Elementary features of Kaluza-Klein theories

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Abstract: In many current unification theories involving gravity, it is natural to add compactified extra dimensions. To make a connection with our four-dimensional universe we need dimensional reduction to link a higher-dimensional theory to a lower-dimensional one. As a first contact with the subject, we study how dimensional reduction works analyzing pure Einstein gravitation in $D = d+1$ dimensions, with an extra dimension compactified on S^1. Finally, motivated by the presence of non-Abelian gauge groups in the Standard Model, we analyze extra dimensions compactified in some internal compact space, which give rise to gauge fields associated to these gauge groups.

I. INTRODUCTION

The idea to unify in physics is old. After A. Einstein presented his theory of Special Relativity, H. Minkowski introduced the Minkowski space-time, a continuous four-dimensional manifold which combines the dimension of time with the three dimensions of space, where special relativity is simply and elegantly formulated. In this background, the notions of space and time were unified for first time on history. But that was not all.

Due to the relativistic invariance of Maxwell's theory of Electromagnetism (EM) it was realized that the unification of electricity and magnetism was intimately connected with this Minkowski space-time. Indeed, Maxwell's theory was rapidly accommodated in this new framework. The three-dimensional objects describing electricity and magnetism, the fields $\mathbf{E}$ and $\mathbf{B}$ joined together as components of a single six-component antisymmetric tensor $F^{\mu\nu}$ and the corresponding potentials Φ and $\mathbf{A}$ unified in the four-vector A^{μ}, as well as space and time, being lower-dimensional objects in themselves, joined together in a higher-dimensional framework.

Once Einstein formulated his relativistic theory of gravitation (GR), in the same line of thinking one could ask himself if it is possible to obtain GR + EM from a single higher-dimensional theory. Indeed, at that time efforts were made to unify gravity with the electromagnetic theory in this way. After all, despite their enormous differences, both theories have a formulation in geometric terms.

The hope to see this unification was initiated by Th. Kaluza [1] and O. Klein [2]. Of course, nowadays with the formulation of Nuclear theories, the landscape has changed. Even though there are existing theories in four dimensions which unify electroweak and strong interactions (GUT's), it seems not to be possible to do the same when gravity wants to be considered.

Here is where Kaluza's idea revives, and although there have been many developments and advances since their days, the general procedure bears their names, leading to what today is known as Kaluza-Klein (KK) theories.

Within the wide range of unification programs, important ingredients in some of the most promising modern unification theories, for instance, supergravity and superstrings are the key ideas of Kaluza and Klein.

In particular, ten-dimensional string theory and eleven-dimensional M-theory are at present our best candidates for providing a unified description of all the fundamental forces in nature. If we hope that one day they may allow us to describe our four-dimensional world, we need to have a way of extracting four-dimensional physics from higher-dimensional theories. Since current higher-dimensional theories are all theories of gravity plus additional fields, a good starting point is to study how the dimensional reduction of pure gravity works.

This TFG (Bachelor's Degree Final Paper) is organized as follows. In the first section we perform a dimensional reduction in which the space-time dimension is reduced by 1, where the extra dimension is curled-up in a tiny spatial circle, S^1.

In this section we recover the result which gave an air of hope in the first attempt to unification. At the end of this section, we examine where the $U(1)$ gauge invariance comes from.

In section II, we just pay attention on how to introduce higher-dimensional fields and how go to low dimensions. In section III, we analyze the possibility to consider other types of internal spaces instead of S^1, with associated fields having a correct behavior from the gauge transformation viewpoint. We discuss on the minimum number (the simplest case) of dimensions needed to obtain as a group of symmetry the gauge group of the Standard Model (SM) of particle physics.

However, as we will see, the original theory does not contains fermions and moreover, the non-Abelian scenario does not admit a d-dimensional flat space-time. These are the two main problems which suggest that original KK attempt can not be sustained.

A. Notation & Conventions

Generally, we work in a D-dimensional space-time manifold, $\mathcal{M}^d \times \mathcal{N}$, with $D = d + n$, where d is the dimension of the first product manifold, and n is the dimension of the internal space. We adopt the following no-

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tation: the fields and magnitudes living in D dimensions are hatted and labeled by latin capital indices, $A, B, \ldots = 0, 1, 2, \ldots, d+n$. The greek indices range over the non-compact d-dimensional manifold with coordinates denoted collectively by x, whereas latin indices range over the compact n-dimensional spacelike manifold, with coordinates written as y. The metric holds like signature $\mathrm{sg}(\hat{\mathbf{g}}) = d-2+n$, i.e. $(-,+,+,\ldots,+)$ (spacelike convention). We set $\hbar = c = 1$.

II. KALUZA-KLEIN FUNDAMENTALS

A. Kaluza-Klein approach: reduction on S^1

Let us start considering Einstein's theory of pure gravity in $D = d+1$ dimensions. Using the Lagrangian formalism, all the theory derives from the Einstein-Hilbert action,

$$S = \frac{1}{16\pi\hat{G}} \int \hat{R}\sqrt{-\hat{g}}\mathrm{d}^{d+1}x, \tag{1}$$

where $\hat{G}$ is the gravitational constant, $\hat{R} = \hat{g}^{MN}\hat{R}_{MN}$ is the Ricci scalar and $\hat{g}$ denotes the determinant of the metric, of course, all of these in D dimensions.

The idea to consider an extra dimension may seem to defy our common sense. Although our physical experience obtained so far did not suggest the existence of an extra dimension, we are free to consider our space-time to be a d-dimensional part of higher-dimensional one. One then has to take into account the fact that we are only aware of field variations respect the d-dimensional space-time part, leading to the supposition that all fields in d dimensions do not depend on y.

Physically, the fact to forbid any dependence on y remains a bit hard to digest. Here is where the compactified dimension plays an interesting role.

We compactify the extra dimension with an identification. That is, we declare that points along y with coordinates that differ by $2\pi R$, are the same point: $y \sim y + 2\pi R$ (R is assumed to be very small, due to the non-observability). With such an identification, the extra-dimension becomes a circle S^1 (compact). It implies that the space-time background is not $\mathcal{M}^{d+1}$, but instead is $\mathcal{M}^d \times S^1$, the direct product of our d-dimensional space-time with a circle of radius R at every point. As we will see in the next section, this fact has powerful consequences.

Due to the periodicity of y, we can expand all the components of $\hat{g}_{MN}$ as Fourier series

$$\hat{g}_{MN}(x,y) = \sum_n g^{(n)}_{MN}(x)e^{iny/R}, \tag{2}$$

getting an infinite number of fields in d dimensions, labelled by the Fourier mode number n. The key point is that the modes with $n \neq 0$ are associated with massive fields, while those with $n = 0$ are massless.

1. *Scalar fields in higher dimensions*

This can be seen by considering a massless scalar theory in $\mathcal{M}^{d+1} = \mathbb{R}^{(1,d-1)} \times S^1$, with $(d+1)$-Minkowski metric, and exploring its Fourier expansion. It is described by the action

$$S = -\frac{1}{2}\int \partial_M\hat{\Psi}\partial^M\hat{\Psi}\mathrm{d}^{d+1}x. \tag{3}$$

Once the coordinate y is compactified on S^1, we can Fourier expand the D-scalar field so that

$$\hat{\Psi}(x,y) = \sum_n \psi_{(n)}(x)e^{iny/R}. \tag{4}$$

Plugging (4) into (3), it is easy to arrive to

$$S \sim \int \Big(-\frac{1}{2}\partial_\mu\psi_{(0)}\partial^\mu\psi_{(0)} - \sum_{n=1}^{\infty}\Big[\partial_\mu\psi_{(n)}\partial^\mu\psi^\dagger_{(n)} + \frac{n^2}{R^2}\psi_{(n)}\psi^\dagger_{(n)}\Big]\Big)\mathrm{d}^d x. \tag{5}$$

Therefore, a higher-dimensional scalar theory with a compactified extra dimension admits a d-dimensional description consisting of an infinite Kaluza-Klein tower of fields. The spectrum of the theory consists of:

1. One real massless scalar field, called the 0-mode.
2. An infinite number of massive scalar fields with quantized masses inversely proportional to the compactification radius, $m_n = |n|/R$.

In fact, if extra dimensions exist, they may be too small to be explored directly, but their existence may be inferred from patterns imprinted on lower-dimensional physics. These particles could be one of them.

The usual Kaluza-Klein spirit assumes a sufficiently small compactification scale, perhaps of the order of the Planck length $\sim 10^{-35}$ m. Then the associated masses become very high, of the order of the Planck mass $\sim 10^{19}$ GeV, unattainable experimentally by the modern particle accelerators, and thus in a low-energy limit it makes no sense to consider the excited modes ($n \neq 0$).

In this case, where the compactifying manifold is simply S^1, we can consistently neglect [3] these higher mass modes and take only the 0-mode. Such a mechanism suggests as to why we can choose to work with fields independent of y, and accounts for why the existence of this extra dimension is not noted in everyday experience.

Similar reasoning holds for the Fourier expansion of the metric (2). Thus, in the conventional KK-theories one takes $\mathcal{M}^d \times S^1$ (with $R \sim \ell_{\mathrm{Pl.}}$) as D-dimensional space-time and the physical spectrum truncated to the massless sector. The fields that appear in (6) are only the 0-modes (we will ignore the superscript zero).

Coming back to our starting point, the Kaluza-Klein metric ansatz will simply be to take the metric $\hat{g}_{MN}$ to be independent of y, i.e. equating $\partial_y(\dots) = 0$ in all computations. The main point is that from the d-dimensional point of view, the index M, which runs over the $d+1$ values, splits into a range lying in the lower d dimensions, or it takes the value associated with the compactified dimension, y. Therefore, the metric $\hat{g}_{MN}$ splits into $\hat{g}_{\mu\nu}$, $\hat{g}_{\mu y}$ and $\hat{g}_{yy}$. From a d-dimensional observer, these look like a 2-covariant symmetric tensor, a one-form and a scalar field.

In particular, we write the D-dimensional metric in terms of d-dimensional fields $g_{\mu\nu}$, A_μ and ϕ as follows [3]:

$$\mathrm{d}\hat{s}^2 = e^{2\alpha\phi}\mathrm{d}s^2 + e^{2\beta\phi}(A_\mu \mathrm{d}x^\mu + \mathrm{d}y)^2, \tag{6}$$

where α and β are constants that we shall choose conveniently. At first glance, we cannot claim that these fields are the metric tensor of GR and the d-potential vector of Maxwell's EM, after all these have to be solutions of the Einstein's and Maxwell's equations, respectively.

To proceed, and obtain the effective low-dimensional theory no new mathematics needs to be considered. It is now mechanical, but rather lengthy to compute the D-dimensional Christoffel symbols and then the Ricci tensor via the usual known expressions.

We will need the inverse metric, which comes from $\hat{g}_{MP}\hat{g}^{PN} = \hat{\delta}^N_M$. The determinant of the metric in terms of the determinant g, is

$$\sqrt{-\hat{g}} = e^{(\beta+d\alpha)\phi}\sqrt{-g}. \tag{7}$$

Our goal is to express the D-dimensional quantities in terms of the d-dimensional ones. We will not present here all the steps because 10 pages would not be enough.

It does not take long to realize that a good choice of the constants is $\beta = -(d-2)\alpha$ and $\alpha^2 = (2(d-1)(d-2))^{-1}$. Schematically, $\hat{R}_{\mu\nu} = R_{\mu\nu} + \dots$, so $\hat{R} = e^{-2\alpha\phi}R + \dots$, and then, according with (7),

$$\mathcal{L} = \hat{R}\sqrt{-\hat{g}} = e^{(\beta+(d-2)\alpha)\phi}R\sqrt{-g} + \dots \tag{8}$$

To ensure that the dimensionally-reduced lagrangian is Einstein framed, we see that we must choose $\beta = -(d-2)\alpha$. Moreover, a ϕ-kinetic term appears in $\hat{R}$, it turns out to be

$$\sim -(d-1)(d-2)\alpha^2(\partial\phi)^2\sqrt{-g}. \tag{9}$$

Fixing $\alpha^2 = (2(d-1)(d-2))^{-1}$, this kinetic term acquires the canonical normalization. With such choices, once we have computed $\hat{R}$ we finally get

$$\begin{aligned} S = \frac{1}{16\pi\hat{G}}\int \Bigg(& R - \frac{1}{2}(\partial\phi)^2 + (d-3)\alpha\Box\phi - \frac{1}{4} \\ & \times e^{-2(d-1)\alpha\phi}F^2\Bigg)\sqrt{-g}\mathrm{d}^{d+1}x. \end{aligned} \tag{10}$$

Since $\Box\phi$ is just a total derivative in $\mathcal{L}$, we drop it. The $\mathcal{L}$ associated to (10) does not depends on y, and as y is compact, $\int \mathrm{d}y = 2\pi R$. We can identify then, the gravitational constant $\hat{G}$ with: $G\cdot(2\pi R)$. In fact, in a space-time with n extra space dimensions the units of the constant $\hat{G}$ are related with the units of G, via $[\hat{G}] = [G]\cdot L^n$. Except a global factor,

$$S \sim \int \left(R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{-2(d-1)\alpha\phi}F^2\right)\sqrt{-g}\mathrm{d}^d x. \tag{11}$$

At that point, we could be tempted to fix $\phi = 0$ in order to obtain in accurate way the Einstein's GR and Maxwell's theory, but the interaction between $F_{\mu\nu}$ and ϕ prevents this configuration. To see explicitly why, let us work out the field equations. We vary the action

$$\delta S \sim \int \sum_{\text{fields}} \frac{1}{\sqrt{-g}}\frac{\delta S}{\delta\Psi}\delta\Psi\sqrt{-g}\mathrm{d}^d x, \tag{12}$$

denoting the fields collectively by Ψ. The variational principle requires $\delta S = 0$, and every functional derivative must be zero, obtaining:

$$\begin{aligned} & R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \left(\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}(\partial\phi)^2\right) + \frac{1}{2} \\ & \qquad\qquad \times e^{-2(d-1)\alpha\phi}\left(F_{\mu\sigma}F^\sigma_\nu - \frac{1}{4}g_{\mu\nu}F^2\right) \\ & \nabla^\mu\left(e^{-2(d-1)\alpha\phi}F_{\mu\nu}\right) = 0 \\ & \Box\phi = -\frac{1}{2}(d-1)\alpha e^{-2(d-1)\alpha\phi}F^2 \end{aligned} \tag{13}$$

where $F_{\mu\nu} = 2!\partial_{[\mu}A_{\nu]}$ is the Faraday 2-form and $\Box = \nabla^\mu\nabla_\mu$ is the d'Alembert covariant operator. Note that if we set $\phi = 0$, the third equation of motion implies $F^2 = 0$ and the hope of recognizing $F_{\mu\nu}$ as the Maxwell's field-strength tensor would vanish.

The main result is we have obtained a coupled system of gravity, electrodynamics and scalar field theory all in d dimensions, from pure gravity in $d+1$, once we have performed consistently the dimensional reduction on S^1.

Now we come back to II A 1 for a moment. If we study the scalar theory in the framework just discussed, an interaction between the scalar Fourier modes and A_μ will come down, giving rise to the charge quantization. Using (4) and (6), the action (3) (slightly modified: we have to add $\sqrt{-\hat{g}}$ to make it covariant) becomes:

$$\begin{aligned} S \sim \int \Bigg(& -\frac{1}{2}\partial_\mu\psi_{(0)}\partial^\mu\psi_{(0)} - \sum_{n=1}^{\infty}\Big[\left(D_\mu\psi_{(n)}\right) \\ & \times \left(D^\mu\psi_{(n)}\right)^\dagger + e^{\xi\phi}\frac{n^2}{R^2}\psi_{(n)}\psi^\dagger_{(n)}\Big]\Bigg)\sqrt{-g}\mathrm{d}^d x, \end{aligned} \tag{14}$$

where $D_\mu = \partial_\mu - i(n/R)A_\mu$ is the covariant gauge derivative and

$$\xi = \sqrt{\frac{2(d-1)}{d-2}}. \tag{15}$$

We can recognize the coupling constant electric charge: $q_n = n/R$. Therefore, the 0-mode is not only massless but also uncharged, whereas the excited modes are charged.

B. $U(1)$ invariance

Now we focus our attention on how the claimed $U(1)$ symmetry of the lower-dimensional theory emerges.

For this, let us compare the KK ansatz (6), with the most general form of metric in D dimensions, namely $\mathrm{d}\hat{s}^2 = \hat{g}_{MN}(x,y)\mathrm{d}x^M\mathrm{d}x^N$. Clearly, the general form is invariant under arbitrary higher-dimensional general coordinate transformations: $\hat{x}^M \to \hat{\zeta}^{M'}(\hat{x}^N)$, i.e. the isometry group is: $G = \mathrm{Diff}(\mathcal{M}^D)$.

This is not true, however, for our KK ansatz (a general y-dependent coordinate transformation could destroy the y-independence of (6)). In such case, the general covariance has been broken once we restricted our coordinate dependence, supported by the reduction on S^1. Nevertheless, our ansatz is invariant under a restricted class of coordinate transformations (inside $\mathrm{Diff}(\mathcal{M}^D)$), which is no more that a remnant of the broken general symmetry.

Indeed, (6) is invariant under $x^\mu \to \zeta^{\mu'}(x)$, $y \to y$. There is also another transformation which leaves invariant $\mathrm{d}\hat{s}^2$, this is $x^\mu \to x^\mu$, $y \to y + \Lambda(x)$. Under the last one, according to the transformation rule for $\hat{g}_{MN}$, we see that $A'_\mu = A_\mu + \partial_\mu\Lambda$. In other words, the KK ansatz is invariant under a shift on the extra coordinate, accompanied by an Abelian gauge transformation on A_μ.

This is an astonishing fact! This gauge freedom, previously considered internal in some unphysical sense, appears naturally here as a diffeomorphic freedom in the extra dimension.

A fruitful way of looking at the origin of this gauge invariance is as a consequence of the fact that shifts on y are isometries of the metric, i.e. $\partial/\partial y$ is a Killing vector of the metric. Then, the isometry group of the internal circle, say $G_{S^1} = SO(2)$, becomes the gauge group $U(1)$ of the d-dimensional theory, since $SO(2) \simeq U(1)$. From this point of view, the gauge transformation of A_μ arises from the Lie derivative of $\hat{g}_{\mu y}$ along the vector field $\hat{\mathbf{v}} = \Lambda(x)\partial_y$: $(\pounds_{\hat{\mathbf{v}}}\,\hat{\mathbf{g}})_{\mu y} = e^{2\beta\phi}\partial_\mu\Lambda \iff \delta A_\mu = \partial_\mu\Lambda$.

The last result sets an intimate relation between the Killing's vectors of the metric and the gauge character of the fields "belonging" to the internal space. This point of view is particularly useful when one wants to obtain non-Abelian gauge symmetries via KK-reduction (see next section).

All in all, the symmetries of S^1 are observed as $U(1)$ gauge symmetries in the effective d-dimensional world. In this case the isometry group becomes: $G = \mathrm{Diff}(\mathcal{M}^d) \times U(1)$. With all we have seen, we can resume it in a "Kaluza-Klein principle":

"Our space-time may have extra dimensions and the space-time symmetries in those dimensions are seen as internal gauge symmetries from the four-dimensional point of view. In that manner, all symmetries (and thus interactions) could then be unified."

Arrived at that point, performing a reduction on internal space $\mathcal{N}$, suitably chosen, may lead to unify an arbitrary gauge group, Abelian or non-Abelian, with ordinary general relativity in a $(d+n)$-dimensional theory.

The simplest generalization is to consider a reduction from $D = d+n$ to d dimensions on a n-torus. The product manifold is of the form $\mathcal{M}^d \times S^1 \times \cdots n \cdots \times S^1$, i.e. $\mathcal{N} = \mathbb{T}^n = (S^1)^n$. We can propose as an anasatz

$$\mathrm{d}\hat{s}^2 = e^{2\alpha\phi}\mathrm{d}s^2 + e^{2\beta\phi}\tilde{g}_{ab}(\mathrm{d}y^a + A^a_\mu\mathrm{d}x^\mu) \times (\mathrm{d}y^b + A^b_\nu\mathrm{d}x^\nu). \tag{16}$$

Note that a is an internal index, runs from $d+1$ to $d+n$.

The dimensional reduction of such a toroidally compactified Kaluza-Klein theory gives rise to d-dimensional gravitation, n Abelian gauge fields plus $n(n+1)/2$ scalar fields. The gauge group becomes $[U(1)]^n$.

The physics is not much different from the theory above, so it is not a fruitful case. Note that this compactification still gives rise to an Abelian gauge group.

To obtain more interesting theories we must go beyond the toroidal compactification. Something more ambitious is try to compactify on a arbitrary higher-dimensional internal space $\mathcal{N}$, which perhaps leads to non-Abelian groups. For instance, $\mathcal{N}$ could be the group manifold of the Lie group $G_\mathcal{N}$ itself, or a homogeneous space $G_\mathcal{N}/H$ for some subgroup $H \subset G_\mathcal{N}$.

III. NON-ABELIAN GENERALIZATION

If we want a non-Abelian generalization we have to replace S^1 by an arbitrary compact space and proceed analogously as before. The higher-dimensional space-time will be of the form $\mathcal{M}^d \times \mathcal{N}$. The main point is that gauge symmetries in low dimensions arise from isometries generated by the Killing vectors of $\mathcal{N}$, let us call them $K^a_i(y)$.

These Killing vectors form the Lie algebra of the isometry group $G_\mathcal{N}$,

$$[K_i, K_j]^a = K^b_i\partial_b K^a_j - K^b_j\partial_b K^a_i = f^k_{ij}K^a_k \tag{17}$$

for some structure constants. Here, $i,j,k = 1,\ldots,\ell$ are label-indices, denote the ℓ linearly independent Killing vectors of the metric $\tilde{g}_{ab}(y)$, i.e. the metric of the internal space, whereas $a = d+1,\ldots,d+n$ is an internal index.

To provide an ansatz, we need some way to relate $\hat{g}_{\mu a}(x,y)$ (the off-block-diagonal components of the higher-dimensional metric) with the $A^i_\mu(x)$'s. This is provided by the Killing vectors just mentioned. We note that the Killing vectors carry label index i, as well as internal index a. Indeed, $K_{ai} \equiv \tilde{g}_{ab}K^b_i$, and then we can write

$$\hat{g}_{\mu a}(x,y) = \tilde{g}_{ab}K^b_i(y)A^i_\mu(x). \tag{18}$$

A reasonable Kaluza-Klein ansatz for the metric is

$$\mathrm{d}\hat{s}^2 = \mathrm{d}s^2 + \tilde{g}_{ab}(\mathrm{d}y^a + K^a_i A^i_\mu\mathrm{d}x^\mu)(\mathrm{d}y^b + K^b_j A^j_\nu\mathrm{d}x^\nu). \tag{19}$$

Now, guided by the subsection II B, we consider an infinitesimal coordinate transformation generated by the vector field $\hat{\mathbf{v}}(x,y) = \Lambda^i(x)K_i^a(y)\partial_a$ on the coordinates of $\mathcal{N}$:

$$y^a \to y^a + \Lambda^i(x)K_i^a(y). \tag{20}$$

Note that in the S^1 reduction case, $\dim(\mathcal{N}) = 1$, so there is just one internal index. We also only had one Killing vector, ∂_y, and hence: $K_{i=1}^{a=1} \equiv K = 1$. We recover the isometry transformation $y \to y + f(x)$.

The transformation (20) leaves invariant the internal metric and moreover $\mathrm{d}\hat{s}^2$ itself, since $\delta\hat{g}_{\mu a} = (\mathcal{L}_{\hat{\mathbf{v}}}\,\hat{\mathbf{g}})_{\mu a}$ implies the transformation rule:

$$\delta A_\mu^i = \partial_\mu\Lambda^i + f_{kj}^i\Lambda^k A_\mu^j \equiv D_\mu\Lambda^i. \tag{21}$$

The last transformation law can be recognized as the usual Yang-Mills gauge transformation, i.e. the A_μ^i's are non-Abelian gauge fields. One is then assured to find a Yang-Mills like term in the reduced dimensional lagrangian,

$$\mathcal{L}_{\text{Y-M}} \sim \tilde{g}_{ab}K_i^aK_j^bF_{\mu\nu}^iF^{j\mu\nu} \tag{22}$$

with $F_{\mu\nu}^i = 2!\partial_{[\mu}A_{\nu]}^i - f_{jk}^iA_\mu^jA_\nu^k$ [4].

The problem with the scenario above is that it implies that our d-dimensional space cannot be chosen to be flat. In the S^1 case, or more generally, in the toroidal compactification, it is easy to find a vacuum solution that satisfies the field equations of the theory, $\hat{R}_{MN} = 0$.

This is not the case, however, in the non-Abelian model. This arises because the dimensional reduction for a compact internal space with non-Abelian isometries, in general, it will be curved.

If the d-space-time is chosen to be flat, $\hat{R}_{\mu\nu} = 0$, but this implies that $\tilde{R}_{ab} = 0$, which is not allowed in this framework. This problem could be solved introducing "by hand" an appropriate (huge) cosmological constant of the opposite sign or matter fields into the higher-dimensional theory, but this last option destroys somehow the beauty of the Kaluza-Klein idea, which essentially is that Yang-Mills fields appear by dimensional reduction and are not present in the fundamental action. Nevertheless, this and other problems have not stopped people from looking for "realistic Kaluza-Klein theories" giving rise to the Standard Model (SM) gauge group $(U(1) \times SU(2) \times SU(3))$ in $d = 4$.

An important observation in this context is that to reproduce the SM of particle physics via the KK mechanism, it is necessary to have (at least) $\dim(\mathcal{N}) = 7$.

Let us count it. $U(1)$ is the symmetry group of the already known circle S^1, with dimension one. The ordinary 2-sphere S^2 ($\dim(S^2) = 2$), is the lowest-dimensional manifold with symmetry $SU(2)$. The $SU(3)$ symmetry is provided by the complex projective space $\mathbb{CP}^2$, which has dimension four. Therefore, the space $S^1 \times S^2 \times \mathbb{CP}^2$ has $U(1) \times SU(2) \times SU(3)$ symmetry, and it has $n = 7$ dimensions. Although this is not the only possibility, is the simplest one [5]. It implies that the number of dimensions of realistic KK theory should be at least $D = 4 + 7 = 11$.

IV. OUTLOOK AND CONCLUSIONS

In this short introduction to Kaluza-Klein theory, we have seen the possibility to obtain Abelian and non-Abelian gauge fields starting from pure gravity in higher dimensions. All gauge fields in low dimensions arise from components of the higher-dimensional metric, and there would not be explicit gauge fields present in the higher-dimensional space-time.

However, there are strong reasons to believe that our physical world cannot be derived from higher dimensions where there are no fields but gravity present. Besides the problem to obtain non-Abelian gauge fields mentioned in the subsection above, there is an even bigger problem of not obtaining fermions, and hence it seems difficult to sustain the original KK position.

A way to solve both problems may be to assume that some gauge fields and matter fields (fermions) are already present in the higher-dimensional theory. Particularly, a promising example is supergravity with the gauge group $E_8 \times E_8$, which is regarded as the low-energy limit of superstring theory. Typically, these string theories live in ten dimensions, and thus one needs to compactify the theory on a small internal six-dimensional space.

Although the original Kaluza-Klein theory is not the final answer, the concept of extra spatial dimensions seems to play a very important role in modern theoretical physics. In all other respects Kaluza's old idea is alive, doing very well because of the usefulness of the techniques of dimensional reduction, and setting an indispensable part of the toolkit of modern theoretical unification physics.

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