* is not correct. That is, to show that there is no tendency for sellers to locate towards the center. Actually, d'Aspremont, Gabszewicz and Thisse stated that there is no equilibrium price solution when both sellers are not far enough from each other, so nothing can be rigorously claimed about the structure of the market.

Using the same notation as in the previous section, necessary and sufficient conditions on a and b for such an equilibrium to exist will be fixed. In addition, the equilibrium points for those locations for which there is price equilibrium will be computed.

Proposition 2.1. *For $a = b$, the unique equilibrium point is given by $p_A^* = p_B^* = 0$. For $a < b$, there is an equilibrium point if and only if,*

$$(2 + a + b)^2 \geq 12(2 + a - 2b), \tag{2.2}$$

$$(4 - a - b)^2 \geq 12(1 + 2a - b), \tag{2.3}$$

and, whenever it exists, an equilibrium point is uniquely determined by

$$p_A^* = \frac{c}{3}(2 + a + b), \tag{2.4}$$

$$p_B^* = \frac{c}{3}(4 - a - b). \tag{2.5}$$

Proof. The case $a = b$ is immediate. When $a = b$, then both sellers are located at the same place, so customers will buy the product from the store that offers it at lowest price. As a consequence, a price war arises due to the tendency to reduce the price. So, the unique equilibrium point is determined by $p_A^* = p_B^* = 0$.

Let's suppose now that $a < b$. Firstly, it needs to be shown that any equilibrium point must satisfy the following condition:

$$|p_A^* - p_B^*| < c(b - a).$$

Let (p_A^*, p_B^*) be an equilibrium point. Suppose that $|p_A^* - p_B^*| > c(b - a)$. Then, according to the profit functions, the seller who charges the highest price gets a null profit and so may gain by charging a positive price equal to the charged price of the other. So there is an incentive for one of the sellers to change its price, and this contradicts the fact that (p_A^*, p_B^*) is an equilibrium point.

Suppose now that $|p_A^* - p_B^*| = c(b - a)$, and, for instance, $p_A^* > p_B^*$ (the opposite case is analogous), so $p_A^* - p_B^* = c(b - a)$. If $p_B^* = 0$, then the profit of firm B is zero and so it would generate a profit by charging less than p_A^*. If $p_B^* > 0$, since sellers A and B are sharing equally the demand of the market segment $[0, a]$, two cases may arise. Either B gets the vast majority of the market and so A, who charges a positive price, can increase its profit by decreasing slightly its price. Or B gets only a small fraction of the market, that is, $q_B = 1 - \frac{a}{2} < 1$, and it is then sufficient for B to charge a slightly lower price to capture the whole market and make a larger profit: indeed for $0 < \varepsilon < \frac{ap_B^*}{2}$ we have $\pi_B(p_A^*, p_B^* - \varepsilon) = (p_B^* - \varepsilon) > \frac{1}{2}p_B^*(2 - a) = \pi_B(p_A^*, p_B^*)$. Both cases contradict the fact that (p_A^*, p_B^*) is an equilibrium point. As a consequence, any equilibrium point (p_A^*, p_B^*) must satisfy the condition $|p_A^* - p_B^*| < c(b - a)$.

Consider now any equilibrium point (p_A^*, p_B^*). Since it is an equilibrium, it is known that p_A^* must maximize $\pi_A(p_A, p_B^*)$ in the open interval $(p_B^* - c(b - a), p_B^* + c(b - a))$ and p_B^* must maximize $\pi_B(p_A^*, p_B)$ in the open interval $(p_A^* - c(b - a), p_A^* + c(b - a))$. Taking first order conditions, that is,

$$\begin{cases} \dfrac{\partial \pi_A}{\partial p_A}(p_A, p_B) = 0 \\ \dfrac{\partial \pi_B}{\partial p_B}(p_A, p_B) = 0, \end{cases}$$

we get the best response function

$$\begin{cases} p_A = \dfrac{1}{2}\left(p_B + c(a+b)\right) \\ p_B = c + \dfrac{1}{2}\left(p_A - c(a+b)\right). \end{cases} \tag{2.6}$$

From (2.6) we obtain the pair of prices of the equilibrium point

$$\begin{cases} p_A^* = \dfrac{c}{3}(2+a+b) \\ p_B^* = \dfrac{c}{3}(4-a-b), \end{cases}$$

in fact, (2.4) and (2.5) are reached.

To be an equilibrium strategy, p_A^* must maximize π_A not only in the above interval but on the whole domain $[0,\infty)$, and similarly for p_B^*. It only happens on a restricted set of possible locations. Indeed, given $a<b$, suppose that $p_A \notin (p_B^* - c(b-a), p_B^* + c(b-a))$, and seller B plays p_B^*. Different cases must be studied.

(i) Assume seller B plays p_B^* and that $p_A > p_B^* + c(b-a)$. Then,

$$\pi_A(p_A^*, p_B^*) \geq \pi_A(p_A, p_B^*),$$

since this last one is zero. Hence, it will be better for seller A to keep p_A^*.

(ii) Suppose that seller B plays p_B^* and that $p_A < p_B^* - c(b-a)$. Let $\varepsilon > 0$ such that $p_A = p_B^* - c(b-a) - \varepsilon$. Then,

$$\pi_A(p_A^*, p_B^*) > \pi_A(p_B^* - c(b-a) - \varepsilon, p_B^*),$$

is equivalent to

$$\frac{c}{18}(2+a+b)^2 > \frac{c}{3}(4+2a-4b-\varepsilon).$$

From the inequality above we get

$$(2+a+b)^2 \geq 12(2+a-2b).$$

Thus, if (2.2) is satisfied, then playing p_A^* is also the best option for seller A. By symmetry, if (2.3) is accomplished, then p_B^* is an equilibrium strategy too.

Finally, it remains to check that $|p_A^* - p_B^*| < c(b-a)$ is verified when (2.2) and (2.3) are satisfied.

Let us focus on π_A. The profit function of seller A if seller B plays $p_B^* = \frac{c}{3}(4-a-b)$ is

$$\pi_A(p_A, p_B^*) = \begin{cases} p_A & \text{if } p_A < \dfrac{2c}{3}(2+a-2b), \\ \dfrac{1}{2}\, p_A(1+b) & \text{if } p_A = \dfrac{2c}{3}(2+a-2b), \\ p_A\left(\dfrac{1}{3}\,(2+a+b) - \dfrac{p_A}{2c}\right) & \text{if } \dfrac{2c}{3}(2+a-2b) < p_A < \dfrac{2c}{3}(2-2a+b), \\ \dfrac{1}{2}\, a p_A & \text{if } p_A = \dfrac{2c}{3}(2-2a+b), \\ 0 & \text{if } p_A > \dfrac{2c}{3}(2-2a+b). \end{cases}$$

Note that its quadratic piece — the domain of which is all $\mathbb{R}$ — has a unique maximum. We will see that this maximum is reached when $p_A^* \in \left(\frac{2c}{3}(2+a-2b), \frac{2c}{3}(2-2a+b)\right)$. It happens if $\frac{\partial \pi_A}{\partial p_A}(p_A, p_B^*)$ is positive on the left extreme of the interval and negative on the right one. Deriving $\pi_A(p_A, p_B^*)$ with respect to p_A in this interval we get

$$\frac{\partial \pi_A}{\partial p_A}(p_A, p_B^*) = \frac{1}{3}(2+a+b) - \frac{p_A}{c}. \tag{2.7}$$

If we evaluate (2.7) at the extremes of the interval we obtain

$$\begin{aligned} \frac{\partial \pi_A}{\partial p_A}(p_A, p_B^*)\Big|_{p_A=\frac{2c}{3}(2+a-2b)} &= \frac{1}{3}(-2-a+5b), \\ \frac{\partial \pi_A}{\partial p_A}(p_A, p_B^*)\Big|_{p_A=\frac{2c}{3}(2-2a+b)} &= \frac{1}{3}(-2-5a-b). \end{aligned}$$

Therefore, if

$$\frac{1}{3}(-2-a+5b) > 0 \quad \text{which is equivalent to} \quad 5b-a > 2, \tag{2.8}$$

and

$$\frac{1}{3}(-2-5a-b) < 0 \quad \text{which is equivalent to} \quad 5a-b < 2, \tag{2.9}$$

then $p_A^* \in \left(\frac{2c}{3}(2+a-2b), \frac{2c}{3}(2-2a+b)\right)$.

Notice that if $a = 0$ and $b = 1$, that is, if sellers A and B are as far as possible from each other, (2.8) and (2.9) are satisfied. Furthermore, (2.2) and (2.3) are also accomplished.

Let us study what happens at the points where (2.8) and (2.9) stop being satisfied, that is, when $5b - a = 2$ and $5a - b = 2$.

(i) If $a = 5b - 2$ then, condition (2.2) can be rewritten as follows

$$(2+5b-2+b)^2 \geq \frac{1}{2}(2+5b-2-2b) \Leftrightarrow b \geq 1.$$

Since by hypothesis $b \leq 1$, it can only be the equality. So, both (2.8) and (2.2) are satisfied if $a \in [0, 5b-2)$.

(ii) If $b = 5a - 2$, then condition (2.3) can be rewritten as follows

$$(4-a-5a+2)^2 \geq 12(1+2a-5a+2) \Leftrightarrow a \leq 0.$$

Since by hypothesis $a \geq 0$, it can only be the equality. So, both (2.9) and (2.3) are satisfied if $b \in [0, 5a-2)$.

Therefore, (2.8) and (2.9) are equivalent to (2.2) and (2.3). Thus, when (2.2) and (2.3) are satisfied, $p_A^* \in \left(\frac{2c}{3}(2+a-2b), \frac{2c}{3}(2-2a+b)\right)$, and so, the global maximum of $\pi_A(p_A, p_B^*)$ is $\pi_A(p_A^*, p_B^*)$. That is, it is guaranteed that p_A^* is the best strategy for seller A. By symmetry, p_B^* is the best strategy for seller B. □

According to Proposition 2.1, note that if $a \neq b$ and we consider only symmetric locations around the center ($a = 1 - b$ or $b = 1 - a$), then the necessary and sufficient conditions (2.2) and (2.3) can be rewritten as

$$(2+1-b+b)^2 \geq 12(2+1-b-2b) \;\;\Rightarrow\;\; 9 \geq 36(1-b) \;\;\Rightarrow\;\; b \geq \frac{3}{4},$$
$$(4-a-1-a)^2 \geq 12(1+2a-1+a) \;\;\Rightarrow\;\; 9 \geq 36a \;\;\Rightarrow\;\; a \leq \frac{1}{4}.$$

That is, firms A and B cannot be located at the center of the market to achieve equilibrium in prices. Indeed, the *Principle of Minimum Differentation* exposed by Hotelling, as stated, is not correct, except for prices equal to zero.

2.3 Quadratic Transport Costs

In 1979, d'Aspremont, Gabszewicz and Thisse, in their article *On Hotelling's "Stability in competition"* [4], modified Hotelling's model in order to get a price equilibrium solution for any pair of locations (a, b). In their version, instead of considering linear transportation costs, these costs are assumed to be quadratic with respect to the distance. That is, the transport cost function is $c(z) = cz^2$, where z is the distance from the consumer to the seller, and $c > 0$ a multiplicative constant. Under this assumption, the development of Hotelling's model will be replayed following Hotelling's original hypothesis — with the exception of the transport cost function.

Let a and b be the locations of sellers A and B, respectively. In order to know the price which maximizes the profit for both stores, the demand functions must be calculated. As with Hotelling's model, it is important to know where the indifferent consumer is located. Let x be this point. At this case, assuming $a < b$, $x \in [0, 1]$ will be indifferent if

$$p_A + c|x-a|^2 = p_B + c|x-b|^2. \tag{2.10}$$

Observe that $|x-a|^2 = (x-a)^2 = (a-x)^2$. Thus, equation (2.10) becomes

$$p_A + c(x-a)^2 = p_B + c(x-b)^2,$$

and we obtain

$$x = \frac{p_B - p_A}{2c(b-a)} + \frac{a+b}{2}.$$

The indifferent consumer will be inside the market as long as $x \in [0, 1]$. That is,

$$0 \leq \frac{p_B - p_A}{2c(b-a)} + \frac{a+b}{2} \leq 1$$

and a simple manipulation yields to

$$c(a^2 - b^2) \leq p_B - p_A \leq c(b-a)(2-a-b).$$

So, if $p_B - p_A$ satisfies the inequalities above, seller A will serve the market segment $[0, x]$, while seller B will serve the market segment $[x, 1]$. That is, $q_A = x$ and $q_B = 1 - x$.

On the other hand, notice that if $x > 1$, that is, $p_B - p_A > 2c(b-a) + c(a^2 - b^2)$, then we see that $p_B + c(1-b)^2 > p_A + c(1-a)^2$ and a consumer placed on 1 will prefer to buy from A. Indeed,

$$\begin{aligned}
&p_B + c(1-b)^2 - p_A - c(1-a)^2 \\
&= p_B - p_A + c(b-a)(-2+a+b) \\
&> c(b-a)(2-a-b) + c(b-a)(-2+a+b) = 0.
\end{aligned}$$

That is, $p_B + c(x-b)^2 > p_A + c(x-a)^2$ in the interval $[0,1]$, and so, demand for seller A is all the market. That is, $q_A = 1$ and $q_B = 0$.

Finally, observe that if $x < 0$, that is, $p_B - p_A < c(a^2 - b^2)$, then for a consumer placed at 0 we have

$$p_B + c(0-b)^2 - p_A - c(0-a)^2 = p_B - p_A + c(b^2 - a^2) < c(a^2 - b^2) + c(b^2 - a^2) = 0.$$

That is, $p_B + c(x-b)^2 < p_A + c(x-a)^2$ in the interval $[0,1]$. Therefore, demand for seller B is all the market. That is, $q_A = 0$ and $q_B = 1$.

With the demand for both sellers determined, it is possible to compute the profit functions as follows:

$$\pi_A(p_A, p_B) = \begin{cases} p_A & \text{if } \ 0 \le p_A < p_B - c(b-a)(2-a-b), \\ \dfrac{p_A p_B - p_A^2}{2c(b-a)} + \dfrac{p_A(a+b)}{2} & \text{if } \ p_B - c(b-a)(2-a-b) \le p_A \le p_B + c(b^2 - a^2), \\ 0 & \text{if } \ p_A > p_B + c(b^2 - a^2), \end{cases}$$

and

$$\pi_B(p_A, p_B) = \begin{cases} p_B & \text{if } \ 0 \le p_B < p_A + c(a^2 - b^2), \\ p_B + \dfrac{p_A p_B - p_B^2}{2c(b-a)} - \dfrac{p_B(a+b)}{2} & \text{if } \ p_A + c(a^2 - b^2) \le p_B \le p_A + c(b-a)(2-a-b), \\ 0 & \text{if } \ p_B > p_A + c(b-a)(2-a-b). \end{cases}$$

In the case of quadratic transport cost function, π_A and π_B are continuous for all $p_A, p_B \in [0, \infty)$. In addition, both π_A and π_B have a global maximum which corresponds to the unique maximum of their quadratic piece — which is strictly concave. As a consequence, there is only a unique Nash equilibrium. So that, for any fixed a and b, any equilibrium point (p_A^*, p_B^*) must satisfy

$$c(a^2 - b^2) \le p_B^* - p_A^* \le c(b-a)(2-a-b).$$

With this information, let us calculate the pair of prices of the equilibrium point. Taking first order conditions,

$$\begin{cases} \dfrac{\partial \pi_A}{\partial p_A}(p_A, p_B) = 0 \\ \dfrac{\partial \pi_B}{\partial p_B}(p_A, p_B) = 0, \end{cases}$$

we get the best response function

$$\begin{cases} p_A = \dfrac{1}{2}\left(p_B + c(b^2 - a^2)\right) \\ p_B = \dfrac{1}{2}\left(p_A - c(2-a-b)(b-a)\right). \end{cases} \tag{2.11}$$

From (2.11) we obtain the profit maximizing prices

$$\begin{cases} p_A^* = \dfrac{c}{3}(2+a+b)(b-a) \\ p_B^* = \dfrac{c}{3}(4-a-b)(b-a). \end{cases}$$

Observe that (p_A^*, p_B^*) is the Nash equilibrium that is true without any condition on locations a and b, as long as $a \neq b$.

With the information above, let us investigate which the best location for each seller is to maximize its profits. We have to find a and b that maximize the values of $\pi_A(p_A^*, p_B^*)$ and $\pi_B(p_A^*, p_B^*)$. We know that

$$\begin{aligned} \pi_A(p_A^*, p_B^*) &= \frac{c}{18}(b-a)(2+a+b)^2, \\ \pi_B(p_A^*, p_B^*) &= \frac{c}{18}(b-a)(4-a-b)^2. \end{aligned}$$

Let us focus on π_A. Deriving $\pi_A(p_A^*, p_B^*)$ with respect to a and b, we get

$$\begin{aligned} \frac{\partial \pi_A}{\partial a}(p_A^*, p_B^*) &= -\frac{c}{18}(2+a+b)(2+3a-b), \\ \frac{\partial \pi_A}{\partial b}(p_A^*, p_B^*) &= \frac{c}{18}(2+a+b)(2-a+3b). \end{aligned}$$

Notice that there is no solution to

$$\begin{cases} \dfrac{\partial \pi_A}{\partial a}(p_A^*, p_B^*) = 0 \\ \dfrac{\partial \pi_B}{\partial b}(p_A^*, p_B^*) = 0, \end{cases}$$

if $a, b \in [0,1]$. So locations that maximize profits must be on the extremes of the interval. By hypothesis, $a < b$, so the best option is $a = 0$ and $b = 1$.

Doing the same process for π_B we obtain the same result. Consequently, at any given pair of locations, each seller gains an advantage from moving away as far as possible from the other. That is, Hotelling's model with quadratic transport costs implies exactly the contrary of the *Principle of Minimum Differentiation.*

Chapter 3

Extension of Hotelling's Model to a Circle

One of the most remarkable variations of Hotelling's model is Salop's circle model, which was introduced in 1979 by Salop in his article *Monopolistic Competition with Outside Goods* (see [17]). This locational model is similar to Hotelling's one since it examines consumer preference with regards to the geographic location. However, there exist two significant differences: firms are located around a circle with no endpoints instead of a line segment, and it allows consumers to choose a second heterogeneous good. Consumers have the opportunity to purchase either one unit or none of this second product — according to preferences, prices, and distribution of brands in product space — , and they spend the rest of their income on a homogeneous good.

Since the circle model was introduced, it has been material of investigation. In particular, de Frutos, Hamoudi and Jarque (1999) [5], studies the existence of a unique perfect equilibrium in the circle model with linear quadratic transport costs. In this work, below, their analysis is adapted to validate the existence of a Nash equilibrium in a duopoly circular model in the case of quadratic travel costs.

3.1 Circular Model With Quadratic Transport Costs

We are going to study the existence of a Nash equilibrium in a circular market with quadratic transportation costs.

Let $a, b \in [0, 1)$ be the location of sellers A and B, situated in a circular market represented by a circumference of length 1. Notice that location 0 and 1 overlap. Without loss of generality, we assume that seller A is located at $a = 0$, and seller B at $b \in (0, 1/2]$. Suppose that they sell a homogeneous good produced at zero cost. Let p_A and p_B denote, respectively, the mill price of the product of sellers A and B.

A continuum of consumers are spread uniformly with unit density on the circumference. Let us reuse the same assumption that each consumer buys one unit of the good from the store with the lowest cost, considering both mill price and travel costs. In this case, travel costs are assumed to be quadratic with respect to the distance. That is, the transport cost function is $c(z) = cz^2$, where z is the closest distance between the consumer and the firm traveling along the perimeter of the circle, and $c > 0$ a multiplicative constant.

Let q_A and q_B be the demand for sellers A and B, respectively. Recall that demand for A is made by the proportion of consumers for whom buying from seller A is more worthwhile (and the analogous applies to seller B). The market boundaries between stores are determined by the indifferent consumers. So, to obtain the demand functions, we first calculate the location of the indifferent consumers in the circle. Remember that the location of an indifferent consumer, $x \in [0, 1)$, is determined as follows:

$$p_A + c|x - a|^2 = p_B + c|x - b|^2. \tag{3.1}$$

Depending on the value of p_A and p_B, and according to Figure 3.1, four different cases must be studied to determine the possible locations of x with respect to the location of the stores.

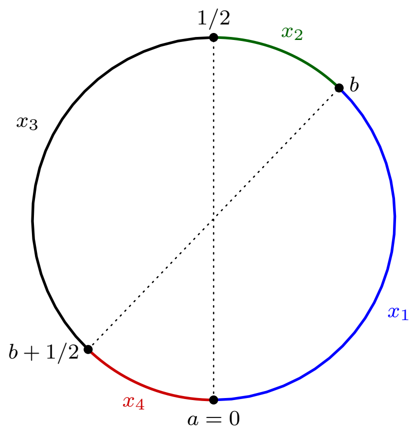

Figure 3.1

(i) Let x_1 be the location of the indifferent consumer situated on $[0, b]$. Since $a = 0$, equation (3.1) becomes

$$p_A + cx_1^2 = p_B + c(x_1 - b)^2.$$

From the equality above we get

$$x_1 = \frac{p_B - p_A}{2cb} + \frac{b}{2}.$$

So, there will be an indifferent consumer in this region of the market if $x_1 \in [0, b]$, that is,

$$-cb^2 \leq p_A - p_B \leq cb^2.$$

If $p_A - p_B$ satisfies the inequalities above, consumers located to the right of x_1 choose seller A, whereas consumers to the left of x_1 will buy from seller B.

(ii) Let x_2 be the location of the indifferent consumer situated on $[b, 1/2]$. Similarly to the case above, from equation (3.1) we obtain

$$x_2 = \frac{p_B - p_A}{2cb} + \frac{b}{2}.$$

Therefore, if $x_2 \in [b, 1/2]$, that is,

$$cb(b-1) \leq p_A - p_B \leq -cb^2,$$

there will be an indifferent consumer in this area of the market.

In this case, if $p_A - p_B$ satisfies the inequalities above, consumers to the right of x_2 will select firm A, and those located to the left of x_2 will buy from seller B.

(iii) Let x_3 be the location of the indifferent consumer situated on $[1/2, b+1/2]$. Recall that location 1 and 0 overlap. Then, equation (3.1) becomes

$$p_A + c(1-x_2)^2 = p_B + c(x_2 - b)^2.$$

Thus,

$$x_3 = \frac{p_B - p_A}{2c(b-1)} + \frac{1}{2}(b+1).$$

There exists an indifferent consumer if $x_3 \in [1/2, b+1/2]$, that is,

$$cb(b-1) \leq p_A - p_B \leq cb(1-b).$$

If $p_A - p_B$ satisfies the inequalities above, those consumers located to the right of x_3 will buy from seller B, while those located to the left of x_3 will select firm A.

(iv) Let x_4 be the location of the indifferent consumer situated on $[b+1/2, 1]$. In this case, equation (3.1) changes into

$$p_A + c(1-x_4)^2 = p_B + c(1-x_4+b)^2.$$

With a simple manipulation we obtain

$$x_4 = \frac{p_B - p_A}{2cb} + \frac{1}{2}(b+2).$$

And $x_4 \in [b+1/2, 1]$ as long as

$$cb^2 \leq p_A - p_B \leq cb(1-b).$$

If $p_A - p_B$ satisfies the inequalities above, consumers to the left of x_3 will buy from seller A, while the remaining consumers select seller B.

We have seen that the location and the existence of indifferent consumers depends on the value of $p_A - p_B$. Therefore, we have to study the following cases:

(a) If $p_A - p_B < cb(b-1)$, there is no indifferent consumer. Observe that

$$p_A + c(b-0)^2 \leq p_B$$

since

$$p_A - p_B + cb^2 < cb(b-1) + cb^2 = cb(2b-1) \leq 0.$$

So, even a consumer located at firm B prefers to buy the product from firm A. That is, demand for seller A is all the market.

(b) If $cb(b-1) \leq p_A - p_B \leq -cb^2$, then there are two indifferent consumers, x_2 and x_3. In this case, demand for seller A is the group of consumers located to the right of x_2 who travel to store A clockwise, plus the consumers located to the left of x_3 travelling to firm A counterclockwise. That is,

$$q_A = x_2 + (1 - x_3) = \frac{p_A - p_B}{2cb(b-1)} + \frac{1}{2}.$$

(c) If $-cb^2 \leq p_A - p_B \leq cb^2$, then there are two indifferent consumers, x_1 and x_3. Since x_1 equals x_2, quantity demanded for seller A is the same as in the previous case.

(d) If $cb^2 \leq p_A - p_B \leq cb(1-b)$, then there are two indifferent consumers, x_3 and x_4. In this case, demand for seller A is the group of consumers located in $[x_3, x_4]$, who travel counterclockwise. That is,

$$q_A = x_4 - x_3 = \frac{p_A - p_B}{2cb(b-1)} + \frac{1}{2}.$$

(e) If $p_A - p_B > cb(1-b)$, there is no indifferent consumer. Observe that

$$p_A + c(b-0)^2 \geq p_B$$

since

$$p_A - p_B - cb^2 > cb(1-b) - cb^2 = cb(1-2b) \geq 0.$$

So, even a consumer located at firm A prefers to buy the product from firm B. That is, demand for firm B is all the market, and so, demand for firm A is null.

Therefore, demand for seller A is determined. And we know that $q_B = 1 - q_A$. Consequently, it is possible to define the profit functions as follows:

$$\pi_A(p_A, p_B) = \begin{cases} p_A & \text{if} \quad 0 \leq p_A \leq p_B + cb(b-1), \\ \dfrac{p_A^2 - p_A p_B}{2cb(b-1)} + \dfrac{p_A}{2} & \text{if} \quad p_B + cb(b-1) \leq p_A \leq p_B + cb(1-b), \\ 0 & \text{if} \quad p_A \geq p_B + cb(1-b), \end{cases}$$

and

$$\pi_B(p_A, p_B) = \begin{cases} p_B & \text{if} \quad 0 \leq p_B \leq p_A + cb(b-1), \\ \dfrac{p_B^2 - p_A p_B}{2cb(b-1)} + \dfrac{p_B}{2} & \text{if} \quad p_A + cb(b-1) \leq p_B \leq p_A + cb(1-b), \\ 0 & \text{if} \quad p_B \geq p_A + cb(1-b). \end{cases}$$

It is easy to see that π_A and π_B are continuous for all $p_A, p_B \in [0, \infty)$.

For any fixed $b \in (0, 1/2]$, a pair (p_A^*, p_B^*) is an equilibrium point if p_A^* maximizes $\pi_A(p_A, p_B^*)$ and p_B^* maximizes $\pi_B(p_A^*, p_B)$ on $[0, \infty)$. Since π_A and π_B have their respective maxima on their quadratic piece — which is strictly concave —, for any given b, they have a unique global maximum. That is, there exists a unique pair of equilibrium prices. Let us calculate it. Taking first order conditions,

$$\begin{cases} \dfrac{\partial \pi_A}{\partial p_A}(p_A, p_B) = 0 \\ \dfrac{\partial \pi_B}{\partial p_B}(p_A, p_B) = 0, \end{cases}$$

we get the best response function

$$\begin{cases} p_B = 2p_A + cb(b-1) \\ p_A = 2p_B + cb(b-1), \end{cases}$$

from which we obtain the equilibrium point

$$\begin{cases} p_A^* = cb(1-b) \\ p_B^* = cb(1-b). \end{cases}$$

As a consequence, for any given $b \in (0, 1/2]$ the maximum profit of each seller is

$$\pi_A(p_A^*, p_B^*) = \pi_B(p_A^*, p_B^*) = \frac{cb(1-b)}{2}.$$

With the information above, let us compute which the best location of b is that maximizes profits for both sellers. Deriving $\pi_A(p_A^*, p_B^*)$ and $\pi_B(p_A^*, p_B^*)$ with respect to b we get

$$\frac{\partial \pi_A}{\partial b}(p_A^*, p_B^*) = \frac{\partial \pi_B}{\partial b}(p_A^*, p_B^*) = \frac{1}{2}(c - 2cb),$$

and if we equate to 0, we obtain $b = 1/2$. Since

$$\frac{\partial^2 \pi_A}{\partial b^2}(p_A^*, p_B^*) = \frac{\partial^2 \pi_B}{\partial b^2}(p_A^*, p_B^*) = -c$$

is always negative, the profit maximizing location for seller B is $b = 1/2$. That is, each seller benefits if firms are as far away as possible one from the other. In other words, if firms are located opposite to each other on the circle.

Competition on Location but Not on Prices

Under the same assumptions, if we assume that sellers do not compete on price but only on location, then the model is very simplified. That is, supposing that $p_A = p_B$, equation (3.1) becomes

$$|x-a|^2 = |x-b|^2. \tag{3.2}$$

In this case, there are two different possible locations of the indifferent consumers, x_1 and x_2.

(i) Let x_1 be the location of the indifferent consumer situated on [0, 1/2]. Assuming that $a = 0$, from equation (3.2) we get

$$x_1 = \frac{b}{2}.$$

(ii) Let x_2 be the location of the indifferent consumer situated on [1/2, 1]. Assuming that $a = 1$, from equation (3.2) we obtain

$$x_2 = \frac{b+1}{2}.$$

Notice that $x_1 \in [0, 1/2]$ and $x_2 \in [1/2, 1]$ as long as $0 \le b \le 1$. So, there always will be two indifferent consumers. Therefore, demand for seller A will be

$$q_A = x_1 + (1 - x_2) = \frac{1}{2}.$$

Consequently, $q_A = q_B$. That is, no matter the location of firm B, each seller gets half of the market.

Chapter 4

Location Games

Fournier and Scarsini, in 2019, in their article *Location Games on Networks: Existence and Efficiency of Equilibria* (see [9] and [10]) extend a variation of Hotelling's model to graphs. They consider a game where a finite number of players sell a homogeneous product at the same price to their consumers, who are uniformly distributed on a network. In their model, sellers do not compete on price but only on location. They show that if the number of retailers is large enough, the game admits a Nash equilibrium. Then, they analyse how efficient the equilibria in location games are, using the Price of Anarchy and the Price of Stability. They apply these results to some particular examples. We will focus on the segment case — which was already studied previously by Eaton and Lipsey (1975) [7] and Pálvölgyi (2011) [16], among others — , the star network, and the circle — also considered by Eaton and Lipsey (1975) [7].

4.1 Existence of Equilibrium on a Segment

In this section, we study the existence of equilibrium on a segment when a finite number of sellers do not compete on price but only on location.

As in Hotelling's model, the market is represented by a line segment, which, without loss of generality, is assumed to be $[0, 1]$. We consider that a finite number of firms simultaneously choose a location where to operate within this interval. All firms produce a unique homogeneous product which is charged at the same price. Let $N = \{1, \ldots, n\}$, with $n \geq 2$, denote the set of sellers and $x = (x_1, \ldots, x_n) \in [0, 1]^n$ the strategy profile they play.

Consumers, represented by a mass of 1, are uniformly distributed along the unit interval. Let us suppose that each customer buys one unit of this good per unit of time. Because products are homogeneous and charged at the same price, transport costs are the only aspect which consumers take into consideration when deciding from which seller to buy. Therefore, each consumer decides to buy from the closest shop. If two or more sellers are in the exact same location, then consumers meant to purchase at that location are split equally between sellers. Firms seek to maximize their profits. The payoff of a firm is equal to the mass of consumers it attracts.

Given a strategy profile $x = (x_1, \ldots, x_n) \in [0, 1]^n$, since it does not matter which firm is in what location, we assume that player 1 is the leftmost point, and player n is the rightmost point. That is, $0 \leq x_1 \leq \cdots \leq x_n \leq 1$. Then, for each seller $i \in N$, we define two lengths $\overleftarrow{p}_i(x)$ and $\overrightarrow{p}_i(x)$ as follows:

$$\overleftarrow{p}_i(x) = \begin{cases} d(0, x_1) & \text{if} \quad i = 1, \\ \dfrac{1}{2} d(x_{i-1}, x_i) & \text{if} \quad i \in \{2, \dots, n\}, \end{cases}$$

and

$$\overrightarrow{p}_i(x) = \begin{cases} \dfrac{1}{2} d(x_i, x_{i+1}) & \text{if} \quad i \in \{1, \dots, n-1\}, \\ d(x_n, 1) & \text{if} \quad i = n. \end{cases}$$

Note that, since consecutive players can share the same location, $\overleftarrow{p}_i(x)$ and $\overrightarrow{p}_i(x)$ can be equal to 0.

Therefore, the payoff of seller $i \in N$ under the strategy profile $x = (x_1, \dots, x_n) \in [0,1]^n$ is

$$\rho_i(x) = \frac{1}{card\{k :\, x_k = x_i\}} \sum_{k:\, x_k = x_i} \left(\overleftarrow{p}_k(x) + \overrightarrow{p}_k(x) \right).$$

The above defined game is called *location game* on $[0,1]$ with $n \geq 2$ players, and is denoted by $\mathscr{L}(n, [0,1])$.

Given a strategy profile $x = (x_1, \dots, x_n) \in [0,1]^n$, we denote by x_{-i} the $(n-1)$ dimensional vector $(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$. Given any $z \in [0,1]$ define

$$(z, x_{-i}) = (x_1, \dots, x_{i-1}, z, x_{i+1}, \dots, x_n),$$

to allow for deviations of player i. Thus, a strategy profile x^* is a *Nash equilibrium* of the game $\mathscr{L}(n, [0,1])$ if for all $i \in N$ and for all $x_i \in [0,1]$ we have

$$\rho_i(x^*) \geq \rho_i(x_i, x^*_{-i}).$$

We first state some general equilibrium properties.

Lemma 4.1. *Let $x^* = (x^*_1, \dots, x^*_n) \in [0,1]^n$ be a Nash equilibrium of $\mathscr{L}(n, [0,1])$. Then,*

(a) There are no players on the extremes of the interval $[0,1]$, i.e., $x^*_i \neq 0$ and $x^*_i \neq 1$ for all $i \in N$.

(b) For every $y \in [0,1]$

$$card\{i :\, x^*_i = y\} \leq 2.$$

(c) There exists $\xi > 0$ such that for all $y \in [0,1]$, if

$$card\{i :\, x^*_i = y\} = 2,$$

and i and $i+1$ are the two players located on y, then

$$\rho_i(x^*) = \rho_{i+1}(x^*) = \overleftarrow{p}_i(x^*) = \overrightarrow{p}_{i+1}(x^*) = \xi.$$

(d) Players closest to the extremes are coupled, i.e., $x^*_1 = x^*_2$ and $x^*_{n-1} = x^*_n$.

Proof. Suppose that $x^* = (x^*_1, \dots, x^*_n)$ is a Nash equilibrium of $\mathscr{L}(n, [0,1])$, with $n \geq 2$.

(a) We show that there are no players placed on 0 nor 1. We focus on the extreme 0, but the analogous arguments can apply to location 1.

First, we suppose that all players are located on 0. That is, there are no more firms to the right of 0. In this case, players on 0 split the entire market amongst themselves in equal parts — each gaining $1/n$. If a firm located on 0 moves to $z \in (0,1)$ it will get a payoff equal to $1 - z/2$. Since $n \geq 2$, we have

$$\frac{1}{n} < 1 - \frac{z}{2}$$

for all $z \in (0,1)$. Therefore, a strategy profile where all players are located on 0 is not a Nash equilibrium since any seller will increase profits by deviating to the right.

Now, we assume there are $k < n$ players on 0. So, there is at least one player to the right of 0. Let j be the closest seller to 0. Hence, the amount of consumers that buy at 0 is $\frac{1}{2}d(0, x_j^*)$, and so each firm on 0 gains $\frac{1}{2k}d(0, x_j^*)$. Observe that, if $k > 1$,

$$\frac{1}{2k}d(0, x_j^*) < \frac{1}{2}d(0, z) + \frac{1}{2}d(z, x_j^*),$$

for all $z \in (0, x_j^*)$. Thus, a seller i located on 0 will benefit from moving to $z \in (0, x_j^*)$, which means that x^* is not an equilibrium strategy.

On the other hand, if $k = 1$, the unique player located on 0 has a payoff equal to $\frac{1}{2}d(0, x_j^*)$. Note that

$$\frac{1}{2}d(0, x_j^*) < d(0, z) + \frac{1}{2}d(z, x_j^*),$$

for all $z \in (0, x_j^*)$. So, the firm located on 0 will get a higher payoff if it moves anywhere between $(0, x_j^*)$. Therefore, there is not an equilibrium if there is only one seller located at 0.

Consequently, there is no Nash equilibrium if any player is located at the extremes of the interval $[0,1]$.

(b) We prove that only two firms can choose the same point in equilibrium. When $n = 2$ it is trivial. Thus, suppose that $n \geq 3$ and assume, by contradiction, that there is a point $y \in (0,1)$ with $k \geq 3$ players. Let $i, i+1, \ldots, i+k-1$ be the sellers located at y.

The amount of consumers that buy at location y is

$$\sum_{l=i}^{i+k-1} \left(\overleftarrow{p}_l(x^*) + \overrightarrow{p}_l(x^*) \right) = \overleftarrow{p}_i(x^*) + \overrightarrow{p}_{i+k-1}(x^*),$$

since $d(x_i^*, x_{i+1}^*) = \cdots = d(x_{i+k-2}^*, x_{i+k-1}^*) = 0$.

Therefore, each of the sellers $i, i+1, \ldots, i+k-1$ on y gains

$$\frac{1}{k} \left(\overleftarrow{p}_i(x^*) + \overrightarrow{p}_{i+k-1}(x^*) \right).$$

This amount satisfies the following inequalities:

$$\frac{1}{k} \left(\overleftarrow{p}_i(x^*) + \overrightarrow{p}_{i+k-1}(x^*) \right) \leq \frac{2}{k} \max \left\{ \overleftarrow{p}_i(x^*), \overrightarrow{p}_{i+k-1}(x^*) \right\} < \max \left\{ \overleftarrow{p}_i(x^*), \overrightarrow{p}_{i+k-1}(x^*) \right\}.$$

Then, for ε small enough, a seller who moves by ε from y in the direction of the $\max\{\overleftarrow{p}_i(x^*), \overrightarrow{p}_{i+k-1}(x^*)\}$ gets a payoff of at least $\max\{\overleftarrow{p}_i(x^*), \overrightarrow{p}_{i+k-1}(x^*)\} - \varepsilon/2$. Note that if $\overleftarrow{p}_i(x^*) = \overrightarrow{p}_{i+k-1}(x^*)$, then the direction the player chooses does not matter. Let us denote

$$\delta = \max\left\{\overleftarrow{p}_i(x^*), \overrightarrow{p}_{i+k-1}(x^*)\right\} - \frac{1}{k}\left(\overleftarrow{p}_i(x^*) + \overrightarrow{p}_{i+k-1}(x^*)\right).$$

If we take $\varepsilon < 2\delta$ this deviation is profitable. That means that x^* is not an equilibrium, so $k \leq 2$.

(c) We prove that all coupled firms of the game get the same payoff. Set $n \geq 2$ and let $y \in [0,1]$ be such that $card\{i : x_i^* = y\} = 2$. Let i and $i+1$ be the only two players located at y. Since $x_i^* = x_{i+1}^* = y$, then

$$\overrightarrow{p}_i(x^*) = \overleftarrow{p}_{i+1}(x^*) = \frac{1}{2}d(x_i^*, x_{i+1}^*) = 0.$$

Therefore, the amount of consumers that buy at location y is $\overleftarrow{p}_i(x^*) + \overrightarrow{p}_{i+1}(x^*)$, and each of both sellers gains

$$\rho_i(x^*) = \rho_{i+1}(x^*) = \frac{1}{2}\left(\overleftarrow{p}_i(x^*) + \overrightarrow{p}_{i+1}(x^*)\right).$$

Moreover, $\overleftarrow{p}_i(x^*) = \overrightarrow{p}_{i+1}(x^*)$. Because, if this was not true, we would have

$$\max\left\{\overleftarrow{p}_i(x^*), \overrightarrow{p}_{i+1}(x^*)\right\} > \frac{1}{2}\left(\overleftarrow{p}_i(x^*) + \overrightarrow{p}_{i+1}(x^*)\right),$$

and then, one of the two sellers located at y could benefit by deviating a small amount ε from y to the largest side, which contradicts that x^* is a Nash equilibrium. So, each player at y gains

$$\rho_i(x^*) = \rho_{i+1}(x^*) = \overleftarrow{p}_i(x^*) = \overrightarrow{p}_{i+1}(x^*).$$

We denote this amount by $\xi(y)$.

Finally, we show that the $\xi(y)$ does not depend on the location y. Suppose that $z \in [0,1]$ is another location with two sellers. Without loss of generality, we suppose that z is to the right of y. Let j and $j+1$ be the sellers located at z. Reapplying the above argument to players j and $j+1$ we get that each player at z gains

$$\rho_j(x^*) = \rho_{j+1}(x^*) = \overleftarrow{p}_j(x^*) = \overrightarrow{p}_{j+1}(x^*) = \xi(z).$$

We have to check that $\xi(y) = \xi(z)$. Suppose, for example, that $\xi(z) < \xi(y)$. This is equivalent to $\rho_j(x^*) = \rho_{j+1}(x^*) < \xi(y)$. In this case, seller j or $j+1$ would deviate to $y - \varepsilon$ and win at least $\xi(y) - \varepsilon/2$, which is an improvement if ε is small enough. So, x^* would not be an equilibrium. The analogous argument applies if $\xi(y) < \xi(z)$. Hence, $\xi(y) = \xi(z)$ and we can simply denote them by ξ.

(d) We show that on occupied locations nearest to the extremes of the interval there are always two firms. Assume that the seller closest to an extreme of the interval is alone. That contradicts the fact that x^* is an equilibrium strategy because this firm could increase its profit by moving away from the extreme, as long as it does not reach the point where the next store is located.

□

Now we state some necessary and sufficient conditions for the strategy profiles to be Nash equilibria of the game $\mathscr{L}(n, [0,1])$.

Proposition 4.1. *Let $x^* = (x_1^*, \dots, x_n^*) \in [0,1]^n$ be a strategy profile of $\mathscr{L}(n, [0,1])$. The following conditions characterize Nash equilibria for $n \neq 3$:*

(i) $x_1^* = x_2^* = 1 - x_{n-1}^* = 1 - x_n^*$.

(ii) $x_3^* - x_2^* = x_{n-1}^* - x_{n-2}^* = 2x_1^*$.

(iii) For all $i \in \{3, \dots, n-3\}$, $x_{i+1}^* - x_i^* \leq 2x_1^*$.

(iv) For all $i \in \{3, \dots, n-2\}$, $x_{i+1}^* - x_{i-1}^* \geq 2x_1^*$.

Proof. First, suppose that $x^* = (x_1^*, \dots, x_n^*) \in [0,1]^n$ is a Nash equilibrium in $\mathscr{L}(n, [0,1])$. We want to see that conditions (i) to (iv) are verified. We denote $x_1^* = \xi > 0$. Note that conditions (i) and (ii) follow directly from Lemma 4.1(c) and (d).

Let us focus on condition (iii). Suppose, by contradiction, that there is at least one $i \in \{3, \dots, n-3\}$ such that $x_{i+1}^* - x_i^* > 2\xi$. Then, any of sellers $1, 2, n-1$ and n, that earn ξ, would have an incentive to deviate anywhere in (x_i^*, x_{i+1}^*) and win

$$\frac{x_{i+1}^* - x_i^*}{2} > \xi.$$

Consequently, x^* would not be a Nash equilibrium. Hence, condition (iii) must be true.

Now, we concentrate on condition (iv). Observe that, for $i \in \{3, \dots, n-2\}$, if i is the only firm located on x_i^*, then

$$\rho_i(x^*) = \overleftarrow{p}_i(x^*) + \overrightarrow{p}_i(x^*) = \frac{x_{i+1}^* - x_{i-1}^*}{2}.$$

If there are two firms on x_i^*, then, without loss of generality, we can assume that $x_i^* = x_{i+1}^*$, for $i \in \{3, \dots, n-3\}$. From Lemma 4.1(c) we know that

$$\rho_i(x^*) = \overleftarrow{p}_i(x^*) = \frac{x_i^* - x_{i-1}^*}{2} = \frac{x_{i+1}^* - x_{i-1}^*}{2}.$$

Suppose, by contradiction, that $x_{i+1}^* - x_{i-1}^* < 2\xi$, which is equivalent to $\rho_i(x^*) < \xi$. In this case, for ε small enough, firm i could improve its payoff by moving to $x_1^* - \varepsilon$, which contradicts that x^* is a Nash equilibrium. Therefore, for all $i \in \{3, \dots, n-2\}$, $x_{i+1}^* - x_{i-1}^* \geq 2\xi$. That is, condition (iv) is achieved.

So far, we have seen that any Nash equilibrium satisfies the conditions of the statement. Now, we prove that if a strategy profile validates conditions above, then it is a Nash equilibrium. That is, we show that any $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n) \in [0,1]^n$ which verifies conditions (i) to (iv) does not allow any profitable deviation. If $n \leq 5$, it is easy to check each particular case. Note that we do not consider $n = 3$, since in this case conditions (i) and (ii) cannot be true at the same time. From now on, we assume that $n \geq 6$.

We observe that $\bar{x}_1 > 0$. Suppose, by contradiction, that $\bar{x}_1 = 0$. Then, from condition (i) to (iii) it follows that all firms are located on 0. But, this contradicts (i) because $\bar{x}_n = 1 - \bar{x}_1 = 1 \neq 0$. So, we denote $\bar{x}_1 = \mu > 0$.

First, we show that at most 2 players can be in the exact same location. Let i be any firm located at $\bar{x}_i$. By conditions (i) and (ii), it is obvious if $i \in \{1, 2, n-1, n\}$. Thus, suppose $i \in \{3, \dots, n-2\}$ and, assume, by contradiction, that at least $\bar{x}_{i-1} = \bar{x}_i = \bar{x}_{i+1}$. By condition (iv), we know that $\bar{x}_{i+1} - \bar{x}_{i-1} = 0 \geq 2\mu$, but this contradicts that μ is strictly positive. Therefore, there can only be one or two firms at an exact same point.

Now, we study the payoff of each firm. Let $i \in N$ be any firm located at $\bar{x}_i$. If i is the only seller located at $\bar{x}_i$, then, by (i), it cannot be located on an extreme. Therefore, its payoff is

$$\rho_i(\bar{x}) = \frac{\bar{x}_{i+1} - \bar{x}_{i-1}}{2}.$$

In addition, observe that if (iii) and (iv) are true, for any single firm i, we get

$$\mu \leq \rho_i(\bar{x}) \leq 2\mu. \tag{4.1}$$

On the other hand, we suppose that i shares location with another seller. From conditions (i) and (ii) it follows that

$$\rho_1(\bar{x}) = \rho_2(\bar{x}) = \rho_{n-1}(\bar{x}) = \rho_n(\bar{x}) = \mu.$$

Moreover, for any $i \in \{3, \dots, n-3\}$ such that $\bar{x}_i = \bar{x}_{i+1}$, we have that

$$\rho_i(\bar{x}) = \rho_{i+1}(\bar{x}) = \frac{1}{2}\left(\frac{\bar{x}_i - \bar{x}_{i-1}}{2} + \frac{\bar{x}_{i+2} - \bar{x}_i}{2}\right). \tag{4.2}$$

Observe that, from conditions (iii) and (iv), for any $i \in \{3, \dots, n-3\}$ such that $\bar{x}_i = \bar{x}_{i+1}$ we have

$$2\mu \leq \bar{x}_{i+1} - \bar{x}_{i-1} = \bar{x}_i - \bar{x}_{i-1} \leq 2\mu,$$

which implies, in particular, that $\bar{x}_i - \bar{x}_{i-1} = 2\mu$. Similarly, we get that $\bar{x}_{i+2} - \bar{x}_i = 2\mu$ for any $i \in \{3, \dots, n-3\}$ such that $\bar{x}_i = \bar{x}_{i+1}$. So, equality (4.2) becomes

$$\rho_i(\bar{x}) = \rho_{i+1}(\bar{x}) = \mu.$$

As a consequence, the payoff of any coupled firm $i \in N$ is

$$\rho_i(\bar{x}) = \mu. \tag{4.3}$$

Finally, we show that no firm can benefit by moving from its position. From (i) we know that the distance between the extremes of the interval and the rightmost and leftmost firms is μ. Furthermore, from (ii) and (iii), we know that the distance between any two firms is no more than 2μ. So, if any firm i moves to any unoccupied location it will get a payoff at most μ, which is not an improvement according to (4.1) and (4.3).

On the other hand, according to equation (4.3), if seller i moves to a location where there is another firm, the payoff will be μ, which neither is an improvement. In addition, since we have seen that only two firms can be at the exact same location, there are no other options for firm i to change its position.

Hence, we have proved that if conditions (i) to (iv) are verified, none of the firms can gain by moving from $(\bar{x}_1, \dots, \bar{x}_n)$. As a consequence, the strategy profile $\bar{x}$ is a Nash equilibrium. □

With all the information above, we analyse the existence of a Nash equilibrium in the game based on the number of players.

Theorem 4.1. *Consider the game $\mathscr{L}(n, [0,1])$.*

(a) For $n = 2, 4, 5$, there exists a unique (modulo permutation of players) Nash equilibrium.

(b) For $n = 3$, there is no Nash equilibrium.

(c) For $n \geq 6$, there is an infinite number of Nash equilibria.

Proof. (a) It follows from conditions (i) and (ii) of Proposition 4.1. When $n = 2$, the only equilibrium is achieved when both sellers are located in the middle of the segment. That is $(x_1^*, x_2^*) = (1/2, 1/2)$. Figure 4.2 illustrates this situation.

If $n = 4$, then $x_1^* = 1/4$ is the only value that satisfies conditions (i) and (ii) of Proposition 4.1. Therefore, the strategy profile $(1/4, 1/4, 3/4, 3/4)$ is the unique Nash equilibrium.

Finally, consider $n = 5$. Similar to the case above, $x_1^* = 1/6$ is the only option which validates the required conditions. Consequently, $(1/6, 1/6, 1/2, 5/6, 5/6)$ is the unique Nash equilibrium.

(b) When $n = 3$, the only way to satisfy Lemma 4.1(d) is if all the firms are in the same location. But this contradicts Lemma 4.1(a). As a consequence, it is impossible to reach the equilibrium when there are three firms in the market.

(c) When $n \geq 6$ there are infinite options that satisfy conditions (i) to (iv). Hence, there is an infinite number of Nash equilibria. Let us construct equilibria to illustrate this.

We consider a strategy profile $x^* = (x_1^*, \ldots, x_n^*) \in [0,1]^n$ which verifies conditions (i) and (ii) of Proposition 4.1. That is, x^* is such that $x_1^* = x_2^* = 1 - x_{n-1}^* = 1 - x_n^*$ and $x_3^* - x_2^* = x_{n-1}^* - x_{n-2}^* = 2x_1^*$. Let $x_1^* = \xi > 0$. We place the remaining sellers individually such that for all $i \in \{3, \ldots, n-3\}$, $x_{i+1}^* - x_i^* = \mu$. The following figure represents this scenario:

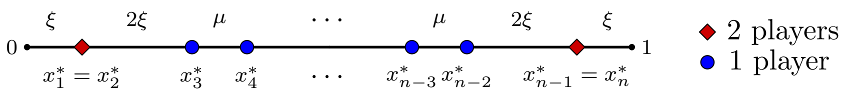

Figure 4.1: Example of Nash equilibrium with n players.

Consequently, we have

$$6\xi + (n-5)\mu = 1. \tag{4.4}$$

The strategy profile x^* is a Nash equilibrium if and only if conditions of Proposition 4.1 are satisfied. By construction, we know that (i) and (ii) are reached. From condition (iii), it is necessary that $\mu \leq 2\xi$. On the other hand, from condition (iv), $\mu \geq \xi$. Hence, we have

$$\xi \leq \mu \leq 2\xi. \tag{4.5}$$

Therefore, by (4.4) and (4.5), we get

$$\frac{1}{2n-4} \leq \xi \leq \frac{1}{n+1}.$$

Then, for every ξ satisfying inequalities above, the strategy profile x^* is a Nash equilibrium.

□

Let us represent some examples of Nash equilibria of $\mathscr{L}(n, [0, 1])$.

If $n = 2$, as we have explained above, we have

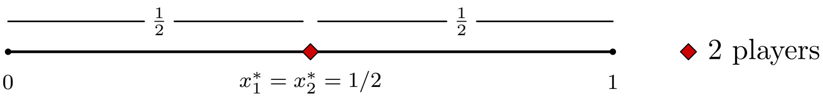

Figure 4.2: Unique Nash equilibrium with 2 players.

Note that, the way to construct Nash equilibria for $n \geq 6$ explained in the proof of Theorem 4.1(c) does not consider all the possible equilibria, since all players are alone in their location, apart from the sellers located closer to the extremes. Below, we study all the Nash equilibria in the game $\mathscr{L}(n, [0, 1])$ for $n = 6$.

Consider a strategy profile $x^* = (x_1^*, \ldots, x_6^*)$ such that satisfies the conditions of Preposition 4.1. Take $\xi > 0$, such that $x_1^* = \xi$. From conditions (i) and (ii), we have $x_1^* = x_2^* = 1 - x_5^* = 1 - x_6^* = \xi$, and $x_3^* - x_2^* = x_5^* - x_4^* = 2\xi$. By μ, we denote the distance between x_3^* and x_4^*. Figure 4.3 represents this situation:

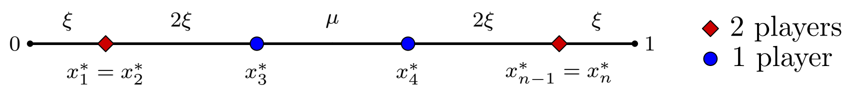

Figure 4.3: Nash equilibria with 6 players.

From condition (iii) we know that $\mu \leq 2\xi$, and from (iv), $\mu \geq 0$. That is,

$$0 \leq \mu \leq 2\xi.$$

Since $6\xi + \mu = 1$, from inequalities above we have

$$1/8 \leq \xi \leq 1/6.$$

Consequently, the strategy profile $(\xi, \xi, 3\xi, 1 - 3\xi, 1 - \xi, 1 - \xi)$ for every $\xi \in [1/8, 1/6]$ is a Nash equilibrium of $\mathscr{L}(n, [0, 1])$ when $n = 6$.

Note that, when $\xi = 1/6$, firms 3 and 4 are placed in the exact same location. On the other hand, when $\xi = 1/8$, players 3 and 4 are as far apart as possible.

4.2 Existence of Equilibrium on a Star

Below, we study the existence of Nash equilibria in location games on a star when a finite number of firms compete only on location.

We assume that the market is represented by a star network. Let S_k denote the star, where $k > 2$ is its number of edges. That is, S_k is a network with $k + 1$ vertices

$\{v_0, v_1, \ldots, v_k\}$, where for $j \in \{1, \ldots, k\}$, every vertex v_j is connected to the central vertex, v_0, and to no other vertex. We assume that, for all j, the length of edges $e_{v_0 v_j}$ is equal to 1. Figure below represents an example of a 6-star:

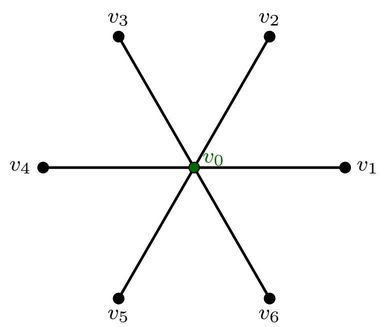

Figure 4.4: Star network S_6.

We consider that a finite number of firms simultaneously choose a location where to operate on the star — where there are numberless consumers uniformly distributed. Each firm's goal is to maximize their profits, and their payoff is equal to the mass of consumers they attract. All firms supply a homogeneous product charged at the same price. Hence, consumers will buy from the closest shop in order to minimize their transport costs. If two or more sellers are in the exact same location, then consumers meant to purchase at that location are split equally between sellers.

We denote by $N = \{1, \ldots, n\}$, with $n \geq 2$, the set of sellers. Each player chooses a location in the star S_k, so the strategy space is $S = S_k^n$. A strategy profile is denoted by $x = (x_1, \ldots, x_n) \in S$. In order to simplify the model, since it does not matter which firm is in what location, we order the players as follows. For each edge, that is, for $j \in \{1, \ldots, k\}$ define

$$N^j = \big\{ i \in N : x_i \in e_{v_0 v_j} \backslash \{v_0\} \big\}$$

and call $h(j)$ the cardinality of N^j. Then, on each edge $e_{v_0 v_j} \backslash \{v_0\}$, we suppose that players are ordered based on their position from v_0, from the closest to the furthest. That is, $v_0 < x_{j,1} \leq x_{j,2} \leq \cdots \leq x_{j,h(j)} \leq v_j$, where $x_{j,1}$ denotes the location of the player closest to v_0 in the edge $e_{v_0 v_j}$, and so on until $x_{j,h(j)}$, which represents the location of the furthest player. Therefore, if a firm i is not located on v_0 it can be referred as player (j, m), where $j \in \{1, \ldots, k\}$ indicates the edge where it is located and $m \in \{1, \ldots, h(j)\}$ its position on this edge.

Given a strategy profile $x = (x_1, \ldots, x_n) \in S$, in order to define the payoff of each firm, first, we define, for $j \in \{1, \ldots, k\}$

$$p_j(x) = \begin{cases} \dfrac{1}{2} d(v_0, x_{j,1}) & \text{if there is at least one player on } e_{v_0 v_j}, \\ 1 & \text{otherwise.} \end{cases}$$

The payoff of any firm depends on whether it is located in the central vertex or not, as well as on whether — according to the strategy profile x — there is a player located in v_0 or not. So, we distinguish between the following cases:

(a) The strategy profile $x = (x_1, \ldots, x_n) \in S$ is such that there is at least one player in v_0. Then,

(i) Payoff of players $i \in N$ such that $x_i = v_0$ is

$$\rho_i(x) = \frac{1}{card\{l : x_l = v_0\}} \sum_{j=1}^{k} p_j(x).$$

(ii) For $j \in \{1, \ldots, k\}$ such that $p_j(x) \neq 1$, we define two lengths as follows:

$$\overleftarrow{p}_{j,m}(x) = \frac{1}{2} d(x_{j,m-1}, x_{j,m}) \quad \text{for } m \in \{1, \ldots, h(j)\},$$

where $x_{j,0} := v_0$ for all $j \in \{1, \ldots, k\}$, and

$$\overrightarrow{p}_{j,m}(x) = \begin{cases} \frac{1}{2} d(x_{j,m}, x_{j,m+1}) & \text{if } m \in \{1, \ldots, h(j) - 1\}, \\ d(x_{j,h(j)}, v_j) & \text{if } m = h(j). \end{cases}$$

Then, payoff of players located in $e_{v_0 v_j} \backslash \{v_0\}$ is

$$\rho_{j,m}(x) = \frac{1}{card\{k : x_{j,k} = x_{j,m}\}} \sum_{k} \left(\overleftarrow{p}_{j,k}(x) + \overrightarrow{p}_{j,k}(x) \right).$$

(b) The strategy profile $x = (x_1, \ldots, x_n) \in S$ is such that there is no player in v_0. Then, the player closest to v_0 may attract some consumers from other edges. For the purpose of our study, it is not necessary to specify the payoff of the firms in this situation.

The information above defines a game called *location game* on S_k. We denote it by $\mathscr{L}(n, S_k)$. Recall that a strategy profile $x^* \in S$ is a *Nash equilibrium* of the game $\mathscr{L}(n, S_k)$ if for all $i \in N$ and for all $x_i \in S_k$ we have

$$\rho_i(x^*) \geq \rho_i(x_i, x^*_{-i})$$

where (x_i, x^*_{-i}) denotes $(x^*_1, \ldots, x^*_{i-1}, x_i, x^*_{i+1}, \ldots, x^*_n)$.

Below, we describe some properties which are verified for any Nash equilibrium.

Lemma 4.2. *Let $x^* = (x^*_1, \ldots, x^*_n) \in S$ be a Nash equilibrium of $\mathscr{L}(n, S_k)$. Then,*

(a) There exists $i \in N$ such that $x^*_i = v_0$.

(b) There are no players on the extremes of the edges, i.e., for all $i \in N$ and for all $j \in \{1, \ldots, k\}$, $x^*_i \neq v_j$.

(c) For every $y \in S_k \backslash \{v_0\}$

$$card\{i : x^*_i = y\} \leq 2.$$

(d) $card\{i : x^*_i = v_0\} \leq k$.

(e) If for some $i \in N$, we have $x^*_i \in S_k \backslash \{v_0\}$, then for each $j \in \{1, \ldots, k\}$, there are at least two players $(j, h(j) - 1), (j, h(j)) \in N^j$ such that $x^*_{j,h(j)-1} = x^*_{j,h(j)}$.

Proof. Suppose that $x^* = (x^*_1, \ldots, x^*_n) \in S$ is a Nash equilibrium of $\mathscr{L}(n, S_k)$.

(a) Suppose, by contradiction, that $x_i^* \neq v_0$ for all $i \in N$. If there is any edge with no firms, the seller located furthest from the center will improve his or her payoff by moving to v_0. Moreover, if there are edges with just one player, the firms that are alone on their respective edges benefit from moving closer to the center. So, we assume that x^* is such that there are at least two firms on each edge. Note that, if $d(v_0, x_{j,1}^*)$ is the same amount for all $j \in \{1, \ldots, k\}$, then for any j, the firm $(j, 1)$ benefits from moving towards v_0, since it will gain customers from all other edges. Thus, suppose now, there is at least one player located nearest to the center than the others. Take $(\hat{j}, 1)$ one of the firms closest to v_0. Observe that player $(\hat{j}, 1)$ attracts customers from all edges such that $d(x_{\hat{j},1}) < d(x_{j,1})$ for all $j \in \{1, \ldots, k\}$. In addition, if firm $(\hat{j}, 1)$ moves by ε to v_0, it will gain $\varepsilon/2$ from all $e_{v_0 v_j}$ such that $j \neq \hat{j}$. Therefore, if firm $(\hat{j}, 1)$ moves towards the center, its payoff increases, since the amount it loses from consumers in its own edge is less than the gains from consumers coming from all other edges. In any case, x^* cannot be a Nash equilibrium, and therefore, there must be at least on player located in v_0.

(b) The argument is similar to the proof of Lemma 4.1(a).

(c) The argument is similar to the proof of Lemma 4.1(b).

(d) Assume $card\{i : x_i^* = v_0\} = k' > k$. Then, each firm in v_0 gets a payoff of

$$\frac{1}{k'} \sum_{j=1}^{k} p_j(x^*) \leq \frac{k}{k'} \max_{j \in \{1,\ldots,k\}} p_j(x^*) < \max_{j \in \{1,\ldots,k\}} p_j(x^*)$$

Therefore, for $\varepsilon > 0$ small enough, if a firm located in v_0 moves by ε to the edge $e_{v_0 v_{\hat{j}}}$, with $\hat{j}$ such that $\max_{j \in \{1,\ldots,k\}} p_j(x^*) = p_{\hat{j}}(x^*)$, it will get a higher payoff. That means that x^* is not an equilibrium, and so $k' \leq k$.

(e) Suppose that there exists $i \in N$ such that $x_i^* \in e_{v_0 v_j} \backslash \{v_0\}$. We show that there are at least two coupled firms on each edge. First, we observe that seller i cannot be alone on $e_{v_0 v_j} \backslash \{v_0\}$. By contradiction, suppose that i is the only firm on its edge. Its payoff is

$$\frac{1}{2} d(v_0, x_{j,1}^*) + d(x_{j,1}^*, v_j).$$

Hence, player i would get a higher payoff moving towards the center. This contradicts that x^* is a Nash equilibrium and, therefore, there must be at least 2 players on $e_{v_0 v_j} \backslash \{v_0\}$. Furthermore, note that there are no empty edges — if this were the case, for any player located on an edge, it would be worthwhile to move to the empty edge, close enough to v_0.

On the other hand, using a similar argument to that used to prove Lemma 4.1(d), we know that all players who are the furthest from the center, on each edge, are in pairs.

□

Now, we determine the payoff of all sellers that share location with at least another firm under a strategy profile. First, we focus on players located on an edge and then, on players placed in the center. In particular, we will see that all the firms that are not alone get the same payoff. We denote by ξ this amount.

Lemma 4.3. *Let x^* be a Nash equilibrium of $\mathscr{L}(n, S_k)$, and let $y \in e_{v_0 v_j} \backslash \{v_0\}$ be such that*

$$card\{i : x_i^* = y\} = 2.$$

Let (j, m) and $(j, m+1)$ be the two players in y. We have

(a) if $h(j) > m+1$, then

$$\rho_{j,m}(x^*) = \rho_{j,m+1}(x^*) = \frac{1}{2} d(x^*_{j,m-1}, x^*_{j,m}) = \frac{1}{2} d(x^*_{j,m+1}, x^*_{j,m+2}) = \xi(y).$$

(b) if $h(j) = m+1$, then

$$\rho_{j,m}(x^*) = \rho_{j,m+1}(x^*) = \frac{1}{2} d(x^*_{j,m-1}, x^*_{j,m}) = d(x^*_{j,m+1}, v_j) = \xi(y).$$

(c) The value $\xi(y)$ does not depend on y (hence, we simply denote it ξ).

Proof. It follows by a similar argument to that used in the proof of Lemma 4.1(c). □

Lemma 4.4. *Let x^* be a Nash equilibrium of $\mathscr{L}(n, S_k)$. If $card\{i : x_i^* = v_0\} = k$, then*

$$\frac{1}{k} \sum_{j=1}^{k} p_j(x^*) = \xi.$$

Proof. Consider a Nash equilibrium x^* such that $card\{i : x_i^* = v_0\} = k$. Then, each of the players located in the center gains $(\sum_{j=1}^{k} p_j(x^*))/k$.

Notice that $p_j(x^*)$ is the same amount for all $j \in \{1, \ldots, k\}$. If this was not the case, we would have that

$$\frac{1}{k} \sum_{j=1}^{k} p_j(x^*) < \max_{j \in \{1,\ldots,k\}} p_j(x^*).$$

Let $\hat{j}$ be such that $\max_{j \in \{1,\ldots,k\}} p_j(x^*) = p_{\hat{j}}(x^*)$. Then, one of the firms located in v_0 would improve its payoff by moving towards $v_{\hat{j}}$ by $\varepsilon > 0$ small enough.

Suppose, now, that there exists a location y that satisfies conditions from Lemma 4.3, and so players in y gain ξ. We will show that players in v_0 also gain ξ. That is, that

$$\frac{1}{k} \sum_{j=1}^{k} p_j(x^*) = \xi.$$

Suppose, by contradiction, that $\xi > (\sum_{j=1}^{k} p_j(x^*))/k$. Then, for ε small enough, any firm located in v_0 could increase its payoff by moving to $y - \varepsilon$ since it would gain ξ. On the other hand, suppose that $\xi < (\sum_{j=1}^{k} p_j(x^*))/k$, which leads to $\xi < \max_{j \in \{1,\ldots,k\}} p_j(x^*)$. Let $\hat{j}$ be such that $\max_{j \in \{1,\ldots,k\}} p_j(x^*) = p_{\hat{j}}(x^*)$. Then any player located in y could improve by moving to $e_{v_0 v_{\hat{j}}}$ closer to v_0. Both cases contradicts that x^* is a Nash equilibrium, so payoff of players in the center must be equal to ξ. □

Below, we study the existence of Nash equilibria in $\mathscr{L}(n, S_k)$ depending on the number of players in the game.

Theorem 4.2. *Consider a location game $\mathscr{L}(n, S_k)$.*

(a) If $2 \leq n \leq k$, a unique equilibrium x^* exists where $x_i^* = v_0$ for all $i \in N$.

(b) If $k < n < 3k-1$, there is no Nash equilibrium.

(c) If $3k-1 \leq n \leq 3k$, there exists a unique equilibrium.

(d) If $3k+1 \leq n$, there exists an infinite number of equilibria.

Proof. (a) Suppose $2 \leq n \leq k$. When all players are located on v_0, each seller gets a payoff equal to $k/n \geq 1$. On the other hand, if any firm i, located on v_0, moved to any point $x \in e_{v_0 v_j} \backslash \{v_0\}$ its payoff would be

$$\frac{1}{2} d(v_0, x_{j,1}) + d(x_{j,1}, v_j),$$

which is strictly less than 1 because of $d(v_0, v_j) = 1$. Hence, a strategy profile $x^* \in S_k$ such that $x_i^* = v_0$ for all $i \in N$ is a Nash equilibrium of $\mathscr{L}(n, S_k)$. Now, we prove its uniqueness. Assume, by contradiction, that there exists an equilibrium such that there is at least one $i \in N$ such that $x_i^* \in e_{v_0 v_j} \backslash \{v_0\}$. Consequently, by Lemma 4.2(d), there must be at least 2 players on each edge, and so there should be at least $n \geq 2k$ players, which contradicts $n \leq k$.

(b) Assume $k < n < 3k-1$, we show that it is not possible to place sellers in an equilibrium strategy profile. Suppose, by contradiction, that $x^* \in S$ is a Nash equilibrium. By Lemma 4.2(a), there exists at least one player $i \in N$ such that $x_i^* = v_0$. In addition, by Lemma 4.2(c), at most k players can be in v_0. Since $k < n$, then there must be at least one seller on an edge. From Lemma 4.2(d) it follows that there are at least two coupled firms on each edge. Therefore, if $k < n < 2k+1$, x^* can not be a Nash equilibrium.

Thus, suppose now that $2k+1 \leq n < 3k+1$ and that there are 2 coupled sellers on each edge. Observe that the remaining $n-2k$ players must be located in v_0. If this was not true, we would have more than 2 firms in at least one edge, and so, according to Lemma 4.3,

$$1 = d(v_0, v_j) = d(v_0, x_{j,1}^*) + \cdots + d(x_{j,h(j)-3}^*, x_{j,h(j)-2}^*) + 3\xi. \tag{4.6}$$

However, since there are at most $k-2$ remaining players, we have that there is at least one edge with only 2 coupled players. Consequently,

$$1 = d(v_0, x_{j,1}^*) + d(x_{j,2}^*, v_j) = 3\xi$$

which contradicts (4.6).

Then, each firm in v_0 would get a payoff of

$$\frac{1}{n-2k} \sum_{j=1}^{k} \frac{1}{2} d(v_0, x_{j,1}^*) = \frac{k\xi}{n-2k}.$$

By Lemma 4.3 we know that coupled players on edges gain ξ. Note that if one firm on an edge moves to v_0 will get a payoff of

$$\frac{k\xi}{n-2k+1},$$

which is larger than ξ as long as $n < 3k - 1$. Therefore, there is no Nash equilibrium if $k < n < 3k - 1$.

(c) If $n \in \{3k - 1, 3k\}$, it is easy to prove that a strategy profile x^* where there are 2 players on each edge, both located at a distance $2/3$ from v_0, and the remaining players located on the central vertex, is a Nash equilibrium. Let us prove that it is the only existing equilibrium. According to Lemma 4.2, we know where $2k + 1$ firms must be located for x^* to be an equilibrium: one on v_0 and two on each edge. Then, there are $k - 2$ remaining players if $n = 3k - 1$, and $k - 1$ if $n = 3k$. Consequently, using the same argument to that of case (b), there is at least one edge with only 2 firms, and so, the only way for x^* to be an equilibrium is that all remaining players are in v_0. Moreover, since $n \geq 3k - 1$ no firm can increase its payoff by changing position. Hence, x^* is a Nash equilibrium.

(d) Suppose $n \geq 3k + 1$. We construct a strategy profile x^* with m players on each edge and r players in the center, assuming that the disposition of players is the same in all k edges. Let $n = ak + b$ be the Euclidean division of n by k. If $b \neq 0$, then $m = a$ and $r = b$. If $b = 0$, $m = a - 1$ and $r = k$. Note that, since $n \geq 3k + 1$, there always will be at least 3 players on each edge. For every $j \in \{1, \ldots, k\}$, we allocate the two furthest players from v_0 at distance ξ from v_j, that is, $d(x_{j,m-1}) = d(x_{j,m}, v_j) = \xi$. We place the remaining sellers individually such that the distance between two consecutive players on edges is 2ξ. Assume $\mu = d(v_0, x_{j,1})$. Consequently, we have $(2m - 3)\xi + \mu = 1$. The following figure clarifies this scenario:

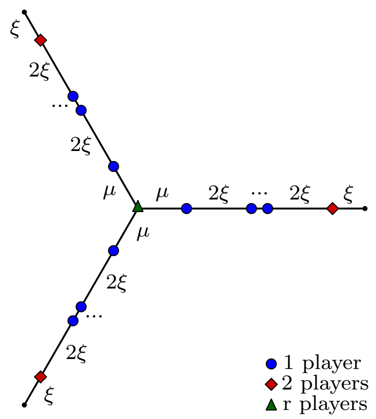

Figure 4.5: Strategy profile x^* on S_3.

Let us determine the payoff of players in each possible position:

- For all $i \in N$ such that $x_i^* = v_0$,

$$\rho_i(x^*) = \frac{1}{r}\sum_{j=1}^{k} p_j(x) = \frac{1}{r}\sum_{j=1}^{k} \frac{d(v_0, x_{j,1})}{2} = \frac{k\mu}{2r}.$$

- For $i \in N$ such that $x_i^* \neq v_0$,

$$\rho_i(x^*) = \frac{\mu}{2} + \xi, \qquad \rho_i(x^*) = 2\xi, \qquad \rho_i(x^*) = \xi,$$

depending on whether x_i^* is such that $d(v_0, x_i^*) = \mu$, $d(x_{i-1}^*, x_i^*) = d(x_i^*, x_{i+1}^*) = 2\xi$ or $d(x_i^*, v_j) = \xi$, respectively.

The strategy profile x^* is a Nash equilibrium if no player can get a higher payoff by moving from its location. That is, x^* is a Nash equilibrium if an only if the following conditions are satisfied:

(i) No player has an incentive to move to an unoccupied location. If a firm moved from its position, its payoff would be $\mu/2$, ξ or $\xi - \varepsilon/2$ with $\varepsilon \in (0, \xi)$, depending on whether the length of the interval is μ, 2ξ or ξ, respectively. Thus, for all $i \in N$, $\rho_i(x^*)$ must be greater than or equal to these amounts. Therefore, it is necessary that
$$\frac{k\mu}{2r} \geq \xi \qquad \text{and} \qquad \mu \leq 2\xi.$$

(ii) No player is incentivized to move to v_0. By construction, we know that $\rho_i(x^*) \geq \xi$ for all $i \in N$ such that $x_i^* \neq v_0$. On the other hand, $\rho_i(x^*) = k\mu/2r$ for all $i \in N$ such that $x_i^* = v_0$. Then, it is required that
$$\frac{k\mu}{2(r+1)} \leq \xi.$$

(iii) No player can gain by moving to an occupied location on the edges. If a firm occupies a location where there is at least another firm, its payoff will be $\mu/4 + \xi/2$, ξ or $2\xi/3$ (depending on the chosen location). Thus, if inequalities of (i) and (ii) are verified, no player would have an incentive to do so.

From conditions above we have that
$$\frac{k}{2(r+1) + 2km - 3k} \leq \xi \leq \frac{k}{2r + 2km - 3k}. \tag{4.7}$$

Then, for every ξ satisfying (4.7), the strategy profile x^* is a Nash equilibrium. Hence, the game has an infinite number of Nash equilibria.

□

Let us illustrate some examples of the Nash equilibria explained above. To represent any star S_k, we consider a star with 6 edges.

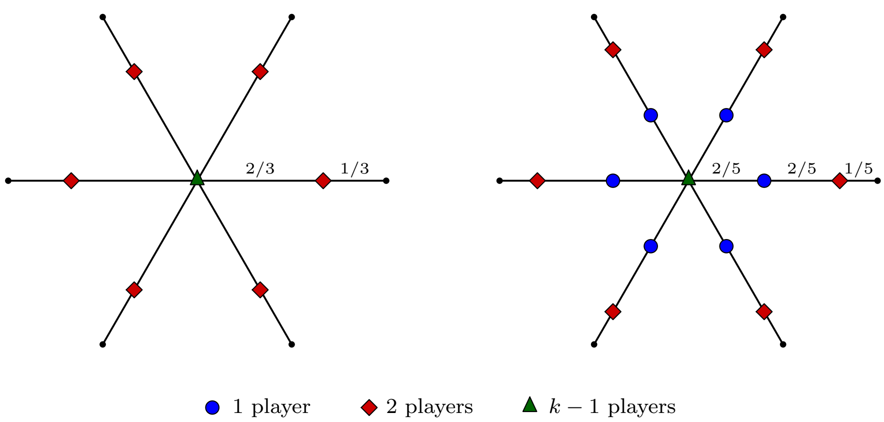

Figure 4.6: Left: equilibrium with $3k-1$ players. Right: equilibrium with $4k-1$ players.

4.3 Existence of Equilibrium on a Circle

In this section, we look into the scenario when firms do not compete on prices but only on location in a circle. Particularly, we analyze the existence of Nash equilibria when a finite number of sellers choose a location in a unit circle.

Consider a circular market of length 1, $\mathscr{C} = [0,1)$, where there are a finite number of players that simultaneously choose a location for their store. All sellers supply a homogeneous product which is charged at the same mill price. Let $N = \{1,\dots,n\}$, with $n \geq 2$, be the set of players and $x = (x_1,\dots,x_n) \in \mathscr{C}^n$ the strategy they play.

We assume that consumers are uniformly distributed along the unit circle, and we suppose that each customer buys one unit of the product from the closest firm. If two or more sellers are in the exact same location, then consumers meant to purchase in that location are split equally between sellers. The payoff of each firm is the number of consumers it attracts.

Given a strategy profile $x = (x_1,\dots,x_n) \in \mathscr{C}^n$, we suppose that player 1 is the seller nearest to 0 (counterclockwise), and seller n the one closest to 1 (clockwise). Recall that location 0 and 1 overlap. In other words, $0 \leq x_1 \leq \cdots \leq x_n < 1$. Then, for each player $i \in N$, we define two lengths $\overleftarrow{p}_i(x)$ and $\overrightarrow{p}_i(x)$ as follows:

$$\overleftarrow{p}_i(x) = \frac{1}{2}d(x_{i-1},x_i) \quad \text{and} \quad \overrightarrow{p}_i(x) = \frac{1}{2}d(x_i,x_{i+1}),$$

where $x_0 := x_n$ and $x_{n+1} := x_1$.

As a consequence, given a strategy profile $x = (x_1,\dots,x_n) \in \mathscr{C}^n$, the payoff of a player $i \in N$ is

$$\rho_i(x) = \frac{1}{card\{k:\, x_k = x_i\}} \sum_{k:\, x_k = x_i} \left(\overleftarrow{p}_k(x) + \overrightarrow{p}_k(x)\right).$$

The above information defines a *location game* on a unit circle. Let us denote it by $\mathscr{L}(n,\mathscr{C})$.

Recall that a strategy profile $x^* \in \mathscr{C}^n$ is a *Nash equilibrium* of the game $\mathscr{L}(n,\mathscr{C})$ if for all $i \in N$ and for all $x_i \in \mathscr{C}$ we have

$$\rho_i(x^*) \geq \rho_i(x_i, x^*_{-i})$$

where (x_i, x^*_{-i}) denotes $(x^*_1,\dots,x^*_{i-1},x_i,x^*_{i+1},\dots,x^*_n)$.

Note that location games on a circle do not differ too much from location games on a segment. The main difference is that on the circle there are no endpoints. Thus, doing a similar analysis to that of the previous sections, we know that some of the conditions which are needed for a strategy profile to be a Nash equilibrium are also true. For example, at most 2 firms can be in the exact same location, and all coupled firms of the game get the same payoff.

Nevertheless, in this section, we focus on demonstrate that for each $n \geq 2$, there always is an equilibrium in the game $\mathscr{L}(n,\mathscr{C})$. To do so, we first construct two different strategy profiles and then we prove that they are indeed a Nash equilibrium.

Below, we define possible strategy profiles of the game $\mathscr{L}(n,\mathscr{C})$ as follows:

(i) Let $\bar{x}$ be the strategy profile such that all firms are equidistant from one another. That is,

$$\bar{x}_i = \frac{i}{n}, \quad \text{for all } i \in N.$$

(ii) Let $\hat{x}$ be the strategy profile such that the distance between two consecutive locations is always the same. Whenever possible, players are placed in pairs. So, if n is even, all players are coupled. If n is odd, all players are in pairs except one. That is,

$$\hat{x}_{2i-1} = \hat{x}_{2i} = \frac{2i}{n}, \quad i \in \{1, \dots, \lfloor n/2 \rfloor\};^{1} \qquad \hat{x}_n = 1, \quad \text{if } \lfloor n/2 \rfloor \neq n/2.$$

Lemma 4.5. *The strategy profiles $\bar{x}$ and $\hat{x}$ are equilibria of $\mathscr{L}(n, \mathscr{C})$.*

Proof. Let us consider each strategy profile separately.

(i) Under the strategy profile $\bar{x}$, firms are placed individually such that $d(\bar{x}_i, \bar{x}_{i+1}) = 1/n$ for all $i \in N$. So, each player $i \in N$ would get a payoff of

$$\rho_i(\bar{x}) = \frac{1}{2} d(\bar{x}_{i-1}, \bar{x}_i) + \frac{1}{2} d(\bar{x}_i, \bar{x}_{i+1}) = \frac{1}{n}.$$

Observe that if a firm i deviates to any unoccupied location, it would get a payoff of $1/2n$. As a consequence, no movement to any interval between two players is profitable. Furthermore, if a firm i moved to a location where there is another player, its payoff would be either $3/4n$ or $1/2n$, depending on whether this player is or is not a consecutive player. In either case, the payoff is smaller than $1/n$, and so it neither is beneficial. Thus, $\bar{x}$ is a Nash equilibrium.

(ii) Under the strategy profile $\hat{x}$, we should differentiate two different cases: n even and n odd.

When n is even, in all occupied location there are 2 players. So, distance between two consecutive locations is $2/n$. Therefore, each firm $i \in N$ gets

$$\rho_i(\hat{x}) = \frac{1}{n}.$$

If a firm i moves to an unoccupied location it will get a payoff at most $1/n$, which is not an improvement. On the other hand, if a firm i moves to an occupied location, its payoff will become $2/3n$, which neither is profitable. As a result, $\hat{x}$ is a Nash equilibrium for n even.

On the other hand, if n id odd, all players are coupled except one. We assume $i = n$ is the player who is alone. The distance between two consecutive locations is $2/(n+1)$. Hence,

$$\rho_i(\hat{x}) = \frac{1}{n+1} \quad \text{for } i \in \{1, \dots, n-1\},$$
$$\rho_n(\hat{x}) = \frac{2}{n+1}.$$

[1]The floor function $\lfloor \cdot \rfloor \colon \mathbb{R} \to \mathbb{Z}$ defined by $\lfloor x \rfloor = \max\{m \in \mathbb{Z} \mid m \leq x\}$ gives the largest integer less than or equal to x.

It is obvious that the single firm would not benefit from changing its location. As per the coupled sellers, if one moves to an unoccupied location or joins the single seller, it will get a payoff of $1/(n+1)$, which is not an improvement. Finally, if a coupled seller decides to occupy a location where there is already a pair of players, the payoff will be $2/3(n+1)$, which is not profitable. As a consequence, $\hat{x}$ is a Nash equilibrium.

□

Finally, with the information above, we are able to prove that for $n \geq 2$ the game $\mathscr{L}(n, \mathscr{C})$ has always a Nash equilibrium.

Proposition 4.2. *For every $n \geq 2$, the set of equilibria of the game $\mathscr{L}(n, \mathscr{C})$ is nonempty.*

Proof. Since for any $n \geq 2$ players can be located according to one of the strategy profiles defined above, it is a direct consequence of Lemma 4.5. □

The following figure illustrates an example of Nash equilibrium for each of the strategy profiles explained above. For instance, we consider the game $\mathscr{L}(6, \mathscr{C})$.

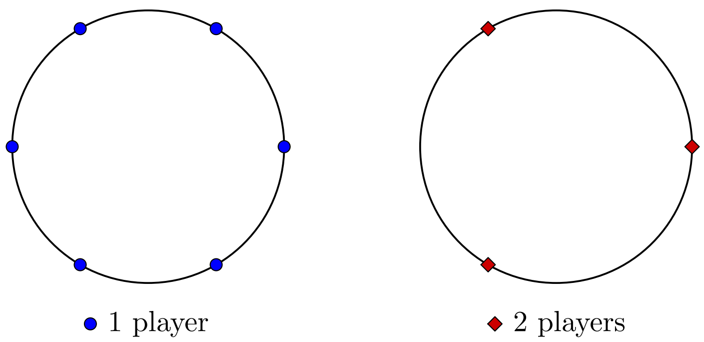

Figure 4.7: Left: equilibrium $\bar{x}$ with 6 players. Right: equilibrium $\hat{x}$ with 6 players.

Chapter 5

Efficiency of Nash Equilibria

Nash equilibria in non-cooperative games do not always represent the most desirable outcomes for all economic agents involved, due to the fact that each agent plays in a selfish way. When this happens, we say that Nash equilibria are not efficient.

Let us review some examples of inefficient Nash equilibria.

To begin with, recall the Prisoners' Dilemma described in Example 1.1. As we mentioned in Example 1.2, the outcome of the game's Nash equilibrium is worse for each player (each prisoner spends more years in jail) than it would have been if they had made a non-selfish decision. Hence, the unique Nash equilibrium in the Prisoners' Dilemma game is inefficient since there is another outcome that is preferred by all players.

Regarding the location games studied in previous chapters, recall that firms' goal is to maximize profits, whereas consumers want to minimize travelled distances. Note that any strategy profile produces the same total payoff for sellers. Therefore, from sellers' point of view, Nash equilibria in location games are efficient. However, these Nash equilibria may not be the best outcome for consumers, since there could be some strategy profiles that decrease travel costs. From consumers' point of view, therefore, Nash equilibria are not always desirable.

This inefficiency of Nash equilibria in location games may explain why governments regulate the location of some business, such as pharmacies. For example, in Barcelona, the location of a new drugstore must be at a distance of at least 250 meters from the nearest existing one.

Let us study some particular examples. First, consider the location game on a segment with 2 firms. As we have seen in Theorem 4.1, the game admits a unique Nash equilibrium which is reached when both firms are located in the middle of the segment. That is, if x_1 and x_2 represents the location of firm 1 and firm 2 in the unit segment respectively, then the strategy profile $(x_1, x_2) = (1/2, 1/2)$ is the Nash equilibrium. Consequently, each firm gets half of the market. Observe that firms would get the same payoff if they were located such that $(x_1, x_2) = (1/4, 3/4)$. If this was the case, total consumers' travelling costs — called social costs — would be lower. Therefore, Nash equilibrium is inefficient from consumers' point of view.

More specifically, when both players are located in the middle of the segment, the total social cost is $1/4$. Instead, when players are located such that $(x_1, x_2) = (1/4, 3/4)$, the total social cost is $1/8$. So, as we stated before, the second strategy profile is more desirable for costumers, although it is not a Nash equilibrium.

The following figure illustrates the two strategy profiles mentioned above. The total area highlighted by shaded triangles represents the total social cost for each situation.

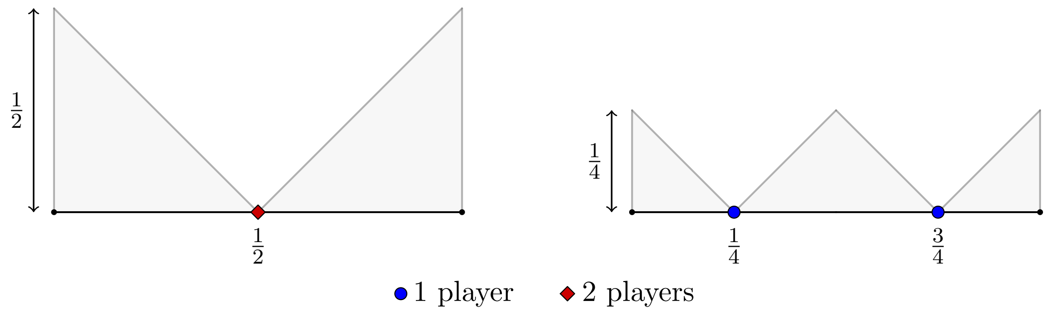

Figure 5.1: Left: $(x_1, x_2) = (1/2, 1/2)$. Right $(x_1, x_2) = (1/4, 3/4)$.

Recall now, the location game on a segment with 6 players. By Theorem 4.1, we know that the game has infinite Nash equilibria. In addition, at the end of Section 4.1, we have seen that the strategy profile

$$x = (\xi, \xi, 3\xi, 1 - 3\xi, 1 - \xi, 1 - \xi)$$

for $\xi \in [1/8, 1/6]$ determines all possible equilibria in $\mathscr{L}(6, [0, 1])$. From sellers' perspective, all possibilities are equally profitable. However, this does not apply to consumers. Let us study which is the best option for them.

First of all, we represent any strategy profile x throughout the following figure. We symbolise the social cost using shaded triangles.

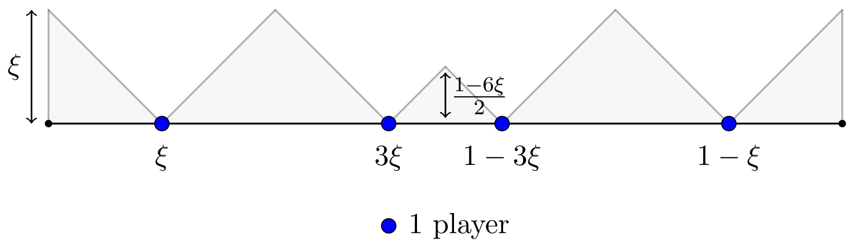

Figure 5.2: $x = (\xi, \xi, 3\xi, 1 - 3\xi, 1 - \xi, 1 - \xi)$.

Observe that, when $\xi = 1/8$, the triangle in the centre is sized as the triangles on its side. In addition, it disappears when $\xi = 1/6$, since $3\xi = 1 - 3\xi$. In any case, the total social cost is given by

$$C(\xi) = 12\xi^2 - 3\xi + 1/4$$

for all $\xi \in [1/8, 1/6]$. Note that C has a global minimum when $\xi = 1/8$. Hence, from consumers' point of view, $(1/8, 1/8, 3/8, 5/8, 7/8, 7/8)$ is the best equilibrium. On the other hand, the strategy profile $(1/6, 1/6, 1/2, 1/2, 7/6, 7/6)$ is their worst equilibrium.

Rather than wanting to reduce travel costs, there are stuations in which players aim to reduce travel time. It is interesting to study the agents' behaviour in these games when they operate freely. Again, due to individuals acting selfishly when deciding their travel plans, the outcome of the game's Nash equilibria may not be the optimal for the entire set of consumers. This behaviour is explained by the Braess' paradox (see [2]), which illustrates situations in which adding roads to a traffic network can increase travel times. Braess' paradox has applications in several fields, such as electricity or telecommunication networks.

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