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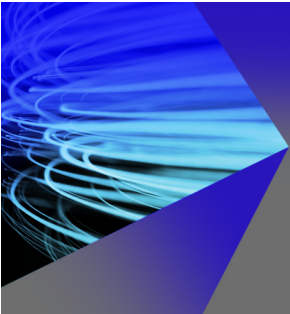
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Cite as: Chaos 33, 093121 (2023); doi: 10.1063/5.0152674

Submitted: 31 March 2023 · Accepted: 21 August 2023 ·

Published Online: 15 September 2023



View Online



Export Citation



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ABSTRACT

We study an agent-based opinion model with two extreme (opposite) opinion states and a neutral intermediate one. We adjust the relative degree of conviction between extremists and neutrals through a dimensionless parameter called the “neutrality parameter” to investigate its impact on the outcome of the system. In our model, agents move randomly on a plane with periodic boundary conditions and interact with each other only when they are within a fixed distance threshold. We examine different movement mechanisms and their interplay with the neutrality parameter. Our results show that in general, mobility promotes the global consensus, especially for extreme opinions. However, it takes significantly less time to reach a consensus on the neutral opinion.

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We propose a modified agent-based three-state opinion model set on a square plane with periodic boundary conditions, where agents can move following distinct strategies. The model consists of N agents with individual velocities v_i , $i = 1, \dots, N$, interacting solely with others within a given distance d . At each time step, agents move in a random direction before updating their opinions through a Glauber Monte Carlo algorithm that favors alignment with neighboring agents. The model includes two extremist opinions and a neutral one, and their relative convictions are tunable using the “neutrality parameter” α . Initially exploring the static scenario, we distribute agents randomly across the plane, resulting in a random geometric graph with a clear community structure, leading to local consensus. However, upon introducing a constant finite velocity v , we observe that global consensus emerges over time, which can be extremist or neutral, depending on α and v values. Interestingly, by linking velocity changes to shifts in neighborhood opinions, the dependence on α and v can be altered, potentially leading to the formation of clusters and local consensus when agents cease movement upon encountering like-minded individuals. While our model simplifies human mobility and opinion dynamics, it provides valuable insights into the intricate interplay of these mechanisms.

I. INTRODUCTION

Sociophysics is an interdisciplinary field that studies the dynamics of social systems using the tools and techniques of statistical physics.¹ In recent years, there has been growing interest in applying sociophysics to a wide range of social phenomena,² from the spread of information and misinformation on social media^{3–5} to the dynamics of political polarization^{6–10} and group decision-making.^{11,12} The goal of this paper is to contribute to this growing body of literature and investigate the interplay between mobility patterns and opinion dynamics by using agent-based simulations.

Discrete opinion models are a class of models used to study the dynamics of opinion formation in society. These models, which consider the opinions of the agents as discrete variables, have been shown to reproduce a rich variety of phenomena, such as the emergence of consensus, polarization, and coexistence of different opinions. A widely used example of discrete opinion models in sociophysics is the voter model. The voter model¹³ is a binary simple model that describes the dynamics of opinion formation through a stochastic process of imitation, where agents adopt the opinion of a randomly chosen neighbor. So far, numerous extensions have been explored, including noise, new update rules for the opinion states, and a wide variety of topologies.^{14,15} The Ising model, on the other hand, is a statistical physics model originally developed to describe

the behavior of magnets. Atomic spins can take two orientations and interact with their nearest neighbors via a potential energy that is a function of the spins' alignment. The Ising model has been adapted to study opinion dynamics by interpreting the spin states as opinions and the coupling between spins as social interactions.^{16–19}

The Blume–Capel^{20,21} model, which is an extension of the Ising model that allows for a third state, has also been applied to the emergence of a neutral state in opinion formation.²² Other three-state opinion models have also been considered in the literature in the past years.^{1,23–26} Opinion models have been also coupled to network evolution in order to characterize the temporal nature of social links.²⁷

The Schelling model²⁸ was originally developed as a formal means to understand the phenomenon of segregation in cities, linking individual behavior with collective emergent behaviors that lead to the formation of ghettos. This model can be mapped onto the Ising model with vacancies,²⁹ and it bears resemblance to the model analyzed in this work when the neutrality parameter α is set to zero. The distinction lies in associating polarized states with people belonging to specific ethnicities, while the neutral state represents the number of vacant houses that allow inhabitants to relocate to other parts of the city.

On the other hand, there is a family of models that consider the opinion space as a continuum spectrum between two extremes.^{30–34} Such models used to have clustered scenarios as the stationary state, so their analysis leads us back to the idea of a discrete opinion space.

Another area of research, in principle unrelated, is the study of collective dynamics of agents moving on a plane. One example of this type of model is the Vicsek model.³⁵ In this model, agents are represented by particles moving on a two-dimensional plane, and the interaction rule is based on alignment. The agents update their direction of motion to align with the average direction of motion of their nearest neighbors. This model has been used to study the emergence of collective motion and flocking behavior in systems of self-propelled agents. Another example is the Cucker–Smale model,³⁶ which is a flocking model that incorporates the effect of attraction and repulsion between agents. The model has been used to study the emergence of flocking behavior in birds, fish, and robots^{37–39} and has served as inspiration for one of the variants we study in this work.

Human mobility patterns have been studied recently, thanks to the access mainly to mobile phone data.^{40,41} It has been found that in some cities, there exist gender gaps; for instance, in Ref. 42, they show that in Santiago de Chile, men tend to visit more diverse places than women, and women tend to spend less time at each location compared to men. Other demographic factors, including age, socio-economic status, and race, can also be linked to heterogeneous mobility patterns,^{39,43} contributing to the persistence of social segregation.

The agent-based model presented in this paper consists of randomly distributed agents on a two-dimensional physical plane. Additionally, each agent has one of three possible opinion states: two extreme states and a neutral one. The agents move on the plane according to a random walk, with their velocities taken as a parameter of the system, and interact only with neighbors placed within a given distance. The agents' opinions are updated in parallel with their movement, using a Markov Chain Monte Carlo algorithm.^{44,45}

The transition probability for an agent to change its opinion depends on the relative number of agents with different opinions in its neighborhood. By coupling the movement of the two-dimensional spatial structure and the opinion changes, we study their interplay and its impact on phenomena such as the formation of the global consensus or the appearance of spatial clusters of agents with similar opinions.

In many real-world social systems, there is a significant proportion of individuals who hold a neutral or centrist opinion, and their behavior can have a crucial impact on the overall dynamics of opinion formation. Our model, already presented in Refs. 46 and 47, aims to emphasize the role of the relative degree of conviction between neutral agents and extremists, by tuning the influence of centrists on the opinion dynamics via a parameter α , called the neutrality parameter.

By varying the parameters of the model, we find that the mobility patterns and the role of neutral agents in the opinion updating rules affect the emergent patterns of opinion formation, leading to non-trivial outcomes. We compare our results with those obtained in Refs. 46 and 47 for static networks, showing that, in general, velocity enhances the appearance of consensus and diminishes the neutral basins of attraction. However, it is shown that the amount of time necessary to reach a consensus is much shorter when it occurs in the neutral state. Our findings have implications for understanding the mechanisms that drive the dynamics of opinion formation in some real-world social phenomena, such as the spread of misinformation on social media, the polarization of political opinions, and the emergence of extremist groups, and will contribute to the ongoing dialog between the fields of sociology and physics.

This paper is organized as follows. In Sec. II, we describe the algorithm used for the system simulations and explain its features, establishing a connection between the different variables and their social interpretation. In Sec. III, we present the main findings for a variety of settings, and we discuss their meaning in a social context. Finally, in Sec. IV, we summarize our results and present the conclusions.

II. THE MODEL

We consider a population of N agents that are placed on a square with side L and periodic boundary conditions (we take $L = 1$, for simplicity). The agents start the simulation scattered randomly on the plane, and each agent i holds one of the three possible opinion states for the model: rightist, leftist, and centrist. We represent each opinion state mathematically by associating a vector as follows:

- $S_i = (1, 0)$; positive opinion/rightist,
- $S_i = (0, \alpha)$; neutral opinion/centrist, and
- $S_i = (-1, 0)$; negative opinion/leftist,

where α is called the neutrality parameter that represents the relative degree of conviction between extremists and neutrals and allows incorporating a certain notion of distance between the opinion states. At each time step, each agent moves in a random direction with velocity v_i and, after moving, the agents update their opinions. The opinion update is carried out by selecting each agent and calculating the sum of the scalar product of their opinion vector with that of all their neighbors. Each agent will adopt the opinion state that

gives the minimum value for this sum and every term calculated as follows:

- $(1, 0) \cdot (1, 0) = -1$,
- $(-1, 0) \cdot (-1, 0) = -1$,
- $(1, 0) \cdot (-1, 0) = +1$,
- $(0, \alpha) \cdot (1, 0) = 0$,
- $(0, \alpha) \cdot (-1, 0) = 0$, and
- $(0, \alpha) \cdot (0, \alpha) = -\alpha^2$.

An agent's neighbors are considered to be those that are located within a maximum distance of R from the agent, therefore the adjacency matrix of the system changes at each time step as a result of the agents' movement. Note that opinions and positions are updated in parallel, hence the sum of opinion products is calculated using the opinion states previous to the movement.

Although the model considers the tendency to align agents' opinions with those of their neighbors as the main opinion change mechanism, we also consider the possibility that an agent adopts a state that minimizes the sum of opinion products with their neighbors with a certain probability (see pseudocode in Sec. 2 of Appendix A). This probability, which we call temperature T because is the same parameter used in the conventional Metropolis dynamics, is considered to be small and the same for all simulations and accounts for a coarse-graining of all mechanisms that may lead an agent to take a stand against their neighbors as, for instance, social agitation.

Our model not only considers opinion dynamics but also incorporates movement rules that reflect a co-evolving process between homophily (i.e., the tendency to align opinions with those around us) and the capacity of agents to change their connections and the people they discuss a topic with. In their book,⁴⁸ Gross and Sayama provide a review of previous studies on the interplay between dynamics and network structure. Here, we analyze the behavior of the system for different movement rules, starting from the static case, where $v_i = 0$ for all agents, following with the case with constant finite velocity $v_i = v$, with two subcases: return to the initial position after every movement and subsequent opinion update and advance to the next position with no return. In Secs. III D–III F, we introduce an acceleration term for the extremists, influenced by the proportion of neighbors sharing the same opinion state. This concept is grounded in the understanding that extremists exhibit a stronger confirmation bias, leading them to actively seek like-minded neighborhoods instead of opting for places with diverse opinions to evade discussions. Conversely, neutrals may demonstrate greater open-mindedness and be content in any kind of neighborhood.

We explore three different scenarios for the accelerated case, taking into account diverse intensities in the extremists' reactions to their neighbors: total halting, non-accumulative acceleration, and accumulative acceleration. In the total halting scenario, extremists come to a complete stop when surrounded by like-minded agents, while neutrals maintain the constant initial velocity v . In the non-accumulative acceleration scenario, the initial velocity of extremists is doubled when surrounded by agents holding opposing opinions, but it is halved when the majority of their neighbors share their own opinion. Centrists continue to move at a constant velocity v throughout the simulation. In the accumulative acceleration

scenario, extremists experience a doubling of their velocity at each time step when they are surrounded by agents with opposing opinions. In the last variant, centrists retain the velocity they acquired in previous time steps if they were extremists at those times.

Agents evolve until either they reach a stability condition that will depend on the particular case of study or they have completed a fixed number of steps. We measure the fraction of neutral agents $n_0 = N_0/N$ at the end of each simulation, as well as the difference between rightists and leftists $m = (N_+ - N_-)/N$, which we call magnetization for the analogy with Ising-like models. We also pay attention to the amount of steps required to achieve the stability condition, when it is reached, and the size of the biggest cluster, in case the system ends up fragmented into several connected components with different opinions.

III. RESULTS AND DISCUSSION

In this section, we conduct a detailed analysis and discussion of several variants of the model. A summary of all the cases can be found in Appendix B (Table I), along with their respective outcomes (Table II).

A. Static case

In this subsection, we assume that all agents remain immobile, i.e., $v_i = 0$; $\forall i$, so we only consider the opinion updating dynamics. To create the network displayed in Fig. 1(a), we randomly place $N_{init} = 155$ agents in a square with side length $L = 1$. We then connect all pairs of nodes that are separated by a distance less than or equal to $R = (1 + \varepsilon)d_c$, where d_c is the critical distance and is defined by the following relation: $(N_{init} - 1)\pi d_c^2/L^2 \approx 4.51$.⁴⁹ Finally, we remove all connected components with a size N_C less than $0.2N_0$.

The critical distance d_c separates the system's regime below percolation, in which it is divided into several small connected components, from the regime above percolation, in which a giant connected component appears. For this case, we choose $\varepsilon = 10^{-5}$, which places us slightly above the critical distance and enables us to take only the nodes belonging to the giant connected component.

As mentioned in Sec. II, we already analyzed this model in Refs. 46 and 47 for a wide variety of graphs, including the complete graph, the one-dimensional chain, Erdős-Rényi graphs, Barabási-Albert networks, the Girvan-Newman communities,³⁰ and other synthetic modular graphs, as well as three particular cases of hashtags and mentions networks extracted from Twitter. In general, we found that networks with a well-defined community structure are unable to reach the global consensus and that the local consensus within communities is the stationary state at low temperatures.

When randomly scattering nodes on a plane and connecting them based on a maximum Euclidean distance between first neighbors, we obtain a random geometric graph. When this distance is close to the percolation threshold the giant connected component of the graph exhibits a strong community structure.⁵¹ We expect that the opinion dynamics we propose will never lead to the global consensus in such networks. Panels (b) and (d) in Fig. 1 were obtained by running 1000 Monte Carlo simulations on the graph shown in

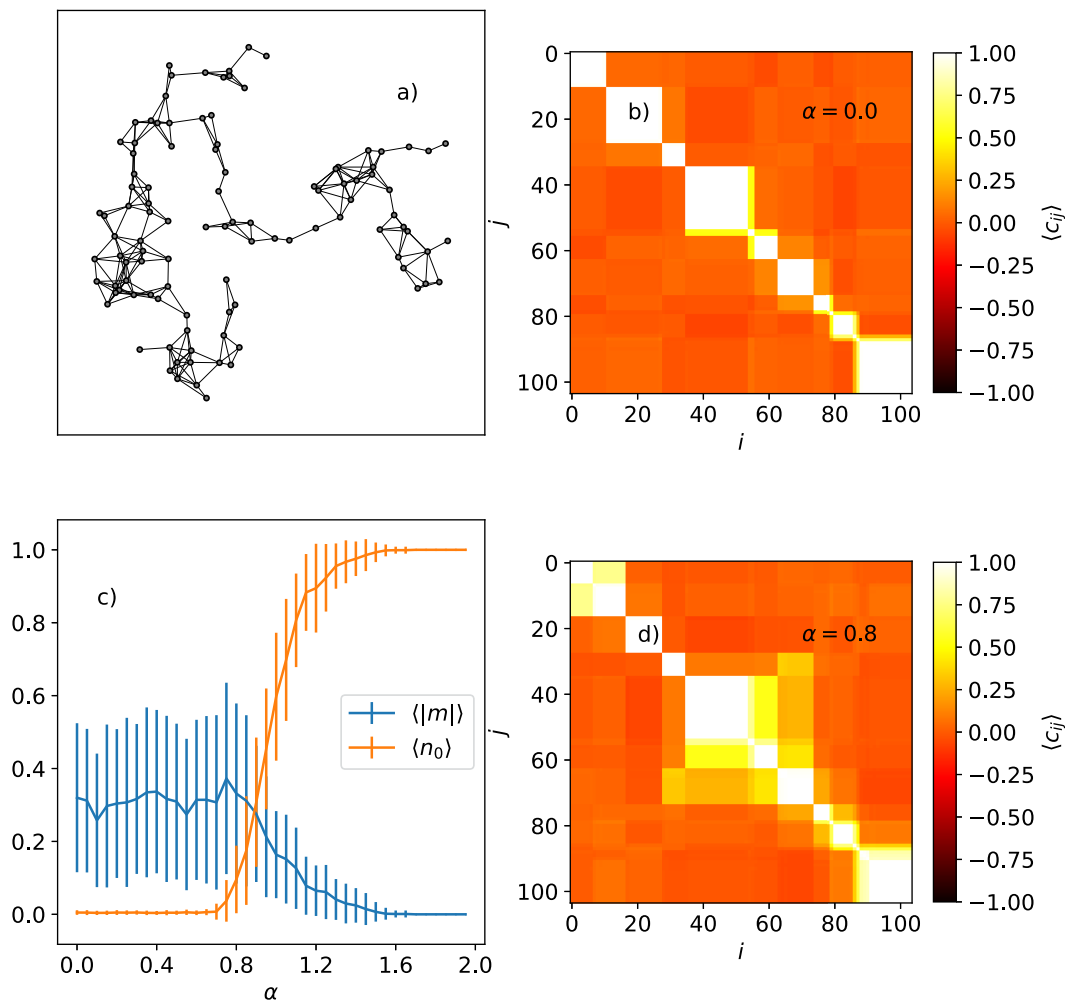


FIG. 1. (a) Random geometric graph with $N = 104$ nodes and an average degree $\langle k \rangle = 4.87$. (b) Opinion correlation matrix for $\alpha = 0.0$, obtained from simulations on the graph represented in (a) 1000 with different initial opinions. (c) Absolute value of the magnetization and fraction of neutrals as a function of the neutrality parameter α , with their standard deviations. Results are averaged over simulations with 100 different initial opinions. (d) Opinion correlation matrix for $\alpha = 0.75$, obtained from simulations with 1000 different initial opinions.

panel (a), starting from different initial opinions each time. The correlation value C_{ij} between any pair of nodes i and j takes the value 1 if the nodes belonging to that pair always finish the simulations in the same opinion state and -1 if they always finish in different states. We identify pairs of nodes with $C_{ij} = 1$ as belonging to the same community, since their opinions at the end of the simulation are totally correlated, while nodes with $C_{ij} \leq 0$ denote pairs of nodes in different communities whose opinions are totally uncorrelated. Intermediate values $0 < C_{ij} < 1$ indicate more levels of community structure, in other words, partitions of the network that achieve local consensus with a certain probability proportional to C_{ij} (see Sec. 1 of Appendix A).

For $\alpha = 0$, we observe mainly one level of community structure, corresponding to the small groups with $C_{ij} = 1$ that appear in

the diagonal. When α increases, the transition between extremist opinions and the neutral one becomes easier, leading to two consequences. First, a second level of community structure emerges, as a consequence, we observe that small communities in the diagonal appear surrounded by nodes with correlation values $C_{ij} > 0$, indicating bigger communities that achieve local consensus in a partial number of simulations. Second, the number of attractors of the dynamics increases, causing the fragmentation of the first level of community. For instance, the nodes that occupy the positions $\{12, \dots, 28\}$ in Fig. 1(b) form a single community with a correlation value equal to one for $\alpha = 0$. However, they rearrange into smaller communities for $\alpha = 0.75$, as the neutral state can now also form a stable local consensus. In fact, panel (c) shows that the fraction of neutral agents n_0 becomes non-zero for $\alpha \approx 0.8$, indicating the

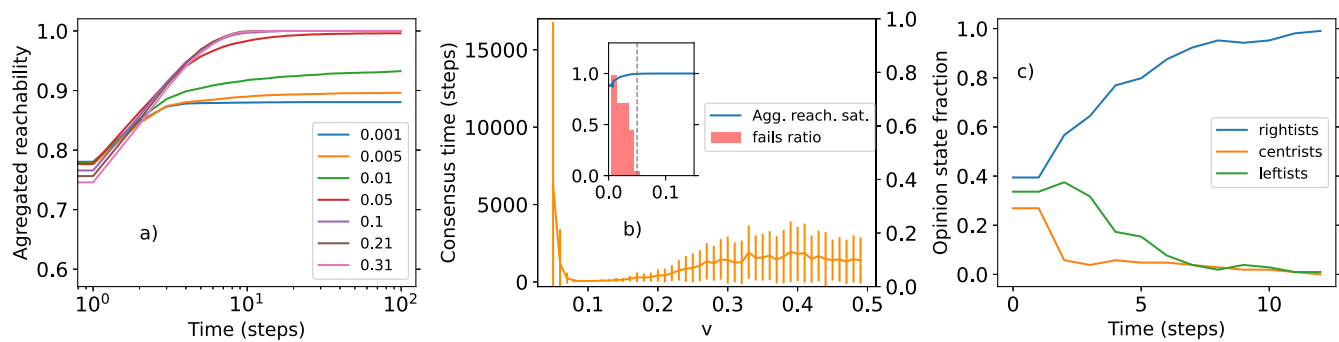


FIG. 2. (a) Aggregated reachability for the elastic case with constant velocity (see main text for explanation). Results averaged over 100 simulations. (b) Number of steps needed to reach the global consensus as a function of the velocity for $\alpha = 0$ (orange), in-plot: number of simulations that do not achieve the global consensus (red bars) and saturation value of the aggregated reachability (blue line; see main text for explanation). The gray dashed line indicates the value of v for which the fails become zero, which is very close to the value of v for which the saturation value of the aggregated reachability becomes one. Results averaged over 100 simulations. (c) Particular example of the temporal evolution of the number of agents in each opinion state for $\alpha = 0$ and $v = 0.1$.

appearance of neutral communities above this value of the neutrality parameter. Below this value, the system becomes fragmented only into extremist communities. We would expect them to be evenly distributed, with an average magnetization $\langle |m| \rangle$ close to zero, by symmetry reasons. However, we observe a non-zero $\langle |m| \rangle$ due to finite-size effects. For higher values of α , the number of neutral communities rapidly increases until the global consensus at the neutral opinion is guaranteed for $\alpha > 1.5$.

B. Constant velocity. Elastic

In this subsection, we continue our study of the same graph analyzed in Sec. III A, but now incorporating movement. At each time step, agents are allowed to move in a random direction with a constant velocity $v_i = v$. They update their opinions in the new position and return to their original position afterward, indicating that agents have a preferred location, akin to a home, to which they periodically return. This movement can be understood as oscillations around the initial positions, with short amplitudes for low velocities. Therefore, for $v \rightarrow 0$, the results are similar to those obtained for the static case, since the agents are not able to change their neighbors.

As we increase the velocity, opinions can be transmitted to different neighbors at each time step, which makes the global consensus more likely to emerge, since it creates connections between more pairs of agents. The emergence of the global consensus is related to the saturation value of the aggregated reachability [see Fig. 2(a)], defined as the accumulated proportion of links between any pair of agents holding different opinions that have been present in the system up to a given moment, in the absence of opinion updates. At the initial instant, the aggregated reachability is equal to the ratio between the number of links connecting agents with different opinions in Fig. 1(a) and the total number of pairs of agents that have different opinions. This number will monotonically increase until it reaches its maximum value, which depends on the velocity. Notice that since we impose periodical boundary conditions for the plane, we do not consider velocities larger than $v = L/2 = 0.5$.

The amount of time (in steps) required to reach consensus, once the velocity is sufficiently large, is shown in Fig. 2(b) and exhibits a non-monotonic behavior. At low velocities, the consensus time is high and has large fluctuations, since it strongly depends on the initial conditions and the extent to which random movements favor changes toward a given majority opinion. For large values of v , we can think of the system as a complete graph with blinking edges that appear at each time step with a certain probability. The consensus time for high velocities stabilizes around 2000 steps, but with large fluctuations because the neighborhood changes completely at every time step, and every movement of the agents can potentially destabilize the local consensus achieved in the previous step. There is a minimum around $v = 0.1 \approx R$, for which opinion diffusion is very efficient, and the system reaches the global consensus rapidly in all the simulations, as shown in the example displayed in Fig. 2(c).

C. Constant velocity. Non-elastic

In this subsection, we again consider a constant velocity of $v_i = v$ for all agents, but now we let the agents evolve following a random walk without returning to their initial position. In this case, the aggregated reachability (as defined in Sec. III B) never saturates to values lower than 1; instead, it always increases with time until it reaches 1, ensuring that the system eventually achieves the global consensus given enough simulation time. However, the growth rate of the aggregated reachability increases with the velocity until it saturates around $v = 0.2$, as shown in Fig. 3. Unexpectedly, the time needed to reach the global consensus increases with the aggregated reachability for low velocities until it reaches a peak that depends on α [see Figs. 3(b) and 3(d)]. In order to understand which features of two consecutive time adjacency matrices cause this behavior, we represent the probability of an agent being a first neighbor of another agent, given that they were the second neighbor in the previous step (represented by the black line), which shows a peak close to the consensus time peak [see Figs. 3(b) and 3(d)]. We show an example of the evolution of the fraction of agents in each opinion state for a system that evolves with a velocity around the maximum in Fig. 3(c).

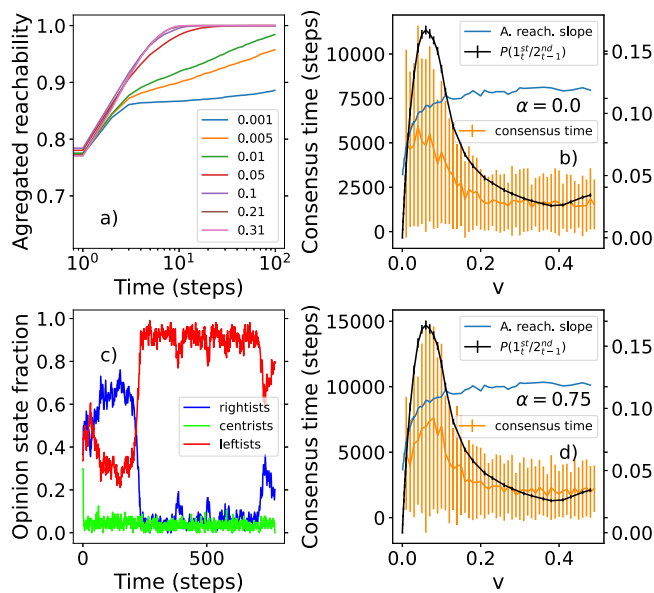


FIG. 3. (a) Aggregated reachability for the non-elastic case with constant velocity. Number of steps needed to reach the global consensus as a function of the velocity (orange bars) for $\alpha = 0$ (b) and $\alpha = 0.75$ (d), along with the slope of the aggregated reachability as a function of the logarithm of time in the first seven steps (blue line), and the probability that an agent is the first neighbor of another agent given that in the previous step they were their second neighbor (black line). (c) Particular example of the temporal evolution of the number of agents in each opinion state for $\alpha = 0$ and $v = 0.05$. All results are averaged over 100 simulations.

We observe switches in the majority opinion that prevent the system from achieving the global consensus quickly. Local consensus is preserved by the motion, but individual agents keep bouncing between opinions for a large number of steps. Note that this behavior is very different from the one observed in the elastic case.

For higher velocities, the behavior is identical to that observed in the elastic case, as the position of the agents is random at each time step in both cases. Similar to the elastic case, we do not consider $v > L/2 = 0.5$ due to the symmetry imposed by the periodic boundary conditions.

The neutrality parameter α plays the most significant role in determining the opinion state of the final consensus, as shown in Fig. 4. These results are obtained for a system of $N = 100$ agents and a value of $R = d_c - 0.1d_c = 0.108$, which is only slightly lower than the one used in Sec. III B, and therefore, we do not expect significant differences. Broadly speaking, the system achieves neutral consensus for $\alpha > 1$ and a polarized global consensus otherwise. With respect to velocity, the minimum value of α required to observe neutral consensus in some simulations is very close to $\alpha = 1$ for low velocities, while for high velocities, it decreases to $\alpha = 0.8$.

This model exhibits a first-order phase transition at $\alpha = 0.8$ in the mean-field limit.⁴⁷ The mean-field approximation assumes that each agent interacts with the average effect of all other agents, rather than taking into account specific interactions with every neighbor,

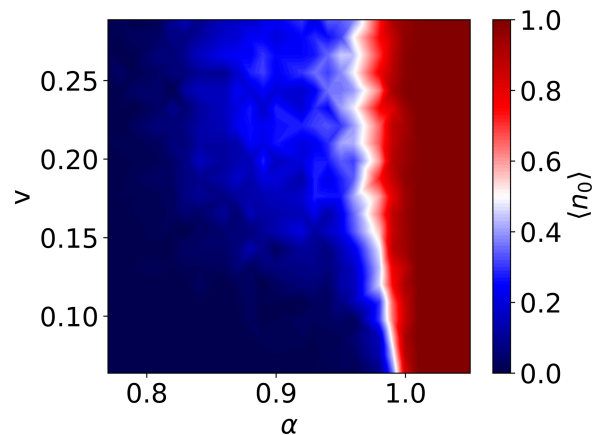


FIG. 4. Phase diagram showing the fraction of simulations that finish with neutral consensus for the non-elastic case as a function of the velocity v and the neutrality parameter α . Results for a system of $N = 100$ agents and averaged over 100 simulations that start from different initial positions and opinions.

and it is exact for the fully-connected graph. Therefore, the previous result is consistent with the notion that, for high velocities, the system behaves like a complete graph with blinking edges, where every possible pair of agents gets connected with a certain probability at each time step, regardless of their previous connections. Above $\alpha = 1$, the absorbing state is neutral consensus, regardless of velocity, as expected.

The average value of the absolute magnetization is complementary to the fraction of neutral agents, since all simulations end in a global consensus. When this consensus is polarized, both rightist and leftist consensus have an equal probability of appearing due to the opinion symmetry of the model.

D. Total halting

In this subsection, we examine how the system behaves when the velocity is adjusted at every time step following the next rule: extremists stop moving if they are surrounded by a majority of neighbors who share the same opinion state as them, and they move in a non-elastic fashion with velocity v if half or less of their neighbors share their same opinion. Neutral agents, on the other hand, are unaffected by this rule and move with a constant velocity of v at all times. The simulation ends either when the system reaches a global consensus or when all the agents remain immobile. With these rules, we assume that extremists are more prone to change their links with people whose opinion is opposite to theirs. On the contrary, neutral agents are sensitive to their neighbors' opinions regarding the opinion update but do not selectively change their links according to their neighbors' opinion.

This type of motion favors local consensus, which by construction is always polarized, over global consensus. Additionally, it penalizes the neutral opinion, since extremists tend to form communities that capture neutral agents when they pass nearby and convince them to change their opinion. Even for values of $\alpha > 1$, the average number of neutral agents is lower than 1, as we can see

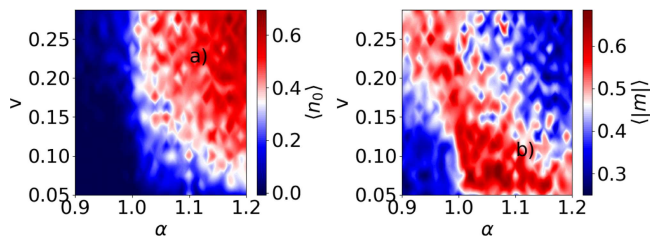


FIG. 5. (a) Final average fraction of neutral agents and (b) final average absolute magnetization as a function of the velocity v and the neutrality parameter α for the total halting model. Results for a system formed by $N = 100$ agents and averaged over 100 simulations starting from different initial positions and opinions.

in Fig. 5(a). For large velocities, neutral agents, who do not stop, are able to propagate their opinion fast enough to become a majority within the first time steps (at least in a certain number of simulations) and as a consequence $\langle n_0 \rangle$ increases. When this happens, the system achieves the neutral global consensus in very few time steps (see Fig. 6), compared to the number of steps necessary to get partial or polarized consensus.

The average absolute magnetization strongly depends on both the velocity and the neutrality parameter α , as shown in Fig. 5(b). For $\alpha < 1$ and low velocities, the system reaches a local polarized consensus with a similar number of clusters in both positive and

negative opinions. Therefore, $\langle |m| \rangle$ is low, and the number of neutral agents $\langle n_0 \rangle$ is approximately zero. However, when the velocity is higher, the agents travel further and the formation of local polarized clusters is faster. Any imbalance between rightists and leftists grows rapidly, leading to an increase in magnetization because the number of clusters in each polarized state is no longer equal, and the fraction of neutral agents remains close to zero.

Counterintuitively, when the neutrality parameter $\alpha > 1$, the tendency is reversed and the average value of the absolute magnetization is larger for lower velocities. This is because the abundance of neutral agents allows the system to achieve polarized global consensus in a significant number of simulations, as shown in Fig. 6. Due to the high value of α , neutral agents persist in time, while extremists stop and form communities. Eventually, neutral agents separate and get caught in the polarized clusters, changing their opinion to align with the community they encounter. Furthermore, the abundance of neutral agents can convert small polarized clusters to the neutral opinion, making its members move again. Later, these agents may be converted to extremism again by a larger polarized community, and eventually, the system can reach the global polarized consensus. When the velocity increases, extremist communities cannot grow enough, and in most simulations, we obtain either neutral global consensus, marked by an increase of $\langle n_0 \rangle$, or polarized local consensus with a similar number of small extremized groups, consistent with a low average magnetization. Figure 6 displays examples of the consensus time histogram for each scenario. However, it is worth noting that for $v = 0.25$ and $\alpha = 0.95$, the system only reaches local polarized consensus, despite having an average magnetization of $\langle |m| \rangle > 0$, indicating an unbalanced distribution of extremists in each state. Global neutral consensus is achieved much faster than local or global polarized consensus. Surprisingly, the value of the initial velocity v does not have a high impact on the consensus time, especially for $\alpha < 1$.

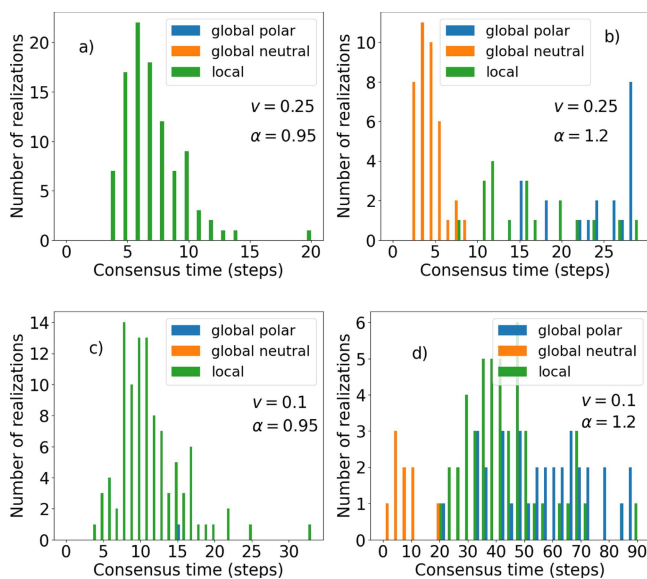


FIG. 6. Histograms for the consensus time for the total halting model for (a) $v = 0.25$ and $\alpha = 0.95$, (b) $v = 0.25$ and $\alpha = 1.2$, (c) $v = 0.15$ and $\alpha = 0.95$, and (d) $v = 0.15$ and $\alpha = 1.2$. Blue bars represent global polarized consensus, green bars are for global neutral consensus, and yellow bars denote local polarized consensus. Results averaged over 100 repetitions, starting from different initial positions and opinions.

E. Non-accumulative acceleration

In this subsection, we consider a case that is similar to the previous one, except that extremists do not stop their motion completely when they have a majority of neighbors sharing their same opinion. Instead, they reduce their velocity to half the initial value ($v/2$) and, in addition, they double the initial velocity if their neighbors with an equal opinion are not a majority. This strategy enables the system to reach a global consensus, just like in the case where the velocity is constant. As a consequence, magnetization is complementary to the fraction of neutral agents.

Although the dynamics still always converges to a global consensus, the phase diagram exhibits slight differences from the case where the velocity is constant. In particular, for low velocities, a value of the neutrality parameter α greater than 1 is required to guarantee neutral consensus (see Fig. 7). This is because extremists adapt their motion to their neighborhood, making them come to an agreement more efficiently. In contrast, the neutrals, which are not affected by this rule, need a stronger interaction (i.e., a higher value of the neutrality parameter α) to be able to form neutral opinion groups. Therefore, for low velocities, the results for the order parameters are resemblant to those obtained for the total halting case.

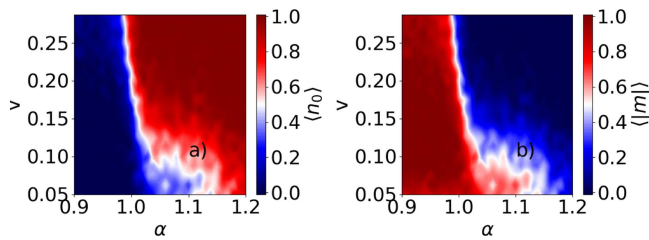


FIG. 7. (a) Average fraction of simulations that finish with consensus in the neutral opinion and (b) final average value for the absolute magnetization as a function of the velocity v and the neutrality parameter α for the non-accumulative accelerated model.

Conversely, for high velocities, the changes in velocity are insufficient to produce this effect, and the average values of the order parameters are equal to those obtained for the constant velocity case.

Regarding the consensus time, the motion rules steer the system toward the global consensus by making the agents sensitive to the agents' opinions, while allowing all of them to move around the plane and potentially interact with any other agent. As a result, the consensus is reached in significantly fewer steps compared to agents moving with a constant velocity, as evidenced by comparing Figs. 3 and 8. Neutral consensus is always faster than polarized consensus, and it is enhanced by high velocities, as observed in the previous scenarios. Both increasing v or α reduce the number of steps required to achieve consensus, similar to the constant velocity case.

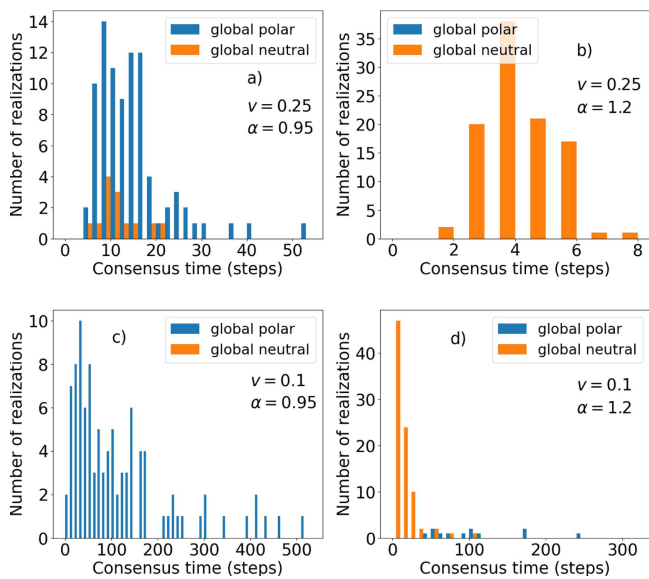


FIG. 8. Histograms for the consensus time for the non-accumulative accelerated model for (a) $v = 0.10$ and $\alpha = 1.0$, (b) $v = 0.10$ and $\alpha = 1.2$, (c) $v = 0.25$ and $\alpha = 1.0$, and (d) $v = 0.25$ and $\alpha = 1.14$. Blue bars represent global polarized consensus and orange bars denote global neutral consensus. Results are averaged over 100 simulations, starting from different initial positions and opinions.

F. Accumulative acceleration

In the case examined in this subsection, extremists modify their velocity based on the proportion of neighbors who share their opinion, as in Secs. III D and III E. However, in this case, an extremist agent i reduces their velocity to $v_i(t) = v_i(t-1)/2$ when they have a majority of neighbors in their same opinion state and increases it to $v_i(t) = v_i(t-1) * 2$ otherwise (with a maximum velocity of $v_{max} = L/2 = 0.5$ to avoid higher velocities that do not make sense with periodic boundary conditions). Although the extremists' velocity can now increase, they still tend to form opinion clusters, similar to the total halting scenario. On the other hand, neutrals no longer move with the initial constant velocity v , but instead, they conserve the velocity they had acquired previously, so if a neutral agent i was polarized at some point in the past time, and they have modified its velocity to a given $v_i \neq v$, they will continue moving at their own v_i . Eventually, agents move so slowly that they are considered immobile, so we stop the simulation when the average velocity is lower than 5×10^{-4} .

In most simulations, the stationary state for this scenario is local consensus for $\alpha < 1$, similar to the total halting case, while for high values of the neutrality parameter α , the system reaches a neutral consensus. The main difference now is that when an extremist changes their opinion to neutral, they conserve their velocity, which can be arbitrarily slow. Therefore, although neutral agents do not react to their neighbors' opinions, partial consensus can still contain neutral clusters, even for $\alpha = 0.95$ (see Fig. 10). Polar cluster formation depletes some regions of extremists, leaving empty zones that can eventually be occupied by slow neutral agents with short average displacement. As in previous scenarios, we find that higher initial velocities v favor the achievement of neutral consensus. For example, in Fig. 9(a), the proportion of neutrals at the end of the simulations when $\alpha = 1.2$ is $\langle n_0 \rangle \approx 1$ only for $v \gtrsim 0.2$.

The phase diagram for the average magnetization Fig. 9(b) is slightly different from the one obtained for the total halting scenario. Specifically, for low α and low v , we obtain higher values for $\langle |m| \rangle$. This is because in this case, extremists do not suddenly stop their movement when surrounded by a majority of agents with the same opinion. Instead, they gradually modify their velocity, adjusting their positions, aligning their opinions, and convincing isolated neutral agents when they pass nearby. This mechanism allows them to form larger clusters, thus increasing the magnetization. For $\alpha < 1$, the system rarely reaches a global neutral consensus. The average absolute magnetization increases with v , as in the total halting case, and the proportion of local consensus with neutral clusters decreases.

Regarding the typical size of opinion clusters, polarized and neutral agents behave differently, which is expected, since the movement rules are still different for both types of agents. Figure 9(c) shows that the largest neutral cluster found in the simulations has a number of agents $N_0 \approx 45$, while the polarized clusters can be the size of the system $N = 100$, as shown in panel (d). On one hand, this is due to the fact that neutral agents' motion is not reduced when they are surrounded by other neutrals, hence any big neutral connected component can break more easily than a polarized one. On the other hand, we do not see such big polarized clusters in the case of total halting, but we do here due to the smoother movement of

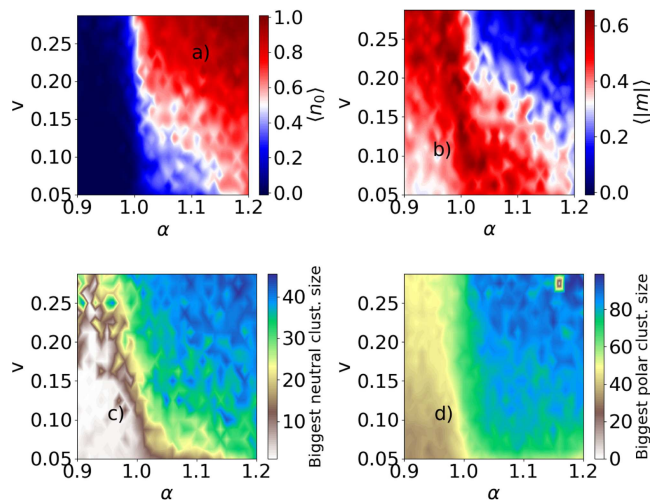


FIG. 9. (a) Average fraction of agents that finish the simulation holding the neutral opinion, and (b) average final value for the magnetization as a function of the velocity v and the neutrality parameter α for the accumulative accelerated model. Size of the biggest (c) neutral-connected component and (d) polarized connected component. The gray dot in the upper right corner of panel (d) ($\alpha = 1.16$ and $v = 0.275$) indicates that there were no polarized clusters in any of the simulations for that particular values of the parameters. Results are averaged over 100 simulations, starting from different initial positions and opinions.

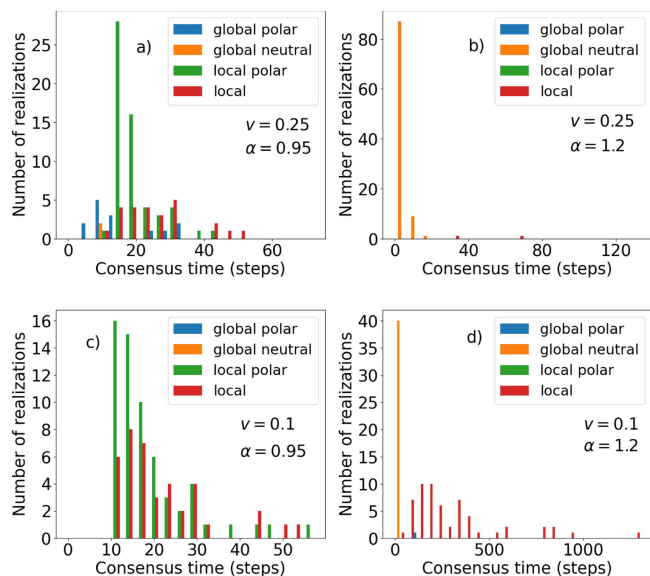


FIG. 10. Histograms for the consensus time for (a) $v = 0.10$ and $\alpha = 1.0$, (b) $v = 0.10$ and $\alpha = 1.2$, (c) $v = 0.25$ and $\alpha = 1.0$, and (d) $v = 0.25$ and $\alpha = 1.14$ for the accumulative accelerated model. Blue bars represent global polarized consensus, orange bars are for global neutral consensus, green bars denote local consensus, which only contains polarized clusters, and red bars represent the local consensus that contains at least one neutral cluster. Results for 100 simulations, starting from different initial positions and opinions.

the extremists. It may seem counterintuitive that these big polarized clusters only appear for high values of α . However, the reason is that it takes time to form these structures and, for $\alpha < 1$, we achieve local polarized consensus too fast to give the agents time to form big opinion groups.

Our casuistic is richer than in other scenarios, particularly for $\alpha = 0.95$ and $v = 0.25$, where we observe all four possible outcomes, although a global consensus is less likely (see Fig. 10). In contrast to the total halting case, the time required to achieve a polarized consensus is sometimes shorter than the time needed to reach a global neutral consensus. Local polarized consensus is only observed for $\alpha = 0.95$, and the time required to achieve it is similar to that of achieving global neutral consensus. The impact of v and α on the consensus time is smoother than in the non-accumulative model.

IV. CONCLUSIONS

We use Monte Carlo simulations to investigate the impact of agents' motion on the final outcomes of a three-state opinion model with a neutrality parameter that adjusts the relevance of the neutral opinion state. Our findings indicate that both the movement and the neutrality parameter play a significant role in the final outcome of the system. In particular, for a given value of α , mobility patterns can change the system's attractors in all cases studied.

We found that motion drives the system toward the global consensus, whereas the local consensus arises when the motion rules involve agent stopping. The composition of this local consensus depends on how agents adjust their velocity according to their neighbors' opinions. In the case of total halting, where polarized agents stop completely when they are surrounded by like-minded individuals, and neutrals move at a constant velocity, the local consensus is always polarized. Only when neutrals inherit the extremists' velocity are they able to occupy the free space left by polarized agents and form stable clusters, even if $\alpha < 1$.

In the limit case of a static network, a strong community structure gives rise to multiple opinion groups that may merge due to mobility effects. Performing a regular random walk at a constant velocity leads to the global consensus in all cases. However, it is interesting to note that periodically returning to a base location, even if agents do not update their opinions there, can reduce the time needed to achieve the global consensus by disrupting the formation of opinion clusters. This is true as long as the velocity is high enough to allow the system to eventually achieve an aggregated reachability of one.

Across all cases studied, high velocities favor the neutral opinion, bringing the system closer to mean-field behavior, where the neutral opinion becomes a dynamic attractor. In general, global neutral consensus is achieved faster than global or local polarized consensus, except for some marginal simulations for the accelerated accumulative model with large velocity and $\alpha < 1$, where we observe local polarized consensus at very short times. Extremists use to take advantage of longer interaction times to form large clusters and establish polarization. Once a large polarized community forms, it is hard to destabilize because opinion transitions become unlikely when agents remain clustered. In contrast, neutral agents are more prone to changing their minds, both because their movement is not influenced by the neighborhood opinion and because

they have a higher probability of changing to either polarized opinion state. Interestingly, in some cases, a larger value of the neutrality parameter α can increase polarization instead of reducing it, not only for mobile agents but also for some static networks, as shown in Ref. 47.

While this work primarily concentrates on the physical plane of agent movement, there is potential to extend the concept to include ideological or emotional spaces, where the dimensions represent beliefs or emotions. In such scenarios, at lower velocities, agents would primarily interact if their ideas or emotional states are closely aligned, following the homophily principle, similar to what is done in bounded confidence models. As an example, we refer to the works by Starnini *et al.*,⁵² which explore related concepts in the context of social dynamics and opinion formation. The work “Asymmetric Multibody Opinion Formation Model of Periodical Interacting Agents” (Ferri, Nicolás-Olivé, Cozzo, Di’az-Guilera, and Prignano), currently in preparation, also focuses on these types of spaces.

ACKNOWLEDGMENTS

The authors acknowledge support from Spanish Grant Nos. PGC2018-094754-B-C22 and PID2021-128005NB-C22, funded by MCIN/AEI/10.13039/501100011033 and “ERDF A way of making Europe,” and from Generalitat de Catalunya (2021SGR00856). I.F.’s work has been supported by Grant No. PRE2019-090279 (No. MCIN/AEI/10.13039/501100011033).

AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

I. Ferri: Conceptualization (equal); Formal analysis (equal); Funding acquisition (equal); Investigation (equal); Methodology (equal); Software (lead); Supervision (supporting); Writing – original draft (lead); Writing – review & editing (equal). **A. Gaya-Ávila:** Formal analysis (supporting); Methodology (supporting); Software (supporting); Supervision (supporting); Writing – original draft (lead); Writing – review & editing (supporting). **A. Díaz-Guilera:** Conceptualization (equal); Formal analysis (supporting); Funding acquisition (equal); Investigation (equal); Methodology (equal); Software (supporting); Supervision (equal); Writing – review & editing (equal).

DATA AVAILABILITY

The data that support the findings of this study are openly available in GitHub at <https://github.com/IreneFerri/Three-state-opinion-model-with-mobile-agents>, Ref. 53.

APPENDIX A: PSEUDOCODES

1. Correlation matrix pseudocode

The following lines describe the algorithm we use to find the correlation between nodes and to identify the community structure of the networks.

ALGORITHM 1. Calculate correlations.

Require: multiple initial set of opinions $\{S_i\}_{initial}$ and the adjacency matrix of the embedding network A_{ij} . Number of repetitions N_{reps}
Ensure: a correlation matrix C_{ij} that contains the number of times a pair of nodes has the same final opinion in the steady state when running a Metropolis algorithm N_{reps} times

```

set  $\hat{C}_{ij} = 0 \forall i, j$ 
set  $rep = 0$ 
repeat
  set initial opinion  $\{S_i\}_{initial}$  uniformly at random
  call a Metropolis algorithm
  define correlation value for each pair of nodes
  for all  $S_i, S_j \in \{S_i\}_{final}$  with  $i, j \in 1, \dots, N$  do
    if  $S_i = S_j$  then
       $\hat{C}_{ij} \leftarrow \hat{C}_{ij} + 1$ 
    else
       $\hat{C}_{ij} \leftarrow \hat{C}_{ij} - 1$ 
    end if
  end for
   $rep \leftarrow rep + 1$ 
until  $rep = N_{reps}$ 
return  $\hat{C}_{ij}/N_{reps}$  as  $C_{ij}$  correlation matrix

```

ALGORITHM 2. Sorting links.

Require: C_{ij} correlation matrix
Ensure: a correlation matrix C_{ij} that contains in the secondary diagonal the pairs of nodes that always share the same opinion in the steady state, regardless the initial set of opinions. Rows and columns are sorted in such a way that the structure of the network can be inferred from its visual representation

```

set  $pivot = C_{ij}$  dimension
sort from lowest to highest correlation value  $C_{pivot,node}$  all nodes
from 1 to pivot
repeat
  set the new pivot
  for all  $i \in C_{pivot,i}$  do
    if  $C_{pivot,i} = 1$  then
       $pivot \leftarrow i - 1$ 
    end if
  end for
until  $pivot = 1$ 
return  $C_{ij}$  sorted correlation matrix

```

2. Mobility pseudocode

ALGORITHM 3. Perform an agent-based simulation using the three-state opinion model and a random walk.

```
Require: Initial set of opinions  $\{S_i\}_{initial}$  and initial set of positions  $\{x_i\}_{initial}$  The number of Monte Carlo steps  $N_{steps}$ . A lower bound  $tol$  for the average velocity  $v$   
Ensure: Getting to one of the attractors of the dynamics, given a sufficiently large  $N_{steps}$   
  
set  $step = 0$   
repeat  
  set initial opinion  $\{S_i\}_{initial}$  uniformly at random  
  set initial positions  $\{x_i\}_{initial}$  uniformly at random  
  update agents positions following a random walk with velocity  $v_i$  for each agent  $i$ , given by each variant of the model  
  call a Metropolis – Hastings algorithm  
  if  $S_i = +1$  or  $S_i = 0$  or  $S_i = -1$  then  
     $global\ consensus = True$   
    Break  
  else  
    if  $\langle v_i \rangle < tol$  then  
       $local\ consensus = True$   
      Break  
    end if  
  end if  
until  $step = N_{steps}$   
return  $\{S_i\}_{final}$  set of final opinion states and number of steps when the simulation stops
```

APPENDIX B: MODEL VARIANTS AND OUTCOMES

TABLE I. Table to summarize the variants of the model.

Variants of the model	Motion rules	Time step
A. Static	$v_i = 0 = ctt \forall i$	Opinion update
B. Elastic	Initial positions: Same as the static case. $v_i = v = ctt \forall i$	Move in a random direction—Opinion update—Return to the initial position
C. Non-elastic	Initial positions: Same as the static case. $v_i = v = ctt \forall i$	Move in a random direction—Opinion update
D. Total halting	<i>Polarized:</i> $v = 0$ if at least half their neighbors share their same opinion state; $v = v_{initial}$ otherwise <i>Neutrals:</i> $v = v_{initial} = ctt$	Move in a random direction—Opinion update
E. Accelerated non-accumulative	<i>Polarized:</i> $v = v_{initial}/2$ if at least half their neighbors share their same opinion state; $v = 2\Delta v_{initial}$ otherwise <i>Neutrals:</i> $v = v_{init} = ctt$	Move in a random direction—Opinion update
F. Accelerated accumulative	<i>Polarized:</i> $v = v_{t-1}/2$ if at least half their neighbors share their same opinion state; $v = 2v_{t-1}$ otherwise ($v_{max} = L/2 = 0.5$) <i>Neutrals:</i> $v = v_{t-1}$	Move in a random direction—Opinion update

TABLE II. Table to summarize the outcomes of the model.

Variants	Time for global consent	Stationary state
A. Static	Only partial consensus	Network divided into opinion clusters according to: (1) the network structural partitions and (2) the value of α
B. Elastic	The system reaches global consensus only if v guarantees an aggregated reachability equal to 1 at any time step. Non-monotonic behavior with a minimum around $v = R$ and higher values with large fluctuations for slower and faster velocities.	Global consensus if $v \geq R$. • Neutral consensus if $\alpha > 1$ • Polarized consensus if $\alpha < 1$ • Partial consensus for low v: the simulation stops at the maximum time steps, and the proportion of agents in each state depends on α
C. Non-elastic	The system reaches global consensus only if $\forall v$. Non-monotonical behavior with a maximum around $v = R$	Global consensus if $v \geq R$. • Neutral consensus if $\alpha > 1$ • Polarized consensus if $\alpha < 0.8 = \alpha_{\text{trcritical}}$ (mean-field) • If $0.8 < \alpha < 1$ opinion state of the consensus depending on v (higher v favors global neutral consensus)
D. Total halting	Partial consensus for all parameter values explored	The number of neutral or polarized agents depends on both α and v , both parameters boosting the neutral proportion of agents.
E. Accelerated non-accumulative	Global consensus, faster than in variant C	Similar to variant C. However, velocity has a greater impact favoring neutral consensus when $0.8 < \alpha < 1$
F. Accelerated accumulative	All possible outcomes: • Neutral consensus • Polarized consensus • Partial polarized consensus • Partial consensus with neutral groups	Rich casuistic. Mixture of variants D and E. See main text for details

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